Stable determination of an inclusion in a layered medium

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Abstract

We consider the inverse problem of detecting an inclusion in a layered conductor through electrostatic measurements taken on the boundary. We analyze in particular the stability issue showing that the solution depends from the available data with a rate of continuity of logarithmic type.

1 Introduction

In this paper we consider the inverse problem of determining an unknown inclusion contained in a layered medium from electrostatic boundary measurements. In particular we focus our analysis on the stability issue, that is the dependance of the inclusion from the measurements performed. This result follows the line started in [3], where it is considered a domain of constant conductivity inside which a region with different unknown conductivity is located. It is shown that the dependance of the inclusion from the boundary measurements is of logarithmic type. This rate of continuity turns out to be optimal as shown by examples in [10].

The approach to get stability has been later applied to different contexts (more general isotropic conductivity [6], a class of anisotropic conductivities [9], inverse scattering [7], thermal imaging [11, 12], elasticity [4]) and it is based mainly on two arguments:

• quantitative estimates of unique continuation;
• singular solutions.

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Here we will use the same approach. As for the quantitative estimates of unique continuation we will take advantage of the recent paper [13], where it has been derived a three-region inequality for a second order elliptic equation with a jump discontinuous coefficient. A crucial tool to get such inequality is the Carlemann estimate proved in [8]. It is possible indeed to provide a precise evaluation of the propagation of smallness measured on the boundary up to the inclusion.

As for the singular solution method, we will use the asymptotic study contained in [3] to get the stability estimates. Let us mention here that, in the present setting, the inclusion is assumed to be located at a positive distance from the interface of the layer. The major difficulty relays on the fundamental solution argument. In particular it is not clear how to write explicitly such a solution when the boundary of the inclusion and the interface intersect each other.

The three region argument, since the background conductivity is known, does not notice the presence of the interface. Therefore the result, we prove, can be obtained for multi layers media no matter the number of interfaces there are, as long as the inclusion does not touch the interface.

The paper is organized as follows. In the next Section 2, after some notations and definitions, we will state our main result, whose proof is presented in the next Section 3. The proof is based on some auxiliary propositions proved in Section 4.

2 Main Result

Let us first premise some notations and definitions. Let the domain $\Omega$ be a bounded open set in $\mathbb{R}^n$ and the layer $\Sigma$ be a closed hyper-surface contained in $\Omega$. With $\Sigma$, the domain $\Omega$ is separated into the union of three parts

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-,$$

where $\Omega_{\pm}$ are open subsets such that $\partial \Omega_- = \partial \Omega \cup \Sigma$ and $\partial \Omega_+ = \Sigma$. We denote by $D$ a subset of $\Omega$ such that $D \subset \Omega_+ \subset \Omega$. We consider $\gamma(x)$ the conductivity of $\Omega$ of the form

$$\gamma(x) = c_1 + (c_2 - c_1)\chi_{\Omega_+} + (k - c_2)\chi_D,$$

where $c_1$ and $c_2$ are given constants and $k$ is an unknown constant.

For points $x \in \mathbb{R}^n$, we will write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$. Moreover, denoted by $\text{dist}(\cdot, \cdot)$ the standard Euclidean distance, we define

$$B_r(x) = \{y \in \mathbb{R}^n | \text{dist}(x, y) \leq r\}, \quad B_r'(x') = \{y' \in \mathbb{R}^{n-1} | \text{dist}(x', y') \leq r\}$$
as the open balls with radius \( r \) centered at \( x \) and \( x' \) respectively. We write \( Q_r(x) = B'_r(x') \times (x_n - r, x_n + r) \) for the cylinder in \( \mathbb{R}^n \). For simplicity, we use \( B_r, B'_r, Q_r \) instead of \( B_r(0), B'_r(0'), Q_r(0) \) respectively. We shall also denote half domain, as well as its associated ball and cylinder

\[
\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}; \quad B^+_r = B_r \cap \mathbb{R}^n_+; \quad Q^+_r = Q_r \cap \mathbb{R}^n_+.
\]

**Definition 2.1.** Let \( \Omega \) be the bounded domain in \( \mathbb{R}^n \). Given \( \alpha \in (0, 1] \), we say a portion \( S \) of \( \partial \Omega \) is of \( C^{1,\alpha} \) class with constants \( r, L > 0 \) if for any point \( p \in S \), there exists a rigid transformation \( \varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) of coordinates under which we have

\[
\Omega \cap B_r = \{(x', x_n) \in B_r | x_n > \varphi(x')\},
\]

where \( \varphi(\cdot) \) is a \( C^{1,\alpha} \) function on \( B'_r \), which satisfies

\[
\varphi(0) = |\nabla \varphi(0)| = 0
\]

and

\[
||\varphi||_{C^{1,\alpha}(B'_r)} \leq Lr,
\]

where the norm is defined as

\[
||\varphi||_{C^{1,\alpha}(B'_r)} := ||\varphi||_{L^\infty(B'_r)} + r||\nabla \varphi||_{L^\infty(B'_r)} + r^{1+\alpha} ||\nabla \varphi|_{\alpha,B'_r}
\]

Assumptions and a priori data

For \( f \in H^{1/2}(\partial \Omega) \), let \( u \) be the solution of the problem

\[
\begin{cases}
\text{div}(\gamma(x) \nabla u) = 0 & \text{in } \Omega, \\
u u \big|_{\partial \Omega} = f & \text{on } \partial \Omega.
\end{cases}
\]

(2.1)

Our inverse problem is addressed to determine the anomalous region \( D \) when the Dirichlet-to-Neumann map \( \Lambda_D \)

\[
\Lambda_D : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)
\]

is given for any \( f \in H^{1/2}(\partial \Omega) \). Here, \( \nu \) denotes the outer unit normal to \( \partial \Omega \), and \( \frac{\partial u}{\partial \nu} |_{\partial \Omega} \) corresponds to the current density measured on \( \partial \Omega \). Thus,
the Dirichlet–to–Neumann map represents the knowledge of infinitely many
boundary measurements.

Given constants \( r_1, M_1, M_2, \delta_1, \delta_2 > 0 \) and \( 0 < \alpha < 1 \), we assume the
domain \( \Omega \subset \mathbb{R}^n \) is bounded
\[ |\Omega| \leq M_2 r_1^n, \]
where \( |\cdot| \) denotes the Lebesgue measure.

The interface \( \Sigma \) is \( C^2 \) and assumed to stay away from the boundary of the
domain, as \( \text{dist}(\Sigma, \partial \Omega) \geq \delta_2 \), and the inclusion \( D \) is assumed to stay away
from \( \Sigma \), as \( \text{dist}(D, \Sigma) \geq \delta_1 \), and also \( \Omega \setminus D \) is connected. Both \( \partial D \) and \( \partial \Omega \)
are of \( C^{1,\alpha} \) class with constants \( r_1, M_1, M_2, \alpha, \delta_1, \delta_2 \).

We refer to \( n, r_1, M_1, M_2, \alpha, \delta_1, \delta_2 \) as the a priori data. To study the
stability, we also denote by \( D_1 \) and \( D_2 \) two inclusions in \( \Omega \), which satisfy the
above properties. The associated Dirichlet-to-Neumann map are \( \Lambda_{D_1} \) and \( \Lambda_{D_2} \).

**Theorem 2.2.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) and we have two known constants \( c_1, c_2 \)
and one unknown constant \( k \), which are given. Let \( D_1, D_2 \) be two inclusions in \( \Omega \) as above. If for any \( \varepsilon > 0 \) we have
\[ \|\Lambda_{D_1} - \Lambda_{D_2}\|_{L(H^{1/2}, H^{-1/2})} \leq \varepsilon, \]
then
\[ d_K(\partial D_1, \partial D_2) \leq \omega(\varepsilon), \]
where \( \omega \) is an increasing function on \( [0, +\infty) \), which satisfies
\[ \omega(t) \leq C|\log t|^{-\eta}, \quad \forall t \in (0, 1) \]
and \( C > 0, 0 < \eta \leq 1 \) are constants depending on the a priori data only.

### 3 Proof of the Main Result

The proof of Theorem 2.2 is based on some auxiliary propositions, and their
proofs are collected in the next Section 4. In what follows we define layers
of our domains. We denote by \( \mathcal{G} \) the connected component of \( \Omega \setminus (D_1 \cup D_2) \),
whose boundary contains \( \partial \Omega \). \( \Omega_D = \Omega \setminus \mathcal{G}, S_{2r} := \{ x \in \mathbb{R}^n \mid r \leq \text{dist}(x, \Omega) \leq 2r \}, S_r := \{ x \in \Omega \mid \text{dist}(x, \Omega) \leq r \} \) and \( \mathcal{G}^h := \{ x \in \mathcal{G} \mid \text{dist}(x, \Omega_D) \geq h \} \).

We recall that the layer \( \Sigma \) separates the domain into two parts known
as \( \Omega_- \) and \( \Omega_+ \). We also define \( \mathcal{F}^\lambda := \{ x \in \Omega_- \mid \text{dist}(x, \Sigma) \geq \lambda \} \), and \( \Sigma_\lambda := \{ x \in \Omega_- \mid \text{dist}(x, \Sigma) = \lambda \} \)

We introduce a variation of the Hausdorff distance called the modified
distance, which simplifies our proof.
Definition 3.1. The modified distance between $D_1$ and $D_2$ is defined as

$$d_m(D_1, D_2) := \max \left\{ \sup_{x \in \partial D_1 \cap \partial D} \text{dist}(x, \partial D_2), \sup_{x \in \partial D_2 \cap \partial D} \text{dist}(x, \partial D_1) \right\}. $$

With no loss of generality, we can assume that there exists a point $O \in \partial D_1 \cap \partial D$ such that the maximum of $d_m = d_m(D_1, D_2) = \text{dist}(O, D_2)$ is attained. We remark here that $d_m$ is not a metric, and in general, it does not dominate the Hausdorff distance. However, under our a priori assumptions on the inclusion, the following lemma holds.

Lemma 3.2. Under the assumptions of Theorem 2.2, there exists a constant $c_0 \geq 1$ only depending on $M_1$ and $\alpha$ such that

$$d_H(\partial D_1, \partial D_2) \leq c_0 d_m(D_1, D_2). \quad (3.1)$$

Proof. See [3, Proposition 3.3]

Another obstacle comes from the fact that the propagation of smallness arguments are based on an iterated application of the three spheres inequality for solutions of the equation over chains of balls contained in $\mathcal{G}$. Therefore, it is crucial to control from below the radii of these balls. In the following Lemma 3.3 we treat the case of points of $\partial D$ that are not reachable by such chains of balls. This problem was originally considered by [5] in the context of cracks detection in electrical conductors.

Let us premise some notations. Given $O = (0, \ldots, 0)$ the origin, $v$ a unit vector, $H > 0$ and $\vartheta \in (0, \pi/2)$, we denote

$$C(O, v, \vartheta, H) = \{ x \in \mathbb{R}^n : |x - (x \cdot v)v| \leq \sin \vartheta |x|, \ 0 \leq x \cdot v \leq H \}$$

the closed truncated cone with vertex at $O$, axis along the direction $v$, height $H$ and aperture $2\vartheta$. Given $R, d, 0 < R < d$ and $Q = -de_n$, where $e_n = (0, \ldots, 0, 1)$, let us consider the cone $C(O, -e_n, \arcsin \frac{R}{d}, \frac{d^2 - R^2}{d})$.

From now on, without loss of generality, we assume that

$$d_m(D_1, D_2) = \max_{x \in \partial D_1 \cap \partial D} \text{dist}(x, \partial D_2)$$

and we write $d_m = d_m(D_1, D_2)$.

We shall make use of paths connecting points in order that appropriate tubular neighborhoods of such paths still remain within $\mathbb{R}^n \setminus \Omega_D$. Let us pick a point $P \in \partial D_1 \cap \partial \Omega_D$, let $\nu$ be the outer unit normal to $\partial D_1$ at $P$ and let $d > 0$ be such that the segment $[(P + d\nu), P]$ is contained in $\mathbb{R}^n \setminus \Omega_D$. Given $P_0 \in \mathbb{R}^n \setminus \Omega_D$, let $\gamma$ be a path in $\mathbb{R}^n \setminus \Omega_D$ joining $P_0$ to $P + d\nu$. We consider
the following neighborhood of $\gamma \cup [(P + d \nu), P] \setminus \{P\}$ formed by a tubular neighborhood of $\gamma$ attached to a cone with vertex at $P$ and axis along $\nu$

\[
V(\gamma) = \bigcup_{S \in \gamma} B_R(S) \cup C \left( P, \nu, \arcsin \frac{R}{d}, \frac{d^2 - R^2}{d} \right). \tag{3.2}
\]

Note that two significant parameters are associated to such a set, the radius $R$ of the tubular neighborhood of $\gamma$, $\cup_{S \in \gamma} B_R(S)$, and the half-aperture $\arcsin \frac{R}{d}$ of the cone $C \left( P, \nu, \arcsin \frac{R}{d}, \frac{d^2 - R^2}{d} \right)$. In other terms, $V(\gamma)$ depends on $\gamma$ and also on the parameters $R$ and $d$. At each of the following steps, such two parameters shall be appropriately chosen and shall be accurately specified. For the sake of simplicity we convene to maintain the notation $V(\gamma)$ also when different values of $R$, $d$ are introduced. Also we warn the reader that it will be convenient at various stages to use a reference frame such that $P = O = (0, \ldots, 0)$ and $\nu = -e_n$.

**Lemma 3.3.** Under the above notation, there exist positive constants $\overline{d}$, $c_1$, where $\overline{d}$ only depends on $M_1$ and $\alpha$, and $c_1$ only depends on $M_1$, $\alpha$, $M_2$, and there exists a point $P \in \partial D_1$ satisfying

\[
c_1 d_m \leq \text{dist}(P, D_2),
\]

and such that, giving any point $P_0 \in S_{2\rho_0}$, there exists a path $\gamma \subset \overline{(\Omega^{\rho_0} \cup S_{2\rho_0}) \setminus \Omega_D}$ joining $P_0$ to $P + d \nu$, where $\nu$ is the unit outer normal to $D_1$ at $P$, such that, choosing a coordinate system with origin $O$ at $P$ and axis $e_n = -\nu$, the set $V(\gamma)$ introduced in (3.2) satisfies

\[
V(\gamma) \subset \mathbb{R}^n \setminus \Omega_D,
\]

provided $R = \frac{\overline{d}}{\sqrt{1 + L_0^2}}$, where $L_0$, $0 < L_0 \leq M_1$, is a constant only depending on $M_1$ and $\alpha$.

**Proof.** See [4, Lemma 4.2].

In order to use the information provided by the boundary measurements to evaluate the distance between two inclusions $D_1$ and $D_2$, we apply the following identity firstly introduced by Alessandrini in [1]. Let $u_i \in H^1(\partial \Omega)$, $i = 1, 2$, be solutions to (2.1) with conductivities $\gamma_{D_i} = c_1 + (c_2 - c_1) \chi_{\Omega_+} + (k - c_2) \chi_{\Omega_i}$, respectively, we have

\[
\int_{\Omega} (\gamma_{D_1} \nabla u_1 \cdot \nabla u_2) - \int_{\Omega} (\gamma_{D_2} \nabla u_1 \cdot \nabla u_2) = \int_{\partial \Omega} u_1 [\Lambda_{D_1} - \Lambda_{D_2}] u_2. \tag{3.3}
\]
However, when we use the method of fundamental solutions, we will only deal with the interface which is close to $\Omega_D$. This means we are only interested in what is happening inside of $\Omega_+$, where $\chi_{\Omega_+} = 1$ is used for conductivities. Thus, we are only interested in the operator $\text{div}((c_2 + (k - c_2)\chi_{D_1})\nabla \cdot)$ and the associated fundamental solutions $\Gamma_{D_i}$ for $i = 1, 2$. We apply (3.3) locally to $\Gamma_{D_1}$ and $\Gamma_{D_2}$, obtains

$$
\int_\Omega (c_2 + (k - c_2)\chi_{D_1}) \nabla \Gamma_{D_1}(:,:,\cdot, y) \cdot \nabla \Gamma_{D_2}(:,:,\cdot, z)
- \int_\Omega (c_2 + (k - c_2)\chi_{D_2}) \nabla \Gamma_{D_1}(:,:,\cdot, y) \cdot \nabla \Gamma_{D_2}(:,:,\cdot, z)
= \int_{\partial \Omega} \Gamma_{D_1}(:,:,\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(:,:,\cdot, z). \tag{3.4}
$$

For $y, z \in G \cap C\Omega$, where $C\Omega$ is the complementary of $\Omega$, we define

$$
S_{D_1}(y, z) = (k - c_2) \int_{D_1} \nabla \Gamma_{D_1}(:,:,\cdot, y) \cdot \nabla \Gamma_{D_2}(:,:,\cdot, z)
S_{D_2}(y, z) = (k - c_2) \int_{D_2} \nabla \Gamma_{D_1}(:,:,\cdot, y) \cdot \nabla \Gamma_{D_2}(:,:,\cdot, z)
$$

$$
f(y, z) = S_{D_1}(y, z) - S_{D_2}(y, z).
$$

Therefore (3.4) can be written as

$$
f(y, z) = \int_{\partial \Omega} \Gamma_{D_1}(:,:,\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(:,:,\cdot, z), \quad \forall y, z \in C\Omega. \tag{3.5}
$$

The following two propositions provide quantitative estimates on $f(y, y)$ and $S_{D_1}(y, y)$, when moving $y$ towards $O$ along $\nu(O)$. Their proof are postponed in the next Section 4.

**Proposition 3.4.** Given $\epsilon > 0$, the domain $\Omega$ and inclusions $D_1, D_2$, and a transformation of coordinates defined as $y = h\nu(O)$, if we have

$$
||\Lambda_{D_1} - \Lambda_{D_2}||_{L(H^{1/2},H^{-1/2})} < \epsilon, \tag{3.6}
$$

then for every $h$ where $0 < h < c\rho, 0 < c < 1$, and $c$ depends on $M_1$, we have

$$
|f(y, y)| \leq C_0 \frac{\epsilon^{B\rho r}}{h^T}. \tag{3.7}
$$

Here $0 < T < 1$ and $C_0, B, F > 0$ are constants that depend only on the a priori data.
Proposition 3.5. Given $\epsilon > 0$, the domain $\Omega$ and inclusions $D_1, D_2$, and a transformation of coordinates $y = hv(O)$ defined as above. Then for every $0 < h < r_0/2$

$$|S_{D_1}(y, y)| \geq C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3,$$

(3.8)

where $r_0 := \frac{\tilde{r}}{2} \min \left\{ \frac{3}{2} (8M_1)^{-1/\alpha}, \frac{1}{2} \right\}$, and $C_1, C_2, C_3$ are positive constants depending only on the a priori data.

Now, we have all the ingredients to conclude this section with the proof of Theorem 2.2.

Proof of Theorem 2.2. We start from the origin of the coordinate system, point $O \in \partial D_1 \cap \partial \Omega_D$, for which the maximum in Definition 3.1 is attained

$$d_m := d_m(D_1, D_2) = \text{dist}(O, D_2).$$

Then with a transformation of coordinates $y = hv(O)$ where $0 < h < h_1$, $h_1 := \min \{d_m, cr, r_0/2\}, 0 < c < 1$, where $c$ depends on $M_1$. By applying [Al-DC] Proposition 3.4 (i); i.e., $|\nabla \Gamma_{D_1}(x, y)| \leq c_1 |x - y|^{1-n}$, where $c_1 > 0$ depending only on $k, n, \alpha, M_1$; we have

$$|S_{D_1}(y,y)| = (k - c_2) \int_{D_2} \nabla \Gamma_{D_1}(\cdot, y) \nabla \Gamma_{D_2} (\cdot, y)$$

$$\leq (k - c_2) \int_{D_2} (c_1 |y|^{1-n})^2 \leq (k - c_2) c_1^2 \int_{D_2} (|d_m - h|^{1-n})^2$$

$$\leq (k - c_2) c_1^2 |d_m - h|^{2-2n} |D_2| \leq C_4 |d_m - h|^{2-2n}.$$

(3.9)

Here $|D_2|$ is the measure of the inclusion $D_2$ which is bounded by $|D_2| \leq |\Omega| \leq M_2 r_1^p$. Thus, $C_4$ depends on $k, n, \alpha, M_1, M_2, r_1$. From (3.7), we already have the upper bound of $f(y, y)$. Moreover, if we apply the triangular inequality, we obtain

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \leq |S_{D_1}(y,y) - S_{D_2}(y,y)| = |f(y,y)| \leq C_6 \frac{\epsilon Bh^F}{h^T}.$$

(3.10)

Meanwhile, (3.8) gives us the lower bound of $S_{D_1}(y,y)$. Therefore, together with (3.9) and (3.10), we obtain

$$C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3 \leq C_4 |d_m - h|^{2-2n} + C_6 \frac{\epsilon Bh^F}{h^T}.$$

We can rearrange terms, and with a bit modification of the notations of the constants, (in particular, let $C_3 = C_2 d_m^{2-2n}$) we have

$$C_1 h^{2-n} \leq C_4 |d_m - h|^{2-2n} + C_6 \frac{\epsilon Bh^F}{h^T}.$$
We can simplify the above as, by setting $C_5 = C_4/C_0$ and $C_6 = C_1/C_0$

$$C_5|d_m - h|^2 - 2n \geq C_0 h^{2-n} - \frac{\epsilon h^{F}}{h^{F}} = C_0 h^{2-n}(1 - \frac{\epsilon^{h^{F}}}{h^{K}}),$$

where $0 < K = n - 2 - T$. Now let $h = h(\epsilon) = \min\{ |\ln \epsilon|^{-1 + \frac{1}{n}}, d_m \}$, for $0 < \epsilon \leq \epsilon_1, \epsilon_1 \in (0, 1)$ such that $\exp(-B|\ln \epsilon_1|^{1/2}) = 1/2$. It is easy to see if $d_m \leq |\ln \epsilon|^{-\frac{1}{n}}$, the main Theorem 2.2 is already proved thanks to Lemma 3.2. Because we can set $\eta = \frac{1}{2F} > 0$, then

$$d_H(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 |\ln \epsilon|^{-\eta} = \omega(\epsilon) \quad (3.11)$$

In another case where $d_m \geq |\ln \epsilon|^{-\frac{1}{n}}$, it is easy to check

$$(d_m - h)^2 - 2n \geq \frac{C_6}{2C_5} h^{2-n} \implies d_m \leq C_7 |\ln \epsilon|^{-\frac{2}{F(n-1)}}.$$ 

Here we can solve $d_m$ because here $h = h(\epsilon) = |\ln \epsilon|^{-\frac{1}{F}}$, and $C_7$ depends only on the a priori data. Therefore we conclude the proof by setting $\eta = \frac{n-2}{F(n-1)}$

$$d_H(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 C_7 |\ln \epsilon|^{-\eta} = \omega(\epsilon) \quad (3.12)$$

and for $\epsilon_1 \leq \epsilon$, we can also include the proof because $d_m \leq |\Omega| \leq M_2 r_1^n$.

$$d_H(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 M_2 r_1^n = \omega(\epsilon). \quad (3.13)$$

We can conclude the proof Theorem 2.2 by (3.11), (3.12) and (3.13)

$$d_H(\partial D_1, \partial D_2) \leq C d_m = \omega(\epsilon),$$

where $C$ only depends on the a priori data.

4 Proof of the Auxiliary Propositions

First, we prove Proposition 3.4 mainly followed by [3, Proposition 3.5], with three sphere inequalities. When it comes to the situation that we need to cross $\Sigma$ during the iterative process, we apply the three-region inequalities. The proof contains two major steps: (1) we need to define our “smallness” outside of the domain $\Omega$, and then we will propagate this smallness until it arrives at any given small ball contained inside $\mathcal{F}^\lambda$; (2) we use three-region inequality to propagate the “smallness” by crossing $\Sigma$; (3) we continue with three-sphere inequalities until the “smallness” arrives arbitrarily close to $O \in \partial D_1$. 

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Before proving Proposition 3.4, let us briefly recall this result contained in [13, Theorem 3.1]. Based on some suitable Carleman estimate (see [8, Theorem 2.1]), the following three region inequality in the \(L^2\) norm across the interface \(y = 0\) holds.

**Theorem 4.1.** There exist \(C\) and \(R\), depending on \(\lambda_0, M_0, n\) such that if \(0 < R_1, R_2 < R\) then

\[
\int_{U_2} |u|^2 \, dx \leq (e^{\tau_0 R_2} + CR_1^{-4}) \left( \int_{U_1} |u|^2 \, dxdy \right)^{\frac{R_2}{2n(1+3)}} \left( \int_{U_3} |u|^2 \, dxdy \right)^{\frac{2R_1 + 2R_2}{2n(1+3)}},
\]

where \(\tau_0\) is the constant derived in the Carleman estimate [8, Theorem 2.1],

\[
\begin{align*}
U_1 &= \{-4R_2 \leq z, \ -\frac{R_1}{8a} < y < \frac{R_1}{a}\}, \\
U_2 &= \{-R_2 \leq z \leq \frac{R_1}{2a}, \ y < \frac{R_1}{8a}\}, \\
U_3 &= \{-4R_2 \leq z, \ y < \frac{R_1}{a}\},
\end{align*}
\]

\[a = \alpha_+/\delta\] and \(z = \frac{\alpha_+}{\delta} y + \frac{\beta}{2\delta^2} y^2 - \frac{1}{2\delta} |x|^2\).

We also compute \(H := \frac{\delta}{\beta} [\alpha_- - \sqrt{\alpha_-^2 - 2R_2\beta}]\) to measure "vertical depth" of the region \(U_2\) below the \(x\)-axis.

**Proof of Proposition 3.4.** Let us consider \(f(y, \cdot)\) with a fixed \(y \in S_{2r}\) then

\[
\Delta_w f(y, w) = 0 \quad \text{in} \quad \mathcal{C}\Pi_D.
\]

For \(x \in S_{2r}\), by (3.5) and (3.6), we have the smallness quantity

\[
|f(y, x)| \leq C(r, M_1, M_2)||\Gamma_{D_1} - \Gamma_{D_2}|| = \epsilon.
\]

Also by [3, Proposition 3.4], the uniform bound of \(f\) is given as

\[
|f(y, x)| \leq ch^{2-2n}, \quad \text{in} \quad \mathcal{G}^h \cup \mathcal{F}^\lambda.
\]

For any \(0 < \tau < r\) and for every \(\overline{x} \in \mathcal{F}^\lambda\), we can have smallness on any arbitrarily small ball \(B_{\tau/2}(\overline{x}) \subset \mathcal{F}^\lambda \subset \mathcal{G}^h\) by iteratively applying three-sphere inequalities on a simple arc \(\gamma \in \overline{\Omega_-} \cup \overline{S}_r \cup \overline{S}_{2r}\) which connects \(\overline{x}\) and \(x\). By [3] (4.21), we can reach \(\overline{x}\) from \(x\) with a finite number \(s\) of balls. Thus we obtain

\[
||f(y, \cdot)||_{L^\infty(B_{\tau/2}(\overline{x}))} \leq C(h^{1-n})^{1-\tau^s} e^{-\tau^s},
\]

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where $C$ depends on the a prior data.

Now let us deal with the situation when smallness is crossing the interface $\Sigma$. We define a coordinator system locally in a small neighbor near $\Sigma$. For any point $O \in \Sigma$, the outer norm $\nu$ from $O$ onto $\Omega_-$ with respect to $\Sigma$ is defined as the $y$-axis; and its tangential direction is defined as the $x$-axis. We choose $\frac{R_s}{8n} = 2\lambda$, and $\lambda \leq \frac{\tau}{15}$ for $\mathcal{F}$. Now by (4.2), there are three regions $U_1, U_2$ and $U_3$ located near the $\Sigma$, where $U_1 \subset \mathcal{F}$ and $U_2 \cap \Omega_+ \neq \emptyset$.

Notice with the choice of $\lambda$, we can use a finite amount of balls to cover the region $U_1$. We pick up $\omega_j \in \Omega_{7/2+\lambda}$, where $\Omega_{7/2+\lambda} := \{ x \in \Omega : \text{dist}(x, \Sigma) = 7/2 + \lambda \}$. Then there exists $J < \infty$ such that $\bigcup_{j=1}^{J} B_{7/2}(\omega_j) \subset U_1$. By standard bound for $L^2$ and $L^\infty$ norms, and (4.6), we obtain

$$
\|f(y, \cdot)\|_{L^2(U_2)} \leq C\|f(y, \cdot)\|_{L^2(U_1)} \cdot \|f(y, \cdot)\|_{L^2(U_3)}
$$

$$
\leq C\|f(y, \cdot)\|_{L^\infty(B_{r_0/2}(x_0))} \cdot (h^{2-2n})^{2(1-A)}
$$

$$
\leq C(h^{-1-n})^{(1-r^*)} \epsilon A^{r^*} \cdot (h^{2-2n})^{2(1-A)}
$$

(4.7)

where $A = 4 - 3A - Ar^*$ and $B = Ar^*$, $C$ depends on $\tau_0, R_1, R_2, \lambda_0, M_0, n, J$ and a prior data. Then we pick up a small ball inside of $U_2 \cap \Omega_+$ as the start for the rest of the propagation. For the above coordinator system $x-O-y$, we choose $x_0 = (0, -H/2)$ and $r_0 < H$, so that $B_{r_0/2}(x_0) \subset U_2 \cap \Omega_+$. By (4.7),

$$
\|f(y, \cdot)\|_{L^\infty(B_{r_0/2}(x_0))} \leq \|f(y, \cdot)\|_{L^2(U_2)} \leq C(h^{-1-n})^A \epsilon B.
$$

(4.8)

The rest of the propagation is similar as (4.6). If we choose any $0 < r_0 < r_0$ and any $\omega_0 \in \Omega_+$, by connecting $x_0$ and $\omega_0$ with a simple arc, we obtain

$$
\|f(y, \cdot)\|_{L^\infty(B_{r_0/2}(\omega_0))} \leq C(h^{-1-n})^A \epsilon B.
$$

(4.9)

Then rest of the proof is followed by [3] (4.22) to (4.25): we define a truncated cone $C(O, \nu(O), \theta, r)$, in which $O \in D_1$ is the point where the maximum of Definition 3.1 is attained. Then we consider $f(y, w)$ as a function of $y$ to obtain similar results. The last step is to choose $y = w = h\nu(O)$, where $\nu(O)$ is the exterior unit normal to $\partial \Omega_D$ in $O$, we can obtain

$$
|f(y, y)| \leq Ch^T(\epsilon B A^{k(h)-1}) \gamma A^{k(h)-1}.
$$

(4.10)

We observe that for $0 < k < cr$, where $0 < c < 1$ depends on $M_1$, $k(h) \leq c|\log A| = -c\log h$, we can rewrite

$$
A^{k(h)} = \exp\{-c\log h \log A\} = h^{-c\log A} = h^{c|\log A|} = h^Q,
$$

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\[(A^{k(h)})^2 = (hQ)^2 = h^F\]

with \(F = 2Q = 2c|\log A|\). Therefore

\[|f(y, y)| \leq C h^{-T} e^{B(A^{k(h)})^2} \leq \exp\{-T \log h\} \exp\{B(A^{k(h)})^2 \log \epsilon\} \leq \exp\{-T \log h + Bh^F \log \epsilon\} = \frac{\epsilon Bh^F}{h^F},\]

where \(B = \frac{\gamma \tilde{B}}{A^2}\)

The proof of Proposition 3.5 is based on the asymptotic behaviour of the fundamental solutions locally in the neighbour of \(O \in D_1\), which is contained complete inside of \(\Omega_+\) because \(\text{dist}(D, \Sigma) \geq \delta_1\)

\[
\text{div}[(c_2 + (k - c_2)\chi_D)\nabla \Gamma_D(\cdot - y)] = -\delta(\cdot - y)
\]

\[
\text{div}[(c_2 + (k - c_2)\chi_+)\nabla \Gamma_+ (\cdot - y)] = -\delta(\cdot - y),
\]

where \(\chi_+\) is the characteristics function if the half-space \(\{x_n > 0\}\). If \(\Gamma\) is the standard fundamental solution of the Laplace operator, and \(y^* = (y', -y_n)\) is the reflecting point, we have the following relationship

\[
\Gamma_+(x, y) = \begin{cases} 
\frac{1}{k} \Gamma(x, y) + \frac{k-c_2}{k(2-k)} \Gamma(x, y^*) & \text{for } x_n > 0, y_n > 0 \\
\frac{2}{k+c_2} \Gamma(x, y) & \text{for } x_n < 0, y_n > 0 \\
\frac{1}{c_2} \Gamma(x, y) - \frac{k-c_2}{c_2(2-k)} \Gamma(x, y^*) & \text{for } x_n < 0, y_n < 0.
\end{cases}
\]

We have the following theorem.

**Theorem 4.2.** Let \(D \in \mathbb{R}^n\) be an open set with \(C^{1,\alpha}\) boundary subjected to constants \(M_1, r\). We have \(\Gamma_D\) and \(\Gamma_+\) the fundamental solutions defined above, respectively. The following asymptotic estimate holds for any \(x, y \in \mathbb{R}^n\)

\[|\nabla \Gamma_D(x, y)| \leq C|x - y|^{1-n}.\]

As for \(x \in D \cap B_r(O)\) and every \(y = hv(O)\), with \(0 < \rho < r_0\) and \(0 < h < r_0\) where \(r_0 = \frac{1}{2} \min\{\frac{1}{2}(8M_1)^{-1/2}, \frac{1}{2}\}\) we have

\[|\Gamma_D(x, y) - \Gamma_+(x, y)| \leq \frac{C}{r^\alpha} |x - y|^\alpha - n + 2\]

\[|\nabla \Gamma_D(x, y) - \nabla \Gamma_+(x, y)| \leq \frac{C}{r^{\alpha^2}} |x - y|^\alpha^2 - n + 1,
\]

where \(C > 0\) only depends on the a priori data.
For both proofs of Theorem 4.2 and Proposition 3.5, we refer to the proofs of Propositions 3.4 and 3.6 in [3] with a slightly modification on the constant coefficients. In fact, during the integration steps, we use $k - c_2$ as the constant instead of $k - 1$. This won’t affect the proofs since both $k - c_2$ and $k - 1$ can be absorbed into a constant $C$ in the final step, where $C$ depends on $k$. We mention that in our paper, $\Gamma_+$ is also represented as a linear combination of standard Laplace $\Gamma$ with constants coefficients.

References


