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BRANCHING RANDOM WALKS AND MULTI-TYPE CONTACT-PROCESSES ON THE PERCOLATION CLUSTER OF $\mathbb{Z}^d$

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In this paper we prove that, under the assumption of quasi-transitivity, if a branching random walk on $\mathbb{Z}^d$ survives locally (at arbitrarily large times there are individuals alive at the origin), then so does the same process when restricted to the infinite percolation cluster $\mathcal{C}_\infty$ of a supercritical Bernoulli percolation. When no more than k individuals per site are allowed, we obtain the k-type contact process, which can be derived from the branching random walk by killing all particles that are born at a site where already k individuals are present. We prove that local survival of the branching random walk on $\mathbb{Z}^d$ also implies that for k sufficiently large the associated k-type contact process survives on $\mathcal{C}_\infty$. This implies that the strong critical parameters of the branching random walk on $\mathbb{Z}^d$ and on $\mathcal{C}_\infty$ coincide and that their common value is the limit of the sequence of strong critical parameters of the associated k-type contact processes. These results are extended to a family of restrained branching random walks, that is, branching random walks where the success of the reproduction trials decreases with the size of the population in the target site.

1. Introduction. The branching random walk is a process which serves as a model for a population living in a spatially structured environment [the vertices of a graph $(X, \mathcal{E}(X))$]. Each individual lives in a vertex, breeds and dies at random times and each offspring is placed (according to some rule) in one of the neighboring vertices. Since for the branching random walk (BRW in short) there is no bound on the number of individuals allowed per site, it is natural to consider a modification of the process, namely the multitype contact process, where, for some $k \in \mathbb{N}$, no more than k particles per site are allowed (if $k = 1$ one gets the usual contact process). The multitype contact processes are more realistic models. Indeed, instead of thinking of the vertices of the graph as small portions of the ecosystem where individuals may pile up indefinitely (like in the BRW), here each vertex can host at most k individuals. This is, in particular, true for patchy habitats (each vertex represents a patch of soil) or in host-symbionts interactions (each vertex represents a host on top of which symbionts may live); see, for instance, [3, 4, 6].

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The need for more realistic models also brings random environment into consideration. BRWs in random environment has been studied by many authors; see, for instance, [13, 15, 18, 24, 25, 28]. In many cases the random environment is a random choice of the reproduction law of the process (in some cases there is no death). In our case we put the randomness into the underlying graph. When choosing $(X, \mathcal{E}(X))$, $\mathbb{Z}^d$ is perhaps the first choice that comes to mind, but other graphs are reasonable options. In particular the BRW and the contact process have been studied also on trees [20–22, 26, 29, 33] and on random graphs as Galton–Watson trees [30]. Although $\mathbb{Z}^d$ has clear properties of regularity, which make it a nice case to study, random graphs are believed to serve as a better model for real-life structures and social networks. It is therefore of interest to investigate the behavior of stochastic processes on random graphs, which possibly retain some regularity properties which make them treatable. An example is the small world, which is the space model in [17] and [5], where each vertex has the same number of neighbors. The percolation cluster of $\mathbb{Z}^d$ given by a supercritical Bernoulli percolation, which we denote by $\mathcal{C}_\infty$, has no such regularity, but has a "stochastic" regularity, and its geometry, if viewed at a large scale, does not differ too much from $\mathbb{Z}^d$ (e.g., it is true that, for large N, in many N-boxes of $\mathbb{Z}^d \cap \mathcal{C}_\infty$, there are open paths crossing the box in each direction and these paths connect to crossing paths in neighboring boxes; see [19], Chapter 7). Indeed $\mathcal{C}_\infty$ shares many stochastic properties with $\mathbb{Z}^d$: the simple random walk is recurrent in $d = 1, 2$, transient in $d \geq 3$ and the transition probabilities have the same space–time asymptotics as those of $\mathbb{Z}^d$ (with different constants, [1]); two walkers collide infinitely many often in $d = 1, 2$ and finitely many times in $d \geq 3$ (see [2]); the voter model clusters in $d = 1, 2$ and coexists in $d \geq 3$ (see [6]), just to mention a few facts.

The aim of this paper is to compare the critical parameters of the BRW and of the multitype contact process on the infinite percolation cluster $\mathcal{C}_\infty$ with the corresponding ones on $\mathbb{Z}^d$ (from now on we tacitly assume that the infinite cluster exists almost surely, i.e., that the underlying Bernoulli percolation is supercritical). In order to define these parameters, let us give a formal definition of the processes involved.

Let $(X, \mathcal{E}(X))$ be a graph and $\mu : X \times X \to [0, +\infty)$ *adapted to the graph*, that is, $\mu(x, y) > 0$ if and only if $(x, y) \in \mathcal{E}(X)$. We require that there exists $K < +\infty$ such that $\zeta(x) := \sum_{y \in X} \mu(x, y) \leq K$ for all $x \in X$. Given $\lambda > 0$, the *λ-branching random walk* (λ-BRW or, when λ is not relevant, BRW) is the continuous-time Markov process $\{\eta_t\}_{t \geq 0}$, with configuration space $\mathbb{N}^X$, where each existing particle at x has an exponential lifespan of parameter 1 and, during its life, breeds at the arrival times of a Poisson process of parameter $\lambda\zeta(x)$ and then chooses to send its offspring to y with probability $\mu(x, y)/\zeta(x)$. Thus we associate to μ a family of BRWs, indexed by λ. With a slight abuse of notation, we will say that (X, μ) is a BRW [$\mu(x, y)$ represents the rate at which existing particles at x breed in y]. The BRW is called *irreducible* if and only if the underlying graph is connected. Clearly, any BRW on $\mathbb{Z}^d$ or $\mathcal{C}_\infty$ is irreducible; we note that in their graph structure we possibly admit loops; that is, every vertex might be a neighbor of itself (thus allowing reproduction from a vertex onto itself). If $(Y, \mathcal{E}(Y))$ is a subgraph of $(X, \mathcal{E}(X))$, we denote by $\mu_{|Y}(x, y)$ the map $\mu \cdot \mathbb{1}_{\mathcal{E}(Y)}$. The associated BRW $(Y, \mu_{|Y})$, indexed by λ, is called the *restriction* of (X, μ) to Y and, to avoid cumbersome notation, we denote it by (Y, μ).

Two critical parameters are associated to the continuous-time BRW: the weak (or global) survival critical parameter λ_w and the strong (or local) survival one λ_s. They are defined as

$$\begin{aligned} \lambda_w(x_0) &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}(\exists t : \eta_t = \underline{0}) < 1\}, \\ \lambda_s(x_0) &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}(\exists \bar{t} : \eta_t(x_0) = 0, \forall t \geq \bar{t}) < 1\}, \end{aligned} \tag{1.1}$$

where x_0 is a fixed vertex, $\underline{0}$ is the configuration with no particles at all sites and $\mathbb{P}^{\delta_{x_0}}$ is the law of the process which starts with one individual in x_0. Note that these parameters do not depend on the initial state $\ell\delta_{x_0}$, provided that $\ell > 0$. Moreover, if the BRW is irreducible, then these values do not depend on the choice of x_0 nor on the initial configuration, provided that this configuration is nonzero and finite (i.e., it has a strictly positive, finite number of individuals). When there is no dependence on x_0, we simply write λ_s and λ_w. These parameters depend also on (X, μ): when we need to stress this dependence, we write $\lambda_w(x_0, X, \mu)$ and $\lambda_s(x_0, X, \mu)$ [or simply $\lambda_w(X, \mu)$ and $\lambda_s(X, \mu)$ in the irreducible case]. We refer the reader to Section 2 for how to compute the explicit value of these parameters.

Given (X, μ) and a nonincreasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$, the *restrained branching random walk* (briefly, RBRW) (X, μ, c) is the continuous-time Markov process $\{\eta_t\}_{t\geq 0}$, with configuration space $\mathbb{N}^X$, where each existing particle at x has an exponential lifespan of parameter 1 and, during its life, breeds, as the BRW, at rate $c(0)\zeta(x)$, then chooses to send its offspring to y with probability $\mu(x, y)/\zeta(x)$, and the reproduction is successful with probability $c(\eta(y))/c(0)$. For the RBRW the rate of successful reproductions from x to y, namely $\mu(x, y)c(\eta(y))$ depends on the configuration; for a formal introduction to RBRWs, see [7].

Restrained branching random walks have been introduced in [7] in order to provide processes where the natural competition for resources in an environmental patch is taken into account (since c is nonincreasing, the more individuals are present at a vertex, the more difficult it is for new individuals to be born there). If we imagine that the vertex can host at most N individuals, a natural example of c is represented by the logistic growth $c_N(i) = \lambda(1 - i/N)\mathbb{1}_{[0,N]}(i)$. A more general choice where the parameter N represents the strength of the competition between individuals (the smaller N, the stronger the competition), is given by fixing a nonincreasing $\widetilde{c}$ and letting $c_N(\cdot) := \widetilde{c}(\cdot/N)$. The usual BRWs and multitype contact processes can be seen as particular cases of RBRWs: if $c \equiv \lambda$, the associated RBRW is the λ-BRW; if $c = \lambda\mathbb{1}_{[0,k-1]}$, we call the corresponding RBRW

$$\begin{tikzcd}
 & \lambda_w(\mathcal{C}_\infty,\mu) \arrow[dl] & \lambda_w^k(\mathcal{C}_\infty,\mu) \arrow[l] \arrow[dl] & \\
\lambda_w(\mathbb{Z}^d,\mu) & \lambda_w^k(\mathbb{Z}^d,\mu) \arrow[l] & \lambda_s(\mathcal{C}_\infty,\mu) \arrow[ul] \arrow[dl] & \lambda_s^k(\mathcal{C}_\infty,\mu) \arrow[l] \arrow[ul] \arrow[dl] \\
 & \lambda_s(\mathbb{Z}^d,\mu) \arrow[ul] & \lambda_s^k(\mathbb{Z}^d,\mu) \arrow[l] \arrow[ul] &
\end{tikzcd}$$

FIG. 1. *Order relation between critical values* ($a \to b$ *means* $a \geq b$).

k-type contact process, and we denote it by $\{\eta_t^k\}_{t\geq 0}$. The critical parameters of the k-type contact process are denoted by λ_s^k and λ_w^k.

The order relations between all these critical values are shown in Figure 1; these relations hold for every μ adapted to $\mathbb{Z}^d$.

It has already been proven in [11] that if μ is quasi-transitive on X (a property of regularity, see Definition 2.1), then $\lambda_s^k(X,\mu) \stackrel{k\to\infty}{\longrightarrow} \lambda_s(X,\mu)$, and, if μ is translation invariant on $\mathbb{Z}^d$, then $\lambda_w^k(\mathbb{Z}^d,\mu) \stackrel{k\to\infty}{\longrightarrow} \lambda_w(\mathbb{Z}^d,\mu)$. Analogous results for discrete-time processes can be found in [34], and recently some progress has been made for discrete-time BRWs on Cayley graphs of finitely generated groups; see [27].

When considering BRWs and multitype contact processes on $\mathcal{C}_\infty$, two natural questions arise. First, we wonder whether the critical parameters of the BRW on $\mathcal{C}_\infty$ can be deduced from the ones of the BRW on $\mathbb{Z}^d$; second, whether the parameters of the k-type contact process converge to the corresponding ones of the BRW. Note that even if the BRW $(\mathbb{Z}^d,\mu)$ has good properties of regularity, like quasi-transitivity, its restriction to $\mathcal{C}_\infty$ has none of these properties, and the aforementioned questions are not trivial.

Our main result answers both questions regarding λ_s: for quasi-transitive BRWs on $\mathbb{Z}^d$ the strong critical parameter coincides with the one on $\mathcal{C}_\infty$ (this result was actually already in [11], Theorem 7.1, but here we provide a different proof which can be extended to answer the second question). Moreover the sequence of the strong critical parameters of k-type contact processes restricted to $\mathcal{C}_\infty$ converge to the one of the BRW on $\mathbb{Z}^d$. We note that here we consider only continuous-time processes, but analogous results hold for discrete-time BRWs as well.

THEOREM 1.1. *Let* $(\mathbb{Z}^d,\mu)$ *be a quasi-transitive BRW and* $C_\infty \subseteq \mathbb{Z}^d$ *be the infinite cluster of a supercritical Bernoulli percolation. Then*:

(1) $\lambda_s(\mathcal{C}_\infty,\mu) = \lambda_s(\mathbb{Z}^d,\mu)$ *a.s. with respect to the realization of* $\mathcal{C}_\infty$;

(2) $\lim_{k\to\infty}\lambda_s^k(\mathcal{C}_\infty,\mu) = \lim_{k\to\infty}\lambda_s^k(\mathbb{Z}^d,\mu) = \lambda_s(\mathbb{Z}^d,\mu)$ *a.s. with respect to the realization of* $\mathcal{C}_\infty$.

We observe that the equality $\lim_{k\to\infty}\lambda_s^k(\mathbb{Z}^d,\mu) = \lambda_s(\mathbb{Z}^d,\mu)$ has already been proven in [11], Theorem 5.1. The result for the weak critical parameter can be obtained when $\lambda_w(\mathbb{Z}^d,\mu) = \lambda_s(\mathbb{Z}^d,\mu)$, which is, for instance, true when μ is quasi-transitive and symmetric; see Section 2.

THEOREM 1.2. *Let $(\mathbb{Z}^d, \mu)$ be a quasi-transitive BRW such that $\lambda_w(\mathbb{Z}^d, \mu) = \lambda_s(\mathbb{Z}^d, \mu)$, and let $C_\infty \subseteq \mathbb{Z}^d$ be the infinite cluster of a supercritical Bernoulli percolation. Then, a.s. with respect to the realization of $\mathcal{C}_\infty$, $\lim_{k\to\infty} \lambda_w^k(\mathcal{C}_\infty, \mu) = \lim_{k\to\infty} \lambda_w^k(\mathbb{Z}^d, \mu) = \lambda_w(\mathcal{C}_\infty, \mu) = \lambda_w(\mathbb{Z}^d, \mu)$.*

The fact that whenever a quasi-transitive BRW on $\mathbb{Z}^d$ is locally supercritical (i.e., $\lambda > \lambda_s$), so are the k-type contact processes restricted to $\mathcal{C}_\infty$, whenever k is sufficiently large, also holds for families of RBRWs, where $c_N(\cdot) := c(\cdot/N)$, and c is a given nonnegative function such that $\lim_{z\to 0^+} c(z) = c(0) > \lambda_s(\mathbb{Z}^d, \mu)$.

THEOREM 1.3. *Let $(\mathbb{Z}^d, \mu)$ be a quasi-transitive BRW and $C_\infty \subseteq \mathbb{Z}^d$ be the infinite cluster of a supercritical Bernoulli percolation. Let c be a nonnegative, nonincreasing function such that $\lim_{z\to 0^+} c(z) = c(0) > \lambda_s(\mathbb{Z}^d, \mu)$, and let $c_N(\cdot) := c(\cdot/N)$. Consider the RBRWs $(\mathbb{Z}^d, \mu, c_N)$ and $(\mathcal{C}_\infty, \mu, c_N)$: they both survive locally whenever N is sufficiently large.*

As an application, we have that [6], Theorem 1(2), can be refined; here is the improved statement.

COROLLARY 1.4. *Let $\mu(x, x) = \alpha$ and $\mu(x, y) = \beta/2d$ for all $x \in \mathbb{Z}^d$ and y such that $|x - y| = 1$, where $\alpha \geq 0$ and $\beta > 0$. Consider the RBRW $(\mathcal{C}_\infty, \mu, c_N)$ where $c_N(i) = (1 - i/N)\mathbb{1}_{[0,N]}(i)$. Then:*

(1) *For all $N > 0$, the process dies out if $\alpha + \beta \leq 1$.*
(2) *If $\alpha + \beta > 1$, then the process survives locally, provided that N is sufficiently large.*

To compare with [6], Theorem 1, we recall that the extinction phase, that is, Corollary 1.4(1), was already stated as [6], Theorem 1(1); to ensure survival when $\alpha + \beta > 1$ and N is large, [6], Theorem 1(2), requires that the parameter of the underlying Bernoulli percolation is sufficiently close to 1. This request has now been proven unnecessary, since it suffices that the Bernoulli percolation is supercritical.

2. Basic definitions and preliminaries. Explicit characterizations of the critical parameters are possible. For the strong critical parameter we have $\lambda_s(x) = 1/\limsup_{n\to\infty} \sqrt[n]{\mu^{(n)}(x, x)}$ (see [10], Theorem 4.1, [12], Theorem 3.2(1)) where $\mu^{(n)}(x, y)$ are recursively defined by $\mu^{(n+1)}(x, y) = \sum_{w\in X} \mu^{(n)}(x, w)\mu(w, y)$ and $\mu^{(0)}(x, y) = \delta_{xy}$. As for $\lambda_w(x)$, it is characterized in terms of solutions of certain equations in Banach spaces (see [10], Theorem 4.2); moreover, $\lambda_w(x) \geq 1/\liminf_{n\to\infty} \sqrt[n]{\sum_{y\in X} \mu^{(n)}(x, y)}$ [10], Theorem 4.3, [12], Theorem 3.2(2). The last inequality becomes an equality in a certain class of BRWs which contains quasi-transitive BRWs (see [10], Proposition 4.5, [12], Theorem 3.2(3)) The definition of quasi-transitive BRW is the following.

DEFINITION 2.1. (X, μ) is a quasi-transitive BRW (or μ is a quasi-transitive BRW on X) if and only if there exists a finite set of vertices $\{x_1, \dots, x_r\}$ such that for every $x \in X$ there exists a bijection $f : X \to X$ such that $f(x_j) = x$ for some j and μ is f-invariant, that is, $\mu(w, z) = \mu(f(w), f(z))$ for all w, z.

Note that if f is a bijection such that μ is f-invariant, then f is an automorphism of the graph $(X, \mathcal{E}(X))$. In many cases λ_s coincides with λ_w. For quasi-transitive and symmetric BRWs [i.e., $\mu(x, y) = \mu(y, x)$ for all x, y], it is known that $\lambda_s = \lambda_w$ is equivalent to amenability ([12], Theorem 3.2, which is essentially based on [10] and [33], Theorem 2.4). Amenability is a slow growth condition; see [33], Section 1, for the definition of amenable graph and [12], Section 2, where m_{xy} stands for $\mu(x, y)$, for the definition of amenable BRW. It is easy to prove that a quasi-transitive BRW is amenable if and only if the underlying graph is amenable. Examples of amenable graphs are $\mathbb{Z}^d$ along with its subgraphs. Therefore, every quasi-transitive and symmetric BRW on $\mathbb{Z}^d$ or $\mathcal{C}_\infty$ has $\lambda_s = \lambda_w$.

Another sufficient condition is the following, where symmetry is replaced by reversibility [i.e., the existence of measure ν on X such that $\nu(x)\mu(x, y) = \nu(y)\mu(y, x)$ for all x, y]. It is a slight generalization of [9], Proposition 2.1 and easily extends to discrete-time BRWs.

THEOREM 2.2. *Let (X, μ) be a continuous-time BRW, and let $x_0 \in X$. Suppose that there exists a measure ν on X and $\{c_n\}_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$*

$$\begin{cases} \nu(y)/\nu(x_0) \le c_n, & \forall y \in B(x_0, n), \\ \nu(x)\mu(x, y) = \nu(y)\mu(y, x), & \forall x, y \in X, \end{cases}$$

where $B(x_0, n)$ is the ball of center x_0 and radius n. If $\sqrt[n]{c_n} \to 1$ and $\sqrt[n]{|B(x_0, n)|} \to 1$ as $n \to \infty$, then $\lambda_s(x_0) = \lambda_w(x_0)$.

PROOF. If we denote by $[x_0]$ the irreducible class of x_0, then it is easy to show that $\mu^{(n+1)}(x_0, x_0) = \sum_{w \in [x_0]} \mu^{(n)}(x_0, w)\mu(w, x_0)$. Note that $\nu(x)\mu^{(n)}(x, y) = \nu(y)\mu^{(n)}(y, x)$ for all $x, y \in X$, $n \in \mathbb{N}$. In particular since $\nu(x_0) > 0$, then ν is strictly positive on $[x_0]$, and $[x_0]$ is a final class. Thus, for all $x, y \in [x_0]$ we have $\mu(x, y) > 0$ if and only if $\mu(y, x) > 0$. This means that the subgraph $[x_0]$ is nonoriented; hence the natural distance is well defined and so is the ball $B(x_0, n)$. Moreover, by the Cauchy–Schwarz inequality, the supermultiplicative property of $\mu^{(n+1)}(x_0, x_0)$ and Fekete's lemma, for all $n \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned} \big(1/\lambda_s(x_0)\big)^{2n} &\ge \mu^{(2n)}(x_0, x_0) = \sum_{y \in [x_0]} \mu^{(n)}(x_0, y)\mu^{(n)}(y, x_0) \\ &= \sum_{y \in B(x_0, n)} \big(\mu^{(n)}(x_0, y)\big)^2 \frac{\nu(x_0)}{\nu(y)} \ge \frac{(\sum_y \mu^{(n)}(x_0, y))^2}{c_n |B(x_0, n)|}. \end{aligned}$$

Hence

$$\frac{1}{\lambda_s(x_0)} \le \frac{1}{\lambda_w(x_0)} \le \liminf_n \sqrt[n]{\sum_y \mu^{(n)}(x_0, y)}$$
$$= \liminf_n \sqrt[2n]{\frac{(\sum_y \mu^{(n)}(x_0, y))^2}{c_n |B(x_0, n)|}} \le \frac{1}{\lambda_s(x_0)}. \qquad \square$$

The condition $\sqrt[n]{|B(x,n)|} \to 1$ is usually called *subexponential growth*. Examples of subexponentially growing graphs are euclidean lattices $\mathbb{Z}^d$ or d-dimensional combs; see [8] for the definition. The assumptions of Theorem 2.2 are, for instance, satisfied, on subexponentially growing graphs, by irreducible BRWs with a reversibility measure ν such that $\nu(x) \le C$ for all $x \in X$ and for some $C > 0$.

One of the tools in the proof of our results is the fact that if the BRW survives locally on a graph X; it also survives locally on suitable large subsets $X_n \subset X$. This follows from the spatial approximation theorems which have been proven in a weaker form in [11], Theorem 3.1, for continuous-time BRWs and in a stronger form in [34], Theorem 5.2, for discrete-time BRWs. The proofs rely on a lemma on nonnegative matrices and their convergence parameters, which in its original form can be found in [32], Theorem 6.8. We restate here both the lemma and the approximation theorem. It is worth noting that the irreducibility assumptions which were present in [11, 32, 34] are here dropped.

Given a nonnegative matrix $M = (m_{xy})_{x,y \in X}$, let $R(x, y) := 1/\limsup_{n\to\infty} \sqrt[n]{m^{(n)}(x, y)}$ be the family, indexed by x and y, of the convergence parameters [$m^{(n)}(x, y)$ are the entries of the nth power matrix M^n]. Note that, as recalled earlier in this section, $\lambda_s(x)$ coincides with the convergence parameter $R(x, x)$ of the matrix $(\mu(x, y))_{x,y\in X}$. Given a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$ let $\liminf_{n\to\infty} X_n := \bigcup_n \bigcap_{k\ge n} X_k$.

LEMMA 2.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a general sequence of subsets of X such that $\liminf_{n\to\infty} X_n = X$, and suppose that $M = (m_{xy})_{x,y\in X}$ is a nonnegative matrix. Consider a sequence of nonnegative matrices $M_n = (m(n)_{xy})_{x,y\in X_n}$ such that $0 \le m(n)_{xy} \le m_{xy}$ for all $x, y \in X_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} m(n)_{xy} = m_{xy}$ for all $x, y \in X$. Then for all $x_0 \in X$ we have ${}_nR(x_0, x_0) \to R(x_0, x_0)$ [${}_nR(x_0, x_0)$ being a convergence parameter of the matrix M_n].*

Clearly, if M is irreducible, then $R(x, y) = R$ does not depend on $x, y \in X$, and for all $x_0 \in X$ we have ${}_nR(x_0, x_0) \to R$. One can repeat the proof of [34], Theorem 5.2, noting that, since $R(x_0, x_0)$ depends only on the values of the irreducible class of $[x_0]$ then ${}_nR(x_0, x_0) \to R(x_0, x_0)$ without requiring the whole matrix M to be irreducible. The following theorem is the application of Lemma 2.3 to the spatial approximation of continuous-time BRWs [an analogous result holds for discrete-time BRWs (see [34], Theorem 5.2), where we can drop the irreducibility assumption].

THEOREM 2.4. *Let (X,μ) be a continuous-time BRW, and let us consider a sequence of continuous-time BRWs $\{(X_n,\mu_n)\}_{n\in\mathbb{N}}$ such that $\liminf_{n\to\infty} X_n = X$. Let us suppose that $\mu_n(x,y)\le\mu(x,y)$ for all $x,y\in X_n$, $n\in\mathbb{N}$ and $\mu_n(x,y)\to\mu(x,y)$ as $n\to\infty$ for all $x,y\in X$. Then, for all $x_0\in X$, $\lambda_s(x_0,X_n,\mu_n)\ge\lambda_s(x_0,X,\mu)$ and $\lambda_s(x_0,X_n,\mu_n)\to\lambda_s(x_0,X,\mu)$ as $n\to\infty$.*

3. Proofs and applications. Before proving our main results, we need to prove some preparatory lemmas. The first lemma gives a useful expression for the expected value of the progeny living at time t at vertex y of a particle which was at x at time 0. Its proof, which can be found in [7], Section 3, is based on the construction of the process by means of its generator as done in [23]. The key to the proof is the fact that the expected value is the solution of a system of differential equations. Neither Bertacchi, Posta and Zucca [7] nor Liggett and Spitzer [23] construct the process in our setting; nevertheless it is not difficult to adapt their construction to our case; the interested reader can find the details in Remark 3.8.

LEMMA 3.1. *For any λ-BRW on a graph X we have that*

$$\mathbb{E}(\eta_t(y)|\eta_0=\delta_x)=e^{-t}\sum_{n=0}^{\infty}\mu^{(n)}(x,y)\frac{(\lambda t)^n}{n!}.$$

The expected number of descendants of generation n at y at time t (of a particle at x at time 0) is

$$e^{-t}\mu^{(n)}(x,y)\frac{(\lambda t)^n}{n!},$$

and the expected number of descendants of generation n at γ_n at time t (of a particle at γ_0 at time 0) along the path $\gamma=(\gamma_0,\dots,\gamma_n)$ is

$$e^{-t}\prod_{i=0}^{n-1}\mu(\gamma_i,\gamma_{i+1})\frac{(\lambda t)^n}{n!}$$

(in this case only the particles of generation $i+1$ at γ_{i+1} which are children of particles of generation i at γ_i are taken into account, for all $i=0,\dots,n-1$).

The following lemma shows that whenever a BRW on $\mathbb{Z}^d$ survives locally [i.e., $\lambda>\lambda_s(\mathbb{Z}^d,\mu)$], it also survives locally if restricted to boxes of sufficiently large radius. We denote by $B(m)=[-m,m]^d\cap\mathbb{Z}^d$ the box centered at 0 and by $x+B(m)$ its translate centered at x.

LEMMA 3.2. *Let μ be a BRW on $\mathbb{Z}^d$. Then for all $\lambda > \lambda_s(\mathbb{Z}^d, \mu)$ and for all $x \in \mathbb{Z}^d$, there exists $m(x) \in \mathbb{N}$ such that for all $m \geq m(x)$, $\lambda > \lambda_s(x + B(m), \mu)$. Moreover, if μ is quasi-transitive, then there exists m_0 such that for all $m \geq m_0$, $\lambda > \sup_{x \in \mathbb{Z}^d} \lambda_s(x + B(m), \mu)$.*

PROOF. Let $X = \mathbb{Z}^d$, $X_n := (x + B(n))$ and $\mu_n := \mu \cdot \mathbb{1}_{X_n \times X_n}$. By Theorem 2.4 there exists m such that $\lambda > \lambda_s(X_n, \mu_n)$ for all $n \geq m$.

If μ is quasi-transitive, there exists a finite set of vertices $\{x_1, \dots, x_r\}$ as in Definition 2.1. It is clear that $\lambda_s(A, \mu) = \lambda_s(f(A), \mu)$ for all $A \subset \mathbb{Z}^d$ and for every automorphism f such that μ is f-invariant. Given $\lambda > \lambda_s(\mathbb{Z}^d, \mu)$, for every i there exists m_i such that $\lambda > \lambda_s(x_i + B(m_i), \mu)$. Take $m \geq m_0 := \max_{i=1,\dots,r} m_i$: by monotonicity $\lambda_s(x_i + B(m_i), \mu) \geq \lambda_s(x_i + B(m), \mu)$ for all i. Thus $\lambda > \max_{i=1,\dots,r} \lambda_s(x_i + B(m), \mu)$. Let $x \in \mathbb{Z}^d$ and f as in Definition 2.1 such that $f(x_j) = x$ for some j. Then $\lambda_s(x + B(m), \mu) = \lambda_s(f(x + B(m)), \mu) = \lambda_s(x_j + B(m), \mu)$ and $\max_{i=1,\dots,r} \lambda_s(x_i + B(m), \mu) = \sup_{x \in \mathbb{Z}^d} \lambda_s(x + B(m), \mu)$. $\square$

The following lemma states that for any λ-BRW on a graph X, with $\lambda > \lambda_s(x)$ the expected value of the number of particles in a given site, grows exponentially in time.

LEMMA 3.3. *Let μ be a BRW on a graph X, $x \in X$ and $\lambda > \lambda_s(x)$. Let $\{\eta_t\}_{t \geq 0}$ be the associated λ-BRW. Then there exists $\varepsilon = \varepsilon(x, X)$, $C = C(x, X)$ such that*

$$\mathbb{E}\big(\eta_t(x) | \eta_0 = \delta_x\big) \geq C e^{\varepsilon t} \qquad \forall t \geq 0. \tag{3.1}$$

PROOF. We follow the idea in the proof of [11], Lemma 5.1. We prove (3.1) for all $t \geq t_1$ for some t_1; the assertion then follows by replacing C with $\min(C, C_1)$, where $C_1 = \min_{t \in [0, t_1]} e^{-\varepsilon t} \mathbb{E}(\eta_t(x) | \eta_0 = \delta_x)$ which exists and it is strictly positive by continuity [since $t \mapsto \mathbb{E}(\eta_t(x) | \eta_0 = \delta_x)$ is a solution of a differential equation].

Since $\lambda > \lambda_s(x)$, then $\lambda \sqrt[n]{\mu^{(n)}(x, x)} > 1$ for some n. Therefore there exist $n_0 \geq 1$ and $\varepsilon_1 > 0$ such that $\mu^{(n_0)}(x, x) > (\frac{1+\varepsilon_1}{\lambda})^{n_0}$. By the supermultiplicativity of the sequence $\mu^{(n)}(x, x)$, for all $r \in \mathbb{N}$,

$$\mu^{(n_0 r)}(x, x) \geq \left(\frac{1 + \varepsilon_1}{\lambda} \right)^{n_0 r}.$$

Recalling Lemma 3.1, we get

$$\mathbb{E}\big(\eta_t(x) | \eta_0 = \delta_x\big) \geq e^{-t} \sum_{r \geq 0} \frac{((1 + \varepsilon_1) t)^{n_0 r}}{(n_0 r)!}.$$

FIG. 2. *The portion of $\mathbb{Z}^d$ where we restrict the BRW.*

Let $\bar{\lambda} := 1 + \varepsilon_1$. We can write a lower bound for the summands in the previous series:

$$\frac{(\bar{\lambda}t)^{n_0 r}}{(n_0 r)!} \geq \frac{\bar{\lambda}t - 1}{(\bar{\lambda}t)^{n_0} - 1} \cdot \left\{ \frac{(\bar{\lambda}t)^{n_0 r}}{(n_0 r)!} + \frac{(\bar{\lambda}t)^{n_0 r + 1}}{(n_0 r + 1)!} + \cdots + \frac{(\bar{\lambda}t)^{n_0(r+1)-1}}{(n_0(r+1)-1)!} \right\},$$

whence, for all $t \geq t_1$ and for some $t_1 > 0$, the following holds:

$$\mathbb{E}\big(\eta_t(x)|\eta_0 = \delta_x\big) \geq e^{-t} \cdot \frac{\bar{\lambda}t - 1}{(\bar{\lambda}t)^{n_0} - 1} \cdot e^{\bar{\lambda}t} \geq \frac{\bar{\lambda}t - 1}{(\bar{\lambda}t)^{n_0} - 1} \cdot e^{\varepsilon_1 t} \geq e^{\varepsilon_1 t/2}. \qquad \square$$

The following lemma states that, for the BRW on $\mathbb{Z}^d$, given two vertices x and y (also at a large distance), the expected progeny at y of a particle at x, can be made arbitrarily large, after a sufficiently large time, even if the process is restricted to a large box centered at x plus a fixed path from x to y; see Figure 2. The idea of the proof is that the BRW can stay inside the box until the expected number of particles at x is large, and then move along the path toward y.

LEMMA 3.4. *Let μ be a BRW on $\mathbb{Z}^d$, $x \in \mathbb{Z}^d$, $\lambda > \lambda_s(\mathbb{Z}^d, \mu)$. Fix $M, \delta > 0$, and choose m such that $\lambda > \lambda_s(x + B(m), \mu)$. Then there exists $T = T(x, m, M, \delta)$ such that*

$$\mathbb{E}\big(\widetilde{\eta}_t(y)|\widetilde{\eta}_0 = \delta_x\big) \geq 1 + \delta, \tag{3.2}$$

for all $t \geq T$, γ path of length $l \leq M$ with $\gamma_0 = x$, $\gamma_l = y$, where $\{\widetilde{\eta}_t\}_{t \geq o}$ is the BRW restricted to $(x + B(m)) \cup \gamma$. Moreover, if μ is quasi-transitive, we can choose m and T independent of x such that (3.2) holds for all $x \in \mathbb{Z}^d$.

PROOF. Fix $t_2 > 0$. We use the Markov property of the BRW (and the superimposition with respect to the initial condition) and apply Lemma 3.1

$$\begin{aligned}\mathbb{E}\big(\widetilde{\eta}_{t_1+t_2}(y)|\widetilde{\eta}_0 = \delta_x\big) &\geq \mathbb{E}\big(\widetilde{\eta}_{t_1}(x)|\widetilde{\eta}_0 = \delta_x\big) \cdot e^{-t_2} \prod_{i=0}^{l-1} \mu(\gamma_i, \gamma_{i+1}) \frac{(\lambda t_2)^l}{l!} \\ &\geq \mathbb{E}\big(\widetilde{\eta}_{t_1}(x)|\widetilde{\eta}_0 = \delta_x\big) \cdot e^{-t_2} \frac{(\lambda t_2 \alpha)^l}{l!} \geq \mathbb{E}\big(\widetilde{\eta}_{t_1}(x)|\widetilde{\eta}_0 = \delta_x\big) \cdot \widetilde{\varepsilon},\end{aligned}$$

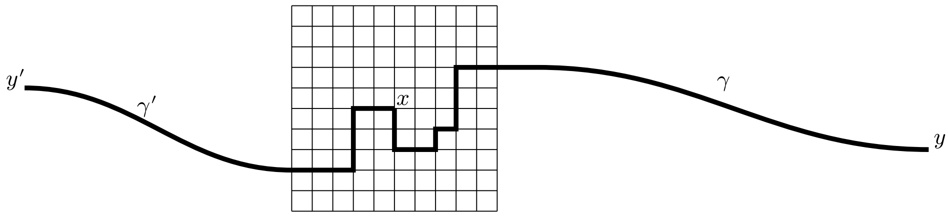

FIG. 3. *From ℓ individuals at x to ℓ individuals at y and y'.*

where $0 < \alpha = \alpha(x, M) = \min\{\mu(\gamma'_i, \gamma'_{i+1}) : i = 0, \dots, l' - 1, \gamma'$ path of length $l' \le M, \gamma'_0 = x\}$ and $0 < \widetilde{\varepsilon} = \widetilde{\varepsilon}(x, t_2, m, M) = \min\{e^{-t_2}(\lambda t_2 \alpha)^l / l! : l \le M\}$. Since $\widetilde{\eta}$ restricted to $x + B(m)$ survives locally, by Lemma 3.3,

$$\mathbb{E}(\widetilde{\eta}_{t_1+t_2}(y) | \widetilde{\eta}_0 = \delta_x) \ge C e^{\varepsilon t_1} \cdot \widetilde{\varepsilon} \ge 1 + \delta,$$

for all sufficiently large t_1 depending on x, m, M and δ. Fix t_1 and define $T(x, m, M, \delta) := t_1 + t_2$.

If μ is quasi-transitive, take $\{x_1, \dots, x_r\}$ and m_i as in the proof of Lemma 3.2. Take $m := \max_{i=1,\dots,r} m_i$ and $T = \max_{i=1,\dots,r} T(x_i, m, M, \delta)$, and the proof is complete. $\square$

In the next lemma we prove that given x, y and y', if we start the process with l particles at x, after a sufficiently large time, with arbitrarily large probability, we will have l particles both at y and at y', even if we restrict the process to a large box centered at x plus a fixed path from x to y and a fixed path from x to y'; see Figure 3. The proof relies on Lemma 3.4 and the central limit theorem.

LEMMA 3.5. *Let μ be a BRW on $\mathbb{Z}^d$, and let x, λ and m as in Lemma 3.4. Fix $M, \varepsilon > 0$. Then choosing $T = T(x, m, M, 1)$ as in Lemma 3.4, for all $t \ge T$ there exists $\ell(\varepsilon, x, m, M, t) \in \mathbb{N}$ and*

$$\mathbb{P}(\widetilde{\eta}_t(y) \ge \ell, \widetilde{\eta}_t(y') \ge \ell | \widetilde{\eta}_0(x) = \ell) > 1 - \varepsilon, \tag{3.3}$$

for all $\ell \ge \ell(\varepsilon, x, t)$, $l, l' \le M$, γ, γ' paths of length l and l' from x to y and to y', respectively, where $\widetilde{\eta}_t$ is the BRW restricted to $(x + B(m)) \cup \gamma \cup \gamma'$. Moreover, if μ is quasi-transitive, we can choose m and T independent of x and $\ell(\varepsilon, m, M)$ such that (3.3) holds for all $x \in \mathbb{Z}^d$ when $t = T$.

PROOF. By monotonicity it suffices to prove the result with the event $(\widetilde{\eta}_0 = \ell\delta_x)$ in place of $(\widetilde{\eta}_0(x) = \ell)$.

Let $X = (x + B(m)) \cup \gamma \cup \gamma'$. Let us denote by $\{\xi_t\}_{t \ge 0}$ the BRW, restricted to X, starting from $\xi_0 = \delta_x$. By Lemma 3.4, there exists T such that $\mathbb{E}(\xi_t(z) | \xi_0 = \delta_x) > 2$ for all $t \ge T$, $z = y, y'$. A realization of our process is $\widetilde{\eta}_t = \sum_{j=1}^{\ell} \xi_{t,j}$ where $\{\xi_{t,j}(y)\}_{j \in \mathbb{N}}$ is an i.i.d. family of copies of $\{\xi_t\}_{t \ge 0}$. Fix $z \in \{y, y'\}$. Since $\xi_{t,j}$ is stochastically dominated by a continuous time branching process with birth rate $\lambda \sup_w \sum_v \mu(w, v) < +\infty$, it is clear that $\mathrm{Var}(\xi_{t,j}(z)) =: \sigma_{t,z}^2 < +\infty$ (note that the variance depends on x). Thus by the central limit theorem, if ℓ is sufficiently large,

$$\frac{\varepsilon}{4} \geq \left| \mathbb{P}\left(\sum_{j=1}^{\ell} \xi_{t,j}(z) \geq s \middle| \xi_{0,j} = \delta_x, \forall j = 1, \ldots, \ell \right) \right.$$
$$\left. - 1 + \phi\left(\frac{s - \ell \mathbb{E}(\xi_t(z) | \xi_0 = \delta_x)}{\sqrt{\ell} \sigma_{t,z}} \right) \right|$$

uniformly with respect to $s \in \mathbb{R}$, where ϕ is the cumulative distribution function of the standard normal. Whence there exists $\ell(\varepsilon, x, m, M, z, t)$ such that, for all $\ell \geq \ell(\varepsilon, x, m, M, z, t)$,

$$\mathbb{P}(\widetilde{\eta}_t(z) \geq \ell | \widetilde{\eta}_0 = \ell \delta_x) \geq 1 - \phi\left(\sqrt{\ell} \frac{1 - \mathbb{E}(\widetilde{\eta}_t(y) | \widetilde{\eta}_0 = \delta_x)}{\sigma_{t,z}} \right) - \frac{\varepsilon}{4} \geq 1 - \frac{\varepsilon}{2},$$

since $\sqrt{\ell}(1 - \mathbb{E}(\widetilde{\eta}_t(z) | \widetilde{\eta}_0 = \delta_x) / \sigma_{t,z} \to -\infty$ as $\ell \to +\infty$. Take $\ell(\varepsilon, x, m, M, t) := \ell(\varepsilon, x, m, M, y, t) \vee \ell(\varepsilon, x, m, M, y', t)$. Hence (3.3) follows.

If μ is quasi-transitive, take $\{x_i\}_{i=1}^r$ and $\{m_i\}_{i=1}^r$ as in the proof of Lemma 3.4. It suffices to choose $m := \max_{i=1,\ldots,r} m_i$ and $T = \max_{i=1,\ldots,r} T(x_i, m, M)$. □

We say that a subset A of $\mathbb{Z}^d$ is contained in $\mathcal{C}_\infty$ if all the vertices are connected to $\mathcal{C}_\infty$ and all the edges (x, y), with $x, y \in A$, are open. The following is a lemma on the geometry of $\mathcal{C}_\infty$ which states that $\mathcal{C}_\infty$ contains a biinfinite open path where one can find large boxes at bounded distance from each other.

LEMMA 3.6. *Let us consider a supercritical Bernoulli percolation on $\mathbb{Z}^d$. For every $m \in \mathbb{N}$ there exists $M = M(m) > 0$ such that, a.s. with respect to the percolation measure, the infinite percolation cluster $\mathcal{C}_\infty$ contains a pairwise disjoint family $\{B_j\}_{j=-\infty}^{+\infty}$ with the following properties*:

(1) *there exists $\{x_j\}_{j=-\infty}^{+\infty}$, $x_j \in \mathbb{Z}^d$ for all j, and $B_j = x_j + B(m)$ for all j;*

(2) *there is a family of open paths $\{\pi_j\}_{j=-\infty}^{+\infty}$ such that $x_j \stackrel{\pi_j}{\longleftrightarrow} x_{j+1}$, and $|\pi_j| \leq M$ for all j.*

PROOF. For every $N \in \mathbb{N} \setminus \{0\}$, we define the N-partition of $\mathbb{Z}^d$ as the collection $\{2Nx + B(N) : x \in \mathbb{Z}^d\}$.

We use [31], Proposition 4.1, which holds also for $d = 2$ according to [14], Proposition 11. In order to achieve in [14], Proposition 11, the same generality of [31], Proposition 4.1, one has to take into account also a general family of events $\{V_\Gamma\}_\Gamma$ (indexed on the boxes of the collection of the N-partitions as $N \in \mathbb{N} \setminus \{0\}$) satisfying equation (4.4) of [31]. This can be easily done by noting that the inequality (4.25) of [31] still holds in the case $d = 2$. From now on, when we refer to [31], Proposition 4.1, we mean this "enhanced" version which holds for $d \geq 2$.

We define $V_\Gamma :=$ "there exists a seed $x_\Gamma + B(N^{1/2}) \subseteq \Gamma$" where by *seed* we mean a box with no close edges in the percolation process (to avoid a cumbersome notation, we omit the integer part symbol $\llcorner \cdot \lrcorner$ in the side length). Note that V_Γ is measurable with respect to the σ-algebra of the percolation process restricted to Γ, thus independent from the rest of the process. Given a box Γ of side length N, by partitioning it into disjoint boxes of side length $N^{1/2}$, we obtain the following upper estimate $\mathbf{\Phi}[(V_\Gamma^c)] \leq (1-p)^{(N/N^{1/2})^d} = (1-p)^{N^{d/2}} \to 0$ as $n \to \infty$, where $\mathbf{\Phi}$ is the law of the Bernoulli percolation on $\mathbb{Z}^d$ with parameter p. This implies that $\{V_\Gamma\}_\Gamma$ satisfies equation (4.4) of [31].

We know from [31], Proposition 4.1, that, for any fixed supercritical Bernoulli percolation on $\mathbb{Z}^d$ ("microscopic" percolation in this context), for every sufficiently large N the renormalized percolation ("macroscopic" percolation from now on) stochastically dominates a Bernoulli site percolation of arbitrarily large parameter. Let us describe briefly, how the macroscopic percolation is constructed from the microscopic one. For every $k = \pm 1, \dots, \pm d$ we define the kth face of the box $B(N)$ as the set $\{y \in \mathbb{Z}^d : y(|k|) = \operatorname{sgn}(k)N\}$, that is, the face in the kth direction. Roughly speaking, in the renormalized macroscopic process a box $\Gamma := 2Nx + B(N)$ ($x \in \mathbb{Z}^d$) of the N-partition is *occupied* if and only if:

(1) there exists a unique *crossing cluster*, that is, a set of open edges containing open paths connecting any two opposite faces of the boxes,

(2) any open path γ such that $\operatorname{diam}(\gamma) \geq N^{1/2}/10$ is connected to the crossing cluster,

(3) for every $k = \pm 1, \dots, \pm d$, if D_k is a translation of the box $B(N/4)$ centered at the middle point of the kth face of the box $2Nx + B(N)$, then there exists a path connecting the k face and the $-k$ face of D_k,

(4) V_Γ holds.

If Γ and $\Gamma' := 2Nx' + B(N)$ are occupied, where $x'(i) - x(i) = \delta_{i,k}$ (i.e., Γ' is adjacent to Γ in the kth direction), then the crossing clusters of these two boxes are connected by (2), (3) and by noting that $D_k = D'_{-k}$ [where D_k and D'_{-k} are the boxes described in (3) related to Γ and Γ', resp.].

We consider $N > m^2 \vee 23$ ($N \geq 24$ is required in [14, 31]). Thus the seed $x_\Gamma + B(N^{1/2}) \subseteq \Gamma$, when it exists, it contains an open path of diameter $2dN^{1/2} > N^{1/2}/10$, and hence it is connected to the crossing cluster in Γ by construction of the renormalized process; see [31], Section 4.2 or [14], Section 5. Moreover it contains a translated box $x_\Gamma + B(m)$. By [31], Proposition 4.1, given a supercritical Bernoulli percolation on $\mathbb{Z}^d$, for all sufficiently large N, there exists an infinite open cluster of boxes in the "macroscopic" renormalized graph; see [31] for details on the definition of occupied box. This implies the existence of an infinite cluster (in the original microscopic percolation) which contains a seed no smaller than the

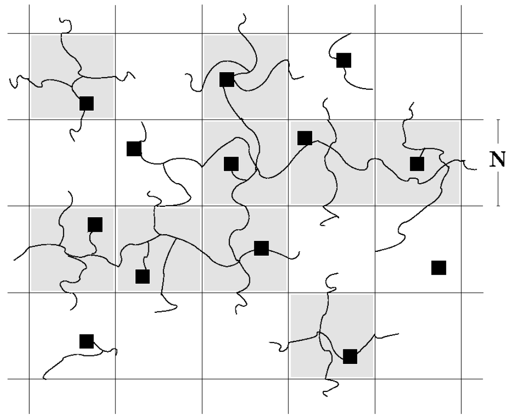

FIG. 4. *The occupied boxes in the renormalized percolation.*

box $B(N^{1/2})$ in each occupied box of the macroscopic cluster; see Figure 4 where the grayed boxes are occupied.

By uniqueness, this infinite microscopic cluster coincides with $\mathcal{C}_\infty$. Clearly, by construction, the centers of the seeds in two adjacent occupied "macroscopic" boxes are connected (in $\mathcal{C}_\infty$) by a path contained into these two boxes; clearly, the length of such a path is bounded from above by $M := 2N^d$. Since the percolation cluster in the renormalized "macroscopic" process contains a bi-infinite self-avoiding path of open boxes, the proof is complete. $\square$

PROOF OF THEOREM 1.1. Even though (1) follows easily from (2) and the diagram in Figure 1, we prove it separately in order to introduce the key idea, which will be used later to prove (2), in a simpler case. (1) Since $\mathcal{C}_\infty$ is a subgraph of $\mathbb{Z}^d$, we have that $\lambda_s(\mathcal{C}_\infty, \mu) \geq \lambda_s(\mathbb{Z}^d, \mu)$ (remember that these critical values do not depend on the finite, nonzero initial condition). Take $\lambda > \lambda_s(\mathbb{Z}^d, \mu)$: our goal is to prove that $\lambda > \lambda_s(\mathcal{C}_\infty, \mu)$. By Lemma 3.2 we know that there exists (a smallest) m such that $\lambda > \lambda_s(x + B(m), \mu)$ for all $x \in \mathbb{Z}^d$. Let M, $\{x_j\}_{j=-\infty}^{+\infty}$, $\{\pi_j\}_{j=-\infty}^{+\infty}$ as in Lemma 3.6. By Lemma 3.5 and by monotonicity, for all $\varepsilon > 0$ there exist T and ℓ such that

$$\mathbb{P}\big(\widetilde{\eta}_T(x_{j-1}) \geq \ell, \widetilde{\eta}_T(x_{j+1}) \geq \ell | \widetilde{\eta}_0(x_j) = \ell\big) > 1 - \varepsilon,$$

where $\{\widetilde{\eta}_t\}_{t\geq 0}$ is the BRW (starting from the initial condition $\ell\delta_{x_0}$) restricted to $\mathcal{A} = \bigcup_{j=-\infty}^{\infty}(x_j + B(m)) \cup \pi_j$ (which, by Lemma 3.6, is a subset of $\mathcal{C}_\infty$ which exists a.s. whenever the cluster is infinite). We recall that the critical parameters of the BRW are independent of $\ell > 0$.

We construct a process $\{\xi_t\}_{t\geq 0}$ on $\mathcal{A}$, by iteration of independent copies of $\{\widetilde{\eta}_t\}_{t\geq 0}$ on time intervals $[nT, (n+1)T)$, and we associate it with a percolation process ϱ on $\mathbb{Z} \times \vec{\mathbb{N}}$ ($\mathbb{Z}$ representing space and $\vec{\mathbb{N}}$ representing time), where $\vec{\mathbb{N}}$ is the oriented graph on $\mathbb{N}$ where all edges are of the type $(n, n+1)$. We index the family

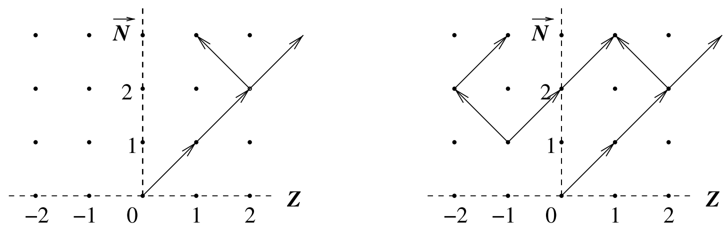

FIG. 5. *A realization of the cluster in the percolation ϱ (left) and ϱ_2 (right).*

of copies needed as $\{\widetilde{\eta}_{(i,j)}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$ and use $\widetilde{\eta}_{(i,j),t}$ when also the dependence on time has to be stressed; moreover $\widetilde{\eta}_{(i,j),0} = \ell\delta_{x_i}$ for all i, j. The construction will be made in such a way that $\widetilde{\eta}_t$ stochastically dominates ξ_t for all $t \ge 0$ and, whenever in the percolation process ϱ we have that $(0,0) \stackrel{\varrho}{\to} (j,n)$, then $\xi_{nT}(x_j) \ge \ell$.

Let us begin our iterative construction with its first step. Start $\{\widetilde{\eta}_{(0,0),t}\}_{t\ge 0}$, and let $\xi_t = \widetilde{\eta}_{(0,0),t}$ for $t \in [0,T]$; thus $\xi_0 = \widetilde{\eta}_{(0,0),0} = \ell\delta_{x_0}$. In the percolation process, the edge $(0,0) \stackrel{\varrho}{\to} (j,1)$, $j = \pm 1$, is open if $\widetilde{\eta}_{(0,0),T}(x_j) \ge \ell$. Now suppose that we constructed $\{\xi_t\}_{t\ge 0}$ for $t \in [0,nT]$; to construct it for $t \in (nT,(n+1)T]$, we put $\xi_t = \sum_{h\in[-n,n]:\, \xi_{nT}(x_h)\ge\ell} \widetilde{\eta}_{(h,n),t-nT}$ for all $t \in (nT,(n+1)T]$. In the percolation ϱ, for all (i,n) such that there is an open path $(0,0) \stackrel{\varrho}{\to} (i,n)$, we connect $(i,n) \stackrel{\varrho}{\to} (j,n+1)$, $j = i \pm 1$, if $\widetilde{\eta}_{(i,n),T}(x_j) \ge \ell$.

In order to show that, by choosing ℓ sufficiently large, with positive probability there is an open path in the percolation ϱ, from $(0,0)$ to $(0,n)$ for infinitely many n (which means that at arbitrarily large times there are at least ℓ individuals at x_0 in the original process), we need a comparison with a one-dependent oriented percolation ϱ_2 on $\mathbb{Z} \times \vec{\mathbb{N}}$. This new percolation ϱ_2 is obtained by "enlarging" ϱ in the following way: for all $(i,n) \in \mathbb{Z} \times \vec{\mathbb{N}}$, we connect $(i,n) \stackrel{\varrho_2}{\to} (j,n+1)$, $j = i \pm 1$, if $\widetilde{\eta}_{(i,n),T}(x_j) \ge \ell$. Note that ϱ differs from ϱ_2 simply in the fact that in ϱ the opening procedure takes place only from sites already connected to $(0,0)$ (see Figure 5). By induction on n, this coupled construction implies that there exists a ϱ_2-open path from $(0,0)$ to (i,n) if and only if there exists a ϱ-open path from $(0,0)$ to (i,n). By Lemma 3.5, for all $\varepsilon > 0$, by choosing ℓ sufficiently large, we have that for ϱ_2 the probability of opening all edges from (i,n) is at least $1-\varepsilon$. Let us choose ε such that the one-dependent percolation ϱ_2 dominates a supercritical independent (oriented) Bernoulli percolation. According to Lemma 3.7, the infinite Bernoulli percolation cluster in the cone $\{(i,j) : j \ge |i|\}$ contains infinitely many sites of type $(0,n)$ almost surely. Hence, by coupling, there is a positive probability that the one-dependent infinite percolation cluster contains infinitely many sites of type $(0,n)$ as well.

The first claim follows since the λ-BRW on $\mathcal{C}_\infty$ (starting with ℓ particles at x_0) stochastically dominates $\{\widetilde{\eta}_t\}_{t\ge 0}$, which in turn dominates $\{\xi_t\}_{t\ge 0}$, and by comparison with ϱ_2 we know that $\xi_{nT}(x_0) \ge \ell$ for infinitely many $n \in \mathbb{N}$.

(2) Let us now consider the k-type contact process $\{\eta_t^k\}_{t\geq 0}$. Take $\lambda > \lambda_s(\mathbb{Z}^d, \mu)$, m as in the previous step, and $\mathcal{A}$ (along with $\{x_j\}_{j=-\infty}^{+\infty}$ and $\{\pi_j\}_{j=-\infty}^{+\infty}$) given by Lemma 3.6 as before. Consider the restriction $\{\widetilde{\eta}_t^k\}_{t\geq 0}$ of the k-type contact process to $\mathcal{A}$. Let us begin by proving that $\lambda > \lambda_s^k(\mathcal{C}_\infty, \mu)$ for all k sufficiently large. To this aim it is enough to prove that for the above fixed λ, $\{\widetilde{\eta}_t^k\}_{t\geq 0}$ survives locally for all k sufficiently large.

Fix $\varepsilon > 0$, and let T and ℓ be given by Lemma 3.5, such that

$$\mathbb{P}\big(\widetilde{\eta}_T(y) \geq \ell, \widetilde{\eta}_T(y') \geq \ell | \widetilde{\eta}_0 = \ell\delta_x\big) > 1 - \varepsilon.$$

Let N_T^x be the total progeny up to time T (including the initial particles), in the BRW $(\mathcal{A}, \mu)$, starting from ℓ individuals at site x. Define N_T as the total number of individuals ever born (including the initial particles), up to time T, in a branching process with rate λK, starting with ℓ individuals at time 0: N_T stochastically dominates N_T^x for all $x \in \mathcal{A}$. We have

$$\begin{aligned}
&\mathbb{P}\big(\widetilde{\eta}_T(y) \geq \ell, \widetilde{\eta}_T(y') \geq \ell, N_T^x \leq n | \widetilde{\eta}_0 = \ell\delta_x\big) \\
&\quad \geq \mathbb{P}\big(\widetilde{\eta}_T(y) \geq \ell, \widetilde{\eta}_T(y') \geq \ell | \widetilde{\eta}_0 = \ell\delta_x\big) + \mathbb{P}\big(N_T^x \leq n | \widetilde{\eta}_0 = \ell\delta_x\big) - 1 \\
&\quad \geq \mathbb{P}\big(\widetilde{\eta}_T(y) \geq \ell, \widetilde{\eta}_T(y') \geq \ell | \widetilde{\eta}_0 = \ell\delta_x\big) + \mathbb{P}(N_T \leq n) - 1 > 1 - 2\varepsilon,
\end{aligned}$$

for all $n \geq \bar{n}$ where $\bar{n}$ satisfies $\mathbb{P}(N_T \leq \bar{n}) > 1 - \varepsilon$ ($\bar{n}$ is independent of x).

Define an auxiliary process $\{\bar{\eta}_t\}_{t\in[0,T]}$ obtained from $\{\widetilde{\eta}_t\}_{t\geq 0}$ by killing all newborns after that the total progeny has reached size $\bar{n}$. This implies that, in the process $\{\bar{\eta}_t\}_{t\in[0,T]}$, the progeny does not reach sites at distance larger than $\bar{n}$ from the ℓ ancestors, nor it goes beyond the $\bar{n}$th generation. In particular, when started from $\ell\delta_x$, the processes $\{\bar{\eta}_t\}_{t\in[0,T]}$ and $\{\widetilde{\eta}_t\}_{t\geq 0}$ coincide, up to time T, on the event $(N_T \leq \bar{n})$. Thus

$$\begin{aligned}
&\mathbb{P}\big(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell | \bar{\eta}_0 = \ell\delta_x\big) \\
&\quad \geq \mathbb{P}\big(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell, N_T^x \leq n | \bar{\eta}_0 = \ell\delta_x\big) \\
&\quad = \mathbb{P}\big(\widetilde{\eta}_T(y) \geq \ell, \widetilde{\eta}_T(y') \geq \ell, N_T^x \leq n | \widetilde{\eta}_0 = \ell\delta_x\big) > 1 - 2\varepsilon.
\end{aligned}$$

The percolation construction of step (1) can be repeated by using i.i.d. copies of $\{\bar{\eta}_t\}_{t\in[0,T]}$ instead of $\{\widetilde{\eta}_{(i,j)}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$. Call $\{\bar{\xi}_t\}_{t\geq 0}$ the corresponding process constructed from these copies as $\{\xi_t\}_{t\geq 0}$ was constructed from $\{\widetilde{\eta}_{(i,j)}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$. As in step (1), by choosing ε sufficiently small, we have that $\bar{\xi}_{nT}(x_0) \geq \ell$ for infinitely many $n \in \mathbb{N}$.

Let H be the number of paths in $\mathbb{Z}^d$ of length $\bar{n}$, containing the origin: H is an upper bound for the number of such paths in $\mathcal{C}_\infty$ or in $\mathcal{A}$. It is easy to show that $\bar{\xi}_t(x) \leq H\bar{n}$ for all t and x. Thus if we take $k \geq H\bar{n}$, then $\widetilde{\eta}_t^k$ stochastically dominates $\bar{\xi}_t$. The supercriticality of the percolation on $\mathbb{Z} \times \vec{\mathbb{N}}$ associated to $\bar{\xi}$ implies that $\{\widetilde{\eta}_t^k\}_{t\geq 0}$ survives locally. The inequality $\lambda > \lambda_s^k(\mathcal{C}_\infty, \mu)$ follows since $\{\eta_t^k\}_{t\geq 0}$ stochastically dominates $\{\widetilde{\eta}_t^k\}_{t\geq 0}$. This implies that, for every sufficiently large k, $\lambda_s(\mathbb{Z}^d, \mu) \le \lambda_s^k(\mathbb{Z}^d, \mu) \le \lambda_s^k(\mathcal{C}_\infty, \mu) < \lambda$ (see Figure 1), and the proof is complete. $\square$

We discuss here an interesting result on oriented percolation which is used in the proofs of Theorem 1.1 and [11], Theorem 5.1.

LEMMA 3.7. *Consider a supercritical Bernoulli oriented percolation in $\mathbb{Z} \times \vec{\mathbb{N}}$: almost every infinite cluster contains an infinite number of vertices of type $(0, n)$. The same holds for a supercritical Bernoulli oriented percolation in $\mathbb{N} \times \vec{\mathbb{N}}$.*

PROOF. Let us begin with the percolation in $\mathbb{Z} \times \vec{\mathbb{N}}$. By $(i, j) \to (i', j')$ we mean that there is an open path in the percolation from (i, j) to (i', j'), while by $(i, j) \to \infty$ we mean that there is an infinite open path from (i, j). By using the translation invariance of the percolation law, the results of [16], Section 3 [in particular equations (7) and (11)] imply that a.s. if $(i, j) \to \infty$, then for all $i' \in \mathbb{Z}$ there exists $j' \ge j$ such that $(i, j) \to (i', j')$. This implies that a.s., with respect to the percolation measure, every vertex satisfying $(i, j) \to \infty$ has the *covering property*; that is, if we project on the first coordinate (i.e., on $\mathbb{Z}$) the vertices in the cluster "branching" from it, we obtain the whole set $\mathbb{Z}$. Let $J := \min\{j' : (i, j') \to \infty \text{ for some } i \in \mathbb{Z}\}$ be the bottom level of the infinite cluster; for all $j \ge J$ there exists $i \in \mathbb{Z}$ such that $(i, j) \to \infty$. Consider the set of infinite clusters which contain just a finite number of vertices of type $(0, n)$; denote by $(0, N)$ the "highest" of such vertices (N depending on the cluster), then there exists i (depending on the cluster) such that $(i, N + 1) \to \infty$. This implies that $(i, N + 1)$ is in the infinite cluster a.s. By the covering property above, the probability that there are no paths from $(i, N + 1)$ to $(0, j')$ (for some $j' \ge N + 1$) is 0. Thus the set of infinite clusters containing a finite number of vertices of type $(0, n)$ has probability 0.

In the case of the oriented Bernoulli percolation on $\mathbb{N} \times \vec{\mathbb{N}}$, we proceed analogously. Observe that this percolation can be obtained from the oriented Bernoulli percolation on $\mathbb{Z} \times \vec{\mathbb{N}}$ by deleting all edges outside $\mathbb{N} \times \vec{\mathbb{N}}$; this defines a coupling between these percolation processes. We use here $(i, j) \rightsquigarrow (i', j')$ for an open path in the oriented percolation in $\mathbb{N} \times \vec{\mathbb{N}}$ and again $(i, j) \to (i', j')$ for a path in the oriented percolation in $\mathbb{Z} \times \vec{\mathbb{N}}$ (clearly the existence of the first one implies the existence of the last one). Since the infinite cluster in $\mathbb{Z} \times \vec{\mathbb{N}}$ is unique a.s., then the infinite cluster $\mathbb{N} \times \vec{\mathbb{N}}$ is a.s. a subset of the previous one. In the supercritical case, for all $j \ge \min\{j' : (i, j') \rightsquigarrow \infty \text{ for some } i \in \mathbb{N}\}$, there exists $i \in \mathbb{N}$ such that $(i, j) \rightsquigarrow \infty$; thus $(i, j) \to \infty$. We proved before that a.s. $(i, j) \to (0, j')$ for some $j' \ge j$. Let us take the smaller of such j's, say j'_0. Hence $(i, j) \to (0, j'_0)$ and the connecting path is entirely contained in $\mathbb{N} \times \vec{\mathbb{N}}$, thus $(i, j) \rightsquigarrow (0, j'_0)$. The conclusion follows as in the previous case. $\square$

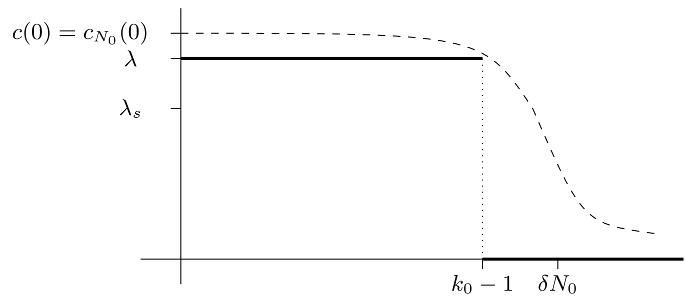

FIG. 6. *Comparison between c_{N_0} (dashed) and the k_0-type contact process (thick).*

PROOF OF THEOREM 1.2. If follows easily from Theorem 1.1, the hypothesis $\lambda_s(\mathbb{Z}^d, \mu) = \lambda_w(\mathbb{Z}^d, \mu)$ and the diagram shown in Figure 1. $\square$

PROOF OF THEOREM 1.3. Let $\varepsilon > 0$ such that $c(0) - \varepsilon > \lambda_s(\mathbb{Z}^d, \mu)$. By the assumptions on c, there exists $\delta > 0$ such that $c(z) > c(0) - \varepsilon$ for all $z \in [0, \delta]$. From Theorem 1.1 we know that there exists k_0 such that the k-type contact process $(\mathcal{C}_\infty, \mu)$ associated with $\lambda := c(0) - \varepsilon$ survives locally, for all $k \geq k_0$. Moreover, there exists N_0 such that $\delta N > k_0 - 1$ for all $N \geq N_0$. Since $c_N(i) \geq \lambda \mathbb{1}_{[0,k-1]}(i)$ for all $i \in \mathbb{N}$ (see Figure 6), by coupling we have local survival for the RBRWs $(\mathcal{C}_\infty, \mu, c_N)$ for all $N \geq N_0$. $\square$

PROOF OF COROLLARY 1.4. (1) It suffices to note that the total number of individuals is dominated by the total number of particles in a continuous-time branching process with breeding parameter $\alpha + \beta$; for the details, see [6], Theorem 1(1).

(2) Note that μ is translation invariant, hence quasi-transitive. The claim follows from Theorem 1.3 since $\lambda_s(\mathbb{Z}^d, \mu) = (\alpha + \beta)^{-1}$ and $c(0) = 1$. $\square$

REMARK 3.8. In [23] the process is constructed by means of a semigroup of operators on $\mathrm{Lip}(\mathcal{W})$ (the space of Lipschitz functions on the configuration space $\mathcal{W}$). In [7] this technique is applied to the construction of the restrained BRWs where $(\mu(x, y))_{x,y \in X}$ is a stochastic matrix adapted to a graph with bounded geometry. Our definition of μ is more general. The only difference between the construction needed here and those in [7, 23] consists in the choice of the configuration space $\mathcal{W}$ and its norm; we refer to [23] for the notation and details. As in [7, 23] we consider the space $\mathcal{W} := \{\eta \in \mathbb{N}^X : \sum_{x \in X} \eta(x)\alpha(x) < +\infty\}$ where its metric is defined by $\|\eta - \bar{\eta}\| := \sum_{x \in X} |\eta(x) - \bar{\eta}(x)|\alpha(x)$. Our choice of the positive function $\alpha : X \to (0, +\infty)$ is made in such a way that $\sum_{y \in X} \mu(x, y)\alpha(y) \leq \widetilde{K}\alpha(x)$ for all $x \in X$ (and some fixed $\widetilde{K} > 0$). There are many ways to do this: a possible choice is

$\alpha(x) := \sum_{n=0}^{\infty} \widetilde{K}^{-n} \sum_{y\in X} \mu^{(n)}(x,y)b(y)$ where $b: X \to (0,+\infty)$ is a fixed positive, bounded function and $\widetilde{K} > \sup_{x\in X} \limsup_{n\to\infty} \sqrt[n]{\sum_{y\in X} \mu^{(n)}(x,y)}$ [take, e.g., $\widetilde{K} > \sup_{x\in X} \sum_{y\in X} \mu(x,y)$]. Once α is chosen, the rest of the construction of the process is carried on as in [7, 23]. In particular the system of differential equations satisfied by $\{\mathbb{E}(\eta_t(y)|\eta_0 = \delta_x)\}_{x\in X}$ can be explicitly derived from [23], Lemmas 2.12 and 2.16(e).

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