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Modal Density of Rectangular Structures in a Wide Frequency Range

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Abstract

A novel approach to investigate the modal density of a rectangular structure in a wide frequency range is presented. First, the modal density is derived, in the whole frequency range of interest, on the basis of sound transmission through the infinite counterpart of the structure; then, it is corrected by means of the low-frequency modal behavior of the structure, taking into account actual size and boundary conditions. A statistical analysis reveals the connection between the modal density of the structure and the transmission of sound through its thickness. A transfer matrix approach is used to compute the required acoustic parameters, making it possible to deal with structures having arbitrary stratifications of different layers. A finite element method is applied on coarse grids to derive the first few eigenfrequencies required to correct the modal density. Both the transfer matrix approach and the coarse grids involved in the finite element analysis grant high efficiency. Comparison with alternative formulations demonstrates the effectiveness of

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the proposed methodology.

Keywords: Modal Density, Statistical Energy Analysis, Transfer Matrix Method

1. Introduction

The modal density of a system is a frequency dependent function defined as the number of modes which lie in a unitary frequency interval. Its knowledge is required in the medium-high frequency range, when the number of modes make inapplicable the modal analysis. In such frequency range the response of a structure under mechanical or acoustic excitation can be deduced from its modal density. The modal density of various structures has been identified both theoretically and experimentally. For common structural components, such as a thin plate, beams, a spherical cap or a circular cylinder, the modal density was established through analytical expressions [1, 2, 3, 4]. More complex configurations have also been studied, *e.g.* an expression for the modal density of honeycomb panels with orthotropic face sheets including transverse shear effects was established under specific hypothesis [5]. Parametric studies have also been performed in order to investigate the influence of various parameters on the modal density of a sandwich panel [6]. However, the analysis of general structures is often difficult.

The most commonly used procedure for deriving the modal density of a structure involves solving the dispersion problem for free-wave propagation in the structure. The mathematics of wave propagation in periodic systems was first discussed by Brillouin [7] in the field of electrical engineering. Afterwards, Orris and Petyt [8, 9] employed the Finite Element (FE) technique for wave propagation analysis in periodic structures. Modal density depends on the group and phase velocity of the wave group considered. Langley [10] derived the modal density of periodically stiffened beam and plate structures in terms of phase constants associated with propagating wave motion. Finnvedan [11] used the wave-guide FE method to calculate the wave propagation characteristics of built-up thin-walled structures; he described the process of deriving the modal density and group velocity from FE input for a beam structure. However, the free-wave approach is reliable at high frequency only, since it considers structures with an infinite extent. Moreover, obtaining the modal density of a structure with several wave groups requires the intervention of the analyst to identify and discriminate dispersion curves.

Alternatively, a modal analysis of the finite structure with real boundary conditions can provide the modal density by means of the mode count. However, a modal approach is viable only at low frequency because of the related computational effort, regardless of the numerical method adopted.

A procedure to evaluate the modal density of a planar, rectangular structure in a wide frequency range is proposed. The key idea on the basis of the present work is the separation of the stacking and the boundary effects on the panel modal density. Such a separation allows to combine the ability of a dispersion problem to catch high-frequency structure dynamics, and the flexibility of a modal analysis in describing boundary effects at low frequency. The original contribution of the present work is twofold: first, an expression relating the modal density of an arbitrarily stratified, planar structure with the diffuse transmission and reflection coefficients of its infinite counterpart is presented, and second, a corrective scheme for the modal density is derived, taking into account the real size and boundary conditions of the structure by means of its low-frequency modal behavior. Involving sound transmission through the structure thickness instead of the dispersion problem makes it possible to bypass any difficulty related to dispersion curves. A Transfer Matrix Method (TMM) is used to evaluate the required acoustic indicators, allowing to deal efficiently with structures with generic stratifications, possibly including in-plane periodic layers [12]. The correction accounting for real size and boundary conditions requires only few eigenfrequencies of the structure, making it possible to stem the computational cost related to the modal analysis.

Section 2 presents the separation of the stacking and the boundary effects on the panel modal density. The stacking contribution is derived in Section 3 by means of a Statistical Energy Analysis (SEA) on sound transmission through the structure. Section 4 reveals the role of the low eigenfrequencies in correcting the modal density. A number of applications are then discussed and compared with alternative formulations.

2. Overview

The scope of the present work is twofold: first, to define the modal density of a planar structure avoiding the solution of the dispersion problem and second, to exploit the modal approach at low frequency in order to effectively take boundary effects into account. The key idea consists in treating stacking and boundary effects independently. To clarify this we consider the example of a simply supported, homogeneous, rectangular thin plate with dimensions

$a \times b \times h$. The modal density of the plate can be expressed as [13]

$$n(\omega) = \frac{S}{4\pi}\sqrt{\frac{m}{B}} - \frac{P}{8\pi\sqrt{\omega}}\sqrt[4]{\frac{m}{B}} \ , \tag{1}$$

where $S = ab$ is the panel area, $P = 2a + 2b$ is the panel perimeter, $m = \rho h$ is the mass per unit area, $B = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity, E is Young's modulus, ν is the Poisson ratio and ρ is the density. The expression for the modal density, Eq. (1), contains information about the stacking properties of the panel, through B and m, and the panel dimensions, through S and P. In order to bring out such a dichotomy, the asymptotic specific modal density is introduced:

$$\mu_\infty(\omega) = \lim_{S\to\infty} \frac{n(\omega)}{S} \ . \tag{2}$$

Since boundary effects quickly vanish moving away from panel edges, and so becoming null in a panel with infinite extent, the asymptotic specific modal density, μ_∞, depends only on the stacking properties of the panel. Such an interpretation is crucial for the proposed methodology, since it allows an evaluation of μ_∞ considering an infinite extent for the panel. This, in turn, allows the use of both a free-wave approach, *i.e.* TMM, and a statistical approach, *i.e.* SEA, in the whole frequency range of interest, even at low frequencies. Invoking the expression of the modal density obtained for the thin plate, Eq. (1), we obtain

$$\mu_\infty(\omega) = \frac{1}{4\pi}\lim_{S\to\infty}\left(\sqrt{\frac{m}{B}} - \frac{P}{2S\sqrt{\omega}}\sqrt[4]{\frac{m}{B}}\right) = \frac{1}{4\pi}\sqrt{\frac{m}{B}} \ . \tag{3}$$

It should be noted that the dependency on the frequency disappears in the obtained expression of the specific modal density, Eq. (3), because shear deformation is neglected in the adopted thin plate model. Dependency on the frequency persists in general and is contemplated by the proposed methodology. Using the expression for the specific modal density, Eq. (3), the modal density, Eq. (1), may be expressed as

$$n(\omega) = S\mu_{\infty}(\omega) - \frac{P}{4}\sqrt{\frac{\mu_{\infty}(\omega)}{\pi\omega}} \ , \tag{4}$$

where the way in which the panel dimensions act on the specific modal density, μ_{∞}, is highlighted. Even though the obtained expression for the modal density, Eq. (4), is valid for a thin plate (shear deformation neglected in the kinematic model) with simply supported boundary conditions, it can be seen as a reliable way of separating asymptotic and boundary contributions to the modal density of a generic panel. In fact, even a simple kinematic model, *e.g.* the thin plate model, can accurately catch panel dynamics in the frequency range in which boundary effects are significant. Nevertheless, a detailed description of the displacement field through the thickness of the structure can be included in the asymptotic contribution, μ_{∞}. On the other hand, a simple way of taking different boundary conditions into account can be found by looking at Eq. (3) and Eq. (4): it can be seen that, in the case of thin plates, only the second term of Eq. (4) is frequency dependent. So, since boundary effects quickly vanish for increasing frequency, only the term related to the panel perimeter, P, can be affected by boundary conditions [13]. As a consequence, a more general expression for the modal density of a rectangular panel is proposed:

$$n(\omega) = S\mu_{\infty}(\omega) - \delta\frac{P}{4}\sqrt{\frac{\mu_{\infty}(\omega)}{\pi\omega}} \ , \tag{5}$$

where the correction factor, δ, may be determined by means of a modal analysis according to the actual boundary conditions. Conclusively, two independent contributions can be identified in the final expression for the modal density, Eq. (5): the first one is the frequency dependent function, μ_∞, which depends only on the stacking properties of the layered structure, the so-called *stacking contribution*, the second one is the correction factor, δ, which depends also on the size of the structure and its boundary conditions, the so-called *boundary contribution*.

3. Stacking Contribution

The stacking contribution of the modal density is defined by the asymptotic specific modal density, Eq. (2). We propose deducing it by means of the acoustic properties of the structure, in particular the power transmission and reflection coefficients. Such a purpose draws legitimacy from the idea that sound transmission through the structure thickness hides and carries the very same information as the dispersion problem for the medium. The passkey for such information is exposed here using a statistical approach, which is reliable at any frequency since the subsystems involved have infinite extent, see the limit in Eq. (2).

3.1. Statistical Approach

Sound transmission through the thickness of a planar structure can be investigated by placing the structure between two rooms. In the context of SEA, two energy paths can be identified between the rooms. The first one links the rooms without involving the resonance of the interposed wall, and depends only on the specific mass of the wall, the so-called non-resonant path. A second path treats the interposed structure as a subsystem so involving its modal density, the so-called reverberant path. As a consequence, the non-resonant path is neglected in the following since it is not sensitive to the panel properties we are looking for, *i.e.* the panel modal density. The conditions under which such a choice may be effective will be investigated afterwards (Section 3.2).

Focusing on the reverberant path, the power balance of a panel perturbed by an incident acoustic power, Π_{inc}, can be expressed as

$$\Pi_{\text{tra}}(n_s, \eta_s) + \Pi_{\text{ref}}(n_s, \eta_s) + \Pi_{\text{dis}}(n_s, \eta_s) = \Pi_{\text{inc}} \ , \tag{6}$$

where the transmitted power, Π_{tra}, the reflected power, Π_{ref}, and the dissipated power, Π_{dis}, depend on both the modal density of the panel, n_s, and the panel loss factor, η_s. The power incident on a wall of the first room when a diffuse field persists within the room is [1]

$$\Pi_{\text{inc}} = \frac{\omega^2 S e_1}{8\pi^2 c^2} \ , \tag{7}$$

where S is the wall area, c is the speed of sound of the fluid filling the room, and the modal energy, $e_i = E_i/n_i$, is introduced by exploiting the main SEA hypothesis concerning equal distribution of the system energy, E_i, among its modes [1]. The expression for the incident power in case of a diffuse acoustic field, Eq. (7), is a very simple analytical formula and leads to easier theoretical developments. Furthermore, the diffuse field is fully compatible with the statistical approach adopted. The power dissipated by the panel can be expressed as [1]

$$\Pi_{\text{dis}} = \omega E_s \eta_s = \omega e_s n_s \eta_s \ . \tag{8}$$

As a result, the power balance for the panel, Eq. (6), can be written as

$$\tau_d(n_s, \eta_s) + r_d(n_s, \eta_s) + \frac{8\pi^2 c^2}{\omega S} \left(\frac{e_s}{e_1} \right) n_s \eta_s = 1 \ , \tag{9}$$

where $\tau_d = \Pi_{\mathrm{tra}}/\Pi_{\mathrm{inc}}$ is the power transmission factor and $r_d = \Pi_{\mathrm{ref}}/\Pi_{\mathrm{inc}}$ is the power reflection factor. By linearizing Eq. (9) with respect to panel damping, η_s, and invoking the power balance for null structural damping ($\tau_d|_{\eta_s=0} + r_d|_{\eta_s=0} = 1$) we obtain

$$\delta\eta_s \left[\frac{\partial \tau_d}{\partial \eta_s} + \frac{\partial r_d}{\partial \eta_s} + \frac{8\pi^2 c^2}{\omega S}\left(\frac{e_s}{e_1}\right)\left(n_s + \eta_s \frac{\partial n_s}{\partial \eta_s}\right)\right]_{\eta_s=0} = 0 \ . \tag{10}$$

In case of null damping, the modal energy of the panel is equal to the mean of the modal energies in the rooms [14]: $e_s|_{\eta_s=0} = (e_1 + e_2)/2$. Moreover, invoking the SEA hypothesis concerning the weak coupling between subsystems [15] ($\eta_{ij} \ll \min(\eta_i, \eta_j)$), we obtain $e_2/e_1 \sim 0$ and consequently $e_s/e_1 \sim 0.5$. Since Eq. (10) has to hold for any arbitrary damping perturbation, $\delta\eta_s$, the desired expression for the asymptotic specific modal density of the panel is obtained:

$$\mu_\infty = -\frac{\omega}{4\pi^2 c^2}\left(\frac{\partial \tau_d}{\partial \eta_s} + \frac{\partial r_d}{\partial \eta_s}\right)_{\eta_s=0} \ . \tag{11}$$

3.2. Weak coupling and non-resonant path

The expression for the asymptotic specific modal density of the panel, Eq. (11), is derived under the hypothesis of i) negligibility of the non-resonant path in the power transmission and ii) weak coupling between subsystems (rooms). The only way to fulfil these hypotheses is to properly choose the properties of the fluid for which the sound transmission is evaluated. In particular, the non-resonant path in the sound transmission is related to the mass-law contribution, which is predominant below the acoustic coincidence. Moreover, a strong coupling between the two semi-infinite fluids (rooms) is due to coincidence phenomena. As a result, moving the coincidence region to low frequencies, well below the frequency range of interest, ensures both a negligibility of the non-resonant contribution to the sound transmission and a weak coupling between the rooms. To this end, the speed of sound, c, must be small enough to fulfil the above discussed hypotheses at the minimum frequency at which the specific modal density, μ_∞, is desired. Additionally, it can be observed that for a diffuse field at a given frequency, ω, the modulus of the projection of the incident wave on the interface, $k_t = \sqrt{k_x^2 + k_y^2} = \omega \sin(\theta)/c$, spans as $0 \leq k_t < \omega/c$. As a consequence, the speed of sound, c, must be set as small as possible to ensure the excitation of all the propagating waves contributing to the modal density of the medium. Moreover, the limit of the mechanical impedance of a thin plate can be expressed as [16]

$$\lim_{c\to 0} Z_p = \mathrm{j}\omega \lim_{c\to 0}\left(m - \frac{Bk_t^4}{\omega^2}\right) = -\mathrm{j}\omega^3 B \frac{\sin^4(\theta)}{c^4} \ , \tag{12}$$

where the panel mass and, consequently, the non-resonant contribution disappear. Instead, the choice of the fluid density, ρ, is less critical. In fact, a low speed of sound of the surrounding fluid yields to $Z = \rho c \ll Z_p$, so granting a weak coupling between the structure and the fluid and, consequently, between the rooms, regardless of the chosen density, ρ. The expression for the asymptotic specific modal density, Eq. (11), can therefore be modified as

$$\mu_\infty(\omega) = -\lim_{c\to 0}\left[\frac{\omega}{4\pi^2 c^2}\left(\frac{\partial \tau_d(\omega,\rho,c)}{\partial \eta_s} + \frac{\partial r_d(\omega,\rho,c)}{\partial \eta_s}\right)_{\eta_s=0}\right] \quad \forall \rho \in \mathbb{R}^+ \ , \tag{13}$$

where the limit ensures fulfilment of the hypotheses involved to derive Eq. (11) in the frequency range of interest.

3.3. Evaluation of the transmission and reflection coefficients

The diffuse transmission factor, τ_d, and reflection factor, r_d, can be defined by expressing the diffuse acoustic field in the reverberant room as a combination of plane waves traveling in all the possible directions [16]. At a given frequency, ω, each plane wave impinging upon the flat structure is defined by its amplitude, I, azimuth, α and elevation, $\pi/2-\theta$. Both a transmitted wave and a reflected wave therefore propagate from the medium and their amplitudes, T and R, depend on the properties of the barrier. Assuming unitary amplitude for every incident wave, the power transmission and reflection factors related to the incident diffuse field can be expressed as

$$\tau_d(\omega) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi/2} |T(\omega,\theta,\alpha)|^2 \cos(\theta)\sin(\theta)\mathrm{d}\theta\mathrm{d}\alpha \ , \tag{14}$$

and

$$r_d(\omega) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi/2} |R(\omega,\theta,\alpha)|^2 \cos(\theta)\sin(\theta)\mathrm{d}\theta\mathrm{d}\alpha \ . \tag{15}$$

A practical and efficient tool for evaluating the transmission and reflection coefficients, T and R, of planar, stratified media is the TMM. This approach easily allows for multilayers made from a combination of elastic, porous and fluid layers. It assumes the multilayer of infinite extent and uses a representation of plane wave propagation in different media in terms of transfer matrices. The transfer matrix of a layered medium is obtained from the transfer matrices of individual layers by imposing continuity conditions at interfaces. Enforcing the impedance condition of the surrounding fluid, at both the excitation and the termination side, allows calculation of the transmission coefficient, T, and the reflection coefficient, R. This methodology is explained in detail in chapter 11 of Ref. [16]. In the frame of linear vibroacoustics, the wave approach on the basis of the TMM provides accuracy and efficiency in defining the sound transmission through planar structures with infinite extent, flat interfaces and homogeneous layers. However, the last two limitations can be overcome by involving a FE model for the periodic unit cell of each heterogeneous layer [12].

3.4. Retrieving dispersion curves

As stated, the above presented theory draws legitimacy from the idea that sound transmission through the thickness of a structure hides and carries the very same information as the dispersion problem for the medium. In order to realize such an idea, a procedure to retrieve dispersion solutions is presented. The acquired expression for the asymptotic specific modal density of the panel, Eq. (13), provides the total modal density of the medium, since the incident diffuse acoustic field adopted excites every possible free-wave in the medium. In order to obtain the modal density of a specific wave, only this one must be involved in the transmission of sound through the medium. To this end, the diffuse field must be replaced with a number of incident acoustic waves contained in a wedge narrow enough to excite a single wave. As a consequence, the expressions of the power transmission and reflection factors, Eq. (14) and Eq. (15), become

$$\tau(\omega,\theta_i,\bar{\alpha}) = 4\frac{\int_{\theta_i-\Delta\theta/2}^{\theta_i+\Delta\theta/2}|T(\omega,\theta,\bar{\alpha})|^2\cos(\theta)\sin(\theta)\mathrm{d}\theta}{\cos(2\theta_i-\Delta\theta)-\cos(2\theta_i+\Delta\theta)}\ , \tag{16}$$

and

$$r(\omega,\theta_i,\bar{\alpha}) = 4\frac{\int_{\theta_i-\Delta\theta/2}^{\theta_i+\Delta\theta/2}|R(\omega,\theta,\bar{\alpha})|^2\cos(\theta)\sin(\theta)\mathrm{d}\theta}{\cos(2\theta_i-\Delta\theta)-\cos(2\theta_i+\Delta\theta)}\ , \tag{17}$$

where $\bar{\alpha}$ is the selected direction of propagation, θ_i defines the selected incidence and the interval width, $\Delta\theta$, must be small enough to ensure the excitation of a single free-wave in the structure. The modal density function can therefore be expressed as

$$n(\omega, k_i, \bar{\alpha}) = -\lim_{c\to 0}\left[\frac{S\omega}{4\pi^2 c^2}\left(\frac{\partial\tau(\omega,\theta_i,\bar{\alpha})}{\partial\eta_s} + \frac{\partial r(\omega,\theta_i,\bar{\alpha})}{\partial\eta_s}\right)_{\eta_s=0}\right] , \qquad (18)$$

where $k_i = \omega\sin(\theta_i)/c$. The pair $(\omega_j - k_i)$ for which $n(\omega, k_i, \bar{\alpha}) > 0$ belongs to the j-th dispersion curve of the medium for the given direction of propagation, $\bar{\alpha}$, and $n(\omega_j, k_i, \bar{\alpha})$ is the related modal density. It must be remarked that only acoustically effective dispersion curves can be recovered, since acoustically ineffective free-waves of the medium cannot be excited by an incident acoustic wave. In other words, all the free-waves (or modes) characterized by null out-of-plane displacement at the interfaces are not involved in the transmission and reflection of sound through the thickness of the structure and, consequently, they are ignored by the present theory, in terms of both dispersion curves and total modal density. Such an intrinsic property of the proposed methodology automatically ensures the proper modal density for applications involving primarily the coupling of the structure with acoustic cavities. However, further analyses may be needed to assess the efficacy of the discarded waves in terms of structural transmission.

4. Boundary Contribution

The boundary contribution of the modal density is defined by the correction factor, δ, which appears in Eq. (5). Simply supported boundary conditions at the whole perimeter imply $\delta = 1$. Otherwise, the low-frequency modal behavior of the structure, accounting for the actual boundary conditions, must be assessed. A modal analysis can provide a number of lowest eigenfrequencies for a system. The sorted eigenfrequencies allow to assess the mode count function, $N(\omega)$, by relating the mode number, i, of the i-th mode to its eigenfrequency, ω_i. The expression for the mode count function is also obtained as the indefinite integral of the general expression for modal density, Eq. (5), with respect to the frequency, ω:

$$N(\omega) = S\xi(\omega) - \delta \frac{P}{2\sqrt{\pi}} \sqrt{\xi(\omega)} + N_0 \ , \tag{19}$$

where $\xi(\omega) = \omega\mu_\infty(\omega)$ and N_0 is the number of rigid modes. As a consequence, the correction factor, δ, can be evaluated by combining modal information, through $N(\omega_i) = i$, and stacking contribution, through $\mu_\infty(\omega_i)$. A least squares approach, relied on a set of eigenfrequencies, leads to

$$\delta = \frac{2\sqrt{\pi}}{P} \cdot \frac{\sum_i \sqrt{\xi_i}(S\xi_i + N_0 - i)}{\sum_i \xi_i} \ . \tag{20}$$

5. Validation Examples

The derivatives of the transmission and reflection coefficients required to compute modal densities are evaluated by means of finite differences. A perturbing damping factor of 10^{-8} ensures satisfactory precision and avoids numerical issues. At each frequency, the speed of sound of the fluid, c, is reduced starting from a guess, until the modal density converges. A fluid density $\rho = 1.225$ kgm^{-3} is used for all applications.

5.1. Modal density of a homogeneous plate

The first application involves a simply supported, rectangular plate made of aluminum alloy ($\rho = 2700$ kgm^{-3}, $E = 71$ GPa, $\nu = 0.3$). The dimensions are $1 \times \sqrt[4]{2} \times 0.02$ m^3 and they were chosen to emphasize both boundary and thickness effects in the frequency range of interest. Integration over the heading angle, α, can be omitted in Eq. (14) and Eq. (15), since the plate is isotropic. Figure 1 shows the modal densities obtained through the proposed methodology, with ($\delta = 1$) and without ($\delta = 0$) finite size correction in Eq. (5), along with the analytical expression for a thin plate, Eq. (1), and the mode count performed for an FE model of the plate. A speed of sound $c = 100$ ms^{-1} grants the convergence of the modal density in the whole frequency range explored. The FE grid consists of 200×200×4 linear 8-node hexahedron elements with consistent mass. Some of the eigenfrequencies obtained are related to mode shapes with null out-of-plane displacements of their external surfaces. These modes are acoustically ineffective and must be removed for sake of comparison with the proposed approach. The modal density is evaluated by fitting the mode numbers as $N(\omega) = d_4\omega^2 + d_3\omega^{3/2} + d_2\omega + d_1\sqrt{\omega}$. All the modal densities are normalized with the analytical asymptotic value for a thin plate, $n_\infty = S/(4\pi)\sqrt{m/B}$. As expected, discrepancies between the thin plate and the other models grow with frequency, because of the shear and thickness effects. Finite size correction, Eq. (5), improves the evaluation of modal density in the whole frequency range explored. Figure 2 shows the dispersion solutions retrieved with the proposed methodology (TMM), along with those evaluated by imposing periodic boundary conditions on an FE model of the unit cell of the medium and solving the related eigenproblem [8, 9]. The FE grid used to model the unit cell of the medium consists of 8 linear 8-node hexahedron elements with consistent mass. Looking at the eigenvectors resulting from the periodic approach, the related dispersion solutions can be classified as related to an acoustically effective or ineffective wave. It can be observed that the proposed methodology, based on the TMM, allows an accurate retrieval of the acoustically effective waves. The acoustically ineffective dispersion curve shown in Figure 2 justifies the discrepancies between the total and effective mode count performed for the FE model of the whole plate, as shown in Figure 1.

5.2. Modal density of a sandwich panel with honeycomb core

The second application involves the comparison with an alternative literature approach providing the modal density of a sandwich panel with isotropic core as [5]:

$$n = \frac{Sm\omega}{4\pi N}\left(1 + \frac{m\omega^2 + 2N^2/D}{\sqrt{m^2\omega^4 + 4m\omega^2 N^2/D}}\right) \ , \tag{21}$$

where $N = Gh(1+t/h)^2$, G is the core shear modulus, h is the thickness of the core, t is the thickness of the face sheet, $D = Et(h+t)^2/(2(1-\nu^2))$, E and ν are the elastic modulus and the Poisson coefficient of the face sheets. The test case refers to a panel of 1 m^2 composed of 0.5 mm-thick isotropic skins made of aluminum ($\rho = 2700$ kgm^{-3}, $E = 71$ GPa, $\nu = 0.3$) and a 8 mm-thick honeycomb core made of nomex. The equivalent material properties of the core are obtained by means of a homogenization technique [17]. The core is then forced to be transversely isotropic for the sake of comparison with the chosen alternative formulation ($\rho = 48$ kgm^{-3}, $E_{xx} = E_{yy} = 1$ MPa, $E_{zz} = 131$ MPa, $G_{yz} = G_{zx} = 30$ MPa, $G_{xy} = 0.26$ MPa, $\nu_{yz} = 0.001$, $\nu_{xy} = 0.9$, $\nu_{zx} = 0.131$). Figure 3 shows the dispersion solutions retrieved with the proposed methodology (TMM) along with the dispersion solutions evaluated by means of the periodic approach [8, 9]. A speed of sound $c = 80$ ms^{-1} grants the convergence of the dispersion solutions in the whole frequency range explored. The FE grid consists of 12 linear 8-node hexahedron

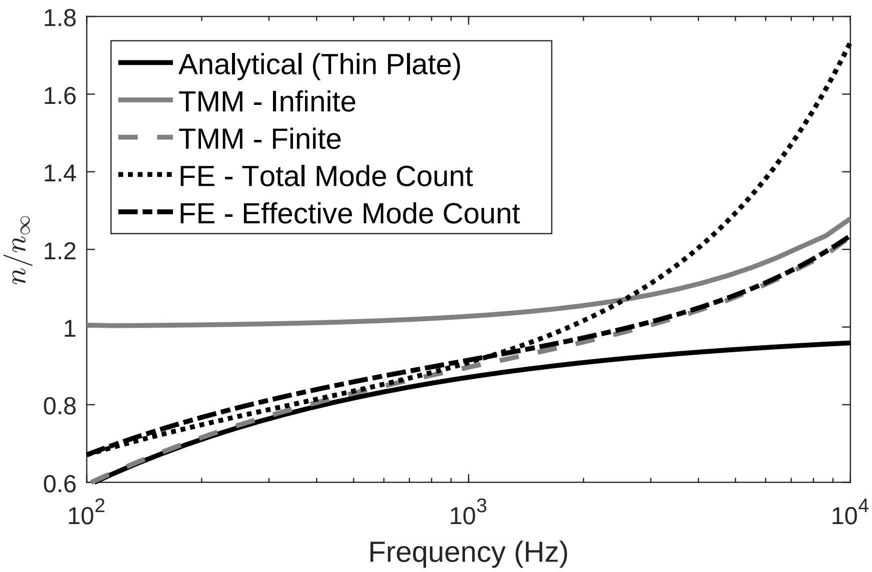

Figure 1: Modal density of a simply supported, homogeneous plate

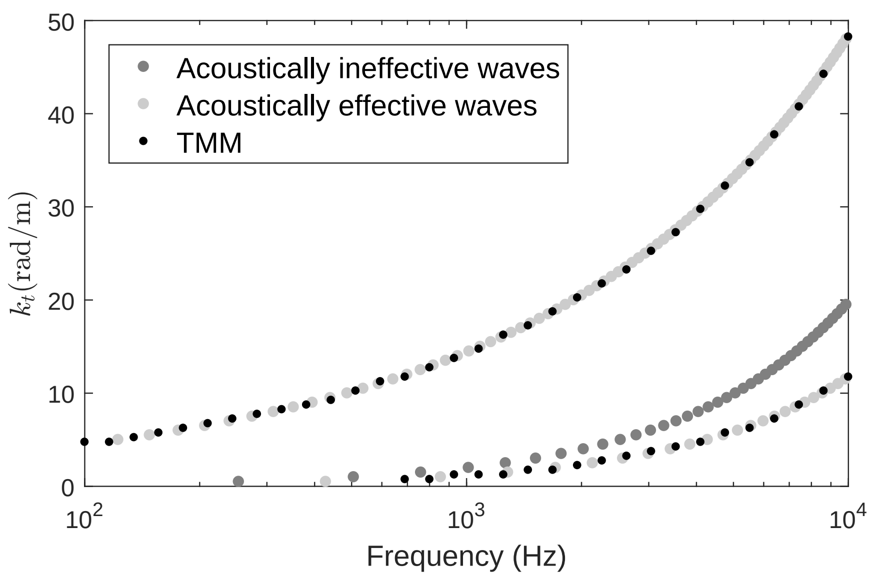

Figure 2: Dispersion curves of a homogeneous plate

elements with consistent mass (2 elements for each skin and 8 elements for the core). Since the panel is transversely isotropic, dispersion solutions do not depend on the direction of propagation, α. Both acoustically effective and ineffective waves are shown. The proposed methodology allows the accurate retrieval of the acoustically effective waves. Figure 4 shows the modal density for the sandwich panel according to Eq. (21), along with the total and the bending modal density obtained with the proposed approach and the modal density evaluated by fitting the bending dispersion curve, $k(\omega)$, with a 5th order polynomial. This latter is obtained with the periodic approach and the related modal density is evaluated by means of the Courant's formula [18]: $n(\omega) = \frac{S\omega}{2\pi c_p c_g}$, where $c_p = \omega/k$ and $c_g = \partial\omega/\partial k$ are the phase velocity and the group velocity of the wave. The finite size correction is not applied to modal density ($\delta = 0$), since Eq. (21) takes into account the real size of the structure only through the panel area, S. Comparison with the modal density obtained by fitting the bending dispersion curve proves the effectiveness of the proposed approach in the whole frequency range explored. Discrepancies observed at high frequencies between the proposed methodology and the literature formula, Eq. (21), are probably due to the cinematics adopted in the latter.

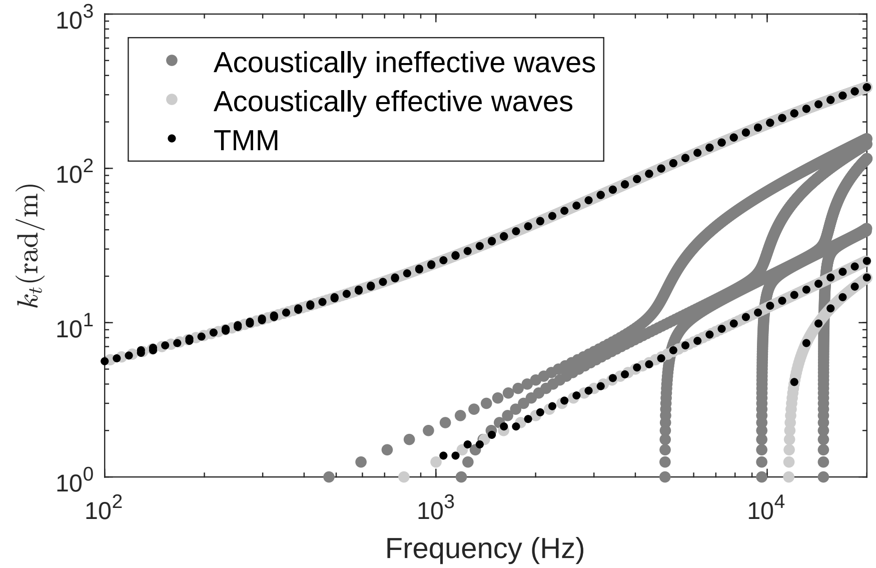

Figure 3: Dispersion curves of a sandwich panel

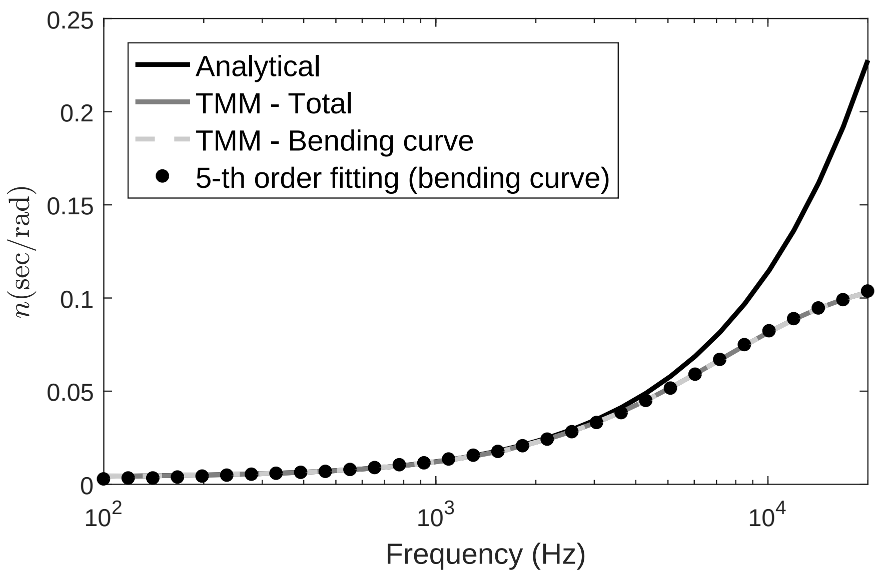

Figure 4: Modal density of a sandwich panel

5.3. Boundary correction factor

The correction factor, δ, and the resultant modal density of a rectangular plate with in-plane dimensions of $\sqrt[4]{2} \times 1$ m^2 were estimated for different boundary conditions. A three-node triangular element, combining a discrete Kirchhoff triangle bending element and a constant strain triangle membrane element, was used to model the plate. The triangles have 10 mm-long and $10\sqrt[4]{2}$ mm-long catheti. Transverse displacement was locked on the perimeter of the plate and an angular stiffness, k, was applied to each rotational degree of freedom of the perimeter, according to edge direction. Such technique make it possible to span between a simply supported plate, $k = 0$, and a clamped one, $k = \infty$. The same analysis has been performed in the case of a plate constrained only along the long edges. Figures 5 and 7 show the correction factors, δ_n, as functions of the nondimensional stiffness, k/B, where B is the plate bending stiffness, and n is the number of eigenfrequencies involved in the least squares solution, Eq. (20), along with the correction factors relied on the whole frequency range explored (300 modes). Figures 6 and 8 show the related modal densities for three values of bounding stiffness, k, as functions of the nondimensional frequency, ω/ω_{SS1}, where ω_{SS1} is the first eigenfrequency of the simply supported plate. The convergence of the modal density functions, with respect the number of modes involved in the least squares solutions, is slower for the plate constrained on two opposite edges. This is possibly due to the additional beam-like modes [13].

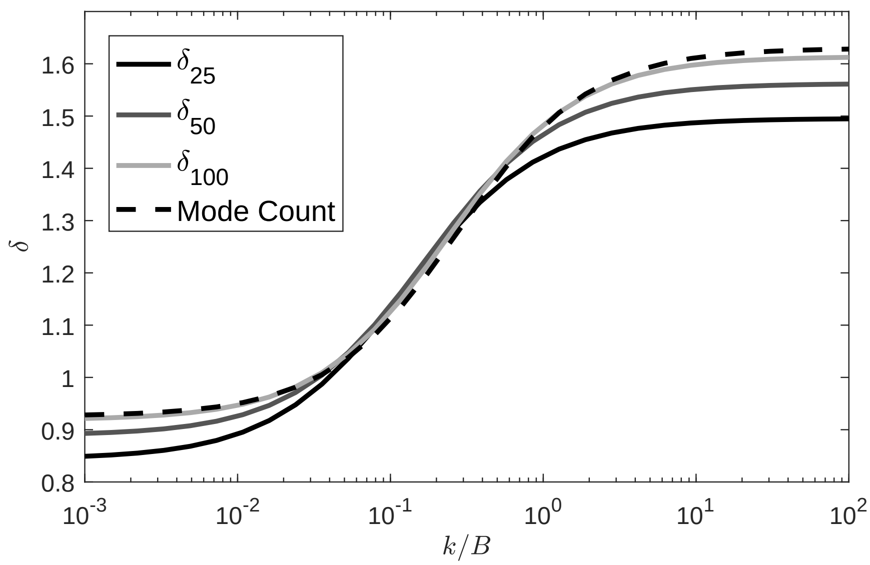

Figure 5: Correction factor of a rectangular plate constrained on all edges

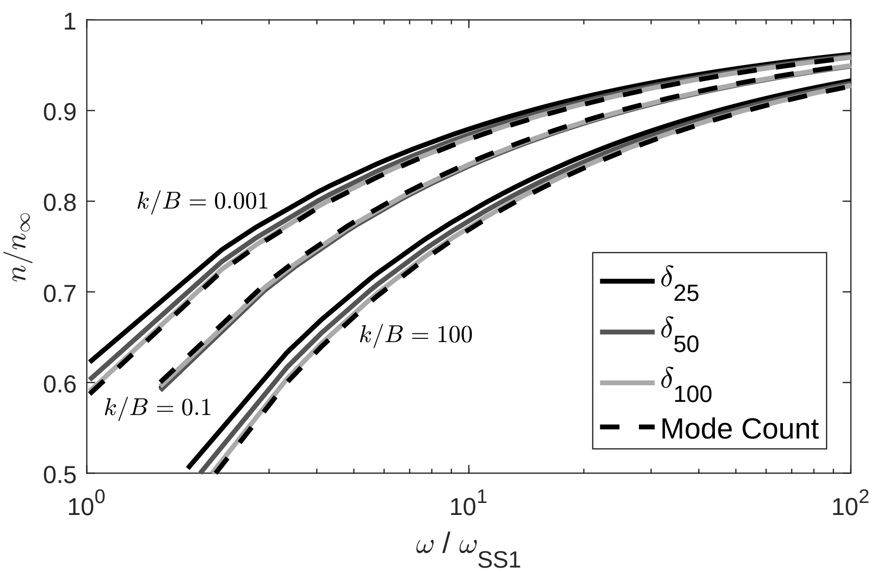

Figure 6: Modal density of a rectangular plate constrained on all edges

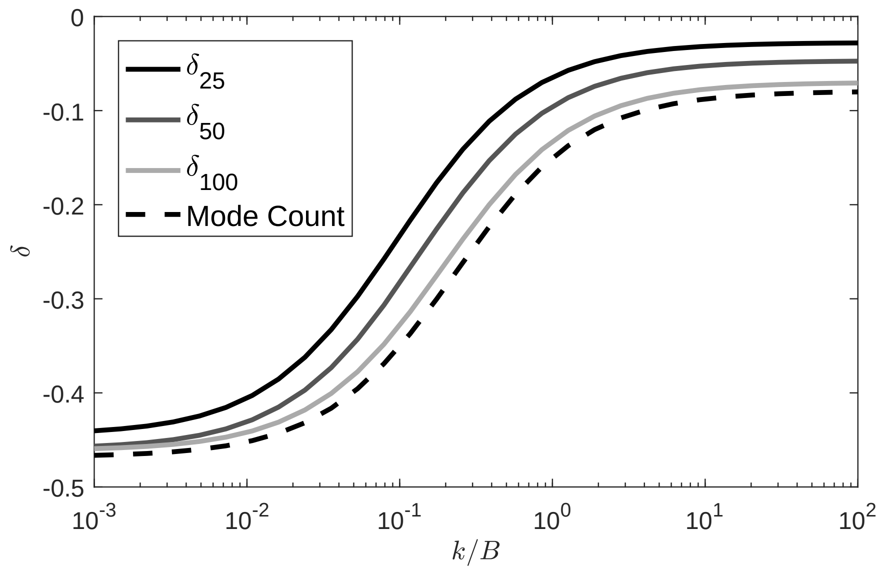

Figure 7: Correction factor of a rectangular plate constrained on long edges

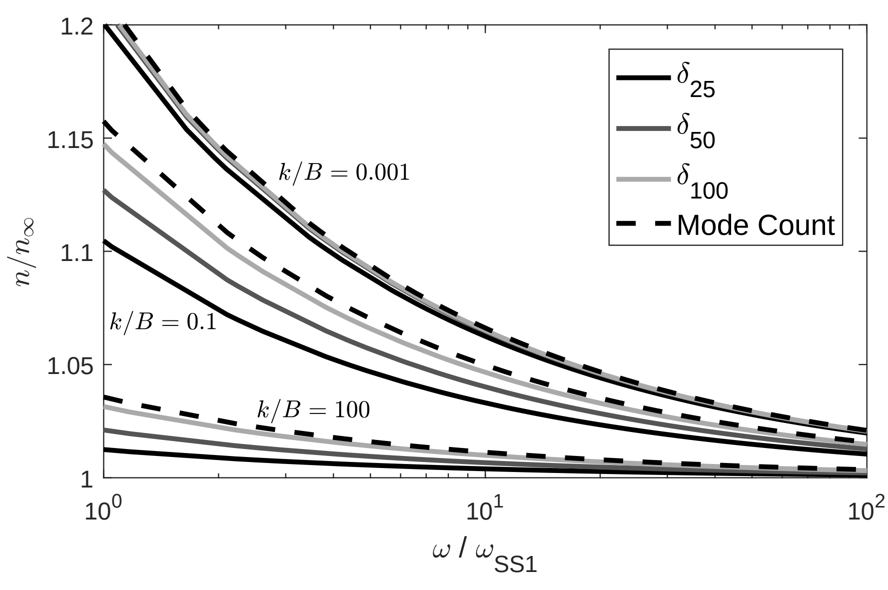

Figure 8: Modal density of a rectangular plate constrained on long edges

6. Conclusions

The modal density of a planar structure was investigated in a wide frequency range by merging low and high-frequency information: the high-frequency dynamics was attributed to the stacking properties of the structure and extracted by means of a statistical analysis of sound transmission through the infinite counterpart of the structure, while the low-frequency dynamics was corrected accounting for the actual boundary conditions of the structure according to information acquired by means of a modal analysis.

The proposed approach provides a modal density corrected for the real size of the structure and purged of acoustically ineffective modes as proven by the comparison with a mode count performed on the finite element model of a homogeneous plate. Furthermore, the comparison with an alternative analytical formulation, suited for the specific case of a three-layer system, proves the effectiveness of the proposed method in a wider frequency range. The retrieved dispersion solutions perfectly match those obtained using a periodic approach applied to the finite element model of a unit cell of the medium. Moreover, the modal density is immediately available at each dispersion solution, so avoiding the need to evaluate the related group velocity. The procedure for estimating the boundary correction factor was also validated for different boundary conditions, including elastic restraints.

Ultimately, the efficiency of a transfer matrix approach in modeling any multi-layered structure accurately was combined with the versatility of a finite element modal analysis in catching the effect of boundaries on the modal density.

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