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Switched Adaptation Strategies for Integral Sliding Mode Control: Theory and Application

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Summary
Integral Sliding Mode control is an interesting approach, as it can maintain the good chattering alleviation property of higher-order sliding modes while making the reaching phase less critical and keeping the controlled system trajectory on a suitably selected sliding manifold since the initial time instant. In order to make such a method more robust and to improve its flexibility by the adaptation of its parameters to the current system condition, in this paper a switched strategy is proposed. Specifically, the Suboptimal Second Order Sliding Mode algorithm is considered as a basis in its integral formulation, and the switching strategy is designed by partitioning the so-called auxiliary system state space in a finite number of regions. The proposed method allows one to improve the transient performance by adapting the gains through these regions, thus implying an energy saving capability. The proposal is theoretically analyzed and, in order to test its performance, the control of the lateral dynamics of ground vehicles is used as a case study. Specifically, yaw-rate tracking is considered, as it is made difficult by parametric uncertainties and nonlinear effects that arise especially with large steering angles. Extensive simulation tests are carried out using standard validation maneuvers, which favorably witness the performance of the new control algorithm.

KEYWORDS:
Sliding mode control, switched control, vehicle dynamics, uncertain systems

1 | INTRODUCTION

Sliding Mode Control (SMC) owes its success and diffusion mainly to its capabilities of rejecting the so-called matched disturbances,1 and to its ease of implementation. Yet, because of the discontinuous nature of the resulting control law, one of its main drawbacks is the so-called chattering effect,2 which results in high frequency oscillations of the controlled variable induced by the discontinuities of the control variable. Such chattering must in general be alleviated as it induces wear and tear that significantly limit the life cycle of the actuators. To overcome the chattering limitation, so-called Higher Order Sliding Mode (HOSM) controllers have been proposed, see e.g., 345678. Such control schemes are designed so that, after a transient phase, a sliding mode is enforced that involves not only the so-called sliding variable, but also its time derivatives. By doing so it confines the discontinuity, which is necessary to steer the sliding variable to zero in a finite time, to a given time derivative of the control variable, so that the control signal actually fed into the plant is continuous. As a by-product, all the time derivatives of the sliding variables involved in the HOSM control laws are also steered to zero in finite time.
As they allow to obtain a continuous control action, HOSM control approaches are applicable in a wide range of practical applications, see e.g., [10][11][12][13][14][15][16][17] and references therein. Yet, as discussed in [18], during the so-called reaching phase, which is of finite but, in general, unpredictable length, the controlled system is still sensitive to the uncertain terms, as the matched disturbances are formally canceled out only when the controller confines the system over the sliding manifold.

In this paper, a new Sliding Mode (SM) control algorithm is proposed. The idea is that of combining two existing methods, the Integral SM (ISM) approach presented in [19] and the Switched Suboptimal Second Order Sliding Mode (S-SSOSM) algorithm introduced in [20] to improve the performance of the base algorithms and yield a robust and flexible control solution. ISM has been widely adopted in the literature thanks to its capability of enforcing a sliding mode since the initial time instant. More precisely, the motivation to introduce the integral manifold was to remove the reaching phase by adjusting the initial values of the output of the system. Its application results in a system that mimics a system controlled via any stabilizing high level controller when all the matched uncertainties are rejected. For instance, in some recent papers, these beneficial effects have been exploited in combination with other strategies such as Model Predictive Control (MPC), giving rise to new robust MPC algorithms. Moreover, the suppression of the reaching phase allows one to evaluate the so-called equivalent control from the initial time, so that the so-called “perturbation estimator” capability of the algorithm can be exploited, becoming powerful for applications such as fault-tolerant controllers for various complex control systems (see for instance [21][22] and the references therein). Finally, particularly interesting is the use of ISM to cope with both matched and unmatched uncertain terms [23]. In [21], for instance, it is proved that, with a suitable sliding manifold, the minimization of the effect of the disturbance terms is guaranteed, that is the matched disturbances are completely rejected and the unmatched ones are not amplified. In [20], instead, a modification of ISM control is discussed and an efficient solution is finally suggested to improve the performance in terms of chattering alleviation.

The control law proposed herein retains all the good properties of the original SSOSM approach introduced in [20] also in terms of chattering alleviation, but thanks to the capability of imposing a desired system dynamics during the reaching phase, a prescribed transient time is achieved. This feature is highly beneficial in practical applications, as it provides the needed degree of freedom to act upon the time span during which the 2-sliding mode is not enforced, thus making the controlled system robust from the initial time instant.

While having such an interesting effect on the initial transient phase, though, the ISM approach has no capability of adapting its action to possible different requirements associated to different regions of the state-space, thus often resulting in an unnecessarily high control authority even when the trajectory is close to the origin. In [19] an alternative sliding variable which takes into account the average value of the discontinuous input is proposed for alleviating chattering, without any reduction of the control gain. Differently from [19], to obtain extra freedom in shaping the convergence phase, the ISM controller is herein extended to encompass a switched formulation, which allows adapting the controller parameters and tuning them differently according to the current region of the state space of the auxiliary system to which the current trajectory belongs, see also [20]. Such a combined control algorithm retains the good features of both approaches [19][20], allowing to outperform them when applied to challenging application case studies. More specifically, since in this paper second-order systems are considered and the main challenge is to provide robustness of the controlled system from the initial time instant, while maintaining the good property of chattering alleviation typical of higher-order sliding modes, a switched version of the Integral Suboptimal Second Order Sliding Mode (ISSOSM) in [20] is proposed. Note that in [20] the algorithm was not capable to adapt the control gains in different regions of the state space, thus not having the flexibility needed to accommodate different design objectives when moving towards the desired equilibrium. In the following we refer to this capability as “switching adaptation mechanism”, which represents the main novelty of the paper, proving the stability properties of the closed-loop switched nonlinear system. Furthermore, devising a switched algorithm is an effective way to achieve performance enhancement with benefits in terms of energy saving, which makes the proposal appealing for practical applications where the use of efficient control methods is required.

To highlight the good performance of the approach, and to offer a detailed comparison with other methods, the proposed control law is designed and tested considering a case study from the automotive field: to this end, the lateral control of ground vehicle dynamics is considered, with special reference to yaw-rate control. As a matter of fact, the vehicle dynamics of interest in this case are speed-dependent, and significant parametric uncertainties may occur. Thus, the controller performance are evaluated on specific maneuvers that are commonly used in lateral vehicle dynamics controller testing, and then they are quantitatively compared with those yielded by ISSOSM and SSOSM alone, to prove the potential of the new approach.

A preliminary version of this work has been presented in [20]. Specifically, with respect to [20] this paper offers the formal analysis of the closed-loop properties and a more extended simulation testing of the proposed solution.

The structure of the paper is as follows. Section 2 introduces the needed preliminaries and notation. Section 3 presents the novel control approach proposed in this work and Section 4 formally analyzes its closed-loop properties. Finally, Section 5 presents the
lateral vehicle dynamics control problem in terms of yaw-rate tracking, and assesses the performance of the proposed algorithm comparing it to other control strategies. Both qualitative and quantitative considerations allow us to highlight the performance of the proposed approach.

2 | PROBLEM FORMULATION

Consider a plant which can be described by the single-input system affine in the control variable, i.e.,

\[
\begin{align*}
\dot{x}(t) &= \phi(x(t)) + \gamma(x(t))u(t) \\
y(t) &= \sigma(x(t))
\end{align*}
\]

where \( x \in \Omega (\Omega \subset \mathbb{R}^n \text{ bounded}) \) is the state vector, the value of which at the initial time instant \( t_0 \) is \( x(t_0) = x_0 \), and \( u \in \mathbb{R} \) is a scalar input, while \( \phi(x(t)) : \Omega \rightarrow \mathbb{R}^n \) and \( \gamma(x(t)) : \Omega \rightarrow \mathbb{R}^n \) are uncertain bounded functions of class \( C^1 \). Define a suitable output function \( \sigma(x(t)) : \Omega \rightarrow \mathbb{R}^n \) of class \( C^2 \). This function will be the so-called sliding variable and, letting \( r \) be the relative degree, i.e., the order of the time derivative of the sliding variable to explicitly obtain the control variable \( u \), the following assumptions hold.

**Assumption 1.** If system (1) is controlled via a suitable designed control law \( u \), then, in a finite time \( t_r \) (the so-called reaching time), \( \sigma(x(t_r)) = 0 \forall x_0 \in \Omega \) and \( \sigma(x(t)) = 0 \forall t > t_r \).

**Assumption 2.** System (1) is complete in \( \Omega \) and, \( \forall x_0 \) and \( \forall u \), \( x(t) \) is defined for almost all \( t \in \mathbb{R} \). Moreover, system (1) has an uniform and time invariant relative degree \( r = 1 \) and admits a normal form in \( \Omega \).

Moreover, by virtue of Assumption 2 there exists a global diffeomorphism of the form \( \Phi(x) : \Omega \rightarrow \mathbb{R}^n \), that is

\[
\Phi(x) = \begin{pmatrix} \zeta \\ z \end{pmatrix}
\]

such that, by artificially increasing the relative degree \( r \), one has

\[
\begin{align*}
\dot{\zeta} &= a(\zeta, z) \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= f(\zeta, z) + g(\zeta, z)w \\
\dot{w} &= w \\
\sigma &= z_1 \\
\end{align*}
\]

with \( w \) being an auxiliary control, and \( \zeta \in \mathbb{R}^{n-r} \) being the internal state. According to [27], the previous system (3b) is called “auxiliary system” and it is assumed that \( a(\cdot), f(\cdot), g(\cdot) \) are continuous functions whose properties are available as declared in the following assumption.

**Assumption 3.** There exist positive known constants \( F, G_{\text{min}}, G_{\text{max}} \) such that

\[
\begin{align*}
|f(\zeta, z)| &\leq F \\
0 &< G_{\text{min}} \leq g(\zeta, z) \leq G_{\text{max}}.
\end{align*}
\]

Furthermore, the internal state is assumed to be not affected by finite escape time phenomena and it is such that the zero dynamics \( \dot{\zeta} = a(\zeta, 0) \) is asymptotically stable.

Remark 1. Note that, according to the HOSM control literature, depending on the natural relative degree of system (1), the auxiliary control \( w \) can be equal to the original control \( u \) or equal to some derivative of \( u \) by artificially increasing the relative degree of the system, with positive effects and performance in terms of “chattering alleviation”[27,31].

Moreover, in order to introduce the proposed approach, making reference to [20,32], we consider also the following assumptions.
**Assumption 4.** Assume that the state space $\mathcal{Z}$ of system (3b) is partitioned into $k$ regions $R_i$, $i = 1, \ldots, k$, i.e.,

$$R_i := \{(z_{1,i}, z_{2,i}) \in \mathbb{R}^2 : \xi_{1,i} \leq z_{1,i} \leq \xi_1, \xi_{2,i} \leq z_{2,i} \leq \xi_2 \}$$

with $\xi_{j,i} < \xi_{j,i+1} < 0$ and $\xi_{j,i} > \xi_{j,i+1} > 0$, $j = 1, 2$, $i = 1, 2, \ldots, k - 1$. Finally, we introduce the regions $\mathcal{Z}_i = R_i \setminus R_{i+1}$, $i = 1, \ldots, k - 1$ and $\mathcal{Z}_k = \mathcal{R}_k$, which are such that $\bigcup_{i=1}^{k} \mathcal{Z}_i = \mathcal{Z}$, and we assume that in each of them it is possible to define different upper and lower bounds for the uncertainties.

This is reasonable in practice where the boundedness of the uncertain terms, due to the finite physical nature of the involved signals, can be proved using for instance, when possible, a trivial trial-and-error procedure. Note that, this approach may cause an over estimation of the bounds, thus possibly implying slightly more conservative control strategies. Moreover, note that only one of the regions, namely the innermost one $\mathcal{Z}_k$, contains the origin. Now, an other assumption on the uncertainty description inside each region needs to be introduced.

**Assumption 5.** In the regions $\mathcal{Z}_i$, $i = 1, \ldots, k$, the uncertain terms are upper-bounded by known constants determined by taking into account the shape of the regions and, making reference to (1)-(5), i.e., $\forall i = 1, \ldots, k$, we can write

$$|f(z(t))| \leq \overline{f}_i,$$  \hspace{1cm} (7)

$$0 < \overline{g}_{\min,i} \leq g(z(t)) \leq \overline{g}_{\max,i}.$$  \hspace{1cm} (8)

As for the outermost region, we need an assumption on the norm of the initial condition in order to ensure that it is bounded, even though arbitrarily large, so that the bounds can be defined also in that region.

Relying on (3b)-(8), one is in the position to introduce the control problem to solve.

**Problem 1.** Design a feedback control law such that $\forall x_0 \in \Omega$, $\exists t_r > 0$ such that $\sigma(x(t)) = 0$, $\forall t \geq t_r$ in spite of the uncertainties.

Before presenting the proposed Switched Integral Suboptimal Second Order Sliding Mode (S-ISSOSM) algorithm, we introduce some preliminary notions on the adopted sliding mode control framework to allow the reader familiarizing with the needed concepts and notation.

### 2.1 Preliminaries on Sliding Mode Control

Since the increased relative degree of the auxiliary system (3b) is equal to 2, in the literature, the original control objective of zeroing the sliding variable in finite time, while guaranteeing a chattering alleviation effect, is typically attained by using HOSM of second order\(^{19,18,28}\). These control laws force the system trajectory to reach in finite time the 2-sliding manifold $z_1 = z_2 = 0$, and there to remain $\forall t \geq t_r$. This problem can be solved by any 2-sliding mode controller of the type

$$u(t) = U_{\max} \Psi(z_1, z_2)$$

where $\Psi$ is a discontinuous function, and $U_{\max} > 0$ is suitably chosen\(^{27}\) so as to ensure the finite time convergence of the state trajectories.

Moreover, in order to improve the robustness property of classical sliding mode control strategies, recently, Integral Sliding Mode (ISM) methods\(^{19,18,28}\), which enable to generate an ideal sliding mode of the controlled system starting from the initial time instant $t_0$, represent an effective solution. These methods consist in the definition of an auxiliary sliding variable $\Sigma$ and of a function $\varphi(t)$, hereafter called “transient function”, suitably chosen to fulfill some restrictions on the transient time, that is the reaching phase. So, thanks to the existence of a sliding mode on the integral sliding manifold since the initial time instant, the controlled system is invariant with respect to matched uncertainties from $t_0$. Analogously, the integral sliding mode can be enforced by using a control law of the same type of (9), but depending on $\Sigma$ and $\Sigma$. 
3 | THE PROPOSED SWITCHED SLIDING MODE CONTROL

In this section, the S-ISSOSM control methodology is presented. The core idea is that of tuning a ISSOSM controller with switching gains \( U_{\text{max},i} \) for each region of the state space, which is determined by different uncertainty levels and/or by possibly different control objectives.

Consider system (3b), with the state partitioned as in (6) and such that for each \( z \in \mathcal{Z}_i, i = 1, \ldots, k, f(z(t)) \) and \( g(z(t)) \) satisfy constraints (7)-(8). Starting from (15), the proposed control approach is essentially based on the definition of the so-called “transient function” as follows

\[
\begin{align*}
\varphi(t) = (t - t_0)^2(c_0 + c_1(t - t_0)), & \quad \forall t, t_0 \leq t \leq t_f \\
\varphi(t) = 0, & \quad \forall t > t_f
\end{align*}
\]

where \( c_0 \) and \( c_1 \) are

\[
\begin{align*}
c_0 &= z_1(t_0)T^{-2} \\
c_1 &= z_2(t_0)T^{-2} + 2z_1(t_0)T^{-3}
\end{align*}
\]

while \( T = t_f - t_0 \) is the so-called “prescribed time”, which allows one to steer the sliding variable \( z_1 \) to zero at the time \( t_f \), and \( z_2(t_0) \) is assumed to be known. Note that, from (11) and (12), the transient function is realized such that the initial conditions are satisfied constraints (7)-(8). Starting from

\[
\begin{align*}
z_1(t_0) &= \varphi(t_0) \\
z_2(t_0) &= \varphi'(t_0)
\end{align*}
\]

Then, the second important ingredient of the algorithm is the definition of an “auxiliary sliding variable”, as follows

\[
\Sigma(t) = z_1(t) - \varphi(t).
\]

Note that, by virtue of the choice of the transient function, it results that \( \Sigma(t_0) = 0 \).

If \( z \in \mathcal{Z}_i, i = 1, \ldots, k \) define the control law as

\[
w_i(t) = -a_iU_{\text{max},i} \text{sgn}
\]

\[
\left( \Sigma(t) - \frac{1}{2}\Sigma_{\text{max}} \right)
\]

where \( U_{\text{max},i} \) is the control gain associated with the \( i \)-th region \( \mathcal{Z}_i \), \( \Sigma_{\text{max}} \) are the “extremal” values, i.e., local minima and maxima of \( \Sigma \), while \( a_i \) is the modulating factor. Moreover, the control parameters \( U_{\text{max},i} \) and \( a_i = a_i^* \) have to satisfy the following constraints:

\[
a_i^* \in (0, 1] \cap \left( 0, \frac{3G_{\text{min},i}}{\bar{G}_{\text{max},i}} \right)
\]

and

\[
U_{\text{max},i} > \max \left( \frac{\bar{G}_i}{a_i^* \bar{G}_{\text{max},i}}, \frac{4\bar{G}_i}{3G_{\text{min},i} - a_i^* \bar{G}_{\text{max},i}} \right)
\]

with \( \bar{G}_i \) being the upperbound of the drift uncertain term in the auxiliary system when the auxiliary sliding variable is chosen as in (15). It depends on \( F_i \) in (7) and on the bound of the transient function \( \varphi(t) \) in (10), and it could be suitably estimated via a trial-and-error procedure. Furthermore, differently from (10), the control law (16) does not change its sign on specific portions of the perimeter of \( \mathcal{R}_i \) when the state trajectory evolves on the left or on the right of the so-called switching line \( z_1 = -\frac{z_2}{2U_{\text{max},i}} \).

This implies that a sliding mode on the borders of \( \mathcal{R}_i \) in Filippov’s sense cannot occur and the definition of the domains of attraction as in (10) is not needed. The proposed “switching adaptation mechanism” refers in fact to the control gains which are adapted depending on the different entire regions of the state space.

Remark 2. Note that, in order to correctly locate at any time instant \( t > t_0 \) the auxiliary system state with respect to the regions \( \mathcal{R}_i \) and, possibly, to detect the maximum \( \Sigma_{\text{max}} \), it is necessary to know the first time derivative of the sliding variable. If this is not measurable, one can use the Levant’s differentiator to estimate it in a finite time. The estimated \( z_2^* \) is however not used to close the feedback, since in the control law (16) only \( z_1 \) is involved. In case the differentiator is used, one needs to consider an initialization time period ending in \( t_0 \), with \( t_0 \geq t_{1d}, t_{1d} \) being the Levant’s differentiator convergence time (see the illustrative example in Figure (16)).

Then, the following algorithm can be written.

S-ISSOSM Algorithm:
1. Set $\Sigma(t) = z_1(x(t)) - \varphi(t)$.
   Repeat for any $t > t_0$, the following steps.

2. Check if $z \in \mathcal{Z}_i$, $i = 1, \ldots, k$.

3. Set $\alpha_i^*$ as in (17).

4. If $\left[ \Sigma(t) - \frac{1}{2}\Sigma_{\text{max}} \right] \left[ \Sigma_{\text{max}} - \Sigma(t) \right] > 0$, then set $\alpha_i = \alpha_i^*$, else set $\alpha_i = 1$.

5. If $\Sigma(t)$ is extremal, the set $\Sigma_{\text{max}} = \Sigma(t)$.

6. If $t_0 \leq t \leq t_r$, then set $\varphi(t) = (t - t_r)^2(c_0 + c_1(t - t_0))$, else set $\varphi(t) = 0$.

7. Apply the control law (16) with (17)-(18).

Since the relative degree of the original system (1) is $r = 1$, the control signal fed into the plant is the output of an integrator and is continuous with beneficial effects in terms of chattering alleviation.

### 4 STABILITY ANALYSIS

In this section the stability properties of the proposed control algorithm will be discussed. It can be verified that the integral approach maintains the good property of the Suboptimal Second Order Sliding Mode (SSOSM) algorithm in terms of chattering alleviation. Moreover, the proposed algorithm, with bounded switching control gains, avoids the reaching phase by keeping the controlled system state on the sliding manifold since the initial time instant $t_0$ (see Figure 1). With reference to the proposed control approach, the following results can be proved.

**Theorem 1.** Given the auxiliary system (3b), with bounds as in (7)-(8), and the switching regions $\mathcal{R}_i$, $i = 1, \ldots, k$, in (6), controlled via the S-SSOSM Algorithm, with transient function (10)-(14), then a sliding mode is enforced on the integral sliding manifold

$$\Sigma(t) = z_1(t) - \varphi(t) = 0$$

$\forall t \geq t_0$, $t_0$ being the initial time instant.

**Proof.** Starting from the basic results in (23) consider the auxiliary sliding variable (15) and compute its first and second-time derivatives, posing $\xi_1 = \Sigma$ and $\xi_2 = \dot{\Sigma}$, respectively. By doing so, one obtains

$$\begin{cases}
\dot{\xi}_1 = \xi_2 \\
\dot{\xi}_2 = \rho(\xi) + g(\xi)w,
\end{cases}$$

where

$$\rho(\xi) = \begin{cases}
0, & \text{if } \xi \in \mathcal{Z}_i, \\
\max(\xi - \xi_{\text{max}}, 0), & \text{if } \xi > \xi_{\text{max}} > 0, \\
\max(-\xi - \xi_{\text{max}}, 0), & \text{if } \xi < -\xi_{\text{max}} < 0.
\end{cases}$$

$w$ and $u$ are discontinuous control and the continuous one, respectively.
where \( p(\cdot) \) is the uncertain term which takes into account the transient function \( \varphi(\cdot) \) and the terms related to the uncertain function \( f(\cdot) \). Since by assumption both \( f(\cdot) \) and \( \varphi(\cdot) \) are bounded with known bounds inside each region \( R_i \), \( i = 1, \ldots, k \), it follows that

\[
|p_i| \leq \Phi_i, \quad \forall i.
\]  

(21)

Hence, system (20) with the control law (16) and constraints (6)-(21) perfectly fits in the system class of the auxiliary system controlled via the SSOSM algorithm presented by Bartolini et al. in 27. It is now possible to apply Theorem 1 which shows that the generation of a sequence of states with coordinates \((\xi_{\text{max}}, 0)\) featuring the contraction property \( |\xi_{\text{max},i}^j| < |\xi_{\text{max},i}^j|, j \in \mathbb{N} \) occurs if the additional constraint

\[
U_{\text{max},i} > \max \left( \frac{\Phi_i}{\alpha_i^* G_{\text{max},i}^j}; \frac{4\Phi_i}{3\overline{G}_{\text{min},i}^j - \alpha_i^* \overline{G}_{\text{max},i}^j} \right)
\]  

(22)

holds. Moreover, the convergence to the origin of the state plane takes place in a finite time \( \tilde{t} \). According to (18), in the considered case the previous additional constraint (22) holds by assumption in any region \( R_i \), \( i = 1, \ldots, k \), so that \( \Sigma(t) = 0, \forall t \geq \tilde{t} \).

It is important to note that, even when the gain switching occurs, since by construction the sign of the control law does not change when crossing specific parts of the perimeter of the different regions and the control gains are assumed to be always able to dominate the uncertainties, the auxiliary sliding variable \( \Sigma(t) \) is progressively steered to zero. The consequence of this fact is the progressive convergence to zero of the sliding variable \( z_i(t) \) without possible limit cycles. What is left to show is that \( \Sigma(t) = 0, \forall t \geq t_0 \). Since the initial conditions of the auxiliary system (3b) are chosen as (13) and (14), it immediately follows that at \( t_0 \) the auxiliary sliding variable \( \Sigma(t_0) = 0 \). This also means that the sliding mode is enforced on the integral sliding manifold \( \Sigma = 0 \) for any \( t > t_0 \) (i.e., \( t_0 \equiv \tilde{t} \)). Then, one can conclude that \( \Sigma = 0, \forall t \geq t_0 \), which ends the proof. \( \square \)

**Remark 3.** Note that, if \( z_2 \) were measurable, the sliding mode would be enforced independently of the choice of the initial time instant \( t_0 \). The Levant’s differentiator needs instead to converge in a time interval such that the sliding mode is enforced for any \( t \geq t_0 \), with \( t_0 \geq t_{\text{ld}} \) and \( t_{\text{ld}} < T \) (see the dashed red line in Figure 1). This last condition can be suitably achieved by tuning the gains of the differentiator in order to speed up its convergence. \( \square \)

By virtue of Theorem 1, noting that the so-called reaching phase, during which the controlled system is still sensitive to the uncertain terms, is in practice “eliminated”, the following theorem can be proved.

**Theorem 2.** Given the auxiliary system (3b) with bounds as in (7)-(8), and the switching regions \( R_i, i = 1, \ldots, k \), in (6), controlled via the S-ISSOSM Algorithm, with transient function (10)-(14), then system (3b) is robust from the initial time instant \( t_0 \) with respect to the uncertainties affecting the system.

**Proof.** Consider the auxiliary sliding variable (15) which is zero from the initial time instant \( t_0 \) and system (20). This implies that \( z_1(t) = \varphi(t) \), \( z_2(t) = \varphi(t) \), \( \forall t \geq t_0 \). Consider now two different cases.

**Case 1:** \( t < t_1 \). In this interval, by construction the dynamics of the transient function \( \varphi(t) \) does not depend on the uncertain terms, implying that also the dynamics of \( z_i(t) \) is insensitive to the uncertain terms.

**Case 2:** \( t \geq t_1 \). In this case \( \varphi(t) = \varphi(t) = 0 \) which means \( z_1(t) = z_2(t) = 0 \), that is a second order sliding mode is enforced.

Then, computing the so-called equivalent control one has

\[
\nu_{\text{eq}} = -\frac{f(\xi, z)}{g(\xi, z)}
\]  

(23)

which, substituted for \( \nu \) into the auxiliary system (3b), implies that the resulting dynamics of \( z(t) \) is again insensitive to the uncertain terms, concluding the proof. \( \square \)

Finally the following theorem can be proved.

**Theorem 3.** Given plant (1), the diffeomorphism (2), and the auxiliary system with the internal dynamics (3a) with bounds as in (7)-(8), and the switching regions \( R_i, i = 1, \ldots, k \), in (6), controlled via the S-ISSOSM Algorithm, then \( \forall t \geq t_1 \) there exist \( x^* \) which is an open-loop equilibrium point of system (1) constrained to \( z_1(x^*(t)) = 0 \).

**Proof.** Given a suitable choice of the diffeomorphism (2), that is of the sliding variable, since it is steered to zero in a finite prescribed time \( T \), the origin is an equilibrium point of the auxiliary state space. Because of the minimum phase property of system (3a) with the internal dynamics in (3a), as stated in Assumption 4, one has that the origin is an asymptotically stable equilibrium point for \((\xi, z)\). Finally, as shown in Chapter 13, the change of variables

\[
x = \Phi^{-1}(\xi, z)
\]  

(24)
maps the origin \((\zeta, z) = (0, 0)\) into the equilibrium point \(x = x^*\), thus concluding the proof.

5 | CASE STUDY

To highlight the potential of the S- ISSOSM control approach, the problem of controlling the lateral dynamics of vehicles by means of yaw-rate control, which is a classical benchmark in automotive applications \((29,36,37,38,39)\), is addressed.

Consider the so-called single track vehicle model \((27,28)\), illustrated in Figure 2 and described by the following equations

\[
mv(r + \dot{\beta}) = F_t + F_f \\
J \dot{r} = F_t l_f + F_r l_r
\]

where \(m\) is the vehicle mass, \(v\) is the longitudinal component of the \(V\) speed vector of the vehicle center of gravity (CoG) along the \(x\)-axis, \(F_t, F_f\) are wheel lateral loads at front and rear axles, \(J\) is the vehicle moment of inertia, \(r = \dot{\psi}\) is the yaw-rate of the vehicle CoG, \(l_f, l_r\) are the distances from the front and rear axles to the CoG, \(\beta\) is the vehicle sideslip angle, and \(\delta\) is the front wheel steering angle.

Assuming that the tire sideslip angles, \(a_f\) and \(a_r\), are small, the front and rear lateral tire loads can be described as

\[
F_f = a_f C_f \\
F_r = a_r C_r
\]
FIGURE 4 Fishhook maneuver and disturbance with $\omega_{\mu_\text{w}} = 1 \text{ rad s}^{-1}$, from the top: time evolution of the yaw rate $r$, its zoomed transient and tracking error $e_r$, when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively.

where $C_f$ and $C_r$ are the front and rear cornering stiffnesses, and the tires sideslip angles are

$$\alpha_f = \delta - \beta - \frac{l_f r}{v}, \quad \alpha_r = -\beta + \frac{l_r r}{v}.$$  

(29)

According to the previous relationships, the vehicle dynamics in (25) can be expressed as a second order system of the form (1) with $x^T = [\beta, r]$, $u = \delta$, $y = r$ and

$$A(v) = \begin{bmatrix} -\left(\frac{C_t + C_{mc}}{m v} \right) & \frac{C_{l_f} - C_{l_f}^2}{J} - 1 \\ \frac{C_{l_f}^2 + C_{r_f}^2}{J^2} & -\left(\frac{C_{r_f}^2 + C_{r_f}^2}{J^2} \right) \end{bmatrix}, \quad B(v) = \begin{bmatrix} \frac{C_t}{m v} \\ \frac{C_{l_f}^2}{J} \end{bmatrix}, \quad C = [0 \quad 1],$$  

(30)

with matrices $A$ and $B$ depending on the (potentially varying) vehicle longitudinal speed $v$. The objective of the lateral dynamics control, is to regulate the yaw-rate $r(t)$ by means of a suitable steering command $\delta(t)$. More specifically, the closed loop system has to track a yaw-rate desired response $r_{\text{ref}}(t)$, computed, see e.g. [25], as

$$r_{\text{ref}} = \frac{\nu_f}{(l_f + l_t + k_{\text{us}} v^2)} \delta_t,$$

$$k_{\text{us}} = m \frac{(C_{l_f} - C_{l_f})}{(l_f + l_t) C_r},$$  

(31)

where $\nu_f$ and $\delta_f$ are low-pass filtered version of vehicle speed $v$ and steering angle $\delta$, while $k_{\text{us}}$ is the vehicle understeering coefficient. [25] All the vehicle parameters are reported in Table [1].
In order to make the setup more realistic and consider parametric uncertainties on the system model, the values of the vehicle mass $m$ and of the tyre cornering stiffnesses $C_f$ and $C_r$ were varied. In the perturbed model, the values of $m$ and $C_f$ were increased by 30%, while that of $C_r$ was decreased by the same amount. Figure 3 shows how these changes impact on the vehicle dynamics of interest: the eigenvalues of matrix (30), obtained for constant values of the vehicle speed $v \in \{50, 100\}$ km/h, are illustrated, both with the nominal and with the perturbed values of the parameters.
FIGURE 6 Fishhook maneuver and disturbance with $\omega_{\text{ref}} = 20 \text{ rad s}^{-1}$, from the top: time evolution of the yaw rate $r$, its zoomed transient and tracking error $e_r$, when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively.

For testing the closed-loop performance, the fishhook and the double lane change, see e.g., were considered. For the fishhook, a handwheel steering angle of 60 deg with time derivative of 400 deg/s was used, while in the double lane change a handwheel steering angle of 90 deg was selected and the sinusoidal variation has angular frequency $\omega = \pi/2 \text{ rad s}^{-1}$.

5.1 | Design of the S-ISSOSM Controller

The proposed S-ISSOSM controller has been analyzed through the two maneuvers, simulated at a constant value of the longitudinal vehicle speed, i.e., $v = 70 \text{ km h}^{-1}$, in the perturbed case. It is now easy to verify that system (30) satisfies Assumption 2 that is it is complete, has a uniform and time-invariant relative degree and admits a normal form. In order to define the auxiliary system, the sliding variable has been set equal to the yaw-rate error as follows $\sigma = r_{\text{ref}} - r$, such that the natural relative degree is equal to 1, while the transient function $\varphi$ is obtained as in (10) by selecting the prescribed convergence time equal to $T = 0.05 \text{ s}$. Note that, as highlighted in the previous sections, in order to solve the problem it is sufficient to suitably select the sliding variable. More precisely, having the dynamics of $\zeta$ related to the state $\beta$, posing $z_1 = \sigma$ and $z_2 = \dot{\sigma}$, one obtains system (3b) with

$$f(\beta, z) = \left( \frac{C_{\beta \beta}^i + C_{\beta u}^i}{J_v} \right) \left( \frac{C_{\beta \beta}^i + C_{\beta u}^i}{J_v} \right) \beta - \left( \frac{C_{\beta \beta}^i + C_{\beta u}^i}{J_v} \right) \dot{\beta} = \left( \frac{C_{\beta \beta}^i + C_{\beta u}^i}{J_v} \right) \left( \frac{C_{\beta \beta}^i + C_{\beta u}^i}{J_v} \right) \beta - \left( \frac{C_{\beta \beta}^i + C_{\beta u}^i}{J_v} \right) \dot{\beta}$$

$$g = \frac{C_{\beta u}^i}{J_v}. \quad (32)$$
Considering the perturbed single track model, with some parameters perturbed as previously discussed, a sinusoidal matched disturbance acting on the discontinuous control signal \( w \) has been injected, i.e., \( w_m = 0.1 \sin(\omega_m t) \), with \( \omega_m = 1 \) rad s\(^{-1} \) or \( \omega_m = 20 \) rad s\(^{-1} \). Making reference to Assumption 5, the S-ISSOSM controller was designed considering as switching parameter the vehicle speed and mapping it onto the sliding variable and its derivatives (see (20) in similar setting), thus achieving switching regions as nested boxes with \(-z_{1,1} = \overline{z}_{1,1} = -z_{2,1} = \overline{z}_{2,1} = 172, -z_{1,2} = \overline{z}_{1,2} = -z_{2,2} = \overline{z}_{2,2} = 57, \) and \(-z_{1,3} = \overline{z}_{1,3} = -z_{2,3} = \overline{z}_{2,3} = 29\). Through a trial-and-error procedure, given the initial conditions, the upperbounds \( \Phi_i \) in (18) are selected so that \( \Phi_i = 1, i = 1, 2, 3 \), while \( U_{\text{max},1} = 860, U_{\text{max},2} = 573, U_{\text{max},3} = 287 \). Note that, in order to verify the robustness and convergence properties of the proposed algorithm, an initial condition equal to \( r(0) = 8.6 \) deg has been considered.

### 5.2 Simulation Results

The closed-loop performances were tested in the two maneuvers for a longitudinal constant vehicle speed of \( v = 70 \) km h\(^{-1} \) and in the perturbed case. In order to understand the beneficial effect of the proposed solution, it has been compared with the existing ISSOSM algorithm, presented in (28), the SSOSM algorithm (27), both tuned with the same control gains of the new S-ISSOSM control, and the \( H_\infty \) control law (43), tuned to the best of our possibilities in order to have a fair comparison. Figure 4, from the top, shows the time histories of the yaw-rate \( r \), also zoomed in during the transient interval, and the tracking error \( e_r \), all in the fishhook maneuver when the frequency of the matched disturbance is \( \omega_m = 1 \) rad s\(^{-1} \), which belongs to the bandwidth \([0, 15]\) rad s\(^{-1} \) of the system controlled via the \( H_\infty \) control. The matched uncertain term effect is more evident when the \( H_\infty \)
FIGURE 8 Double lane maneuver and disturbance with $\omega_m = 1 \text{ rad s}^{-1}$, from the top: time evolution of the yaw rate $r$, its zoomed transient and tracking error $e_r$, when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively. Figure 5 illustrates the discontinuous control variable $w$ when the considered sliding mode algorithms are applied. In the same figure also the continuous front wheel steering angle, $\delta$, directly fed into the plant, and its transient interval up to $0.1 \text{ s}$ are illustrated. Figures 6 and 7 show the same time histories of the yaw-rate $r$, and the tracking error $e_r$, when the frequency of the matched disturbance is $\omega_m = 20 \text{ rad s}^{-1}$, which is out of the bandwidth $[0, 15] \text{ rad s}^{-1}$ of the system controlled via the $H_\infty$ control. Note that, the proposed S-ISSOSM evidently outperforms the others in terms of control energy. Figures 8, 9, 10 and 11 show the same time histories with the double lane change maneuver.

Finally, in order to compare the algorithms, two performance indices have been introduced: i) the root mean square (RMS) value of the tracking error (i.e., the sliding variable), $e_{\text{RMS}}$; ii) the RMS of the control signal $E_c$. These latter are computed as

$$e_{\text{RMS}} = \sqrt{\frac{1}{n_s} \sum_{i=1}^{n_s} e_i^2}, \quad E_c = \sqrt{\frac{1}{n_s} \sum_{i=1}^{n_s} w_i^2}$$

(33)

where $n_s$ is the number of integration steps during the simulation; $e_i$ and $w_i$ are the tracking error and the auxiliary control at the $i$-th integration step, respectively. In Table 2 the performance indices, obtained by considering all the simulation scenarios, are reported, while the results normalized with respect to the worst case are reported in Table 3 and illustrated in Figure 12. While the SSOSM and ISSOSM control laws are more precise in terms of RMS value of the tracking error, the proposed S-ISSOSM control law guarantees, as expected and desired, better performance in terms of control energy reduction with respect to the other algorithms.
FIGURE 9 Double lane maneuver and disturbance with $\omega_{\text{in}}=1 \text{ rad s}^{-1}$, from the top: time evolution of the discontinuous control $w$, the continuous control $\delta$ and its zoomed transient when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively.

<table>
<thead>
<tr>
<th>Maneuver</th>
<th>Control</th>
<th>$\epsilon_{\text{RMS}}$ (deg)</th>
<th>$E_c$ (deg/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fishhook</td>
<td>$H_\infty$</td>
<td>0.023</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>SSOSM</td>
<td>0.0049</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>ISSOSM</td>
<td>0.0049</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>S-ISSOSM</td>
<td>0.0098</td>
<td>4.97</td>
</tr>
<tr>
<td>Double Lane</td>
<td>$H_\infty$</td>
<td>0.021</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>SSOSM</td>
<td>0.0049</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>ISSOSM</td>
<td>0.0049</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>S-ISSOSM</td>
<td>0.01</td>
<td>4.96</td>
</tr>
</tbody>
</table>

6 | CONCLUSIONS

This paper proposed a novel adaptive formulation of integral sliding mode control, combining the two ingredients to obtain a more flexible and robust control solution. Indeed, thanks to the integral formulation, it eliminates the sensitiveness to matched
uncertainties during the reaching phase, ensuring that the system trajectory lies on a prescribed sliding manifold depending on the selected sliding variable since the initial time instant, and adding to it the extra degrees of freedom provided by the switched adaptation policy. The paper presented the full algorithm formulation, together with the formal proof of the closed-loop stability properties. Further, the validity of the approach was assessed considering a challenging case study, that is the yaw-tracking control of ground vehicles. Extensive simulations using standard testing maneuvers for lateral vehicle control design proved that

![Figure 10](image-url)  
**FIGURE 10** Double lane maneuver and disturbance with $\omega_{w_{\infty}} = 20 \text{ rad s}^{-1}$, from the top: time evolution of the yaw rate $r$, its zoomed transient and tracking error $e_r$, when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively.

**TABLE 3** Performance indices with respect to the worst case

<table>
<thead>
<tr>
<th>Steering</th>
<th>Control</th>
<th>$e_{\text{RMS}}$ (%)</th>
<th>$E_c$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fishhook</td>
<td>$H_\infty$</td>
<td>100</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>SSOSM</td>
<td>4.76</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>ISSOSM</td>
<td>4.76</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>S-ISSOSM</td>
<td>9.51</td>
<td>33.13</td>
</tr>
<tr>
<td>Double Lane</td>
<td>$H_\infty$</td>
<td>100</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>SSOSM</td>
<td>4.45</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>ISSOSM</td>
<td>4.45</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>S-ISSOSM</td>
<td>9.1</td>
<td>33.07</td>
</tr>
</tbody>
</table>
FIGURE 11 Double lane maneuver and disturbance with $\omega_{\omega_{m}} = 20 \text{ rad s}^{-1}$, from the top: time evolution of the discontinuous control $\omega$, the continuous control $\delta$ and its zoomed transient, when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively.

FIGURE 12 Performance indices normalized with respect to the worst case, obtained when $H_\infty$, SSOSM, ISSOSM and S-ISSOSM are used, respectively.

the closed-loop system offers properties that are superior to the ISSOSM and SSOSM approaches, thus showing the potential of the proposed control method.
Conflict of interest
The authors declare no potential conflict of interests.

References


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