Early Proceedings of the Alterman Conference on Geometric Algebra and Summer School on Kähler Calculus

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Early Proceedings of the Alterman Conference on Geometric Algebra and Summer School on Kähler Calculus

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Dear Colleagues,

We wish to thank all authors for their contributions to these Early Proceedings and especially to Mr. Eric Alterman for his financial support to the Conference and Summer School.

For the second time, I am having the pleasant task of editing electronic Early Proceedings. The aim is to make the writings of other participants promptly available to attendants. Book editions of proceedings of many conferences are usually so much delayed that interest in them has substantially decreased by the time they finally appear. In the Early Proceedings, new authors have the opportunity to publish papers that may very well not be accepted in scientific journals, which usually have a limited size and different selection criteria.

We thank very much Professors Rafał Abłamowicz, Nikolay Marchuk, Zbigniew Oziewicz, Waldyr Jr. Rodrigues, José G. Vargas and Dr. Lucy Kabza for reviewing the submitted papers and giving their scientific advice. In some cases, members of the scientific committee and authors have begun a fruitful dialogue like between master and student. The beauty of teaching lies in the fact that the experienced (generally elder) masters can transmit to the inexperienced (generally younger) students knowledge that cannot be found in the literature. There are occasions when the breaking of this chain causes knowledge to be lost, and later generations must then discover what older generations already knew. Therefore, I want to thank Prof. José G. Vargas for his main role as master of the Kähler calculus during the Summer School of the Alterman Conference. Without him, it might fall into oblivion. Fortunately, he has removed the dust that covered the writings of Erich Kähler on differential calculus, which were never translated into English. He has summarized their contents for us in the *Notes on the Kähler calculus for the Summer School* included in these Early Proceedings, which we hope will be very useful material for our learning. Moreover, he has assured that there will be a more polished and efficient version of the same material in some other venue, be it a book or a second Alterman Summer School, or both.

Ramon González Calvet,

September 21st, 2016

Foreword to the *Notes on the Kähler calculus for the Summer School*

During months prior to the event *Alterman Conference on Clifford Algebra and Summer School on Kähler Calculus*, I wrote the chapters that follow in response to the non-availability in English of comparable contents. There were meant to be instrumental in preparing knowledgeable mathematicians to absorb the material and share with me the task of teaching them at the Summer School.

Things did not go as planned, owing to different reasons, which need not be reported here. Through polishing, simplification and eliminations, future versions of this material will be presented almost as if for use in a college course. For the moment and in spite of the many typos that I am sure still remain, nothing can be lost from having them as an alternative presentation of a large part of the original German papers.

Chapter 1 is an introduction that can be ignored without consequence. In the remaining chapters, we observe that no differential forms valued in tangent structures are used, observation which is of the essence in understanding this calculus. The material presented is certainly contained as particular case of his calculus of tensor-valued differential forms, but he gave no application for this more general valuedness other than proposing a very cumbersome "Kähler-Dirac" equation, for which we have no use.

I want to thank you all for your participation and I apologize for any deficiencies that should not have occurred. This event encountered severe intrinsic difficulties almost until a few weeks prior to its taking place. In order to avoid them, a second Alterman Summer School would not include a Conference in parallel. It would rather consist of four days of exclusive teaching of as many related topics: (a) the required basic Clifford algebra, (b) Kähler algebra (it is Clifford algebra, but the fact that its elements are differential forms makes a great difference), (c) Kähler calculus proper and (d) Kähler's quantum mechanics. I hope that you do not fail to realize the opportunity you may have to learn this most powerful calculus now that you are aware of its existence. Of particular relevance for the occurrence of an Alterman 2 event is that we find mathematicians who can teach this calculus without resort to non-scalar valuedness. If you can think of such candidates, please let me know as soon as possible.

It was a great pleasure to interact with you. Kind regards.

José G. Vargas

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ON THE STRUCTURE THEOREM OF CLIFFORD ALGEBRAS

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Abstract. In this paper, theory and construction of spinor representations of real Clifford algebras $C\ell_{p,q}$ in minimal left ideals are reviewed. Connection with a general theory of semisimple rings is shown. The actual computations can be found in, for example, [2].

Keywords. Artinian ring, Clifford algebra, division ring, group algebra, idempotent, minimal left ideal, semisimple module, Radon-Hurwitz number, semisimple ring, Wedderburn-Artin Theorem

Mathematics Subject Classification (2010). Primary: 11E88, 15A66, 16G10; Secondary: 16S35, 20B05, 20C05, 68W30

Contents

1. Introduction

Theory of spinor representations of real Clifford algebras $C\ell_{p,q}$ over a quadratic space (V, Q) with a nondegenerate quadratic form Q of signature (p, q) is well known [11, 18, 19, 24]. The purpose of this paper is to review the structure theorem of these algebras in the context of a general theory of semisimple rings culminating with Wedderburn-Artin Theorem [26].

Section 2 is devoted to a short review of general background material on the theory of semisimple rings and modules as a generalization of the representation theory of group algebras of finite groups [17, 26]. While it is well-known that Clifford algebras $C\ell_{p,q}$ are associative finite-dimensional unital semisimple $\mathbb{R}$-algebras, hence the representation theory of semisimple rings [26, Chapter 7] applies to them, it is also possible to view

these algebras as twisted group algebras $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ of a finite group $(\mathbb{Z}_2)^n$ [5–7, 9, 13, 23]. While this last approach is not pursued here, for a connection between Clifford algebras $C\ell_{p,q}$ and finite groups, see [1, 6, 7, 10, 20, 21, 27] and references therein.

In Section 3, we state the main Structure Theorem on Clifford algebras $C\ell_{p,q}$ and relate it to the general theory of semisimple rings, especially to the Wedderburn-Artin theorem. For details of computation of spinor representations, we refer to [2] where these computations were done in great detail by hand and by using `CLIFFORD`, a Maple package specifically designed for computing and storing spinor representations of Clifford algebras $C\ell_{p,q}$ for $n = p + q \leq 9$ [3, 4].

Our standard references on the theory of modules, semisimple rings and their representation is [26]; for Clifford algebras we use [11, 18, 19] and references therein; on representation theory of finite groups we refer to [17, 25] and for the group theory we refer to [12, 14, 22, 26].

2. Introduction to Semisimple Rings and Modules

This brief introduction to the theory of semisimple rings is based on [26, Chapter 7] and it is stated in the language of left R-modules. Here, R denotes an associative ring with unity 1. We omit proofs as they can be found in Rotman [26].

Definition 1. Let R be a ring. A **left R-module** is an additive abelian group M equipped with **scalar multiplication** $R \times M \to M$, denoted $(r, m) \mapsto rm$, such that the following axioms hold for all $m, m' \in M$ and all $r, r' \in R$:

(i) $r(m + m') = rm + rm'$,
(ii) $(r + r')m = rm + r'm$,
(iii) $(rr')m = r(r'm)$,
(iv) $1m = m$.

Left R-modules are often denoted by ${}_RM$.

In a similar manner one can define a **right R-module** with the action by the ring elements on M from the right. When R and S are rings and M is an abelian group, then M is a (R, S)**-bimodule**, denoted by ${}_RM_S$, if M is a left R-module, a right S-module, and the two scalar multiplications are related by an associative law: $r(ms) = (rm)s$ for all $r \in R, m \in M$, and $s \in S$.

We recall that a spinor left ideal S in a simple Clifford algebra $C\ell_{p,q}$ by definition carries an irreducible and faithful representation of the algebra, and it is defined as $C\ell_{p,q}f$ where f is a primitive idempotent in $C\ell_{p,q}$. Thus, as it is known from the Structure Theorem (see Section 3), that these ideals are (R, S)-bimodules where $R = C\ell_{p,q}$ and $S = fC\ell_{p,q}f$. Similarly, the right spinor modules $fC\ell_{p,q}$ are (S, R)-bimodules. Notice that the associative law mentioned above is automatically satisfied because $C\ell_{p,q}$ is associative.

We just recall that when k is a field, every finite-dimensional k-algebra A is both **left** and **right noetherian**, that is, any ascending chain of left and right ideals stops (the **ACC ascending chain condition**). This is important for Clifford algebras because, eventually, we will see that every Clifford algebra can be decomposed into a finite direct sum of left spinor $C\ell_{p,q}$-modules (ideals). For completeness we mention that every finite-dimensional k-algebra A is both **left** and **right artinian**, that is, any descending chain of left and right ideals stops (the **DCC ascending chain condition**).

Thus, every Clifford algebra $C\ell_{p,q}$, as well as every group algebra kG, when G is a finite group, which then makes kG finite dimensional, have both chain conditions by a dimensionality argument.

Definition 2. A left ideal L in a ring R is a **minimal left ideal** if $L \neq (0)$ and there is no left ideal J with $(0) \subsetneq J \subsetneq L$.

One standard example of minimal left ideals in matrix algebras $R = \mathrm{Mat}(n, k)$ are the subspaces $\mathrm{COL}(j)$, $1 \leq j \leq n$, of $\mathrm{Mat}(n, k)$ consisting of matrices $[a_{i,j}]$ such that $a_{i,k} = 0$ when $k \neq j$ (cf. [26, Example 7.9]).

The following proposition relates minimal left ideals in a ring R to simple left R-modules. Recall that a left R-module M is **simple** (or **irreducible**) if $M \neq \{0\}$ and M has no proper nonzero submodules.

Proposition 1 (Rotman [26])**.**

(i) *Every minimal left ideal L in a ring R is a simple left R-module.*
(ii) *If R is left artinian, then every nonzero left ideal I contains a minimal left ideal.*

Thus, the above proposition applies to Clifford algebras $C\ell_{p,q}$: every left spinor ideal S in $C\ell_{p,q}$ is a simple left $C\ell_{p,q}$-module; and, every left ideal in $C\ell_{p,q}$ contains a spinor ideal.

Recall that if D is a division ring, then a left (or right) D-module V is called a **left** (or **right**) **vector space** over D. In particular, when the division ring is a field k, then we have a familiar concept of a k-vector space. Since the concept of linear independence of vectors generalizes from k-vector spaces to D-vector spaces, we have the following result.

Proposition 2 (Rotman [26])**.** *Let V be a finitely generated*[1] *left vector space over a division ring D.*

(i) *V is a direct sum of copies of D; that is, every finitely generated left vector space over D has a basis.*
(ii) *Any two bases of V have the same number of elements.*

Since we know from the Structure Theorem, that every spinor left ideal S in simple Clifford algebras $C\ell_{p,q}$ ($p - q \neq 1 \bmod 4$) is a right $\mathbb{K}$-module where $\mathbb{K}$ is one of the division rings $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the above proposition simply tells us that every spinor left ideal S is finite-dimensional over $\mathbb{K}$ where $\mathbb{K}$ is one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

In semisimple Clifford algebras $C\ell_{p,q}$ ($p-q = 1 \bmod 4$), we have to be careful as the faithful double spinor representations are realized in the direct sum of two spinor ideals $S \oplus \hat{S}$ which are right $\mathbb{K} \oplus \hat{\mathbb{K}}$-modules, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$.[2] Yet, it is easy to show that $\mathbb{K} \oplus \hat{\mathbb{K}}$ is not a division ring.

Thus, Proposition 2 tells us that every finitely generated left (or right) vector space V over a division ring D has a left (a right) dimension, which may be denoted $\dim V$. In [16] Jacobson gives an example of a division ring D and an abelian group V, which is both a right and a left D-vector space, such that the left and the right dimensions are not equal. In our discussion, spinor minimal ideal S will always be a left $C\ell_{p,q}$-module and a right $\mathbb{K}$-module.

[1]The term "finitely generated" means that every vector in V is a linear combination of a finite number of certain vectors $\{x_1, \ldots, x_n\}$ with coefficients from R. In particular, a k-vector space is finitely generated if and only if it is finite-dimensional [26, Page 405].

[2]Here, $\hat{S} = \{\hat{\psi} \mid \in S\}$, and similarly for $\hat{\mathbb{K}}$, where $\hat{}$ denotes the grade involution in $C\ell_{p,q}$.

Since semisimple rings generalize the concept of a group algebra $\mathbb{C}G$ for a finite group G (cf. [17, 26]), we first discuss semisimple modules over a ring R.

Definition 3. A left R-module is **semisimple** if it is a direct sum of (possibly infinitely many) simple modules.

The following result is an important characterization of semisimple modules.

Proposition 3 (Rotman [26])**.** *A left R-module M over a ring R is semisimple if and only if every submodule of M is a direct summand.*

Recall that if a ring R is viewed as a left R-module, then its submodules are its left ideals, and, a left ideal is minimal if and only if it is a simple left R-module [26].

Definition 4. A ring R is **left semisimple**[3] if it is a direct sum of minimal left ideals.

One of the important consequences of the above for the theory of Clifford algebras, is the following proposition.

Proposition 4 (Rotman [26])**.** *Let R be a left semisimple ring.*

(i) *R is a direct sum of finitely many minimal left ideals.*
(ii) *R has both chain conditions on left ideals.*

From a proof of the above proposition one learns that, while $R = \bigoplus_i L_i$, that is, R is a direct sum of finitely-many left minimal ideals, the unity 1 decomposes into a sum $1 = \sum_i f_i$ of mutually annihilating primitive idempotents f_i, that is, $(f_i)^2 = f_i$, and $f_i f_j = f_j f_i = 0, i \neq j$. Furthermore, we find that $L_i = Rf_i$ for every i.

We can conclude from the following fundamental result [15, 26] that every Clifford algebra $C\ell_{p,q}$ is a semisimple ring, because every Clifford algebra is a twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ for $n = p + q$ and a suitable twist [1, 7, 9].

Theorem 1 (Maschke's Theorem)**.** *If G is a finite group and k is a field whose characteristic does not divide $|G|$, the kG is a left semisimple ring.*

For characterizations of left semisimple rings, we refer to [26, Section 7.3].

Before stating Wedderburn-Artin Theorem, which is all-important to the theory of Clifford algebras, we conclude this part with a definition and two propositions.

Definition 5. A ring R is **simple** if it is not the zero ring and it has no proper nonzero two-sided ideals.

Proposition 5 (Rotman [26])**.** *If D is a division ring, then $R = \mathrm{Mat}(n, D)$ is a simple ring.*

Proposition 6 (Rotman [26])**.** *If $R = \bigoplus_j L_j$ is a left semisimple ring, where the L_j are minimal left ideals, then every simple R-module S is isomorphic to L_j for some j.*

The main consequence of this last result is that every simple, hence irreducible, left $C\ell_{p,q}$-module, that is, every (left) spinor module of $C\ell_{p,q}$, is isomorphic to some minimal left ideal L_j in the direct sum decomposition of $R = C\ell_{p,q}$.

Following Rotman, we divide the Wedderburn-Artin Theorem into the existence part and a uniqueness part. We also remark after Rotman that Wedderburn proved

[3]One can define a **right semisimple** ring R if it is a direct sum of minimal right ideals. However, it is known [26, Corollary 7.45] that a ring is left semisimple if and only if it is right semisimple.

the existence theorem 2 for semisimple k-algebras, where k is a field, while E. Artin generalized this result to what is now known as the Wedderburn-Artin Theorem.

Theorem 2 (Wedderburn-Artin I)**.** *A ring R is left semisimple if and only if R is isomorphic to a direct product of matrix rings over division rings $D_1, \ldots, D_m$, that is*

$$R \cong \mathrm{Mat}(n_1, D_1) \times \cdots \times \mathrm{Mat}(n_m, D_m). \tag{1}$$

A proof of the above theorem yields that if $R = \bigoplus_j L_j$ as in Proposition 6, then each division ring $D_j = \mathrm{End}_R(L_j), j = 1, \ldots, m$, where $\mathrm{End}_R(L_j)$ denotes the ring of all R-endomorphisms of L_j. Another consequence is the following corollary.

Corollary 1. *A ring R is left semisimple if and only if it is right semisimple.*

Thus, we may refer to a ring as being **semisimple** without specifying from which side.[4] However, we have the following result which we know applies to Clifford algebras $C\ell_{p,q}$. More importantly, its corollary explains part of the Structure Theorem which applies to simple Clifford algebras. Recall from the above that every Clifford algebra $C\ell_{p,q}$ is left artinian (because it is finite-dimensional).

Proposition 7 (Rotman [26])**.** *A simple left artinian ring R is semisimple.*

Corollary 2. *If A is a simple left artinian ring, then $A \cong \mathrm{Mat}(n, D)$ for some $n \geq 1$ and some division ring D.*

Before we conclude this section with the second part of the Wedderburn-Artin Theorem, which gives certain uniqueness of the decomposition (1), we state the following definition and a lemma.

Definition 6. Let R be a left semisimple ring, and let

$$R = L_1 \oplus \cdots \oplus L_n, \tag{2}$$

where the L_j are minimal left ideals. Let the ideals $L_1, \ldots, L_m$, possibly after re-indexing, be such that no two among them are isomorphic, and so that every L_j in the given decomposition of R is isomorphic to one and only one L_i for $1 \leq i \leq m$. The left ideals

$$B_i = \bigoplus_{L_j \cong L_i} L_j \tag{3}$$

are called the **simple components** of R relative to the decomposition $R = \bigoplus_j L_j$.

Lemma 1 (Rotman [26])**.** *Let R be a semisimple ring, and let*

$$R = L_1 \oplus \cdots \oplus L_n = B_1 \oplus \cdots \oplus B_m \tag{4}$$

where the L_j are minimal left ideals and the B_i are the corresponding simple components of R.

(i) *Each B_i is a ring that is also a two-sided ideal in R, and $B_i B_j = (0)$ if $i \neq j$.*
(ii) *If L is any minimal left ideal in R, not necessarily occurring in the given decomposition of R, then $L \cong L_i$ for some i and $L \subseteq B_i$.*
(iii) *Every two-sided ideal in R is a direct sum of simple components.*
(iv) *Each B_i is a simple ring.*

[4]Not every simple ring is semisimple, cf. [26, Page 554] and reference therein.

Thus, we will gather from the Structure Theorem, that for simple Clifford algebras $C\ell_{p,q}$ we have only one simple component, hence $m = 1$, and thus all 2^k left minimal ideals generated by a complete set of 2^k primitive mutually annihilating idempotents which provide an orthogonal decomposition of the unity 1 in $C\ell_{p,q}$ (see part (c) of the theorem and notation therein). Then, for semisimple Clifford algebras $C\ell_{p,q}$ we have obviously $m = 2$.

Furthermore, we have the following corollary results.

Corollary 3 (Rotman [26])**.**

(1) *The simple components $B_1, \ldots, B_m$ of a semisimple ring R do not depend on a decomposition of R as a direct sum of minimal left ideals;*
(2) *Let A be a simple artinian ring. Then,*
   (i) *$A \cong \mathrm{Mat}(n, D)$ for some division ring D. If L is a minimal left ideal in A, then every simple left A-module is isomorphic to L; moreover, $D^{\mathrm{op}} \cong \mathrm{End}_A(L)$.*[5]
   (ii) *Two finitely generated left A-modules M and N are isomorphic if and only if $\dim_D(M) = \dim_D(N)$.*

As we can see, part (1) of this last corollary gives a certain invariance in the decomposition of a semisimple ring into a direct sum of simple components. Part (2i), for the left artinian Clifford algebras $C\ell_{p,q}$ implies that simple Clifford algebras ($p - q \neq 1 \bmod 4$) are simple algebras isomorphic to a matrix algebra over a suitable division ring D. From the Structure Theorem we know that D is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending on the value of $p - q \bmod 8$. Part (2ii) tells us that any two spinor ideals S and S', which are simple right $\mathbb{K}$-modules (due the right action of the division ring $\mathbb{K} = fC\ell_{p,q}f$ on each of them) are isomorphic since their dimensions over $\mathbb{K}$ are the same.

We conclude this introduction to the theory of semisimple rings with the following uniqueness theorem.

Theorem 3 (Wedderburn-Artin II)**.** *Every semisimple ring R is a direct product,*

$$R \cong \mathrm{Mat}(n_1, D_1) \times \cdots \times \mathrm{Mat}(n_m, D_m), \tag{5}$$

where $n_i \geq 1$, and D_i is a division ring, and the numbers m and n_i, as well as the division rings D_i, are uniquely determined by R.

Thus, the above results, and especially the Wedderburn-Artin Theorem (parts I and II), shed a new light on the main Structure Theorem given in the following section. In particular, we see it as a special case of the theory of semisimple rings, including the left artinian rings, applied to the finite dimensional Clifford algebras $C\ell_{p,q}$.

We remark that the above theory applies to the group algebras kG where k is an algebraically closed field and G is a finite group.

3. The Main Structure Theorem on Real Clifford Algebras $C\ell_{p,q}$

We have the following main theorem that describes the structure of Clifford algebras $C\ell_{p,q}$ and their spinorial representations. In the following, we will analyze statements in that theorem. The same information is encoded in the well–known Table 1 in [19, Page 217].

[5]By D^{op} we mean the **opposite ring** of D: It is defined as $D^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in D\}$ with multiplication defined as $a^{\mathrm{op}} \cdot b^{\mathrm{op}} = (ba)^{\mathrm{op}}$.

Structure Theorem. *Let $C\ell_{p,q}$ be the universal real Clifford algebra over (V,Q), Q is non-degenerate of signature (p,q).*

(a) *When $p-q \neq 1 \bmod 4$ then $C\ell_{p,q}$ is a simple algebra of dimension 2^{p+q} isomorphic with a full matrix algebra $\mathrm{Mat}(2^k,\mathbb{K})$ over a division ring $\mathbb{K}$ where $k = q - r_{q-p}$ and r_i is the Radon-Hurwitz number.*[6] *Here $\mathbb{K}$ is one of $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ when $(p-q) \bmod 8$ is $0,2,$ or $3,7,$ or $4,6$.*

(b) *When $p - q = 1 \bmod 4$ then $C\ell_{p,q}$ is a semisimple algebra of dimension 2^{p+q} isomorphic to $\mathrm{Mat}(2^{k-1},\mathbb{K}) \oplus \mathrm{Mat}(2^{k-1},\mathbb{K})$, $k = q - r_{q-p}$, and $\mathbb{K}$ is isomorphic to $\mathbb{R}$ or $\mathbb{H}$ depending whether $(p-q) \bmod 8$ is 1 or 5. Each of the two simple direct components of $C\ell_{p,q}$ is projected out by one of the two central idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots n})$.*

(c) *Any element f in $C\ell_{p,q}$ expressible as a product*

$$f = \frac{1}{2}(1 \pm \mathbf{e}_{i_1})\frac{1}{2}(1 \pm \mathbf{e}_{i_2}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{i_k}) \tag{6}$$

where $\mathbf{e}_{i_j}$, $j = 1,\ldots,k$, are commuting basis monomials in $\mathcal{B}$ with square 1 and $k = q - r_{q-p}$ generating a group of order 2^k, is a primitive idempotent in $C\ell_{p,q}$. Furthermore, $C\ell_{p,q}$ has a complete set of 2^k such primitive mutually annihilating idempotents which add up to the unity 1 of $C\ell_{p,q}$.

(d) *When $(p-q) \bmod 8$ is $0,1,2,$ or $3,7,$ or $4,5,6,$ then the division ring $\mathbb{K} = fC\ell_{p,q}f$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and the map $S \times \mathbb{K} \to S$, $(\psi,\lambda) \mapsto \psi\lambda$ defines a right $\mathbb{K}$-module structure on the minimal left ideal $S = C\ell_{p,q}f$.*

(e) *When $C\ell_{p,q}$ is simple, then the map*

$$C\ell_{p,q} \xrightarrow{\gamma} \mathrm{End}_{\mathbb{K}}(S), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi \tag{7}$$

gives an irreducible and faithful representation of $C\ell_{p,q}$ in S.

(f) *When $C\ell_{p,q}$ is semisimple, then the map*

$$C\ell_{p,q} \xrightarrow{\gamma} \mathrm{End}_{\mathbb{K}\oplus\hat{\mathbb{K}}}(S \oplus \hat{S}), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi \tag{8}$$

gives a faithful but reducible representation of $C\ell_{p,q}$ in the double spinor space $S \oplus \hat{S}$ where $S = \{uf \,|\, u \in C\ell_{p,q}\}$, $\hat{S} = \{u\hat{f} \,|\, u \in C\ell_{p,q}\}$ and $\hat{}$ stands for the grade involution in $C\ell_{p,q}$. In this case, the ideal $S \oplus \hat{S}$ is a right $\mathbb{K} \oplus \hat{\mathbb{K}}$-module structure, $\hat{\mathbb{K}} = \{\hat{\lambda} \,|\, \lambda \in \mathbb{K}\}$, and $\mathbb{K}\oplus\hat{\mathbb{K}}$ is isomorphic to $\mathbb{R}\oplus\mathbb{R}$ when $p-q = 1 \bmod 8$ or to $\mathbb{H} \oplus \hat{\mathbb{H}}$ when $p - q = 5 \bmod 8$.

Parts (a) and (b) address simple and semisimple Clifford algebras $C\ell_{p,q}$ which are distinguished by the value of $p-q \bmod 4$ while the dimension of $C\ell_{p,q}$ is 2^{p+q}. For simple algebras, the Radon-Hurwitz number r_i defined recursively as shown, determines the value of the exponent $k = q - r_{q-p}$ such that

$$C\ell_{p,q} \cong \mathrm{Mat}(2^k,\mathbb{K}) \quad \text{when } p - q \neq 1 \bmod 4. \tag{9}$$

Then, the value of $p - q \bmod 8$ ("Periodicity of Eight" cf. [8, 19]) determines whether $\mathbb{K} \cong \mathbb{R},\mathbb{C}$ or $\mathbb{H}$. Furthermore, this automatically tells us, based on the theory outlined above, that

$$C\ell_{p,q} = L_1 \oplus \cdots \oplus L_N, \quad N = 2^k, \tag{10}$$

that is, that the Clifford algebras decomposes into a direct sum of $N = 2^k$ minimal left ideals (simple left $C\ell_{p,q}$-modules) L_i, each of which is generated by a primitive idempotent. How to find these primitive mutually annihilating idempotents, is determined in Part (c).

[6] The Radon-Hurwitz number is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0$, $r_1 = 1$, $r_2 = r_3 = 2, r_4 = r_5 = r_6 = r_7 = 3$.

In Part (b) we also learn that the Clifford algebra $C\ell_{p,q}$ is semisimple as it is the direct sum of two simple algebras:

$$C\ell_{p,q} \cong \mathrm{Mat}(2^{k-1}, \mathbb{K}) \oplus \mathrm{Mat}(2^{k-1}, \mathbb{K}) \quad \text{when } p - q = 1 \bmod 4. \tag{11}$$

Thus, we have two simple components in the algebra, each of which is a subalgebra. Notice that the two algebra elements

$$c_1 = \frac{1}{2}(1 + \mathbf{e}_{12\ldots n}) \quad \text{and} \quad c_2 = \frac{1}{2}(1 - \mathbf{e}_{12\ldots n}) \tag{12}$$

are central, that is, each belongs to the center $Z(C\ell_{p,q})$ of the algebra.[7] This requires that $n = p + q$ be odd, so that the unit pseudoscalar $\mathbf{e}_{12\ldots n}$ would commute with each generator $\mathbf{e}_i$, and that $(\mathbf{e}_{12\ldots n})^2 = 1$, so that expressions (12) would truly be idempotents. Notice, that the idempotents c_1, c_2 provide an **orthogonal decomposition** of the unity 1 since $c_1 + c_2 = 1$, and they are mutually annihilating since $c_1 c_2 = c_2 c_1 = 0$. Thus,

$$C\ell_{p,q} = C\ell_{p,q} c_1 \oplus C\ell_{p,q} c_2 \tag{13}$$

where each $C\ell_{p,q} c_i$ is a simple subalgebra of $C\ell_{p,q}$. Hence, by Part (a), each subalgebra is isomorphic to $\mathrm{Mat}(2^{k-1}, \mathbb{K})$ where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{H}$ depending on the value of $p - q \bmod 8$, as indicated.

Part (c) tells us how to find a complete set of 2^k primitive mutually annihilating idempotents, obtained by independently varying signs $\pm$ in each factor in (6), provide an orthogonal decomposition of the unity. The set of k commuting basis monomials $\mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k}$, which square to 1, is not unique. Stabilizer groups of these 2^k primitive idempotents $f_1, \ldots, f_N$ ($N = 2^k$) under the conjugate action of Salingaros vee groups are discussed in [6, 7]. It should be remarked, that each idempotent in (6) must have exactly k factors in order to be primitive.

Thus, we conclude from Part (c) that

$$C\ell_{p,q} = C\ell_{p,q} f_1 \oplus \cdots \oplus C\ell_{p,q} f_N, \quad N = 2^k, \tag{14}$$

is a decomposition of the Clifford algebra $C\ell_{p,q}$ into a direct sum of minimal left ideals, or, simple left $C\ell_{p,q}$-modules.

Part (d) determines the unique division ring $\mathbb{K} = f C\ell_{p,q} f$, where f is any primitive idempotent, prescribed by the Wedderburn-Artin Theorem, such that the decomposition (9) or (11) is valid, depending whether the algebra is simple or not. This part also reminds us that the left spinor ideals, while remaining left $C\ell_{p,q}$ modules, are right $\mathbb{K}$-modules. This is important when computing actual matrices in spinor representations (faithful and irreducible). Detailed computations of these representations in both simple and semisimple cases are shown in [2]. Furthermore, package `CLIFFORD` has a built-in database which displays matrices representing generators of $C\ell_{p,q}$, namely $\mathbf{e}_1, \ldots, \mathbf{e}_n$, $n = p+q$, for a certain choice of a primitive idempotent f. Then, the matrix representing any element $u \in C\ell_{p,q}$ can the be found using the fact that the maps γ shown on Parts (e) and (f), are algebra maps.

Finally, we should remark, that while for simple Clifford algebras the spinor minimal left ideal carries a **faithful** (and irreducible) representation, that is, $\ker \gamma = \{1\}$, in the case of semisimple algebras, each $\frac{1}{2}$ spinor space S and $\hat{S}$ carries an irreducible but not faithful representation. Only in the double spinor space $S \oplus \hat{S}$, one can realize the semisimple algebra faithfully. For all practical purposes, this means that each element u in a semisimple algebra must be represented by a pair of matrices, according to the

[7] The center $Z(A)$ of an k-algebra A contains all elements in A which commute with every element in A. In particular, from the definition of the k-algebra, $\lambda 1 \in Z(C\ell_{p,q})$ for every $\lambda \in k$.

isomorphism (11). In practice, the two matrices can then be considered as a single matrix, but over $\mathbb{K}\oplus\hat{\mathbb{K}}$ which is isomorphic to $\mathbb{R}\oplus\mathbb{R}$ or $\mathbb{H}\oplus\mathbb{H}$, depending whether $p-q = 1 \bmod 8$, or $p - q = 5 \bmod 8$. We have already remarked earlier that while $\mathbb{K}$ is a division ring, $\mathbb{K} \oplus \hat{\mathbb{K}}$ is not.

4. Conclusions

In this paper, the author has tried to show how the Structure Theorem on Clifford algebras $C\ell_{p,q}$ is related to the theory of semisimple rings, and, especially of left artinian rings. Detailed computations of spinor representations, which were distributed at the conference, came from [2].

5. Acknowledgments

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A TUTORIAL ON CLIFFORD[1] WITH eClifford[2] AND BIGEBRA[3] A MAPLE PACKAGE FOR CLIFFORD AND GRASSMANN ALGEBRAS

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Various computations in Clifford algebras $C\ell(V,B)$ of an arbitrary bilinear form B in $\dim V \leq 9$ can be performed with a free package CLIFFORD for Maple. Here, the bilinear form B is arbitrary, not necessarily symmetric, or, it could be purely symbolic. Since the package is based on Chevalley's definition of Clifford algebra as a subalgebra of an endomorphism algebra of Grassmann algebra, the underlying basis in $C\ell(B)$ is an undotted Grassmann basis, although a dotted Grassmann basis can be used when the antisymmetric part of B is non-zero. A new experimental package eClifford for $C\ell_{p,q,r}$ uses a different database and extends computations to vector spaces V of arbitrary dimension. Using CLIFFORD, one can solve, for example, algebraic equations when searching for general elements satisfying certain conditions, solve an eigenvalue problem for a Clifford number, and find its minimal polynomial, or compute algebra representations, such as spinor or regular. One can compute with Clifford algebras $C\ell_{p,q}$ viewed as twisted group rings $\mathbb{R}^t[G_{p,q}]$ of Salingaros vee groups $G_{p,q}$. Also, computations with quaternions, split quaternions, octonions, and matrices with entries in a Clifford algebra can easily be completed. Due to the fact that CLIFFORD is a Maple package based on Maple programming language, that is, it runs inside Maple, all Maple packages are available. Thus, CLIFFORD can be easily made to work with new special-purpose packages written by the user. Some examples of algorithms used in the package and computations will be presented.

Keywords: Bigebra, Clifford algebra, CLIFFORD, contraction, dotted wedge product, grade involution, Grassmann algebra, group algebra, twisted group algebra, multivector, octonions, quaternions, reversion, transposition, spinors, vee group, wedge product

Table of Contents (tentative):

[1] CLIFFORD is a Maple package developed and maintained jointly by R. Ablamowicz and Bertfried Fauser.

[2] eClifford is a Maple package developed and maintained by R. Ablamowicz.

[3] BIGEBRA is a Maple package for computing with tensors and Hopf algebras. It has been developed by Bertfried Fauser.

7. A fast algorithm `ecmul` for Clifford product in $C\ell_{p.q.r}$ based on Walsh functions (a modified `cmulWalsh3`)
8. Algebraic operations in $C\ell(B)$ including:
   (a) reversion, grade involution, conjugation, transposition $T_\varepsilon{}^\sim$ (in $C\ell_{p,q}$)
   (b) spinor representations of $C\ell_{p,q}$
   (c) computations with matrices with entries in a Clifford algebra
9. Research with `CLIFFORD` and related packages, such as `SymGroupAlgebra`:
   (a) Deriving and proving properties of the transposition anti-involution $T_\varepsilon{}^\sim$ in $C\ell_{p,q}$.
   (b) Computations with Salingaros vee groups $G_{p,q}$
   (c) Relating Clifford algebra $C\ell_{p,q}$ to the group algebra of Salingaros vee group $G_{p,q}$:
      (i) As a homomorphic image of the group algebra $\mathbb{R}[G_{p,q}]$ modulo a two dimensional ideal (Chernov [13])
      (ii) As a twisted group ring (Albuquerque and Majid [14])
10. On parallelizing the Clifford product in `CLIFFORD`
11. Using periodicity theorems in higher dimensional Clifford algebras or using `eCLIFFORD`
12. Appendix:
   (a) Sample help pages
   (b) Sample Maple worksheets created to derive the above papers, go to `http://math.tntech.edu/rafal/publications.html`

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APPLICATION OF GEOMETRIC ALGEBRA TO PLANE AND SPACE GEOMETRY

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ABSTRACT. Here we will give some examples of application of geometric algebra to solve geometric problems of plane geometry but also of the three-dimensional Euclidean space. Although scalar and exterior products suffice to solve many geometric problems, other emblematic problems such as Euler's line of a triangle or of a tetrahedron, Fermat's and Morley's theorems can only be solved by using geometric product. In this way, it is displayed how close the geometric algebra is to geometry.

1. BASIC NOTIONS OF PLANE GEOMETRIC ALGEBRA

The geometric algebra of the Euclidean plane $Cl_{2,0}$ is generated by two orthogonal unitary vectors e_1 and e_2:

$$e_1^2 = e_2^2 = 1 \qquad e_1 e_2 = -e_2 e_1 \tag{1}$$

A vector v in the Euclidean plane is a linear combination of both vectors:

$$v = v_1\, e_1 + v_2\, e_2 \qquad v_1,\, v_2 \in \mathbb{R} \tag{2}$$

On the other hand, the product of two vectors v and w yields a complex number:

$$(v_1\, e_1 + v_2\, e_2)(w_1\, e_1 + w_2\, e_2) = v_1\, w_1 + v_2\, w_2 + (v_1\, w_2 - v_2\, w_1) e_{12} \tag{3}$$

that we can write in short form as:

$$v\, w = v \cdot w + v \wedge w \tag{4}$$

The bivector e_{12} is the imaginary unit because its square is -1:

$$e_{12}^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1 \tag{5}$$

Therefore complex numbers z are elements of $Cl_{2,0}$ of the form:

$$z = a + b\, e_{12} \qquad a,\, b \in \mathbb{R} \tag{6}$$

A rotation of a vector v through an angle α is described as the multiplication of v on the right by the unitary complex number 1_α:

$$v' = v\ 1_\alpha = (v_1\ e_1 + v_2\ e_2)\ (\cos\alpha + e_{12}\ \sin\alpha) \tag{7}$$

$$v'_1\ e_1 + v'_2\ e_2 = (v_1 \cos\alpha - v_2 \sin\alpha)\ e_1 + (v_2 \cos\alpha + v_1 \sin\alpha)\ e_2 \tag{8}$$

which is the expression for a rotation:

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{9}$$

In order of brevity, we will denote the imaginary unit $e_{12} = i$. In this way, rotations (7) through an angle α will be written as:

$$v' = v\ \exp(i\alpha) \tag{10}$$

This expression of rotations is not general and can only be applied to vectors, which suffices for the geometric problems we will deal with. The general expression is:

$$v' = \exp\left(-\frac{i\alpha}{2}\right) v\ \exp\left(\frac{i\alpha}{2}\right) \tag{11}$$

which leaves complex numbers invariant.

Three coplanar vectors u, v, and w satisfy the *permutative property* ([1] p. 6) although the geometric product is not commutative:

$$u\ v\ w = w\ v\ u \tag{12}$$

The product of two vectors is a complex number. Under reversion (change of their order) the complex number becomes conjugate because the angle between vectors changes the sign. Therefore, the permutative property for a product of a vector v and a complex number z is written as ([1] p. 18):

$$v\ z = z^*\ v \qquad\qquad z \in \mathbb{C} \tag{13}$$

where $z^* = a - b\ e_{12}$ is the complex conjugate of $z = a + b\ e_{12}$.

Finally, an axial symmetry with respect to an axis with direction vector d is obtained as:

$$v' = d^{-1} v\ d \tag{14}$$

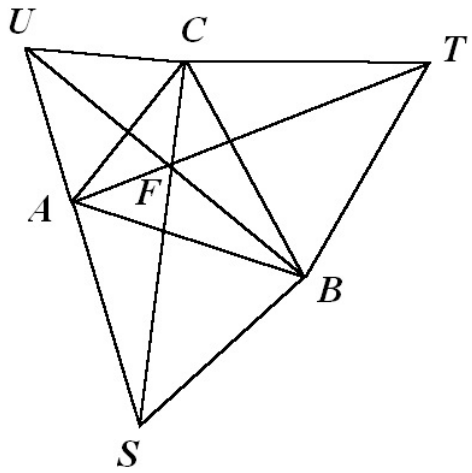

FIGURE 1. Fermat's theorem.

2. APPLICATION TO THEOREMS OF PLANE GEOMETRY

Let us see the application to some theorems of elemental geometry.

Theorem 2.1 (Fermat)**.** *Over each side of a triangle ΔABC draw an equilateral triangle. Let T, U and S be the vertices of the equilateral triangles that are respectively opposite to A, B and C (fig. 1). Then, segments AT, BU and CS have the same length, form angles of $2\pi/3$ and interesect at a unique point F, called the Fermat point. If P is any point in the plane, the addition of the three distances from P to the vertices of the triangle is minimal when $P = F$ provided that no interior angle of ΔABC is higher than $2\pi/3$ (fig. 2).*

Proof. ([1], p. 77)

Let us prove that BU is obtained by turning AT through $2\pi/3$:

$$AT\ t = (AC + CT)\ t = AC\ t + CT\ t = CU + BC = BU \qquad t = \exp\left(\frac{2\pi i}{3}\right) \tag{15}$$

because CU and BC are obtained from AC and CT respectively by means of a rotation thorugh $2\pi/3$. In the same way we find CS=$BU\ t$ and $AT = CS\ t$. Therefore CS, BU and AT have the same lenght and each of them is obtained from each other by successive rotations through $2\pi/3$. In fact, if we make the exterior product, we have:

$$AT \wedge BU = (AC + CT) \wedge (BA + AU) = (AC + BC\ t^*) \wedge (BA + CA\ t^*) \tag{16}$$

$$= AC \wedge BA + AC \wedge (CA\ t^*) + (BC\ t^*) \wedge BA + (BC\ t^*) \wedge (CA\ t^*)$$

$$= AC \wedge BA + \frac{AC\ CAt^* - CA\ t^* AC + BC\ t^* BA - BA\ BC\ t^*}{2} + BC \wedge CA$$

In the last term, one takes into account that a rotation of two vectors does not change their product. By applying the permutative property (13), we have:

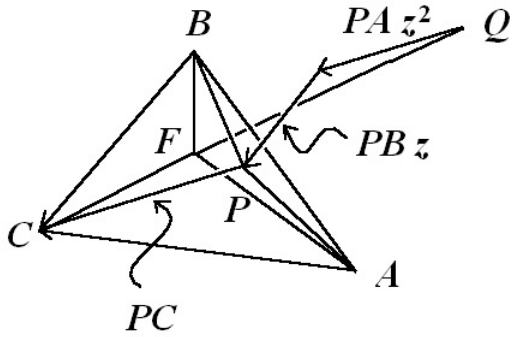

FIGURE 2. The addition of distances from any point P to the three vertices is minimal when P is the Fermat point F provided that the three interior angles are lower than $2\pi/3$.

$$AT \wedge BU = CA \wedge AB + BC \wedge CA + \frac{-AC^2 t^* + AC^2 t + BC\ BA\ t - BA\ BC\ t^*}{2} \tag{17}$$

Since $t = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$$AT \wedge BU = 2AB \wedge BC + \frac{AC^2\sqrt{3} - BC \wedge BA + BA \cdot BC\ i\sqrt{3}}{2} \tag{18}$$

According to the cosine theorem, $2BA \cdot BC = AB^2 + BC^2 - AC^2$. Then we finally arrive at:

$$AT \wedge BU = \frac{3}{2} AB \wedge BC + \frac{\sqrt{3}}{4}(AB^2 + BC^2 + CA^2) i \tag{19}$$

Since $AT \wedge BU = AT^2 i\sqrt{3}/2$, we finally have:

$$AT^2 = \sqrt{3}\|AB \wedge BC\| + \frac{AB^2 + BC^2 + CA^2}{2} = BU^2 = CS^2 \tag{20}$$

an expression that is invariant under cyclic permutation of the vertices because $AB \wedge BC = BC \wedge CA = CA \wedge AB$ is twice the area of the triangle.

Let us see that the addition of distances from any point P to the three vertices is minimal when P is the Fermat point. We must firstly prove that the vector sum of PA turned through $4\pi/3$, PB turned through $2\pi/3$ and PC is constant and independent of the point P (fig. 2), that is, for every two points P and P' the following equality always holds:

$$PA\ t^2 + PB\ t + PC = P'A\ t^2 + P'B\ t + P'C \tag{21}$$

A fact that is easily proven by arranging all the terms on one side of the equation:

$$PP'(t^2 + t + 1) = 0 \tag{22}$$

This product is always zero because $t^2+t+1=0$. Then, there is a unique point Q such that:

$$PA\ t^2 + PB\ t + PC = QC \tag{23}$$

For any point P, the three segments form a broken line as shown in fig. 2. Therefore, by the triangular inequality we have:

$$\|PA\| + \|PB\| + \|PC\| \geq \|QC\| \tag{24}$$

When P is the Fermat point F, these segments form a straight line. Then, the addition of the distances from F to the three vertices is minimal provided that no angle of the triangle is greater than $2\pi/3$:

$$\|FA\| + \|FB\| + \|FC\| = \|QC\| \leq \|PA\| + \|PB\| + \|PC\| \tag{25}$$

Otherwise, some vector among $FA\ t^2$, $FB\ t$ or FC has a direction opposite to the others, so that its length is subtracted from the others instead of added to them, and their sum is not minimal. □

Theorem 2.2 (Morley about interior trisectors)**.** *The intersections of the trisectors of the interior angles of a triangle form an equilateral triangle.*

Proof. Let A, B and C be the vertices of a generic triangle and 3α, 3β and 3γ the interior angles of the triangle (fig. 3). The trisectors divide each interior angle into three equal angles. The lengths of the sides will be denoted by a, b and c:

$$a = \|BC\| \qquad b = \|CA\| \qquad c = \|AB\| \tag{26}$$

Since the point P is the intersection of the line $\overline{BP}$ (whose direction vector is obtained from BC by turning it through an angle β), and $\overline{CP}$ (whose direction vector is obtained from CB by turning it through an angle $-\gamma$), we have:

$$P = B + \lambda\ u_{BC}\exp(i\beta) = C - \mu\ u_{BC}\exp(-i\gamma) \qquad \lambda,\mu \in \mathbb{R} \tag{27}$$

where u_{BC} is the unitary direction vector of $\overline{BC}$. By arranging terms we obtain:

$$BC = u_{BC}(\lambda\exp(i\beta) + \mu\ \exp(-i\gamma)) \tag{28}$$

$$a = \lambda\ \exp(i\beta) + \mu\ \exp(-i\gamma) \tag{29}$$

because $a = \|BC\|$. This complex equation yields a system of two real equations whose solution is:

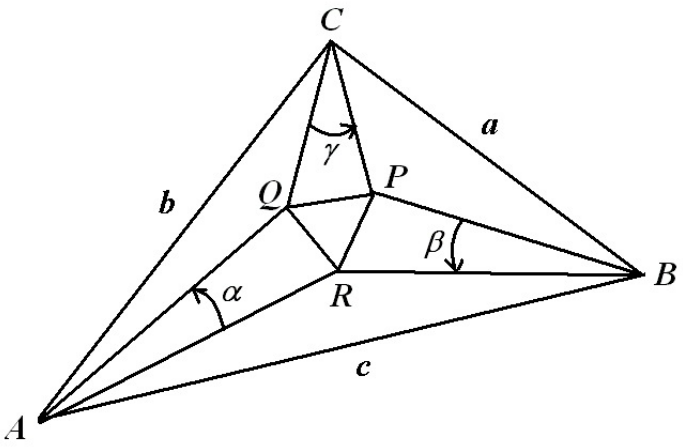

FIGURE 3. Morley's theorem. The intersections of the trisectors of the interior angles form an equilateral triangle.

$$\lambda = \frac{a\ \sin\gamma}{\sin(\beta+\gamma)} \qquad \mu = \frac{a\ \sin\beta}{\sin(\beta+\gamma)} \tag{30}$$

Now, by introducing this solution into (27) we have:

$$P = B + \frac{a\ \sin\gamma}{\sin(\beta+\gamma)} u_{BC} \exp(i\beta) \tag{31}$$

By cyclic permutation, the other points of Morley's triangle are:

$$Q = C + \frac{b\ \sin\alpha}{\sin(\gamma+\alpha)} u_{CA} \exp(i\gamma) \tag{32}$$

$$R = A + \frac{c\ \sin\beta}{\sin(\alpha+\beta)} u_{AB} \exp(i\alpha) \tag{33}$$

whence we build the vector PQ:

$$PQ = a\ u_{BC}\left(1 - \frac{\sin\gamma}{\sin(\beta+\gamma)}\exp(i\beta)\right) + \frac{b\sin\alpha}{\sin(\gamma+\alpha)} u_{CA}\exp(i\gamma)$$

$$= \frac{a\ \sin\beta}{\sin(\beta+\gamma)} u_{BC}\exp(-i\gamma) + \frac{b\sin\alpha}{\sin(\gamma+\alpha)} u_{CA}\exp(i\gamma) \tag{34}$$

The unitary vector u_{CA} is obtained from u_{CB} by a rotation through -3γ:

$$u_{CA} = -u_{BC}\exp(-3i\gamma) \tag{35}$$

whence we can write PQ only as function of u_{BC}:

$$PQ = u_{BC}\left[\frac{a\ \sin\beta}{\sin(\beta+\gamma)}\exp(-i\gamma) - \frac{b\sin\alpha}{\sin(\gamma+\alpha)}\exp(-2i\gamma)\right] \tag{36}$$

The sine theorem links the lengths of the three sides:

$$\frac{a}{\sin 3\alpha} = \frac{b}{\sin 3\beta} = \frac{c}{\sin 3\gamma} = 2\rho \tag{37}$$

where ρ is the radius of the circumcircle. Substitution of the sine theorem yields:

$$PQ = 2\rho\ u_{BC}\exp(-i\gamma)\left[\frac{\sin 3\alpha\sin\beta}{\sin(\frac{\pi}{3}-\alpha)} - \frac{\sin 3\beta\sin\alpha}{\sin(\frac{\pi}{3}-\beta)}\exp(-i\gamma)\right] \tag{38}$$

where in the denominators we have taken into account that $\alpha+\beta+\gamma=\pi/3$. In order to remove the denominators we make use of the trigonometric identity [4]:

$$\sin 3\alpha = 4\sin\alpha\ \sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{2\pi}{3}-\alpha\right) \tag{39}$$

whence we obtain:

$$PQ = 8\rho\ \sin\alpha\sin\beta\ u_{BC}\exp(-i\gamma)\left[\sin\left(\frac{2\pi}{3}-\alpha\right) - \sin\left(\frac{2\pi}{3}-\beta\right)\exp(-i\gamma)\right] \tag{40}$$

We now write α as function of β and γ only inside the parenthesis:

$$PQ = 8\rho\ \sin\alpha\sin\beta\ u_{BC}\exp(-i\gamma)\left[\sin\left(\beta+\gamma+\frac{\pi}{3}\right) - \sin\left(\frac{2\pi}{3}-\beta\right)\exp(-i\gamma)\right] \tag{41}$$

Since $\sin(\beta+\pi/3) = \sin(2\pi/3-\beta)$ because they are supplementary angles we have:

$$\begin{aligned} PQ &= 8\rho\ \sin\alpha\sin\beta\ u_{BC}\exp(-i\gamma)\left[\sin\left(\beta+\gamma+\frac{\pi}{3}\right) - \sin\left(\beta+\frac{\pi}{3}\right)\exp(-i\gamma)\right] \\ &= 8\rho\ \sin\alpha\sin\beta\ \sin\gamma\ u_{BC}\exp(-i\gamma)\exp\left(i\left(\beta+\frac{\pi}{3}\right)\right) \end{aligned} \tag{42}$$

after using the angle addition identity. This result can also be written as:

$$PQ = 8\rho\sin\alpha\sin\beta\sin\gamma\ u_{BC}\exp\left(i\left(\alpha+2\beta\right)\right) \tag{43}$$

The norm of this vector is invariant under cyclic permutation, which is already a proof of Morley's theorem:

$$\|PQ\| = \|QR\| = \|RP\| = 8\rho\sin\alpha\sin\beta\sin\gamma \tag{44}$$

□

There are many proofs of Morley's theorem [2, 3, 4, 5, 6], but the novelty of the final result (43) is to know the direction of each side of Morley's equilateral triangle, which neither proof gives. By cyclic permutation of the vertices in (43) we also have:

$$QR = 8\rho \sin\alpha \sin\beta \sin\gamma \, u_{CA} \exp(i(\beta + 2\gamma)) \tag{45}$$

$$RP = 8\rho \sin\alpha \sin\beta \sin\gamma \, u_{AB} \exp(i(\gamma + 2\alpha)) \tag{46}$$

In order to check these expressions, let us turn PQ through $-\pi/3$ to obtain $-QR$:

$$\begin{aligned} PQ \exp\left(-\frac{i\pi}{3}\right) &= 8\rho \; u_{BC} \sin\alpha \sin\beta \sin\gamma \exp(i(\alpha + 2\beta)) \exp(-i(\alpha + \beta + \gamma)) \\ &= 8\rho \; u_{BC} \exp(-3i\gamma) \sin\alpha \sin\beta \sin\gamma \exp(i(\beta + 2\gamma)) \end{aligned} \tag{47}$$

Since u_{CA} is obtained from u_{CB} by a rotation through -3γ, the substitution of eq. (35), gives:

$$PQ \exp\left(-\frac{i\pi}{3}\right) = -8\rho \; u_{CA} \sin\alpha \sin\beta \sin\gamma \exp(i(\beta + 2\gamma)) = -QR \qquad \surd \tag{48}$$

In the same way, turning the vector PQ through $\pi/3$ yields $-RP$:

$$\begin{aligned} PQ \exp\left(\frac{i\pi}{3}\right) &= 8\rho \; u_{BC} \sin\alpha \sin\beta \sin\gamma \exp(i(\alpha + 2\beta)) \exp(i(\alpha + \beta + \gamma)) \\ &= 8\rho \; u_{BC} \exp(3i\beta) \sin\alpha \sin\beta \sin\gamma \exp(i(\gamma + 2\alpha)) \end{aligned} \tag{49}$$

Since u_{BA} is obtained from u_{BC} by a rotation through 3β, we have:

$$PQ \exp\left(\frac{i\pi}{3}\right) = -8\rho \; u_{AB} \sin\alpha \sin\beta \sin\gamma \exp(i(\gamma + 2\alpha)) = -RP \qquad \surd \tag{50}$$

Theorem 2.3 (Morley about exterior trisectors)**.** *The intersections of the trisectors of the exterior angles of a triangle form an equilateral triangle. Moreover, these intersection together with the intersections of the exterior trisectors with the interior trisectors form three additional equilateral triangles.*

Proof. From now on and in order to simplify notation, we will write:

$$\alpha' = \frac{\pi}{3} - \alpha \qquad \beta' = \frac{\pi}{3} - \beta \qquad \gamma' = \frac{\pi}{3} - \gamma \tag{51}$$

These angles are indicated in fig. 4 and satisfy:

$$\alpha' + \beta' + \gamma' = \pi - \alpha - \beta - \gamma = \frac{2\pi}{3} \tag{52}$$

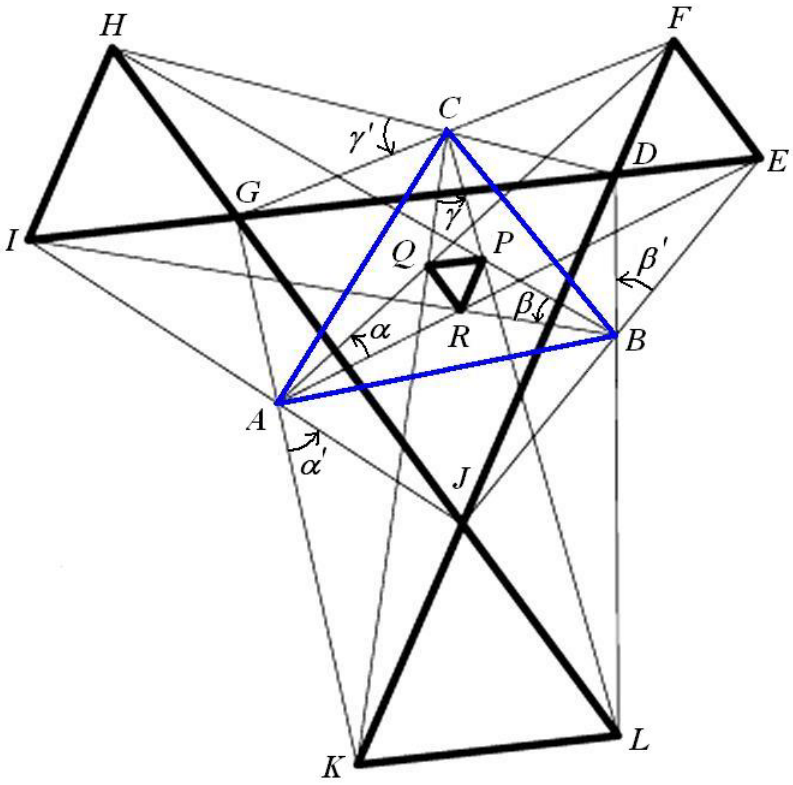

FIGURE 4. Morley's theorem. The exterior angles between sides are divided in three equal angles α', β' and γ' by their trisectors. The intersections of exterior and interior trisectors also form equilateral triangles.

Since the point D is the intersection of the exterior trisectors $\overline{CD}$ and $\overline{BD}$, and the unitary vector u_{CD} is obtained by turning u_{CB} through γ', and u_{BD} is obtained by turning u_{CB} through $-\beta'$, we have:

$$D = B + \lambda\ u_{BC}\exp(-i\beta') = C - \mu\ u_{BC}\exp(i\gamma') \tag{53}$$

By arranging points on the *lhs* we obtain:

$$BC = u_{BC}\left(\lambda\exp(-i\beta') + \mu\exp(i\gamma')\right) \qquad \lambda,\mu \in \mathbb{R} \tag{54}$$

The equation (54) is now similar to (28) with a solution analogous to (30) and (31)

$$\lambda = \frac{a\ \sin\gamma'}{\sin(\beta'+\gamma')} \qquad \mu = \frac{a\ \sin\beta'}{\sin(\beta'+\gamma')} \tag{55}$$

$$D = B + \frac{a\ \sin\gamma'}{\sin(\beta'+\gamma')}u_{BC}\exp(-i\beta') \tag{56}$$

The point E belongs to the interior trisector $\overline{AE}$ and the exterior trisector $\overline{BE}$ so that:

$$E = A + \xi\ u_{AB}\exp(i\alpha) = B + \psi\ u_{AB}\exp(i\beta') \tag{57}$$

By arranging points on the *lhs* we have:

$$AB = u_{AB}\left(\xi\exp(i\alpha) - \psi\exp(i\beta')\right) \quad \Rightarrow \quad c = \xi\exp(i\alpha) - \psi\exp(i\beta') \tag{58}$$

whose solution is:

$$\xi = \frac{c\sin\beta'}{\sin(\beta'-\alpha)} = \frac{c\sin\beta'}{\sin\gamma} \qquad \psi = \frac{c\sin\alpha}{\sin(\beta'-\alpha)} = \frac{c\sin\alpha}{\sin\gamma} \tag{59}$$

$$E = A + \frac{c\sin\beta'}{\sin\gamma}\ u_{AB}\exp(i\alpha) \tag{60}$$

because $\beta' - \alpha = \pi/3 - \beta - \alpha = \gamma$. Then we build the vector ED:

$$ED = AB - \frac{c\sin\beta'}{\sin\gamma}u_{AB}\exp(i\alpha) + \frac{a\sin\gamma'}{\sin(\beta'+\gamma')}u_{BC}\exp(-i\beta') \tag{61}$$

$$= -\frac{c\sin\alpha}{\sin\gamma}u_{AB}\exp(i\beta') + \frac{a\sin\gamma'}{\sin(\beta'+\gamma')}u_{BC}\exp(-i\beta')$$

Since $u_{AB} = u_{BC}\exp(-3i\beta')$:

$$ED = u_{BC}\left(-\frac{c\sin\alpha}{\sin\gamma}\exp(-2i\beta') + \frac{a\sin\gamma'}{\sin(\frac{\pi}{3}+\alpha)}\exp(-i\beta')\right) \tag{62}$$

and after applying the sine theorem (37) we have:

$$ED = 2\rho\ u_{BC}\exp(-i\beta')\left(-\frac{\sin 3\gamma\sin\alpha}{\sin\gamma}\exp(-i\beta') + \frac{\sin 3\alpha\sin\gamma'}{\sin(\frac{\pi}{3}+\alpha)}\right) \tag{63}$$

We now apply the trigonometric identity (39) taking into account that $\pi/3+\alpha$ and $2\pi/3-\alpha$ are supplementary angles and have the same sine:

$$ED = 8\rho\ u_{BC}\sin\alpha\exp(-i\beta')\left(-\sin\left(\frac{\pi}{3}-\gamma\right)\sin\left(\frac{2\pi}{3}-\gamma\right)\exp(-i\beta')\right. \tag{64}$$

$$\left. + \sin\left(\frac{\pi}{3}-\alpha\right)\sin\gamma'\right)$$

$$= 8\rho\ u_{BC}\sin\alpha\sin\gamma'\exp(-i\beta')\left(-\sin\left(\frac{2\pi}{3}-\gamma\right)\exp(-i\beta') + \sin\alpha'\right)$$

Since:

$$\frac{2\pi}{3}-\gamma=\pi-\alpha'-\beta' \qquad \Rightarrow \qquad \sin\left(\frac{2\pi}{3}-\gamma\right)=\sin(\alpha'+\beta') \tag{65}$$

we have:

$$ED = 8\rho\ u_{BC}\sin\alpha\sin\gamma'\exp(-2i\beta')\left(-\sin(\alpha'+\beta')+\sin\alpha'\exp(i\beta')\right) \tag{66}$$

and after applying the angle addition identity we arrive at:

$$ED = -8\rho\ u_{BC}\sin\alpha\sin\beta'\sin\gamma'\exp(-i(\alpha'+2\beta'))$$

$$= 8\rho\ u_{BC}\sin\alpha\sin\beta'\sin\gamma'\exp(i(\alpha+2\beta)) \tag{67}$$

where the Euler identity $\exp(-i\pi)=-1$ has been applied. Now we see that ED has the same direction as PQ (43):

$$u_{ED}=u_{PQ}=u_{BC}\exp(i(\alpha+2\beta)) \tag{68}$$

and its length is:

$$\|ED\| = 8\rho\sin\alpha\sin\beta'\sin\gamma' \tag{69}$$

In the same way, the point F is the intersection of the trisectors $\overline{AF}$ and $\overline{CF}$:

$$F = A+\phi\ u_{AC}\exp(-i\alpha) = C-\chi\ u_{AC}\exp(-i\gamma') \qquad \phi,\chi\in\mathbb{R} \tag{70}$$

which leads to:

$$AC = u_{AC}(\phi\exp(-i\alpha)+\chi\exp(-i\gamma')) \quad \Rightarrow \quad b=\phi\exp(-i\alpha)+\chi\exp(-i\gamma') \tag{71}$$

whose solution is:

$$\phi=\frac{b\sin\gamma'}{\sin(\gamma'-\alpha)}=\frac{b\sin\gamma'}{\sin\beta} \qquad \chi=-\frac{b\sin\alpha}{\sin(\gamma'-\alpha)}=-\frac{b\sin\alpha}{\sin\beta} \tag{72}$$

$$F = A+\frac{b\sin\gamma'}{\sin\beta}\ u_{AC}\exp(-i\alpha) \tag{73}$$

whence we obtain:

$$EF = -\frac{c\sin\beta'}{\sin\gamma}u_{AB}\exp(i\alpha)+\frac{b\sin\gamma'}{\sin\beta}u_{AC}\exp(-i\alpha) \tag{74}$$

Since u_{AC} is obtained by turning u_{AB} through 3α:

$$u_{AC} = u_{AB}\exp(3i\alpha) \tag{75}$$

we have:

$$\begin{aligned} EF &= u_{AB}\left(-\frac{c\sin\beta'}{\sin\gamma}\exp(i\alpha)+\frac{b\sin\gamma'}{\sin\beta}\exp(2i\alpha)\right) \\ &= u_{AB}\left(-\frac{c\sin(\alpha+\gamma)}{\sin\gamma}\exp(i\alpha)+\frac{b\sin(\alpha+\beta)}{\sin\beta}\exp(2i\alpha)\right) \\ &= 2\rho u_{AB}\left(-\frac{\sin 3\gamma\sin(\alpha+\gamma)}{\sin\gamma}\exp(i\alpha)+\frac{\sin 3\beta\sin(\alpha+\beta)}{\sin\beta}\exp(2i\alpha)\right) \end{aligned} \tag{76}$$

Applying the trigonometric identity (39) we have:

$$\begin{aligned} EF = 8\rho\, u_{AB}&\left(-\sin\left(\frac{\pi}{3}-\gamma\right)\sin\left(\frac{2\pi}{3}-\gamma\right)\sin(\alpha+\gamma)\exp(i\alpha)\right. \\ &\left.+\sin\left(\frac{\pi}{3}-\beta\right)\sin\left(\frac{2\pi}{3}-\beta\right)\sin(\alpha+\beta)\exp(2i\alpha)\right) \\ = 8\rho u_{AB}&\sin(\alpha+\beta)\sin(\alpha+\gamma)\left(-\sin\left(\frac{2\pi}{3}-\gamma\right)\right. \\ &\left.+\sin\left(\frac{2\pi}{3}-\beta\right)\exp(i\alpha)\right)\exp(i\alpha) \\ = 8\rho u_{AB}&\sin(\alpha+\beta)\sin(\alpha+\gamma)\left(-\sin\left(\frac{\pi}{3}+\alpha+\beta\right)\right. \\ &\left.+\sin\left(\frac{2\pi}{3}-\beta\right)\exp(i\alpha)\right)\exp(i\alpha) \end{aligned} \tag{77}$$

Since $\beta+\pi/3$ and $2\pi/3-\beta$ are supplementary angles and they have the same sine:

$$\begin{aligned} EF = 8\rho\, u_{AB}&\sin(\alpha+\beta)\sin(\alpha+\gamma)\left(-\sin\left(\frac{\pi}{3}+\alpha+\beta\right)\right. \\ &\left.+\sin\left(\frac{\pi}{3}+\beta\right)\exp(i\alpha)\right)\exp(i\alpha) \end{aligned} \tag{78}$$

Now applying the angle addition identity we have:

$$EF = -8\rho\, u_{AB}\sin(\alpha+\beta)\sin(\alpha+\gamma)\sin\alpha\exp\left(-i\left(\pi/3+\beta\right)\right)\exp(i\alpha) \tag{79}$$

This result can be written in a briefer way:

$$EF = -8\rho\ u_{AB} \sin\alpha \sin\beta' \sin\gamma' \exp(-i(2\beta+\gamma)) \tag{80}$$

Since u_{AB} is obtained from u_{AC} by turning through -3α (eq. (75)), we have:

$$EF = -8\rho\ u_{CA} \sin\alpha \sin\beta' \sin\gamma' \exp(i(\beta+2\gamma)) \tag{81}$$

where Euler's identity $\exp(i\pi) = -1$ has also been applied. Now we see that EF (80) and QR (45) have opposite directions:

$$u_{EF} = -u_{QR} = -u_{CA} \exp(i(\beta+2\gamma)) \tag{82}$$

and:

$$\|EF\| = 8\rho \sin\alpha \sin\beta' \sin\gamma' = \|ED\| \tag{83}$$

which is coincident with $\|ED\|$ (69). Since ED and EF have the same norm and their directions are those of PQ and $-QR$, it proves that the triangle DEF is equilateral. We can also check that ED is obtained from DF by means of a rotation through $\pi/3$, or equivalently, that $EF\ ED^{-1} = \exp(i\pi/3)$:

$$EF\ ED^{-1} = -u_{AB} \exp(-i(2\beta+\gamma) \exp(-i(\alpha+2\beta)) u_{BC} \tag{84}$$

$$= u_{BA} \exp\left(-i\left(\frac{\pi}{3}+3\beta\right)\right) u_{BC} = u_{BA} u_{BC} \exp\left(i\left(\frac{\pi}{3}+3\beta\right)\right) = \exp\left(\frac{i\pi}{3}\right)$$

where the permutative property (13) has been applied. Therefore the triangle ΔDEF is equilateral.

In order to show that the triangle ΔDGJ is equilateral, let us apply a cyclic permutation to the expresion (56) for D:

$$G = C + \frac{b\ \sin\alpha'}{\sin(\gamma'+\alpha')} u_{CA} \exp(-i\gamma') \tag{85}$$

$$J = A + \frac{c\ \sin\beta'}{\sin(\alpha'+\beta')} u_{AB} \exp(-i\alpha') \tag{86}$$

From (56) and (85) we find:

$$DG = u_{CA} \frac{b\ \sin\alpha' \exp(-i\gamma')}{\sin(\gamma'+\alpha')} + a\ u_{BC} \left(1 - \frac{\sin\gamma' \exp(-i\beta')}{\sin(\beta'+\gamma')}\right) \tag{87}$$

$$= u_{CA}\frac{b\ \sin\alpha'\exp(-i\gamma')}{\sin(\gamma'+\alpha')} + a\ u_{BC}\frac{\sin\beta'\exp(i\gamma')}{\sin(\beta'+\gamma')} \tag{88}$$

Since $u_{CA} = u_{BC}\exp(3i\gamma')$ we have:

$$DG = \ u_{BC}\left[\frac{b\ \sin\alpha'\exp(2i\gamma')}{\sin(\gamma'+\alpha')} + \frac{a\sin\beta'\exp(i\gamma')}{\sin(\beta'+\gamma')}\right] \tag{89}$$

Application of the sine theorem (37) yields:

$$DG = 2\rho\ u_{BC}\exp(i\gamma')\left[\frac{\sin 3\beta\sin\alpha'\exp(i\gamma')}{\sin(\frac{\pi}{3}+\beta)} + \frac{\sin 3\alpha\sin\beta'}{\sin(\frac{\pi}{3}+\alpha)}\right] \tag{90}$$

and with the trigonometric identity (39) we find:

$$DG = 8\rho u_{BC}\exp(i\gamma')\left[\sin\left(\frac{\pi}{3}-\beta\right)\sin\beta\sin\alpha'\exp(i\gamma') + \sin\left(\frac{\pi}{3}-\alpha\right)\sin\alpha\sin\beta'\right] \tag{91}$$

$$= 8\rho u_{BC}\sin\alpha'\sin\beta'\exp(i\gamma')\left(\sin\beta\exp(i\gamma') + \sin(\gamma'-\beta)\right)$$

because $\alpha = \gamma' - \beta$. After applying the angle subtraction identity:

$$DG = 8\rho u_{BC}\sin\alpha'\sin\beta'\sin\gamma'\exp(i\gamma')\exp(i\beta) \tag{92}$$

$$= 8\rho u_{BC}\sin\alpha'\sin\beta'\sin\gamma'\exp(i(\alpha+2i\beta))$$

which has the same direction as PQ (43) with the norm:

$$\|DG\| = 8\rho u_{BC}\sin\alpha'\sin\beta'\sin\gamma' \tag{93}$$

Therefore, by cyclic permutation we see that ΔDGH is an equilateral triangle with sides parallel to the sides of ΔPQR. Summarizing all the results we have:

$$\frac{\Delta DGJ}{\Delta PQR} = \frac{\sin\alpha'\sin\beta'\sin\gamma'}{\sin\alpha\sin\beta\sin\gamma} \qquad \frac{\Delta DEF}{\Delta PQR} = -\frac{\sin\beta'\sin\gamma'}{\sin\beta\sin\gamma} \tag{94}$$

$$\frac{\Delta GHI}{\Delta PQR} = -\frac{\sin\alpha'\sin\gamma'}{\sin\alpha\sin\gamma} \qquad \frac{\Delta JKL}{\Delta PQR} = -\frac{\sin\alpha'\sin\beta'}{\sin\alpha\sin\beta} \tag{95}$$

□

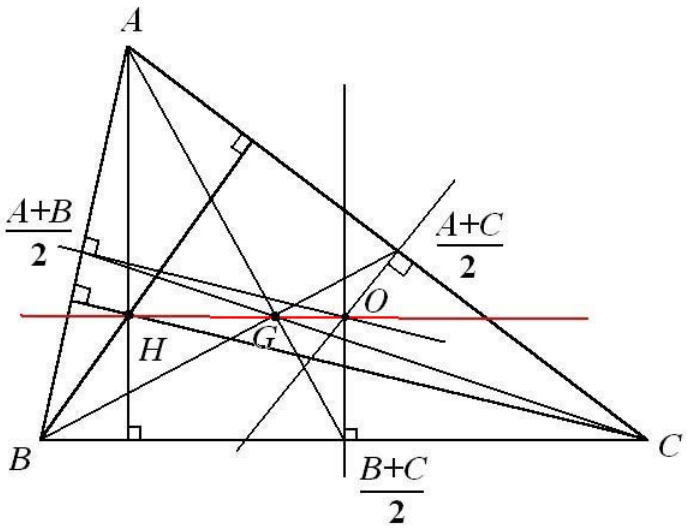

FIGURE 5. Euler's theorem. The orthocentre H, the centroid G and the circumcentre O lie on Euler's line indicated in red.

Theorem 2.4 (Euler)**.** *The centroid G, the circumcentre O and the orthocenter H of a triangle are collinear, and they satisfy* $OH = 3OG$ *(fig. 5).*

Proof. If A, B and C are the vertices of a triangle, then the circumcentre O is found from the equations:

$$OA^2 = OB^2 = OC^2 \tag{96}$$

After developing them taking $OA = A - O$ and so on, we obtain ([1], p. 71):

$$O = -(A^2BC + B^2CA + C^2AB)(2AB \wedge BC)^{-1} \tag{97}$$

Notice that there is a geometric product of a vector by an imaginary number whose norm is inverse of four times the area of the triangle.

The orthocentre H is obtained as the intersection of altitudes, which pass through each vertex and are perpendicular to the opposite side (fig. 5).

$$H = A + z\,BC = B + t\,CA \qquad z = \lambda\, i \qquad t = \mu\, i \qquad \lambda, \mu \in \mathbb{R} \tag{98}$$

Multiplication by the imaginary numbers z and t warrants perpendicularity to the corresponding base. By arranging terms we have a vectorial equality:

$$z\,BC - t\,CA = AB \tag{99}$$

This equation resembles the resolution of a vector into linear combination of two vectors, but the coefficients are imaginary instead of real. In order to solve this equation, we multiply by CA on the left and on the right:

$$z\,BC\;\;CA - t\,CA^2 = AB\;\;CA \tag{100}$$

$$CA\,z\,BC - CA\,t\,CA = CA\;\;AB \qquad \Rightarrow \qquad -z\,CA\;\;BC + t\,CA^2 = CA\;\;AB \tag{101}$$

where the permutative property has been applied in the last step. Addition of both equations yields:

$$z\,(BC\;\;CA - CA\;\;BC) = AB\;\;CA + CA\;\;AB \tag{102}$$

whence we find the solution for z. The solution for t is found in the same way:

$$z = \frac{AB \cdot CA}{BC \wedge CA} \qquad\qquad t = \frac{AB \cdot BC}{BC \wedge CA} \tag{103}$$

Substitution of (103) in (98) yields:

$$H = A + (AB\;\;CA + CA\;\;AB)(BC\;\;CA - CA\;\;BC)^{-1}BC \tag{104}$$

and after some steps ([1], p. 75) we arrive at:

$$H = (A\;\;A \cdot BC + B\;\;B \cdot CA + C\;\;C \cdot AB)(BC \wedge CA)^{-1} \tag{105}$$

This formula is invariant under cyclic permutation of vertices. Therefore, all the altitudes of the triangle are concurrent at the orthocentre H. Now, in order to prove that the centroid, the circumcentre and the orthocentre are collinear, we write the centroid with the same right factor as (97) and (105):

$$G = \frac{A+B+C}{3} = (A+B+C)\;BC \wedge CA\;(3BC \wedge CA)^{-1} \tag{106}$$

Since $BC \wedge CA = A \wedge B + B \wedge C + C \wedge A$, and after some steps we find:

$$G = (-A^2BC + A\;\;BC\;\;A - B^2CA + B\;\;CA\;\;B - C^2AB + C\;\;AB\;\;C)(6\;BC \wedge CA)^{-1} \tag{107}$$

whence the relation between the three centres follows:

$$G = \frac{H + 2O}{3} \qquad \Rightarrow \qquad OH = 3\;OG \tag{108}$$

From equations (97) and (105) it is easy to get [7]

$$OH = -(AB\;\;BC\;\;CA + BC\;\;CA\;\;AB + CA\;\;AB\;\;BC)(2\;AB \wedge BC)^{-1} \tag{109}$$

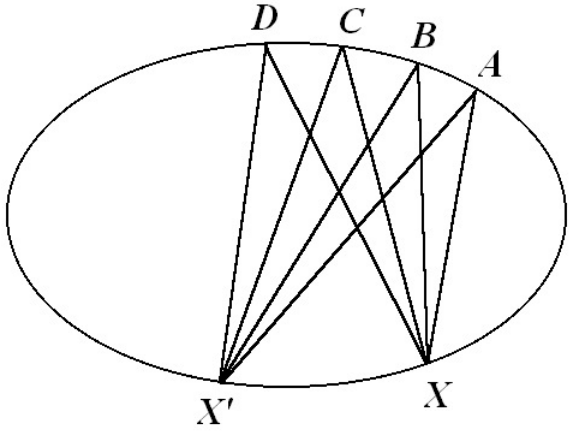

FIGURE 6. Chasles's theorem. The cross ratio of the pencil of lines from a point X to four given points A, B, C and D on a proper conic is independent of the choice of X on this conic.

Notice that all the products are geometric products, and that the right factor is imaginary. The vector OH has the direction of Euler's line, so that this result not only proves that O, G and H are collinear (which was already done by the synthetic method) but it also gives the direction of the line where these centres (and other centres) lie, and this is a unique and exclusive expression obtained from geometric algebra. □

Theorem 2.5 (Chasles)**.** *The cross ratio of a pencil of lines from a point X to four given points A, B, C, D on a proper conic is constant for every point X on the same conic (see figure 6).*

Proof. From the formula we deduced for the cross ratio ([7],[1] p. 100):

$$\{X,ABCD\} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC} \tag{110}$$

we will prove that it is constant. The vectorial equation of a proper conic obtained from its polar equation is ([1], p.122):

$$FX = \frac{1+\varepsilon}{1+\varepsilon\cos\chi} FQ \, \exp(i\chi) = \frac{1+\varepsilon}{1+\varepsilon\cos\chi} \|FQ\|(e_1\cos\chi + e_2\sin\chi) \tag{111}$$

where ε is the eccentricity, F is the focus of the conic, Q is the point of the conic that is nearest to its focus F, which in astronomy is known as perihelion, and χ is the angle $\angle QFX$ between the perihelion Q and any other point X. In the second expression, $FQ = \|FQ\|e_1$ has been taken without loss of generality, that is, the major axis of the conic is taken horizontal. Then we have (fig. 7):

$$XA = FA - FX = \|FQ\|(1+\varepsilon)\left[\frac{e_1\cos\alpha + e_2\sin\alpha}{1+\varepsilon\cos\alpha} - \frac{e_1\cos\chi + e_2\sin\chi}{1+\varepsilon\cos\chi}\right] \tag{112}$$

$$XA \wedge XC = FQ^2 i(1+\varepsilon)^2 \frac{\sin(\gamma-\alpha)+\sin(\chi-\gamma)+\sin(\alpha-\chi)}{(1+\varepsilon\cos\alpha)(1+\varepsilon\cos\gamma)(1+\varepsilon\cos\chi)} \tag{113}$$

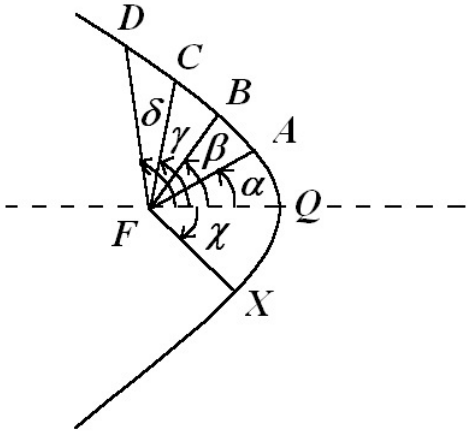

FIGURE 7. F is the focus of the conic and the vertex of all the angles, which are measured with regard to the major axis of the conic.

By applying half-angle identity, the addition of sines is converted into a product:

$$XA \wedge XC = -4FQ^2 i(1+\varepsilon)^2 \frac{\sin\frac{\gamma-\alpha}{2}\sin\frac{\chi-\gamma}{2}\sin\frac{\alpha-\chi}{2}}{(1+\varepsilon\cos\alpha)(1+\varepsilon\cos\gamma)(1+\varepsilon\cos\chi)} \tag{114}$$

whence we obtain from (110) a cross ratio that does not depend on χ:

$$\{X, ABCD\} = \frac{\sin\frac{\gamma-\alpha}{2}\sin\frac{\delta-\beta}{2}}{\sin\frac{\delta-\alpha}{2}\sin\frac{\gamma-\beta}{2}} \tag{115}$$

This result is trivial if the conic is a circle because an inscribed angle is half of the central angle that subtends the same arc (for instance $\angle AXB = \angle AFB/2$). This is no longer satisfied by other proper conics. In spite of this, the cross ratio continues to being the quotient of the sines of the half angles with vertices at the focus F. This is another example of how geometric algebra not only proves theorems but also yields a quantitative estimation of the result, in this case, the cross ratio. □

Theorem 2.6 (Simson)**.** *The feet of the perpendiculars of a point to the sides of a triangle are collinear if and only if the point lies on its circumcircle.*

Proof. Let D be a point on the circumcircle of the triangle ΔABC (fig. 8), and let P, Q and R be the feet of the perpendiculars to the sides AB, AC and BC. Notice that the triangles ΔADP and ΔCDR are similar because $\angle DAB = \pi - \angle BCD = \angle DCR$. From the similarity of both triangles:

$$DC\ \ DA^{-1} = DR\ \ DP^{-1} \tag{116}$$

ΔADQ and ΔBDR are also similar because the inscribed angles $\angle CAD$ and $\angle CBD$ are equal:

$$DB\ \ DA^{-1} = DR\ \ DQ^{-1} \tag{117}$$

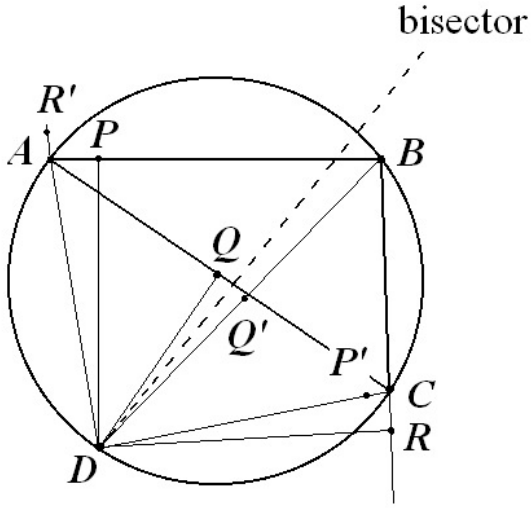

FIGURE 8. Simson's theorem: The feet P, Q and R of the perpendiculars from a point D to the sides of a triangle ΔABC are collinear iff D lies on the circle circumscribed to the triangle ΔABC.

Since $\angle RDC = \angle PDA$ and $\angle RDB = \angle QDA$, the bisector of $\angle BDQ$ is also a bisector of $\angle CDP$ and $\angle RDA$, whose direction vector is:

$$v = \frac{DA}{\|DA\|} + \frac{DR}{\|DR\|} = \frac{DB}{\|DB\|} + \frac{DQ}{\|DQ\|} = \frac{DC}{\|DC\|} + \frac{DP}{\|DP\|} \tag{118}$$

Let P', Q' and R' be the symmetric points of P, Q and R respectively with respect to the bisector. Then:

$$DP' = v^{-1} DP\, v \qquad DQ' = v^{-1} DQ\, v \qquad DR' = v^{-1} DR\, v \tag{119}$$

$$DR' DA = v^{-1} DR\, v\, DA = \|DR\|\, \|DA\| \tag{120}$$

because the application to R of the axial symmetry with respect to the bisector of $\angle RDA$ always yields a point R' collinear with A. Anyway, the result can also be checked algebraically. We also have:

$$DQ' DB = v^{-1} DQ\, v\, DR\, DQ^{-1} DA = v^{-1} DR\, v\, DA = \|DR\|\, \|DA\| \tag{121}$$

$$DP' DC = v^{-1} DP\, v\, DR\, DP^{-1} DA = v^{-1} DR\, v\, DA = \|DR\|\, \|DA\| \tag{122}$$

where substitution of equation (117) and the permutative property have been applied. In the same way, substitution of eq. (116) yields:

$$DP'DC = v^{-1}DP\ v\ DR\ DP^{-1}DA = v^{-1}DR\ v\ DA = \|DR\|\ \|DA\| \tag{123}$$

Now we see that R', Q' and P' are the transformed points of A, B and C obtained under an inversion with centre D:

$$DR'DA = DQ'DB = DP'DC \in \mathbb{R} \tag{124}$$

Since D lies on the same circle as A, B and C, this inversion transform them into the collinear points P', Q' and R', and the axial symmetry preserves their collinearity, so that P, Q and R are also collinear. □

3. Basic notions of the geometric algebra of the three-dimensional Euclidean space

The geometric algebra of the Euclidean space $Cl_{3,0}$ is generated by three orthogonal unitary vectors e_1, e_2 and e_3:

$$e_i^2 = 1 \qquad e_i e_j = -e_j e_i \qquad i \neq j \tag{125}$$

A vector v in the Euclidean space is a linear combination of them:

$$v = v_1\ e_1 + v_2\ e_2 + v_3\ e_3 \qquad v_1,\ v_2,\ v_3 \in \mathbb{R} \tag{126}$$

On the other hand, the product of two vectors v and w yields a quaternion:

$$(v_1\ e_1 + v_2\ e_2 + v_3\ e_3)(w_1\ e_1 + w_2\ e_2 + w_3\ e_3) = v_1\ w_1 + v_2\ w_2 + v_3\ w_3 \tag{127}$$

$$+(v_1\ w_2 - v_2\ w_1)e_{12} + (v_2 w_3 - v_3 w_2)e_{23} + (v_3 w_1 - v_1 w_3)e_{31}$$

that we can write in short form as:

$$v\ w = v \cdot w + v \wedge w \tag{128}$$

The bivectors e_{ij} are imaginary units because its square is -1:

$$e_{ij}^2 = e_i e_j e_i e_j = -e_i e_i e_j e_j = -1 \qquad i \neq j \tag{129}$$

Therefore quaternions $q \in \mathbb{H}$ are elements of $Cl_{3,0}$ of the form:

$$\mathbb{H} \ni q = a + b\ e_{23} + c\ e_{31} + d\ e_{12} \qquad a,\ b,\ c,\ d \in \mathbb{R} \tag{130}$$

The rotation of a vector v through an angle α can no longer be described only as the multiplication of v on the right by a quaternion as done for plane geometry in (7), but a general expression analogous to (11) with half-angle factors holds:

$$v' = \exp\left(-\frac{u\alpha}{2}\right) v \, \exp\frac{u\alpha}{2} \qquad u^2 = -1 \qquad u \in \langle Cl_{3,0} \rangle_2 \tag{131}$$

which can also be written as:

$$v' = q^{-1} v \, q \qquad\qquad q \in \mathbb{H} \tag{132}$$

since it is not necessary quaternion q to be unitary. The composition of rotations is obtained by successive application of this operator.

$$v'' = q^{-1} v' \, q = q^{-1} p^{-1} v \, p \, q = r^{-1} v \, r \quad \Rightarrow \quad r = p \, q \qquad p, \, q, \, r \in \mathbb{H} \tag{133}$$

$$\exp\frac{u_r \rho}{2} = \exp\frac{u_p \omega}{2} \exp\frac{u_q \theta}{2} \qquad u_p^2 = u_q^2 = u_r^2 = -1 \qquad u_r = \frac{u_p u_q - u_q u_p}{\| u_p u_q - u_q u_p \|} \tag{134}$$

Since three vectors u, v, and w in the space are not usually coplanar, the permutative property changes from (12) and becomes ([1], p. 172):

$$u \, v \, w - w \, v \, u = 2 \, u \wedge v \wedge w \tag{135}$$

Finally, an axial symmetry with respect to an axis with direction vector d is obtained as:

$$v' = d^{-1} v \, d \tag{136}$$

4. Application to geometric problems of the three-dimensional space

4.1. **Euler's line of a tetrahedron.** It is very easy to show in an algebraic way that the medians of the four faces of a tetrahedron $ABCD$ meet at a unique point, the centroid G (fig. 9). The proof is available at ([10], p. 87) and is omitted, and the result is:

$$G = \frac{A + B + C + D}{4} \tag{137}$$

The centre of the sphere circumscribed to the tetrahedron $ABCD$, the circumcentre O, can be found from:

$$OA^2 = OB^2 = OC^2 = OD^2 \tag{138}$$

After developing by means of the scalar product, $OA^2 = (A - O)^2 = A^2 - 2O \cdot A + A^2$, we arrive at the equation system:

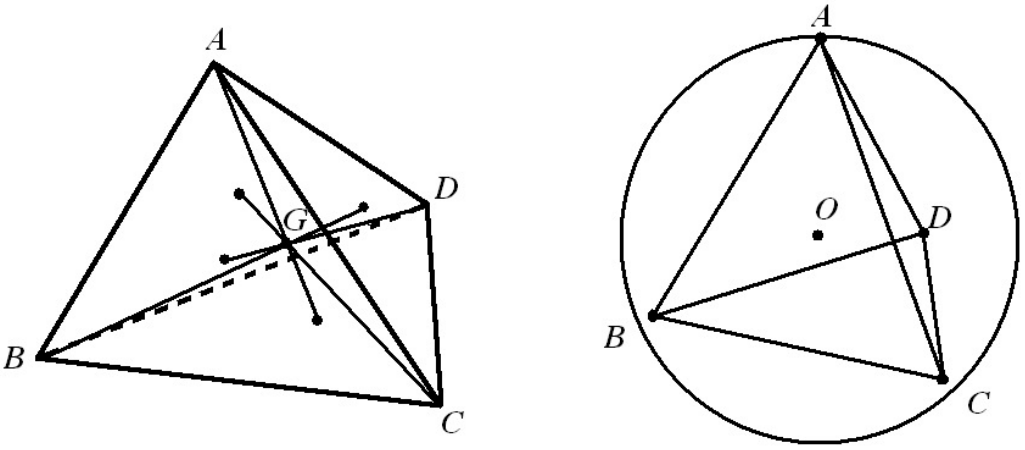

FIGURE 9. The centroid G is the intersection of the medians. The circumcentre O is the centre of the sphere circumscribed to the tetrahedron.

$$O \cdot AB = \frac{B^2 - A^2}{2} \qquad O \cdot BC = \frac{C^2 - B^2}{2} \qquad O \cdot CA = \frac{A^2 - C^2}{2} \tag{139}$$

which can be solved by means of Cramer's rule and whose solution is ([9],[10] p. 89):

$$O = [(B^2 - A^2)BC \wedge CD + (C^2 - B^2)CD \wedge AB + (D^2 - C^2)AB \wedge BC](2AB \wedge BC \wedge CD)^{-1} \tag{140}$$

When giving a form invariant under cyclic permutation, one finds:

$$O = [-A^2 BC \wedge CD + B^2 CD \wedge DA - C^2 DA \wedge AB + D^2 AB \wedge CD](AB \wedge BC \wedge CD)^{-1} \tag{141}$$

Monge's point M is the intersection of the planes that pass through the midpoint of each edge and are perpendicular to the opposite edge. Three of these planes intersect at one point, but the other three also intersect at the same point M as we will now prove. Since M belongs to the plane that passes through $(C+D)/2$ and is perpendicular to AB, the plane that passes through $(A+D)/2$ and is perpendicular to BC, and the plane that passes through $(A+B)/2$ and is perpendicular to CD, we have:

$$\frac{MC + MD}{2} \cdot AB = 0 \qquad \frac{MA + MD}{2} \cdot BC = 0 \qquad \frac{MA + MB}{2} \cdot CD = 0 \tag{142}$$

Isolation of Monge's point yields the equation system:

$$M \cdot AB = \frac{C+D}{2} \cdot AB \qquad M \cdot BC = \frac{A+D}{2} \cdot BC \qquad M \cdot CD = \frac{A+B}{2} \cdot CD \tag{143}$$

Addition of couples of equations yields the equations corresponding to the other planes:

$$M \cdot AC = \frac{B+D}{2} \cdot AC \qquad M \cdot AD = \frac{C+D}{2} \cdot AD \qquad M \cdot BD = \frac{A+C}{2} \cdot BD \tag{144}$$

Therefore, the six planes intersect at Monge's point M, which is the solution to the former equation systems:

$$M = \left(\frac{C+D}{2} \cdot AB\ BC \wedge CD + \frac{A+D}{2} \cdot BC\ CD \wedge AB + \frac{A+B}{2} \cdot CD\ AB \wedge BC \right) (AB \wedge BC \wedge CD)^{-1} \tag{145}$$

From the semisum of the first equalities of (139) and (144), we obtain:

$$\frac{O+M}{2} \cdot AB = \frac{B^2 - A^2}{4} + \frac{C+D}{4} \cdot AB = \frac{A+B+C+D}{4} \cdot AB = G \cdot AB \tag{146}$$

In the same way, the semisum of the second and third equalities of (139) and (144) are:

$$\frac{O+M}{2} \cdot BC = G \cdot BC \qquad \frac{O+M}{2} \cdot CD = G \cdot CD \tag{147}$$

which means that the centroid is the midpoint of Monge's point and the circumcentre:

$$\frac{O+M}{2} = G \tag{148}$$

and that O, G and M are collinear. The line $\overline{OGH}$ is called *Euler's line* of the tetrahedron $ABCD$. Now, let us subtract (139) and (144):

$$OM \cdot AB = \frac{-DA+BC}{2} \cdot AB \qquad OM \cdot BC = \frac{-AB+CD}{2} \cdot BC \cdot CD \tag{149}$$

$$OM \cdot CD = \frac{-BC+DA}{2}$$

whose solution is:

$$OM = \left(\frac{-DA+BC}{2} \cdot AB\ BC \wedge CD + \frac{-AB+CD}{2} \cdot BC\ CD \wedge AB + \frac{-BC+DA}{2} \cdot CD\ CD \wedge AB \right) (AB \wedge BC \wedge CD)^{-1} \tag{150}$$

By using geometric product instead of scalar and exterior products in the first factor we obtain an expression for OM invariant under cyclic permutation:

$$OM = (AB\ \ BC\ \ CD\ \ DA - BC\ \ CD\ \ DA\ \ AB + CD\ \ DA\ \ AB\ \ BC - DA\ \ AB\ \ BC\ \ CD)(4\ AB \wedge BC \wedge CD)^{-1} \tag{151}$$

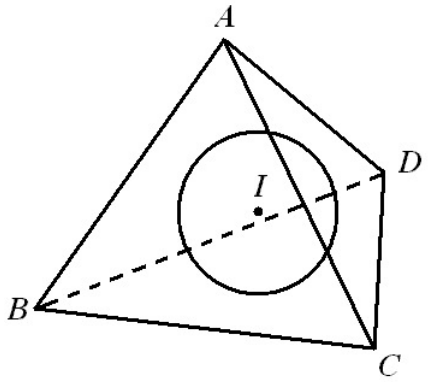

FIGURE 10. Inscribed sphere and incentre I of a tetrahedron.

This result is very instructive: the vector OM of Euler's line can only be expressed as a function of the edges of tetrahedron by means of geometric product and geometric algebra. Of course, the fact that O, G and M are collinear was proven long time ago, but only geometric algebra allows the quantitative computation of OM, the director vector of Euler's line.

4.2. **Incentre of a tetrahedron.** The power of geometric algebra is displayed again when calculating the incentre I of a tetrahedron, which is the point equidistant from its four faces. The incentre I of a triangle ΔABC is given by ([1], p. 72):

$$I = \frac{A\,\|BC\| + B\,\|CA\| + C\,\|AB\|}{\|AB\| + \|BC\| + \|CA\|} \tag{152}$$

When generalizing this formula to three dimensions, the following theorem is obtained ([9],[10] p. 92).

Theorem 4.1 (Incentre of a tetrahedron)**.** *The incentre I of a tetrahedron ABCD is given by:*

$$I = \frac{A\,\|BC \wedge CD\| + B\,\|CD \wedge DA\| + C\,\|DA \wedge AB\| + D\,\|AB \wedge BC\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{153}$$

Proof. Let us build the vectors AI and BI:

$$AI = I - A = I = \frac{AB\,\|CD \wedge DA\| + AC\,\|DA \wedge AB\| + AD\,\|AB \wedge BC\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{154}$$

$$BI = I - B = I = \frac{BA\,\|BC \wedge CD\| + BC\,\|DA \wedge AB\| + BD\,\|AB \wedge BC\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{155}$$

and let us now compute the following exterior products:

$$AB \wedge AC \wedge AI = \frac{AB \wedge AC \wedge AD\,\|AB \wedge BC\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{156}$$

$$AI \wedge AC \wedge AD = \frac{AB \wedge AC \wedge AD \, \|CD \wedge DA\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{157}$$

$$AB \wedge AI \wedge AD = \frac{AB \wedge AC \wedge AD \, \|DA \wedge AB\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{158}$$

$$BC \wedge BI \wedge BD = \frac{AB \wedge BC \wedge BD \, \|BC \wedge CD\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} \tag{159}$$

Since $AB \wedge AC \wedge AD = AB \wedge BC \wedge CD = AB \wedge BC \wedge CD$ is six times the volume of the tetrahedron, we now see that the four distances from I to each face of the tetrahedron are equal to the same value r:

$$r = \frac{\|AB \wedge AC \wedge AI\|}{\|AB \wedge AC\|} = \frac{\|AI \wedge AC \wedge AD\|}{\|CD \wedge DA\|} = \frac{\|AB \wedge AI \wedge AD\|}{\|DA \wedge DB\|} = \frac{\|BC \wedge BI \wedge BD\|}{\|BC \wedge CD\|} \tag{160}$$

Again, geometric algebra not only gives the proof but also yields the value of the radius r of the inscribed sphere:

$$r = \frac{\|AB \wedge BC \wedge CD\|}{\|BC \wedge CD\| + \|CD \wedge DA\| + \|DA \wedge AB\| + \|AB \wedge BC\|} = \frac{3V}{S} \tag{161}$$

which is three times the quotient of the volume V and total area S of the tetrahedron. □

4.3. **Parallelogram in a tetrahedron.** Let us see an old theorem recalled by Hongbo Li [11].

Theorem 4.2 (Parallelogram in a tetrahedron)**.** *Let $ABCD$ be a tetrahedron, and let π be a plane intersecting lines $\overline{AB}$, $\overline{AC}$, $\overline{DC}$, $\overline{DB}$ at M, N, E, F respectively (fig. 11). If for different positions of π, $MNEF$ is a parallelogram, then the center O of the parallelogram is on a fixed straight line.*

Proof. Since $M \in \overline{AB}$, $N \in \overline{AC}$, $E \in \overline{DC}$ and $F \in \overline{DB}$ we have:

$$\begin{aligned} &M = (1-\lambda)A + \lambda B && N = (1-\mu)A + \mu C && \lambda, \mu \in \mathbb{R} \\ &E = (1-\chi)C + \chi D && F = (1-\psi)B + \psi D && \chi, \psi \in \mathbb{R} \end{aligned} \tag{162}$$

If $MNEF$ is a parallelogram then its centre O is the midpoint of the opposite vertices:

$$O = \frac{M+E}{2} = \frac{N+F}{2} \tag{163}$$

Substitution of (162) in (163) gives:

$$(1-\lambda)A + \lambda B + (1-\chi)C + \chi D = (1-\mu)A + \mu C + (1-\psi)B + \psi D \tag{164}$$

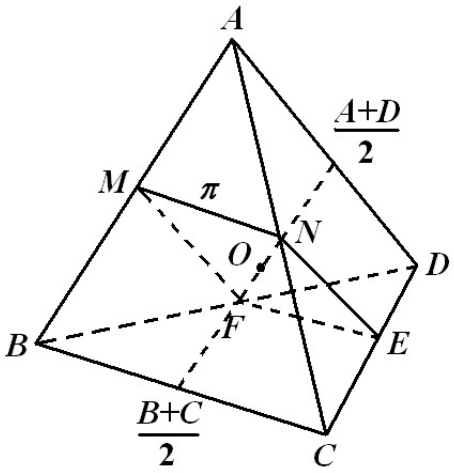

FIGURE 11. Parallelogram in a tetrahedron.

Since A, B, C and D are four noncoplanar points, they are linearly independent, and this means that the coefficients on the left hand side and on the right hand side are equal:

$$\lambda = \mu \qquad \psi = \chi = 1 - \lambda \tag{165}$$

so that, instead of (162) we have:

$$\begin{aligned} M &= (1-\lambda)A + \lambda B & N &= (1-\lambda)A + \lambda C \\ E &= \lambda C + (1-\lambda)D & F &= \lambda B + (1-\lambda)D \end{aligned} \tag{166}$$

whence the center (163) of the parallelogram is given by:

$$O = (1-\lambda)\frac{A+D}{2} + \lambda\frac{B+C}{2} \tag{167}$$

that is, the centre O lies on the line passing through the midpoints of the edges AD and BC. □

The proof in [11] is cumbersome and spends two pages, while our proof is very short. In order to see again the power of geometric algebra, let us find the area of this parallelogram. If we build the vectors MN and MF we have:

$$MN = N - M = \lambda BC \qquad MF = F - M = (1-\lambda)AD \tag{168}$$

It means that the parallelogram is parallel to the sides BC and AD. The area s of the parallelogram is then:

$$s = \|MN \wedge MF\| = \lambda(1-\lambda)\|BC \wedge AD\| = \lambda(1-\lambda)\sqrt{BC^2 AD^2 - (BC \cdot AD)^2} \tag{169}$$

Since:

$$BC \cdot AD = \frac{-AB^2 + AC^2 - CD^2 + BD^2}{2} \tag{170}$$

we finally have:

$$s = \frac{\lambda(1-\lambda)}{2}\sqrt{4AD^2BC^2 - (AB^2 - AC^2 + CD^2 - BD^2)^2} \tag{171}$$

Now we see that the maximal area is achieved for $\lambda = 1/2$, finding the result given in [12].

4.4. **Spherical trigonometry.** Geometric algebra allows us to easily deduce the equations of spherical trigonometry. Since the cross product of bivectors is not associative, only geometric algebra can unlock the algebraic steps for simplifying expressions containing a combination of scalar, exterior or cross porducts ([1] p.172, [9]) thanks to the associative property. Associativity of the geometric product is essential when going from stating geometric equations to their solutions, which are also algebraic expressions containing geometric products many times. We also showed how to apply geometric algebra to Lobachevsky's geometry and trigonometry with many advantages ([1], p. 189-223). This will not be developed here since this article is too long, but the reader can go to the given references.

5. Conclusions

Here we have proven some theorems of Euclidean plane geometry (Fermat, Morley, Euler, Chasles, Simson) and space geometry (Euler's line, incentre and parallelogram of a tetrahedron). These theorems have previous proofs by means of synthetic geometry. However, geometric algebra always yields something new that other proofs lack: a quantitative result that supersedes and goes beyond the previous qualitative proofs. The direction or size of every geometric element can be calculated with geometric algebra, which is not obtained from synthetic geometry. Although being a help, synthetic reasonings are not enough for giving quantitative results. For instance, we remember that Chris Doran asked, during the AGACSE 2010 conference, whether anybody could calculate the centre of a sphere passing through four points in the space. The answer is the equation (97). During the same conference, at the end of our presentation [9], Joan Lasenby asked whether there was any proof of Morley's theorem with geometric algebra. Our answer at that time was that we had not yet tried it. We then outlined the geometric equations of Morley's theorem, but the work of solving them was harder than we thought. Anyway, geometric algebra does not exhibit any intrinsic limitation. The only limitations are our little abilities to play with geometric algebra. In this way, we are discovering new algebraic procedures, such as the imaginary resolution of a vector (100) into components (103). Perhaps, the lessening of geometric algebra by a significant part of the mathematical community comes from a biassed mathematical education where only proofs of existence are important but not their quantitative evaluation.

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THE EXTERIOR CALCULUS

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ABSTRACT. The exterior differentiation, defined in the way Cartan did, and its algebraic properties are reviewed. The Poincaré lemma about integrability of differential forms is explained and applied to some closed differential forms, whose primitives are calculated in order to exemplify the steps of the process of integration. Stokes' theorem is outlined and applied to specific cases in the three-dimensional space. As another application of exterior differentiation, the differential operators of the three-dimensional space (gradient, curl and divergence) in curvilinear orthogonal coordinates are deduced. Finally, its application to electromagnetism is discussed finding some troubles whose satisfactory solution can only be obtained from the Kähler calculus.

1.DEFINITION OF EXTERIOR DIFFERENTIAL

Let $f(x_1,\cdots x_n)$ be a real function depending on n real variables x_i. Its differential is:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \qquad x^i \in \mathbb{R} \qquad f \in \Lambda_x^0 \tag{1}$$

This expression has vectorial character, and df is an element of the module spanned by $\{dx^1,\cdots dx^n\}$. In general an element ω of this module, which is called a *differential 1-form*, can be written as:

$$\omega = \sum_{i=1}^{n} a_i dx^i \qquad a_i = a_i(x^i,\cdots x^n) \in \Lambda_x^0 \tag{2}$$

Not all differential 1-forms ω satisfy $\omega = df$; they only do so under certain conditions that will be seen later. The module of all the differential 1-forms ω is indicated as Λ_x^1. We are going to have other types of differential forms with higher grades, which will be indicated as Λ_x^k.

É. Cartan [1], [2 p. 33] defined the exterior differential of the 1-form (2) as:

$$d\omega = \sum_{i=1}^{n} da_i \wedge dx^i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial a_i}{\partial x_j} dx^j \wedge dx^i \qquad \omega \in \Lambda_x^1 \tag{3}$$

$a_i \in \Lambda_x^0$

where:

(4) $$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

The assumption $d\left(dx^i\right)=0$ is implicit in Cartan's definition. By means of (4), eq. (3) becomes:

(5) $$d\omega = \sum_{i=1}^{n}\sum_{j>i}^{n}\left(\frac{\partial a_i}{\partial x_j}-\frac{\partial a_j}{\partial x_i}\right)dx^j \wedge dx^i$$

where all the $dx^j \wedge dx^i$ are therefore linearly independent. This expression is a bivector of the exterior algebra of $\left\{dx^i\right\}$, which play the same role as $\left\{e_i\right\}$ in a geometric vector space. The module spanned by $\left\{dx^j \wedge dx^i\right\}$ is indicated as Λ_x^2. In general, a differential 2-form can be written as:

(6) $$\chi = \sum_{i=1}^{n}\sum_{j>i}^{n} b_{ij}dx^i \wedge dx^j \qquad \chi \in \Lambda_x^2 \qquad b_{ij} \in \Lambda_x^0$$

where b_{ij} are not necessarily like the coefficients of the differential form (5). By using Einstein's summation convention where the sum is always implicit for repeated indices, we can write (6) as:

(7) $$\chi = b_{ij}\left(dx^i \wedge dx^j\right) = -b_{ji}\left(dx^j \wedge dx^i\right)$$

where the parentheses are a notation of Jose Vargas indicating that all the terms differing by a permutation of $\left\{dx_i\right\}$ have been gathered. For instance, in 3 dimensions we will have

(8) $$\chi = b_{12}dx^1 \wedge dx^2 + b_{23}dx^2 \wedge dx^3 + b_{31}dx^3 \wedge dx^1$$

Again, not all the differential 2-forms χ satisfy $\chi = d\omega$, but if this equality is satisfied we will then say that ω is a *primitive* of χ. The module spanned by products of k differentials of $\left\{dx_i\right\}$ is indicated as Λ_x^k and its elements are said to be *differential k-forms*, which in general are not differentials of a $(k-1)$-form. Therefore, the exterior algebra of differential forms includes exterior products of $\left\{dx^i\right\}$ up to the number n of them. In this exterior algebra, the exterior product of two differential forms α_k and β_l of grades k and l in dimension n satisfies a rule [3, p. 18] similar to that for the exterior product of multivectors:

(9) $$\alpha_k \wedge \beta_l = (-1)^{k\,l}\beta_l \wedge \alpha_k \qquad 0 \le k,l \le n$$

From now on and for the applications we will deal with, only homogeneous elements will be considered, that is, differential forms will always be linear combinations of basis

elements with the same grade. Linear combinations of elements with different grades will be considered in the Kähler calculus.

2. PROPERTIES OF THE EXTERIOR DIFFERENTIATION

The exterior differentiation satisfies the following properties:

$$\begin{aligned} &1)\ d\left(f\alpha_k\right) = df \wedge \alpha_k + f\ d\alpha_k && f \in \Lambda_x^0 && \alpha_k \in \Lambda_x^k \\ &2)\ d\left(\alpha_k \wedge \beta_l\right) = d\alpha_k \wedge \beta_l + (-1)^k \alpha_k \wedge d\beta_l && && \beta_l \in \Lambda_x^l \\ &3)\ d\left(d\alpha_k\right) = 0 \end{aligned} \tag{10}$$

The proofs of properties 1 and 2 can be found in [3, p. 26]. Let us prove the property 3. A differential form of any grade is a sum which has this aspect when indicated with Einstein's summation convention:

$$\alpha = a_{i\cdots j} dx^i \wedge \cdots \wedge dx^j \tag{11}$$

Here Vargas' notation is not applied. Therefore $1 \le i \le n$, $1 \le j \le n$ and the same for all the other indices. By differentiation we have $d\alpha$:

$$d\alpha = \frac{\partial a_{i\cdots j}}{\partial x^k} dx^k \wedge dx^i \wedge \cdots \wedge dx^j \tag{12}$$

where $1 \le k \le n$. A new differentiation yields:

$$d\left(d\alpha\right) = \frac{\partial a_{i\cdots j}}{\partial x^k \partial x^l} dx^l \wedge dx^k \wedge dx^i \wedge \cdots \wedge dx^j \tag{13}$$

If k or l are included in the sequence $\{i \cdots j\}$ the term vanishes. Otherwise, the term can be non-null. Since each term obtained by differentiation firstly with respect to x^k and later with respect to x^l has another corresponding term obtained by differentiation in the opposite order, firstly with respect to x^l and then with respect to x^k:

$$\frac{\partial a_{i\cdots j}}{\partial x^l \partial x^k} dx^l \wedge dx^k \wedge dx^l \wedge \cdots \wedge dx^k, \quad \frac{\partial a_{i\cdots j}}{\partial x^k \partial x^l} dx^k \wedge dx^l \wedge dx^l \wedge \cdots \wedge dx^k \tag{14}$$

we can couple them in a unique term to see that they cancel:

$$d\left(d\alpha\right) = \left(\frac{\partial a_{i\cdots j}}{\partial x^k \partial x^l} - \frac{\partial a_{i\cdots j}}{\partial x^l \partial x^k} \right) \left(dx^k \wedge dx^l\right) \wedge dx^i \wedge \cdots \wedge dx^j = 0 \tag{15}$$

owing to Schwarz's theorem about the equality of crossed partial derivatives. Notice that now $dx^k \wedge dx^l$ are affected by Vargas' notation. Therefore, it is proven that $d\left(d\omega\right) = 0$ under the conditions of validity of Schwarz's theorem. Q.E.D.

3. CHANGE OF VARIABLES IN DIFFERENTIAL FORMS

In a change of variables we have:

$$x^i = x^i\left(y^1, \cdots, y^n\right) \qquad 1 \le i \le n \qquad \left|\frac{\partial x^i}{\partial y^j}\right| \neq 0 \tag{16}$$

with a non-null Jacobian. In the points where the Jacobian vanishes, the change of variable does not work properly. A change of variable is basically a point-dependent linear transformation in the module of differential forms because:

$$dx^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j} dy^j \tag{17}$$

In matrix form:

$$\begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^1} \end{pmatrix} \begin{pmatrix} dy^1 \\ \vdots \\ dy^n \end{pmatrix} \tag{18}$$

A change of variable in a differential 1-form is straightforward:

$$\omega_1 = \sum_{i=1}^{n} a_i dx^1 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \frac{\partial x^i}{\partial y^j} dy^j = \sum_{j=1}^{n} b_j dy^j \quad \Rightarrow \quad b_j = \sum_{i=1}^{n} a_i \frac{\partial x^i}{\partial y^j} \tag{19}$$

In matrix form:

$$\omega_1 = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^n} \end{pmatrix} \begin{pmatrix} dy^1 \\ \vdots \\ dy^n \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \begin{pmatrix} dy_1 \\ \vdots \\ dy_n \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^n} \end{pmatrix}$$

The question is how it applies to forms of any grade. It is immediate from the definition of determinant that:

$$(21)\qquad dx^1 \wedge \cdots \wedge dx^n = \begin{vmatrix} \dfrac{\partial x^1}{\partial y^1} & \cdots & \dfrac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x^n}{\partial y^1} & \cdots & \dfrac{\partial x^n}{\partial y^n} \end{vmatrix} dy^1 \wedge \cdots \wedge dy^n = \frac{\partial\left(x^1 \cdots x^n\right)}{\partial\left(y^1 \cdots y^n\right)} dy^1 \wedge \cdots \wedge dy^n$$

where the usual bridged notation for the Jacobian has been introduced. For forms of a grade less than the number n of variables involved in the change, the expression is intermediate between (18) and (21), but the entries of the matrix of change of basis are always Jacobians of an order equal to the grade of the differential forms. For instance:

$$(22)\qquad dx^i \wedge dx^j = \left(\frac{\partial x^i}{\partial y^k} dy_k + \frac{\partial x^i}{\partial y^l} dy^l\right) \wedge \left(\frac{\partial x^j}{\partial y^k} dy^k + \frac{\partial x^j}{\partial y^l} dy^l\right)$$

$$= \left(\frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l} - \frac{\partial x^i}{\partial y^l}\frac{\partial x^j}{\partial y^k}\right)\left(dy^k \wedge dy^l\right) = \begin{vmatrix} \dfrac{\partial x^i}{\partial y^k} & \dfrac{\partial x^i}{\partial y^l} \\ \dfrac{\partial x^j}{\partial y^k} & \dfrac{\partial x^j}{\partial y^l} \end{vmatrix} \left(dy^k \wedge dy^l\right) = \frac{\partial\left(x^i x^j\right)}{\partial\left(y^k y^l\right)}\left(dy^k \wedge dy^l\right)$$

where Einstein's summation convention has been applied. In the same way:

(23)

$$dx^i \wedge dx^j \wedge dx^k = \begin{vmatrix} \dfrac{\partial x^i}{\partial y^l} & \dfrac{\partial x^j}{\partial y^m} & \dfrac{\partial x^k}{\partial y^n} \\ \dfrac{\partial x^i}{\partial y^l} & \dfrac{\partial x^j}{\partial y^m} & \dfrac{\partial x^k}{\partial y^n} \\ \dfrac{\partial x^i}{\partial y^l} & \dfrac{\partial x^j}{\partial y^m} & \dfrac{\partial x^k}{\partial y^n} \end{vmatrix} \left(dy^l \wedge dy^m \wedge dy^n\right) = \frac{\partial\left(x^i x^j x^k\right)}{\partial\left(y^l y^m y^n\right)}\left(dy^l \wedge dy^m \wedge dy^n\right)$$

and so on.

4. POINCARÉ LEMMA

A differential form ω is said to be *closed* if $d\omega = 0$, and *exact* if there exists another differential form ψ such that $\omega = d\psi$. As we have seen above, every exact differential form is closed. However, the question is to know under which conditions a closed differential form is exact. The answer is the Poincaré lemma: *If $A \subset \mathbb{R}^n$ is a star-shaped open set, then every closed differential form is locally exact* [3, p. 40], [4, p.86].

An important fact is that the primitive ψ of ω that the Poincaré lemma ensures to exist is not unique. That is, any other differential form of the kind $\psi + d\alpha$ is also a primitive of ω:

$$\left.\begin{array}{c} d\psi = \omega \\ \psi' = \psi + d\alpha \end{array}\right\} \quad \Rightarrow \quad d\psi' = \omega \tag{24}$$

5. PRIMITIVES OF CLOSED DIFFERENTIAL FORMS

Laurent Schwartz [5] provided an iterative method to calculate a primitive ψ_1 of an exact differential form ω. Let us suppose that ω is a linear function of exterior products of $dx_1, \cdots dx_n$. Then, we can write ω as:

$$\omega = dx_1 \wedge L_1 + M_1 \tag{25}$$

where L_1 and M_1 are uniquely defined by the requirement that they do not contain dx_1. Now we calculate a differential form Λ_1 such that $L_1 = \dfrac{\partial \Lambda_1}{\partial x_1}$ through the integration of L_1 from 0 to x_1 keeping $x_2, \cdots x_n$ constant. When subtracting the differential of Λ_1 from ω, one obtains a differential form not containing dx_1 and not dependent on x_1. If the subtraction yields zero we have finished. If the subtraction yields a differential form, we repeat the process by separating those terms containing any of the surviving dx_i, which we assume to be dx_2, from those not containing it:

$$\omega - d\Lambda_1 = dx_2 \wedge L_2 + M_2 \tag{26}$$

where L_2 and M_2 only contain exterior products of $dx_3, \cdots dx_n$ and do not depend on x_1. In order to obtain Λ_2, one integrates L_2 from 0 to x_2 keeping $x_3, \cdots x_n$ constant. Subtraction of $d\Lambda_2$ from $\omega - d\Lambda_1$ yields another differential form only containing exterior products of $dx_3, \cdots dx_n$, and so on until exhausting ω:

$$\omega - d\Lambda_1 - d\Lambda_2 - \cdots d\Lambda_k = 0 \qquad k \leq n \tag{27}$$

When this subtraction equals to zero, we have found a primitive $\psi_1 = \Lambda_1 + \Lambda_2 + \cdots + \Lambda_k$. Then, the indefinite integral ψ will be $\psi = \psi_1 + d\chi$, where χ is a differential form whose grade is a unit lower than the grade of ψ because $d^2\chi = 0$. $d\chi$ plays the same role as the constant of integration in the differential calculus of one variable.

Let us apply, for instance, Schwartz's method to the differential form ω (28), which is closed because $d\omega = 0$, According to the Poincaré lemma a local primitive ψ exists such that $d\psi = \omega$. Let us write ω in the form of equation (25):

$$\omega = 2\,x\,y^2 z\,dx \wedge dy - x^2 y^2\,dy \wedge dz + x\,z\,dz \wedge dx \tag{28}$$

$$= dx \wedge \left(2\,x\,y^2 z\,dy - x\,z\,dz\right) - x^2 y^2\,dy \wedge dz$$

By integration of L from 0 to x we find Λ:

$$\Lambda = \int_0^x d\xi \wedge L = \left[\int_0^x 2\xi\ y^2 z\, d\xi \right] \wedge dy - \left[\int_0^x \xi\, z\ d\xi \right] \wedge dz = x^2\ y^2 z\, dy - \frac{x^2 z}{2}\, dz \tag{29}$$

whose differential is:

$$d\Lambda = 2\, x\, y^2 z\, dx \wedge dy - x^2 y^2\ dy \wedge dz + x\, z\ dz \wedge dx \tag{30}$$

Notice that dx must be placed on the left of the exterior product of differentials in (29), which has implied a change of sign of the second integral. Integration is the inverse operation of exterior differentiation (3), so that the differential of the variable with respect to which the integration is carried out must be placed on the left like in (3).

We now see that $\omega = d\Lambda$ and Λ (30) is therefore a primitive. If $\omega - d\Lambda \neq 0$ we would repeat the process for dy. The indefinite integral of ω will be:

$$\psi = \frac{\partial f}{\partial x} dx + \left(x^2\ y^2 z + \frac{\partial f}{\partial y} \right) dy + \left(\frac{\partial f}{\partial z} - \frac{x^2 z}{2} \right) dz \tag{31}$$

For instance, for $f = \frac{x^2 z^2}{8}$ we find another primitive ψ_2:

$$\psi_2 = \frac{x\, z^2}{4} dx + x^2\ y^2 z\, dy - \frac{x^2 z}{4}\, dz \tag{32}$$

Another example is provided in the integration of the differential form (45).

6. DEFINITE INTEGRALS OF DIFFERENTIAL FORMS

A differential form that is not exact can nevertheless be integrated in the sense that follows. The definite integral so obtained will depend on the manifold where the integration is carried out. Let us see, for instance, a line integral, the integral of a differential 1-form, following two distinct paths L_1 and L_2 between two given points:

$$\int_L \omega = \int_{(0,0,0)}^{(1,1,1)} \left(x^2\ dx + (x+y)\, dy + (z+y)\, dz \right) \tag{33}$$

If we follow the path $L_1(x,y,z) = (t,t,t)$ we have:

$$\int_{L_1}\omega = \int_0^1 \left(t^2 + 4t\right)dt = \left[\frac{t^3}{3} + 2t^2\right]_0^1 = \frac{7}{3} \tag{34}$$

But, if we follow the path $L_2(x,y,z) = \left(t, t^2, t^3\right)$ we have:

$$\int_{L_2}\omega = \int_0^1 \left(3t^2 + t + t^3\right)dt = \left[t^3 + \frac{t^2}{2} + \frac{t^4}{4}\right]_0^1 = \frac{7}{4} \tag{35}$$

The results of both integrations differ because the differential form is not exact:

$$\omega = x^2\ dx + \left(x + y\right) dy + \left(z + y\right) dz \quad \Rightarrow \quad d\omega = dx \wedge dy + dy \wedge dz \neq 0 \tag{36}$$

7. STOKES' THEOREM

Stokes' theorem states that *if ω is a smooth differential (k-1)-form with compact support on a smooth manifold D of dimension k with boundary ∂D then it satisfies*:

$$\oint_{\partial D}\omega = \int_D d\omega \qquad \omega \in \Lambda_x^{k-1} \qquad \dim\left(D\right) = k \tag{37}$$

7.1 Case $k = 1$: line integrals. When it applies to exact differential 1-forms, line integrals can be computed in a very easy way:

$$\int_{L_{A\to B}}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) = f\left(B\right) - f\left(A\right) \tag{38}$$

where $L_{A\to B}$ is any path. In vectorial notation:

$$\int_{L_{A\to B}} \text{grad} f \cdot dl = f\left(B\right) - f\left(A\right) \tag{39}$$

where dl is the line element:

$$dl = dx\ e_1 + dy\ e_2 + dz\ e_3 \tag{40}$$

and the gradient operator is:

$$\text{grad} f = \frac{\partial f}{\partial x}e_1 + \frac{\partial f}{\partial y}e_2 + \frac{\partial f}{\partial z}e_3 \tag{41}$$

Going to a concrete example, let us consider the line integral:

$$\int_{(0,1,0)}^{(1,0,1)} \omega = \int_{(0,1,0)}^{(1,0,1)} \left((2xyz + 2z)dx + \left(x^2 z + 1\right)dy + \left(x^2 y + 2x\right)dz\right) \tag{44}$$

Let us check that the differential 1-form ω is closed:

$$d\omega = d\left((2xyz + 2z)dx + \left(x^2 z + 1\right)dy + \left(x^2 y + 2x\right)dz\right) = 0 \tag{45}$$

According to the Poincaré lemma, ω has a primitive to be obtained by Schwartz's method. By integrating from 0 to x we have:

$$\Lambda_1 = \int_0^x (2\xi\ yz + 2z)dx = x^2 yz + 2xz \tag{46}$$

$$d\Lambda_1 = (2xyz + 2z)dx + x^2 z\ dy + \left(x^2 y + 2x\right)dz$$

$$\omega - d\Lambda_1 = dy \qquad \Lambda_2 = y$$

$$\omega - d\Lambda_1 - d\Lambda_2 = 0$$

$$\int \omega = \Lambda_1 + \Lambda_2 = x^2 yz + 2xz + y + C \tag{47}$$

$$\int_{(0,1,0)}^{(1,0,1)} \omega = \left[x^2 yz + 2xz + y\right]_{(0,1,0)}^{(1,0,1)} = 2 - 1 = 1 \tag{48}$$

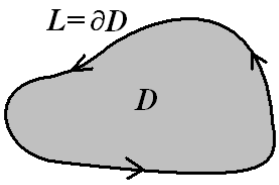
As an alternative, try to integrate the line integral (48) for a path given by $(x, y, z) = (\sin t, \cos t, \sin^2 t)$. Then the line integral becomes:

$$\int_{(0,1,0)}^{(1,0,1)} \omega = \int_0^{\pi/2} \left(4\sin^3 t \cos^2 t + 4\sin^2 t \cos t - \sin^5 t - \sin t\right) dt \tag{49}$$

The computations that follow take much more time.

7.2 Case $k = 2$. Surface integrals. In this case Stokes' theorem relates surface integrals to line integrals along the boundary of the integration surface. For the concrete case of the plane, we have Green's theorem:

$$(50)\qquad \iint_D \left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dx\wedge dy = \oint_{L=\partial D}\left(p\,dx+q\,dy\right)$$



where D is a closed region in the plane and L is the curve that bounds D. The line integral on the right must be taken with a counterclockwise orientation. An elemental proof of Green's theorem can be found in handbooks such as [6, p.440]. Following the case $k=2$ for the three-dimensional space, we have Stokes' original theorem:

$$(51)\qquad \iint_\Sigma \left[\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dx\wedge dy+\left(\frac{\partial r}{\partial y}-\frac{\partial q}{\partial z}\right)dy\wedge dz+\left(\frac{\partial p}{\partial z}-\frac{\partial r}{\partial x}\right)dz\wedge dx\right]$$

$$= \oint_{\partial\Sigma}\left(p\,dx+q\,dy+r\,dz\right)$$

where Σ is a region with boundary in a surface of the three-dimensional space, and $\partial\Sigma$ is the closed curve that bounds Σ. In this way, the components of the *curl* of the vector field $v=(p,q,r)$ are:

$$(52)\qquad \operatorname{curl} v=\left(\frac{\partial r}{\partial y}-\frac{\partial q}{\partial z},\frac{\partial p}{\partial z}-\frac{\partial r}{\partial x},\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)$$

7.3 Case $k=3$. Gauss' theorem. If Stokes' general theorem is applied to differential 3-forms in the space we have Gauss' theorem:

$$(53)\qquad \iiint_V \left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}\right)dx\wedge dy\wedge dz = \oiint_{\partial V}\left(p\,dy\wedge dz+q\,dz\wedge dx+r\,dx\wedge dy\right)$$

which relates a volume integral in a volume V with a surface integral over the closed surface ∂V that bounds the volume V. In this way, the divergence of a vector field $w=(p,q,r)$ is:

$$(54)\qquad \operatorname{div} w=\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}$$

8. THE DIFFERENTIAL OPERATORS IN ORTHOGONAL CURVILINEAR COORDINATES

Another application of exterior differential is the deduction of expressions for the differential operators gradient, curl and divergence. The line element dl in curvilinear orthogonal coordinates of the three-dimensional space is:

(55) $$dl = h_1 du^1 e_1 + h_2 du^2 e_2 + h_3 du^3 e_3 \qquad e_i^2 = 1 \qquad e_i \cdot e_j = 0 \text{ if } i \neq j$$

where h_i are called *scale factors*. Locally, we can always write the line element as an Euclidean line element [7, p. 95]:

(56) $$dl = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3 \qquad \omega^i = h_i du^i$$

The differential of a function f is:

(57) $$df = \frac{\partial f}{\partial u^1} du^1 + \frac{\partial f}{\partial u^2} du^2 + \frac{\partial f}{\partial u^3} du^3$$

which can be written in the local Euclidean basis:

(58) $$df = \frac{1}{h_1}\frac{\partial f}{\partial u^1}\omega^1 + \frac{1}{h_2}\frac{\partial f}{\partial u^2}\omega^2 + \frac{1}{h_3}\frac{\partial f}{\partial u^3}\omega^3$$

The components in the basis ω_i are called *physical* components because they are the components that a local observer would see. The vector having the physical components of the differential of a function (58) is the gradient operator [8, p. 250], [9, p. 269]:

(59) $$\text{grad} f = \left(\frac{1}{h_1}\frac{\partial f}{\partial u^1}, \frac{1}{h_2}\frac{\partial f}{\partial u^2}, \frac{1}{h_3}\frac{\partial f}{\partial u^3} \right)$$

In the same way, the differential 1-form of a vector field[1] $A = (a_1, a_2, a_3)$ is:

(60) $$\alpha = a_1\omega^1 + a_2\omega^2 + a_3\omega^3 = h_1 a_1 du^1 + h_2 a_2 du^2 + h_3 a_3 du^3$$

The differentiation of this 1-form yields a 2-form:

(61) $$d\alpha = \left(\frac{\partial(h_2 a_2)}{\partial u^1} - \frac{\partial(h_1 a_1)}{\partial u^2} \right) du^1 \wedge du^2 + \left(\frac{\partial(h_3 a_3)}{\partial u^2} - \frac{\partial(h_2 a_2)}{\partial u^3} \right) du^2 \wedge du^3$$
$$+ \left(\frac{\partial(h_1 a_1)}{\partial u^3} - \frac{\partial(h_3 a_3)}{\partial u^1} \right) du^3 \wedge du^1$$

which can be written in the local Euclidean basis ω_i (56) as:

(62) $$d\alpha = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 a_3)}{\partial u^2} - \frac{\partial(h_2 a_2)}{\partial u^3} \right) \omega^2 \wedge \omega^3 + \frac{1}{h_1 h_3}\left(\frac{\partial(h_1 a_1)}{\partial u^3} - \frac{\partial(h_3 a_3)}{\partial u^1} \right) \omega^3 \wedge \omega^1$$

[1] Here, the vector field A is a polar vector such as linear momentum or the electric field, which changes the sign under the inversion of coordinates.

$$+\frac{1}{h_1 h_2}\left(\frac{\partial(h_2 a_2)}{\partial u^1}-\frac{\partial(h_1 a_1)}{\partial u^2}\right)\omega^1\wedge\omega^2$$

Then, the physical components of the curl are [8, p.250], [9, p. 275]:

$$\text{(63)}\qquad \operatorname{curl}(a_1,a_2,a_3)=\left(\frac{1}{h_1 h_2}\left(\frac{\partial(h_2 a_2)}{\partial u^1}-\frac{\partial(h_1 a_1)}{\partial u^2}\right),\frac{1}{h_2 h_3}\left(\frac{\partial(h_3 a_3)}{\partial u^2}-\frac{\partial(h_2 a_2)}{\partial u^3}\right),\right.$$

$$\left.\frac{1}{h_3 h_1}\left(\frac{\partial(h_1 a_1)}{\partial u^3}-\frac{\partial(h_3 a_3)}{\partial u^1}\right)\right)$$

Let us consider the differential 2-form associated to a vector field[2] $B=(b_1,b_2,b_3)$:

$$\text{(64)}\qquad \beta=b_1\omega_2\wedge\omega_3+b_2\omega_3\wedge\omega_1+b_3\omega_1\wedge\omega_2$$

$$=h_2h_3b_1du^2\wedge du^3+h_3h_1b_2du^3\wedge du^1+h_1h_2b_3du^1\wedge du^2$$

After differentiation we find:

$$\text{(65)}\qquad d\beta=\left(\frac{\partial(h_2h_3b_1)}{\partial u^1}+\frac{\partial(h_3h_1b_2)}{\partial u^2}+\frac{\partial(h_1h_2b_3)}{\partial u^3}\right)du^1\wedge du^2\wedge du^3$$

and by writing the result in the local Euclidean basis:

$$\text{(66)}\qquad d\beta=\frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3b_1)}{\partial u^1}+\frac{\partial(h_3h_1b_2)}{\partial u^2}+\frac{\partial(h_1h_2b_3)}{\partial u^3}\right)\omega^1\wedge\omega^2\wedge\omega^3$$

the divergence operator [7, p. 77], [8, p. 250], [9, p. 272] is obtained:

$$\text{(67)}\qquad \operatorname{div}B=\frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3b_1)}{\partial u^1}+\frac{\partial(h_3h_1b_2)}{\partial u^2}+\frac{\partial(h_1h_2b_3)}{\partial u^3}\right)$$

9. APPLICATION OF THE FORMULAS OF DIFFERENTIAL OPERATORS TO ELECTROMAGNETISM

Let us apply the differential operators to electromagnetism. For instance, the electric field E is obtained from the vector potential A and the scalar potential ϕ as:

$$\text{(68)}\qquad E=-\operatorname{grad}\phi-\frac{\partial A}{\partial t}$$

[2] Here, the vector field B is an axial vector such as angular momentum or the magnetic field, which remain invariant under the inversion of coordinates.

On the other hand, the magnetic field B is obtained from Lorentz's equation:

(69) $B = \text{curl}A$

The expressions for gradient (59) and curl (63) can be applied to equations (68) and (69) because E is a polar vector and B an axial vector. Maxwell's equations in the vacuum are:

(70) (a) $\text{curl}E = -\dfrac{\partial B}{\partial t}$ (b) $\text{div}B = 0$

(71) (a) $\text{div}E = \dfrac{\rho}{\varepsilon_0}$ (b) $\text{curl}B = \mu_0\, j + \dfrac{1}{c^2}\dfrac{\partial E}{\partial t}$

where ρ is the electric charge density, j is the electric current density, c is the speed of light in the vacuum, ε_0 is the vacuum electric permittivity and μ_0 the vacuum magnetic permeability. There is a fundamental difference between eqns. (70) and (71). In (70a) the curl is applied to a polar vector yielding an axial vector so that the expression (63) for curl holds. In (70b) the divergence is applied to an axial vector and therefore the expression (67) for divergence is also valid. However, in (71) (a) the divergence is applied to a polar vector, which is not the exterior differentiation of (64), and in (71 b) the curl is applied to an axial vector, which is not the exterior differentiation of (60). Then the formulas (63) and (67) cannot be applied (although this happens currently) to eqns. (71). In order to have a wider view of the insufficiency of the exterior differential, let us see Maxwell's equations in the space-time. The electromagnetic field F has grade 2 [10, 11] and corresponds to the differential 2-form [12]:

(72) $F = E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$

The vector potential A is a differential 1-form:

(73) $A = A_x dx + A_y dy + A_z dz - \phi\, dt$

Equations (68) and (69) are included in the exterior differentiation:

(74) $dA = F$

Obviously, a second exterior differentiation yields zero:

(75) $dF = d^2 A = 0$

which accounts for Maxwell's equations (70), but not for (71). Clearly, Kähler differentiation is needed if we want to get the current density and the charge by differentiation of the electromagnetic field F. Firstly, we can write instead of (74):

(76) $d \vee A = F$

which yields additionally the Lorentz normalization condition:

$$d \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{77}$$

Here we have applied Kähler's signature (3,1), that is, $dx^2 = dy^2 = dz^2 = 1$ and $c^2 dt^2 = -1$ for lengths measured in meters and time in seconds. In a second step we can write instead of (75):

$$d \vee F = d \cdot F = -k\, J \tag{78}$$

where J is the current density vector of the special relativity:

$$J = j_x dy \wedge dz \wedge dt + j_y dz \wedge dx \wedge dt + j_z dx \wedge dy \wedge dt - \rho\, dx \wedge dy \wedge dz \tag{79}$$

whence we deduce that the constant k is a pseudoscalar, because $d \cdot F$ is a differential 1-form:

$$k = \frac{1}{\varepsilon_0}\, dx \wedge dy \wedge dz \wedge dt \tag{80}$$

Equations (76) and (78) yield d'Alembert's equation:

$$d \vee d \vee A = \Delta\, A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -k\, J \tag{81}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian. The inner differentiation of (81) results in the equation of continuity:

$$d \cdot \left(\Delta\, A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \right) = \Delta \left(d \cdot A \right) - \frac{1}{c^2}\frac{\partial^2 \left(d \cdot A \right)}{\partial t^2} = d \cdot \left(k\, J \right) \tag{82}$$

because the d'Alembertian commutes with the inner differentiation. Now, taking into consideration the Lorentz normalization condition $d \cdot A = 0$ (77), we finally arrive at the continuity equation:

$$d \cdot \left(kJ \right) = 0 \quad \Rightarrow \quad \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \tag{83}$$

Therefore, the Kähler calculus provides a full description of electromagnetism, from potentials to currents, which cannot be given by the exterior calculus.

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Summer School, phases 2, 3:

Kähler Calculus and its Applications

José G. Vargas

Notes on the Kähler calculus for the Summer School

1. ***Perspective on the Kähler calculus***

2. ***Kähler algebra***

3. ***Kähler differentiations***

4. ***Relativistic Quantum Mechanics***

5. ***Lie differentiation and angular momentum***

6. ***Conservation in Quantum Mechanics. Beyond Hodge's theorem***

Notes on the Kähler Calculus for the Summer School

Jose G. Vargas

1 CHAPTER 1: Perspective on the Kähler Calculus for Experts on Clifford Analysis

1.1 Reconsidered Program of the Kähler part of the summer school

This is the first of a few chapters that I plan to post in this web site for potential participants in the Summer School, phases II and III. It provides an illustration for (mainly) Clifford mathematicians who would like to know what the Kähler calculus is about and what it offers. Starting with the second chapter, these notes will be used at the school for the actual teaching.

For the moment, the table of contents will not be removed, so that the change will be easier to appreciate. I have come to the conclusion that rather than teaching as much theory as needed for its multiple applications, I shall present some materials and produce immediately applications of it. Very occasionally I shall borrow a theorem (like a uniqueness theorem) to be proved in a future chapter. The chapters to be posted here will be only part of what will be lectured in Brasov, just enough to provide the flavor of it. Those who are in a position to understand these notes and plan to attend the conference might be able to start reading the pertinent Kähler papers on their own and ask, during the lectures, questions more profound that they might be asking otherwise.

As an example, the second chapter will deal with Kähler algebra of scalar-valued differential forms. It will be followed by a pair of applications: the theorem of residues and, unrelated to that, a new perspective

on the issue of superposition, a concept to be replaced with decomposition, since it is through decomposition that the objects that one tries to superimpose are actually generated. Let me be more explicit. If the wave function is a member of the Kähler algebra, it can be written as a sum of pieces that belong to the ideals and also are solutions, but this is not sufficient to make them particle states. If you start with particle states (even mixing particles with antiparticles both with both options for handness) you do not get that intermediate state where you decompose but do not quite get particles, as one still needs something else for this. Of course, one can introduce concepts like matrix densities or something of the sort to deal with issues of this kind. But this is not as natural, or as rich or as reliable as starting with a solution of a wave equation that you decompose to get spinors which are not yet wave functions for leptons living in the same ideals.

The third chapter will deal with, among other topics, Kähler differentiation of scalar-valued differential forms, strict harmonic and harmonic differentials, Dirac type spinors (before we even consider a Dirac type Kähler equation, since their form has to do with theory of solutions with symmetry of exterior systems regardless of specifics), and so on. Applications: Real complex-like calculus in the plane, Helmholtz theorem for differential $1-$forms and $2-$forms in 3-D Euclidean space, and a hint at why there is room for theory on quantum collapse of the wave function and quantum teleportation in this calculus. These issues are intimately related to the issue of decomposition. One could say that these topics are aspects of the same theory, which grows in sophistication as we keep building more basic theory.

A fourth chapter will deal with the "Kähler-Dirac equation", or simply Kähler equation. A variety of applications for relativistic quantum mechanics follows: emergence of momentum operators, Pauli-Dirac equation for particles and antiparticles, post Pauli-Dirac Hamiltonian and an additional couple of topics.

At about this point, we shall have reached the third phase of the Summer School. It will deal with symmetry and conservation, a uniqueness theorem for differential forms, integration at different levels of generality for differentials whose exterior derivative and co-derivative are known, Lie and Killing operators, unified spin and orbital angular momentum, total argument momentum and its square, antiparticles, leptons and quarks.

1.2 Introduction

The Kähler calculus is a calculus of tensor-valued differential forms. So, we have a product of structures. The restriction of the Kähler calculus

(KC) to scalar-valued differential forms is the generalization of the exterior calculus that results from replacing the underlying exterior algebra with Clifford algebra. Clifford analysts are used to the replacement of exterior with Clifford. Then, what is special here?

In the KC, the interior derivative of a scalar-valued differential $1-$form is like the divergence, and the same derivative of a vector field is zero, not the divergence, as we shall see. That is such an obvious difference that one can understand it before learning the KC. This difference has its origin in the fact that there is a greater variety of elements in the full Kähler calculus than in other calculi. But the real important differences consist in what the KC can do and the Dirac calculus cannot. Of course, this chapter is an exception. It is meant to provide a perspective to those who can understand it. Everybody else should start with the next chapter. So, if upon reading this chapter you start not to understand, abandon it and go to the next one. Just before moving to the next chapter, please read the last four sections of the present one. I have given the exact places where the claims were proved by Kähler.

The files I may put in the web site of this summer school will be referred as chapters because they will be so, up to minor modifications in a book I am writing about what I shall call the Cartan-Kähler calculus. It was announced in my previous book "Differential Forms for Physicists and Mathematicians". I know some readers are looking forward to read it. There will be prior chapters on Clifford algebra and the exterior calculus. But, for the purposes of the summer school, this will be chapter one.

An outstanding feature of the KC is the ease with which one gets deep into the world of quantum mechanics, immediately acquiring new vistas since one enters if through a different door. So much so that we shall speak of the quantum mechanics, ab initio relativistic, that emerges as a virtual concomitant of the KC. At the end of this chapter, we shall briefly illustrate some deep results for quantum mechanics that Dirac's theory does not match. For the moment let us just say that the wave function in the "Kähler-Dirac" equation for scalar-valued differential forms is for members of the algebra in general, not only specifically for spinors (i.e. for members of ideals in this algebra).

Of course, there will be a lot of work to be done to revisit every piece of physics from the new perspective. Better yet, just a little bit of Kähler algebra will allow us, at the end of the next chapter, to start seeing how old problems can be seen in a new light. For the moment, I shall deal only with some computational issues to respond to the question, what is new?

1.3 Kähler and Cartan-Kähler calculi

As I mentioned in the second half of the biographical note of this web site, the KC was formulated in three papers. The translation of the title of the first of those papers is *Interior and exterior differential calculus*, and the translation of the third is *The interior calculus.* The author is dealing with the same calculus at the same level of generality. The only main difference is that the last of these two is far more comprehensive than the first one. So, it appears that he was not too sure at some time(s) about what title he should have given to his calculus. The issue we have with titles far transcends this one because of a far more important reason, as we are about to explain.

We are fully interested in Kähler's calculus for scalar-valued differential forms, not for tensor-valued differential forms, which we shall replace with Clifford-valued ones. In the last two of these three cases, we are dealing with tensors products of algebras. Then, sometimes, we shall consider a restriction to "mirror elements" (concept not needed at this point), which constitute a fourth structure. All four have to do with Clifford structure but only one is a Clifford algebra. Hence, to minimize clutter, I shall often use the term algebra not in the technical sense but in the more general sense of the dictionary, namely as a mathematical system that uses symbols and specially letters to generalize certain arithmetical operations and relationships. In this wide sense, all four of those structures qualify as algebras.

Eventually, we shall use the terms Kähler's algebra and KC to when we deal only with scalar-valued differential forms. When dealing with Clifford valuedness, we shall use the terms Cartan-Clifford algebra and calculus. It is worth noticing that Cartan considered curvatures as bivector valued differential 2-forms. We do not risk ignoring any applications with tensor-valued differential forms, since Kähler did not produce any application for them.

1.4 Kähler's differential forms

Kähler wrote his general differential forms as

$$u_{i_1\ldots i_\lambda}^{k_1\ldots k_\mu} = \frac{1}{p!} a_{i_1\ldots i_\lambda\, l_1\ldots l_p}^{k_1\ldots k_\mu} dx^{l_1} \wedge \ldots \wedge dx^{l_p}, \tag{1}$$

where we are using Einstein's convention of summation over repeated indices. His excessive use of components is not a desirable feature for a calculus, but has a very useful consequence. It shows explicitly that we must consider two types of subscripts. They refer to two essentially different concepts: scalar-valued differential r-forms (quantities with just a l series of indices) and tensor-valued differential 0-forms or tensor fields

(quantities with only k and or i series of indices). Hence, his calculi are ab initio different from all other known calculi that are based on Clifford algebra. But Kähler did not exhibit the basis elements that pertain to the i and k indices.

The affine curvature is a very well known example of the rare quantities that have indices of all three types. Consider some vector-valued differential 1-form. It need not be closed. For simplicity let us assume that it were what Cartan and Kaehler would call the exterior derivative of a vector field, though practitioners might refer to it with the name of covariant derivative, name which is here reserved for a different purpose. We have

$$d\mathbf{v} = dv^i\mathbf{e}_i + v^k d\mathbf{e}_k = (dv^i + v^k\omega^i_k)\mathbf{e}_i. \tag{2}$$

Differentiating next $d\mathbf{v}$, we obtain a vector-valued differential 2−form, $dd\mathbf{v}$ (which happens to be zero in Euclidean space), but we shall ignore this (you may assume that the manifold is only approximately a Euclidean space). The components of $dd\mathbf{v}$ are not what in the tensor calculus one calls covariant derivatives. For that, we would have to differentiate $v^j_{;k}\phi^k\mathbf{e}_j$, where (ϕ^i) is the basis of covariant vector fields dual to the basis field $(\mathbf{e}_j)$, i.e. $\phi^i \llcorner \mathbf{e}_k = \delta^i_k$. Let us not overlook that, in the Kähler calculus, a differential 1−form is not a covariant vector (i.e. a linear function of vectors) field, but a function of curves, evaluated by integration on a given curve.

We get five terms for $dd\mathbf{v}$, two from the differentiation of $dv^i\mathbf{e}_i$, and three more from the differentiation of $v^k\omega^i_k\mathbf{e}_i$. The first term, ddv^i is obviously zero. The second and third terms cancel each other out. We thus have

$$dd\mathbf{v} = v^k d(\omega^i_k\mathbf{e}_i) = v^k(d\omega^i_k - \omega^j_k \wedge \omega^i_j)\mathbf{e}_i. \tag{3}$$

Since $d\omega^k_i - \omega^j_i \wedge \omega^k_j$ is a differential 2−form, it is of the form. Hence,

$$dd\mathbf{v} = v^i R^k_{i l_1 l_2}\omega^{l_1} \wedge \omega^{l_2}\mathbf{e}_k. \tag{4}$$

This says that the components of the vector-valued differential 2−form $dd\mathbf{v}$ are $v^i R^k_{i l_1 l_2}$. We then define curvatures $\mho$ by

$$\mho \equiv R^k_{i l_1 l_2}\omega^{l_1} \wedge \omega^{l_2}\phi^i\mathbf{e}_k, \tag{5}$$

where all three types of indices are involved. Clearly $dd\mathbf{v}$ results from evaluating $\mho$ in $\mathbf{v}$,

$$\mho \llcorner \mathbf{v} = R^k_{i l_1 l_2}\omega^{l_1} \wedge \omega^{l_2}\phi^i\mathbf{e}_k \llcorner v^m\mathbf{e}_m = v^k(d\omega^i_k - \omega^j_k \wedge \omega^i_j)\mathbf{e}_i = dd\mathbf{v}. \tag{6}$$

For more on this approach to affine curvature and the use of bases ω of differential 1−forms (which are of the essence further below), check in

your library my book "Differential geometry for physicists and mathematicians". I do not know of any other book which stays close to this Cartanian way of doing modern differential geometry, or to the Kähler calculus for that matter.

1.5 Kähler's differentiation

In Kähler, all differentiations except Lie differentiation are based on his concept of covariant differential. He gives it ab initio as

$$\begin{aligned} d_h a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots l_p} = & \frac{\partial}{\partial x^h} a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots l_p} + \\ & + \Gamma^{k_1}_{hr} a^{r\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots l_p} + \ldots + \Gamma^{k_\mu}_{hr} a^{k_1\ldots r}_{i_1\ldots i_\lambda\ l_1\ldots l_p} \\ & - \Gamma^{r}_{hi_1} a^{k_1\ldots k_\mu}_{r\ldots i_\lambda\ l_1\ldots l_p} + \ldots + \Gamma^{r}_{hi_r} a^{k_1\ldots k_\mu}_{i_1\ldots r\ l_1\ldots l_p} \\ & - \Gamma^{r}_{hl_1} a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ r\ldots l_p} + \ldots + \Gamma^{r}_{hl_r} a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots r_p}, \end{aligned} \tag{7}$$

where the gammas are the Christoffel symbols. Since this equation will probably put many readers off, let me start by saying that the author of these notes does not keep anything like this in his memory, nor does he look for this formula when needed. There is a better way of doing his differentiations. At this point, let me just make some helpful comments.

The differential of scalar-valued differential forms written in terms of Cartesian coordinates requires only the first of those four lines. That is good enough for many applications. If the differential form is scalar-valued but the coordinates are not Cartesian, d_h involves only the first and fourth lines, regardless of whether the metric is the Euclidean metric disguised by the use of an arbitrary coordinate system or whether it is a proper post-Euclidean Riemannian metric. Lines two and three are for the non-scalar valuedness but only when the affine connection is the Levi-Civita (LC) connection. If the connection were another one, those symbols would have to be replaced with the components of the given (metric compatible) affine connection of the space. But the fourth line would not change in this respect, as it does not depend on the connection but only on the metric and its derivatives through the Christoffel symbols. In addition to generalizing this formula for arbitrary connection, one could also replace the tensor-valuedness with Clifford valuedness. And let us not ignore either that vector-valuedness can take place in exterior algebra, Clifford algebra and tensor algebra contexts. It does not make a difference.

A more efficient, less cumbersome way to deal with "Kähler differentiation" resorts to inferring the specific form of the covariant derivative in each case from the equations of structure of the manifold. That is what we are about to do.

Let us give the basic ideas about Kähler differentiation, represented by the symbol ∂. We seek to find an operator ∂ such that $\partial \vee u$ $(= \partial \wedge u + \partial \cdot u)$ will become

$$\partial u = du + \delta u, \tag{8}$$

with $\partial \wedge u = du$, and such that δu will be intimately connected with the divergence. In Cartesian coordinates, ∂ could be simply $dx^i \dfrac{\partial}{\partial x^i}$. But this is not good enough for general coordinates, as the form of the divergence in curvilinear coordinate attests to. For this reason, we shall rather seek ∂u in the form

$$\partial u = dx^i \vee d_i u. \tag{9}$$

for some covariant derivatives $d_i u$ canonically determined by the structure of the manifold and such that $dx^i \wedge d_i u = du$. The notation

$$\partial u = \partial \vee u = \partial \wedge u + \partial \cdot u$$

still is justified if we understand it to mean

$$\partial \wedge u = dx^i \wedge d_i u = du, \qquad \partial \cdot u = dx^i \cdot d_i u = \delta u.$$

We have just connected with the exterior calculus, which we are extending with this δu, once we define $d_i u$. This will be much richer than Ricci, or de Rham or Dirac theory since the context is much larger than in those theories. The proof will be in the pudding. We shall thus refer to δu as the interior derivative since it is a concept more comprehensive than divergence depending on what objects u the operator δ is applied to.

1.6 Kähler's differentiation through geometric structure

Assuming we had already computed du by the standard formula in the exterior calculus, we might try to infer $d_i u$ from $du = dx^i \wedge d_i u$. But du does not determine d_i or ∂, which is the reason why we have not displayed $dx^i \partial/\partial_i$. It would work for the exterior derivative. but it would not yield the right divergence. But there is one such solution that is canonically determined by the equations of the structure of the manifold endowed with a metric, regardless of whether the manifold has an affine structure or not.

Define a set of n differential forms ω^i's such that

$$ds^2 = \sum_{i=1}^{n} (\omega^i)^2, \tag{10}$$

for any specifically given quadratic symmetric differential form $ds^2 = g_{ij}dx^i dx^j$ $(i = i, ..., n)$. These ω^i are defined up to the most general rotations in dimension n.

As any good book on differential geometry shows, the system of equations

$$d\omega^i = \omega^j \wedge \omega^i_j, \qquad \omega_{ij} + \omega_{ji} = 0 \tag{11}$$

defines a set of ω^i_j's. For the same metric, we may consider the system

$$0 = d(dx^i) = \omega^j \wedge \alpha^i_j, \qquad d_h g_{kl} = 0,$$

the second of these equations being known as the statement that the covariant derivatives the metric is zero. The Christoffel symbols are defined by $\alpha^i_j = \Gamma^i_{jl}dx^l$.

The last two systems define the same mathematical object. This does not mean that the α^i_j and the ω^i_j are equal, or even the components of some object of grade two. The Γ^i_{jl} and the Γ'^i_{jl} defined by $\omega^i_j = \Gamma^i_{jl}\omega^l$ are related by non-tensorial equations known as the transformations of connections. The mathematical object of which the α^i_j's and the ω^i_j's are components is a differential form valued in the Lie algebra of the Euclidean group (actually of the affine extension of the Lie algebra of the Euclidean group; do not bother about these details) of dimension n. All those other differential forms are components. It is legal tender nevertheless to call them differential forms by making them so as a matter of definition. I shall try to explain this with what should amount to a simpler example.

The first element of orthonormal vector bases at some point of Euclidean space is not a vector, but a set of them: $\cos\phi\ \mathbf{i} - \sin\phi\ \mathbf{j}$. It is basis dependent. Once a basis has been chosen, say for $\phi = \pi/4$, we have a vector $(2)^{-1/2}\ \mathbf{i} - (2)^{-1/2}\ \mathbf{j}$. We can now take this vector and express it in terms of any other basis (in particular for $\phi = \pi/3$), where it is not its first element. This type of idea is involved at a far more sophisticated level in the difference between α^i_j and the ω^i_j. For a deep understanding of this, see my book "Differential geometry for physicists and mathematicians" if you do not know of any other book dealing with this subject (I do not any, except for information scattered over a variety of E. Cartan's papers). The book is now in 500 libraries, hopefully enough of them to have one in a library of your country, where institution could get it for you if it does not have it.

In order to do differentiation in terms of bases that are not coordinate bases, we first define $f_{/k}$ as given by $df = f_{/k}dx^k$. We then have

$$\begin{aligned} dv &= a_{i/j}\omega^j \wedge \omega^i + a_l \wedge d\omega^l = \omega^j \wedge a_{i/j}\omega^i + a_l\omega^i \wedge \omega_i{}^l = \\ &= \omega^j \wedge a_{i/j}\omega^i - a_l\Gamma_i{}^l{}_j\omega^j \wedge \omega^i = \omega^j \wedge (a_{i/j}\omega^i - a_l\Gamma_i{}^l{}_j\omega^i). \end{aligned} \tag{12}$$

By virtue of the relation of this formula to the structure of a manifold, it pertains to define Kähler's covariant derivative $d_j v$ of a differential 1−form $v = a_i \omega^i$ as

$$d_j v = a_{i/j}\omega^i - a_l \Gamma_{i\ j}^{\ l}\omega^i \tag{13}$$

If ω^i is dx^i, the covariant derivative $d_j v$ becomes

$$d_j v = a_{i,j}\, dx^i - a_l \Gamma_{ij}^l dx^i. \tag{14}$$

For rectilinear coordinates, thus Cartesian in particular, d_j reduces to the partial derivative with respect to x^j since the Christoffel symbols then become zero. For the interior derivative, we then have

$$\delta v = dx^j \cdot d_j v = dx^j \cdot dx^i (a_{i,j} - a_l \Gamma_{ij}^l) = g^{ij}(a_{i,j} - a_l \Gamma_{ij}^l) = a^j{}_{,j} - a_l \Gamma^{jl}_{\ j}. \tag{15}$$

We have not written δv as $\partial \cdot v$ since d_j now is $\partial_j - \underline{\ }_l \Gamma_j^{jl}$. We use the underbar because a has different subscripts on the two terms of $d_j v$.

Let us return to $d\omega^i = \omega^j \wedge \alpha_j^i$. For differential 1−forms dx^j, the exterior derivative, $d(dx^i)$, is zero, and so is $dx^j \wedge \alpha_j^i$. On the other hand, the connection equations are

$$d\mathbf{e}^i = -\omega_j^i \mathbf{e}^j, \tag{16}$$

from which we would infer $d_h \mathbf{e}^i$, and similarly for $d_h \mathbf{e}_i$. The computing of interior and Kähler derivatives will be easily achieved from these equations. It is really simple.

1.7 Perspective on two physical applications developed by Kähler himself

This preview for Clifford analyst would be only one half of an important picture without mentioning a couple of very important physical implications of the quantum mechanics that appears to spring out of it, without obvious alternative. Both of them are related to the fact that, unlike Dirac's theory, Kähler's quantum mechanics is not about spinors but about differential forms. Spinors, in the form of members of ideals of the Kähler algebra, are a concomitant of the treatment of solutions with symmetry of exterior systems, as any system of differential equations can be shown to be.

In his 1960 paper, Kähler starts with a profound treatment of partial differentiation of differential forms with respect to the angular coordinate associated with rotational symmetry. Eventually, both spin and angular momentum emerge from it, at par. Tracing spin from the end of Kähler's derivation to the beginning of his argument, we realize that spin starts its life in Kähler 's theory inside a partial derivative of a field conceived

not as a probability amplitude, but as a more conventional type of field; not as a spinor or member of an ideal in the algebra, but as a member of the algebra itself independently of any ideals.

In addition to the elegance and reliability of Kähler's argument, this reading in reverse shows that a particle is "*some special part of the field*", not just something that mediates among the particles, since these are simply special configurations of it. This is consistent with Einstein's view, though not exactly, as expressed by his words in correspondence with Einstein: "To realize the essential point of atomistic theory, it is sufficient to have a *field* of high intensity *in a spatially small region* ...". This idea of Einsteinean pedigree finds natural implementation in the Kähler calculus.

As profound as this result is in connection with spin, there is another one as profound, namely the treatment of not charge, not antiparticles, not energy, but all three at the same time. Let me give an inkling of why this is so. A solution with time translation symmetry of a quantum equation is known to require a phase factor with time and energy in the exponent. But, if the equation is written as an exterior system (as in the Cartan-Kähler theory of systems of differential equations), one also needs as factors two idempotents that define complementary ideals. Each of these belong to the signs of charge for each value of the energy in the phase factor.

Why may we make such a statement about charge? Kähler already explained why. It is a consequence of the decomposition into two terms of the wave function of a Kähler-Dirac equation under time translation symmetry, one from each of two complementary ideals. The corresponding conservation law for the wave function is then composed of two parts, one for each of those two terms. The left hand side of this law becomes the sum of the left hand sides of two conservation laws, not the sum of two conservation laws. The small difference between the two pairs of similar terms emerging in the process led Kähler to identify a physical magnitude, the electromagnetic charge, which may have two signs. The same energy but two signs of something when the coupling is electromagnetic coupling led him to view the two terms as representing particle and antiparticle. There is no need for an infinite sea of negative energy solutions, the weirdness of this concept being kept virtually silent in modern quantum physics. If Kähler had done his work in the late 1920's, we would be asking in 2016: "Dirac who?" Of course, Dirac was a genius since he provided a solution, though imperfect, to a problem for which the mathematics was not yet ripe.

CHAPTER 2: Kähler Algebra

Jose G. Vargas

Participants in this summer school will at this point know enough Clifford algebra not to be confused if we sometimes represent Clifford products with the inverted wedge product sign and at other times by juxtaposition.

The Kähler calculus is a little bit strange. This may be because it is an approximation to something much deeper and to which I shall refer as the Cartan-Kähler calculus and which reconciles the ways of Cartan in differential geometry with the ways of Kähler on matters of calculus. One gets the impression that Kähler did not totally absorb Cartan's teachings in differential geometry. For this reason we shall use the term Kähler geometry and calculus when dealing only with scalar-valued differential forms. This less comprehensive version is enough to produce a great simplification of proofs and mathematical arguments when compare with the same issues in the standard mathematical and physical paradigms.

1 A practical approach to Kähler algebra

Hervibores in the African savanna are born knowing how to walk and soon learn how to run. They must have acquired some concept of running by the time one of them spots a lion and sends a signal to the herd and everybody starts to run. Of course, they do not know the anatomy and physiology involved in running but they surely know how to run. I do not need to explain how this example applies to this section. Thus, after reading section 1, one can jump to section 3 of this chapter and proceed with the integrations that are usual in an undergraduate course in complex variable. But, for the developments in the next chapters, one will have to read sections 1, 2 and 4. Readers who may not like the Kähler calculus are invited to develop alternatives that will efficiently match Kähler's results.

1.1 Definition of basic products

Kähler algebra (of scalar-valued differential forms) on a differentiable manifold (not on a tangent space or cotangent space!!!) endowed with a metric is the Clifford algebra defined by the relation

$$dx^i dx^j + dx^j dx^i = 2g^{ij}. \tag{1}$$

Hence

$$dx^i \wedge dx^j + dx^j \wedge dx^i = 0, \qquad dx^i \cdot dx^j = g^{ij} = \frac{1}{2}(dx^i dx^j + dx^j dx^i), \tag{2}$$

which come together as

$$dx^i dx^j = dx^i \wedge dx^j + dx^i \cdot dx^j. \tag{3}$$

If the differentiable manifold is a Euclidean vector space and the coordinates are Cartesian, we simply replace g^{ij} with δ^{ij}. All this can be said in a very elegant manner without using bases, but the definition would be very abstract. We have said emphasized the "of scalar-valued differential forms" to make sure we avoid confusions. Kähler also considered a more general structure consisting of tensor-valued differential forms. These do not constitute a Clifford algebra but the tensor product of a tensor algebra by the Kähler algebra just defined.

Let (A, B) be an ordered pair of two points in a differentiable manifold. Let γ be any curve with ends at those points. By definition dx^i is the function of curves such that

$$\int_\gamma dx^i = x^i_B - x^i_A. \tag{4}$$

Notice that we have not invoked either linear functions (i.e. covariant vectors) or covariant vector fields.

Consider next the differential form $3dx+xdy$ on curves between those same points. The integration depends on curves since this differential form does not have a potential function. Finally compute (To avoid confusion, I did not say evaluate) $3dx+xdy$ at any point with coordinate x equal to 2. We obtain $3dx + 2dy$. This is not something that we can evaluate on γ because it is not legal to first compute it at a point and then evaluate it on a curve, unless the differential form to be evaluated had been defined as $3dx + 2dy$ in the first place. The most we can do in this respect is to associate with $3dx + xdy$ equal to 2 points the linear function of tangent vectors $3\phi^1 + 2\phi^2$ where $\phi^i \llcorner \mathbf{a}_i = 1$, $\phi^i \llcorner \mathbf{a}_j = 0$ for $i \neq j$.

The pair (A, B) is the boundary of the manifold γ. So, Eq. (4) is a particular case of Stokes generalized theorem, which is here used as a definition, the reversion of roles of theorems and definitions being permissible. The formula for the exterior derivative would be a theorem rather than a definition. This may not be the most expedient course of action for the development of a calculus, but it is the most fundamental since integration requires less restrictive conditions than differentiation. Notice that we have not given any rule about differentiation yet, since none was needed.

One would proceed in similar manner for differential $r-$forms, i.e. as functions of $r-$surfaces. The use of Stokes theorem as a definition is nothing new. Elie Cartan already used it almost a century ago for defining the exterior derivative of a differential form whose coefficients are not differentiable functions.

Although we should not care too much at this point about the significance of the dot product of two differential $1-$forms, let us make a few remarks that may be helpful and a preview of arguments to come. In Cartesian coordinates, products $dx^i \cdot dx^j$ are zero unless $i = j$. But $dx^i \cdot dx^i$ does not have an invariant meaning; $\sum_1^n dx^i \cdot dx^i$ does. One would then have to see these products in context. $\sum_1^n dx^i \cdot dx^i$. This expression is related to some beautiful canonical Kaluza-Klein geometry which supersedes standard differential geometry. Again, we shall have to wait for Cartan-Kähler calculus in order to go deeper into it in a future chapter. Without going into any of that and not even raising the issue, Kähler obtained great results with the structure just mentioned. They eliminate significant problems of standard quantum mechanics.

Kähler used bases of differentials of arbitrary systems of coordinates, Cartesian coordinates not existing in non-Euclidean or non-pseudo-Euclidean spaces. Hence, we would have

$$dx^i dx^j + dx^j dx^i = 2\delta^{ij}. \tag{5}$$

in the Cartesian case. We would have to also replace g^{ij} with δ^{ij} in (2).

We prefer to use bases (ω^l) constituted by linear combinations of the differentials of the coordinates, whether these are Cartesian or not. We shall again have

$$\omega^i \omega^j + \omega^j \omega^i = 2g^{ij} \tag{6}$$

But we can always orthonormalize the metric so that we get

$$\omega^i \omega^j + \omega^j \omega^i = 2\delta^{ij}, \tag{7}$$

regardless of whether the manifold is a Euclidean space or not.

The ω^i notation is common in differential geometry developed with differential forms, usually in context of theory of connections. But this notation has nothing to do with connections, just with the metric. Of course, one can get Christoffel symbols and the Levi-Civita connection from the metric and its derivatives, but they are not needed at all for present purposes.

Let ω_i be defined by $\omega^j \cdot \omega_i = \delta_i^j$ for all pairs of indices. The ω_i's are not meant to be linear functions on the module spanned by the basis (ω^j) (The concept of module is more general than that of vector space, but you will not have problems here if you think of a module as if it were a vector space). Nothing could be more misleading than to think of the ω_i's as linear functions of anything. They are specific cases of functions of curves, just as legitimate as the ω^i's. They constitute just alternative bases in the module of differential forms.

Kähler likes to define the symbol e_i where we use the left dot multiplication "$\omega_i\cdot$". The action of e_i has the distributive property since, as we know,

$$\omega^i\cdot(\omega^j\wedge\omega^l\wedge\omega^k) = (\omega^i\cdot\omega^j)(\omega^l\wedge\omega^k)-(\omega^i\cdot\omega^l)(\omega^j\wedge\omega^k)+(\omega^i\cdot\omega^k)(\omega^i\wedge\omega^j. \quad (8)$$

And similarly for "$\omega_i\cdot(\omega^j\wedge\omega^l\wedge\omega^k)$", since ω_i is as legitimate a differential $1-$form as ω^i. We thus have

$$\omega_i\cdot(\omega^j\wedge\omega^l\wedge\omega^k) = (\omega_i\cdot\omega^j)(\omega^l\wedge\omega^k)-(\omega_i\cdot\omega^l)(\omega^j\wedge\omega^k)+(\omega_i\cdot\omega^k)(\omega^i\wedge\omega^j \quad (9)$$

or, equivalently,

$$e_i(\omega^j\wedge\omega^l\wedge\omega^k) = (e_i\omega^j)(\omega^l\wedge\omega^k)-(e_i\omega^l)(\omega^j\wedge\omega^k)+(e_i\omega^k)(\omega^i\wedge\omega^j. \quad (10)$$

Some learned readers would prefer to base the calculus and even differential geometry on vector field equations that largely parallel those we have given. A participant in this summer school already did so. The advantage of the Kähler way is that he deserves vector fields for other purposes. The parallelism ceases as soon as we consider differentiation. In any case and for the moment, I am reproducing what Kähler did and shall later show my view of where one should go, guided by the applications that ensue from the course of action to be followed. At this point I advocate the Kähler course of action because of the results he obtained and which nobody else appears to have even matched.

1.2 Using D=2 to get used to Kähler algebra

In terms of polar coordinates in 2-D Euclidean space E_2, we have

$$((d\rho)^2 = 1, \qquad (d\phi)^2 = d\phi\cdot d\phi = \frac{1}{\rho^2}, \qquad d\rho\cdot d\phi = 0. \quad (11)$$

$$dxdy = -dydx, \qquad (dxdy)^2 = -1. \tag{12}$$

We use the abbreviation i for $dxdy$. The complex-like inhomogeneous differential form z,

$$z \doteq x + ydxdy \doteq x + yi, \tag{13}$$

emerges from the relation between $(d\rho, d\phi)$ and (dx, dy):

$$d\phi = \frac{xdy - ydx}{x^2 + y^2} = \frac{x - ydxdy}{x^2 + y^2}dy = \frac{1}{x + ydxdy}dy = z^{-1}dy, \tag{14}$$

$$d\rho = \frac{xdx + ydy}{(x^2 + y^2)^{1/2}} = \rho\frac{x - ydxdy}{x^2 + y^2}dx = \frac{\rho}{x + ydxdy}dx = \rho z^{-1}dx. \tag{15}$$

By virtue of (13), it is clear that

$$z^{\pm m} = (x + yi)^{\pm m} = \rho^{\pm m}e^{m\phi i} = \rho^{\pm m}(\cos m\phi \pm i \sin m\phi), \tag{16}$$

for integer m.

There is a laudable effort on the part of Clifford mathematicians to replace the imaginary unit with elements of real Clifford algebra, an effort with which this author wholly agrees. But I see that there has been an abuse of the replacement of the imaginary unit with the unit pseudo scalar of the algebra when some other element in the algebra of square minus one is a more natural choice. So, rather than replace, it is a better process to let the right element emerge without resort to a replacement, which is what we have done in this case and shall be doing time and time again. And nothing impedes to proceed in reverse, and abbreviate (once found) $dxdy$ as i, and write $\cos m\phi \pm dxdy \sin m\phi$ as $e^{im\phi}$. e shall sometimes use $dxdy$ and i simultaneously, choosing one or the other in each specific case depending on what we wish to emphasize Finally, here is a remark for those without much Clifford experience who may have read this paragraph. The square of the unit pseudo-scalar may be plus or minus one. It depends on dimension and signature.

Let α be a differential $1-$form and let u and v be scalar functions. We have

$$(u + vi)\alpha = \alpha(u + vi)^*, \tag{17}$$

where

$$(u + vi)^* \equiv u - vi, \tag{18}$$

and, in particular,

$$z^* = x - yi, \quad z^* = \rho^2 z^{-1} \quad (z^*)^{-1} = \rho^{-2}z, \tag{19}$$

with $i = dxdy$. Clearly

$$u = \frac{(u + vi) + (u + vi)^*}{2}, \qquad v = \frac{(u + vi) - (u + vi)^*}{2i}, \tag{20}$$

as in the calculus of complex variable but again with $i = dxdy$.

1.3 The angular integrand

For the purpose of certain integrations,we wish to have the angular integrand part $j(\rho,\phi)d\phi$ of a differential $1-$form $\alpha = h(\rho,\phi)d\rho + j(\rho,\phi)d\phi$, when it is given in terms of Cartesian coordinates

$$\alpha = k(x,y)dx + g(x,y)dy. \tag{21}$$

We clearly have

$$j = \rho^2 \ \alpha \cdot d\phi \tag{22}$$

and, therefore,

$$\alpha = wdx, \qquad w \equiv k - gdxdy = k - gidy. \tag{23}$$

We proceed to compute j. For that purpose, we express the dot product in terms of Clifford products:

$$j = \rho^2(wdx)\cdot(z^{-1}dy) = \frac{\rho^2}{2}\left[wdxz^{-1}dy + z^{-1}dywdx\right] =$$

$$= \frac{\rho^2}{2}\left[w(z^*)^{-1}i - z^{-1}w^*i\right] = \frac{1}{2}\left[wz - w^*z^*\right]i = -(wz)^{(2)}, \tag{24}$$

where the superscript refers to the coefficient of the differential $2-$form part (of wz in this case). For the last step, we have used the last of (19).

2 Algebraic background for (algebraic) neophytes

A legitimate question is. Since Kähler algebra is just one more Clifford algebra, why should one give a name to it? One does not give a name to every possible Clifford algebra. The point is that the concept of differential form in 99% of the literature (though not in Rudin's classic book "Principles of Mathematical Analysis") is not as an integrand but as antisymmetric multilinear functions of vector fields. The appellative Kähler algebra is meant to remind us of this feature, of the need to remember that these are $r-$integrands, i.e. functions of $r-$surfaces.

2.1 Splits and pseudo-splits of a Clifford algebra

As you already know, the elements of even grade of a Clifford algebra constitute an algebra by themselves, called the even subalgebra. The set of the odd elements is not an algebra, since it is not closed (The product of two odd element is even). But there are other subalgebras of the same dimension, 2^{n-1}. Choose just an element of a basis of differential $1-$forms. Call it dx^i for any given i. Any differential form u in the Kähler algebra can be written as

$$u = u' + dx^i \wedge u'', \tag{25}$$

where both u' and u'' are both uniquely defined if we demand that none of them contains dx^i as a factor. It is easy to prove that this decomposition splits the algebra into a subalgebra of dx^i's and the set of all the other elements in the algebra. If the signature of the algebra is definite, the signature of the subalgebra does not depend on which dx^i we choose. All of them are isomorphic. But, if it is not definite, we get different subalgebras depending on whether $(dx^i)^2$ is 1 or -1. For simplicity, we took a member dx^i of a basis of differential 1-forms. But any differential $1-$form can be chosen as a member of any such basis. Vice versa, we could always express the dx^i in the displayed formula as a linear combination of the differential of some other coordinate system.

In the following, we proceed very slowly, since we are in a hurry. Consider the "identity"

$$1 = \frac{1}{2}(1+a) + \frac{1}{2}(1-a), \tag{26}$$

where a is a member other than a scalar of Kähler algebra. Premultiplying by arbitrary elements u of the algebra, we get

$$u = u\frac{1}{2}(1+a) + u\frac{1}{2}(1-a) \tag{27}$$

This looks like a decomposition. Call it that if you wish, but let us play with it in order to distinguish between two situations.

Assume we had a two dimensional space with signature (1,1). To be specific, $(dx)^2 = 1$ and $(dt)^2 = -1$. In connection with the decomposition

$$1 = \frac{1}{2}(1+dx) + \frac{1}{2}(1-dx), \tag{28}$$

let us premultiply $\frac{1}{2}(1 \pm dx)$ by dx. We obtain

$$dx\frac{1}{2}(1 \pm dx) = \pm\frac{1}{2}(1 \pm dx). \tag{29}$$

The right and left of this equation are of the same type in the sense that they both have the factor $(1 + dx)$ on the right. We shall later see that they cannot be written with the factor $(1 - dx)$ as last factor on the right.

Consider on the other hand

$$1 = \frac{1}{2}(1+dt) + \frac{1}{2}(1-dt), \tag{30}$$

and premultiply $\frac{1}{2}(1 + dt)$ by dt. We obtain

$$dt\frac{1}{2}(1+dt) = -\frac{1}{2}(1-dt). \tag{31}$$

This element can be written in both ways, meaning that in one case the last factor on the left is of the type $(1 + a)$ and the last factor on the right is of the type $(1 - a)$ for the same a.

Let us see another example. Simple operations show that whereas

$$\frac{1}{2}(1 + dx^i)\frac{1}{2}(1 + dx^i) = \frac{1}{2}(1 + dx^i), \tag{32}$$

we, on the other hand, have

$$\frac{1}{2}(1 + dt)\frac{1}{2}(1 + dt) = \frac{1}{4}(1 + dt) - \frac{1}{4}(1 - dt). \tag{33}$$

Whereas on the right hand side of (32) the factor $(1 + dx^i)$ remains, we have both $(1 + dt)$ and $-\frac{1}{4}(1 - dt)$ in the second. It makes a great difference whether a^2 equals 1 or -1. The cases $a^2 = +1$ has advantages that no alternatives have.

We shall refer to the equation

$$u = u\frac{1}{2}(1 + dx^i) + u\frac{1}{2}(1 - dx^i), \tag{34}$$

as a split of u, and, since u is arbitrary, it splits the algebra into two subalgebras without unit. On the other hand, the equation

$$u = u\frac{1}{2}(1 + dt) + u\frac{1}{2}(1 - dt) \tag{35}$$

does not represent a split since one can write any member of the algebra with both $(1+dt)$ and $(1-dt)$ as last factor. All this will become increasingly obvious as we familiarize ourselves with this type of computation in the next subsection.

Assume now that $\frac{1}{2}(1 \pm dx^i)$ were associated with space translation symmetry in the x^i direction. We would then expect that $\frac{1}{2}(1 \pm dt)$ would be associated with time translation symmetry. The sign of the square will make great difference vis a vis the decompositions that we are about to consider. Since one needs square $+1$ for interesting and fruitful results, Kähler used the decomposition

$$u = u\frac{1}{2}(1 + idt) + u\frac{1}{2}(1 - idt) \tag{36}$$

to treat time translation symmetry, with i as the usual imaginary unit of the calculus of complex variable. Notice that, since $(idt)^2$ equals $+1$, the last equation represents a split into two subalgebras. In order to achieve this behavior (which is desired because of what we shall see in the next section), Kähler resorted to the field of the complex numbers, if we were in the field of the reals in the first place. We shall see in a future chapter that the usual i can and should be represented by real elements of some structure. These elements will emerge spontaneously rather than be introduced ad hoc.

2.2 Idempotents and ideals

A subset A of some Clifford algebra Cl is said to be a left ideal if and only if

$$Cl\ A = A. \tag{37}$$

In words, A is a subset of Cl such that multiplying it on the left by any element of Cl returns an element of A. We are thus saying that A is closed under multiplication by Cl on the left.

An idempotent is defined as an element of Cl whose square is equal to itself. By trivial recursion, one sees that any integer power of an idempotent returns it. Some examples of idempotents are $\frac{1}{2}(1 \pm dx^i)$, $\frac{1}{2}(1 \pm idt)$ and $\frac{1}{2}(1 \pm idxdy)$. The two idempotents in each pair annul each other, i.e.

$$\frac{1}{2}(1+a)\frac{1}{2}(1-a) = \frac{1}{2}(1-a)\frac{1}{2}(1+a) = 0. \tag{38}$$

Any such pair naturally defines a complementary pair of left ideals

$$Cl = Cl\frac{1}{2}(1+a) + Cl\frac{1}{2}(1-a) \tag{39}$$

These ideals are subalgebras, but without a unit. let us start by showing that no element of the algebra except zero can be in both ideals at the same time. Indeed imagine you had

$$u\frac{1}{2}(1+a) = v\frac{1}{2}(1-a). \tag{40}$$

If we right multiply by $1-a$, we find that v is zero; and, if by $1+a$, we find that u is zero. The unit is a linear combination of $(1+a)$ from one ideal and $(1-a)$ from the other.

We now get some important practice with idempotents. Because of our future use of them, we shall adopt the same terminology as Kähler. Define idempotents

$$\epsilon^{\pm} \equiv \frac{1}{2}(1 \mp idt), \qquad \tau^{\pm} \equiv \frac{1}{2}(1 \pm idxdy). \tag{41}$$

Notice, but do not worry, about the inversion of sign between the left and right hand sides of the definition of the $\epsilon^{\pm}$. We have

$$\epsilon^+\epsilon^- = \epsilon^-\epsilon^+ = 0, \qquad \tau^+\tau^- = \tau^-\tau^+ = 0 \tag{42}$$

and

$$\epsilon^+ + \epsilon^- = 1, \qquad \tau^+ + \tau^- = 1. \tag{43}$$

We shall refer with the term of complementary idempotents to any pair of them that add up to one and mutually annul. Notice also the most important feature that the ϵ's commute with the τ's.

A remark about notation. Whereas $\epsilon^{\pm}\tau^{\pm}$ means the two options $\epsilon^{+}\tau^{+}$ and $\epsilon^{-}\tau^{-}$, we shall use the asterisk, as in $\epsilon^{\pm}\tau^{*}$, to mean the four options $\epsilon^{\pm}\tau^{\pm}$ and $\epsilon^{\pm}\tau^{\mp}$. The four $\epsilon^{\pm}\tau^{*}$ mutually annul. This is to be compared with the idempotents jointly generated by the non commuting elements idt and dx^l for given l. The four idempotents $\epsilon^{\pm}\frac{1}{2}(1*dx^i)$ do not mutually annul. For example, we have ,

$$\left[\frac{1}{2}(1+idt)\frac{1}{2}(1+dx^i)\right]\left[\frac{1}{2}(1+idt)\frac{1}{2}(1-dx^i)\right]=\frac{1}{8}(1+idt)(1-dx^{\cdot i}). \tag{44}$$

The right hand side of

$$1=\epsilon^{+}\tau^{+}+\epsilon^{+}\tau^{-}+\epsilon^{-}\tau^{+}+\epsilon^{-}\tau^{-} \tag{45}$$

is a sum of mutually annulling idempotents, but the right hand side of

$$1=\epsilon^{+}\frac{1}{2}(1+dx^i)+\epsilon^{+}\frac{1}{2}(1-dx^i)+\epsilon^{-}\frac{1}{2}(1+dx^i)+\epsilon^{-}\frac{1}{2}(1-dx^i) \tag{46}$$

is not.

We have extended the splits

$$Cl=Cl\ \epsilon^{+}+Cl\ \epsilon^{-},\qquad Cl=Cl\ \tau^{+}+Cl\ \tau^{-} \tag{47}$$

into the more comprehensive split

$$Cl=Cl\ \epsilon^{+}\tau^{+}+Cl\ \epsilon^{+}\tau^{-}+Cl\ \epsilon^{-}\tau^{+}+Cl\ \epsilon^{-}\tau^{-}. \tag{48}$$

It is legitimate to ask whether we can continue this extension. For that, we cannot count on $\frac{1}{2}(1\pm dx^i)$ because of what we said above. We may wonder, however, whether we could find some idempotents other than the $\frac{1}{2}(1\pm dx^i)$ in order to make the split even more comprehensive. For instance, $dtdxdy$ is of square $+1$. But we then have, for example,

$$\epsilon^{-}\tau^{+}\frac{1}{2}(1+dtdxdy)=0,\qquad \epsilon^{-}\tau^{+}\frac{1}{2}(1-dtdxdy)=\epsilon^{-}\tau^{+}, \tag{49}$$

so that we do not get new idempotents and thus not an extended split.

We could keep trying to find some other (pair of) idempotent(s) to multiply $\epsilon^{\pm}\tau^{*}$ and which would commute with them. None exists. We then say that the $\epsilon^{\pm}\tau^{*}$ are primitive as they do not comply with the following definition. An idempotent is said to be primitive if it cannot be decomposed into a sum $A+B$ of two commuting, mutually annulling

idempotents, i.e. such that $AB = BA = 0$. Since the dimension, 4, of spacetime is rather low for present purposes, one can readily find by trial and error that the $\epsilon^{\pm}\tau^*$ are primitive. For higher dimension one resorts to a so called Radon-Hurwitz theorem to find the number of such idempotents as a function of dimension and signature of the metric. A detailed exposition of this subject does not pertain here and would also take too much room. We refer interested readers to the book "Clifford Algebras and Spinors" by Pertti Lounesto. For the case in point, application of the theorem confirms that the $\epsilon^{\pm}\tau^*$ are primitive. To be clear as to the meaning of the theorem, let us say that there are more primitive idempotents in 3-D Euclidean space and in spacetime, like the $\frac{1}{2}(1+dx)\frac{1}{2}(1*idydz)$, but this is not a split that extends the one by $\epsilon^{\pm}\tau^*$. For our purposes, the Radon-Hurwitz theorem has to do with the process of continuing to split idempotents.

3 Application 1: Theorems of residues and Cauchy's

This section is a typical one on applications, i.e. one where the application is completed. In the KC , we distinguish two parts in what goes by the name of calculus of complex variables as taught in an undergraduate course in the subject. There is a major difference between the two. It is for this reason that we treat the subject in two different chapters, the two treatments in the sense that one does not require the other. They are two different topics. In this application, we do not need a concept of differentiation additional to the one in any course on the real calculus of several variables. Algebra suffices, namely the algebra that we have just seen in section 1. We shall perform certain real integrals for which the standard calculus of complex variable is typically used. Instead of the complex plan, we shall have a Clifford algebra $Cl_{(0,1)}$ of differential forms viewed as even subalgebra of $Cl_{(2,0)}$. We intend to present her a polished and more comprehensive version of the presentation of the theorems of residues and of Cauchy presented in a paper in arXiv. Just type Jose G Vargas on the top right hand corner and scroll a little bit down to find the pdf file of the paper "Real Calculus of Complex Variable: Weierstrass Point of View". Since readers already have this available to them, we temporarily skip the writing of this section. We shall thus use our time to get to the core of the Kähler calculus as fast as possible.

In an application in the next chapter, on the other hand, we begin the representation of the theory of complex variable without complex variable. It can deal in principle with any problem which does not even exist in standard real analysis, although it will be focussed on integrals because of the interest of the targeted audience. We shall thus define integrations which, in terms of differential forms, would put a differential

2−form as integrand of a line integral. Of course, this is a totally new game. What will correspond to an integral with complex integrand will require some new concepts in calculus with real differential forms.

4 Application 2: Algebraic template for concepts like collapse, entanglement, confinement and teleportation

All the physical applications of the KC revolve around the split

$$1 = \epsilon^{+}\tau^{+} + \epsilon^{+}\tau^{-} + \epsilon^{-}\tau^{+} + \epsilon^{-}\tau^{-}, \tag{50}$$

or the still simpler splits considered below. This particular one is related to the pair spin and rest mass, as we shall learn in the electromagnetic environment. Neither the electromagnetic differential 2−form nor the electromagnetic potential are members of any of the four ideals defined by the four idempotents $\epsilon^{\pm}\tau^{*}$, but certainly can be decomposed into members of them. Kähler showed in an argument in his 1961 paper —complemented with work in 1962 on charge and antiparticles— that electrons and positrons of both chiralities relate to those idempotents in a one to one correspondence. These play an even larger role than the phase factors —which also are essential— in determining the treatment of solutions of quantum mechanical equations involving particles.

In the Dirac theory, idempotents do not play the core role that they play in the Kähler calculus for decomposing wave functions that do not belong to an ideal. The reason is that the Dirac equation is ab initio about elements of ideals, and the Kähler equation is about members of the whole algebra; the members of the ideals are but a very important development. This is just but one of the reasons why Kähler's quantum mechanics supersedes Dirac's.

The split (50) will later be extended to more comprehensive splits, which he did not pursue. For that, he first should have geometrized the imaginary unit, its role then being played not only by $dxdy$, but also by $dydz$ and $dzdx$, all three simultaneously. But, at this point, we would already be outside the realm of algebra and calculus with scalar-valued differential forms. The extension of (50) would take place through primitive idempotents consisting of three factors, which include one each of the pairs $\epsilon^{\pm}$ and $\tau^{\pm}$. We shall later explain how this is possible.

For the moment, let us go into what (50) has to offer. This split implies that any element of the algebra can be written as a sum

$$u = \ {}^{+}u^{+}\,\epsilon^{+}\tau^{+} + \ {}^{+}u^{-}\,\epsilon^{+}\tau^{-} + \ {}^{-}u^{+}\,\epsilon^{-}\tau^{+} + \ {}^{-}u^{+}\,\epsilon^{-}\tau^{-}. \tag{51}$$

This decomposition is unique if we demand that the coefficients ${}^{\pm}u^{*}$ are

elements of the Kähler algebra that depend on $d\rho$ and dz but not on $d\phi$ and dt. This demand does not entail lack of generality.

Since the four idempotents $\epsilon^{\pm}\tau^{*}$ are mutually annulling, we multiply this equation by $\epsilon^{+}\tau^{+}$ on the right and obtain

$$u\epsilon^{+}\tau^{+} = {}^{+}u^{+}\ \epsilon^{+}\tau^{+}, \tag{52}$$

and similarly for products of (51) on the right with the other three idempotents. We find the ${}^{+}u^{+}$ by performing operations in (52) that move the factors dt and $d\phi$ in u to the right to be absorbed by the idempotents. In computations of later chapters, we shall encounter examples of these absorptions. But how can these ideals represent leptons?

The ideals represented in (51) through an arbitrary element of the algebra are intimately connected with the form that solutions with time translation and rotational symmetry take. The association of $\epsilon^{\pm}\tau^{*}$ with electrons and positrons is but a first step in the association of ideals with particles, including muons, taus and quarks. We also need a corresponding phase factor. The $\epsilon^{\pm}$ idempotents take care of the dependence on dt and the corresponding phase factor then takes care of the dependence on t, which has to do with differentiation, not with algebra. We then need the concept of constant differentials (chapter 3). The $\epsilon^{\pm}\tau^{*}$'s are constant differentials. This has the implication that differentiation of a member of any of those ideals remains in the ideal. This propagates to the Kähler equation (chapter 4), which may then be viewed as an entanglement of four equations. Solutions will correspond to actual leptons when some external factor like an electric field or a measuring instrument produces the *collapse* of that entangled system. Until that happens, the spinors are not necessarily particles. Notice the expanded meaning that collapse has here. What is additional in Kähler's relative to Dirac's quantum mechanics is that we have a richer variety of superpositions because the superimposed elements are entangled. Teleportation phenomena are a case of collapse in this extended sense. The Cartan-Kähler extension of Kähler theory will provide us with a better tool kit to interpret what is it that travels in opposite directions and that ends in a correlated collapsed that apparently violates causality. A violation does not actually occur, as we shall explain further below.

The simpler split

$$u = {}^{+}u\ \epsilon^{+} + {}^{-}u\ \epsilon^{-}. \tag{53}$$

already has the major implication of yielding a concept of charge which comes in types positive and negative, but both for the same sign of the energy. This result will apply in particular to the ideals generated by $\epsilon^{\pm}\tau^{*}$ and, more particularly, to pair creation and annihilation. The

emergence of negative energy solutions in Dirac's theory is an spurious effect. Both particles and antiparticles are in the same footing.

The split

$$u = \ u^+ \ \tau^+ + \ u^- \ \tau^- \tag{54}$$

also is deeply involved with the foundations of quantum mechanics. Not only do orbital and spin components of angular momentum come together, but they actually are born together, not as twins, but as non-invariant terms which only become invariant after we take something from one of them and add it to the other. Spin is as external as orbital, and orbital is as internal as spin. This confirms at a very profound level that the concept of particle emerges from the concept of field solutions of basic equations. The field is not exchange currency (quantized or not) among merchants (particles).

The use of Clifford algebra to deal with rotations makes into bivectors the imaginary units in the exponents of the phase factors. Because of the intimate correspondence between idempotents and phase factors in solutions with symmetry of exterior systems, the same observation applies to the imaginary units in their associated idempotents. The $\epsilon^{\pm}\tau^{*}$'s then become $\epsilon^{\pm}I^{*}_{ij}$, where $I^{\pm}_{ij}$ is $\frac{1}{2}(1 + \mathbf{a}_i\mathbf{a}_j dx^i dx^j)$ with no sum over repeated indices. It is then natural to see here three different generations of leptons. Spontaneously broken anisotropy of 3-space is needed to justify this difference, three special directions being associated with three generations. For this to be consistent with experiment, the Lorentz transformations must remain physically relevant. They do, this consistency having been known to philosophers of science (Reichenbach, Grünbaum) and physicists like David Bohm (see his book on special relativity). But much more remains to be done. Because of the possibilities opened by this compatibility, something like the weak interactions appears to be present in Kähler's quantum mechanics with geometrized imaginary unit.

Consideration of anisotropy is not an ad hoc assumption for particular programs by physicists, though it may have been so at some time. It is brought to the fore by the evolution of the theory of connections, as explained in our $U(1) \times SU(2)$ paper (google "$U(1) \times SU(2)$ from the tangent bundle" to find and freely download it).

The aforementioned geometrization of the imaginary unit leads to a canonical Kaluza-Klein space, where there are obvious classical and quantum sectors associated with two $4-$D subspaces. In the classical sector, the constant speed of light reigns supreme. But there is no such obvious limitation in a purely quantum sector, where information within a pure field configuration u might travel as superluminal speeds. I said might, not may, since we do not know better at this point; but this is

enough to start looking at the issue of teleportation in a new light.

The connection of the last three paragraphs with the split (50) is that we have three copies of the same, but with each of the three independents for each copy.

We finally develop the consequences of the existence of a commutative substructure in the Cartan-Kähler structure. It looks like a Clifford algebra of sorts, but is not so because it is commutative. Nevertheless products in two Clifford algebras are involved in these commutative products. Since idempotents involve only mirror elements (dx mirrors $\mathbf{i}$, but dy does not) there is an additional extension of the split (51) by virtue of this commutativity. For brevity reasons, we shall again speak of the newly relevant idempotents instead of speaking of the split itself. The latter involve three idempotent factors, extending the $\epsilon^{\pm} I^*_{ij}$ with space translation ones, which thus come in triples, one for each generation (It might look as six, but there is redundancy when the products are actually performed). The signature, however, does not allow for decreasing exponentials, i.e. for phase factors. This implies that the particles that would correspond to these idempotents do not reach very far; they start dying as they are born. There can never be enough energy if the wave function does not decrease with distance, much less if it increases. Call this confinement. It will look as a surface effect, not without reason. Translational symmetry when matter is involved must include surface effects, since the symmetry does not extend to infinity. Confinement follows.

Much of what has been said in this section may be viewed as speculation. It certainly is. Speculation stars to become theory as it gains ever more sophisticated mathematical representation. The bridge between theory and experiment is very large, as Einstein said. No single person can build alone the large bridges that the advanced state of the physics makes necessary when its foundations are concerned. But how would you interpret the mathematics we have developed in the hypothetical physical situations to which this mathematics might apply.

CHAPTER 3: Kähler Differentiations

Jose G. Vargas

This chapter will be used for the second day of the second phase of the summer school. Said better, this chapter together with some complementary work posted in the arXiv will be the source from which my collaborators will choose what can and should be taught in view of the available time. I shall try to leave others do the choice so that, in the end, the instructors of this subject will jointly decide on what participants will listen to.

These notes are longer than I intended them to be because they address, in addition, issues that readers with early access to them have raised. One recurring theme is that the concept of differential $r-$form does not quite sink. It is an $r-$integrand, thus a function of $r-$surfaces You evaluate the differential $r-$form by integrating it on the on the $r-$surface. Take $3xdx + dy$. This does not mean $3x\Delta x + \Delta y$ regardless of how small Δx and Δy may be. We rather have

$$\int_{x,\,y}^{x+\Delta x,\,y+\Delta y} 3xdx + dy = \left[\frac{3x^2}{2} + y\right]_{x,\,y}^{x+\Delta x,\,y+\Delta y} = 3x\Delta x + \frac{3}{2}(\Delta x)^2 + \Delta y.$$

It is at this point that, if and only if Δx is arbitrarily small we can validate the expression $3x\Delta x + \Delta y$. A differential form is nothing like an infinitesimal change or anything of the sort. The traditional interpretation of a differential form as an infinitesimal begs the definition of infinitesimal.

If there is anything that could be a little bit difficult to understand is what does it mean that $dx^2 = dy^2 = 1$ (also written as $dx \vee dx = dy \vee dy = 1$). But this is just a particular case of what does it mean that the dot product of a differential $r-$form and a differential $1-$form is a differential $(r-1)$-form? This question, specially in the particular case of $dx^2 = dy^2 = 1$ because it is so simple, is clamoring for something more sophisticated and to which we have referred in the past as the Cartan-Kähler calculus. But we are not yet ready for that!

We insist on a point made in the previous chapter, namely that we write $dx^2 = dy^2 = 1$ only if x and y are Cartesian coordinates. In general, the differential $1-$forms whose square is ± 1 are those which orthonormalize the metric when it is viewed as a quadratic differential form. This quadratic differential form is a $2-$tensor built upon a module of differential 1-forms by tensor product. It does not belong to the Kähler algebra, but one can have it in parallel to it. You would have to wait for a Cartan-Kähler calculus for a better course of action.

As a concession to those who cannot live with formulas like $dx^2 = dy^2 = 1$, consider the following example:

$$(du)^2 = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

You might wish to stop at ,

$$(du)^2 = \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial u}{\partial y}\right)^2 dy^2 = \left(\frac{\partial u}{\partial x}\right)^2 dx \cdot dx + \left(\frac{\partial u}{\partial y}\right)^2 dy \cdot dy$$

and keep going until you would have to actually use it for something.

1 Differentiations

In Kähler's work, all differentiations other than partial differentiation are obtained from the so called covariant derivative. We refer readers to chapter one for the basic concepts at a deeper level than here, as we do not presently require as deep a knowledge of differential geometry as is the case there.

In this chapter, we shall deal only with scalar-valued differential forms. And we shall refer to greater valuedness only occasionally, for the purpose of achieving deep understanding of some important issue.

In order to provide early perspective to those who have already worked at a deep level with differential forms or Clifford analysis, we used the term divergence to what, from this point on, will be referred to as interior derivative. The reason is that the term divergence does not capture the richness and difference with other calculi of the Kähler calculus (KC). In this calculus, the interior derivative of a vector field is zero. So, it is not desirable to use the term divergence for the interior derivative. On the other hand, the interior derivative of a differential 1-form takes the form of the divergence of what is referred to as the divergence of a vector field in the standard literature. From the perspective of the KC, concepts that in the vector calculus are assigned to $r-$tensors should be attributed to differential $r-$forms. That frees the valuedness structures (built upon contravariant and covariant vector fields) for other purposes.

1.1 Kähler's approach to covariant differentiation

Readers may start this chapter by simply accepting formula (1.5) below, and looking at the text up to that formula only anecdotally. This amounts to accepting without proof Eq. (1.5) rather than doing so with equation (1.2). If they want to know better, they should read as far as they can the first chapter and then returning here.

Kähler represented tensor-valued differential forms with the notation

$$u^{j_1\ldots j_p}_{i_1\ldots i_p} = a^{j_1\ldots j_q}_{i_1\ldots i_p k_1\ldots k_m} dx^{k_1} \wedge \ldots \wedge dx^{k_m}. \tag{1.1}$$

He then proceeded to introduce in ad hoc manner a concept of covariant differentiation of tensor-valued differential forms as follows:

$$\begin{aligned} d_h a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots l_p} = & \frac{\partial}{\partial x^h} a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots l_p} + \\ & + \Gamma^{k_1}_{hr} a^{r\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots l_p} + \ldots + \Gamma^{k_\mu}_{hr} a^{k_1\ldots r}_{i_1\ldots i_\lambda\ l_1\ldots l_p} \\ & - \Gamma^{r}_{hi_1} a^{k_1\ldots k_\mu}_{r\ldots i_\lambda\ l_1\ldots l_p} + \ldots + \Gamma^{r}_{hi_r} a^{k_1\ldots k_\mu}_{i_1\ldots r\ l_1\ldots l_p} \\ & - \Gamma^{r}_{hl_1} a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ r\ldots l_p} + \ldots + \Gamma^{r}_{hl_r} a^{k_1\ldots k_\mu}_{i_1\ldots i_\lambda\ l_1\ldots r_p}, \end{aligned} \tag{1.2}$$

where the Γ's are the Christoffel symbols. There is here not only a matter of contents but also of notation. The use of components to such a large extent is reminiscent of the tensor calculus.

For several chapters, we shall be interested only in scalar-valued differential forms. Hence, we may ignore lines 2 and 3 and write

$$d_h a_{\ l_1\ldots l_p} = \frac{\partial}{\partial x^h} a_{l_1\ldots l_p} - \Gamma^{r}_{hl_1} a_{\ r\ldots l_p} + \ldots + \Gamma^{r}_{hl_r} a_{\ l_1\ldots r}. \tag{1.3}$$

Kähler invoked the equation

$$\omega^k_i = \Gamma^k_{ij} \cdot dx^j, \tag{1.4}$$

which may mean different things to those who are not familiar with equations of structure. He used (1.4) to rewrite (1.3) as

$$d_m u = \frac{\partial u}{\partial x^h} - \omega_m{}^r \wedge e_r u. \tag{1.5}$$

So, there is a term additional to the partial derivative. It is zero if we have Cartesian or pseudo-Cartesian (if you prefer to use this term when dealing with pseudo-Euclidean spaces). We shall barely need to use it in this course; we can do very much without it.

In deriving Eq. (1.5), Kähler paid the price of going against the natural order of things, namely the one that we proposed in chapter 1; the

foregoing formula (1.4) should precede formula (1.3). More importantly, we showed that one does not need to derive (1.5), much less memorize this and many other formulas. We shall now show —only for those really interested— how (1.5) can be derived by resorting to the equations of structure of a manifold endowed with a metric but not with an affine structure. But even if we have an affine structure where the affine connection is not Levi-Civita's, there are issues for which the affine structure is not relevant, but the Christoffel symbols are. When the symbols Γ^k_{ij} are the Christoffel symbols (most of the time from now on), we should not forget that they were born half a century before they were seen as a tool for parallel transport, simultaneously by Levi-Civita, Hessenberg and Schouten.

1.2 Cartan's-like approach to covariant differentiation

Cartan did not deal explicitly with the issue of covariant differentiation. His exterior derivative of scalar-valued differential forms and of vector-valued differential forms are equivalent to what, in the modern literature, are known as exterior and as exterior covariant derivatives (or simply covariant derivatives for tensor fields). Something similar is the case with Kähler, although the latter paid more attention to tensor-valuedness and less attention to structure than Cartan. Let us then proceed like the latter would have done if they had explicitly considered covariant and interior derivative.

At its most general, a differential form u will be a sum of terms of the form $\alpha = a\ \omega^1 \wedge ... \wedge \omega^s \wedge ... \wedge \omega^m$ on some differentiable manifold of dimension $n \geq m$. Let us exterior differentiate a monomial $\alpha = a\ \omega^1 \wedge ... \wedge \omega^s \wedge ... \wedge \omega^m$, and then postulate the distributive property of d_h. Proceeding in this way, we recover the exterior calculus and the essence of a reinterpreted vector calculus.

Before differentiating each ω^i, we move it to the front and introduce alternating factors 1 and -1. So, with obvious simplification of notation, and with $1...\bar{s}...m$ meaning the absence of s,we have

$$d\alpha = da \wedge \omega^{1...m} + \sum_{s=1}^{s=m} a(-1)^{s-1} d\omega^s \wedge \omega^{1...\bar{s}...m} = da\ \omega^{1...m} + \sum_{s=1}^{s=m} d\omega^s \wedge e_s\alpha. \tag{1.6}$$

But

$$\sum_{s=1}^{s=m} d\omega^s \wedge e_s\alpha = d\omega^r \wedge e_r\alpha, \tag{1.7}$$

with summation from $r = 1$ to $r = n$ and not just to m. This is because $e_r\alpha$ is $\omega_r \cdot \alpha$ and $\omega_r \cdot \omega^i = \delta^i_r$. terms on the right where i is greater than m

do not contribute to the sum $\omega_r \cdot \alpha$. Equation (1.1) can thus be written further as

$$d\alpha = a_{/h}\ \omega^h \wedge \omega^{1\ldots m} + d\omega^r \wedge e_r\alpha, \tag{1.8}$$

with $a_{/h}$ defined by $da = a_{/i}\ \omega^i$. We next use

$$d\omega^r = \omega^k \wedge \omega_k{}^r = -\Gamma_k{}^r{}_h\omega^h \wedge \omega^k = \omega^h \wedge (-\Gamma_k{}^r{}_h\omega^k). \tag{1.9}$$

Hence

$$d\alpha = \omega^h \wedge a_{/h}\ \omega^{1\ldots m}\ -\ \omega^h \wedge (-\Gamma_k{}^r{}_h\omega^k) \wedge e_r\alpha, \tag{1.10}$$

We define the covariant derivative $d_h\alpha$ as

$$d_h\alpha = a_{/h}\ \omega^{1\ldots m} - \Gamma_k{}^r{}_h\omega^k \wedge e_r\alpha. \tag{1.11}$$

Under the distributive property of differential operators with respect to the sum, we further have

$$d_h u = \frac{\partial u}{\partial x^h} - \omega_h{}^r \wedge e_r u, \tag{1.12}$$

for an arbitrary differential form u, whether of homogeneous grade or not. That is what Cartan would have done.

We are now interested in connecting with Kähler's work, i.e. in specializing this to coordinate bases. In general bases, the $\Gamma_k{}^r{}_h$ are defined by $\omega_k{}^r = \Gamma_k{}^r{}_h\omega^h$ and are not in general equal to the Christoffel symbols, which constitute the particular cases for coordinate bases. In terms of them, we have. In terms of coordinate bases

$$\Gamma_k{}^r{}_h = \Gamma_h{}^r{}_k \qquad \ldots\Gamma_k{}^r{}_h dx^k = \Gamma_h{}^r{}_k dx^k = \omega_h{}^r. \tag{1.13}$$

1.3 The Kähler derivative

The Kähler differential is defined as

$$\partial u = dx^h \vee d_h u = dx^h \wedge d_h u\ +\ dx^h \cdot d_h u = du + \delta u, \tag{1.14}$$

where

$$du = dx^h \wedge d_h u, \qquad \delta u = dx^h \cdot d_h u. \tag{1.15}$$

Clearly

$$\partial u = dx^h \vee \left(\frac{\partial u}{\partial x^h} - \omega_h{}^r \wedge e_r u\right) = dx^h \vee \frac{\partial u}{\partial x^h} - dx^h \vee (\omega_h{}^r \wedge e_r u). \tag{1.16}$$

We further have

$$\partial u = dx^h \vee \frac{\partial u}{\partial x^h}\ -\ dx^h \wedge \omega_h{}^r \wedge e_r u\ -\ dx^h \cdot (\omega_h{}^r \wedge e_r u). \tag{1.17}$$

Since $dx^h \wedge \omega_h{}^r$ equals zero, we finally have

$$\partial u = dx^h \vee \frac{\partial u}{\partial x^h}\ -\ dx^h \cdot (\omega_h{}^r \wedge e_r u) = dx^h \vee \frac{\partial u}{\partial x^h}\ -\ e^h(\omega_h{}^r \wedge e_r u). \tag{1.18}$$

1.4 Leibniz rules

You should be able to show that

$$d_h(u\vee v\vee w\vee ...) = d_h u\vee v\vee w\vee ... \ + \ u\vee d_h v\vee w\vee ... \ + \ u\vee v\vee d_h w\vee ... \ + \ ... \tag{1.19}$$

and that

$$d_h(u\wedge v\wedge w) = d_h u\wedge v\wedge w\vee ... \ + \ u\wedge d_h v\wedge w\vee ... \ + u \ \wedge v\wedge d_h w\vee .... \ + \ ... \tag{1.20}$$

We have used so many terms in order to emphasize the lack of alternation of positive and negative signs. The proofs, essentially the same in both cases, are obvious by showing that, at each point, we can make ${\omega_h}^r$ equal to zero. The justification of making this annulment is specially simple for the Christoffel symbols by resorting to the equation for the geodesics. It can also be achieved —in a different and less easy way— in more general cases, as per section 8.4 of my book. We then have the Leibniz rule for the partial derivative . Since the proof is valid at each point independently of what happens at other points, and since the result involves only invariant terms, the proof is complete.

An important rule to accompany the exterior differential of an exterior product of differential forms is the Kähler derivative of the Clifford product of differential forms. The rule is

$$\partial(u\vee v) = \partial u\vee v + \eta u\vee \partial v + 2e^h u\vee d_h v \tag{1.21}$$

The symbol η stands for changing every differential $1-$form factor by its opposite. So $\eta\alpha_r$ equals α_r or $-\alpha_r$ depending on whether the grade of α_r is even or odd. The symbol e^h is defined by its action $e^h u = \omega^h \cdot u$.

We proceed to prove it. In terms of coordinate bases, the differentiation ∂u was introduced as

$$\partial u = dx^h \vee d_h u. \tag{1.22}$$

Hence

$$\partial(u\vee v) = dx^h\vee d_h(u\vee v) = dx^h\vee d_h u\vee v \ + \ dx^h\vee u\vee d_h v \tag{1.23}$$

In the last term, we want to move dx^h past u, for which purpose, we solve for $dx^h\vee u$ in

$$dx^h\vee u \ - \ \eta u\vee dx^h = 2dx^h\cdot u = 2e^h u \tag{1.24}$$

and replace it in (1.24). Equation (1.21) follows.

It is worth considering other "Leibniz", not so much because they are going to be used, but to illustrate that one has to be careful and not extrapolate unduly. Without proof:

$$d(u \vee v) = du \vee v \ + \ \eta u \vee dv \ + \ e^h u \vee d_h v \ - \ \eta d_h u \vee e^h v. \qquad (1.25)$$

Notice the absence of exterior products.

Let us do an example. Let z be defined as $z = x + ydxdy$. Then

$$z^2 = x^2 - y^2 + 2xydxdy. \qquad (1.26)$$

Its exterior derivative is

$$d(z^2) = 2xdx - 2ydy. \qquad (1.27)$$

Assume now that we wish to apply the Leibniz rule that we have just derived. The sum of the first two terms can be written as $2dx \wedge (x + ydxdy)$ which is $2xdx$. For the last two terms (actually four when developed) we use

$$e^x z = ydy, \quad e^y z = -ydx, \quad d_x z = 1, \quad d_y z = dxdy,$$

where $e^x = e^1$ and $e^y = e^2$. We use these equations to get the sum of those four terms, which yield $ydy - ydy - ydy - ydy = -2ydy$. This checks the formula. It was not pretty. In this case we could first compute the product, so that we did not need to apply the rule, but there will be cases where there may be no alternative to use that "Leibniz rule".

A similar complex rule bearing structural similarity is

$$\partial(u \wedge v) = \partial u \wedge v \ + \ \eta u \wedge \partial \wedge v \ + \ e^h u \wedge d_h v \ + \ \eta d_h u \wedge e^h v. \quad (1.28)$$

In the derivations of the formulas for $d(u \vee v)$ and $\partial(u \wedge v)$, one simply resorts to the contractions of components of covariant derivatives with the elements of a basis of differential $1-$forms.

1.5 Examples of covariant and interior differentiations

We proceed to provide examples of interior differentiation of scalar-valued differential 1-forms. Under a different notation, you can find these formulas in his 1961 paper. I post them here since one is not used to hear or speak of covariant derivatives of the differentials of the coordinates. It is only a matter of applying the formulas that we have given. Instead of indices 1, 2, 3 and 4, I shall use ρ, ϕ, z and t.

The only non-null Christoffel symbols are

$$\Gamma_{\phi\ \phi}^{\ \rho} = -\rho, \qquad \Gamma_{\rho\ \phi}^{\ \phi} = -\frac{1}{\rho} = \Gamma_{\phi\ \rho}^{\ \phi}.$$

The ω's that we require are

$$\omega_{\phi}{}^{\rho} = -\rho d\phi, \qquad \omega_{\rho}{}^{\phi} = \frac{d\phi}{\rho}, \qquad \omega_{\phi}{}^{\phi} = \frac{d\rho}{\rho}. \tag{1.29}$$

The covariant derivatives then are

$$d_\rho u = \frac{\partial u}{\partial \rho} - \frac{\partial \phi}{\partial \rho} \wedge e_\phi u, \qquad d_z u = \frac{\partial u}{\partial z}, \qquad d_t u = \frac{\partial u}{\partial t},$$

$$d_\phi u = \frac{\partial u}{\partial \phi} + \rho d\phi \wedge e_\rho u - \frac{d\rho}{\rho} \wedge e_\phi u. \tag{1.30}$$

Exercise. Find the divergence in cylindrical coordinates in 3-D Euclidean space. Needless to say that you could follow the same process for arbitrary coordinate systems.

We want to make an important clarification to preempt confusions. When one computes the interior derivative —in say cylindrical coordinates— of a differential 1−form, one gets the same expression that one gets in the literature for the divergence of a vector field. We are not dealing here with tensor-valued differential forms, and with vector-valued ones in particular. If one used information given in chapter 1 or the beginning of chapter 2 to compute the interior derivative of a vector field, you would get zero. the reason is simple. The covariant derivative of a vector field in the sense of KK is another vector-valued 0−form. Its interior contraction with differentials 1-forms is zero (you get scalars with interior contraction of differential 1−forms with differential 1−forms, not of scalars with differential 1−forms, regardless of valuedness. Does it contradict the physics? No, in the KC, the so called magnetic vector field is a differential 2−form, and the vector potential is a differential 1−form. The divergence of the magnetic vector field is obtained as the exterior derivative of the magnetic differential 2−form. In the next section, we shall understand this a little bit better in the next section when we deal with the concepts of Hodge duals and coderivative.

Consider now the metric. There are definitions for all tastes. The symmetric quadratic differential form $g_{ij}dx^i dx^j$ is a tensor product of differential 1-forms is a tensor product, $g_{ij}dx^i \otimes dx^j$ and does not belong to what we have called Kähler algebra, which is a Clifford algebra of scalar-valued differential forms, not a tensor algebra. In the KC, the metric tensor of which one says that its covariant derivative is $g^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, equivalently $g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$. In other words it is a 2-tensor-valued differential 0−form.

2 Hodge duality, co-derivative and Laplacians

The term duality has several meanings in algebra, calculus and geometry. Given a vector space, one speaks of its dual vector space. Given an n-dimensional Euclidean space, one speaks of the $(n-1)$-plane dual of a point. Hodge duality is a third concept not too different from the previous one but more comprehensive. It is important because the concept of co-derivative in computationally less comprehensive calculi resorts to it to deal with situations which are better addressed with the interior derivative of the Kähler's calculus.

The "co-derivative version" of the interior derivative form allows one to understand how versions of the Laplacian which apparently have different numbers of terms are all part of the same general formula which takes simpler forms depending of the object to which it is being applied.

2.1 Hodge duality

Let us use the symbol z to refer to the unit differential form in the Kaehler algebra. It is undefined up to the sign, depending on the order in which the differential $1-$form factors are chosen. As defined before, let (ω^i) be a set of n differential $1-$forms that orthonormalize the metric. Let the symbol z denote the unit hypervolume, i.e. the differential $n-$form

$$z = \omega^1 \wedge \omega^2 \wedge ... \wedge \omega^n. \tag{2.1}$$

This is undefined by a sign depending on the order in which we pick the elements of the basis of differential $1-$forms. We shall choose the order so that this expression coincides with

$$z = |g_{ij}|^{1/2} \ dx^1 \wedge dx^2 \wedge ... \wedge dx^n. \tag{2.2}$$

If you were hesitating as to whether the exponent of $|g_{ij}|$ is $+1$ or -1, just check with the element of area in polar coordinates.

We define the Hodge dual (in the following, called simply the dual) $*u$ of a differential form u as

$$*u = u \vee z, \tag{2.3}$$

For positive definite signature, we have

$$** = zz = (-1)^{\binom{n}{2}}, \qquad z^{-1} = (-1)^{\binom{n}{2}} z, \qquad *^{-1} = (-1)^{\binom{n}{2}} *. \tag{2.4}$$

2.2 The co-derivative

The interior derivative of a scalar-valued differential form u is defined in the literature as $*^{-1} d * u$ or as $* \ d * u$, which may differ by the sign.

We retrospectively choose the first of these options, for the same reason that Kähler must have chosen it. Indeed, let us compute. The formula for the exterior derivative of the Clifford product $u \vee z$ reduces to just two terms:

$$d(u \vee z) = du \vee z - \eta d_h \vee e^h z. \tag{2.5}$$

Hence, it is clear that

$$*^{-1} d * u = (-1)^{\binom{n}{2}} d(u \vee z) \vee z = du \ - \ (-1)^{\binom{n}{2}} \eta d_h u \vee e^h z \vee z. \tag{2.6}$$

On the other hand,

$$(-1)^{\binom{n}{2}} e^h z \vee z = (-1)^{\binom{n}{2}} \omega^h \vee z \vee z = \omega^h. \tag{2.7}$$

Hence,

$$*^{-1} d * u = du \ - \eta d_h u \vee \omega^h \tag{2.8}$$

Consider next the dot product $\omega^h \cdot d_h u$,

$$2\omega^h \cdot d_h u = \omega^h \vee d_h u - \eta d_h u \vee \omega^h. \tag{2.9}$$

We thus have

$$*^{-1} d * u = du + 2\delta u - \partial u = \delta u. \tag{2.10}$$

We have thus proved that the co-derivative of a scalar-valued differential form is nothing but the inverse Hodge dual of the exterior derivative of the Hodge dual. Notice that, in proving this result, we did not need to assume anything about the specific valued ness of u. The Hodge dual here is defined in the Kähler algebra.

2.3 Laplacians

The Laplacian of a differential $r-$form is defined as $\partial\partial$. We then have

$$\partial\partial = (d + \delta)^2 = dd + d\delta + \delta d + \delta\delta. \tag{2.11}$$

In general, none of these terms can be ignored. However, one is used to see simpler forms. Indeed, if u is scalar valued, $ddu = 0$. On the other hand, we have, with an abuse of parentheses for greater clarity,

$$\delta\delta u = \left(d\left\{[d(uz)]\, z^{-1}\right\} z\right) z = (dd(uz))\, z. \tag{2.12}$$

If $ddu = 0$, then $\delta\delta u$ also is zero. Hence the Laplacian of scalar-valued differential forms can be written simply as

$$\partial\partial = d\delta + \delta d. \tag{2.13}$$

Since the interior derivative of scalar-valued differential forms is zero, the Laplacian for these forms is then given by just the term δd. This is similar for what in the vector calculus is the divergence of the gradient.

Exercise. Show that $\partial\partial r = \frac{2}{r}$.

3 Harmonic, strict harmonic and constant differentials

We now deal with certain types of differential forms of special interest from a perspective of differentiation. We go from the east to the most specialized.

3.1 Harmonic and strict harmonic differentials

A differential form α is called harmonic if its Laplacian is zero. It is called *strict harmonic* if $\partial\alpha = 0$. The statement $\partial\alpha = 0$ is clearly equivalent to $d\alpha + \delta\alpha = 0$. Clearly, a strict harmonic differential is harmonic. In general, harmonic differentials are not strict harmonic. There are exceptions. If a differentiable manifold is oriented and compact and its metric is positive definite, harmonicity implies strict harmonicity (See Kähler 1962, section 21). Hence, in Euclidean spaces, finding harmonic and strict solutions is the same problem. Structurally, the problem of finding solutions of $\partial\alpha = 0$ is cleaner than finding solutions of $\partial\partial\alpha = 0$. But, the richness of solutions is such that one needs far more theory than this in order to solve the problem without resorting to the method of separation of variables.

Strict harmonic differential forms play a major role in the subalgebra of even differential forms of the Kähler algebra of 2-D Euclidean space.

As an immediate consequence of the definitions, an even differential form $u + vdxdy$ is strict harmonic if and on it satisfies the Cauchy-Riemann conditions

$$u_{,x} = v_{,y} \qquad\qquad u_{,y} = -v_{,x}\,. \tag{3.1}$$

In particular, $x + ydxdy$ is harmonic. It is an immediate consequence of these equations that the equations

$$U = \int udx - vdy, \qquad\qquad V = \int udy + vdx \tag{3.2}$$

define forms U and V, since the integrability conditions are the Cauchy-Riemann equations. We thus have

$$dU = udx - vdy, \qquad\qquad dV = udy + vdx \tag{3.3}$$

and, therefore,

$$U_{,x} = u = V_{,y} \qquad\qquad U_{,y} = -v = -V_{,x}\,. \tag{3.4}$$

These equations for the partial derivatives of U and V take the form of Cauchy-Riemann conditions. Hence, the differential forms $U + Vdxdy$

also are strict harmonic. U and V are determined only up to additive differential forms whose covariant derivative is zero, since they do not change the Kähler derivative.

Functions of strict harmonic differential forms need not be strict harmonic, but most common functions are. Whether they are or not is the same issue in the literature of whether or not a function of a complex variable is analytic or not.

3.2 Constant differentials

It seems that the term *constant differential form* has not been used by anybody but Kähler. So his use is justified when it does not overlap and conflict with prior use of the same term in the literature. His choice of that term is not very fortunate in that he uses it not only for scalar-valued differential forms —where there is no problem— but also for tensor-valued differential forms, where the conflict arises.

Define constant differential forms, c, as those whose covariant derivatives $d_h u$ are zero. This is a more restrictive concept than $\partial u = 0$. It is easy to show that any exterior polynomial with constant coefficients in the differentials of the Cartesian coordinates is a constant differential form, but not if those coefficients are not constant. This is easy to prove by building the argument upon observation of what happens when we build the differentials of, say, $f(x,y)dx \wedge dy$ and $f(x,y,z)dx \wedge dy$. As for polynomials on the differential of other coordinates, suffice to observe that dr has interior derivative $2/r$. Hence $d_h r$ cannot be zero. But this does not impede that polynomials in those differentials with non-constant coefficients may happen to be constant differentials. An example of this is $\rho d\rho d\phi$, which is the unit surface element, $dxdy$, but expressed in terms of cylindrical coordinates. The *unit volume differential* is a constant differential, since we can always choose coordinates such that at any specific point they behave as if they were Cartesian coordinates, this is to say that the metric reduces to the form $\sum (dx^i)^2$ at the given point.

The rules for the different versions of the Leibniz differentiation readily yield

$$d(u \wedge c) = du \wedge c, \qquad \partial(u \vee c) = \partial u \vee c, \tag{3.5}$$

$$\partial(u \wedge c) = \partial u \wedge c + \eta d_h u \wedge e^h c, \qquad d(u \vee c) = du \vee c - \eta d_h u \vee e^h c. \tag{3.6}$$

The equation $\partial(u \vee c) = \partial u \vee c$ is a most important one vis a vis the foundations of quantum mechanics. Recall that we had the equation

$$u = {}^{+}u^{+}\,\epsilon^{+}\tau^{+} + {}^{+}u^{-}\,\epsilon^{+}\tau^{-} + {}^{-}u^{+}\,\epsilon^{-}\tau^{+} + {}^{-}u^{+}\,\epsilon^{-}\tau^{-}, \tag{3.7}$$

whose importance cannot be overemphasized The operators $\epsilon^{\pm}$, $\tau^{\pm}$ and $\epsilon^{\pm}\tau^{*}$ are constant idempotents. Hence, we have in particular

$$\partial u = \ (\partial\ ^{+}u^{+})\ \epsilon^{+}\tau^{+} + (\partial\ ^{+}u^{-})\ \epsilon^{+}\tau^{-} + \ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{+} + \ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{-}, \tag{3.8}$$

and similarly for decompositions involving only the $\epsilon^{\pm}$ or the $\tau^{\pm}$. Let us post-multiply by $\epsilon^{+}\tau^{+}$. We get

$$(\partial u)\epsilon^{+}\tau^{+} = \ (\partial\ ^{+}u^{+})\epsilon^{+}\tau^{+}. \tag{3.9}$$

We could similarly have multiplied by any other of the $\epsilon^{\pm}\tau^{*}$ and have obtained a parallel result. Because of the form of the Kähler calculus, profound implications of this are around the corner. Several applications will be shown in the next chapter.

Consider now the metric tensor, $g_{ij}\mathbf{e}^{i} \otimes \mathbf{e}^{j}$, in order to illustrate the issue of terminology to which we referred above. Under Kähler's terminology, this tensor field is a constant tensor-valued differential $0-$form. He could also consistently call constant tensor field since he uses tensor-valued differential $0-$form and tensor field as synonymous. So, we would be using the term constant tensor field for tensor fields which are not constant. We have described the actual situation in a simplified way since, for his argument, he actually creates a tensor field that he denotes as $(dx \vee dx)$ whose components are given by $(dx \vee dx)^{ik} = dx^{i} \vee dx^{k}$. Be as it may, the fact remains that he uses the term constant to refer to something where nothing appears to be constant; the constancy is a combination of terms, derivatives among them. Fortunately, we do not have to deal with tensor-valuedness in this course. So, we do not have to worry with matters of notation.

4 Application 1: Helmholtz theory in 3-D Euclidean space

We now derive the differential $1-$forms version of Helmholtz theorem. It amounts to the integration of a differential $1-$form whose exterior and interior derivatives are given and satisfies the right conditions. We then proceed to obtain from this result the integration of a differential $2-$form also given by its exterior and interior derivatives. Similar theorems in non-Euclidean spaces and higher dimensions will be proved in later chapters, where we prove a uniqueness theorem which is required as a prerequisite for the validity of the proof. this requirement also applies here. but we proceed without it since the results parallel those of Helmholtz theorem in the vector calculus where the uniqueness theorem is well known. So, for the time being, we trust that such a theorem also exists for differential $1-$forms. The contents of this section, is virtually

the same as in the substantive part of our paper "Helmholtz theorem for differential forms in 3-D Euclidean space", posted in arXiv. We would not reproduce it here except for the following.

Readers of books dealing with differential forms will have noticed that they often have a $\frac{1}{r!}$ at the front of expressions. This is because those are summations over all permutations of indices. As an example, we cannot write $dx^1 \wedge dx^2$ as $\sum dx^i \wedge dx^j$ with $i = 1, 2$ and also $j = 1, 2$. We have to introduce the factor $1/2$ and the minus sign, i.e. $dx^1 \wedge dx^2 = (1/2)(x^1 \wedge dx^2 - x^1 \wedge dx^2)$. Once we have taken care of this issue, we still have to contend with the fact that we may have summation over the different elements in a basis of $r-$forms. This is taken care of with the convention that summations over repeated indices are only over independent elements, meaning that the notation applies to the form that differential forms take after we have reduced say $(1/2)(x^1 \wedge dx^2 - x^1 \wedge dx^2)$ to $dx^1 \wedge dx^2$. Once we have done it, we still have other $dx^i \wedge dx^j$ pertaining to other pairs of indices to cover all options when dealing with differential $r-$forms. And then we may have to sum over the different grades. So, we shall use the notation $a_R dx^R$ to sum from $R = 1$ to $R = 2^n$, which is the dimension of the algebra. If some u is a differential $r-$form, it means that all a_R are zero except, at most, a number $\binom{n}{r}$ of them, which is the dimension of the subspace of differential $r-$forms. In this section, we shall acquire plenty of practice with this.

Finally, we have virtually shown earlier in this chapter that the definition of the Laplacian of a scalar-valued differential $1-$form yields an operator that is the same as in the standard vector calculus. We thus take advantage of knowledge by readers of, for example, what the Laplacian of $\partial\partial\frac{1}{r}$ is.

4.1 Helmholtz Theorem for Differential 1-Forms in 3-D Euclidean Space

We compute with Cartesian coordinates. The results obtained being invariant, they retain their form in arbitrary bases of differential forms.

Let w denote the unit volume differential form in dimension three. (We reserve the letter z for arbitrary dimension). And let us define r_{12} as

$$r_{12} = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2}$$

Theorem: Differential $1-$forms that are smooth and vanish sufficiently fast at infinity can be written as

$$\alpha(\boldsymbol{r}) = -\frac{1}{4\pi} dI^0 - \frac{1}{4\pi}\delta(dx^j dx^k I^i), \tag{4.1}$$

$$I^0 \equiv \int \frac{1}{r_{12}}(\delta'\alpha')w', \qquad I^i \equiv \int \frac{1}{r_{12}} d'\alpha' \wedge dx'^i, \tag{4.2}$$

with summation over the three cyclic permutations of 1,2,3.

Proof: By the uniqueness theorem and the annulment of dd and $\delta\delta$, the proof reduces to showing that δ and d of espectively I^0 and I^i yield $d\alpha$ and $\delta\alpha$.

Since $\delta dI^0 = \partial\partial I^0$, we write $-(1/4\pi)\delta dI^0$ as

$$\frac{-1}{4\pi}\partial\partial I^0 = \frac{-1}{4\pi}\int_{E'_3}(\partial\partial\frac{1}{r_{12}})(\delta'\alpha')w' = \frac{-1}{4\pi}\int_{E'_3}(\partial'\partial'\frac{1}{r_{12}})(\delta'\alpha')w' = \delta\alpha, \tag{4.3}$$

after using the relation of $\partial\partial$ to the Dirac distribution.

For the second term, we use that $d\delta = \partial\partial - \delta d$ when acting on $dx^j dx^k I^i$. We move $\partial\partial$ past $dx^j dx^k$. Let α be given as $a_l(x)dx^l$ in terms of the same coordinate system. We get $d'\alpha' \wedge dx'^i = (a'_{k,j} - a'_{j,k})w'$. The same property of $\partial\partial$ now allows us to obtain $d\alpha$.

For the second part of the second term, we apply δd to $dx^j dx^k I^i$:

$$\delta d(dx^j dx^k I^i) = \delta\left(w\frac{\partial I^i}{\partial x^i}\right) = wd\left(\frac{\partial I^i}{\partial x^i}\right) = wdx^l\frac{\partial^2 I^i}{\partial x^i \partial x^l} =$$

$$= wdx^l\int_{E'_3}\left[\frac{\partial^2}{\partial x'^i\partial x'^l}\left(\frac{1}{r_{12}}\right)\right](a'_{k,j} - a'_{j,k})w'. \tag{4.4}$$

We integrate by parts with respect to x'^i. One of the two resulting terms is:

$$wdx^l\int_{E'_3}\frac{\partial}{\partial x'^i}\left[\frac{\partial\left(\frac{1}{r_{12}}\right)}{\partial x'^l}(a'_{k,j} - a'_{j,k})\right]w'. \tag{4.5}$$

Application to this of Stokes theorem yields

$$wdx^l\int_{\partial E'_3}\frac{\partial\left(\frac{1}{r_{12}}\right)}{\partial x'^l}(a'_{k,j} - a'_{j,k})dx'^j dx'^k. \tag{4.6}$$

It vanishes for sufficiently fast decay at infinity.

The other term resulting from the integration by parts is

$$-wdx^l\int_{E'_3}\frac{\partial\left(\frac{1}{r_{12}}\right)}{\partial x'^l}\frac{\partial}{\partial x'^i}(a'_{k,j} - a'_{j,k})w', \tag{4.7}$$

which vanishes identically (perform the $\frac{\partial}{\partial x'^i}$ differentiation and sum over cyclic permutations). The theorem has been proved.

4.2 Helmholtz Theorem for Differential 2−forms in E_3

The theorem obtained for differential 1−forms, here denoted as α, can be adapted to differential 2−forms, β, by defining α for given β as

$$\alpha \equiv w\beta, \qquad \beta = -w\alpha. \tag{4.8}$$

Then, clearly,

$$w\delta(w\beta) = -d\beta, \qquad wd\beta = \delta(w\beta). \tag{4.9}$$

Helmholtz theorem for differential 1−forms can then be written as

$$w\beta = -\frac{1}{4\pi}d\left(\int_{E_3}\frac{\delta'(w'\beta')}{r_{12}}w'\right) - \frac{1}{4\pi}\delta\left(dx^{jk}\int_{E_3}\frac{d'(w'\beta')\wedge dx'^i}{r_{12}}\right), \tag{4.10}$$

and, therefore,

$$\beta = \frac{1}{4\pi}wd\left(\int_{E_3}\frac{\delta'(w'\beta')}{r_{12}}w'\right) + \frac{1}{4\pi}w\delta\left(dx^{jk}\int_{E_3}\frac{d'(w'\beta')\wedge dx'^i}{r_{12}}\right). \tag{4.11}$$

The integrals are scalar functions of coordinates x. We shall use the symbol f to refer to them in any specific calculation. In this way, steps taken are more easily identified.

The first term in the decomposition of β, we transform as follows:

$$wdf = (\partial f)w = \partial(fw) = \delta(fw), \tag{4.12}$$

where we have used that w is a constant differential.

For the second term, we have:

$$w\delta(dx^{jk}f) = w\partial[wdx^i f)] - wd[fdx^{jk})]. \tag{4.13}$$

The first term on the right is further transformed as

$$w\partial(wdx^i f) = wwdx^i\partial f = -dx^i df, \tag{4.14}$$

where we have used that wdx^i is a constant differential, which can be taken out of the ∂ differentiation. For the other term, we have

$$-wd(fdx^{jk})] = -wdf\wedge dx^{jk} = -wf_{,i}\,w = f_{,i} = dx^i\cdot df. \tag{4.15}$$

From the last three equations, we get

$$w\delta(dx^{jk}f) = -dx^i df + dx^i\cdot df = -dx^i\wedge df = d(dx^i f). \tag{4.16}$$

In order to complete the computation, we have to show that $d(w\beta) \wedge dx^i$ can be written as $\delta\beta \wedge dx^{jk}$. This can be shown easily by direct calculation. Let α be given as $a_i dx^i$. Then $d(w\beta) \wedge dx^1 = d\alpha \wedge dx^1 = (a_{3,2} - a_{2,3})w$. On the other hand, $\beta = -a_i dx^{jk}$ and

$$\delta\beta = (a_{3,2} - a_{2,3})dx^1 + \textit{cyclic permutations}. \tag{4.17}$$

Hence $\delta\beta \wedge dx^{23} = (a_{3,2} - a_{2,3})w$ and, therefore,

$$d(w\beta) \wedge dx^1 = d\alpha \wedge dx^1 = \delta\beta \wedge dx^{23}, \tag{4.18}$$

and similarly for the cyclic permutations of the indices.

5 Application 2: Cauchy like calculus with real differential forms

In chapter 2, we dealt with algebraic issues which were sufficient by themselves to compute without resort to complex variable theory certain types of real integrals which are usually solved using that theory. We shall now produce theory that involve operations which differential forms and that give results as if we were computing complex integrals.

When this author did all this work on this subject (two papers in arXiv), it was not too clear to him what was the most important feature of what he was doing. He was distracted by whether an analytic function was defined by a power series (Weierstrass point of view) or by compliance with the Cauchy-Riemann conditions (Cauchy's point of view). But the difference is more profound; it lies somewhere else. We proceed to explain this at length in subsection 5.3, after we had introduced the basis concepts.

The theory of complex variable not only deals with the solution of those real integrals, but is also used to solve integrals which are not real. The question then is whether one can extend the theory of real differential forms so that it also comprises this new situation. One can extend it, and that is what this section is about.

5.1 Representation with real differential forms of complex integrals

This section supersedes subsections 2.3 and 3.2 of our paper "Real Version of Calculus of Complex Variable: Cauchy's Point of View". That subsection 2.3 is confusing and may be even misleading. It will now be made unnecessary. As for subsection 3.2, it is clear but irrelevant since it does not clearly address the issue that will be considered here under the term antivaluation. This is the operation opposite of valuation, rightly

considered there as in here. We think that its naturalness is here made much more clearly.

5.1.1 Valuations

The equivalence between i and $dxdy$ is only algebraic. The unit i is a constant and, as such, it can be taken out of an integral sign. But one cannot do so with $dxdy$, since it would change the nature of the integral. So, one has to introduce a theory with real differential forms that represents the calculus of complex variable through expressions where i is outside the integral. This takes place as follows.

With $f(z) = u + iv$ and $dz = dx + idy$ (which is correct, but which we might also write as $\partial z = \partial x + i\partial y$), we have

$$\int f(z)dz = \int f(z)dx + i\int f(z)dy = \int (u+iv)dx + i\int (u+iv)dy =$$
$$= \int udx - vdy + i\int udy + vdx = U + iV. \tag{5.1}$$

for analytic functions, i.e. satisfying the Cauchy-Riemann conditions. Hence, we ignore the first equalities and focus on the last one. In this section, we use the symbol w to refer to $u + vdxdy$. The last equation can then be written as

$$\int udx - vdy + dxdy \int udy + vdx = U + Vdxdy, \tag{5.2}$$

We replace integration $\int_c f(z)dz$ on a curve of the complex plane with "valuation" $\langle w \rangle_c$ of an edif, $u + vdxdy$, on a curve c of the real plane:

$$\langle w \rangle_c \equiv \left[\int_c wdx\right] + dxdy \left[\int_c wdy\right], \tag{5.3}$$

The integrability conditions for these integrals to not depend on c but only on its end points are the Cauchy-Riemann relations. For $u+vdxdy$, this is the strict harmonic differential condition. "Valuation potentials"

$$\langle w \rangle = U + Vdxdy = \int udx - vdy \ + dxdy \int (udy + vdx) \tag{5.4}$$

then exist. Of course, U and V are undefined up to integration constants. The valuation on a closed curve (on simply connected manifolds) is zero. This is the Cauchy-Goursat theorem for strict harmonic differentials. In domains that are not simply connected, we surround the poles enclosed by closed curves C with equally oriented circles c_i, all of them with the

same orientation as C and containing one and only one pole each. We then have

$$\langle w \rangle_C = \sum_i \langle w \rangle_{c_i} . \tag{5.5}$$

From now on, let us denote $U + V dxdy$ as W. We have

$$\partial W = dU + dV dxdy. \tag{5.6}$$

Since

$$dU = udx - vdy, \qquad dV = udy + vdx, \tag{5.7}$$

we obtain

$$U_{,x} = u = V_{,y} \qquad U_{,y} = -v = V_{,x} \ . \tag{5.8}$$

W is, therefore, strict harmonic. Notice that we have not spoken of $f(x+ydxdy)$ inside of an integral sign. The Cauchy theory of differential forms belongs to the elements of the algebra of even differential forms, not to the functions of some other variable.

5.1.2 Antivaluations

The valuation plays the role played by integration in the calculus of complex variable. We shall refer to $u + vdxdy$ as the antivaluation of $U + V dxdy$. It is a simple matter to show that

$$dxdU = dydV \qquad dydU = -dxdV, \tag{5.9}$$

and

$$dUdx = dVdy \qquad dUdy = -dVdx. \tag{5.10}$$

The antivaluation can be given in a variety of ways:

$$u = dx \cdot dU = dy \cdot dV = idy \wedge dU \ = -idx \wedge dV \tag{5.11}$$

and

$$v = dx \cdot dV = -dy \cdot dU = idx \wedge dU \ = idy \wedge dV. \tag{5.12}$$

i referring, of course, to $dxdy$.

To strengthen ideas, let us see what $f(z) = z = x + iy$, $df/dz = 1$ corresponds to. We compute the antivaluation for $U = x$, $V = y$. We get $u = dx \cdot dU = dx \cdot dx = 1$ and $v = dx \cdot dV = dx \cdot dy = 0$. So, $x+ydxdy$ is a primitive of 1. We check it computing the valuation:

$$\int 1dx + dxdy \int 1dy = x + ydxdy. \tag{5.13}$$

5.1.3 Cauchy calculus versus exterior and Kähler calculi

The contents of the previous two subsubsections can be taken as the foundations of a Cauchy calculus of strict harmonic differential forms. The Cauchy calculus is different from the Kähler calculus of differential forms, since it is based on a different concept of integration and differentiation. What goes by the concept of integration and differentiation in the calculus of complex variable has a correspondence in valuations and antivaluations in the Cauchy calculus of differential forms. The fact that this calculus uses the Kähler algebra might lead one into confusion. Readers might be right in thinking that this section belongs to the previous chapter. This is logically correct, but consider this. A Kähler calculus on a Kähler algebra is far more important than a Cauchy calculus on the same algebra. It is better to be confused learning this Cauchy calculus after having learned the Kähler calculus than to be confused learning the latter after having learned the former.

Let me insist on the point just made. In the Kähler calculus, differential $r-$forms are evaluated (i.e. integrated) on $r-$surfaces. In the Cauchy calculus of differential forms presented so far, we "valuate" even differential forms on curves. The valuation on curves involves the integrating or evaluation on those curves of associated differential $1-$forms, specifically the differential $1-$forms $udx - vdy$ and $udy + vdx$.

$$dU = udx - vdy, \qquad dV = udy + vdx. \tag{5.14}$$

Properly speaking, an expression such as $dz = dx + idy$ does not pertain to the complex variable calculus proper, but to the representation in the plane of differential forms over the complex numbers. The obtaining of dz through differentiation belongs to the exterior calculus of these differential forms, not to the calculus of complex variable, where differentiations take place with respect to z.

Let us use define $Z = x+idxdy$. It is certainly the case that $dz = dx+idy$ and that, in contrast, $dZ = dx$. But, as we have explained, $d(z) = dx + idy$ is not a differentiation in the calculus of complex variable and $d(Z) = dx$ is not a differentiation in the Cauchy calculus of differential forms. We cannot make anything of this difference.

The differentiation of z in the calculus of complex variable is simply dz. End of story. We can of course write $dz = (dz/dz)dz = dz$. In the calculus of complex variable, the differentiation of x takes place through the differentiation of $x = (z+z^*)/2$. This poses problems when trying to differentiate z^*, for which purpose one resorts to a representation in the plane. See, for instance, H. Cartan's "Elementary Theory of Analytic Functions of One or Several Complex Variables". He introduces z and z^* for the first time in his book as follows: "Recall that a complex number

$z = x + iy$ ... is represented by a point on the plane ... If we associate with each complex number $z = x + iy$ its 'conjugate' $\bar{z} = x - iy$...". Of course, H. Cartan knows better. Later in his book he considers differentiations with respect to the complex variables z and its conjugate $\bar{z}$ and introduces the Laplacian as $\partial^2 f/\partial z \partial \bar{z}$. And later, when dealing with n complex variables, he writes

$$df = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial z_k} dz_k + \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k \right), \tag{5.15}$$

for the differential of a function of so called n complex variables. When this is done, $d(x+iy) = dz$ through a process where x and y are expressed in complex variable terms.

This discussion was worth writing for an indirect purpose: as reformulated, the exterior calculus of one complex variable can be seen as a theory of strict harmonic differentials. Forget about functions of $x + iy$. That may be included in the game, specially as a bridge to start the game, like even the most notable mathematicians do. But, once again, we did not use $x+ydxdy$ above. And, if one uses it and the likes of it, we shall have to take into account that the relation between "the complex variables" $x + ydxdy$ and $x + zdxdz$ will not be the same as the relation between $x + ydxdy$ and $z + tdzdt$, assuming of course positive definite metric. Indeed $dxdydxdz$ equals a differential $2-$form; $dxdydzdt$ is a differential $4-$form. Its interior derivative is not a differential $1-$form.

We have set the stage for a Cauchy theory involving strict harmonic differential forms in Euclidean spaces of arbitrary dimension, not spaces of several complex variables. It is a theory much richer than standard theory of several complex variables.

5.2 Cauchy's theorems

This subsection should be about Cauchy's integral formula and Cauchy's integral formula for derivatives. They are clearly explained in subsections 3.1 and 3.3 of our paper "Real Version of Calculus of Complex Variable: Cauchy's Point of View". Although we have said that the Cauchy theory of differential forms belongs to the elements of the algebra of even differential forms, not to the functions of some other variable, using the notation $f(z)$ instead of $f(x + ydxdy)$, which in turn we use there instead of $u + vdxdy$, has notational advantages. The nature of those theorems is at the root of those notational advantages.

See the same reference for a couple of exercises.

CHAPTER 4: Relativistic Quantum Mechanics

Jose G. Vargas

Once again, lecturers will choose from this chapter what they will teach on day 6 of the summer school.

This chapter was going to be called Kähler equations. I changed the title when I recently learned that a group of billionaires (like Ambani, Bezos, Branson, Gates, Ma, Soros, Zuckerberg, etc. is launching a major clean energy initiative (For fair treatment of those that I have not mentioned, please go breakthroughtenergycoalition.com/en) Energy is an issue for exceptionally good theoreticians, like Einstein in the first half of the twentieth century (energy of photons, energy equals mc^2, the energy-momentum tensor equals the Einstein tensor) and Julian Schwinger (See the last section of this chapter for more on Schwinger, the other Feynman or even better) in the second half. The ideas of both, Einstein and Schwinger, in the last decades of their lives were dismissed. In the last section, we also say a little bit about the connection between Kähler's calculus and Einstein's failed at unification with teleparallelism.

Kähler's quantum mechanics (actually, the Kähler equation by itself) has to do with the initiative of the coalition, even though its proponents do not know it. It prompts and promises a better understanding of energy at the quantum level. Present day quantum physics is being built with inadequate arguments, which are at the root of the spurious emergence of negative energy solutions in Dirac's theory. It is resolved by means of so called "hole theory", where all states with negative energy are filled and thus constitutes a sea of infinite energy density. Holes (or lack of negative energy solutions) are interpreted as positrons (A very good book for all of this is "Relativistic Quantum Mechanics" by J. D. Bjorken and S. D. Drell, both from the Stanford Linear Accelerator Center"; its chapter five is about hole theory). The infinite energy density of the sea is reduced to still huge finite ones through a couple of independent artifacts which do not even give the same results (They actually

give results that differ by dozens of orders of magnitude). In contrast, Kähler's equation entails the emergence of positrons with the same sign of energy as electrons. Since we reserve that derivation for chapter six, we send impatient readers to the very end of section 26 of Kähler's 1962 paper "Der innere Differentialkalkül".

Also recently —while writing these chapters— I heard again the often repeated statement that quantum physics and general relativity are mathematically and conceptually incompatible. But, with the Kähler calculus, the mathematics of general relativity are contained in the mathematics of quantum mechanics. They are not, therefore, mathematically incompatible. As for conceptual incompatibility, it is not so because of the nature of the subject but because of the Dirac equation. In his time, it was a tremendous conquest, but provisional and not quite as intelligible as physics used to be). Its limitations led to misinterpretations as to the true nature of the field. If, as this author claims, the Kähler calculus is the right equation, the quantum mechanics that ensues is more about a field in the classical mold rather than about particles and probability amplitudes. The latter (certainly as well as their square) should be viewed as derived or emergent concepts. But does one really believe that the mathematics for both sectors of physics they will incompatible for ever? We should not replace "we do not know how to do ..." with "it cannot be done". And, fortunately, Kähler's version of quantum mechanics is such that the day to day practice of most of quantum physics will not change; only the foundations will.

In order to provide an early glimpse of Kähler's approach to quantum physics, we shall alter for the summer school what should have been the order of chapters in the presentation of the theory. In this chapter, we shall mention, though only briefly, Kähler's obtaining of the strict harmonic differentials in 3-D Euclidean space (minus the origin), where they coincide with the harmonic differentials. Such a study should logically follow the study of rotational symmetry (chapter 5), which is instrumental in solving the fine structure of the hydrogen atom. We shall leave for readers to follow the long calculations directly from the Kähler papers, which they should be able to read with little knowledge of German after we have provided here the basic foundations. Chapter 6 will deal with the conservation law and related applications (the two types of charge, a uniqueness theorem for integration of Helmholtz type systems, etc.).

Starting with chapter 7, I would present my own work on the Cartan-Kähler calculus of differential forms, where physics and mathematics look as if they were the same thing. Fortunately for participants in this summer school, you will not have to suffer me much longer since, by then, it will be the early morning of the last day of the school.

What does all this have to do with the breakthrough energy coalition? A better platform for quantum mechanics is a better platform for understanding energy. One could bring the Kähler calculus —and thus quantum physics and the understanding of energy— to greater heights with an amount of funds that would be pocket money for billionaires, specially if the funds are used in countries where eager scholars draw meager salaries. I have written this for the record so that one day, perhaps not too far for me to see it, mathematically gifted acquaintances of those billionaires let them know of this opportunity to carry on their project with the much more sophisticated Kähler theory. It may lead to less costly and more efficient and environmentally friendly way of obtaining clean energy. Schwinger may have pointed the way with his source theory, and Kähler mathematics is the right tool to make his theory more accessible.

1 Kähler equations

With the explicit purpose of accelerating the bringing about of a more complete picture of what the Kähler calculus has to offer, I am starting to skip details of computations present in Kähler's papers.

1.1 Solutions with time symmetry of exterior systems

The form of solutions with symmetry of exterior systems is independent of the system itself. We mean what types of factors represent the symmetry in solutions that have it. The dependence on coordinates that are parameters of symmetries is given by a phase factor, and the dependence on their differentials is given by idempotents that are constant differential forms.

Recall a decomposition such as

$$u = {}^{+}u\,\epsilon^{+} + {}^{-}u\,\epsilon^{-}. \tag{1.1}$$

It corresponds to time translation symmetry. The phase factor must, therefore, be e^{-iEt}. As we shall son start to understand and with $\hbar = 1$, a negative charge solution with time translation symmetry should be given by

$$e^{-iEt}R(x, dx)\epsilon^{-}. \tag{1.2}$$

By virtue of the fact that the $\epsilon^{\pm}$ are constant differentials, we shall have equations

$$\partial u = \quad (\partial\ {}^{+}u)\ \epsilon^{+} + (\partial\ {}^{-}u)\ \epsilon^{-}, \tag{1.3}$$

implicit in the more comprehensive equation

$$\partial u = \ (\partial\ {}^{+}u^{+})\ \epsilon^{+}\tau^{+} + (\partial\ {}^{+}u^{-})\ \epsilon^{+}\tau^{-} + \ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{+} + \ (\partial^{-}u^{+})\ \epsilon^{-}\tau^{-} \tag{1.4}$$

for when, in addition to time translation symmetry, one also has rotational symmetry (See chapter 2 for all these idempotents). Suffice to rewrite this equation as

$$\partial u = \ [(\partial\ {}^{+}u^{+})\ \tau^{+} + (\partial\ {}^{+}u^{-})\tau^{-}]\ \epsilon^{+} + [(\partial^{-}u^{+})\ \tau^{+} + (\partial^{-}u^{+})\tau^{-}]\epsilon^{-} \tag{1.5}$$

and then as

$$\partial u = \ \partial({}^{+}u^{+}\ \tau^{+} + \ {}^{+}u^{-}\tau^{-})\ \epsilon^{+} + \ \partial({}^{-}u^{+}\ \tau^{+} + \ {}^{-}u^{+}\tau^{-})\epsilon^{-} \tag{1.6}$$

$\tau^{\pm}$ has to do with issues that come under the name of handedness, chirality and helicity, intrinsically associated with spin. This will be understood when, in the next chapter, we deal with angular momentum.

For a system where the rest mass of the electron were the dominant energy, it makes practical sense to write E as $m+\Delta E$. This is well known from standard relativistic quantum mechanics. In the Kähler calculus, the dominant energy representation of electron systems is given us

$$u = e^{-imt}R(t, x, dx)\epsilon^{-}, \tag{1.7}$$

where $R(t, x, dx)$ depends the more slowly on time the more close to each other m and E are. Compare with (1.2). This expression for solutions will be very much used in the following.

1.2 Kähler equations for stationary solutions

Let u be any element of the Kähler algebra (again, of scalar-valued differential forms). Any equation of the form

$$\partial u = au \tag{1.8}$$

will be referred to as a Kähler equation. A particular case is $\partial u = 0$. In Euclidean space of dimension 2 we get even differential forms, already considered in the two previous chapters. In dimension 3, Kähler derived a magnificent theory of harmonic differentials without resort to separation of variables. That is very interesting but momentarily, it is not as important as the equation (1.8) in spacetime, with a different from zero and the algebra being over the complex field. We shall also make $c = 1$.

$$(\partial^{\pm}u)\epsilon^{\pm} = \ a\ ({}^{\pm}u\ \epsilon^{\pm}). \tag{1.9}$$

Solutions ${}^{\pm}u\ \epsilon^{\pm}$ may correspond to particle states of undefined chirality.

From the perspective of the number of components in the solutions and what they represent, neither Eq. (1.8) nor Eqs. (1.9) have a close analog in the Dirac equation. To start with, they do not have the same number of components. The electron and positrons of both chiralities are directly associated with the equations

$$(\partial\ {}^{\pm}u^{*})\epsilon^{\pm}\tau^{*} = \ a\ ({}^{\pm}u^{*}\ \epsilon^{\pm}\tau^{*}). \tag{1.10}$$

The parentheses have been put there simply to emphasize that the differentiation of $\epsilon^{\pm}\tau^{*}$ can be ignored since it yields zero. The asterisk means that correlation between the superscripts of ϵ and τ is not implied. The $\epsilon^{\pm}$ would be for pairs of particle and antiparticle, and the $\tau^{\pm}$ would be for the two opposite handedness in each of those two cases. Phase factors are inside the ${}^{\pm}u^{*}$ when solutions proportional to ${}^{\pm}u^{*}\ \epsilon^{\pm}\tau^{*}$ represent specific particle states.

By writing a as $\alpha + i\beta dt$ and replacing ${}^{\pm}u$ with $e^{-iEt}p^{\pm}$, the development of Eqs. (1.9) by Kähler yields the following equations for stationary solutions

$$\partial p \pm (E+\beta)\eta p\ - \alpha p = 0, \tag{1.11}$$

where the signs $\pm$ corresponds to $p^{\pm}$ respectively. This is obtained without resort to dominant energies or specific couplings. Kähler used this equation just for the fine structure of the hydrogen atom, for which it takes a very simple form. We shall later see another important application.

1.3 Kähler equations for Dirac spinors

We may be interested in what we shall denote as Dirac spinors that are proper "functions" for given energy and chirality. Kähler wrote such spinor solutions as

$$u = e^{is\phi - iEt}p\tau^{\pm}\epsilon^{*}, \tag{1.12}$$

where p depends only on ρ, z, $d\rho$ and dz, and where s and E are the angular momentum and energy of the system. We shall replace the symbol $\vee$ for juxtaposition when we judge that it makes equations more transparent.

Differentiating (1.12), we get

$$\partial u = e^{is\phi - iEt}(isd\phi \vee p - iEdt \vee p + \partial p)\tau^{\pm}\epsilon^{*}. \tag{1.13}$$

We then use

$$d\phi \vee p = d\phi \wedge p = \eta p \wedge d\phi = \eta p \vee d\phi, \tag{1.14}$$

and

$$dt \vee p = dt \wedge p = \eta p \wedge dt = \eta p \vee dt. \tag{1.15}$$

Thus,

$$\partial u = e^{is\phi - iEt}(is \vee \eta p \vee d\phi - iE\eta p \vee dt + \partial p)\tau^{\pm}\epsilon^{*}. \tag{1.16}$$

We now use that

$$\tau^{-} - \tau^{+} = -idxdy = -i\rho d\rho d\phi \tag{1.17}$$

to obtain $d\phi$ and then

$$d\phi \vee \tau^{\pm} = -i\frac{d\rho}{\rho}(\tau^{-} - \tau^{+})\tau^{\pm} = -i\frac{d\rho}{\rho}\tau^{\pm}. \tag{1.18}$$

Equation (1.18) together with $idt\epsilon^{\pm} = \mp\epsilon^{\pm}$ allows us to finally write (1.16) as

$$\partial u = e^{is\phi - iEt}(\partial p \pm s\eta p\frac{d\rho}{\rho} - *E\eta p)\tau^{\pm}\epsilon^{*}, \tag{1.19}$$

which could be used on the left hand side of Kähler equations, specially if we do not want to resort to dominant energies in reducing the simplicity of some Kähler equations.

2 Dominant energy and electromagnetic coupling

In the first subsection of this section as well as everything that went above is original work of Kähler. If you do not like it, go to church and talk to him. From now on, I take the blame for what follows. I say that because some learned people will retort that this is not in the spirit of modern physics and mathematics, or they will say even something worse. I do not care. The mathematical and physics community should at least know what Kähler did. He wrote in German. I am doing it in English and in a slightly more Cartanian way. One has to understand geometry à la Cartan in order to really appreciate the Kähler's calculus and even go beyond it. With my book "Differential geometry for physicists and mathematicians" I tried to do with some of Cartan's work what I am now trying to do with some of Kähler's work.

2.1 Charge

With electromagnetic coupling, the Kähler equation reads

$$\partial u = (-m + ieA)u, \tag{2.1}$$

where m is the rest mass, and e (positive or negative) represents the charge of positrons and electrons. In going from (1.8) to (2.1), there are unstated or undefined assumptions. We are trying to say that, whereas (1.8) does not assume a situation where particles and antiparticles are of

the essence, (2.1) does, even though this is not explicit. It is contained in the mere presence of a mass competing with an electromagnetic field. For the split (1.1),we have

$$(\partial^{\pm} u)\ \epsilon^{\pm} = (-m + ieA)\ {}^{\pm}u\ \epsilon^{\pm}. \tag{2.2}$$

Let us now deal with the announced existence of what, in the electromagnetic case, becomes charge of opposite signs. We shall reproduce in chapter 6 the derivation of the conservation equation of this calculus and its specialization to when u is written as (1.1). After the appropriate natural interpretations, the conservation law takes a form which translated to the language of the vector calculus reads like this

$$\left(\frac{\partial \rho_+}{\partial t} + div\ j_+\right) + \left(\frac{\partial \rho_-}{\partial t} + div\ j_-\right) = 0, \tag{2.3}$$

where ρ_+ and j_+ are determined only by ${}^{+}u$ and its complex conjugate, and similarly for ρ_-, j_+ and j^-. Hence the two options in $\epsilon^{\pm}$ should be assigned to positrons and electrons. Probability densities should in turn be assigned directly to charge and indirectly to particles, but certainly not yet at this point.

We shall not go until chapter 6 into the rather laborious derivation of how $\rho_{\pm}$ and $j_{\pm}$ depend on ${}^{\pm}u$. Equation (2.3) implies the apparent existence of two types of charge, and that only total charge (understood as their sum) and not each individual one is conserved when both are present. Their individual conservation is an afterthought when the two types stay apart, like a nucleus and its surrounding cloud of electrons.

In obtaining (2.3), Kähler does not make full use of the Kähler equation with electromagnetic coupling, but simply that a for that coupling has a specific property, which can also be satisfied in principle by other ones. And even if that property were not satisfied, we might still get conservation equations with the flavor of (2.3), but where the ρ's and j's would depend on the ${}^{\pm}u$'s in a different way. Results like this do not make part of the books on the Dirac theory from which I, and probably you, learned.

The equations we derived in the previous section can easily be adapted to the electromagnetic coupling. We are about to see that the equations for positrons and electrons have nothing to do with small and large components of the solutions of one equation, solutions misinterpreted in the Foldy-Wouthuysen treatment of Dirac's theory as respectively representing positron and electron amplitudes. Large and small components belong equally to solutions for particles and antiparticles. These two are in the same footing at this level of theory.

2.2 The Pauli equation

Of practical interest is the case when the rest energy m of the electron is the dominant energy. We use (1.7) in (2.1) for an electron in a state of undefined handedness in an electromagnetic field. Given that ∂u is $dt\ u_{,t} + dx^i\ u_{,i}$, we get, after some simplification,

$$\left[dt(-imR + R_{,t}\,) + dx^i R_{,i}\,\right]\epsilon^- = (-m - ie\Phi dt + ieA_i dx^i)R\epsilon^-. \quad (2.4)$$

Premultiplying by $-dt$, using that $(dt)^2$ is -1 and leaving $R_{,t}\,\epsilon^-$ as the sole term on the left, we get

$$R_{,t}\,\epsilon^- = \left[-dx^i dt R_{,i} + (-ieA_i dx^i dt + im + mdt - ie\Phi)R\right]\, dt\epsilon^- \quad (2.5)$$

We cannot simplify ϵ^- until we absorb all dt factors into ϵ^-, which we do by means of $dt\epsilon^- = -i\epsilon^-$ (Check what would happen and compare with what we are about to get). We thus obtain

$$R_{,t} = -P\eta R - ie\Phi + im(R - \eta R), \quad (2.6)$$

where we have replaced $dx^j(-i\partial_j - eA_j)$ with the symbol P and where we have used that R does not contain dt as a factor. Recall from previous chapters that η changes the sign of terms of odd grade. We apply η to (2.6). Combining the resulting equation with (2.6) itself, we get

$$\varphi_{,t} = P\chi - ie\Phi\varphi, \quad (2.7)$$

$$\chi_{,t} = -P\varphi - ie\Phi\chi + 2im\chi, \quad (2.8)$$

where

$$\varphi = \frac{1}{2}(R + \eta R), \qquad \chi = \frac{1}{2}(R - \eta R). \quad (2.9)$$

We now proceed following a step to be found in the standard literature. Among the books on relativistic or advanced quantum mechanics that we have come across, there is one which deserves special attention because of the topics it deals with, namely the aforementioned "Relativistic Quantum Mechanics" by J. D. Bjorken and S. D. Drell". This book is particularly interesting not only because of the institutional affiliation of its authors and the contributions they made to the high energy physics of their time, but also because of the year in which their work was published, 1963. That was the time when high energy physics was becoming corporate physics, meaning that advances were produced as if they were products of a corporation, one of the main headquarters being Stanford's linear accelerator. It was also the time when problems in the very foundations of quantum mechanics were addressed rather than swept under

the rug, as, more often than not happens nowadays. This is why there was a chapter on hole theory in their book. We have mentioned all this because we are about to take a step right now which is conceptually the very same step as one of those that they took in their first chapter. In the paragraph just before the last, they had connected with the Pauli equation. They then opened the last paragraph of the chapter as follows: "Fortified by this successful non-relativistic reduction of the Dirac equation, we go on ..."

The title of their chapter 4 is "The Foldy-Wouthuysen transformation". And chapter five's title is "Hole theory", designed to deal with the issue of negative energy solutions of Dirac's theory. These solutions are a consequence of what is spurious in Dirac's theory. The association of antiparticles with negative energy solutions does not emerge when doing relativistic quantum mechanics with Kähler's mathematics.

In the process of deriving the Pauli equation, Bjorken and S. D. Drell took a step which is totally equivalent to the one that we are about to take with our equation (2.8). Of their equation that parallels this equation (2.8), they said that "it may be approximated for kinetic energies and field interaction energies small in comparison with $mc^2, ...$". We return to this further below, where we shall see that this amounts to neglecting terms $\chi_{,t}$ and $-ie\Phi\chi$ in our equation (2.8). The assumption $\chi_{,t}$ is unwarranted, since $\chi_{,t}$ may be huge and can thus not be ignored even if $\chi_{,t}$ were small. This will have a very important consequence that is worth mentioning with a view to trying to understand at least some obscure point in the Dirac theory, as we shall explain further below. Let us assume that we ignore this difficulty and proceed in parallel to what they did. If the said approximation were correct, we would have

$$\chi_1 = -\frac{i}{2m}P\varphi, \qquad\qquad i\varphi'_{,t} = iPP\varphi + i\Phi\varphi. \tag{2.10}$$

Let us assume that we knew φ. We would then approximately know χ. We refer to this approximated χ as χ_1. With χ replaced with χ_1, the right equation (2.7) gives only an approximate $\varphi_{,t}$ even if we knew the right φ and left it on the right. This is the reason why we have used φ' on the left and φ on the right. This remark will be useful for the approximation in the next section

We expand $PP\varphi$ and use that

$$P_kP_j\varphi = -i\partial_k(ieA_j) = ieA_{j,k}\,. \tag{2.11}$$

We then perform some manipulations given in more detail in our "The Foundations of Quantum Mechanics and the Evolution of the Cartan-

Kaehler Calculus" (Foundations of Physics **38**, 610-647 (2008)), and get

$$i\frac{\partial}{\partial t}\varphi' = \frac{1}{2m}P^2\varphi + \frac{ie}{2m}B_k dx^i dx^j \varphi + e\Phi\varphi, \tag{2.12}$$

where the summation is over the three cyclic permutations of $(1, 2, 3)$. This is the Pauli equation in terms of differential forms, except that Pauli's φ on the right hand side also is φ'. At this stage and for the present purpose, "this difference does not make a difference".

One should not overlook that this equation is for φ, not for R, here as in the corresponding formula in the Dirac theory. In spite of the objection raised about (2.10), equation (2.12) is of great interest,. though not for the same reason as in the paradigm, where it was welcome because it gave confidence in the Dirac equation to construct a relativistic theory of the electron. For us, it will be important for another reason, which we shall discuss at length later on.

2.3 The Foldy-Wouthuysen Hamiltonian

In the literature, one uses the term Foldy-Wouthuysen transformation more often than Foldy-Wouthuysen transformation. But its main purpose is the transformation of the Hamiltonian, which Bjorken and Drell perform in the every same section. In fact, they do not even speak of the transformed wave function on which the new Hamiltonian acts, though this is obvious by gauge invariance considerations.

We continue the process by which (2.12) was obtained. We write the sought χ as the old one, χ_1, plus some χ_1',

$$\chi = -\frac{i}{2m}P\varphi + \chi_1'. \tag{2.13}$$

This together with (2.8) yields

$$\chi_{,t} = -ie\Phi\chi + 2im\chi_1'. \tag{2.14}$$

Hence, solving for χ', we have

$$\chi_1' = \frac{i}{2m}\chi_{,t} + \frac{e\Phi}{2m}\chi. \tag{2.15}$$

From (2.7), we get that χ_1' adds the terms $iP\chi_1'$, i.e.

$$iP\chi_1' = \frac{-1}{2m}P\chi_{,t} + i\frac{e\Phi}{2m}P\chi. \tag{2.16}$$

It is only at this point that we replace χ with χ_1 on the right hand side of (2.16). The computations are straightforward but rather laborious. One obtains

$$iP\chi_1' = \frac{-1}{8m^3}p^4\varphi - i\frac{e}{4m^2}PE^C\varphi - \frac{e}{4m^2}E_{i\ ,\ j}dx^j \wedge dx^i\varphi, \tag{2.17}$$

where we have eliminated higher order terms for comparison purposes with how this would be written in the paradigm. The small p is $-idx^j\partial_j$, and E is here notation for dx^jE_j. The superscript is used to indicate that E in $PE^C\varphi$ is treated as if it were a constant in this term. The parenthesis around $dx^j \wedge dx^i$ is meant to say that we add over cyclic permutations in $1^k dx^i \wedge dx^j$.We thus further get

$$-\frac{ie}{4m^2}PE^C\varphi = -\frac{e}{4m^2}dx^i dx^j E_j\partial_i\varphi, \tag{2.18}$$

which corresponds to the $\sigma \cdot (\mathbf{E} \times \mathbf{p})$ term in the standard version of relativistic quantum mechanics, i.e. with vector-valued quantities. Please be informed that we had an inadvertent replacement of a "$\vee$" product with a "$\wedge$" product. That in turn gave rise to a spurious term, which is the reason for a discrepancy with (2.18).

The last term in (2.17) yields

$$-\frac{e}{4m^2}E_{i\ ,\ j}dx^j \vee dx^i\varphi = \frac{e}{4m^2}E^i{}_{,i}\ \varphi - \frac{e}{8m^2}E_{j\ ,i}dx^j \wedge dx^i\varphi, \tag{2.19}$$

equivalently, the $div\mathbf{E}$ and $\sigma \cdot curl\mathbf{E}$ terms with appropriate factors. In all terms on the right of the $i\frac{\partial}{\partial t}\varphi'$ post-Pauli equation, all operators are acting on φ the original wave differential form φ.

Let us look at what we have got. We have developed to order $1/m^3$ the Kähler equation with electromagnetic coupling and mass as dominant energy term for the even grade part of solutions whose odd grade part satisfies $\chi_{,t} \ll -P \vee \varphi$. The development was directed towards letting $i\frac{\partial}{\partial t}\varphi$ alone on the left hand side. Hence, the right hand side defines a Hamiltonian, namely

$$\begin{aligned} H = &\frac{1}{2m}P^2 + \frac{ie}{2m}B_k dx^i dx^j + e\Phi - \frac{1}{8m^3}p^4 - \frac{e}{4m^2}dx^i dx^j E_j\partial_i \\ &+ \frac{e}{4m^2}E^i{}_{,i} - \frac{e}{8m^2}E_{j\ ,i}dx^j dx^i, \end{aligned} \tag{2.20}$$

acting on the original wave function, whereas the energy operator, $i\partial/\partial t$ is acting on a slightly different wave function, i.e.

$$i\frac{\partial}{\partial t}\varphi' = H\varphi. \tag{2.21}$$

Of course, we could think of φ' as a φ, and vice versa, thus $i\frac{\partial}{\partial t}\lambda = H\lambda'$, where we have used λ's instead of φ's in case it would create confusion.

Observe how, by virtue of the little bit of Kähler equations that we have learned, we have reached deep into relativistic quantum mechanics, to a depth even greater than most advanced books on quantum mechanics.

2.4 Leftovers

TO BE COMPLETED. This subsection would deal with subjects that we left over because of the pressure for this author to write about other jewels of this calculus and concomitant quantum mechanics.

Readers can now do exactly the same for positrons. One only has to repeat with

$$u = e^{-imt}R(t, x, dx)\epsilon^{+}, \tag{2.22}$$

what we did with $e^{-imt}R(t, x, dx)\epsilon^{-}$. You will find a system just like (2.7)-(2.8), except that the even, $\bar{\varphi}$, and odd, $\bar{\chi}$, parts of u will satisfy the equations satisfied by χ and φ. Thus $\bar{\varphi}$ will be small and $\bar{\chi}$ will be large under similar assumptions. But notice that both χ and φ belong to the electron and both $\bar{\varphi}$ and $\bar{\chi}$ belong to the positron. Whether we have one particle or another depends only on whether we are dealing with ϵ^{+} or ϵ^{-}.

Another topic that will be treated in this subsection is the explanation from the Kähler theory of why the mass of the positron emerges as $-m$. In this and following chapters, I shall provide you with entries into the world of Kähler, exploring areas that nobody has jet explored.

3 An entry point for research on relativistic quantum mechanics with the Kähler calculus

What follows might be felt as being derogatory on Dirac, whom many will rightly consider the second best physicist of the twentieth century. His equation presently is more relevant for theoretical physics than Einstein equations. Dirac did his work when the math was not ready for the task. So, thanks are due to Dirac. But now is now and the mathematics allows for a better job. We proceed to describe the situation with respect to post-Pauli-Dirac electromagnetic Hamiltonians. For the discussion that will follow, we write the Dirac equation with explicit universal constants

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\alpha\cdot\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) + \beta mc^2 + e\Phi\right]\psi, \tag{3.1}$$

where α and β represent well known matrices in relativistic quantum mechanics. Clearly, a development in powers of $1/m$, which is what Bjorken and Drell did, is not equivalent to a development in powers of $1/c$. The reason to make this remark will be seen later below.

3.1 Post-Pauli-Dirac and Foldy-Wouthuysen equations

The term Foldy-Wouthuysen Hamiltonian is not used in the literature. One uses instead the term Foldy-Wouthuysen to refer to transformations through which one reaches the Hamiltonian:

$$H = \beta\left(m + \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - \frac{\mathbf{p}^4}{8m^3}\right) + e\Phi \; - e\frac{1}{2m}\beta\sigma\cdot\mathbf{B}$$
$$- \frac{ie}{8m^2}\sigma\cdot\, curl\ \mathbf{E} \; - \frac{e}{4m^2}\sigma(\mathbf{E}\times\mathbf{p}) \; - \frac{e}{8m^2}div\mathbf{E}, \qquad (3.2)$$

where $\mathbf{p}$ has components p^i, where Φ and $\mathbf{A}$ are components of the 4-vector potential and where σ has as components the three Pauli matrices. Equation (3.2) results in the approximation (kinetic energy/m)^3 and (kinetic energy)(field energy)/m^2. The Hamiltonian (3.2) is the tangent vector field equivalent of (2.20), except for the absence of the mass term in the latter and for a factor of two in the divergence of $\mathbf{E}$ term. The three terms in the parenthesis in (3.2) and the five terms outside it add to eight. Terms 1, 2, 3, 4, 5,6 and 7 of (2.20) correspond to terms 2, 5, 4, 3, 7, 8 and 6 of (3.2). The reason for the absence of the m term has already being explained. What is surprising tis that all the other terms are the same except for the factor of two to which we have just made reference

We follow the presentation by Bjorken and Drell. This is a Hamiltonian that displays the different interaction terms between the electron and an applied field in an easily interpretable form. They do not give the form of the equation or on what "function" (actually a two component spinor) this operator is acting upon. We shall see further below why this remark is important.

They apply a unitary transformation three times. The purpose in doing so is to remove all from the equation all operators such as α which couple the large to the small components. It is implicit in this that the equation where (3.2) would be the Hamiltonian is an equation for the positive energy state. More on this below. They describe a Foldy-Wouthuysen transformation as "a canonical transformation which decouples the Dirac equation into two two-component equations; one reduces to the Pauli description in the nonrelativistic limit; the other describes the negative energy states."

In speaking of negative energies, they state that they "and four-component wave functions are the price we must pay in order to have a factorization of H' in (4.1) into a linear Dirac equation." H' is the operator

$$H' = \beta\sqrt{m^2 + p^2.} \qquad (3.3)$$

It is worth noticing that the Kähler equation was born linear, and that negative energy solutions need not accompany the positive ones. There is not an ab initio need to separate them if we start with a spinor in the right ideal, and small components, when dealing with the electron, do not then belong to the positron.

We now cite on the Foldy-Wouthuysen transformation from the book "Relativistic Quantum Mechanics and Introduction to Field Theory" by F. J. Yndurain. Let us start with "... of the four solutions of the Dirac equation (with or without a potential) only two are physical". Make what you want of such a statement. You would find in every book on the subject some statement who makes you raise your eyebrows, though any such statement does not raise the same eyebrows.

Yndurain performs two Foldy-Wouthuysen transformations on "the Dirac for a particle in a potential, the sum of a Minkowski vector ($\mathbf{A}$, A_0), a Minkowski fourth component, V_0, and a scalar one, βV_S :

$$H = mc^2\beta + V + V_S\beta + c\alpha\cdot\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right); \tag{3.4}$$

we have here defined $V \equiv A_0 + V_0$". Here $\mathbf{P}$ is the usual $\mathbf{p}$. His result for "the Foldy-Wouthuysen Hamiltonian" is

$$H_{FW} = mc^2\beta + V + V_S\beta + \beta\frac{(\mathbf{P} - e\mathbf{A})^2}{2m} - e\frac{1}{2m}\beta\sigma\cdot\mathbf{B} - \beta\frac{\mathbf{P}^4}{8m^3c^2} - \frac{1}{8mc^2}\left[\alpha\mathbf{P},[\alpha\mathbf{P},V]\right] - \frac{1}{8mc^2}\beta\left\{\alpha\mathbf{P},\{\alpha\mathbf{P},V_s\}\right\}. \tag{3.5}$$

The author has performed two Foldy-Wouthuysen transformations. Acting on operators (call them h) to the left of the spinor, each Foldy-Wouthuysen transformation yields take $e^T h e^{-T}$. Under the two transformations, the positive energy solutions of the Dirac equation are represented by wave functions

$$\psi_{FW} = e^{T'}e^{T}\psi + O(c^{-3}). \tag{3.6}$$

($O(c^{-3})$ as in the original). We were interested in this comment, which certainly we are not criticizing. He is certainly treating Foldy-Wouthuysen transformations as gauge transformations. $H\psi$ will go into

$$(e^{T'}e^{T}He^{-T}e^{-T'})(e^{T'}e^{T}\psi) = e^{T'}e^{T}H\psi. \tag{3.7}$$

Hence, the Dirac equation written as $i\hbar\partial\psi/\partial t = H\psi$ goes to

$$H_{FW}\psi_{FW} = i\hbar e^{T'}e^{T}\frac{\partial\psi}{\partial t} = i\hbar\frac{\partial\psi_{FW}}{\partial t}. \tag{3.8}$$

Let us finally consider one more approach to the Hamiltonian in a post-Pauli approximation. In the first part of their volume (IV) on Relativistic Quantum mechanics, Landau and Lifchitz reportedly reproduce the treatment of A. Akhieser an V. Berestetski to reach the next approximation in the development of (3.1), They say that they assume that there is only an electric field. They set $\mathbf{A} = \mathbf{0}$ and obtain

$$H = \frac{\mathbf{p}^2}{2m} + e\Phi - \frac{\mathbf{p}^4}{8m^3} \ - \frac{e}{4m^2}\sigma \cdot (\mathbf{E} \times \mathbf{p}) - \frac{e}{8m^2} div\mathbf{E}. \tag{3.9}$$

Again, the method followed causes the absence of the m term. The absence of the m term, as was the case with (2.20) has to do with the fact that they intended to obtain the next approximation for a Schrödinger type equation.

Also absent in (3.9) are the terms 5 and 6 of (3.2). The absence of term 5 is not surprising given that they have assumed $\mathbf{A} = \mathbf{0}$. But the absence of term 6 might be attributed to the fact that making $\mathbf{A} = \mathbf{0}$ makes the electric field incomplete. There is no discrepancy between the $div\mathbf{E}$ term in (3.2) and (3.9). We do not know where the discrepancy by a factor of two with a similar term in (2.20) is born.

3.2 Research suggestions for involvement with the Kähler calculus

Without the need for further comparisons, it is fair to say that these developments of the Dirac equation are not very transparent. Our development of the Kähler equation is, but it remains without explanation why we should be satisfied with an equation for φ rather than an equation for u. It would be desirable to approach the problem without the early separation of the equation for u into φ and χ I suggest readers have here an entry point for research on relativistic quantum mechanics with the Kähler calculus. I suggest the following two problems. I have not even started to do what I am suggesting for fear of getting too involved with them at a "time when I do not have time". So, I do not know whether one gets something of interest and/or easy to handle. Much will depend on the ability of those who may try.

Research problem. Since the β of equation (1.11) is a $0-$form, we can solve for ηp. This can be replaced in the equation obtained by applying η to (1.11). At this point, it will be useful to see what this equation says after replacing α the β for electromagnetic coupling read from (2.10). It seems clear that this will not be directly comparable to (2.20), (3.2) and (3.3) since it will involve the potential rather than the electric and magnetic fields. One can then proceed with different types of approximations only limited by your imagination.

Research problem. One can then try to follow the same process now with (1.19), where the chirality or handedness of the state differential form is defined.

4 Breakthrough energy, Kähler calculus and Schwinger's source theory

Dirac's theory carries a baggage that Kähler's does not. The former has been loaded since birth with the concept of spinor as fundamental, rather than emergent (Notice that u does not even represent a particle in Eq. (1.8), and a need not contain the mass). We want to extract clean energy from wherever. matter holds the greatest density of energy. But, at the nuclear level, energy is not sufficiently well understood. What we have seen in previous sections should be enough to make the point that Kähler's theory holds promise in this regard because it is more powerful, more structured and more comprehensive. Its connection with nuclear physics may be in source theory.

4.1 Schwinger's source theory

The modern expansion of Dirac's theory, with concepts like spin connection, comes at a stiff price and does not even have the right generality. In the first of the already posted chapters, one sees that the Kähler derivative of vector-valued differential forms also depends on the connection on the manifold. Unlike the spinor connection, it will not generally involve Christoffel symbols, either directly or exclusively, unless the affine connection of the manifold is the Levi-Civita connection. To make matters worse, dealing with the generalization of the Dirac operator requires years of specialization by those who eventually work with it. And what has been achieved with it?

As in Dirac's, the concept of mass is not yet emergent in Kähler's theory, but the concept of mechanical and generalized momentum is. Concepts like Foldy-Wouthuysen transformations are unnecessary and cloud the issues. Also in the Kähler calculus, the treatment of the energy operator (we leave the imaginary unit factor for the last day of the summer school) and of Hamiltonians is just a matter of computing with differential equations without resort to operator theory. And most significant is the fact that Kähler's treatment of angular momentum is just a matter of Killing symmetry and properly handling sums of partial derivatives, rather than replacing operators for classical particle magnitudes. Our treatment of relativistic quantum mechanics without operator theory should have made plausible the idea that cutting edge quantum physics need not be quantum field theory, which is operator

based. Nor is the by now forgotten S-matrix theory a suitable alternative. We anticipate that a third option, Schwinger's source theory, constitutes a proxy for what the Kähler calculus will become when used to address the same issues. But it has received far less attention than it deserves.

Source theory is difficult to define. Its major attractiveness is that "the results of quantum electrodynamics are reproduced without the irrelevance of divergences, or renormalizations". It has spacetime emphasis like quantum field theory, but it differs from it in that it is not operator based. Like S-matrix theory, it also has phenomenological emphasis, which we do not view with enthusiasm. But the phenomenology might look less so when approached with the more formal perspective that the Kähler calculus provides.

Schwinger points out that "... in general, particles must be **created** in order to study them, since most of them are unstable. In a general sense this is also true of high-energy stable particles, which must be created in that situation by some device, i.e. an accelerator. One can regard all such creation acts as collisions, in which the necessary properties are transferred from other particles to the one of interest... The other particles in the collision appear only to supply these attributes. They are, in an **abstract** sense, the **source** of the particle in question... We try to represent this abstraction of realistic processes numerically...".And further down, he writes: "Unstable particles eventually decay and the decay process is a detection device. More generally, any detection device can be regarded as removing or **annihilating** the particle. Thus the **source concept** can again be used as an **abstraction** ... with the source acting negatively, as a sink." Bold face has been added.

At a much simpler level, consider the wealth of results we obtained from writing down $\partial u = au$, which is like $y' = f(x)y$ but for a calculus based on Clifford algebra. The formalism leading to it did was not based on physical considerations except for geometry of the spacetime manifold. The Kähler equation does not have physical history. Certainly it borrows from history, but it is one in which it did not take part. It thus acquires physical ascendancy only when we claim that it represents reality. If it could talk, it would advertise itself as follows "If I am given the sum of a constant scalar and a scalar-valued differential $1-$form, I return to you these solutions ...". It does not have its roots in classical physics, which usually takes place through operators.

4.2 Breakthrough energy

The types of Hamiltonians used in nuclear reactions appear not to be clean cut. A translation into a Kähler calculus description of source

theory might could go a long way towards enrichment of the former, and towards a systematic and clean study of Hamiltonians for nuclear reactions. So far, this young but languishing calculus has not yet reached the necessary degree of development.

A case in point of this fussiness is precisely the reaction of deuterium with hydrogen to yield ^{3}He plus heat in a hydrolysis process that uses as cathode a palladium lattice (see next paragraph). After Schwinger argued that the interaction with phonons in the palladium lattice enhanced the p-d fusion rate, O. H. Crawford counterargued that this mechanism does not enhance it (Fusion Technology, Vol 21, March 1992, pp 161-162). This is a beautiful example of a controversy about Hamiltonians for nuclear reactions that, by its very nature, an extended Kähler calculus might help resolve. The core issue is if and how the palladium lattice can diminish the width of the Coulomb barrier so that the reaction could take place in significant amounts. It is precisely in connection with such controversies that an extended Kähler calculus could play a crucial role.

Let us focus on the experimental evidence. Silver, Dash and Keefe (Fusion Technology, Vol 24, Dec. 1993, pp 423-430) examined "with the scanning electron microscope, scanning tunnelling microscope, and atomic force microscope" unusual surface characteristics of an electrolyzed palladium cathode but not on palladium that had not been electrolyzed. More specifically, the unusual features happened when the electrolyte contained hydrogen and deuterium atoms, but not, for instance, when it was constituted by heavy water and sulfuric acid.

There is another important point, which I serendipitously learned about. In the mid to late eighties, physicist Edbertho Leal-Quiros was doing experimental plasma physics with a machine that he himself had helped build as Ph. D. student in the group of Professor M. Prelas at the University of Missouri in Columbia, and that he used for his doctoral thesis. He was creative at building gadgets to measure properties of plasmas given the experience he had acquired at doing so while obtaining his master's degree at the National University of Colombia in Bogota under a German program to help develop physics in that country. He once proudly told me that at some point he was able to measure more plasma parameters than anybody else, and that he had put that knowledge to work when helping build that machine.

Years later, when the program in Columbia was terminated, its university donated the machine to the Metropolitan University of Puerto Rico, where Leal had become a professor. Many years ago he told me that while working with this machine, he used to find from time to time a gamma ray of about the expected energy of about the 5.5 MeVs to be expected if such a reaction took place. Leal, who now holds with his son

David at least a patent on the production of the rare and expensive element ^{3}He, did not have a lattice. He rather had an erratic high energy photon, but not an explanation of where it came from. Since he was working with plasmas, it should not be surprising that the formidable Coulomb barrier might have been jumped from time to time. More recently, they claim to have produced ^{3}He yielding phonons on a crystal, as predicted by Schwinger, but not in palladium crystal.

Finally, there is a very intimate relation between Einstein's failed attempt at unification with teleparallelism —this need not have been so much of a failure— and the Kähler calculus. Hint: Finslerian connections on pseudo-Riemannian metrics show preference for Kähler's over Dirac's theory by virtue of where the two indices of the electromagnetic field fit in geometric quantites of interest in tangent bundle related geometrization of the physics. The last ideas by Einstein, Schwinger and Kähler may have been their best. They were ignored without regard for their merit. One cannot appreciate what one does not care to understand!

CHAPTER 5: Lie Differentiation and Angular Momentum

Jose G. Vargas

1 Lie differentiation

Kähler's theory of angular momentum is a specialization of his approach to Lie differentiation. We could deal with the former directly, but we do not want to miss this opportunity to show you both, as they are jewels. As an exercise, readers can at each step specialize the Lie theory to rotations.

1.1 Of Lie differentiation and angular momentum

For rotations around the z axis, we have

$$\frac{\partial}{\partial \phi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{1.1}$$

The partial derivative equals an example of what Kähler defines as a Lie operator, i.e.

$$X = \alpha^i(x^1, x^2, ...x^n)\frac{\partial}{\partial x^i}, \tag{1.2}$$

without explicitly resorting to vector fields and their flows (See section 16 of his 1962 paper). Incidentally, $\partial/\partial x^i$ does not respond to the concept of vector field in Cartan and Kähler (For more on these concepts, see section 8.1 of my book "Differential Geometry for Physicists and Mathematicians"). Contrary to what one may read in the literature, not all concepts of vector field are equivalent, but simply related (See section 3.5 of that book).

One would like to make (1.2) into a partial derivative. When I had already written most of this section, I realized that it was not good enough to refer readers to Kähler's 1960 paper in order to know how to do that; until one gets hold of that paper (in German, by the way), many readers would not be able to understand this section. So, we have

added the present subsection 1.8 to effect such a change into a partial derivative.

Following Kähler, we write the operator (1.1) as χ_3 since we may extend the concept to any plane. We shall later use

$$\chi_k = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \tag{1.3}$$

where (i, j, k) constitutes any of the three cyclic permutations of $(1, 2, 3)$, including the unity. Here, the coordinates are Cartesian.

Starting with chapter 2 posted in this web site (the first one to be taught in the Kähler calculus phases (II and III) of the summer school), we have not used tangent-valued differential forms, not even tangent vector fields. Let us be more precise. We will encounter expressions that can be viewed as components of vector-valued differential $1-$forms because of the way they transform when changing bases. But those components are extractions from formulas arising in manipulations, without the need to introduce invariant objects of which those expressions may be viewed as components. The not resorting to tangent-valued quantities will remain the case in this chapter, even when dealing with total angular momentum; the three components will be brought together into just one element of the algebra of scalar-valued differential forms.

1.2 Lie operators as partial derivatives

Cartan and Kähler defined Lie operators by (1.2) (in arbitrary coordinate systems!) and applied them to differential forms. A subreptitious difficulty with this operator is that the partial derivatives take place under different conditions as to what is maintained constant for each of them. This has consequences when applied to differential forms.

In subsection 1.8, we reproduce Kähler's derivation of the Lie derivative as a single partial derivative with respect to a coordinate y^n from other coordinate systems,

$$X = \alpha^i(x) \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^n}. \tag{1.4}$$

His proof of (1.4) makes it obvious why he chose the notation y^n

Let u be a differential form of grade p,

$$u = \frac{1}{p!} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \tag{1.5}$$

in arbitrary coordinate systems. Exceptionally, summation does not take place over a basis of differential $p-$forms, but over all values of the

indices. This notation is momentarily used to help readers connect with formulas in in Kähler's 1960 paper.

Our starting point will be

$$Xa_{i_1\ldots i_p} = \alpha^i(x)\frac{\partial a_{i_1\ldots i_p}}{\partial x^i} = \frac{\partial a_{i_1\ldots i_p}}{\partial y^n}. \tag{1.6}$$

1.3 Non-invariant form of Lie differentiation

In subsection 1.8, we derive

$$Xu = \frac{1}{p!}\alpha^i\frac{\partial a_{i_1\ldots i_p}}{\partial x^i}dx^{i_1}\wedge\ldots\wedge dx^{i_p} + d\alpha^i\wedge e_i u, \tag{1.7}$$

with the operator e_i as in previous chapters.

Assume that the α^i's were constants. The last term would drop out. Hence, for X_i given by $\partial/\partial x^i$ and for u given by $a_{i_1\ldots i_p}dx^{i_1}\wedge\ldots\wedge dx^{i_p}$, we have

$$X_i(a_{i_1\ldots i_p}dx^{i_1}\wedge\ldots\wedge dx^{i_p}) = \frac{\partial(a_{i_1\ldots i_p}dx^{i_1}\wedge\ldots\wedge dx^{i_p})}{\partial x^i} = \frac{\partial a_{i_1\ldots i_p}}{\partial x^i}dx^{i_1\ldots i_p}, \tag{1.8}$$

where $dx^{i_1\ldots i_p}$ stands for $dx^{i_1}\wedge\ldots\wedge dx^{i_p}$. This allows us to rewrite (1.7) as

$$Xu = \frac{1}{p!}\alpha^i\left[\frac{\partial(a_{i_1\ldots i_p}dx^{i_1\ldots i_p})}{\partial x^i}\right] + d\alpha^i\wedge e_i u, \tag{1.9}$$

It is then clear that

$$Xu = \alpha^i\frac{\partial u}{\partial x^i} + d\alpha^i\wedge e_i u, \tag{1.10}$$

In 1962, Kähler used (1.10) as starting point for a comprehensive treatment of lie differentiation.

The first term on the right of (1.10) may look as sufficient to represent the action of X on u, and then be overlooked in actual computations. In subsection 1.8, we show that this is not so. We now focus on the first term since it is the one with which one can become confused in actual practice with Lie derivatives.

Notice again that, if the α^i's are constants —and the constants $(0, 0, ..1, 0, ...0)$ in particular— the last term in all these equations vanishes. So, we have

$$X(cu) = c\frac{\partial u}{\partial x^i}, \tag{1.11}$$

for a equal to a constant c. But

$$X\left[a(x)u\right] = a(x)\frac{\partial u}{\partial x^i} \qquad \text{(Wrong!)}$$

is wrong. When in doubt with special cases of Lie differentiations, resort to (1.10).

The terms on the right of equations (1.7) to (1.10) are not invariant under changes of bases. So, if u were the state differential form for a particle, none of these terms could be considered as properties of the particle, say its orbital and spin angular momenta.

1.4 Invariant form of Lie differentiation

Kähler subtracted $\alpha^i \omega_i{}^k \wedge e_k u$ from the first term in (1.10) and simultaneously added it to the second term. Thus he obtained

$$Xu = \alpha^i d_i u + (d\alpha)^i \wedge e_i u, \tag{1.12}$$

since

$$\alpha^i \frac{\partial u}{\partial x^i} - \alpha^i \omega_i{}^k \wedge e_k u = \alpha^i d_i u, \tag{1.13}$$

and where we have defined $(d\alpha)^i$ as

$$(d\alpha)^i \equiv d\alpha^i + \alpha^i \omega_i{}^k. \tag{1.14}$$

One may view $d\alpha^i + \alpha^k \omega_k{}^i$ as the contravariant components of what Cartan and Kähler call the exterior derivative of a vector field of components α^i. By "components as vector", we mean those quantities which contracted with the elements of a field of vector bases yield the said exterior derivative. Both differential-form-valued vector field and vector-field-valued differential $1-$form are legitimate terms for a quantity of that type. The corresponding covariant components are

$$(d\alpha)_i = d\alpha_i - \alpha_h \omega_i{}^h. \tag{1.15}$$

If you do not find (1.15) in the sources from which you learn differential geometry, and much more so if your knowledge of this subject is confined to the tensor calculus, please refer again to my book "Differential Geometry for Physicists and Mathematicians". Of course, if you do not need to know things in such a depth, just believe the step from (1.14) to (1.15). We are using Kähler's notation, or staying very close to it. Nevertheless, there is a more Cartanian way of dealing with the contents of this and the next subsections. See subsection 1.7.

In view of the considerations made in the previous sections, we further have

$$Xu = \alpha^i d_i u + (d\alpha)_i \wedge e^i u \tag{1.16}$$

All three terms in (1.12) and (1.16) are invariant under coordinate transformations. The two terms on the right do not mix when performing a change of basis. This was not the case with the two terms on the right of (1.7) and (1.10), even though their form might induce one to believe otherwise.

1.5 Action of a Lie operator on the metric's coefficients

Following Kähler we introduce the differential 1−form α with components α_i, i.e.

$$\alpha = \alpha_k dx^k = g_{ik}\alpha^i dx^k. \tag{1.17}$$

If the α^i were components of a vector field, the α_k would be its covariant components. But both of them are here components of the differential form α. We define $d_i\alpha_k$ by

$$(d\alpha)_i = (d_i\alpha_k)dx^k. \tag{1.18}$$

Hence, on account of (1.15),

$$d_i\alpha_k \equiv \alpha_{i,k} - \alpha_h\Gamma_i{}^h{}_k. \tag{1.19}$$

Therefore,

$$d_i\alpha_k + d_k\alpha_i = \alpha_{i,k} + \alpha_{k,i} - \alpha_h\Gamma_i{}^h{}_k - \alpha_h\Gamma_k{}^h{}_i \tag{1.20}$$

In a coordinate system where $\alpha^i = 0$ $(i < n)$ and $\alpha^n = 1$, we have

$$\alpha_{i,k} = (g_{pi}\alpha^p)_{,k} = g_{pi,k}\,\alpha^p = g_{ni,k}\,, \tag{1.21}$$

and, therefore,

$$\alpha_{i,k} + \alpha_{k,i} = g_{ni,k} + g_{nk,i}\,. \tag{1.22}$$

On the other hand,

$$\alpha^l\Gamma_{ilk} + \alpha^l\Gamma_{kli} = 2\Gamma_{ink} = g_{ni,k} + g_{nk,i} - g_{ik,n}, \tag{1.23}$$

From (1.20), (1.22) and (1.23), we obtain

$$d_k\alpha_i + d_i\alpha_k = \frac{\partial g_{ik}}{\partial x^n}. \tag{1.24}$$

1.6 Killing symmetry and the Lie derivative

When the metric does not depend on x^n, (1.24) yields

$$d_k\alpha_i + d_i\alpha_k = 0. \tag{1.25}$$

We then have that

$$e_i d\alpha = -2(d\alpha)_i. \tag{1.26}$$

Indeed,

$$e_i d\alpha = e_i d(\alpha_k dx^k)] = e_i[(\alpha_{k,m} - \alpha_{m,k})(dx^m \wedge dx^k)] = (\alpha_{k,i} - \alpha_{i,k})dx^k, \tag{1.27}$$

where the parenthesis around $dx^m \wedge dx^k$ is meant to signify that we sum over a basis of differential 2−forms, rather than for all values of i and k. By virtue of (1.18), (1.19) and (1.25), we have

$$2(d\alpha)_i = (d_i\alpha_k - d_k\alpha_i)dx^k = [(\alpha_{i,k} - \alpha_h\Gamma_i{}^h{}_k) - (\alpha_{k,i} - \alpha_h\Gamma_k{}^h{}_i)]dx^k. \tag{1.28}$$

We now use that $\Gamma_i{}^h{}_k = \Gamma_k{}^h{}_i$ in coordinate bases, and, therefore,

$$2(d\alpha)_i = (\alpha_{i,k} - \alpha_{k,i})dx^k = -e_i d\alpha. \tag{1.29}$$

Hence (1.26) follows, and (1.16) becomes

$$Xu = \alpha^i d_i u - \frac{1}{2} e_i d\alpha \wedge e^i u. \tag{1.30}$$

Notice that we have just got Xu in pure terms of differential forms, unlike (1.16), where $(d\alpha)_i$ makes implicit reference to the differentiation of a tensor field.

An easy calculation (See Kähler 1962) yields

$$-2e_i d\alpha = d\alpha \vee u - u \vee d\alpha. \tag{1.31}$$

Hence,

$$Xu = \alpha^i d_i u + \frac{1}{4} d\alpha \vee u - \frac{1}{4} u \wedge d\alpha, \tag{1.32}$$

which is our final expression for the Lie derivative of a differential form if that derivative is associated with a Killing symmetry.

1.7 Remarks for improving the Kähler calculus

The Kähler calculus is a superb calculus, and yet Cartan would have written it without coordinate bases. We saw in chapter one the disadvantage that these bases have relative to the orthonormal ω^i's, which are differential invariants that define a differentiable manifold endowed with a metric. In this section, the disadvantage lies in that one needs to have extreme care when raising and lowering indices, which is not a problem with orthonormal bases since one simply multiplies by one or minus one. Add to that the fact that dx_i does not make sense since there are not such a thing as "covariant curvilinear coordinates". On the other hand, ω_i is well defined.

Consider next the Killing symmetry, (1.25). The $d_k\alpha_i$ are associated with the covariant derivative of a vector field. But they could also be associated with the covariant derivatives of a differential 1-form. Indeed, we define $(d_i\alpha)_k$ by

$$d_i\alpha = (\alpha_{k,i} - \alpha_l\Gamma_k{}^l{}_i)dx^k \equiv (d_i\alpha)_k dx^k. \tag{1.33}$$

But

$$d_k\alpha_i \equiv \alpha_{k,i} - \alpha_h \Gamma_k{}^h{}_i. \tag{1.34}$$

Thus

$$(d_i\alpha)_k = d_k\alpha_i \tag{1.35}$$

and the argument of the previous two sections could have been carried out with covariant derivatives of differential forms without invoking components of vector fields.

1.8 Derivation of Lie differentiation as partial differentiation

Because the treatment of vector fields and Lie derivatives in the modern literature is what it is, we now proceed to show how a Lie operator as defined by Kähler (and by Cartan, except that he did not use this terminology but infinitesimal operator) can be reduced to a partial derivative.

Consider the differential system

$$\frac{\partial x^i}{\partial \lambda} = \alpha_i(x^1, \ ... \ x^n), \tag{1.36}$$

the α_i not depending on λ. One of n independent "constant of the motion" (i.e. line integrals) is then additive to λ. It can then be considered to be λ itself. Denote as y^i $(i = 1, \quad n-1)$ a set of $n-1$ such integrals, independent among themselves and independent of λ, to which we shall refer as y^n. The y^i's $(i = 1, \quad n)$ constitute a new coordinate system and we have

$$x^i = x^i(y^1, \quad , \ y^n). \tag{1.37}$$

In the new coordinate system, the Lie operator reads $X = \beta^i \partial/\partial y^i$. Its action on a scalar function is

$$\beta^i \frac{\partial f}{\partial y^i} = \alpha^l \frac{\partial f}{\partial x^l} = \frac{\partial x^l}{\partial \lambda}\frac{\partial f}{\partial x^l} = \frac{\partial f}{\partial y^n}. \tag{1.38}$$

We rewrite u (given by (1.5)), as

$$u = \frac{1}{p!} a_{i_1 \ldots i_p} \frac{\partial x^{i_1}}{\partial y^{i_1}} \quad \frac{\partial x^{i_p}}{\partial y^{i_p}} dy^{k_1} \wedge ... \wedge dy^{k_p}, \tag{1.39}$$

and then

$$\begin{aligned} \frac{\partial u}{\partial y^n} = & \frac{1}{p!} \frac{\partial a_{i_1 \ldots i_p}}{\partial y^n} \frac{\partial x^{i_1}}{\partial y^{i_1}} \quad \frac{\partial x^{i_p}}{\partial y^{i_p}} dy^{k_1} \wedge ... \wedge dy^{k_p} + \\ & + \frac{1}{(p-1)!} a_{i_1 \ldots i_p} \frac{\partial}{\partial y^n} \left(\frac{\partial x^{i_1}}{\partial y^{k_1}} dy^{k_1} \right) \frac{\partial x^{i_2}}{\partial y^{k_2}} \quad \frac{\partial x^{i_p}}{\partial y^{k_p}} dy^{k_2} \wedge ... \wedge dy^{k_p}. \end{aligned} \tag{1.40}$$

We now use that

$$\frac{\partial a_{i_1\ldots i_p}}{\partial y^n} = \frac{\partial a_{i_1\ldots i_p}}{\partial x^i}\frac{\partial x^i}{\partial y^n} = \alpha^i \frac{\partial a_{i_1\ldots i_p}}{\partial x^i} \tag{1.41}$$

and that

$$\frac{\partial}{\partial y^n}\left(\frac{\partial x^{i_1}}{\partial y^{k_1}}dy^{k_1}\right) = \frac{\partial}{\partial y^{k_1}}\left(\frac{\partial x^{i_1}}{\partial y^n}\right)dy^{k_1} = d\left(\frac{\partial x^{i_1}}{\partial y^n}\right) = d\alpha^{i_1}. \tag{1.42}$$

Hence

$$Xu = \frac{\partial u}{\partial y^n} = \frac{1}{p!}\alpha^i \frac{\partial a_{i_1\ldots i_p}}{\partial x^i}dx^{i_1\ldots i_p} + \frac{1}{p!}a_{i_1\ldots i_p}d\alpha^i \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}. \tag{1.43}$$

and finally

$$Xu = \frac{1}{p!}\alpha^i \frac{\partial u}{\partial x^i} + d\alpha^i \wedge e_i u, \tag{1.44}$$

2 Angular momentum

The components of the angular momentum operators acting on scalar functions are given by (1.3), and therefore

$$\alpha_k = -x^j dx^i + x^i dx^j, \tag{2.1}$$

and

$$d\alpha_k = -dx^j \wedge dx^i + dx^i \wedge dx^j = 2dx^i \wedge dx^j \equiv 2w_k. \tag{2.2}$$

Hence

$$\chi_k u = x^i \frac{\partial u}{\partial x^j} - x^j \frac{\partial u}{\partial x^i} + \frac{1}{2}w_k \vee u - \frac{1}{2}u \wedge w_k. \tag{2.3}$$

The last two terms constitute the component k of the spin operator. It is worth going back to (1.7) and (1.10), where we have the entangled germs of the orbital and spin operator, if we replace $\chi$with χ_k. It does not make sense to speak of spin as intrinsic angular momentum until u represents a particle, which would not be the case at this point.

Kähler denotes the total angular momentum as $K + 1$, which he defines as

$$(K+1)u = \sum_{i=1}^{3} \chi_i u \vee w_i. \tag{2.4}$$

He then shows by straightforward algebra that

$$-K(K+1) = \chi_1^2 + \chi_2^2 + \chi_3^2. \tag{2.5}$$

He also develops the expression for $(K + 1)$ until it becomes

$$(K+1)u = -\sum_i \frac{\partial u}{\partial x^i} \vee dx^i \vee r dr + \sum_i x^i \frac{\partial u}{\partial x^i} + \frac{3}{2}(u - \eta u) + g\eta u \tag{2.6}$$

and also

$$(K+1)u = -\zeta\partial\zeta u \vee r dr + \sum_i x^i \frac{\partial u}{\partial x^i} + \frac{3}{2}(u - \eta u) + g\eta u, \qquad (2.7)$$

where η is as in previous chapters, where ζ reverses the order of all the differential $1-$form factors in u and where $g \equiv dx^i \wedge e_i$. This expression for $(K+1)u$ is used in the next section.

3 Strict harmonic differential forms in $E_3 - \{0\}$

This section is a somewhat abbreviated form of Kähler's treatment of strict harmonic differential forms in $E_3 - \{0\}$, meaning the 3-D Euclidean space punctured at the origin of coordinates.

Kähler starts his argument with considerations on Laurent series of harmonic functions. He states that "... one can prove that every time differentiable harmonic function in $E_3 - \{0\}$ can be written as a series"

$$f = \sum_{h=-\infty}^{\infty} f^{(h)}, \qquad (3.1)$$

where $f^{(h)}$ is a homogeneous polynomial of degree h of homogeneity, for $h \geq 0$, and its the product of polynomial by r^{2h+1} for $h < 0$ (A theorem along similar lines in the modern literature can be found in the book "Harmonic Function Theory" by S. Axler, P. Bourdon and W. Ramey, copyrighted in 2001). From there, Kähler argues that one can expand a strict harmonic differential form u as

$$u = \sum_{-\infty}^{\infty} u^{(h)}, \qquad (3.2)$$

where $u^{(h)}$ is (a) a homogeneous of degree h with respect to the Cartesian coordinates, and (b) also being polynomic for $h \geq 0$, and finally the product of a polynomial by r^{-2h-1} for $h < 0$.

We shall consider 3.2 as an ansatz with $u^{(h)}$'s of type (a). As for (b), we shall deal with this in due time. Be aware of the fact that 3.1 is not contained in 3.2, since the former is for harmonic functions and 3.2 is for strict harmonic differential forms. Obviously, non-trivial strict harmonic functions do not exist.

3.1 Simplification by reduction

Kähler shows that there is isomorphisms between modules M_h and M_{-h-2}, for $-\infty < h < \infty$, the subscript h being the degree of homogeneity h

of the members of the module. One can show that the module M_{-1} is empty.

Let u be strict harmonic of degree h. Then,

$$\partial(r^{-2h-2}dr \vee u) = \partial(r^{-2h-2}dr) \vee u + 2[e^l(r^{-2h-2}dr)] \vee d_l u, \tag{3.3}$$

by virtue of the rule for $\partial(v \vee u)$ and the assumption $\partial u = 0$.

Let us compute the first term on the right hand side

$$\partial(r^{-2h-2}dr) = -(2h+2)r^{-2h-3} + r^{-2h-2}\partial dr. \tag{3.4}$$

For ∂dr, we need a little bit of computations which we address using Cartan's notation (If not familiar with it, see this author's "Differential Geometry for Physicists and Mathematicians"). Clearly $\partial dr = \omega^l \cdot d_l dr$, where

$$\omega^1 = dr, \quad \omega^2 = rd\theta, \quad \omega^3 = r\sin\theta d\phi. \tag{3.5}$$

Using

$$d\omega^i = \omega^j \wedge \omega^i_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{3.6}$$

one readily obtains

$$\omega^2_1 = d\theta \quad \omega^3_1 = \sin\theta d\phi, \quad \omega^3_2 = \cos\theta d\phi. \tag{3.7}$$

Then,

$$d_l(dr) = d_l\omega^1 = -\Gamma^1_{l\,i}\omega^i = -\omega^1_l. \tag{3.8}$$

Hence

$$\begin{aligned}\partial(dr) = \omega^l \cdot d_l(dr) = \omega^2 \cdot (-\omega^1_2) + \omega^3 \cdot (-\omega^1_3) = \\ = rd\theta \cdot d\theta + r\sin\theta d\phi \cdot \sin\theta d\phi = \frac{1}{r} + \frac{1}{r} = \frac{2}{r}\end{aligned} \tag{3.9}$$

which will go into the last term of (3.4) and then, therefore, into the first term of (3.3). For the second term of (3.3), we use that

$$e^l(r^{-2h-2}dr) = e^l(r^{-2h-3}rdr) = e^l(r^{-2h-3}x^i dx^i) = r^{-2h-3}x^l \tag{3.10}$$

so that it becomes

$$2r^{-2l-3}x^l\frac{\partial u}{\partial x^l} = 2hr^{-2h-3}u, \tag{3.11}$$

after using the homogeneity of u in $d_l u$.

From (3.3), (3.4), (3.9) and (3.11), we get

$$\partial(r^{-2h-2}dr \vee u) = 0. \tag{3.12}$$

hence r^{-2h-2} is strict harmonic. Its degree of homogeneity is $-h-2$. We write the result just obtained as

$$r^{-2h-2}dr \vee u_h = v_{-h-2}. \tag{3.13}$$

No assumption was made as to whether h is positive, or negative, or zero. We proceed to confirm this.

We multiply (3.13) by $r^{2h+2}dr$ and obtain

$$r^{2h+2}dr \vee v_{-h-2} = u_h \tag{3.14}$$

which is equivalent to referring to $-h-2$ as l, in which case, the exponent $2h+2$ becomes

$$2h+2 = 2(-l-2)+2 = -2l-2 \tag{3.15}$$

and

$$r^{-2l-2}dr \vee v_l = u_{-l-2}, \tag{3.16}$$

in agreement with (3.13).

It follows from all this that it suffices to compute M_h for $h \geq 0$ in order to readily obtain M_{-h-2}. It also suffices to compute the subset of even differential forms in M_h since the even ones will be obtained through Clifford multiplication by the unit differential form of grade n.

3.2 Eigen differential forms of total angular momentum

We seek as expansion of strict harmonic differentials by eigen differential forms of the total angular momentum operator, since the equation $\partial_3 u = 0$ has spherical symmetry.

The elements of M_h are of the form $a+v$ where a and v are respectively differential 0-form and 2-form respectively. The formula for the action of K on $a+v$ is

$$Ku = -(h+1)a + [(h+1)v - 2da \wedge rdr] \tag{3.17}$$

Kähler shows that a necessary and sufficient condition for $a+v$ to be an eigen differential of K with proper value k is that the equations

$$-(h+1)a = ka, \qquad (h+1)v - 2da \wedge rdr = kv \tag{3.18}$$

be satisfied. This can be achieved in any of the two following ways:

$$I : a = 0, \qquad k = h+1 \tag{3.19}$$

$$II : k = -h-1, \qquad (h+1)v - da \wedge rdr = kv. \tag{3.20}$$

The solutions of type I are of the form

$$u = df \vee w, \quad \partial\partial f = 0, \tag{3.21}$$

where the annulment of $\partial\partial f$ follows from $\partial u = 0$.

The solutions of type II can be written as

$$u = a + \frac{1}{h+1} da \wedge r dr, \quad \partial\partial a = 0, \tag{3.22}$$

(for the proof of $\partial\partial a = 0$, see further below) and further as

$$\begin{aligned} u &= a + \frac{1}{h+1}(da \vee r dr - da \cdot r dr) = \\ &= a + \frac{1}{h+1}(da \vee r dr - \frac{\partial a}{\partial x^i} dx^i \cdot x_j dx^j) = a + \frac{1}{h+1}\partial a \vee r dr - \frac{h}{h+1} a. \end{aligned}$$

Hence we have

$$u = \frac{1}{h+1}(a + \partial a \vee r dr), \qquad \partial\partial a = 0, \tag{3.23}$$

as alternative to (3.22).

We now prove that $\partial\partial a = 0$. We use (3.23) and

$$\partial(u \vee v) = \partial u \vee v + (\eta u \vee \partial v) + 2 e^i u \vee d_i v,$$

to obtain

$$\begin{aligned} 0 = \partial u &= \partial((h+1)u) = \partial(a + \partial a \vee r dr) = \\ &= \partial a + \partial\partial a \vee r dr - \partial a \vee \partial(r dr) + 2 \frac{\partial a}{\partial x^i} \vee d_i(r dr). \end{aligned} \tag{3.24}$$

But the sum of the first, third and fourth term on the right of (3.24) cancel out, since $d_i(r dr) = dx^i$ and

$$\partial(r dr) = \partial(x_i dx^i) = \sum dx^i \vee dx^i = 3. \tag{3.25}$$

So, (3.24) becomes

$$0 = \partial u = \partial\partial a \vee r dr.$$

Clifford multiplication by $\frac{1}{r} dr$ on the right yields $\partial\partial a = 0$.

Notice that we could have used the fact that $h \geq 0$, but have not done so.

The solutions of type II can still be given another form. First notice that if we multiply the right hand side of (3.22) by $h+1$ we reproduce the submodule. So let us write the u in (3.22) and (3.23) further as

$$u = (h+1)a + da \wedge r dr. \tag{3.26}$$

Now observe the following:

$$-r^{2h+2} dr \vee d(r^{-2h-1}a) = (2h+1)a - r dr \vee da, \tag{3.27}$$

but

$$dr \vee da = dr \wedge da + \sum \frac{x^i}{r} \frac{\partial a}{\partial x^i} = -da \wedge dr + \frac{h}{r} a. \tag{3.28}$$

Hence, from (3.27) and (3.28)

$$-r^{2h+2} dr \vee d(r^{-2h-1}a) = (2h+1)a + da \wedge r dr - ha = (h+1)a + da \wedge r dr = u, \tag{3.29}$$

where we have used (3.26) for the last step..

From this equation follows a very important result. Notice that equation (3.13) states that the Clifford product by $r^{-2h-2}dr$ of a strict harmonic differential form of degree of homogeneity h gives a strict harmonic differential form of degree $-h-2$. In (3.29), we solve for $d(r^{-2h-1}a)$ through multiplication by $-r^{-2h-1}dr$ and, since u is homogeneous of degree h, we conclude that $d(r^{2h-2}a)$ is strict harmonic of degree $-h-2$.

Kähler obtains the same result in a different way. Assume that the a's are harmonic functions. Since $r^{-2h-1}a$ is obtained from a by replacement of x^i by x^i/r^2 followed by division by r, it also is harmonic. Hence, it follows from $\partial\partial(r^{-2h-1}a) = 0$ that $\partial(r^{-2h-1}a)$, which equals $d(r^{-2h-1}a)$, is strict harmonic of degree $-h-2$. This leads Kähler to write the submodule of even differential forms as

$$\frac{1+\eta}{2} M_h = dF_{h+1} \vee w + r^{2h+2} dr \vee dF_{-h-1}, \qquad h \geq 0, \tag{3.30}$$

where F_h is the module, say over the complex field, of homogeneous harmonic polynomials of degree h, and F_{-h-1} is the set of harmonic functions $r^{-2h-1}a$.

The members of the module (3.30), but without the input of F_h for $h \geq 0$ being harmonic polynomials, would simply be written as

$$u_h = da_h \vee w + r^{2h+2} dr \vee d(r^{-2h-1} b_h), \tag{3.31}$$

where a_h and b_h are arbitrary harmonic functions. As we just said after Eq.(3.25), h could be ≥ 0 or < 0. We have not assumed one or the other. As for the odd differential forms, they would be given as

$$u_h = de_h + r^{2h+2} dr \vee d(r^{-2h-1} f_h) \vee v, \tag{3.32}$$

where e_h and f_h are arbitrary harmonic functions.

3.3 Angular factor of strict harmonic differential form solutions of $\partial_3 u = 0$

In chapter 2, we already saw that one can accommodate less symmetry in solutions of equations than on the equations themselves. We now look at a similar situation from another perspective.

The Laplacian operator is isotropic. Its solutions are not so in general. In other words, they are not spherically symmetric because of their dependence on θ and ϕ. So, how does the symmetry of an equation reflect itself in its solutions. The answer for Kähler equations with, for example, spherical symmetry,

$$\partial u = a \vee u, \tag{3.33}$$

means that, when we obtain ∂u for any solution, we not recover the solution left multiplied by some spherical symmetric u.

With this observation, we can understand Kähler's construction of solutions from homogeneous harmonic polynomials, namely

$$F_k = \sum_{m=-l}^{l} C \cdot r^k Y_k^m, \qquad\qquad l = |k| \tag{3.34}$$

where C stands for complex coefficients C_l^k. The $d(r^k Y_k^m)$ are strict harmonic differentials, as per the first term on the right of (3.32) or because $0 = \partial\partial(r^k Y_k^m) = \partial[d(r^k Y_k^m)]$. We do not want solutions of $\partial_3 u = 0$, but of the Kähler equation. So, we seek solutions whose Kähler derivative will behave as specified in the previous paragraph.

He then defines the "spherical differentials", S_k^m,

$$S_k^m = r^{1-k} d(r^k Y_k^m). \tag{3.35}$$

These are differential 1-forms of degree of homogeneity zero. Using that $d(r^k Y_k^m)$ is strict harmonic, we readily obtain

$$\partial S_k^m = \frac{1-k}{r} dr \vee S_k^m. \tag{3.36}$$

It is not as easy to obtain the following important result for later use:

$$d_r S_k^m = 0, \tag{3.37}$$

for later use. The argument goes as follows. The d_h operator is covariant, i.e. it transforms tensorially. Use a superscript to denote the coordinate system with which one is computing. Denote $d_h^{(x)}$ and $d_h^{(y)}$ as v_h and v'_h.

Clearly, $v_i' = (\partial x^l/\partial y^i)v_l$. Thus $d_i^{(y)} = (\partial x^l/\partial y^i)d_l^{(x)}$ and with d_r denoting $d_i^{(y)}$ for $y^i = r$, we have

$$d_r u = \frac{\partial x^i}{\partial r} d_i^{(x)} u = \frac{x^i}{r}\frac{\partial u}{\partial x^i} = \frac{h}{r}u. \tag{3.38}$$

Since S_k^m is homogeneous of degree zero, Eq. (3.37) follows.

3.4 Inclusion of the radial factor

We now seek radial factors R that will satisfy the rotational symmetry around all three axes,

$$X_i R = 0. \tag{3.39}$$

We show below that

$$\partial(R \vee S_k^m) = (\partial R + \eta\zeta R \vee \frac{1-k}{r}dr) \vee S_k^m, \tag{3.40}$$

the angular dependence on the right hand side thus being confined to S_k^m.

In order to satisfy (3.39), one requires R to be of the form $R = R_1 + R_2$, where

$$R_1 = \rho_0 + \rho_1 dr_1, \qquad R_2 = (\rho_2 + \rho_3 dr) \vee w, \tag{3.41}$$

with $\rho_\nu = \rho_\nu(r)$. In turn, we can rewrite this as

$$R = R_1 + R_2 \vee w, \qquad e_\theta R_{1,2} = e_\phi R_{1,2} = 0. \tag{3.42}$$

Using (3.28) and (3.42), we get

$$\partial(R_1 \vee S) = \partial R_1 \vee S + \eta R_1 \vee \partial S, \tag{3.43}$$

and recalling that w is a constant differential and commutes with the whole algebra,

$$\begin{aligned} \partial(R_2 \vee w \vee S) = \partial(R_2 \vee S \vee w) = \partial(R_2 \vee S) \vee w = \\ = \partial R_2 \vee S \vee w + \eta R_2 \vee \partial S \vee w \\ = \partial R_2 \vee w \vee S + \eta R_2 \vee w \vee \partial S. \end{aligned} \tag{3.44}$$

Needless to say that indices k and m are understood everywhere. Since $R_1 - R_2 \vee w = \tau R$, we further get, using (3.43) and (3.44)

$$\partial(R \vee S) = \partial R \vee S + \eta\zeta R \vee \partial S, \tag{3.45}$$

which, together with (3.36), implies (3.40).]

3.5 The general solution

The S_k defined by

$$S_k = \sum_{m=-l}^{+l} C \cdot S_k^m \qquad (l = |k|) \tag{3.46}$$

constitutes a C-module. Then, by virtue of (3.34) and (3.35),

$$dF_k = r^{k-1} S_k \tag{3.47}$$

and, then, following (3.30) and that

$$\frac{1-\eta}{2} M_h = \frac{1+\eta}{2} M_h \vee w, \tag{3.48}$$

(equivalently eqs, (3.31) and (3.32)), we finally get

$$M_h = r^h S_{h+1} + r^h dr \vee S_{-h-1} + r^h S_{h+1} \vee w + r^h dr \vee S_{-h-1} \vee w. \tag{3.49}$$

(Notice that the first and last terms on the right are from (3.32) and the other one from (3.31). This is, as we argued, valid for $h \geq 0$ and $h < 0$. Kähler had assumed $h \geq 0$. One then Clifford-multiplies (3.49) by $r^{-2h-2}dr$ and obtains

$$\begin{aligned} M_{-h-2} = {} & r^{-h-2} dr \vee S_{h+1} + r^{-h-2} S_{-h-1} + \\ & + r^{-h-2} dr \vee S_{h+1} \vee w + r^{-h-2} w \vee S_{-h-1}. \end{aligned} \tag{3.50}$$

Of course, the form of the right hand side of (3.49) is identical to the form of the right hand side of (3.50). Hence, once again, (3.49) is valid regardless of whether h is positive, zero or negative.

The expansion of M_h leads to the following expansion over the indices K and m

$$u = \sum_{k,m} R_m^k \vee S_k^m \tag{3.51}$$

$$R_k^m = r^k (a_{km} + b_{km} dr + c_{km} w + f_{km} dr \vee w) \tag{3.52}$$

with constant coefficients a_{km}, b_{km}, c_{km} and f_{km}.

4 The fine structure of the hydrogen atom

It takes no extra effort to let the charge of the nucleus be $Z|e|$. This amounts to neglecting the interaction of the electrons in the atom.

The electromagnetic potential then takes the simple form

$$\omega = -c\Phi dt, \qquad \Phi = \frac{Z|e|}{r}. \tag{4.1}$$

The Kähler equation now is

$$\partial u = \frac{1}{hc}(-E_0 + \frac{Zc^2}{r} icdt) \vee u, \tag{4.2}$$

where E_0 is the mass of the electron. We apply to (4.2) the treatment for stationary solutions reported in the previous chapter with

$$u = p \vee e^{-\frac{iEt}{\hbar c}} \epsilon^- \tag{4.3}$$

where E is the energy of the electron in the external field. α and β as defined in chapter 4 now take the form

$$\alpha = -\frac{E_0}{\hbar c}, \qquad \beta = \frac{1}{\hbar c}\frac{Ze^2}{r} \tag{4.4}$$

and the equation for p becomes

$$\partial p - \frac{1}{\hbar c}(E + \frac{Ze^2}{r}) \vee \eta p + \frac{1}{\hbar c} E_0 p = 0. \tag{4.5}$$

Because of spherical symmetry, we use the ansatz

$$p = R \vee S_k^m, \tag{4.6}$$

where R is spherically symmetric. From (4.5), (4.6) and (3.40), we obtain

$$[\partial R + \eta\zeta R \vee \frac{1-k}{r} + \frac{1}{\hbar c}(E + \frac{Ze^2}{r}) \vee \eta R + \frac{1}{\hbar c} E_0 R] \vee S_k^m = 0. \tag{4.7}$$

Let us refer to the contents of the square brackets as $\mathcal{R}$ and let us solve (4.7) by setting $\mathcal{R} = 0$. On account of the first of equations (3.42), we write this as

$$\mathcal{R}_1 + \mathcal{R}_2 \vee w = 0, \tag{4.8}$$

where $\mathcal{R}_i$ is $\mathcal{R}$ with R replaced with $R_i (i = 1, 2)$. Hence we proceed to solve the equation

$$\partial R_i + \eta R_i \vee \frac{1-k}{r} dr \pm \frac{1}{\hbar c}(E + \frac{Ze^2}{r})\eta R_i + \frac{1}{\hbar c} E_0 R_i = 0, \tag{4.9}$$

where the top and bottom signs correspond to R_1 and R_2 respectively.

Both R_1 and R_2 are sums of 0-form and 1-form that only depend on r and dr. For later comparison with equations in the physics literature, we write:

$$R_i = f(r)dr - g(r), \tag{4.10}$$

where the subscript i for f and g is understood but not made explicit because the sign $\pm$ to be used now makes it unnecessary.

When computing ∂R_i we encounter ∂dr, which we found to be $2/r$ (see eq. (3.9). We take ∂R_i into (4.9) and set equal to zero both the scalar part and the coefficient of dr. We obtain the two systems

$$\frac{df}{dr} + \frac{1+k}{r} f = \frac{1}{\hbar c}[E_0 \pm (E + \frac{Ze^2}{r})]g, \tag{4.11}$$

$$\frac{dg}{dr} + \frac{1-k}{r} g = \frac{1}{\hbar c}[E_0 \mp (E + \frac{Ze^2}{r})]f, \tag{4.12}$$

respectively for the upper and lower signs. For each of the two systems, we have the sign plus in front of k in the first equation and the minus sign in the second equation. This difference arises from the action of η on the 0-form and 1-form parts of each of the R_i's, and similarly for the change from $\pm$ to $\mp$. The two systems corresponds to the two solutions

$$u = [f(r)dr - g(r)] \vee S_k^m \vee T^- \tag{4.13}$$

with f and g solutions of the first system, and

$$u = [f(r)dr - g(r)] \wedge S_k^m \vee w \vee T^-, \tag{4.14}$$

with f and g solutions of the second system and where

$$T^- \equiv e^{-\frac{iEt}{\hbar c}} \epsilon^-. \tag{4.15}$$

Notice that we can obtain the (first) system

$$\frac{df}{dr} + \frac{1+k}{r} f = \frac{1}{\hbar c}[E_0 + (E + \frac{Ze^2}{r})]g, \tag{4.16}$$

$$\frac{dg}{dr} + \frac{1-k}{r} g = \frac{1}{\hbar c}[E_0 + (E + \frac{Ze^2}{r})]f, \tag{4.17}$$

from the second one

$$\frac{df}{dr} + \frac{1+k}{r} f = \frac{1}{\hbar c}[E_0 - (E + \frac{Ze^2}{r})]g, \tag{4.18}$$

$$\frac{dg}{dr} + \frac{1-k}{r} g = \frac{1}{\hbar c}[E_0 + (E + \frac{Ze^2}{r})]f, \tag{4.19}$$

by the exchange $(f, g) \to (g, f)$ and $k \to -k$. We use this fact to reduce the problem of solving two systems to solving just one by further observing that, up to the sign, we can also obtain this replacement $(f, g) \to (g, f)$ in (4.13) through multiplication by dr, i.e.

$$[f(r)dr - g(r)]dr = -[g(r)dr - f(r)]. \tag{4.20}$$

The factor -1 can be ignored since it is absorbed into the coefficient of S_k^m. And the change in sign of k is absorbed because $-\infty < k < \infty$, or $k = |k|, 0, -|k|$. Hence, we remove the subscript in (4.10) and use R to refer to R_2. We thus write the solutions (4.13) and (4.14) as

$$u = R \vee S_k^m \vee T^- \quad \text{and} \quad u = R \vee dr \vee w \vee S_k^m \vee T^-. \tag{4.21}$$

The system (4.18)-(4.19) is well known from the treatment of one-electron atoms with the Dirac equation, as per section 151 of volume XXXV of the Handbuch der Physik, or as the treatment in the book "Quantum Mechanics of One – and Two – electron Atoms" by E. E. Salpeter and H. Bethe. See also many other books on quantum mechanics (Notice that, in those references, the role of k is played by $-\varkappa$). We thus need not go further as the obtaining of the fine structure continues in the standard way.

Chapter 6

Conservation in Quantum Mechanics, and Beyond Hodge's Theorem

6.1 Introduction

In this chapter, we deal with results in physics and mathematics that arise from Kähler's first Green's identity for differential forms. On the left hand side of this identity, there is a exterior differentiation. If the right hand side is zero, so is the left hand side and, by Stoke's theorem, the conservation law of the exterior calculus follows. We shall see that, in the KC as in Dirac's theory, the right hand side of the Green identity that gives rise to the conservation law is quadratic in the wave function or, more precisely, linear in it and its conjugate. Of great interest is the specialization of this right hand side when we write the wave function as a sum of two parts, respectively associated with the two ideals in the algebra that are defined by time translation symmetry.

When the coupling is electromagnetic, what is conserved – aside from energy – is some magnitude which comes with both signs. The magnitude is conserved, not what each of the parts refers to. The obvious interpretation or the conserved magnitude is charge, not probability, which does not come with both signs and can thus only be a derived concept. This implies a radically new vision of quantum physics.

Another important application of Green's first identity is a uniqueness theorem for a differential form on a manifold (or region thereof) when both its exterior and interior derivatives are given, as well as the specification of the differential form at the boundary. This theorem is instrumental in obtaining a Helmholtz theorem for differential $r-$forms in Euclidean spaces E_n and in regions or r-surfaces thereof. This results immediately generalizes by embedding to any differentiable manifold, Riemannian ones in particular, that can be embedded in Euclidean spaces.

The generalization of Helmholtz theorem is at the same time a generalization of Hodge's decomposition theorem, since one not only derives the latter theorem, but one actually obtains their forms as integrals, in the guise of Helmholtz theorem. In the Helmholtz theorem of the vector calculus, as in its translation to differential forms and generalization to any grade of the differential form and any dimension of the Euclidean space, one of the two terms is closed and the other one is co-closed. The appearance in Hodge's theorem of a term – the harmonic one – is due to the fact that certain terms that cancel at the boundary under the more restricted conditions of applicability of Helmholtz

theorem no longer do so.

In conclusion, high power results in both physics and mathematics arise from Green's first identity of the Kähler calculus.

6.2 Green's identities

Kähler defines scalar products of different grades for arbitrary differential forms. For arbitrary differential forms, they are written as $(u, v)_r$, where r is the complement to n of the grade of the product. The notations $(u, v)_0$ and $(u, v)_1$ will be used to refer to scalar products of grades n and $n - 1$ respectively, which we are about to define. We shall confine ourselves to defining scalar products of those grades, which are the only ones that enter Kähler's Green identity. Due to the fact that there is no possibility for confusion, Kähler uses the symbol (u, v) for what we have momentarily called $(u, v)_0$, and reserves the subscript zero to refer to the 0-grade part of differential forms of, in general, "inhomogeneous grade".

Following Kähler, we shall use the symbol ζ for the operator that reverses the order of vectors in all products. The scalar product of grade n, or simply scalar product, is defined as

$$(u, v) = (\zeta u \vee v) \wedge z, \tag{6.1}$$

although, as Kähler himself points out, it is better to use the term scalar product to refer to the actual evaluation, i.e. integration, of the differential form (u, v). Since the right hand side of (6.1) obviously vanishes unless $\zeta u \vee v$ is a 0-form, it can be rewritten as

$$(u, v) = (\zeta u \vee v)_0 z = (\zeta u \vee v) \wedge z. \tag{6.2}$$

One similarly defines:

$$(u, v)_1 = e_i(dx^i \vee u, v) = e_i[(\zeta u \vee dx^i \vee v) \wedge z]. \tag{6.3}$$

We shall later use that

$$(u, v) = (v, u) \tag{6.4}$$

i.e. that $(\zeta u \vee v)_0$ equals $(v \vee \zeta u)_0$, which is trivial.

We shall also use that

$$de_i = d_i - e_i d. \tag{6.5}$$

Indeed

$$de_i u = de_i(dx^i \wedge u' + u'') = du' \tag{6.6}$$

$$d_i u = d_i(dx^i \wedge u' + u'') = dx^i \wedge d_i u' + d_i u'', \tag{6.7}$$

and

$$-e_i du = -e_i(-dx^i \wedge du' + du'') = du' - dx^i \wedge e_i du' - e_i du''. \tag{6.8}$$

But

$$e_i du' = d_i u', \qquad e_i du'' = d_i u''. \tag{6.9}$$

Using (6.9) in (6.8) and bringing (6.6), (6.7) and (6.8) together, we get (6.5).

We now prove Green's first identity, which reads

$$d(u, v)_1 = (\partial u, v) + (u, \partial v). \tag{6.10}$$

We differentiate (6.3), use (6.5) and the fact that the square bracket on which the operator d is acting is a differential n-form and that, therefore, its exterior derivative is zero. We thus have:

$$\begin{aligned} d(u,v)_1 &= de_i[(\zeta u \vee dx^i \vee v) \wedge z] = d_i[(\zeta u \vee dx^i \vee v) \wedge z] - \\ -e_i d[(\zeta u \vee dx^i \vee v) \wedge z] &= d_i[(\zeta u \vee dx^i \vee v) \wedge z]. \end{aligned} \tag{6.11}$$

But $d_i z = 0$. Hence

$$d(u,v)_1 = d_i(...) \wedge z = (d_i \zeta u) \vee dx^i \vee v \wedge z + (\zeta u \vee dx^i \vee d_i v) \wedge z, \tag{6.12}$$

where we have used the distributive property of the d_i operator and that d_i of dx_i is zero, where "..." stands for $\zeta u \vee dx^i \vee v$.

We now use that

$$(d_i \zeta u) \vee dx^i = (\zeta d_u u) \vee dx^i = \zeta (dx^i \vee d_i u) = \zeta \partial u. \tag{6.13}$$

The first term on the right hand side of (6.10) follows from the first term on the right hand side of (6.12). For the second term, just notice that $dx^i \wedge d_i v$ equals ∂v. End of proof.

A second Green identity results by first replacing first v and later u with ∂v and ∂u respectively. Thus

$$d(u, \partial v)_1 = (\partial u, \partial v) + (u, \Delta v), \tag{6.14}$$

$$d(v, \partial u)_1 = (\partial v, \partial u) + (v, \Delta u). \tag{6.15}$$

where Δ stands for $\partial\partial$. Subtracting (6.15) from (6.14), we get

$$(u, \Delta u) - (v, \Delta u) = d[(u, \partial v)_1 - (v, \partial u)_1]. \tag{6.16}$$

Needless to say that there are other Green identities, like, for instance, if we replace both u and v with ∂u and ∂v.

6.3 The two signs of charge

6.3.1 The conjugate Kähler equation

The first Green identity prompts us to find a conjugate Kähler equation such that its solutions v will give rise to a conservation law through scalar multiplication with the solutions u of the "direct Kähler equation". We seek it in the form $\partial u = bu$, and try to find b as a function of a. We do not yet assume electromagnetic coupling. We shall solve the equation

$$(u, \partial v) = -(v, \partial u). \tag{6.17}$$

We have

$$(u, \partial v) = (\partial v, u) = (bv, u) = [\zeta v \vee \zeta b \vee u] \wedge z = (v, \zeta b \vee u). \tag{6.18}$$

For (6.18) to become (6.17), we want

$$(\zeta b)u = -\partial u = -au. \tag{6.19}$$

Hence

$$b = -\zeta a, \tag{6.20}$$

and the conjugate Kähler equation therefore is

$$\partial u = -(\zeta a)u. \tag{6.21}$$

We have thus shown that, if u and v are respective solutions of a direct and its conjugate Kähler equation, then $d(u,v)_1 = 0$ because $(\partial u, v) + (u, \partial v) = 0$.

6.3.2 The electromagnetic conservation law

We now show that if u is a solution of the electromagnetic Kähler equation, $\eta\overline{u}$ is a solution of its conjugate equation. For electromagnetic coupling,

$$a = iE_0 + e\phi, \tag{6.22}$$

with $e = \mp|e|$ and with ϕ as the electromagnetic 1-form. Let overbar denote complex conjugation. Since $\partial\eta = -\eta\partial$ and $\eta\bar{a} = a = \zeta a$, we have

$$\partial(\eta\overline{u}) = -\eta\partial\overline{u} = -\eta\overline{\partial u} = -\eta(\overline{a} \vee \overline{u}) = -a \vee \eta u = -(\zeta a) \vee \eta u. \tag{6.23}$$

The conservation law then takes the form

$$d(u, \eta\overline{v})_1 = 0, \tag{6.24}$$

and, in particular,

$$d(u, \eta\overline{u})_1 = 0. \tag{6.25}$$

6.3.3 Computations with scalar products

We produce some formulas needed for the computation of $(u, \eta\overline{u})_1$ when we split u into members of complementary ideals associated with time translation symmetry.

Taking into account (6.4), the equation

$$(\zeta u, \zeta v) = (u, v) \tag{6.26}$$

readily follows since

$$(\zeta u, \zeta v) = (u \vee \zeta v)_0 z = (u \vee v)_0 z = (u, v). \tag{6.27}$$

(6.26) is used in

$$\begin{aligned}(u \vee w, v) = (w, \zeta u \vee v) = (\zeta u \vee v, w) = (\zeta(\zeta u \vee v), \zeta w) = \\ = (\zeta u \vee v \vee \zeta w)_0 z = (u, v \vee \zeta w),\end{aligned} \tag{6.28}$$

where the first step follows from the definition of scalar product of grade n; we have then used (6.4) and (6.26).

Since $\zeta(w,u) \vee v = \zeta u \vee (\zeta u \vee v)$, it readily follows that

$$(w \vee u, v) = (u, \zeta w \vee v), \tag{6.29}$$

which is in turn used to obtain

$$(dx^\mu \vee u, v) = (u, \zeta dx^\mu \vee v) = (u, dx^\mu \vee v) = (dx^\mu \vee v, u) \tag{6.30}$$

and, therefore,

$$(v, u)_1 = (u, v)_1. \tag{6.31}$$

We shall later need

$$(u \vee w, v)_1 = (u, v \vee \zeta w)_1, \tag{6.32}$$

which we prove as follows

$$\begin{aligned} (u \vee w, v)_1 &= e_\mu\{[\zeta(u \vee w) \vee dx^\mu \vee v] \wedge z\} = e_\mu(u \vee w, dx^\mu \vee v) \\ &= e_\mu(u, dx^\mu \vee v \vee \zeta w) = (u, v \vee \zeta w)_1, \end{aligned} \tag{6.33}$$

where we have used (6.28).

We are using Greek symbols to emphasize that we are not restricting ourselves to 3-space. We shall later use spacetime indices (Greek) and 3-space indices (Latin) in the same argument.

6.3.4 The current $(u, \overline{\eta v})_1$ in terms of elements of the ideals generated by $\varepsilon^\pm$

Recall

$$u = {}^+u \vee \epsilon^+ + {}^-u \vee \epsilon^-, \quad v = {}^+v \vee \epsilon^+ + {}^-v \vee \epsilon^-. \tag{6.34}$$

In the next few lines, let ϵ be ϵ^+ or ϵ^- but not both at the same time. Since ϵ is an idempotent, $\epsilon \vee \epsilon = \epsilon$. Also, $\zeta\epsilon = \epsilon$. Hence

$$(u \vee \epsilon, \eta\overline{v} \vee \overline{\epsilon})_1 = (u \vee \epsilon \vee \epsilon, \eta\overline{v} \vee \overline{\epsilon})_1 = (u \vee \epsilon, \eta\overline{v} \vee \overline{\epsilon} \vee \zeta\epsilon)_1 = 0 \tag{6.35}$$

where we have used (6.32) and that

$$\overline{\epsilon} \vee \zeta\epsilon = \overline{\epsilon} \vee \epsilon = 0 \tag{6.36}$$

since $\overline{\epsilon}^\pm = \epsilon^\mp$ and $\epsilon^+\epsilon^- = \epsilon^-\epsilon^+ = 0$.

In order to simplify notation, let us define

$$[u, v] = (u, \eta\overline{v})_1. \tag{6.37}$$

We shall now use (6.34), (6.35) and (.6.37) to obtain

$$[u, v] = [{}^+u \vee \epsilon^+, {}^+v \vee \epsilon^+] + [{}^-u \vee \epsilon^-, {}^-v \vee \epsilon^-]. \tag{6.38}$$

The ${}^+u$, ${}^-u$, ${}^+v$, ${}^-v$ are spatial differentials since the dt dependence of u and v has been replaced through $dt = (1/i)(\epsilon^+ - \epsilon^-)$. They are not "strict" since, in general, they will depend on t. Let such differentials be represented as p and q. The following then applies to both terms in (6.38)

$$4[p \vee \epsilon^\pm, q \vee \epsilon^\epsilon] = [p, q] \mp [p, q \vee idt] \mp [p \vee idt, q] + [p \vee idt, q \vee idt]. \tag{6.39}$$

Using (6.31), we readily prove that the first and second terms are respectively equal to the fourth and third terms. Hence

$$[p \vee \epsilon^\pm, q \vee \epsilon^\pm] = \frac{1}{2}[p, q] \ \mp p, q \vee idt]. \tag{6.40}$$

6.3.5 The emergence of the terms in the continuity equation

We proceed to develop $[p, q]$. For this purpose, we notice that $(\zeta p \vee dt \vee \eta\overline{q})_0 = 0$, since there is no dt factor in ζp and $\eta\overline{q}$. We also notice an alternative way of writing $(u, v)_1$

$$(u, v)_1 = e_\mu(dx^\mu \vee u, v) = e_\mu[(\zeta u \vee dx^\mu \vee v)_0 z] = (\zeta u \vee dx^\mu \vee v)_0 e_\mu z. \tag{6.41}$$

Let z and w represent the unit differential 4-form and 3-form respectively. Then

$$e_k z = e_k(w \wedge idt) = e_k w \wedge idt. \tag{6.42}$$

Hence

$$[p, q] = [(\zeta p \vee dx^k \vee \eta\overline{q})_0 e_k w] \wedge idt = \{p, \eta\overline{q}\}_1 \wedge idt, \tag{6.43}$$

where $\{p, \eta\overline{q}\}_1$ is the symbol used to represent the scalar product of grade $n-1$ in the Kähler algebra for $n = 3$.

We similarly have

$$\begin{aligned}[p, q \vee idt] &= (\zeta p \vee dt \vee \eta\overline{q} \vee idt)_0 e_t z = -i(\zeta p \vee dt \vee dt \vee \overline{q})_0 wi = \\ &= -(\zeta p \vee \overline{q})_0 w = -\{p, \overline{q}\}.\end{aligned} \tag{6.44}$$

Notice the presence of the first dt, instead of dx^i, inside the parenthesis, based on the same type of argument as before.

In view of (6.43) and (6.44), Eq. (6.39) now reads

$$2[p \vee \varepsilon^\pm, q \vee \varepsilon^\pm] = \{p, \eta\overline{q}\}_1 \wedge idt + pm\{p, \overline{q}\}. \tag{6.45}$$

We next make $q = p = {}^+u$ in (6.46) and obtain

$$[{}^+u \vee \varepsilon^+, {}^+u \vee \varepsilon^+] = \frac{1}{2}\{{}^+u, {}^+\overline{u}\} + \frac{1}{2}\{{}^+u, \eta^+ u\}_1 \wedge idt. \tag{6.46}$$

Next we make $q = p = {}^-u$ and obtain an equation almost equal in form to (6.46), except that, in addition to the replacement ${}^+ \to {}^-u$, there will be a change in sign in one of the terms. Thus

$$\begin{aligned}[u, u] &= [{}^+u \vee \varepsilon^+, {}^+u \vee \varepsilon^+] + [{}^-u \vee \varepsilon^-, {}^-u \vee \varepsilon^-] = \\ &= \frac{1}{2}\{{}^+u, {}^+\overline{u}\} + \frac{1}{2}\{{}^+u,\ \eta^+\overline{u}\}_1 \wedge idt \\ &- \frac{1}{2}\{{}^-u, {}^-\overline{u}\} + \frac{1}{2}\{{}^-u,\ \eta^-\overline{u}\}_1 \wedge idt.\end{aligned} \tag{6.47}$$

Each of the two lines has the form of a scalar-valued space time current where the spatial 3-forms are volume densities, ρw, and where the spacetime 3-forms are the currents in the sense similar to the "vector current".

In section (§15) of his 1962 paper, Kähler had already made the remark that, when the metric is positive definite, the product (u, u) for arbitrary u is a number that is everywhere ≥ 0 times the volume differential, and it is positive definite at every point P where $u(P) \neq 0$. In (6.47), the metric at work is the Euclidean metric and, therefore, both $\{{}^+u, {}^+\overline{u}\}$ and $\{{}^-u, {}^-\overline{u}\}$ are nowhere negative.

In view of the foregoing considerations and of Eq. (6.47), Kähler concludes that the characterization of the negative electrons state differential form u by $u \vee \varepsilon^- = u$, $u \vee \varepsilon^+ = 0$ brings about a density

$$\rho w = -\frac{|e|}{2}\{{}^-u, {}^-u\}$$

with $\rho \leq 0$ everywhere.

This is a tremendously important result for the foundations of quantum mechanics. It shows that, in Kähler's theory, the wave "function" is not a probability amplitude but, so to speak, a "charge amplitude".

6.4 A uniqueness for differential $k-$forms of definite grade under Helmholtz type conditions

Let R be a differentiable manifold and let ∂R be its boundary. Let (u_1, u_2) be differential $k-$forms in R such that $du_1 = du_2$, $\delta u_1 = \delta u_2$ on R, and that u_1 equals u_2 on ∂R. The uniqueness theorem states that the differential form is uniquely defined.

β defined as $u_1 - u_2$ satisfies

$$d\beta = 0 = \delta\beta \text{ on } R, \qquad \beta = 0 \text{ on } \partial R, \tag{6.48}$$

and, locally,

$$(\beta = d\alpha,\ \delta d\alpha = 0) \quad \text{on } R, \qquad d\alpha = 0 \quad \text{on } \partial R. \tag{6.49}$$

Equation (6.10) with $u = \alpha$ and $v = d\alpha$ reads

$$d(\alpha, d\alpha)_1 = (\alpha, \partial d\alpha) + (d\alpha, \partial\alpha). \tag{6.50}$$

We use (6.49) to obtain

$$(\alpha, \partial d\alpha) = (\alpha, dd\alpha) + (\alpha, \delta d\alpha) = 0 + 0. \tag{6.51}$$

Consider next $(d\alpha, \partial\alpha)$. If α is of definite grade, so are $d\alpha$ and $\delta\alpha$, but their grades differ by two units. Their scalar product is, therefore, zero. On the other hand, we have, with a_A defined by $d\alpha = a_A dx^A$ (with summation over the algebra as a module),

$$(d\alpha, \partial\alpha) = (d\alpha, \delta\alpha) + (d\alpha, d\alpha) = 0 + \sum |a_A|^2 . \tag{6.52}$$

Substituting (6.51) and (6.52) in (6.50), applying Stokes theorem and using $d\alpha = 0$ on R, we get

$$\int_R \sum |a_A|^2 = \int_R d(\alpha, d\alpha)_1 = \int_{\partial R} (\alpha, d\alpha)_1 = 0. \tag{6.53}$$

Hence all the a_R's are zero in R itself and so is, therefore, α and β $(=d\alpha)$. It follows from the defintion of β as $u_1 - u_2$ that $u_1 = u_2$. The theorem has thus been prooved.

6.5 Helmholtz Theorems for $k-$forms

6.5.1 Helmholtz Theorem for $k-$forms in E_3

In this section, we shall try to avoid potential confusion by replacing the symbol z with the symbol w for the unit differential 3−form.

With $r_{12} \equiv [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$, the standard Helmholtz theorem of the vector calculus states

$$\mathbf{v} = -\frac{1}{4\pi}\nabla \int_{E_3'} \frac{\nabla' \cdot \mathbf{v}(\mathbf{r}')}{r_{12}} dV' + \frac{1}{4\pi}\nabla \times \int_{E_3'} \frac{\nabla' \times \mathbf{v}(\mathbf{r}')}{r_{12}} dV'. \tag{6.54}$$

has an immediate translation to the Helmholtz theorem for differential 1−forms in E_3. It reads

$$\alpha = -\frac{1}{4\pi} d \int_{E_3'} \frac{(\delta' \alpha') w'}{r_{12}} - \frac{1}{4\pi} \delta \left(dx^j dx^k \int_{E_3'} \frac{d' \alpha' \wedge dx'^i}{r_{12}} \right), \tag{6.55}$$

This theorem is a particular case of the theorem proved in the next subsection. It implies a similar theorem for differential 2−forms, as follows from using that the equation $\alpha = w\beta$, uniquely defines β. We substitute it in (6.55) and solve for β:

$$\beta = \frac{1}{4\pi} w d \left(\int_{E_3'} \frac{\delta'(w'\beta')}{r_{12}} w' \right) + \frac{1}{4\pi} w \delta \left(dx^{jk} \int_{E_3'} \frac{d'(w'\beta') \wedge dx'^i}{r_{12}} \right). \tag{6.56}$$

Denote the first integral in (6.56) as I and the second one as I^i. We have $wdI = \delta(wI)$ and $w'\delta'(w'\beta') = w'(w'd\beta') = -d\beta' = -d\beta' \wedge 1$. The exterior product by 1 is superfluous, except for the purpose for making later Eq. (6.60) clear. Similarly, $w\delta(dx^{jk} I^i) = d(w dx^{jk} I^i) = -d(dx^i I^i)$ and

$$d'(w'\beta') \wedge dx'^i = dx'^i \wedge d'(w'\beta') = \frac{1}{2} \left[dx'^i w' \delta' \beta' + w' \delta' \beta' dx'^i \right] =$$

$$= \frac{1}{2} (dx'^{jk} \delta' \beta' + \delta' \beta' dx'^{jk}) = dx'^{jk} \wedge \delta' \beta' = \delta' \beta' \wedge dx'^{jk}. \tag{6.57}$$

We use these results in (6.56), change the order of the terms and get

$$\beta = -\frac{1}{4\pi} d \left(dx^i \int_{E_3'} \frac{\delta' \beta' \wedge dx'^{jk}}{r_{12}} \right) - \frac{1}{4\pi} \delta \left(w \int_{E_3'} \frac{\delta' \beta' \wedge 1}{r_{12}} \right), \tag{6.58}$$

Write the first term in (6.55) as

$$-\frac{1}{4\pi} d \left[1 \wedge \int_{E_3'} \frac{(\delta' \alpha') \wedge w'}{r_{12}} \right]. \tag{6.59}$$

Let the index A label a Cartesian basis of the algebra as module. Let $dx^{\bar{A}}$ be the unique element in the basis such that $dx^A \wedge dx^{\bar{A}} = w$. Define $\int_{E_3} \gamma_r$ if the grade r of γ is different from 3. All four terms on the right of (6.55) and (6.58) are thus of the form

$$-\frac{1}{4\pi} d \left[dx^A \int_{E_3'} \frac{(\delta' \_\_) \wedge dx'^{\bar{A}}}{r_{12}} \right] \quad \text{or} \quad -\frac{1}{4\pi} \delta \left[dx^A \int_{E_3'} \frac{(d' \_\_) \wedge dx'^{\bar{A}}}{r_{12}} \right]. \tag{6.60}$$

Take, for instance, the first of the two expressions in (6.60). We sum over all A, equivalently, over all $\bar{A}'$. The grade of $(\delta' \_\_)$ determines the grade of the only $dx'^{\bar{A}}$ that may yield not zero integral since the sum of the respective grades must be 3. For each surviving value of the index $\bar{A}$, the value of the index A —thus the specific dx^A at the front of the integral— is determined. We shall later show for ulterior generalization that we may replace the Cartesian basis with any other basis, which we shall choose to be orthonormal since they are the "canonical ones" of Riemannian spaces.

6.5.2 Helmholtz Theorem for Differential k-forms in E_n

Let ω^A ($\equiv \omega^{i_1}\omega^{i_2}...\omega^{i_r}$) denote elements of a basis in the Kähler algebra of differential forms such that the ω^μ are orthonormal. The purpose of using an orthonormal basis is that exterior products can be replaced with Clifford products. Let $\omega^{\bar{A}}$ be the monomial (uniquely) defined by $\omega^A\omega^{\bar{A}} = z$, with no sum over repeated indices.

The generalized Helmholtz theorem in E_n reads as follows

$$\alpha = -\frac{1}{(n-2)S_{n-1}}[d(\omega^A I^\delta_A) + \delta(\omega^A I^d_A)], \tag{6.61}$$

with summation over a basis in the algebra and where

$$I^\delta_A \equiv \int_{E'_n} \frac{(\delta'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{n-2}}, \qquad I^d_A \equiv \int_{E'_n} \frac{(d'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{n-2}}. \tag{6.62}$$

r_{12} is defined by $r_{12}^2 = (x_1 - x'_1)^2 + ... + (x_n - x'_n)^2$ in terms of Cartesian coordinates.

It proves convenient for performing differentiations to replace ω^i, ω^A and $\omega^{\bar{A}}$ with dx^i, dx^A and $dx^{\bar{A}}$. If the results obtained are invariants, one can re-express the results in terms of arbitrary bases.

We proceed again via the uniqueness theorem, as in the vector calculus, with specification now of $d\alpha$, $\delta\alpha$ and that α goes sufficiently fast at ∞. vanishing of α at infinity. Because of the annulment of dd and $\delta\delta$, the proof reduces to showing that $\delta d(dx^A I^\delta_A)$ and $d\delta(dx^A I^d_A)$ respectively yield $\delta\alpha$ and $d\alpha$, up to the factor at the front in (6.61). Since the treatment of both terms is the same, we shall carry them in parallel, as in

$$\begin{pmatrix} \delta \\ d \end{pmatrix} \alpha \to \begin{pmatrix} \delta d \\ d\delta \end{pmatrix} dx^A I_A^{\binom{\delta}{d}} = \partial\partial dx^A I_A^{\binom{\delta}{d}} - \begin{pmatrix} d\delta \\ \delta d \end{pmatrix} dx^A I_A^{\binom{\delta}{d}}. \tag{6.63}$$

In the first term on the right hand side of (6.63), we move $\partial\partial$ to the right of dx^A, insert it inside the integral with primed variables, multiply by $-\frac{1}{(n-2)S_{n-1}}$ and treat the integrand as a distribution. We easily obtain that the first term yields $\binom{\delta\alpha}{d\alpha}$.

For the last term in (6.63), we have

$$\begin{pmatrix} d\delta \\ \delta d \end{pmatrix} dx^A I_A^{\binom{\delta}{d}} = \begin{pmatrix} d\left[dx^i \cdot dx^A \frac{\partial I^\delta_A}{\partial x^i}\right] \\ \delta\left[(\eta dx^A) \wedge dx^i \frac{\partial I^d_A}{\partial x^i}\right] \end{pmatrix}. \tag{6.64}$$

For the first line in (6.64), we have used that $d_h u = \frac{\partial}{\partial x^h}$ in Cartesian coordinates, and that $\delta u = dx^h \cdot d_h u$. For the development of the second line, we have used the Leibniz rule.

We use the same rule to also transform the first line in (6.64),

$$d\left(dx^i \cdot dx^A \frac{\partial I^\delta_A}{\partial x^i}\right) = [\eta(dx^i \cdot dx^A)] \wedge dx^l \frac{\partial^2 I^\delta_A}{\partial x^l \partial x^i} = (dx^A \cdot dx^i) \wedge dx^l \frac{\partial^2 I^\delta_A}{\partial x^l \partial x^i}. \tag{6.65}$$

For the second line, we get

$$\delta\left[(\eta dx^A) \wedge dx^i \frac{\partial I^d_A}{\partial x^i}\right] = dx^l \cdot \left[\frac{\partial^2 I^d_A}{\partial x^l \partial x^i}(\eta dx^A) \wedge dx^i\right]. \tag{6.66}$$

We shall use here that

$$dx^l[(\eta dx^A) \wedge dx^i] = -\eta[\eta(dx^A \wedge dx^i)] \cdot dx^l = (dx^A \wedge dx^i) \cdot dx^l, \tag{6.67}$$

thus obtaining

$$\delta\left[(\eta dx^A) \wedge dx^i \frac{\partial I^d_A}{\partial x^i}\right] = (dx^A \wedge dx^i) \cdot dx^l \frac{\partial^2 I^d_A}{\partial x^l \partial x^i}. \tag{6.68}$$

Getting (6.65) and (6.66) into (6.64), we obtain

$$\begin{pmatrix} d\delta \\ \delta d \end{pmatrix} dx^A I_A^{\binom{\delta}{d}} = \left[dx^A(\overset{\cdot}{\wedge})dx^i\right](\overset{\wedge}{\cdot})dx^l \int_{E'_n} \frac{\partial^2}{\partial x'^i \partial x'^l} \frac{1}{r_{12}^{n-2}} \begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}. \tag{6.69}$$

Integration by parts with respect to x'^i yields two terms. The total differential term is

$$\left[dx^A(\overset{\cdot}{\wedge})dx^i\right](\overset{\wedge}{\cdot})dx^l \int_{E'_n} \frac{\partial}{\partial x'^i}\left[\left(\frac{\partial}{\partial x'^l}\frac{1}{r_{12}^{n-2}}\right)\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}\right]. \tag{6.70}$$

Application of Stokes theorem yields

$$\left[dx^A(\overset{\cdot}{\wedge})dx^i\right](\overset{\wedge}{\cdot})dx^l \int_{\partial E'_n} \left(\frac{\partial}{\partial x'^l}\frac{1}{r_{12}^{n-2}}\right)\left\{dx'^i \cdot \left[\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}\right]\right\}, \tag{6.71}$$

where we have indulged in the use of parentheses for greater clarity. This term is null if the differentiations of α go sufficiently fast to zero at infinity.

The other term resulting from the integration by parts is

$$-\left[dx^A(\overset{\cdot}{\wedge})dx^i\right](\overset{\wedge}{\cdot})dx^l \int_{E'_n} \left(\frac{\partial}{\partial x'^l}\frac{1}{r_{12}^{n-2}}\right)\frac{\partial}{\partial x'^i}\left(\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}\right). \tag{6.72}$$

This is zero because of cancellations that take place in groups of three different indices, as shown in the next subsection.

In terms of Cartesian bases, we have, on the top line of the left hand side of (6.63)

$$dx^A \int_{E'_n} \frac{(\delta'\alpha') \wedge dx'^{\bar{A}}}{r_{12}^{n-2}}. \tag{6.73}$$

It is preceded by invariant operators, which we may ignore for present purposes. We move dx^A inside the integral, where we let $(\delta'\alpha')_A$ be the notation for the coefficients of $\delta'\alpha'$. We thus have, for that first term,

$$\int_{E'_n} \frac{dx^A \wedge [(\delta'\alpha')_A dx'^A] \wedge dx'^{\bar{A}}}{r_{12}^{n-2}}. \tag{6.74}$$

The numerator can be further written as $(\delta'\alpha')_A dx^A z'$. It is clear that z and $(\delta'\alpha')_A dx'^A$ are invariants, but not immediately clear that $(\delta'\alpha')_A dx^A$ also is so. Whether we have the basis dx^A or dx'^A as a factor is immaterial. since the invariance of $(\delta'\alpha')_A dx'^A$ can be seen as following from the matching of the transformations of $(\delta'\alpha')_A$ and dx'^A each in accordance with its type of covariance. The same matching applies if we replace ω'^A with dx^A, since dx'^A and dx^A transform in unison.

We have shown that (6.61)-(6.62) constitutes the decomposition of α into closed and co-closed differential forms. It solves the problem of integrating the system $d\alpha = \mu$, $\delta\alpha = \nu$, for given μ and ν, and with the stated boundary condition

6.5.3 Identical vanishing of some integrals

As we are about to show, expressions (6.72) cancel identically (Notice that (6.71) cancels at infinity for fast vanishing; identical vanishing is not needed).

Consider the first line in (6.72). Let α be of grade $h \geq 2$ (If h were one, the dot product of dx^A with dx^i would be zero). Let p and q be a specific pair of indices in a given term in α, i.e. in its projection $a'_{pqC,pq}\, dx^A$ upon some specific basis element dx^A. Such a projection can be written as

$$(a'_{pqC} dx'^p \wedge dx'^q \wedge dx'^C,$$

where dx'^A is a unit monomial differential 1−form (there is no sum over repeated indices. We could also have chosen to write the same term as

$$(a'_{qpC} dx'^q \wedge dx'^p \wedge dx'^C,$$

with $a'_{qpC} = -a'_{pqC}$. Clearly, dx'^C is uniquely determined if it is not to contain dx^p and dx^q. We then have

$$\delta'(a_{'pqC} dx'^p \wedge dx'^q \wedge dx'^C) = a'_{pqC,p}\, dx'^q \wedge dx'^C - a_{pqC,q}\, dx'^p \wedge dx'^C. \tag{6.75}$$

The two terms on the right are two different differential 2−forms. They enter two different integrals, corresponding to $dx'^q \wedge dx'^C$ and $dx'^p \wedge dx'^C$ components of $\delta'\alpha'$. To avoid confusion, we shall refer to the basis elements in the integrals as dx'^B since they are $(h-1)$−forms, unlike the dx'^A of (6.75), which are differential h−forms

When taking the first term of (6.75) with $i = p$ into the top line of (6.72), the factor at the front of the integral is

$$-(dx^B \cdot dx^p) \wedge dx^l.$$

But this factor is zero since dx^B is $dx^q \wedge dx^C$, which does not contain dx^P as a factor. Hence, for the first term, we need only consider $i = q$. By the same argument, we need only consider $i = q$ for the second term in (6.75). Upon multiplying the dx'^B's by pertinent dx'^B's, we shall obtain the combination

$$(a'_{pqC,pq} - a'_{pqC,qp})z'$$

as a factor inside the integral for the first line of (6.72). We could make this statement because the factor outside also is the same one for both terms: $(dx^q \wedge dx^C) \cdot dx^q$ and $(dx^p \wedge dx^C) \cdot dx^p$ are equal. The contributions arising from the two terms on the right hand side of (6.75) thus cancel each other out. We would proceed similarly with any other pair of indices, among them those containing either p or q. The annulment of the top line of (6.72) has been proved.

In order to prove the cancellation of the second line in (6.72), the following considerations will be needed. A given dx^A determines its corresponding $dx'^{\bar{A}}$, and vice versa. It follows then that only the term proportional to dx'^A in $d'\alpha'$ exterior multiplies $dx'^{\bar{A}}$, which is of the same grade as $d'\alpha'$, i.e. $h+1$. Hence $dx^A \wedge dx^i$ is of grade 3 or greater for $h > 0$. If $dx^A \wedge dx^i$ is not to be null, dx^i cannot be in dx^A. Hence, $dx'^{\bar{A}}$ contains dx^i as a factor.

Let (p, q, r) be a triple of three different indices in $dx^A \wedge dx^i$. When i is p or q or r, the respective pairs (q, r), (r, p) and (p, q) are in dx^A. We may thus write

$$dx'^A = dx'^C \wedge dx'^q \wedge dx'^r, \qquad dx'^{\bar{A}} = dx'^p \wedge dx'^B. \tag{6.76}$$

The coefficient of dx'^A in $d'\alpha'$ will be the sum of three terms, one of which is

$$(a'_{Cr,q} - a'_{Cq,r})dx'^q \wedge dx'^C \wedge dx'^r, \tag{6.77}$$

and the other two are cyclic permutations. We partial-differentiate (6.77) with respect to dx'^p and multiply by $dx'^p \wedge dx'^B$ on the right. We proceed similarly with $i = q$ and $i = r$, and add all these contributions. We thus get

$$(a'_{Cr,qp} - a'_{Cq,rp} + a'_{Cp,rq} - a'_{Cr,pq} + a'_{Cq,pr} - a'_{Cp,qr})z'. \tag{6.78}$$

By virtue of equality of second partial derivatives, terms first, second and third inside the parenthesis cancel with terms fourth, fifth and sixth. To complete the proof, we follow the same process with another dx'^C and the same triple (p, q, r) until we exhaust all the options. We then proceed to choose another triple and repeat the same process until we are done with all the terms, which completes the proof of identical vanishing of the second term arising from one of the two integrations by parts of the previous subsection.

6.6 Hodge's Theorems

The "beyond" in the title of this chapter responds to the fact that we shall be doing much more than reproducing Hodge's theorem. As is the case with Helmholtz theorem, we are able to specify in terms of integrals what the different terms are.

We shall later embed Riemannian spaces R_n in Euclidean spaces E_N, thus becoming $n-$ surfaces. As an intermediate step, we shall apply the traditional Helmholtz approach to regions of Euclidean spaces, i.e. R_n's ab initio embedded in E_n. The harmonic form —which is of the essence in Hodge's theorem— emerges from the Helmholtz process in the new venues.

6.6.1 Transition from Helmholtz to Hodge

Though visualization is not essential to follow the argument, it helps for staying focused. For that reason, we shall argue in 3-D Euclidean space. It does not interfere with the nature of the argument.

On a region R of E_3, including the boundary, define a differential $1-$form or $2-$form α. Let A denote any continuously differentiable prolongation of α that vanish sufficiently fast at infinity. On R, we have $dA = d\alpha$ and $\delta A = \delta\alpha$. We can apply Helmholtz theorem to the differential forms A. In order to minimize clutter, we write it in the form

$$\begin{aligned} -4\pi A &= d... \int_{R'} \frac{\delta' A'...}{r_{12}} + \delta \int_{R'} ... \frac{d' A'...}{r_{12}} + \\ &\quad + d... \int_{E'_3 - R'} \frac{\delta' A'...}{r_{12}} + \delta... \int_{E'_R - R'} \frac{d' A'...}{r_{12}}, \end{aligned} \tag{6.79}$$

where $r_{12} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$. We shall keep track of the fact, at this point obvious, that in the first two integrals on the right, r' is in R'. It is outside R' in the other two integrals, which will depend on the prolongation. By representing those terms simply as $\mathcal{F}$, we have

$$-4\pi A = d... \int_{R'} \frac{\delta' \alpha'...}{r} + \delta... \int_{R'} \frac{d' \alpha'...}{r} + \mathcal{F}. \tag{6.80}$$

Since these equations yield A everywhere in E_3 (i.e. r not limited to R), they yield in particular what A and $\mathcal{F}$ are in R . We can thus write

$$-4\pi\alpha = d... \int_{R'} \frac{\delta'\alpha'...}{r} + \delta... \int_{R'} \frac{d'\alpha'...}{r} + \mathcal{F}, \tag{6.81}$$

$\mathcal{F}$ not having changed except that $\mathcal{F}$ in (6.81) refers only to what it is in R but it remains a sum of integrals in $E_3' - R$. The prolongations will be determined as different solutions of a differential system to be obtained as follows.

By following the same process as in Helmholtz theorem, we obtain, in particular,

$$-4\pi d\alpha = d\delta... \int_{R'} \frac{d'\alpha'...}{r} + d\mathcal{F}, \tag{6.82}$$

and similarly for $-4\pi\delta\alpha$ (just exchange d and δ).

Now, the first term on the right hand side of (6.82) will not become simply $-4\pi d\alpha$ as was the case in the previous section. It will yield two terms. One of them is $-4\pi d\alpha$, and the other one is made to cancel with $d\mathcal{F}$, thus determining a differential equation to be satisfied by $\mathcal{F}$. To this we have to add another differential equation arising from application of δ to (6.81). Together they determine the differential system to be determined by $\mathcal{F}$. Thus $-4\pi\alpha$ will be given by the three term decomposition (6.81). Notice that, in the process, we avoid integrating over $E_3' - R'$ and instead solving a differential system in R, since the left hand side and the first term on the right hand side of (6.82) pertain to α.

From now one, we shall make part of the theorems that the prolongations are solutions of a certain differential systems, later to be made explicit.

6.6.2 Hodge theorem in regions of $\mathbf{E}_n$

Let α be a differential $k-$form satisfying the equations $d\alpha = \mu$ and $\delta\alpha = \nu$, and given at the boundary of a region of E_n. We proceed to integrate this system. (6.81) now reads

$$-(n-2)S_{n-1}\alpha = d\left[\int_{R'} \frac{\delta'\alpha'...}{r_{12}}\right] + \delta\left[\int_{R'} \frac{d'\alpha'...}{r_{12}}\right] + \mathcal{F}, \tag{6.83}$$

where R is a region of Euclidean space that contains the origin and where r_{12} is the magnitude of the Euclidean distance between hypothetical points of components $(x, y, ...u, v)$.and $(x', y', ...u', v')$, all the coordinates chosen as Cartesian to simplify visualization. We said hypothetical because the interpretation as distance only makes sense when we superimpose E_n and E_n'.

When we apply either d or δ to (6.83), we shall use, as before, $d\delta + \delta d = \partial\partial$, with one of the terms on the left moved to the right ($d\delta = ...$, $\delta d = ...$ respectively). By developing the $\partial\partial$ term, it becomes the same as term on the right (i.e $d\alpha$ or $\delta\alpha$). It will cancel with the term on the left. The terms that vanished identically also vanish now, precisely because this is an identical vanishing. We are thus left with the total differential terms. If apply Stokes theorem, as before. these terms no longer disappear at the boundary. Hence, we are left with the two equations

$$\left[dx^A(\dot{\wedge})dx^i\right](\dot{\wedge})dx^l \int_{R'} \left(\frac{\partial \frac{1}{r_{12}^{n-2}}}{\partial x'^l}\right) dx'^i \cdot \left[\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}\right] + \begin{pmatrix} d \\ \delta \end{pmatrix} \mathcal{F} = 0 \tag{6.84}$$

(Refer to (6.81)). Hence, the solution to Helmholtz problem is given by the pair of equations (6.83)-(6.84).

We shall now show that $\mathcal{F}$ is harmonic, i.e. $(d\delta + \delta d)\mathcal{F} = 0$..We shall apply δ and d to the first and second lines of (6.94). Start by rewriting the first terms in (6.93) in the form, (6.80), they took before applying Stoke's theorem. Upon applying the δ operator to the first line, we have, for $\delta d\mathcal{F}$,

$$dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^l] \int_{R'} \frac{\partial^2}{\partial x'^h \partial x'^i} \left[\left(\frac{\partial}{\partial x'^l} \frac{1}{r_{12}^{n-2}} \right) \begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}} \right]. \tag{6.85}$$

Since this term happens to vanish, the computation will take place up to the factor -1, provided it is common to all terms in a development into explicit terms. We do so because (6.85) will be shown to vanish identically.

For $dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^l]$ to be different from zero, h and i must be different and contained in A. Since dx^l is not in dx^A, the product $dx^h \cdot dx^l$ is zero. Hence

$$dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^l] = [dx^h \cdot (dx^A \cdot dx^i)] \wedge dx^l]. \tag{6.86}$$

We can always write dx^A as

$$dx^h \wedge dx^j \wedge dx^C \wedge dx^i. \tag{6.87}$$

This is antisymmetric in the pair (i, h), which combines with the symmetry inside the integral to annul this term. Notice that we did not have to assign specific values for (i, h), but we had to "go inside" dx^A. We mention this for contrast with the contents for the next paragraph. We have proved so far that $\delta d\mathcal{F} = 0$.

We rewrite the left hand side of (6.84) as in (6.70) and proceed to apply d to it. We shall now have

$$dx^h \wedge [(dx^A \wedge dx^i) \cdot dx^l] \int_{R'} \frac{\partial^2}{\partial x'^h \partial x'^i} \left[\left(\frac{\partial}{\partial x'^l} \frac{1}{r_{12}^{n-2}} \right) d'\alpha' \wedge dx'^{\bar{A}} \right]. \tag{6.88}$$

It is clear that, when l takes a value different from the value taken by i, we again have cancellation due to the same combination of antisymmetry-symmetry as before. But the terms $dx^i \cdot dx^l$ would seem to interfere with the argument, but it does not. We simply have to be more specific than before with the groups of terms that we put together. We put together only terms where we have $dx^r \wedge dx^s$ arising from ($h = r$, $i = s$) and ($h = s$, $i = r$). When the running index l takes the values r or s, the resulting factor at the front of the integral will belong to a different group. We have thus shown that (6.88) cancels out and, therefore, $d\delta\mathcal{F} = 0$. To be precise, we have not only proved that $\mathcal{F}$ is harmonic, but that it is "hyper-harmonic", meaning precisely that: $\delta d\mathcal{F} = 0$ and $d\delta\mathcal{F} = 0$.

6.6.3 Hodge's theorem for hypersurfaces of $\mathbf{E}_N$

A manifold embedded in a Euclidean space of the same dimension will be called a region thereof. A hypersurface is a manifold of dimension n embedded in a Euclidean space E_N where $N > n$. The treatment here is the same as in subsection 6.1, the hypersurface playing the role of the region. The only issue that we need to deal with is a practical one having to do with the experience of readers. Helmholtz magnificent theorem belongs to an epoch where vector (and tensor) fields often took the place of differential forms. This can prompt false ideas as we now explain.

Let $\mathbf{v}$ be a vector field $\mathbf{v} \equiv a^{\lambda}(u,v)\hat{\mathbf{a}}_{\lambda}$ $(\lambda = 1,2)$ on a surface $x^{i}(u,v)$ $(i = 1,2,3)$ embedded in E_3, the frame field $\hat{\mathbf{a}}_{\lambda}$ being orthonormal. It can be tangent or not tangent. By default, the vector field is zero over the remainder of E_3. In its present form , Helmholtz theorem would not work for this field since the volume integrals over E_3 would be zero. This is a spurious implication because the theorem should be about algebras of differential forms, not tangent spaces.

Let μ be the differential $1-$form $a_{\lambda}(u,v)\hat{\omega}^{\lambda}$, the basis $\hat{\omega}^{\lambda}$ being dual to the constant orthonormal basis field $\mathbf{a}_{\lambda}$. This duality yields $a_{\lambda} = a^{\lambda}$. No specific curve is involved in the definition of μ, which is a function of curves, function determined by its coefficients $a_{\lambda}(u,v)$ The specification of a vector field on a surface, $\mathbf{v}$, on the other hand needs to make reference to a surface for its definition. And yet the components of $d\mu$ and $\delta\mu$ (which respectively are a $2-$form and a $0-$form) enter non-null volume integrals, which pertain to $3-$forms . The fact that most components (in the algebra) of an $k-$form are zero is totally irrelevant. The Helmholtz theorem for, say, a differential $1-$form μ can be formulated in any sufficiently high dimensional Euclidean space regardless of whether the "associated" vector field $\mathbf{v}$ is zero outside some surface.

Similarly, Helmholtz theorem for a differential $n-$form in E_N involves the integration of differential $N-$forms, built upon the interior differential $(n-1)-$form and the exterior differential $(n+1)-$form. In considering simple examples (say a plane in $3-$space), one can be misled or confused if one does not take into account the role of $1/r$, or else we might be obtaining an indefinite integral. Assume finite $\int \lambda(x,y)dx\wedge dy$ when integrating over the xy plane. The integral $\int \lambda(x,y)dx \wedge dy \wedge dz$ would be divergent, but need not be so if there is some factor that goes to zero sufficiently fast at infinity of z and $-z$.

6.6.4 Helmholtz-Hodge's and Hodge's theorem for Riemannian spaces

We shall consider a Helmholtz-Hodge extension of Hodge's theorem (i.e. a theorem of integration) and the standard Hodge theorem, which is a consequence of the former.

Consider now a differentiable manifold R_n endowed with a Euclidean metric. By the Schläfli-Janet-Cartan theorem [1],[2],[3], it can be embedded in a Euclidean space of dimension $N = n(n+1)/2$. Hence, a Helmholtz-Hodge theorem follows for orientable Riemannian manifolds that satisfy the conditions for application of Stokes theorem by viewing them as hypersurfaces in Euclidean spaces. At this point in our argument, the positive definiteness of the metric is required, or else we would have to find a replacement for the Laplacians considered in previous sections. The result is local, meaning non global, remark made in case the term local might send some physicists in a different direction. For clarity, the evaluation of the Laplacian now satisfies

$$1 = \frac{1}{(N-2)S_{N-1}} \int_{E_N} \partial\partial \frac{1}{r^{N-2}} z, \tag{6.89}$$

where r is the radial coordinate in $N-$dimensional space. Needless to say that it also applies to regions and hypersurfaces of E_N that contain the origin. As a consequence of the results in the previous subsections, we have the following.

Helmholtz-Hodge's theorem:

Hodge's theorem is constituted by Eqs. (6.90)-(6.91): For differential $k-$forms in

Riemannian spaces R_n

$$-(N-2)S_{N-1}\alpha = d\left[\omega^A \int_{R'_n} \frac{(\delta'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{N-2}}\right] + \delta\left[\omega^A \int_{R'_n} \frac{(d'\alpha') \wedge \omega'^{\bar{A}}}{r_{12}^{N-2}}\right] + \mathcal{F}, \qquad (6.90)$$

$$\begin{pmatrix} d \\ \delta \end{pmatrix} \mathcal{F} = -\left[dx^A(\dot{}\,\dot{\wedge})dx^i\right](\dot{\wedge}\,\dot{})dx^l \int_{R'_n} \left(\frac{\partial \frac{1}{r_{12}^{n-2}}}{\partial x'^l}\right) dx'^i \cdot \left[\begin{pmatrix} \delta'\alpha' \\ d'\alpha' \end{pmatrix} \wedge dx'^{\bar{A}}\right], \qquad (6.91)$$

with r_{12} being defined in any Euclidean space of dimension $N \geq n(n+1)/2$ where we consider R_n to be embedded.

As previously discussed, r_{12} represents a chord. We insist once more that $\omega'^{\bar{A}}$ is determined by the specific term in $\delta'\alpha'$ and $d'\alpha'$ that it multiplies. $\mathcal{F}$ is undetermined by solutions of the system $\delta\alpha = 0$, $d\alpha = 0$. So is, therefore α.

Hodge's theorem, as opposed to Helmholtz-Hodge theorem, is about decomposition. Hence, once again, uniqueness refers to something different from the uniqueness in the theorem of subsection (3.2), which refers to a differential system.

One might be momentarily tempted to now apply (6.90) to (6.91). We would get an identity, $\mathcal{F} = \mathcal{F}$, by virtue of the orthogonality of the subspace of the harmonic differential forms to the subspaces of closed and co-closed differential forms.

Hodge's theorem:

Any differential $k-$form, whether of homogeneous grade or not, can be uniquely decomposed into closed, co-closed and hyper-harmonic terms. For differential $k-$forms, the theorem is an immediate consequence of (6.100). For differential forms which are not of homogeneous grade, the theorem also applies because one only needs to add the decompositions of the theorem for the different homogeneous $k-$forms that constitute the inhomogeneous differential form.

This is obviously contained in Helmholtz-Hodge's theorem.

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PROJECTIVE, CLIFFORD AND GRASSMANN ALGEBRAS AS DOUBLE AND COMPLEMENTARY GRADED ALGEBRAS (PRELIMINARY VERSION)

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ABSTRACT. This article establishes the concept of projective algebra Λ_n and of Clifford double algebra Γ_n and compares both (a) to what is usually known as Grassmann algebra and (b) to Grassmann algebra in the approach of John Brown. [Bro12]

The 2^n-dimensional projective algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ and the 2^n-dimensional Clifford double algebra $\Gamma_n(+,\cdot,\ ,*)$ both carry the imprint of an algebra twice, i. e. they have a dual axiomatic structure. Projective algebra is the more fundamental concept than Clifford double algebra, since any Clifford double algebra also shows the structure of projective algebra whereas projective algebra is standing on its own.

John Browne used the term *Grassmann algebra* in [Bro12] to describe the body of algebraic theory and results based on Graßmann's *Ausdehnungslehre* from 1844 and from 1862. This Grassmann algebra shows a dual axiomatic structure as projective algebra and Clifford double algebra do. We will compare Grassmann algebra in the approach of John Browne with the complementary graded projective algebra Λ_n and the Clifford double algebra Γ_n.

1. INTRODUCTION

The purpose of this article is to establish the 2^n-dimensional projective algebra Λ_n and the 2^n-dimensional Clifford double algebra Γ_n. Both structures carry the imprint of an algebra twice. Projective algebra is the more fundamental concept than Clifford double algebra, since any Clifford double algebra also shows the structure of projective algebra whereas projective algebra is standing on its own.

In section 2 basic concepts and the concepts of double algebra and of complementary graded algebra are defined. Here double algebra is the more fundamental concept compared to complementary graded algebras.

On the basis of these concepts and by introducing a second Clifford product, in section 3 we start with a non-degenerate Clifford algebra and extend it to a non-degenerate Clifford double algebra Γ_n.

It will turn out that binary indices are of utmost significance in the context of projective and Clifford double algebras. This is why section 4 provides a short introduction to binary indices and several binary operators.

In section 5 we define projective algebra Λ_n and introduce the harmonic model of projective algebra. Projective algebra Λ_n represents a complementary graded algebra.

On the basis of projective algebra Λ_n we define in section 6 Clifford double algebra Γ_n. This time such, that it includes not only the non-degenerate Clifford double algebras derived in section 3, but also Clifford double algebras with degenerate quadratic forms. This is why the definition in section 6 is regarded as the more fundamental charaterisation.

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Section 7 investigates the relation of Grassmann algebras to projective and Clifford double algebras respectively.

The closing section 8 provides a short outlook to applications of projective and double Clifford algebras in the field of geometry.

2. Double and complementary graded algebras

Definition 2.1 (multi vector, homogeneous multi vector)**.** Let $G(+)$ be an abelian group with a decomposition into the direct sum of subgroups G_k,

$$G = \bigoplus_{k\in\mathbb{Z}} G_k. \tag{2.1}$$

The elements $M \in G$ are called *multi vectors*, the elements $X \in G_k$ *homogeneous multi vectors*. The terms *multi vector* and *homogeneous multi vector* will also be used in the context of rings, vector spaces, algebras or any structure, which shows as a substructure an additive group decomposing into a direct sum of subgroups like in equation (2.1).

Definition 2.2 (graded ring, graded algebra, k-vector)**.** A ring $R(+,\cdot)$ is called *graded* or, more precisely, $\mathbb{Z}$-graded, if there exists a family of additive subgroups $\{R_k\}_{k\in\mathbb{Z}}$ of R such that

$$R = \bigoplus_{k\in\mathbb{Z}} R_k \qquad \text{and} \tag{2.2}$$

$$R_k \cdot R_l \subseteq R_{k+l} \qquad \forall\, k,l \in \mathbb{Z}. \tag{2.3}$$

A graded ring R is called *nonnegatively* graded or $\mathbb{N}$-graded, if $R_k = 0$ for all $k < 0$. A graded ring R is called *n-graded* with $n \in \mathbb{N}$, if $R_k = 0$ for all $k < 0$ and for all $k > n$. A homogeneous multi vector $X \in R_k$ is called *k-vector* or *vector of grade k*. In order to indicate the grade, a k-vector X may be endowed with an overlined subscript k,

$$X = X_{\bar{k}}. \tag{2.4}$$

The terms *graded*, *k-vector* etc. will also be used in the context of algebras provided they carry the structure of a graded ring.

Definition 2.3 (k-projection)**.** Let $R(+)$ be an additive group with a decomposition into the direct sum of subgroups R_l.

$$R = \bigoplus_{l\in\mathbb{Z}} R_l. \tag{2.5}$$

The mapping

$$\begin{aligned} \langle\ \ \rangle_k : \quad R &\longrightarrow R_k \qquad\qquad k \in \mathbb{Z} \\ X = \sum_{l\in\mathbb{Z}} X_{\bar{l}} &\longmapsto \langle X\rangle_k := X_{\bar{k}} \end{aligned} \tag{2.6}$$

is called a *k-projection* and denoted by angle brackets with subscript k.

Definition 2.4 (double ring, double algebra, isomorphic double ring, isomorphic double algebra)**.** A set $R(+,\cdot,\circ)$ endowed with the operations of addition $(+)$, of major multiplication $(\cdot)$ and of minor multiplication $(\circ)$ is called a *double ring*, if and only if $R(+,\cdot)$ and $R(+,\circ)$ both represent rings and share the same additive group $R(+)$. A double ring $R(+,\cdot,\circ)$ is called *isomorphic*, if and only if there exists a ring isomorphism ϕ inside the double ring R, such that

$$\begin{aligned} \phi : \quad R(+,\cdot) &\longrightarrow R(+,\circ) \\ X &\longmapsto \phi(X) \end{aligned} \tag{2.7}$$

with

$$\phi(A+B) := \phi(A) + \phi(B), \tag{2.8}$$

$$\phi(A \cdot B) := \phi(A) \circ \phi(B). \tag{2.9}$$

A set $A(+,\ ,\cdot,\circ)$ endowed with the operations of addition $(+)$, of scalar multiplication (no sign), of major multiplication $(\cdot)$ and of minor multiplication $(\circ)$ is called a *double* $\mathbb{F}$*-algebra*, if and only if $A(+,\ ,\cdot)$ and $A(+,\ ,\circ)$ both represent $\mathbb{F}$-algebras and share the same vector space $A(+,\ )$. A double $\mathbb{F}$-algebra $A(+,\ ,\cdot,\circ)$ is called *isomorphic* double $\mathbb{F}$-algebra, if and only if the double ring $A(+,\cdot,\circ)$ is isomorphic.

Definition 2.5 (plus approach, minus approach)**.** Let $R(+,\cdot,\circ)$ be a double ring. The ring $R(+,\cdot)$ is called *plus approach* to the double ring $R(+,\cdot,\circ)$; the ring $R(+,\circ)$ is called *minus approach* to the double ring $R(+,\cdot,\circ)$. Let $A(+,\ ,\cdot,\circ)$ be a double $\mathbb{F}$-algebra. The $\mathbb{F}$-algebra $A(+,\ ,\cdot)$ is called *plus approach* to the double $\mathbb{F}$-algebra $A(+,\ ,\cdot,\circ)$; the $\mathbb{F}$-algebra $A(+,\ ,\circ)$ is called *minus approach* to the double $\mathbb{F}$-algebra $A(+,\ ,\cdot,\circ)$.

Notation 2.6 (plus-minus-notation)**.** The multi vectors and subsets of a double ring or double algebra may be supplied with a superscript plus sign in order to indicate that they belong to the plus approach (plus-notation) or with a superscript minus sign in order to indicate that they belong to the minus approach (minus-notation).

Example. Let $A(+,\ ,\cdot,\circ)$ denote a double $\mathbb{F}$-algebra. If it is not clear out of the context to which approach the identity element 1 is belonging, we supply it with a plus or minus sign. 1^+ represents the identity element in the $\mathbb{F}$-algebra $A(+,\ ,\cdot)$. It also belongs to the $\mathbb{F}$-algebra $A(+,\ ,\circ)$, but not necessarily as identity element. 1^- represents the identity element in the minus approach.

Definition 2.7 (complementary graded ring, complementary graded algebra)**.** An isomorphic double ring $R(+,\cdot,\circ)$ is called *complementary n-graded* or, shorter, *complementary graded* if and only if it meets the following conditions:

(a) The ring $R(+,\cdot)$ is n-graded with decomposition into the direct sum

$$R = \bigoplus_{k=0}^{n} R_k^+, \tag{2.10}$$

(b) The ring $R(+,\circ)$ is n-graded with decomposition into the direct sum

$$R = \bigoplus_{k=0}^{n} R_k^-, \tag{2.11}$$

(c) The decompositions (2.10) and (2.11) are connected by

$$R_k^- := R_{n-k}^+ \qquad \forall\, k \in \mathbb{N},\ 0 \le k \le n. \tag{2.12}$$

An isomorphic double $\mathbb{F}$-algebra $A(+,\ ,\cdot,\circ)$ is called a *complementary n-graded* $\mathbb{F}$-algebra or, shorter, a *complementary graded* $\mathbb{F}$-algebra, if and only if $A(+,\cdot,\circ)$ represents a complementary graded ring.

Example. Let R be a complementary 4-graded ring (n=4). We then have

$$R = R_0^+ \oplus R_1^+ \oplus R_2^+ \oplus R_3^+ \oplus R_4^+ = R_4^- \oplus R_3^- \oplus R_2^- \oplus R_1^- \oplus R_0^- \tag{2.13}$$

with $R_0^+ = R_4^-$, $R_1^+ = R_3^-$, $R_2^+ = R_2^-$, $R_3^+ = R_1^-$ and $R_4^+ = R_0^-$.

As already mentioned, the plus-minus-notation is optional. Out of the context it is often clear in which approach one is working. We do not need to supply the sets and multi vectors in this case with an extra superscript plus or minus sign. The plus-minus-notation is also left out in the case a certain expression is valid in both approaches.

Example. Let A be a complementary n-graded algebra. Then for a homogeneous multi vector $X \in A_k^+ = A_{n-k}^-$ we may write

$$X_k^+ = X = X_{\overline{n-k}}^- . \tag{2.14}$$

Example. Let $P_i = \langle P_i \rangle_1^+$ denote n 1-vectors in the plus approach and $E_i = \langle E_i \rangle_1^-$ n 1-vectors in the minus approach. The sets of these homogeneous multi vectors,

$$\mathscr{P}^+ = \{P_i^+ \mid i \in \mathbb{N},\ 1 \le i \le n\}, \tag{2.15}$$

$$\mathscr{E}^- = \{E_i^- \mid i \in \mathbb{N},\ 1 \le i \le n\}, \tag{2.16}$$

may then be supplied by a superscript plus or minus sign respectively.

3. Clifford double algebra Γ_n with non-degenerate quadratic form

Let us recall the definition of a non-degenerate Clifford algebra $Cl(V,Q)$ as it was for example given by Pertti Lounesto in [Lou01, p. 190].

Definition 3.1 (Clifford algebra with non-degenerate quadratic form)**.** Let $\mathbb{F}$ be a field with a characteristic not equal to 2 and V be a n-dimensional vector space over $\mathbb{F}$. A *Clifford algebra* $Cl(V,Q)$ of a non-degenerate quadratic form Q on V is an associative $\mathbb{F}$-algebra with identity element 1 if and only if it contains V and $\mathbb{F} \equiv \mathbb{F} \cdot 1$ as distinct subspaces such that

(a) $X^2 = Q(X) \cdot 1$ for any $X \in V$,
(b) V generates $Cl(V,Q)$ as a $\mathbb{F}$-algebra,
(c) $Cl(V,Q)$ is not generated by any proper subspace of V.

Notation 3.2 (inner product space)**.** For a vector space V and a quadratic form Q let (V,Q) denote the inner product space given by the symmetric bilinear form $\mathfrak{b}(X,Y) = (Q(X+Y) - Q(X) - Q(Y))/2$ associated with Q.

Theorem 3.3. *Let $Cl(V,Q)$ be a Clifford $\mathbb{F}$-algebra according to Definition 3.1 and let $\{P_i \mid i \in \mathbb{N},\ 1 \le i \le n\}$ be an orthogonal basis of the inner product space (V,Q). We then have:*

(a) $P_iP_j = -P_jP_i$ for $i \ne j$, $i,j \in \mathbb{N}$.
(b) The vectors

$$P_{i_1 i_2 \ldots i_k} := \begin{cases} 1 & k = 0 \\ P_{i_1} P_{i_2} \ldots P_{i_k} & \begin{cases} 1 \le i_1 < i_2 < \cdots < i_k \le n \\ \text{and } 0 < k \le n \end{cases} \end{cases} \tag{3.1}$$

form a basis of $Cl(V,Q)$ as a vector space.
(c) The Clifford algebra $Cl(V,Q)$ decomposes into the direct sum

$$Cl(V,Q) = \bigoplus_{k=0}^{n} Cl^k(V,Q) \tag{3.2}$$

where each subspace $Cl^k(V,Q)$ is generated by the $\binom{n}{k}$ homogeneous multi vectors $P_{i_1 i_2 \ldots i_k}$ of equation (3.1). The dimensions of the Clifford algebra $Cl(V,Q)$ and its vector subspaces are

$$\dim Cl(V,Q) = \sum_{k=0}^{n} \dim Cl^k(V,Q) = \sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{3.3}$$

Proof. (a) From $Q(P_i+P_j) = (P_i+P_j)^2 = Q(P_i) + P_iP_j + P_jP_i + Q(P_j)$ and $\mathfrak{b}(P_i,P_j) = 0$ we get $P_iP_j = -P_jP_i$ for $i \ne j$.
(b) The 2^n homogeneous multi vectors of equation (3.1) are linearly independent and span the whole algebra $Cl(V,Q)$ as a vector space.

(c) Equation (3.2) and equation (3.3) are a consequence of the fact that the $\binom{n}{k}$ homogeneous multi vectors $P_{i_1 i_2 \ldots i_k}$ from equation (3.1) with fixed k form a basis for the sub space $Cl^k(V,Q)$ and that the 2^n homogeneous multi vectors $P_{i_1 i_2 \ldots i_k}$ from equation (3.1) for all k form a basis for the whole space $Cl(V,Q)$.

□

Definition 3.4 (inner and outer product)**.** Let $X_{\overline{k}}$ denote the homogeneous multi vectors of the vector subspaces $Cl^k(V,Q)$. The inner product

$$A_{\overline{r}} \cdot B_{\overline{s}} := \langle A_{\overline{r}} B_{\overline{s}} \rangle_{|r-s|}, \qquad 0 \leq r,s \leq n, \tag{3.4}$$

and the outer product

$$A_{\overline{r}} \wedge B_{\overline{s}} := \begin{cases} \mathbf{0} & r = 0 \text{ or } s = 0 \\ \mathbf{0} & r+s > n \\ \langle A_{\overline{r}} B_{\overline{s}} \rangle_{r+s} & \text{else} \end{cases} \tag{3.5}$$

are defined in terms of the Clifford product and expanded to generic multi vectors by distributivity.

Notation 3.5 (pseudo scalar)**.** The homogeneous multi vector $P_{12\ldots n}$ with $k = n$ in equation (3.1) is denoted by $I := \prod_{i=1}^{n} P_i$ and called *pseudo scalar*.

Theorem 3.6. *The outer product $\wedge$ of $Cl(V,Q)$ is associative and has no identity element.*

Proof. By definition, all scalars are zero divisors with respect to the outer product. Compare equation (3.5). Associativity with respect to generic multi vectors follows from

$$(A_{\overline{r}} \wedge B_{\overline{s}}) \wedge C_{\overline{t}} = \langle \langle A_{\overline{r}} B_{\overline{s}} \rangle_{r+s} C_{\overline{t}} \rangle_{r+s+t} = \langle A_{\overline{r}} \langle B_{\overline{s}} C_{\overline{t}} \rangle_{r+s} \rangle_{r+s+t} = A_{\overline{r}} \wedge (B_{\overline{s}} \wedge C_{\overline{t}}) \tag{3.6}$$

with $A_{\overline{r}} \in Cl^r(V,Q)$, $B_{\overline{s}} \in Cl^s(V,Q)$, $C_{\overline{t}} \in Cl^t(V,Q)$. □

Theorem 3.7. *Let $G(+,\cdot,\ )$ represent the unital Clifford $\mathbb{F}$-algebra $Cl(V,Q)$ with the operations of addition (+), of scalar multiplication ($\cdot$) and of Clifford multiplication (no sign). The $\mathbb{F}$-vector space $G(+,\cdot)$ extended by the outer product $\wedge$, i. e. the $\mathbb{F}$-algebra $G(+,\cdot,\wedge)$ is n-graded and has no identity element.*

Proof. With Theorem 3.6 we know, that $G(+,\cdot,\wedge)$ represents a $\mathbb{F}$-algebra without identity element. $G(+,\wedge)$ is a n-graded ring, since the decomposition of $G(+)$ into a direct sum is given by equation (3.2) and since the definition of the outer product in equation (3.5) meets the condition for the multiplication in a graded ring. Compare the equations (2.2) and (2.3). □

The Clifford $\mathbb{F}$-algebra $G(+,\cdot,\ )$ of Theorem 3.7 is *not* graded, since the Clifford multiplication does not meet the condition of equation (2.3). The Clifford product of the two 1-vectors $P_1 + P_2 = \langle P_1 + P_2 \rangle_1$ and $P_1 = \langle P_1 \rangle_1$ for example results in the direct sum of a scalar and 2-vector,

$$(P_1 + P_2)P_1 = P_1^2 + P_2 P_1 = \langle P_1^2 \rangle_0 + \langle P_2 P_1 \rangle_2. \tag{3.7}$$

Following DAVID HESTENES and RENATUS ZIEGLER [HZ91] we define dual multi vectors as follows:

Definition 3.8 (dual multi vector)**.** Let A be a generic multi vector of a Clifford algebra $Cl(V,Q)$ with non-degenerate quadratic form Q. Then

$$\widetilde{A} := AI^{-1} = A \cdot I^{-1} \tag{3.8}$$

is the *dual multi vector* with respect to A.

We then have

$$I^2 \in \mathbb{F} \setminus \{\mathbf{0}\}, \tag{3.9}$$

$$I^{-1} = (I^{-2})I, \tag{3.10}$$

$$\widetilde{X_{\overline{k}}} = \langle \widetilde{X_{\overline{k}}} \rangle_{n-k}, \tag{3.11}$$

$$\widetilde{\widetilde{X_{\overline{k}}}} = I^{-2} X_{\overline{k}} = \langle I^{-2} X_{\overline{k}} \rangle_k \tag{3.12}$$

The dual of a k-vector is a $(n-k)$-vector and the dual of a $(n-k)$-vector is a k-vector. Thus, any multiplication with a pseudo scalar mediates between complementary subspaces.

We will now define a second Clifford product in any non-degenerate Clifford algebra $Cl(V,Q)$. The original Clifford product will then be called *major Clifford product* and the second one *dual* or *minor Clifford product*.

Definition and Theorem 3.9. Using the commutative diagram of mappings

$$\begin{array}{ccc} Cl(V,Q) \otimes Cl(V,Q) & \xrightarrow{*\text{-product}} & Cl(V,Q) \\ (A\,,B) & & A*B \\ \Big\downarrow{\scriptstyle I\otimes I} & & \Big\downarrow{\scriptstyle I} \\ Cl(V,Q) \otimes Cl(V,Q) & \xrightarrow{\text{Clifford product}} & Cl((V,Q) \\ (AI\,,BI) & & AIBI = (A*B)I \end{array}$$

we define the *dual* or *minor Clifford product* $*$ as

$$A*B := AIB. \tag{3.13}$$

It meets the conditions

$$(A*B)*C = A*(B*C), \qquad A,B,C \in Cl(V,Q), \tag{3.14}$$

$$A*(B+C) = A*B + A*C, \tag{3.15}$$

$$(B+C)*A = B*A + C*A, \tag{3.16}$$

$$X_{\overline{n}} * A = A * X_{\overline{n}} = \lambda A, \qquad X_{\overline{n}} = \lambda I^{-1}, \tag{3.17}$$

$$X_{\overline{n-1}} * X_{\overline{n-1}} = \langle X_{\overline{n-1}} * X_{\overline{n-1}} \rangle_n, \qquad X_{\overline{n-1}} \in Cl^{n-1}(V,Q). \tag{3.18}$$

Proof. The minor Clifford product $*$ is associative,

$$(A*B)*C = (AIB)*C = AIBIC = A*(BIC) = A*(B*C), \tag{3.19}$$

distributive with respect to addition,

$$A*(B+C) = AI(B+C) = AIB + AIC = A*B + A*C, \tag{3.20}$$

$$(B+C)*A = (B+C)IA = BIA + CIA = B*A + C*A, \tag{3.21}$$

any pseudo scalar $X_{\overline{n}} = \lambda I^{-1}$ commutes with any multi vector A and behaves like a scalar factor,

$$X_{\overline{n}} * A = X_{\overline{n}} IA = AX_{\overline{n}} I = AIX_{\overline{n}} = A * X_{\overline{n}} = \lambda A, \tag{3.22}$$

and the square of any $(n-1)$-vector $X_{\overline{n-1}}$,

$$X_{\overline{n-1}} * X_{\overline{n-1}} = \widetilde{X}_{\overline{n-1}} \widetilde{X}_{\overline{n-1}} I^4 I^{-1} = Q(\widetilde{X}_{\overline{n-1}}) I^4 I^{-1} \tag{3.23}$$

results in a pseudo scalar. □

Theorem 3.10. *Let the set $G(+,\cdot,\ )$ endowed with the operations of addition $(+)$, of scalar multiplication $(\cdot)$ and of major Clifford multiplication (no sign) denote the Clifford $\mathbb{F}$-algebra $Cl(V,Q)$ with non-degenerate quadratic form Q and decomposition into the direct sum of homogeneous multi vector subspaces $Cl^k(V,Q)$,*

$$G=\bigoplus_{k=0}^{n} Cl^k(V,Q). \tag{3.24}$$

Let the set $G(+,\cdot,*)$ endowed with the operations of addition $(+)$, of scalar multiplication $(\cdot)$ and of minor Clifford multiplication $(*)$ denote the Clifford $\mathbb{F}$-algebra $Cl(\widetilde{V},\widetilde{Q})$ with

$$\widetilde{V}:=Cl^{(n-1)}(V,Q), \tag{3.25}$$

$$\widetilde{Q}(X):=X*X=Q(\widetilde{X})I^4I^{-1},\qquad X\in\widetilde{V},\qquad \widetilde{Q}(X)\in Cl^0(\widetilde{V},\widetilde{Q}):=Cl^n(V,Q) \tag{3.26}$$

and with decomposition into the direct sum of homogeneous multi vector subspaces $Cl^k(\widetilde{V},\widetilde{Q})$,

$$G=\bigoplus_{k=0}^{n} Cl^k(\widetilde{V},\widetilde{Q}). \tag{3.27}$$

The set $G(+,\cdot,\ ,*)$ endowed with the operations of addition $(+)$, of scalar multiplication $(\cdot)$, of major Clifford multiplication (no sign) and of minor Clifford multiplication $(*)$ represents an isomorphic Clifford double $\mathbb{F}$-algebra.

Proof. We consider the decomposition of equation (3.24) as plus approach and the decomposition of equation (3.27) as minus approach. By precondition $\mathbb{F}$ is a field with characteristic not equal to 2. The vector space $\widetilde{V}$ has dimension n,

$$\dim\widetilde{V}=\dim Cl^{(n-1)}(V,Q)=\binom{n}{n-1}=n, \tag{3.28}$$

and the mapping

$$\widetilde{Q}:\quad \widetilde{V}=Cl^1(\widetilde{V},\widetilde{Q})\quad\longrightarrow\quad \mathbb{F}\cong Cl^0(\widetilde{V},\widetilde{Q}) \tag{3.29}$$

is by definition of equation (3.26) a non-degenerate quadratic form on $\widetilde{V}$. According to Theorem 3.9 the minor Clifford product $*$ is associative, distributive, has the identity element $1^-=(I^+)^{-1}$ and coincides with scalar multiplication for all scalars $X\in Cl^0(\widetilde{V},\widetilde{Q})$ in the minus approach. The n homogeneous multi vectors

$$\widetilde{V}\ni E_i^-:=\widetilde{P_i}=(-1)^{i-1}I^{-2}P_i^2P_1\ldots P_{i-1}P_{i+1}\ldots P_n,\qquad i\in\{1,\ldots,n\}=:\mathscr{L}\subset\mathbb{N}, \tag{3.30}$$

form an orthogonal basis of and with respect to the inner product space $(\widetilde{V},\widetilde{Q})$ and the 2^n homogeneous multi vectors

$$\begin{aligned} E^-_{i_1i_2\ldots i_k}&:=\begin{cases} 1^-=(I^+)^{-1}=\widetilde{1^+} & k=0 \qquad \{i_1,i_2,\ldots,i_k\}\subset\mathscr{L}\\ E^-_{i_1}*E^-_{i_2}*\ldots*E^-_{i_k} & 0<k\le n \qquad 1\le i_1<i_2<\cdots<i_k\le n\end{cases}\\ &=\widetilde{P_{i_1i_2\ldots i_k}}\\ &=(-1)^{-\frac{k(k+1)}{2}+\sum_{l=1}^k i_l}I^{-2}\left[\prod_{l=1}^k P_{i_l}^2\right]\prod_{j\in\mathscr{M}}P_j\qquad \mathscr{M}:=\mathscr{L}\setminus\{i_1,i_2,\ldots,i_k\}\\ &=\left\langle(-1)^{-\frac{k(k+1)}{2}+\sum_{l=1}^k i_l}I^{-2}\left[\prod_{l=1}^k P_{i_l}^2\right]\prod_{j\in\mathscr{M}}P_j\right\rangle_{n-k}^{+}\end{aligned} \tag{3.31}$$

form a basis of the Clifford $\mathbb{F}$-algebra $Cl(\widetilde{V},\widetilde{Q})$ as a vector space. Thus, $G(+,\cdot,\ ,*)$ represents a Clifford double $\mathbb{F}$-algebra. $G(+,\cdot,\ ,*)$ represents even an isomorphic Clifford double $\mathbb{F}$-algebra since the linear mapping defined by

$$\begin{array}{lccc} \phi: & Cl(V,Q) & \longrightarrow & Cl(\widetilde{V},\widetilde{Q}) \\ & P_i & \longmapsto & \phi(P_i) := E_i \end{array} \tag{3.32}$$

with

$$\phi(\lambda A+\beta B) = \lambda\phi(A)+\beta\phi(B) \tag{3.33}$$

$$\phi(AB) = \phi(A)*\phi(B) \tag{3.34}$$

is an algebra-isomorphism. □

Notation 3.11 (Clifford double algebra Γ_n)**.** In section 6, the Clifford double $\mathbb{F}$-algebra $G(+,\cdot,\ ,*)$ of Theorem 3.10 will be denoted by $\Gamma_n(+,\cdot,\ ,*)$.

Definition 3.12 (inner and outer product in the minus approach)**.** Let $X_{\bar{k}}^-$ denote the homogeneous multi vectors of the vector subspaces $Cl^k(\widetilde{V},\widetilde{Q})$. The inner product in the minus approach

$$A_{\bar{r}}^- \circ B_{\bar{s}}^- := \langle A_{\bar{r}}^- * B_{\bar{s}}^- \rangle_{|r-s|}, \qquad 0\le r,s\le n, \tag{3.35}$$

and the outer product in the minus approach

$$A_{\bar{r}}^- \vee B_{\bar{s}}^- := \begin{cases} \mathbf{0} & r=0 \text{ or } s=0 \\ \mathbf{0} & r+s>n \\ \langle A_{\bar{r}}^- * B_{\bar{s}}^- \rangle_{r+s} & \text{else} \end{cases} \tag{3.36}$$

are defined in terms of the minor Clifford product and extended to generic multi vectors by distributivity.

Corollary 3.13. *The outer product $\vee$ of $Cl(\widetilde{V},\widetilde{Q})$ is associative and has no identity element.*

Corollary 3.14. *Let $G(+,\cdot,*)$ represent the unital Clifford $\mathbb{F}$-algebra $Cl(\widetilde{V},\widetilde{Q})$ with the operations of addition (+), of scalar multiplication ($\cdot$) and of minor Clifford multiplication ($*$). The $\mathbb{F}$-vector space $G(+,\cdot)$ extended by the outer product $\vee$, i. e. the $\mathbb{F}$-algebra $G(+,\cdot,\vee)$ is n-graded and has no identity element.*

Theorem 3.15. *Let $\Gamma_n(+,\cdot,\ ,*)$ be a non-degenerate, isomorphic Clifford double $\mathbb{F}$-algebra and let $G(+,\cdot,\wedge,\vee)$ denote the $\mathbb{F}$-vector space $\Gamma_n(+,\cdot)$ extended by the two outer products $\wedge$ and $\vee$. $G(+,\cdot,\wedge,\vee)$ represents a complementary n-graded $\mathbb{F}$-algebra.*

Proof. According to Theorem 3.7 and Corollary 3.14 the $\mathbb{F}$-algebras $G(+,\cdot,\wedge)$ and $G(+,\cdot,\vee)$ with the decompositions $G=\sum_{k=0}^n G_k^+ = \sum_{k=0}^n G_k^-$ are n-graded. From equation (3.31) we get $G_k^+ = G_{n-k}^-$ for all k with $1\le k\le n$. □

In section 6 we will generalize the concept of a double Clifford algebra Γ_n on the basis of projective algebra Λ_n to generic, i. e. also degenerate quadratic forms.

4. Binary indices

Other authors have already used binary numbers to label elements of Clifford algebras in various ways. See [Mar04] or [Lou01, p. 279ff]. We adopted the main idea and shortly review binary numbers and several binary operators in this section.

Notation 4.1 (m-digit binary numbers)**.** In order to label basis elements and other homogeneous multi vectors we introduce binary numbers

$$\mathbf{b} \equiv b_{m-1}\dots b_1 b_0 = \left[\sum_{k=0}^{m-1} b_k 2^k\right]_{10}, \quad m \in \mathbb{N}, \quad b_k \in \{0,1\}, \tag{4.1}$$

with m digits b_{m-1}, …, b_0. The binary numbers will be noted by small latin letters in bold face, its m digits with the same letter but normal face and the digits will be numbered from 0 to $m-1$.

A m-digit number $\mathbf{b}$ translated into a real figure lies between $[0]_{10}$ and $[2^m-1]_{10}$.

Definition 4.2 (sum of binary digits)**.** The sum $S(\mathbf{b})$ of the digits of the binary number (4.1) is

$$S(\mathbf{b}) := \left[\sum_{k=0}^{n-1} b_k\right]_{10}. \tag{4.2}$$

Theorem 4.3. *The m-digit binary number* $\mathbf{b}$

(a) is zero, if and only if $S(\mathbf{b}) = 0$*;*
(b) represents a decimal power of two, if and only if $S(\mathbf{b}) = 1$*;*
(c) is the decimal number $[2^m-1]_{10}$*, if and only if* $S(\mathbf{b}) = m$*.*

Proof. (a) $S(\mathbf{b}) = 0 \Leftrightarrow b_i = 0 \;\forall\; 0 \le i \le m-1 \Leftrightarrow \mathbf{b} = [0]_2 = [0]_{10}$.
(b) We have $S(\mathbf{b}) = 1$, if and only if there is one index $l \in \mathbb{N}$, such that the binary number

$$\begin{aligned} \mathbf{b} &= b_{m-1}\dots b_1 b_0 & b_i &= \begin{cases} 1, & \text{if } i = l, \\ 0, & \text{if } i \neq l, \end{cases} \\ &= [2^l]_{10}, & 0 &\le l \le m-1, \end{aligned} \tag{4.3}$$

represents a decimal power of two.
(c) $S(\mathbf{b}) = m \Leftrightarrow b_i = 1 \;\forall\; 0 \le i \le m-1 \Leftrightarrow \mathbf{b} = [2^m-1]_{10}$.

□

Definition 4.4 (binary complement)**.** The binary complement of a m-digit binary number $\mathbf{b}$ is

$$\overline{\mathbf{b}} = \overline{b_{m-1}\dots b_1 b_0} := \left[\sum_{k=0}^{m-1} \overline{b}_k 2^k\right]_{10} \qquad \text{with} \qquad \overline{0} := 1, \qquad \overline{1} := 0. \tag{4.4}$$

The binary complement translated into a decimal number lies again between $[0]_{10}$ and $[2^n-1]_{10}$. In addition we always have

$$\mathbf{b} + \overline{\mathbf{b}} = \mathbf{u} = [2^m-1]_{10}, \tag{4.5}$$

$$\overline{\overline{\mathbf{b}}} = \mathbf{b}, \tag{4.6}$$

where $\mathbf{u}$ is the m-digit binary number with $S(\mathbf{u}) = m$.

Definition 4.5 (indices on the left side)**.** Let $\mathbf{b}$ be a m-digit binary number and $\mathscr{I} := \{i_1, \dots, i_k\}$ an ordered index set of length k with

$$0 \le i_1 < i_2 < \dots < i_k \le m-1, \qquad k, j, i_j \in \mathbb{N}, \qquad 1 \le j \le k, \tag{4.7}$$

such that

$$\mathbf{b} = b_{m-1}\dots b_1 b_0, \qquad b_i = \begin{cases} 1, & \text{if } i \in \mathscr{I}, \\ 0, & \text{if } i \notin \mathscr{I}. \end{cases} \tag{4.8}$$

In addition, let $l_1, l_2, \ldots, l_p$ be p natural numbers with

$$1 \leq l_1 < l_2 \ldots < l_p \leq k, \qquad l_j \in \mathbb{N}, \qquad 1 \leq j \leq p. \tag{4.9}$$

These numbers determine a subset of $\mathscr{I}$,

$$\mathscr{L} := \{i_{l_1}, \ldots, i_{l_p}\} \subset \mathscr{I}, \tag{4.10}$$

and thus establish an ordered index subset $\mathscr{L}$ of length p.

The natural numbers $l_1, l_2, \ldots, l_p$ represent as *lower left indices in brackets* the binary number

$$\begin{aligned} {}_{(l_1 l_2 \ldots l_p)}\mathbf{b} &:= \mathbf{c} \\ &= c_{m-1} \ldots c_1 c_0, \qquad\qquad c_i = \begin{cases} 1, & \text{if } i \in \mathscr{L}, \\ 0, & \text{if } i \notin \mathscr{L}, \end{cases} \end{aligned} \tag{4.11}$$

and as *upper left indices in brackets* the binary number

$${}^{(l_1 l_2 \ldots l_p)}\mathbf{b} := \overline{{}_{(l_1 l_2 \ldots l_p)}\mathbf{b}}. \tag{4.12}$$

The brackets may be omitted, if there is only one lower left or upper left index.

Example. As an example for the left indices let us take the number $\mathbf{b} = [01011]_2$. Then the index set $\mathscr{I}$ is $\{0,1,3\}$, i. e. $i_1 = 0$, $i_2 = 1$ und $i_3 = 3$ and we have:

$S(\mathbf{b}) = 3$		$S(\overline{\mathbf{b}}) = 2$
$\mathbf{b} = 01011$		$\overline{\mathbf{b}} = 10100$
${}_1\mathbf{b} = 00001$	${}^1\mathbf{b} = 11110$	${}_1\overline{\mathbf{b}} = 00100$
${}_2\mathbf{b} = 00010$	${}^2\mathbf{b} = 11101$	${}_2\overline{\mathbf{b}} = 10000$
${}_3\mathbf{b} = 01000$	${}^3\mathbf{b} = 10111$	${}_{(12)}\overline{\mathbf{b}} = 10100 = \overline{\mathbf{b}}$
${}_{(12)}\mathbf{b} = 00011$	${}^{(12)}\mathbf{b} = 11100$	
${}_{(13)}\mathbf{b} = 01001$	${}^{(13)}\mathbf{b} = 10110$	${}^1\overline{\mathbf{b}} = 11011$
${}_{(23)}\mathbf{b} = 01010$	${}^{(23)}\mathbf{b} = 10101$	${}^2\overline{\mathbf{b}} = 01111$
${}_{(123)}\mathbf{b} = 01011 = \mathbf{b}$	${}^{(123)}\mathbf{b} = 10100 = \overline{\mathbf{b}}$	${}^{(12)}\overline{\mathbf{b}} = 01011 = \mathbf{b}$

Theorem 4.6. *For any m-digit binary number* $\mathbf{b} = b_{m-1} \ldots b_1 b_0$ *with check sum* $S(\mathbf{b}) = k$, $0 < k \leq m$, *and*

$$\mathbf{b} = \sum_{l=1}^{k} {}_l\mathbf{b}, \tag{4.13}$$

we always have

$${}_l\mathbf{b} < {}_{l+1}\mathbf{b}, \tag{4.14}$$

$$S({}_l\mathbf{b}) = 1. \tag{4.15}$$

Proof. By precondition there are exactly k indices

$$0 \leq i_1 < i_2 < \ldots < i_k \leq m-1, \tag{4.16}$$

the binary digits of which do not vanish

$$b_i = \begin{cases} 1, & i \in \{i_1, \ldots, i_k\}, \\ 0, & i \notin \{i_1, \ldots, i_k\}. \end{cases} \tag{4.17}$$

Thus we have

$$\mathbf{b} = b_{m-1}\dots b_1 b_0 = \left[\sum_{l=1}^{k} 2^{i_l}\right]_{10} = \sum_{l=1}^{k} {}_l\mathbf{b} \tag{4.18}$$

with ${}_l\mathbf{b} = [2^{i_l}]_{10}$ and ${}_{l+1}\mathbf{b} = [2^{i_{l+1}}]_{10}$., i. e. ${}_l\mathbf{b} < {}_{l+1}\mathbf{b}$. The equation $S({}_l\mathbf{b}) = 1$ is a consequence of Theorem 4.3. □

Notation 4.7 (nested sequence of left indices)**.** For a binary number with several left indices, the nested sequence of left indices has to be read from inside outwards as in the following example,

$${}^{l}_{4\,(23)}\mathbf{b} = {}_4\Big[{}^{l}\Big[{}_{(23)}\mathbf{b}\Big]\Big]. \tag{4.19}$$

Theorem 4.8. *Let V be a vector space, $A_i \in V$, $i \in \{1,\dots,m\} \subset \mathbb{N}$ and $\mathbf{b}$ a binary variable with m digits. We may then label the m elements A_i with*

$$V \ni A_{\mathbf{b}}, \qquad S(\mathbf{b}) = 1, \tag{4.20}$$

i. e. we use all binary numbers with $S(\mathbf{b}) = 1$ to label the vectors $A_{\mathbf{b}}$.

Proof. With $\mathbf{b}_i = \left[2^{i-1}\right]_{10}$ we get $S(\mathbf{b}_i) = 1$ and $A_i = A_{\mathbf{b}_i}$, $1 \le i \le m$. □

In this article we will also use the bitwise binary operators AND and XOR. Example: 1001 AND 0101 = 0001 and 1001 XOR 0101 = 1100.

5. Projective algebra Λ_n

Notation 5.1 (multiple outer product signs)**.** The two outer products of projective algebra Λ_n are denoted by $\wedge$ and $\vee$. For multiple outer products we use

$$\bigwedge_{l=1}^{m} X_l := X_1 \wedge X_2 \wedge \dots \wedge X_m, \qquad \bigvee_{l=1}^{m} X_l := X_1 \vee X_2 \vee \dots \vee X_m. \tag{5.1}$$

Definition 5.2 (projective algebra Λ_n)**.** A *projective $\mathbb{F}$-algebra* $\Lambda_n(+,\cdot,\wedge,\vee)$ is a set Λ_n with four operations:

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \stackrel{+}{\longrightarrow} & \Lambda_n \\ (A,B) & \longmapsto & A+B \end{array} \qquad \begin{array}{ccc} \mathbb{F} \times \Lambda_n & \stackrel{\cdot}{\longrightarrow} & \Lambda_n \\ (\alpha,A) & \longmapsto & \alpha \cdot A \end{array} \tag{5.2}$$

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \stackrel{\wedge}{\longrightarrow} & \Lambda_n \\ (A,B) & \longmapsto & A \wedge B \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \stackrel{\vee}{\longrightarrow} & \Lambda_n \\ (A,B) & \longmapsto & A \vee B \end{array} \tag{5.3}$$

The operations are called addition $(+)$, scalar multiplication (no sign or $\cdot$), major outer product $(\wedge)$ und minor outer product $(\vee)$. The obey the following conditions:

(P1) $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$.

(P2) The projective $\mathbb{F}$-algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ is a complementary n-graded $\mathbb{F}$-algebra with decompositions into the direct sums

$$\Lambda_n = \bigoplus_{k=0}^{n} \Lambda_n^{k+} = \bigoplus_{k=0}^{n} \Lambda_n^{k-}, \qquad k,n \in \mathbb{N}, \tag{5.4}$$

and with the following dimensions for the whole algebra and for its subspaces,

$$\dim \Lambda_n = \sum_{k=0}^{n} \dim\left(\Lambda_n^k\right) = \sum_{k=0}^{n} \binom{n}{k}, \qquad 0 \le k \le n. \tag{5.5}$$

(P3) Further requirements for the outer products are:

- All scalars $X_{\bar{0}} \in \Lambda_n^0(+,\cdot)$ are left and right zero divisors,

$$X_{\bar{0}}^+ \wedge M = M \wedge X_{\bar{0}}^+ = \mathbf{0}, \qquad \forall\, M \in \Lambda_n, \tag{5.6}$$

$$X_{\bar{0}}^- \vee M = M \vee X_{\bar{0}}^- = \mathbf{0}, \qquad \forall\, M \in \Lambda_n. \tag{5.7}$$

In particular, the two algebras Λ_n^+ and Λ_n^- do not have an identity element.

- For 1-vectors $A_i \in \Lambda_n^{1+}$ or $B_i \in \Lambda_n^{1-}$ we have with $l > 1$

$$\bigwedge_{i=1}^{l} A_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} A_1,\, A_2,\, \ldots,\, A_l \text{ are} \\ \text{linearly independent.} \end{array} \right. \tag{5.8}$$

$$\bigvee_{i=1}^{l} B_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} B_1,\, B_2,\, \ldots,\, B_l \text{ are} \\ \text{linearly independent.} \end{array} \right. \tag{5.9}$$

Definition 5.3 (combined outer product)**.** Any mathematical term which contains the combined outer product $\lozenge$ can be read twice: Firstly with respect to the plus approach as major outer product $\wedge$ and secondly with respect to the minus approach as minor outer product $\vee$.

Example. The expression

$$X_{\bar{1}} \lozenge Y_{\bar{1}} = -Y_{\bar{1}} \lozenge X_{\bar{1}} \qquad \forall\, X_{\bar{1}}, Y_{\bar{1}} \in \Lambda_n^1 \tag{5.10}$$

means

$$X_{\bar{1}}^+ \wedge Y_{\bar{1}}^+ = -Y_{\bar{1}}^+ \wedge X_{\bar{1}}^+ \qquad \forall\, X_{\bar{1}}^+, Y_{\bar{1}}^+ \in \Lambda_n^{1+} \tag{5.11}$$

$$X_{\bar{1}}^- \vee Y_{\bar{1}}^- = -Y_{\bar{1}}^- \vee X_{\bar{1}}^- \qquad \forall\, X_{\bar{1}}^-, Y_{\bar{1}}^- \in \Lambda_n^{1-} \tag{5.12}$$

Notation 5.4 (multiple combined outer product sign)**.** Analogous to the notation for multiple outer products we use for multiple combined outer products the sign

$$\bigdiamond_{l=1}^{m} X_l := X_1 \lozenge X_2 \lozenge \cdots \lozenge X_m. \tag{5.13}$$

Definition and Theorem 5.5 (basis of projective algebra in the plus approach)**.** Let $\mathbf{b}$ be a binary variable with n digits, $\{P_{\mathbf{b}}\}$ with $S(\mathbf{b}) = 1$ a set of n basis 1-vectors from Λ_n^{1+} and $P_0^+ \in \Lambda_n^{0+} \setminus \{\mathbf{0}\}$ a vector of grade 0. Then the homogeneous multi vectors

$$P_{\mathbf{b}} := \left\{ \begin{array}{ll} P_0^+, & S(\mathbf{b}) = 0, \\ \bigwedge_{l=1}^{S(\mathbf{b})} P_{l\mathbf{b}}, & 0 < S(\mathbf{b}) \leq n, \end{array} \right. \tag{5.14}$$

form a basis for the 2^n-dimensional vector space of projective algebra Λ_n.

Definition and Theorem 5.6 (basis of projective algebra in the minus approach)**.** Let $\mathbf{b}$ be a binary variable with n digits, $\{E_{\mathbf{b}}\}$ with $S(\mathbf{b}) = 1$ a set of n basis 1-vectors from Λ_n^{1-} and $E_0^- \in \Lambda_n^{0-} \setminus \{\mathbf{0}\}$ a vector of grade 0. Then the homogeneous multi vectors

$$E_{\mathbf{b}} := \left\{ \begin{array}{ll} E_0^-, & S(\mathbf{b}) = 0, \\ \bigvee_{l=1}^{S(\mathbf{b})} E_{l\mathbf{b}}, & 0 < S(\mathbf{b}) \leq n, \end{array} \right. \tag{5.15}$$

form a basis for the 2^n-dimensional vector space of projective algebra Λ_n.

Notation 5.7 (basis n-vectors)**.** For the n-vectors of the basis we also use the notation

$$I^+ := P_{\mathbf{b}}^+, \qquad I^- := E_{\mathbf{b}}^-, \qquad S(\mathbf{b}) = n. \tag{5.16}$$

The superscript plus or minus sign can be omitted, if I stands for both approaches or if it is clear from the context in which approach one is working.

Notation 5.8 (basis of projective algebra)**.** If an expression is true in the plus and in the minus approache to projective algebra, the basis (5.14) and (5.15) are denoted by $\{B_{\mathbf{b}}\}$, i. e. $\{B^+_{\mathbf{b}}\} = \{P_{\mathbf{b}}\}$ and $\{B^-_{\mathbf{b}}\} = \{E_{\mathbf{b}}\}$.

Proof. In order to proof Theorem 5.5 and 5.6 we need to show that the 2^n basis vectors $B_{\mathbf{b}}$ are linearly independent. According to Definition 5.2 homogeneous multi vectors of different grades are linearly independent. It remains to show, that homogeneous multi vectors $B_{\mathbf{b}}$ of the same grade $S(\mathbf{b}) = k$ are linearly independent. Trivially this is the case for $S(\mathbf{b}) = 0$ and $S(\mathbf{b}) = n$. The 1-vectors $B_{\mathbf{b}}$ are by precondition linearly independent. And for $2 \leq S(\mathbf{b}) \leq n-1$ we get with the coefficients $\lambda_{\mathbf{b}} \in \mathbb{F}$ and with the binary n-digit numbers $\mathbf{c}$

$$\mathbf{0} = \sum_{S(\mathbf{b})=k} \lambda_{\mathbf{b}} B_{\mathbf{b}} \tag{5.17}$$

$$\begin{aligned} \iff \quad \mathbf{0} &= \left[\sum_{S(\mathbf{b})=k} \lambda_{\mathbf{b}} B_{\mathbf{b}} \right] \lozenge B_{\bar{\mathbf{c}}} \qquad \forall\, \mathbf{c} \text{ with } S(\mathbf{c}) = k \\ &= \sum_{S(\mathbf{b})=k} \lambda_{\mathbf{b}} \left(B_{\mathbf{b}} \lozenge B_{\bar{\mathbf{c}}} \right) \\ &= \lambda_{\mathbf{c}} \left(B_{\mathbf{c}} \lozenge B_{\bar{\mathbf{c}}} \right) \end{aligned} \tag{5.18}$$

$$\iff \quad 0 = \lambda_{\mathbf{c}}, \qquad \forall\, \mathbf{c} \text{ with } S(\mathbf{c}) = k. \tag{5.19}$$

□

The sets $\{P_{\mathbf{b}}\}$ and $\{E_{\mathbf{b}}\}$ both form a basis for the same 2^n-dimensional vector space $\Lambda_n(+,\cdot)$. This is why there is always a regular transformation changing the vectors of one basis into the vectors of the other basis.

Notation 5.9 (basis transformation)**.** Let $\zeta_{\mathbf{bc}}$ denote the basis transformation with which we can express the basis vectors of $\{E_{\mathbf{b}}\}$ in terms of the basis vectors of $\{P_{\mathbf{b}}\}$,

$$E_{\mathbf{b}} = \sum_{S(\mathbf{c})=n-k} \zeta_{\mathbf{bc}} P_{\mathbf{c}}, \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{5.20}$$

and let $\zeta^{-1}_{\mathbf{bc}}$ denote the inverse basis transformation with which we can express the basis vectors of $\{P_{\mathbf{b}}\}$ in terms of the basis vectors of $\{E_{\mathbf{b}}\}$

$$P_{\mathbf{b}} = \sum_{S(\mathbf{c})=n-k} \zeta^{-1}_{\mathbf{bc}} E_{\mathbf{c}}, \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{5.21}$$

where $\mathbf{b}$ and $\mathbf{c}$ represent n-digit binary numbers.

We then get with the n-digit binary number $\mathbf{d}$ with $S(\mathbf{b}) = S(\mathbf{d}) = k$

$$\sum_{S(\mathbf{c})=n-k} \zeta_{\mathbf{bc}} \zeta^{-1}_{\mathbf{cd}} = \sum_{S(\mathbf{c})=n-k} \zeta^{-1}_{\mathbf{bc}} \zeta_{\mathbf{cd}} = \delta_{\mathbf{bd}}, \tag{5.22}$$

where $\delta_{\mathbf{bd}}$ is the Kronecker-delta-symbol,

$$\delta_{\mathbf{bd}} := \begin{cases} 1, & \mathbf{b} = \mathbf{d}, \\ 0, & \mathbf{b} \neq \mathbf{d}. \end{cases} \tag{5.23}$$

By means of the basis transformation $\zeta_{\mathbf{bc}}$ it is now possible to compute the major outer product $\wedge$ also in the minus approach and the minor outer product $\vee$ also in the plus approach. Thus, the two outer products of the projective algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ are available for any multi vectors of the algebra.

Theorem 5.10. *Let* $\mathbf{b}, \mathbf{c}, \mathbf{d}$ *and* $\mathbf{e}$ *be binary n-digit numbers with*

$$\mathbf{d} = \mathbf{b} \text{ AND } \mathbf{c} \tag{5.24}$$

$$\mathbf{e} = \mathbf{b} \text{ XOR } \mathbf{c} \tag{5.25}$$

Then we have

$$B_{\mathbf{b}} \lozenge B_{\mathbf{c}} = \begin{cases} \alpha_{\mathbf{bc}} B_{\mathbf{e}}, & S(\mathbf{d}) = 0,\ S(\mathbf{b}) \neq 0,\ S(\mathbf{c}) \neq 0, \\ \mathbf{0}, & S(\mathbf{d}) \neq 0, \\ \mathbf{0}, & S(\mathbf{b}) \neq 0 \text{ or } S(\mathbf{c}) \neq 0, \end{cases} \tag{5.26}$$

with the coefficients

$$\alpha_{\mathbf{bc}} = (-1)^{\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m} = (-1)^{\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m}. \tag{5.27}$$

Proof. We determine two sequences of transpositions such that the factors, i. e. the 1-vectors of the outer product $B_{\mathbf{b}} \lozenge B_{\mathbf{c}}$ get in an order where equal 1-vectors are direct neighbors. We depict the order of 1-vectors in the product $B_{\mathbf{b}} \lozenge B_{\mathbf{c}}$ by

$$(b_0 b_1 \dots b_{n-2} b_{n-1})(c_0 c_1 \dots c_{n-2} c_{n-1}). \tag{5.28}$$

First sequence of transpositions: We move the 1-vectors of $B_{\mathbf{c}}$ and shift them in terms of transpositions from the right side to the left side. Take first c_0 and shift it to the left side until it stands between b_0 and b_1, then take c_1 and shift it to the left side until it stands between b_1 and b_2 and so forth. The last transposition of this first sequence is then the shift of c_{n-2} to the left side such that it stands between b_{n-2} and b_{n-1}. The resulting order of the 1-vectors

$$(b_0 c_0 b_1 c_1 \dots b_{n-2} c_{n-2} b_{n-1} c_{n-1}) \tag{5.29}$$

was achieved by altogether

$$\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m \tag{5.30}$$

transpositions.

Second sequence of transpositions: We move the 1-vectors of $B_{\mathbf{b}}$ and shift them in terms of transpositions from the left side to the right side. Take first b_{n-1} and shift it to the right side until it stands between c_{n-2} and c_{n-1}, then take b_{n-2} and shift it to the right side until it stands between c_{n-3} and c_{n-2} and so forth. The last transposition of this second sequence is then the shift of b_1 to the right side such that it stands between c_0 and c_1. The resulting order of the 1-vectors

$$(b_0 c_0 b_1 c_1 \dots b_{n-2} c_{n-2} b_{n-1} c_{n-1}). \tag{5.31}$$

was achieved by altogether

$$\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m \tag{5.32}$$

transpositions.

In both sequences the number of transpositions is the same

$$\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m = \sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m. \tag{5.33}$$

The outer product $B_{\mathbf{b}} \lozenge B_{\mathbf{c}}$ vanishes if and only if there is at least one pair (b_l, c_l), $0 \leq l \leq n-1$, with $b_l c_l = 1$, i. e. if and only if $S(\mathbf{d}) \neq 0$. □

Corollary 5.11. *Let* $\mathbf{b}$ *be a n-digit binary number with* $S(\mathbf{b}) \neq 0$ *and* $S(\mathbf{b}) \neq n$*. Then we have*

$$B_{\mathbf{b}} \lozenge B_{\overline{\mathbf{b}}} = \begin{cases} \alpha_{\mathbf{b}\overline{\mathbf{b}}} I & S(\mathbf{b}) \neq 0 \text{ and } S(\mathbf{b}) \neq n \\ \mathbf{0} & S(\mathbf{b}) = 0 \text{ or } S(\mathbf{b}) = n \end{cases} \tag{5.34}$$

Proof. It is a consequence of Theorem 5.10 with $\mathbf{c} = \overline{\mathbf{b}}$. □

Theorem 5.12. *Let* $\mathbf{b}$ *and* $\mathbf{c}$ *be n-digit binary numbers. We then have*

$$\alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}} = (-1)^{S(\mathbf{b})S(\mathbf{c}) - S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \tag{5.35}$$

Proof.

$$\begin{aligned} \alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}} &= (-1)^{\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m + \sum_{l=1}^{n-1} c_l \sum_{m=0}^{l-1} b_m} \\ &= (-1)^{c_0 \sum_{m=1}^{n-1} b_m + \sum_{l=1}^{n-2} c_l \left[\sum_{m=l+1}^{n-1} b_m + \sum_{m=0}^{l-1} b_m\right] + c_{n-1} \sum_{m=0}^{n-2} b_m} \\ &= (-1)^{c_0 S(\mathbf{b}) + S(\mathbf{b}) \sum_{l=1}^{n-2} c_l + c_{n-1} S(\mathbf{b}) - \left[c_0 b_0 + \sum_{l=1}^{n-2} c_l b_l + c_{n-1} b_{n-1}\right]} \\ &= (-1)^{S(\mathbf{b})S(\mathbf{c}) - S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \end{aligned} \tag{5.36}$$

□

Theorem 5.13. *For homogeneous multi vectors of grade r and s we have,*

$$X_{\overline{r}} \lozenge Y_{\overline{s}} = (-1)^{rs} \cdot Y_{\overline{s}} \lozenge X_{\overline{r}}. \tag{5.37}$$

Proof. Let $\mathbf{b}$ and $\mathbf{c}$ be n-digit binary numbers with sum of digits $S(\mathbf{b}) = r$ and $S(\mathbf{c}) = s$. Then there are coefficients $\lambda_{\mathbf{b}}$ and $\mu_{\mathbf{c}}$ such that

$$\begin{aligned} X_{\overline{r}} \lozenge Y_{\overline{s}} &= \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}}\right] \lozenge \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}}\right] \\ &= \sum_{\substack{S(\mathbf{b})=r \\ S(\mathbf{c})=s}} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot B_{\mathbf{b}} \lozenge B_{\mathbf{c}} \\ &= \sum_{\substack{S(\mathbf{b})=r \\ S(\mathbf{c})=s}} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot \alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}} B_{\mathbf{c}} \lozenge B_{\mathbf{b}} \\ &= \sum_{\substack{S(\mathbf{b})=r \\ S(\mathbf{c})=s}} (-1)^{rs} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot B_{\mathbf{c}} \lozenge B_{\mathbf{b}} \\ &= (-1)^{rs} \cdot \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}}\right] \lozenge \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}}\right] \\ &= (-1)^{rs} \cdot Y_{\overline{s}} \lozenge X_{\overline{r}}. \end{aligned} \tag{5.38}$$

where we used Theorem 5.12 in the step from the third to the fourth line. □

Definition 5.14 (harmonic model of projective algebra)**.** Let $\mathbf{b}$ and $\mathbf{c}$ be n-digit binary numbers. The *harmonic model of projective algebra* Λ_n is determined by the basis transformation

$$\zeta_{\mathbf{bc}} := \alpha_{\mathbf{b}\overline{\mathbf{b}}}\delta_{\mathbf{c}\overline{\mathbf{b}}}. \tag{5.39}$$

Since $\alpha_{\mathbf{b}\overline{\mathbf{b}}}\delta_{\mathbf{c}\overline{\mathbf{b}}}$ is a regular transformation, the harmonic model of projective algebra is well defined.

Theorem 5.15. *In the harmonic model of projective algebra we have*

$$E_{\mathbf{b}} = \alpha_{\mathbf{b}\bar{\mathbf{b}}} P_{\bar{\mathbf{b}}} \quad \Longleftrightarrow \quad P_{\mathbf{b}} = \alpha_{\bar{\mathbf{b}}\mathbf{b}} E_{\bar{\mathbf{b}}}, \qquad 0 \leq S(\mathbf{b}) \leq n, \tag{5.40}$$

with

$$\alpha_{\mathbf{b}\bar{\mathbf{b}}} = (-1)^{\sum_{l=0}^{n-2} \bar{b}_l \sum_{m=l+1}^{n-1} b_m} = (-1)^{\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} \bar{b}_m}, \tag{5.41}$$

$$\alpha_{\bar{\mathbf{b}}\mathbf{b}} = (-1)^{k(n-k)} \alpha_{\mathbf{b}\bar{\mathbf{b}}}. \tag{5.42}$$

Proof. Insert equation (5.39) into equation (5.20). □

Theorem 5.16. *In the harmonic model of projective algebra we have for the k-vectors*

$$X_{\bar{k}}^{+} = \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}}, \qquad Y_{\bar{k}}^{-} = \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}} E_{\mathbf{b}}, \qquad 0 < k < n, \tag{5.43}$$

the two equivalent equations

$$X_{\bar{k}}^{+} \wedge Y_{\bar{k}}^{-} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] I^{+} \Leftrightarrow X_{\bar{k}}^{+} \vee Y_{\bar{k}}^{-} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] I^{-}. \tag{5.44}$$

Proof.

$$\begin{aligned} & X_{\bar{k}}^{+} \wedge Y_{\bar{k}}^{-} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] I^{+} && (5.45) \\ \Longleftrightarrow \quad & \zeta_{\mathbf{bc}} = \alpha_{\mathbf{b}\bar{\mathbf{b}}} \delta_{\mathbf{c}\bar{\mathbf{b}}} \\ \Longleftrightarrow \quad & \zeta_{\mathbf{cb}}^{-1} = \alpha_{\mathbf{b}\bar{\mathbf{b}}} \delta_{\mathbf{b}\bar{\mathbf{c}}} \\ \Longleftrightarrow \quad & X_{\bar{k}}^{+} \vee Y_{\bar{k}}^{-} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] I^{-} \end{aligned}$$

□

The harmonic model of projective algebra is strongly related to what HANNS-JÖRG STOSS calls *Harmonisches Punkt-Geraden-System* [Sto09, Bezeichnung 2.21 on p. 29] and more general *Harmonisches Punkt-Hyperebenen-System* [Sto09, Bezeichnung 3.36 on p. 51].

6. CLIFFORD DOUBLE ALGEBRA Γ_n

Definition 6.1 (symmetric bilinear forms)**.** Let

$$\begin{aligned} \mathfrak{b}: \quad \Lambda_n^1 \otimes \Lambda_n^1 \quad &\longrightarrow \quad \Lambda_n^0 \\ (X,Y) \quad &\longmapsto \quad \mathfrak{b}(X,Y) = \mathfrak{b}(Y,X) \end{aligned} \tag{6.1}$$

be two symmetric bilinear forms, one in the plus, the other in the minus approach of projective algebra Λ_n.

Notation 6.2 (multiple Clifford product signs)**.** The two Clifford products of double Clifford algebra Γ_n are denoted by 'no sign' and $*$. For multiple Clifford products we use

$$\prod_{l=1}^{m} X_l := X_1 X_2 \cdots X_m, \qquad \overset{m}{\underset{l=1}{\ast}} X_l := X_1 * X_2 * \cdots * X_m. \tag{6.2}$$

Definition 6.3 (orthogonal basis)**.** Let $\Lambda_n(+,\cdot,\wedge,\vee)$ be a projective $\mathbb{F}$-algebra. A basis $\{B_{\mathbf{b}}\}$ of Λ_n is called an *orthogonal basis*, if and only if the set of all 1-vectors

$$\mathscr{O} := \{B_{\mathbf{b}} \mid S(\mathbf{b}) = 1\} \tag{6.3}$$

is a system of orthogonal vectors with respect to the bilinear form $\mathfrak{b}$,

$$\mathfrak{b}(B_{\mathbf{b}}, B_{\mathbf{b}_1}) = 0 \quad \text{for all } \mathbf{b} \neq \mathbf{b}_1 \text{ with } B_{\mathbf{b}}, B_{\mathbf{b}_1} \in \mathscr{O}. \tag{6.4}$$

Definition 6.4 (Clifford double algebra Γ_n)**.** Let $\Lambda_n(+,\cdot,\wedge,\vee)$ be a projective $\mathbb{F}$-algebra and let $\mathfrak{b}$ denote two symmetric bilinear forms according to equation (6.1). A *Clifford double* $\mathbb{F}$*-algebra* $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ is the projective algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ extended by the operations

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \longrightarrow & \Lambda_n \\ (A,B) & \longmapsto & AB \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \stackrel{*}{\longrightarrow} & \Lambda_n \\ (A,B) & \longmapsto & A*B \end{array} \tag{6.5}$$

called *major Clifford product* (no sign) and *minor Clifford product* $(*)$. The Clifford products obey the following conditions:

(C1) They are associative and distributive,

$$(AB)C = A(BC), \qquad A,B,C \in \Lambda_n, \tag{6.6}$$

$$A(B+C) = AB+AC, \qquad (A+B)C = AC+BC, \tag{6.7}$$

$$(A*B)*C = A*(B*C), \tag{6.8}$$

$$A*(B+C) = A*B+A*C, \qquad (A+B)*C = A*C+B*C. \tag{6.9}$$

(C2) If at least one factor is a scalar $X_{\overline{0}} = \alpha 1$, $\alpha \in \mathbb{F}$, the Clifford products reduce to scalar multiplication,

$$X_{\overline{0}}^{+} M = M X_{\overline{0}}^{+} = \alpha M, \qquad \forall\, X_{\overline{0}}^{+} \in \Lambda_n^{0+}, \qquad \forall\, M \in \Lambda_n, \tag{6.10}$$

$$X_{\overline{0}}^{-} * M = M * X_{\overline{0}}^{-} = \alpha M, \qquad \forall\, X_{\overline{0}}^{-} \in \Lambda_n^{0-}, \qquad \forall\, M \in \Lambda_n. \tag{6.11}$$

(C3) Contraction rule for the 1-vectors.

$$X_{\overline{1}}^{+} X_{\overline{1}}^{+} = \mathfrak{b}^{+}(X_{\overline{1}}^{+}, X_{\overline{1}}^{+}) \in \Lambda_n^{0+} \qquad \forall\, X_{\overline{1}}^{+} \in \Lambda_n^{1+} \tag{6.12}$$

$$X_{\overline{1}}^{-} * X_{\overline{1}}^{-} = \mathfrak{b}^{-}(X_{\overline{1}}^{-}, X_{\overline{1}}^{-}) \in \Lambda_n^{0-} \qquad \forall\, X_{\overline{1}}^{-} \in \Lambda_n^{1-} \tag{6.13}$$

(C4) For an orthogonal set $\mathscr{O} = \{X_1, X_2, \ldots, X_m\} \subset \Lambda_n^1$ of m homogeneous multi vectors of grade 1 the Clifford and outer products are the same,

$$\prod_{l=1}^{m} X_l = \bigwedge_{l=1}^{m} X_l \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \mathscr{O}^{+} \text{ is a orthogonal system with} \\ \text{respect to the bilinear form } \mathfrak{b}^{+}. \end{array} \right. \tag{6.14}$$

$$\mathop{\ast}_{l=1}^{m} X_l = \bigvee_{l=1}^{m} X_l \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \mathscr{O}^{-} \text{ is a orthogonal system with} \\ \text{respect to the bilinear form } \mathfrak{b}^{-}. \end{array} \right. \tag{6.15}$$

Definition 6.5 (combined Clifford product)**.** Any mathematical term which contains the combined Clifford product $\circledast$ can be read twice: Firstly with respect to the plus approach as major Clifford product (no sign) and secondly with respect to the minus approach as minor Clifford product $*$.

Notation 6.6 (multiple combined Clifford product sign)**.** For the multiple combined Clifford product we use the sign

$$\mathop{\circledast}_{l=1}^{m} X_l := X_1 \circledast X_2 \circledast \cdots \circledast X_m. \tag{6.16}$$

Definition 6.4 of a Clifford double algebra $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ coincides in the case of a non-degenerate quadratic form

$$Q(X) := \mathfrak{b}(X,X), \quad X \in \Gamma_n^1, \tag{6.17}$$

with the Clifford double algebra mentioned in Theorem 3.10 and Notation 3.11. Compared to Theorem 3.10 the Definition 6.4 of a Clifford double algebra is more general, since it includes Clifford double algebras with a degenerate quadratic form Q. This is why we regard Definition 6.4 as the standard definition of a Clifford double algebra.

Theorem 6.7. *Let $\{B_{\mathbf{b}}\}$ be an orthogonal basis of the Clifford double $\mathbb{F}$-algebra $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$. We then have*

$$B_{\mathbf{b}} = B_0 \in \Gamma_n^0, \qquad S(\mathbf{b}) = 0, \tag{6.18}$$

$$B_{\mathbf{b}} = \overset{S(\mathbf{b})}{\underset{l=1}{\circledast}} B_{l\mathbf{b}} \in \Gamma_n^{S(\mathbf{b})}, \qquad 1 \le S(\mathbf{b}) \le n. \tag{6.19}$$

Proof. The proposition follows immediately from equations (5.14), (5.15), (6.14) and (6.15). □

Theorem 6.8. *Let $\{B_{\mathbf{b}}\}$ be an orthogonal basis of $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ and the variables $\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}$ and $\mathbf{u}$ n-digit binary numbers with*

$$\mathbf{d} = \mathbf{b} \text{ AND } \mathbf{c}, \tag{6.20}$$

$$\mathbf{e} = \mathbf{b} \text{ XOR } \mathbf{c}, \tag{6.21}$$

$$S(\mathbf{u}) = n. \tag{6.22}$$

We then have

$$B_{\mathbf{b}} \circledast B_{\mathbf{c}} = \begin{cases} B_{\mathbf{b}} B_{\mathbf{c}}, & \begin{cases} S(\mathbf{b}) = 0 \\ \text{or} \\ S(\mathbf{c}) = 0, \end{cases} \\ \alpha_{\mathbf{bc}} B_{\mathbf{e}}, & \begin{cases} S(\mathbf{d}) = 0, \\ S(\mathbf{b}) \neq 0, \\ S(\mathbf{c}) \neq 0, \end{cases} \\ \alpha_{\mathbf{bc}} \left[\prod_{p=1}^{S(\mathbf{d})} \mathfrak{b}(B_{p\mathbf{d}}, B_{p\mathbf{d}}) \right] B_{\mathbf{e}}, & S(\mathbf{d}) > 0, \end{cases} \tag{6.23}$$

with

$$S(\mathbf{b}) + S(\mathbf{c}) = 2 \cdot S(\mathbf{d}) + S(\mathbf{e}) \tag{6.24}$$

$$0 \le |S(\mathbf{b}) - S(\mathbf{c})| \le S(\mathbf{e}) \le \mathscr{D}_n(S(\mathbf{b}) + S(\mathbf{c})) \le n \tag{6.25}$$

$$\mathscr{D}_n(i) = \begin{cases} i & 0 \le i \le n \\ 2n - i & n < i \le 2n \end{cases} \tag{6.26}$$

and in the special case $\mathbf{c} = \mathbf{u}$ and $I = B_{\mathbf{u}}$

$$B_{\mathbf{b}} \circledast I = (-1)^{S(\mathbf{b})(n-1)} I \circledast B_{\mathbf{b}} = \begin{cases} \alpha_{\mathbf{bu}} B_{\overline{\mathbf{b}}} & S(\mathbf{d}) = 0, \\ \alpha_{\mathbf{bu}} \left[\prod_{p=1}^{S(\mathbf{b})} \mathfrak{b}(B_{p\mathbf{b}}, B_{p\mathbf{b}}) \right] B_{\overline{\mathbf{b}}}, & S(\mathbf{d}) > 0, \end{cases} \tag{6.27}$$

Proof. According to the proof of Theorem 5.10 we need

$$\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m = \sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m \tag{6.28}$$

transpositions, in order to bring the 1-vectors of $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$ into such an order that equal 1-vectors are standing side by side. This is how we get the coefficient $\alpha_{\mathbf{bc}}$ in equation (6.23). There are $S(\mathbf{d})$ 1-vectors $P_{p\mathbf{b}}$, which show up twice in the product $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$ and where the Clifford product reduces according to the contraction rule (6.12) or (6.13) to the scalar $\mathfrak{b}(P_{p\mathbf{b}}, P_{p\mathbf{b}})$. The remaining 1-vectors are collected in the homogeneous multi vector $B_{\mathbf{e}}$. Thus equation (6.23) is true.

The $S(\mathbf{b})+S(\mathbf{c})$ 1-vectors of the product $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$ contain $S(\mathbf{d})$ pairs of identical 1-vectors. This is why the homogeneous multi vector $B_{\mathbf{e}}$ is made up of exactly $S(\mathbf{b})+S(\mathbf{c})-2\cdot S(\mathbf{d})$ 1-vectors. This is equivalent to equation (6.24).

The lower limit for $S(\mathbf{e})$ in equation (6.25) is realised, if all $S(\mathbf{b})$ 1-vectors of $B_{\mathbf{b}}$ show up in $B_{\mathbf{c}}$ or if all $S(\mathbf{c})$ 1-vectors of $B_{\mathbf{c}}$ show up in $B_{\mathbf{b}}$. The upper limit for $S(\mathbf{e})$ in equation (6.25), i. e. the expression of equation (6.26) becomes clear, if we consider that there are at most n different 1-vectors. So, if the sum $S(\mathbf{b})+S(\mathbf{c})$ is bigger than n, there must be at least $S(\mathbf{b})+S(\mathbf{c})-n$ pairs of identical 1-vectors.

Inserting $\mathbf{c}=\mathbf{u}$ in equation (6.23) we get the special case of equation (6.27). □

Theorem 6.9. *For the Clifford product between the homogeneous multi vectors $X_{\bar{r}}$ and $Y_{\bar{s}}$ we get*

$$X_{\bar{r}} \circledast Y_{\bar{s}} = \sum_{k=0}^{m} \langle X_{\bar{r}} \circledast Y_{\bar{s}} \rangle_{|r-s|+2k} \tag{6.29}$$

with

$$m = \frac{\mathscr{D}_n(r+s)-|r-s|}{2}, \tag{6.30}$$

and the index function $\mathscr{D}_n$ of equation (6.26).

Proof. We can write the homogeneous multi vectors $X_{\bar{r}}$ und $Y_{\bar{s}}$ as linear combinations with respect to the basis $B_{\mathbf{b}}$,

$$X_{\bar{r}} = \sum_{S(\mathbf{b})=r} \mu_{\mathbf{b}} B_{\mathbf{b}}, \tag{6.31}$$

$$Y_{\bar{s}} = \sum_{S(\mathbf{c})=s} \nu_{\mathbf{c}} B_{\mathbf{c}}, \tag{6.32}$$

an then get

$$X_{\bar{r}} \circledast Y_{\bar{s}} = \sum_{\substack{S(\mathbf{b})=r \\ S(\mathbf{c})=s}} \mu_{\mathbf{b}} \nu_{\mathbf{c}} B_{\mathbf{b}} \circledast B_{\mathbf{c}}. \tag{6.33}$$

According to Theorem 6.8, the grades of the product $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$ lie between $|r-s|$ and $\mathscr{D}_n(r+s)$ and differ by a multiple of 2 from the limits $|r-s|$ or $\mathscr{D}_n(r+s)$, since, if there are identical pairs present in the product $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$, always two 1-vectors disappear. □

We are now in the position to compute the Clifford product of any two generic multi vectors M and N,

$$M \circledast N = \sum_{r,s=0}^{n} \langle M \rangle_r \circledast \langle N \rangle_s, \qquad M,N \in \Gamma_n. \tag{6.34}$$

Theorem 6.10. *For the homogeneous multi vectors $X_{\bar{r}}$ and $Y_{\bar{s}}$ we have*

$$X_{\bar{r}} \lozenge Y_{\bar{s}} = \begin{cases} \langle X_{\bar{r}} \circledast Y_{\bar{s}} \rangle_{r+s}, & r+s \le n,\ r \ne 0,\ s \ne 0 \\ \mathbf{0}, & \text{else.} \end{cases} \tag{6.35}$$

Proof. For $r+s>n$, $r=0$ or $s=0$ the outer product $\lozenge$ vanishes. In the cases $r+s\leq n$, $r\neq 0$ and $s\neq 0$ the outer product $X_{\overline{r}}\lozenge Y_{\overline{s}}$ has not to vanish. Using the equations (6.33), (6.23) and (5.26) we get

$$\begin{aligned}\langle X_{\overline{r}}\circledast Y_{\overline{s}}\rangle_{r+s} &= \sum_{\substack{S(\mathbf{b})=r\\ S(\mathbf{c})=s}} \mu_{\mathbf{b}}\nu_{\mathbf{c}}\,\langle B_{\mathbf{b}}\circledast B_{\mathbf{c}}\rangle_{r+s} = \sum_{\substack{S(\mathbf{b})=r\\ S(\mathbf{c})=s}} \mu_{\mathbf{b}}\nu_{\mathbf{c}}\,\langle \alpha_{\mathbf{bc}}B_{\mathbf{b}\,\mathrm{XOR}\,\mathbf{c}}\rangle_{r+s}\\ &= \sum_{\substack{S(\mathbf{b})=r\\ S(\mathbf{c})=s}} \mu_{\mathbf{b}}\nu_{\mathbf{c}}B_{\mathbf{b}}\lozenge B_{\mathbf{c}} = X_{\overline{r}}\lozenge Y_{\overline{s}}.\end{aligned} \tag{6.36}$$

□

Definition 6.11 (inner products)**.** Let $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ be a Clifford double $\mathbb{F}$-algebra. The operations

$$\begin{array}{ccc}\Lambda_n\times\Lambda_n & \stackrel{\cdot}{\longrightarrow} & \Lambda_n\\ (A,B) & \longmapsto & A\cdot B\end{array}\qquad\qquad\begin{array}{ccc}\Lambda_n\times\Lambda_n & \stackrel{\circ}{\longrightarrow} & \Lambda_n\\ (A,B) & \longmapsto & A\circ B\end{array} \tag{6.37}$$

are called *major inner product* $(\cdot)$ and *minor inner product* $(\circ)$ and obey the following conditions:

$$X_{\overline{r}}\cdot Y_{\overline{s}} := \langle X_{\overline{r}}Y_{\overline{s}}\rangle_{|r-s|}, \qquad X_{\overline{r}}\in\Gamma_n^{r+},\ Y_{\overline{s}}\in\Gamma_n^{s+}, \tag{6.38}$$

$$X_{\overline{r}}\circ Y_{\overline{s}} := \langle Y_{\overline{r}}*X_{\overline{s}}\rangle_{|r-s|} \qquad X_{\overline{r}}\in\Gamma_n^{r-},\ Y_{\overline{s}}\in\Gamma_n^{s-}, \tag{6.39}$$

$$A\cdot(B+C)=A\cdot B+A\cdot C, \qquad A,B,C\in\Gamma_n, \tag{6.40}$$

$$(A+B)\cdot C=A\cdot C+B\cdot C, \tag{6.41}$$

$$A\circ(B+C)=A\circ B+A\circ C, \tag{6.42}$$

$$(A+B)\circ C=A\circ C+B\circ C. \tag{6.43}$$

Definition 6.12 (combined inner product)**.** Any mathematical term which contains the combined inner product $\odot$ can be read twice: Firstly with respect to the plus approach as major inner product $\cdot$ and secondly with respect to the minus approach as minor inner product $\circ$.

The inner product $\odot$ is well defined in Definition 6.11 since the Clifford product $\circledast$ is also distributive.

Theorem 6.13. *In the case $r=0$ or $s=0$ the inner product and the scalar multiplication are the same,*

$$A_{\overline{r}}\odot B_{\overline{s}} = A_{\overline{r}}\circledast B_{\overline{s}} = A_{\overline{r}}B_{\overline{s}}, \qquad r=0 \quad \text{or} \quad s=0. \tag{6.44}$$

Proof. Compare equations (6.10) or (6.11) with Definition 6.11. □

Theorem 6.14. *For homogeneous multi vectors of grade r and s with $r\geq s$ we have,*

$$X_{\overline{r}}\odot Y_{\overline{s}} = (-1)^{s(r-1)}\,Y_{\overline{s}}\odot X_{\overline{r}}. \tag{6.45}$$

Proof. Let $\mathbf{b}$ and $\mathbf{c}$ be n-digit binary numbers with sum of digits $S(\mathbf{b})=r$ and $S(\mathbf{c})=s$. Then there are coefficients $\lambda_{\mathbf{b}}$ and $\mu_{\mathbf{c}}$ such that

$$\begin{aligned}X_{\overline{r}}\odot Y_{\overline{s}} &= \left\langle\left[\sum_{S(\mathbf{b})=r}\lambda_{\mathbf{b}}B_{\mathbf{b}}\right]\circledast\left[\sum_{S(\mathbf{c})=s}\mu_{\mathbf{c}}B_{\mathbf{c}}\right]\right\rangle_{r-s}\\ &= \left\langle\sum_{\substack{S(\mathbf{b})=r\\ S(\mathbf{c})=s}}\lambda_{\mathbf{b}}\mu_{\mathbf{c}}\cdot B_{\mathbf{b}}\circledast B_{\mathbf{c}}\right\rangle_{r-s}\end{aligned} \tag{6.46}$$

$$
\begin{aligned}
&= \left\langle \sum_{\substack{S(\mathbf{b})=r \\ S(\mathbf{c})=s}} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot \alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}} B_{\mathbf{c}} \circledast B_{\mathbf{b}} \right\rangle_{r-s} \\
&= \left\langle \sum_{\substack{S(\mathbf{b})=r \\ S(\mathbf{c})=s}} (-1)^{rs-s} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot B_{\mathbf{c}} \circledast B_{\mathbf{b}} \right\rangle_{r-s} \\
&= \left\langle (-1)^{rs-s} \cdot \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}} \right] \circledast \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}} \right] \right\rangle_{r-s} \\
&= (-1)^{s(r-1)} \cdot Y_{\overline{s}} \odot X_{\overline{r}}.
\end{aligned}
$$

where we used equation (5.35) for the step from the third to the fourth line. □

Theorem 6.15. *For* 1*-vectors* X *and* Y *we have*

$$X \circledast Y = X \odot Y + X \lozenge Y, \qquad X, Y \in \Gamma_n^1, \tag{6.47}$$

where the inner product is symmetric and the outer product is antisymmetric. The inner product is given by

$$X \odot Y = \mathfrak{b}(X, Y), \qquad X, Y \in \Gamma_n^1. \tag{6.48}$$

Proof. Equation (6.47) is a consequence of Theorem 6.9. According to equation (5.37) the outer product between two 1-vectors is antisymmetric and according to equation (6.45) the inner product between two 1-vectors is symmetric. With $X = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} B_{\mathbf{b}}$ and $Y = \sum_{S(\mathbf{c})=1} \mu_{\mathbf{c}} B_{\mathbf{c}}$ we get equation (6.48),

$$
\begin{aligned}
X \odot Y &= \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}\mu_{\mathbf{b}}\, \mathfrak{b}(B_{\mathbf{b}}, B_{\mathbf{b}}) \\
&= \mathfrak{b}\left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} B_{\mathbf{b}}\,,\, \sum_{S(\mathbf{c})=1} \mu_{\mathbf{c}} B_{\mathbf{c}} \right) \\
&= \mathfrak{b}(X, Y).
\end{aligned} \tag{6.49}
$$

□

Theorem 6.16. *Let* Γ_n *be a Clifford double* $\mathbb{F}$*-algebra, let* $\{B_{\mathbf{b}}\}$ *be the orthogonal bases in the two approaches and let*

$$
\begin{array}{lccc}
\phi : & \Gamma_n^+ & \longrightarrow & \Gamma_n^- \\
& P_{\mathbf{b}} & \longmapsto & \phi(P_{\mathbf{b}}) := E_{\mathbf{b}}
\end{array} \tag{6.50}
$$

with

$$\phi(A \wedge B) := \phi(A) \vee \phi(B) \tag{6.51}$$

represent an algebra isomorphism mapping the plus approach $\Lambda_n(+,\cdot,\wedge)$ *onto the minus approach* $\Lambda_n(+,\cdot,\vee)$*. The isomorphism* ϕ *preserves the Clifford products*

$$\phi(AB) = \phi(A) * \phi(B), \qquad A, B \in \Gamma_n, \tag{6.52}$$

if and only if it preserves the bilinear forms

$$\phi(\mathfrak{b}^+(X, Y)) = \mathfrak{b}^-(\phi(X), \phi(Y)), \qquad X, Y \in \Gamma_n^{1+}. \tag{6.53}$$

Proof.

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \phi(AB) = \phi(A) * \phi(B), \quad A, B \in \Gamma_n, \\
\Longleftrightarrow &\qquad\qquad\qquad\qquad \phi(XY) = \phi(X) * \phi(Y), \quad X, Y \in \Gamma_n^{1+},
\end{aligned}
$$

$$\begin{aligned}
&\iff && \phi(X\cdot Y)+\phi(X\wedge Y)=\phi(X)\circ\phi(Y)+\phi(X)\vee\phi(Y),\\
&\iff && \phi(X\cdot Y)=\phi(X)\circ\phi(Y),\\
&\iff && \phi(\mathfrak{b}^+(X,Y))=\mathfrak{b}^-(\phi(X),\phi(Y)).
\end{aligned}$$

□

With Theorem 6.16 we know, that a generic Clifford double $\mathbb{F}$-algebra Γ_n does not need to be isomorphic.

Theorem 6.17. *Let Γ_n be a Clifford double $\mathbb{F}$-algebra with*

$$\mathfrak{b}(X,Y)=0 \qquad \text{for all } X,Y\in\Gamma_n^1. \tag{6.54}$$

In this case Γ_n represents a Grassmann double $\mathbb{F}$-algebra, i. e. $\Gamma_n^+(+,\cdot,\ )$ and $\Gamma_n^-(+,\cdot,*)$ are both Grassmann algebras.

Proof. According to equations (6.10) and (6.11) of Definition 6.4 there is an identity element in both approaches. With the vanishing bilinear form of equation (6.54) we have

$$X_{\bar{r}}\circledast Y_{\bar{s}}=X_{\bar{r}}\lozenge Y_{\bar{s}} \qquad \text{for all } r\neq 0, s\neq 0. \tag{6.55}$$

□

7. Grassmann algebras

Since projective and Grassmann algebra look very similar, let us compare the two structures. A Grassmann algebra $\bigwedge V$ of a vector space V with $\dim V=n$ is an associative, unital, graded and antisymmetric algebra of dimension 2^n. Projective algebra Λ_n is a double n-graded algebra, this is why we can only check whether the plus approach $\Lambda_n(+,\cdot,\wedge)$ or the minus approach $\Lambda_n(+,\cdot,\vee)$ show the structure of a Grassmann algebra.

There is no difference between the algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ and $\bigwedge V$ inasmuch as they are all associative, graded, antisymmetric and inasmuch as they have the same dimensions on the level of the whole algebra as well as on the level of their subspaces. The difference between the algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ and $\bigwedge V$ is that there is no identity element present in projective algebra — all scalars are zero divisors — and the Grassmann algebra is unital.

John Browne used the term *Grassmann algebra* in [Bro12] to describe the body of algebraic theory and results based on Graßmann's *Ausdehnungslehre* from 1844 and from 1862. This form of Grassmann algebra is very similar to projective algebra and Clifford double algebra. We will now compare the approach of John Browne with an appropriate Clifford double algebra.

To do this we use two colums. On the left side you will find the concepts and notations from John Brownes book *Grassmann algebra. Volume 1: Foundations* [Bro12]. On the right side we enlist the corresponding concepts in the language of Clifford double algebra.

John Browne's Grassmann algebra	**Clifford double algebra**
Grassmann algebra: $[\Lambda :=]\bigoplus_{k=0}^{n}\underset{k}{\Lambda}$	Grassmann algebra: plus approach of Γ_n with a bilinear form $\mathfrak{b}(X,Y)=0$ as in eq. (6.54).
$\dim\left(\bigoplus_{k=0}^{n}\underset{k}{\Lambda}\right)=\sum_{k=0}^{n}\dim\underset{k}{\Lambda}=\sum_{k=0}^{n}\binom{n}{k}=2^n$	$\dim\Gamma_n=\sum_{k=0}^{n}\dim\Gamma_n^k=\sum_{k=0}^{n}\binom{n}{k}=2^n$
Operations in the Grassmann algebra:	Operations in the Grassmann algebra Γ_n:

• addition $(+)$ • scalar multiplication (no sign) • exterior product $(\wedge)$ • regressive product $(\vee)$	• addition $(+)$ • scalar multiplication $(\cdot)$ • major outer product (no sign) • minor outer product $(*)$
$\Lambda(+,\ ,\wedge)$ and $\Lambda(+,\ ,\vee)$ are both Grassmann algebras.	$\Gamma_n^+(+,\cdot,\ )$ and $\Gamma_n^+(+,\cdot,*)$ with a bilinear form $\mathfrak{b}(X,Y)=0$ as in eq. (6.54) are both Grassmann algebras.
Identity element $1=\underset{0}{1}\in\Lambda(+,\ ,\wedge)$ Identity element $\underset{n}{1}\in\Lambda(+,\ ,\vee)$	Identity element $1^+\in\Gamma_n^+(+,\cdot,\ )$ Identity element $\langle 1^-\rangle^+\in\Gamma_n^+(+,\cdot,*)$
Common factor axiom and theorem	Basis transformation $\zeta_{\mathbf{bc}}$
To be continued ...	...

8. Outlook

Projective algebra Λ_n and double Clifford algebra Γ_n may be applied in the field of geometry. A system of axioms for the 2^n-dimensional projective geometry is given in terms of projective algebra Λ_n in [Con16]. Cayley-Klein geometries can be expressed in terms of double Clifford algebra Γ_n. See [Con00],[Con08, Chapter 5] and [Gun11]. In both cases the structure of a double graded algebra is needed to incorporate the projective principle of duality, which is present in projective geometry as well as in the non-degenerate Cayley-Klein geometries. [Kow09]

Because of the deep geometric significance of projective and double Clifford algebra, we close this article with

Definition 8.1 (geometric algebra)**.** The term *geometric algebra* is used for projective algebra Λ_n and for double Clifford algebra Γ_n.

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PROJECTIVE GEOMETRY WITH PROJECTIVE ALGEBRA (PRELIMINARY VERSION)

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ABSTRACT. This article provides a system of axioms for projective geometry $\mathscr{P}_n$ in terms of projective algebra Λ_n. Basic concepts of projective geometry $\mathscr{P}_n$ such as the principle of duality, primitive geometric forms, the cross ratio of four basic elements and simple projective transformations are then determined on the base of the above mentioned system of axioms.

1. INTRODUCTION

After the discovery of the principle of duality by the French mathematicians JEAN VICTOR PONCELET (1788-1867) and JOSEPH-DIAZ GERGONNE (1771-1859) in the first quarter of the nineteenth century projective geometry experienced an impulse of development for about one century. This impulse expressed itself in a synthetic as well as in an analytic form. In the course of the twentieth century the synthetic approach became less and less known although synthetic projective geometry had led to a wealth of new concepts and an unsurpassed beauty. An example for the new concepts rising out of synthetic projective geometry is the notion of counterspace [Con08, pp. 55].

Nevertheless, the synthetic approach to projective geometry was further developed in the twentieth century by authors like e. g. OSWALD VEBLEN (1880-1960) [VY10, Veb18], GEORGE ADAMS (1894-1963) [Ada65], LOUIS LOCHER (1906-1962) [Loc80b, Loc80a, Loc70b, Loc70a], HAROLD SCOTT MACDONALD COXETER (1907-2003) [Cox74], LAWRENCE EDWARDS (1912-2004) [Edw03], HANNS-JÖRG STOSS [Sto95, Sto99], GERHARD KOWOL [Kow09] and RENATUS ZIEGLER [Zie12].

The purpose of this paper is to provide a system of axioms for projective geometry $\mathscr{P}_n$ in terms of projective algebra Λ_n [Con16]. The aim thereby was to analytically represent into every detail the wealth of synthetic projective geometry as it is described e. g. in the books by LOUIS LOCHER [Loc80b, Loc80a, Loc70b].

Section 2 introduces the system of axioms for projective geometry $\mathscr{P}_n$ in terms of projective algebra Λ_n. The sections 3-6 provide basic concepts of projective algebra $\mathscr{P}_n$ such as the principle of duality, primitive geometric forms, the cross ratio of four basic elements and simple projective transformations.

This paper uses the concepts, notations, definitions and theorems given in [Con16] *without repeating them here again.*

2. A SYSTEM OF AXIOMS FOR PROJECTIVE GEOMETRY $\mathscr{P}_n$

Definition 2.1 (equivalence relation)**.** Two multi vectors A and B of a projective $\mathbb{F}$-algebra Λ_n are called *equivalent*, if and only if A and B differ in a number $\xi \in \mathbb{F} \setminus \{0\}$,

$$A \simeq B \quad :\Longleftrightarrow \quad A = \xi B. \tag{2.1}$$

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Definition 2.2 (projective geometry $\mathscr{P}_n$)**.** Let $\Lambda_n(+,\cdot,\wedge,\vee)$ be a projective $\mathbb{F}$-algebra. Projective geometry $\mathscr{P}_n$ of dimension 2^n is determined in terms of projective algebra Λ_n by the following axioms:

(A1) *Elements of projective geometry.*
  (a) There are $n+1$ different types of *basic elements* corresponding to the $n+1$ different vector subspaces Λ_n^k of projective algebra Λ_n. The basic elements of a certain type (called *k-elements*) are represented by the homogeneous multi vectors X_k of one of the $n+1$ different vector subspaces Λ_n^k.
  (b) A multi vector M of the vector space $\Lambda_n(+,\ )$ represents an *element*, i. e. in general of each type of basic element exactly one.

$$M = \sum_{k=0}^{n} \langle M \rangle_k. \tag{2.2}$$

  (c) Equivalent multi vectors represent the same geometric element.

(A2) *Incidence relation.* Two elements A und B are incident if and only if the corresponding multi vectors A and B meet the conditions

$$A \wedge B = 0 \quad \text{and} \quad A \vee B = 0. \tag{2.3}$$

(A3) *Intersection and connection.* The geometric operation of connection corresponds to the major outer product ($\wedge$), the geometric operation of intersection to the minor outer product ($\vee$).

Definition 2.3 (real, complex or finite projective geometry)**.** Depending on the field $\mathbb{F}$ of the projective $\mathbb{F}$-algbera Λ_n we have a) for $\mathbb{F} = \mathbb{R}$ real projective geometry, b) for $\mathbb{F} = \mathbb{C}$ complex projective geometry and c) for finite $\mathbb{F}$ finite projective geometry.

Definition 2.4 (names of basic elements)**.** In case of the projective geometries $\mathscr{P}_2$, $\mathscr{P}_3$ and $\mathscr{P}_4$ we use the in Table 1 listed names for the different types of basic elements.

Definition 2.5 (space and counterspace)**.** Inasmuch as projective geometry $\mathscr{P}_n$ is expressed in terms of the plus approach Λ_n^+ it is called *space* and inasmuch as it is expressed in terms of the minus approach Λ_n^- it is called *counterspace*.

Remark 2.6. The terms *space* and *counterspace* are in a similar way also used in the Cayley-Klein geometries. The Cayley-Klein geometries may be expressend in terms of double Clifford algebra Γ_n [Con16]. And again, the plus approach is called *space* and the minus approach *counterspace* in these metric geometries.

3. Principle of duality

Definition 3.1 (major dual)**.** The *major dual* S' of any expression S from projective geometry $\mathscr{P}_n$ such as e. g. an equation, a theorem, or a definition is obtained by interchanging $\wedge$ with $\vee$ and by reversing the sign of the plus-minus notation.

Theorem 3.2 (major principle of duality)**.** *Any statement S from projective geometry $\mathscr{P}_n$ is true if and only if the major dual statement S' from projective geometry $\mathscr{P}_n$ is true.*

Proof. According to Theorem 5.14 of [Con16] projective algebra Λ_n is a double n-graded algebra, i. e. there is always an isomorphism between Λ_n^+ and Λ_n^-. □

Definition 3.3 (minor dual)**.** The *minor dual* S'' of any expression S from projective geometry $\mathscr{P}_n$ such as e. g. an equation, a theorem, or a definition is obtained by interchanging $\wedge$ with $\vee$ and by replacing the grades of a multi vector with the complementary grades ($k \to n-k$).

$\mathscr{P}_2$	**line**	
	$\Lambda_2^{0+} = \Lambda_2^{2-}$	line
	$\Lambda_2^{1+} = \Lambda_2^{1-}$	points or planes
	$\Lambda_2^{2+} = \Lambda_2^{0-}$	line
$\mathscr{P}_2$	**incident point-plane-pair**	
	$\Lambda_2^{0+} = \Lambda_2^{2-}$	incident point-plane-pair
	$\Lambda_2^{1+} = \Lambda_2^{1-}$	lines
	$\Lambda_2^{2+} = \Lambda_2^{0-}$	incident point-plane-pair
$\mathscr{P}_3$	**planar field**	
	$\Lambda_3^{0+} = \Lambda_3^{3-}$	field of lines as such
	$\Lambda_3^{1+} = \Lambda_3^{2-}$	points
	$\Lambda_3^{2+} = \Lambda_3^{1-}$	lines
	$\Lambda_3^{3+} = \Lambda_3^{0-}$	field of points as such
$\mathscr{P}_3$	**centric bundle**	
	$\Lambda_3^{0+} = \Lambda_3^{3-}$	bundle of planes as such
	$\Lambda_3^{1+} = \Lambda_3^{2-}$	lines
	$\Lambda_3^{2+} = \Lambda_3^{1-}$	planes
	$\Lambda_3^{3+} = \Lambda_3^{0-}$	bundle of lines as such
$\mathscr{P}_4$	**space**	
	$\Lambda_4^{0+} = \Lambda_4^{4-}$	space of planes as such
	$\Lambda_4^{1+} = \Lambda_4^{3-}$	points
	$\Lambda_4^{2+} = \Lambda_4^{2-}$	linear complexes (including the lines)
	$\Lambda_4^{3+} = \Lambda_4^{1-}$	planes
	$\Lambda_4^{4+} = \Lambda_4^{0-}$	space of points as such

TABLE 1. The names for the different types of basic elements in the projective geometries $\mathscr{P}_2$, $\mathscr{P}_3$ and $\mathscr{P}_4$.

Space Λ_n^+		Counterspace Λ_n^-
X_k^+	$\leftrightarrow$	X_k^-
$\wedge$	$\leftrightarrow$	$\vee$
$\vee$	$\leftrightarrow$	$\wedge$
$A \wedge B = 0$ and $A \vee B = 0$	$\leftrightarrow$	$A \vee B = 0$ and $A \wedge B = 0$

TABLE 2. Major principle of duality in projective geometry $\mathscr{P}_n$.

Theorem 3.4 (minor principle of duality)**.** *Any statement S from projective geometry $\mathscr{P}_n$ is true if and only if the minor dual statement S'' from projective geometry $\mathscr{P}_n$ is true.*

Proof. $S \Longleftrightarrow S' \Longleftrightarrow S''$ □

4. PRIMITIVE GEOMETRIC FORM

Definition 4.1 (k-primitive geometric form of grade m)**.** Let $X_i = \langle X_i \rangle_k$ denote $m+1$ linear independent k-vectors in projective geometry $\mathscr{P}_n$ with $1 \leq i \leq m+1 \leq \binom{n}{k}$ and $0 \leq k \leq n$.

Space Λ_n^+ Counterspace Λ_n^-		Space Λ_n^+ Counterspace Λ_n^-
$X_{\overline{k}}$	$\leftrightarrow$	$X_{\overline{n-k}}$
$\wedge$	$\leftrightarrow$	$\vee$
$A \wedge B = 0$ and $A \vee B = 0$	$\leftrightarrow$	$A \vee B = 0$ and $A \wedge B = 0$

TABLE 3. Minor principle of duality in projective geometry $\mathscr{P}_n$.

Then the *k-primitive geometric form of grade m* is the sub vector space

$$U := \left\{ \sum_{i=1}^{m+1} \xi_i X_i \,\middle|\, \xi_i \in \mathbb{F} \right\} \tag{4.1}$$

of projective algebra Λ_n with $\dim U = m+1$.

The Tables 4 and 5 list all primitive geometric forms of the projective geometries $\mathscr{P}_2$, $\mathscr{P}_3$ and $\mathscr{P}_4$.

Definition 4.2 (general position of l basic elements)**.** Let $\mathscr{L} := \{i \mid i,l \in \mathbb{N}, 1 \le i \le l\}$ be the set of the l first natural numbers. l basic elements X_i, $i \in \mathscr{L}$, of a k-primitive geometric form of grade m are said to be in general position

(a) in case of $l \le m+1$ if and only if the l k-vectors X_i are linearly independent.
(b) in case of $l > m+1$ if and only if for any variation $\{i_1, \ldots, i_{m+1}\}$ of length $m+1$ without repetitions

$$\{i_1, \ldots, i_{m+1}\} \subset \mathscr{L} \qquad \text{and} \qquad 1 \le i_1 \le \ldots \le i_{m+1} \le l \tag{4.2}$$

the k-vectors $X_{i_1}, \ldots, X_{i_{m+1}}$ are linearly independent.

Theorem 4.3. *Let $\mathscr{L} := \{i \mid i,l \in \mathbb{N}, 1 \le i \le l\}$ be the set of the l first natural numbers and let the l basic elements*

$$X_i, \qquad i \in \mathscr{L}, \tag{4.3}$$

of a k-primitive geometric form of grade m be in general position. Further we choose

$$l > m+1, \qquad k,m \in \mathbb{N}, \tag{4.4}$$

$$\mathscr{I} := \{i_1, i_2, \ldots, i_{m+1}\}, \qquad 1 \le i_1 < i_2 < \cdots < i_{m+1} \le l, \tag{4.5}$$

$$\overline{\mathscr{I}} = \{i_{m+2}, i_{m+3}, \ldots, i_l\} := \mathscr{L} \setminus \mathscr{I}, \tag{4.6}$$

$$1 \le i_{m+2} < i_{m+3} < \ldots < i_l \le l. \tag{4.7}$$

The claim is that for any set of indices $\mathscr{I}$ and for any index $r \in \overline{\mathscr{I}}$ the coefficients λ_{rs} of the linear combination

$$X_r = \sum_{s \in \mathscr{I}} \lambda_{rs} X_s, \qquad r \in \overline{\mathscr{I}}, \tag{4.8}$$

do not vanish,

$$\lambda_{rs} \neq 0, \qquad \forall\, \mathscr{I}, \quad \forall\, r \in \overline{\mathscr{I}}, \quad \forall\, s \in \mathscr{I}. \tag{4.9}$$

Proof. By precondition, the basic elements

$$X_s, \qquad s \in \mathscr{I}, \tag{4.10}$$

		space Λ_n^+	counterspace Λ_n^-
$\mathscr{P}_2$	**line**		
$k=0$	$m=0$	line	line
$k=1$	$m=0$	point or plane	plane or point
	$m=1$	pencil of points or of planes	pencil of planes or of points
$k=2$	$m=0$	line	line
$\mathscr{P}_2$	**incident point-plane-pair**		
$k=0$	$m=0$	incident point-plane-pair	incident point-plane-pair
$k=1$	$m=0$	line	line
	$m=1$	pencil of lines	pencil of lines
$k=2$	$m=0$	incident point-plane-pair	incident point-plane-pair
$\mathscr{P}_3$	**planar field**		
$k=0$	$m=0$	field of lines as such	field of points as such
$k=1$	$m=0$	point	line
	$m=1$	pencil of points	pencil of lines
	$m=2$	field of points	field of lines
$k=2$	$m=0$	line	point
	$m=1$	pencil of lines	pencil of points
	$m=2$	field of lines	field of points
$k=3$	$m=0$	field of points as such	field of lines as such
$\mathscr{P}_3$	**centric bundle**		
$k=0$	$m=0$	bundle of planes as such	bundle of lines as such
$k=1$	$m=0$	line	plane
	$m=1$	pencil of lines	pencil of planes
	$m=2$	bundle of lines	bundle of planes
$k=2$	$m=0$	plane	line
	$m=1$	pencil of planes	pencil of lines
	$m=2$	bundle of planes	bundle of lines
$k=3$	$m=0$	bundle of lines as such	bundle of planes as such

TABLE 4. k-primitive geometric forms of grade m. The basic elements of a k-primitive geometric form are homogeneous multi vectors of grade k.

are linearly independent for any set of indices $\mathscr{I}$. Thus, with the linear combination of equation (4.8) we can describe all basic elements of the given k-primitive geometric form of grade m and especially the basic element X_r with $r \in \overline{\mathscr{I}}$.

We will proof the assertion by contradiction an thus assume that there is for a certain choice of $\mathscr{I} = \mathscr{I}_1$, $r = r_1$ and $s = s_1$ a vanishing coefficient $\lambda_{r_1 s_1} = 0$. Then there is the set of indices

$$\mathscr{I}_2 := (\mathscr{I}_1 \setminus \{s_1\}) \cup \{r_1\}, \tag{4.11}$$

where the $m+1$ basic elements X_j, $j \in \mathscr{I}_2$ are linearly dependent. This is in contradiction to the precondition that the l basic elements X_i, $i \in \mathscr{L}$ are in general position. □

5. CROSS RATIO

Definition 5.1 (cross ratio)**.** Four different basic elements

$$A = \langle A \rangle_k, \qquad B = \langle B \rangle_k, \qquad C = \langle C \rangle_k, \qquad D = \langle D \rangle_k, \tag{5.1}$$

of a k-primitive geometric form with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B \tag{5.2}$$

$\mathscr{P}_4$	**space**		
$k=0$	$m=0$	space of planes as such	space of points as such
$k=1$	$m=0$	point	plane
	$m=1$	pencil of points	pencil of planes
	$m=2$	field of points	bundle of planes
	$m=3$	space of points	space of planes
$k=2$	$m=0$	complex	wood
	$m=1$	pencil of complexes	pencil of woods
	$m=2$	bundle of complexes	bundle of woods
	$m=3$	3-manifold of complexes	3-manifold of woods
	$m=4$	4-manifold of complexes	4-manifold of woods
	$m=5$	5-manifold of complexes	5-manifold of woods
$k=3$	$m=0$	plane	point
	$m=1$	pencil of planes	pencil of points
	$m=2$	bundle of planes	field of points
	$m=3$	space of planes	spaces of points
$k=4$	$m=0$	space of points as such	space of planes as such

TABLE 5. k-primitive geometric forms of grade m. The basic elements of a k-primitive geometric form are homogeneous multi vectors of grade k.

form the cross ratio

$$CR(ABCD) := \frac{\lambda}{\mu}. \tag{5.3}$$

In order to show, that the cross ratio is well defined and does not depend on the weight factors of the basic elements A, B, C and D, we replace the latter by

$$A = \alpha' A', \qquad B = \beta' B', \qquad C = \gamma' C', \qquad D = \delta' D' \tag{5.4}$$

with $\alpha', \beta', \gamma', \delta' \in \mathbb{F} \setminus \{0\}$. Inserting the expressions of equation (5.4) into equation (5.2),

$$\gamma\gamma' C' = \alpha' A' + \lambda \beta' B', \qquad \delta\delta' D' = \alpha' A' + \mu \beta' B', \tag{5.5}$$

and dividing by α',

$$\frac{\gamma\gamma'}{\alpha'} C' = A' + \lambda \frac{\beta'}{\alpha'} B', \qquad \frac{\delta\delta'}{\alpha'} D' = A' + \mu \frac{\beta'}{\alpha'} B', \tag{5.6}$$

we get

$$CR(A'B'C'D') = \frac{\lambda}{\mu} = CR(ABCD). \tag{5.7}$$

Theorem 5.2. *The cross ratio of four different basic elements*

$$T_i = \lambda_i X + \mu_i Y, \qquad i \in \{1,2,3,4\}, \tag{5.8}$$

of a k-primitive geometric form is given by

$$CR\,(T_1 T_2\, T_3 T_4) = \frac{\left(\frac{\lambda_1\mu_3 - \mu_1\lambda_3}{\lambda_2\mu_3 - \mu_2\lambda_3}\right)}{\left(\frac{\lambda_1\mu_4 - \mu_1\lambda_4}{\lambda_2\mu_4 - \mu_2\lambda_4}\right)}. \tag{5.9}$$

Proof. We first compute T_3 and T_4 as a function of T_1 and T_2,

$$T_3 = \lambda_3 X + \mu_3 Y \stackrel{!}{=} \alpha T_1 + \beta T_2, \qquad T_4 = \lambda_4 X + \mu_4 Y \stackrel{!}{=} \gamma T_1 + \delta T_2, \tag{5.10}$$

then by comparison of the coefficients get

$$\alpha = \frac{\lambda_2\mu_3 - \mu_2\lambda_3}{\lambda_2\mu_1 - \mu_2\lambda_1}, \qquad \beta = -\frac{\lambda_1\mu_3 - \mu_1\lambda_3}{\lambda_2\mu_1 - \mu_2\lambda_1}, \tag{5.11}$$

$$\gamma = \frac{\lambda_2\mu_4 - \mu_2\lambda_4}{\lambda_2\mu_1 - \mu_2\lambda_1}, \qquad \delta = -\frac{\lambda_1\mu_4 - \mu_1\lambda_4}{\lambda_2\mu_1 - \mu_2\lambda_1}, \tag{5.12}$$

and divide in the third step the two equations (5.10) by α and γ respectively,

$$\left(\frac{1}{\alpha}\right) T_3 = T_1 + \left(\frac{\beta}{\alpha}\right) T_2, \qquad \left(\frac{1}{\gamma}\right) T_4 = T_1 + \left(\frac{\delta}{\gamma}\right) T_2. \tag{5.13}$$

According to Definition 5.1 the cross ratio is

$$CR(T_1T_2T_3T_4) = \frac{\left(\frac{\beta}{\alpha}\right)}{\left(\frac{\delta}{\gamma}\right)} = \frac{\left(\frac{\lambda_1\mu_3-\mu_1\lambda_3}{\lambda_2\mu_3-\mu_2\lambda_3}\right)}{\left(\frac{\lambda_1\mu_4-\mu_1\lambda_4}{\lambda_2\mu_4-\mu_2\lambda_4}\right)}. \tag{5.14}$$

□

Theorem 5.3. *In projective geometry $\mathscr{P}_n$ intersection and connection maintain the cross ratio.*

Proof. We first confirm the assertion with respect to the operation of connection ($\wedge$) in the plus approach. By precondition the four basic elements

$$A = \langle A\rangle_k^+, \qquad B = \langle B\rangle_k^+, \qquad C = \langle C\rangle_k^+, \qquad D = \langle D\rangle_k^+, \tag{5.15}$$

of a k-primitive geometric form with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B \tag{5.16}$$

can be connected with the basic element $Z = \langle Z\rangle_l^+$, $l \in \mathbb{N}$, $0 < l < n$,

$$A_1 = A \wedge Z \neq 0, \qquad B_1 = B \wedge Z \neq 0, \tag{5.17}$$

$$C_1 = C \wedge Z \neq 0, \qquad D_1 = D \wedge Z \neq 0. \tag{5.18}$$

We then have

$$\gamma C_1 = A_1 + \lambda B_1 \qquad \delta D_1 = A_1 + \mu B_1 \tag{5.19}$$

and

$$CR(ABCD) = CR(A_1B_1C_1D_1). \tag{5.20}$$

The assertion of Theorem 5.3 with respect to the operation of intersection ($\vee$) follows with the major principle of duality (Theorem 3.2). □

6. Simple projective transformations

Definition 6.1 (simple projective transformation of grade m, simple collineation, simple correlation)**.** Let $\mathscr{P}_n$ represent the 2^n-dimensional projective geometry and let U and V be two primitive geometric forms of grade m, i. e.

$$\dim U = \dim V = m + 1. \tag{6.1}$$

A *simple projective transformation of grade m* is a linear and bijective mapping

$$\phi: \quad U \longrightarrow V. \tag{6.2}$$

Furthermore let $X \in U$ and $Y \in V$ be two basic elements of the two primitive geometric forms respectively. In the below listed four special cases, the simple projective transformation ϕ gets the following name:

(6.3) simple k-collineation in the plus approach $\iff$ $X = \langle X \rangle_k^+$, $Y = \langle Y \rangle_k^+$

(6.4) simple k-correlation from the plus to the minus approach $\iff$ $X = \langle X \rangle_k^+$, $Y = \langle Y \rangle_k^-$

(6.5) simple k-correlation from the minus to the plus approach $\iff$ $X = \langle X \rangle_k^-$, $Y = \langle Y \rangle_k^+$

(6.6) simple k-collineation in the minus approach $\iff$ $X = \langle X \rangle_k^-$, $Y = \langle Y \rangle_k^-$

The grade m of the simple projective transformation ϕ and the grades m of the primitive geometric forms U and V are according to the Definitions 4.1 and 6.1 always the same. The grades of the basic elements $X \in U$ and $Y \in V$ need not to be the same. A projective transformation ϕ e. g. from the pencil of points in space to the pencil of lines in space does neither represent a simple collineation nor a simple correlation in the sense of Definition 6.1.

Theorem 6.2. *In projective geometry $\mathscr{P}_n$ any simple projective transformation*

$$\phi : \quad U \longrightarrow V \tag{6.7}$$

maintains the cross ratio.

Proof. The four different basic elements A, B, C and D of the primitive geometric form U with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B \tag{6.8}$$

form according to Definition 5.1 the cross ratio

$$CR(ABCD) = \frac{\lambda}{\mu}. \tag{6.9}$$

The images $\phi(A)$, $\phi(B)$, $\phi(C)$ and $\phi(D)$ of these four basic elements under the simple projective transformation ϕ belong to the primitive geometric form V. Because of the linearity of ϕ we have

$$\gamma\phi(C) = \phi(A) + \lambda\phi(B) \qquad \delta\phi(D) = \phi(A) + \mu\phi(B) \tag{6.10}$$

and thus

$$CR(\phi(A)\phi(B)\,\phi(C)\phi(D)) = \frac{\lambda}{\mu} = CR(ABCD). \tag{6.11}$$

□

Theorem 6.3 (fundamental theorem of projective geometry)**.** *Let $\mathscr{L} := \{i \mid i, m \in \mathbb{N}, 1 \leq i \leq m+2\}$ be the set of the $m+2$ first natural numbers. A simple projective transformation ϕ of grade m,*

$$\phi : \quad U \longrightarrow V, \tag{6.12}$$

is determined by $m+2$ pairs of basic elements

$$(X_i, Y_i), \qquad X_i \in U, \qquad Y_i \in V, \qquad i \in \mathscr{L}, \tag{6.13}$$

if the $m+2$ basic elements X_i of the primitive geometric form U and the $m+2$ basic elements Y_i of the primitive geometric form V are in general position respectively.

Proof. According to Theorem 4.3 the precondition leads to

$$X_{m+2} = \sum_{i=1}^{m+1} \mu_i X_i, \qquad \text{with} \qquad \mu_i \neq 0, \tag{6.14}$$

$$Y_{m+2} = \sum_{i=1}^{m+1} \nu_i Y_i, \qquad \text{with} \qquad \nu_i \neq 0. \tag{6.15}$$

For the simple projective transformation ϕ with

$$\phi(X_i) = \lambda_i Y_i, \qquad \lambda_i \neq 0, \qquad i \in \mathscr{L}, \tag{6.16}$$

and

$$\begin{aligned} \phi(X_{m+2}) &= \sum_{i=1}^{m+1} \mu_i \cdot \phi(X_i) = \sum_{i=1}^{m+1} \mu_i \lambda_i Y_i \\ &= \lambda_{m+2} Y_{m+2} = \lambda_{m+2} \sum_{i=1}^{m+1} \nu_i Y_i = \sum_{i=1}^{m+1} \lambda_{m+2} \nu_i Y_i \end{aligned} \tag{6.17}$$

we get

$$\lambda_i = \frac{\nu_i}{\mu_i} \lambda_{m+2}, \qquad i \in \mathscr{L}. \tag{6.18}$$

Thus, up to the factor $\lambda_{m+2} \neq 0$ the simple projective transformation ϕ is uniquely determined. □

7. Outlook

Using the in section 2 defined system of axioms for projective geometry $\mathscr{P}_n$ we determined in the section 3-6 some basic concepts of projective geometry.

In future papers we will continue to determine the foundations of projective geometry $\mathscr{P}_n$ and will elaborate projective geometry $\mathscr{P}_n$ for some specific values of n.

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PEANO READER OF H. GRAßMANN'S *AUSDEHNUNGSLEHRE*

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In this talk I propose an analysis of Peano's studies about the geometric calculus. In the volume of 1888 (*Geometric Calculus according to H. Grassmann's Ausdehnungslehre preceded by the operations of deductive logic*), Peano presents Grassmann's ideas (*Ausdehnungslehre*, 1844) in an original way: he gives an Euclidean interpretation to the fundamental Grassmannian notions. Hence by means of his geometric calculus, Peano is able to show theorems of projective geometry. Therefore, Peano's geometrical calculus (which has an intrinsic mathematical interest in order to the applications to the geometry and to the mechanics) has an implicit foundational role. The disciple of Peano who devoted himself above all to the studies of geometric calculus was Cesare Burali Forti (1861-1931); but also Filiberto Castellano (1860-1919), Tommaso Boggio (1877 - 1963) and Mario Pieri (1860-1904) took an interest in the subject.

THE AFFINE AND PROJECTIVE GEOMETRIES FROM GRASSMANN'S POINT OF VIEW

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ABSTRACT. Grassmann's powerful but largely undefined approach to affine and projective geometries has roots in Möbius' barycentric calculus. In his appraisal of the former work, Peano showed explicitly the relation of barycentric coordinates with Cartesian coordinates. In our *Treatise of Plane Geometry through Geometric Algebra* we went beyond Peano's work by displaying the advantage of barycentric coordinates when dealing with the main theorems of projective geometry (Desargues, Pappus, etc.). By giving the equations of lines and planes with barycentric coordinates, we explain the geometric duality in a purely algebraic way. The generalization of the barycentric coordinates leads to projective frames and coordinates, which allows us to work with the full projective geometry of an n-dimensional space without defining the projective space $\mathbb{PR}_n$ as a projection of an $n+1$ dimensional space. We will also display the advantages of expressing the equations of quadrics with projective coordinates of the three-dimensional space. According to Grassmann, the product of two points is a line, the product of three points is their plane and the product of four points is the whole space. In the same way, the successive products of dual points in the dual space generate geometric elements having decreasing dimensions. Then, Grassmann's products of points and dual points are identified respectively with the operators *join* and *meet* of the projective geometry. Finally, let us emphasize that all these conclusions can be generalized to n-dimensional spaces.

1. INTRODUCTION

Hermann Grassmann's seminal ideas were so fruitful that even now after one and half centuries after his death we are still discussing [1] about their interpretations and applications to many branches of mathematics, physics and technique, leaving aside his prolific work on phylology, such as comparative gramatics, his translation of the *Rig Veda* or his first dictionary of Sanskrit.

Geometry was one of the branches where his contribution was fundamental. The two editions of *Die Ausdehnungslehre* (1844 and 1862) gave such a large amount of geometric and algebraic ideas that they were not understood well in his time, and we still find trouble to read him nowadays. The first edition was translated into Spanish in the excellent collection *Historia y Filosofía de la Ciencia* led by Julio Rey Pastor [2], and both editions have been translated more recently into English by Lloyd C. Kannenberg [3, 4].

Although the exterior algebra of vectors has earned recognition nowadays, one cannot say the same for the exterior algebra of points. Grassmann's work has its bases in Möbius' *Barycentrische Calcul* [5]. In fact, Grassmann outlined an algebra of points resorting to the barycentric calculus. Peano studied Grassmann's work [7] and developed it more explicitly [8] by means of barycentric coordinates [8]. However, some confusion exists about what products of points mean, whether they are representing finite or infinite extensions.

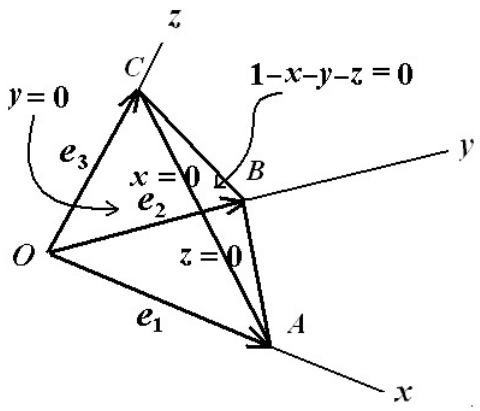

FIGURE 1. Affine frame defining, in the geometric space E, a coordinate system with origin in O and vector basis $\{e_1,e_2,e_3\}$. $\{O,A,B,C\}$ is a basis for points, $\{OA,OB,OC,AB,AC,BC\}$ is a basis for lines, and $\{OAB,OAC,OBC,ABC\}$, whose equations are indicated, is a basis for planes and dual space E^*.

2. AFFINE COORDINATES AND FRAMES

Let O, A ,B, C be four non-coplanar points in the three-dimensional geometric space (figure 1). Let us call O the *origin of coordinates* and let us define the vector basis:

$$e_1 = OA = A - O \qquad e_2 = OB = B - O \qquad e_3 = OC = C - O \tag{1}$$

Then $\{O, e_1e_2, e_3\}$ is said to be an *affine frame* of the three-dimensional geometric space. The position vector of every point P can be written as linear combination of the vectors e_1, e_2 and e_3:

$$OP = x\, e_1 + y\, e_2 + z\, e_3 \tag{2}$$

where (x,y,z) are the affine coordinates of point P. Cartesian coordinates is a special case of affine coordinates for which $\|e_1\| = \|e_2\| = \|e_3\|$ and $e_1 \perp e_2$, $e_1 \perp e_3$ and $e_2 \perp e_3$. However, these last conditions need the introduction of scalar product and metric geometry, which this paper will not deal with. Therefore (x,y,z) will only be affine coordinates, which in general do not need any orthonormal basis of vectors.

According to Cartan ([9], p. 387), an affine frame can be formed by four points, three points and one vector, two points and two vectors, or one point and three vectors (this is the option usually taken in affine geometry). The only required condition is that there does not exist any linear relation among them, that is, they must be geometrically (linearly) independent. In the particular case of affine space we can write:

$$P(x,y,z) = O + x\, e_1 + y\, e_2 + z\, e_3 \tag{3}$$

that is, every point P is a linear combination of the four elements of the affine frame $\{O, e_1, e_2, e_3\}$.

3. BARYCENTRIC COORDINATES

If we make the substitution of (1) in (3) we obtain:

$$P(x,y,z) = (1-x-y-z)O + x\,A + y\,B + z\,C \tag{4}$$

This is just the first option mentioned by Cartan. Now, every point of the three-dimensional space is a linear combination of four non-coplanar points $\{O,A,B,C\}$. The coefficients of the linear combination $(1-x-y-z,x,y,z)$ are called *barycentric coordinates*. It makes no sense to call $\{O,A,B,C\}$ a *barycentric frame* since it is the same frame as $\{O,e_1,e_2,e_3\}$ only taking points instead of vectors. Then, it is preferable to call it a *point frame*. Barycentric coordinates are not linearly independent since their addition equals the unity. They were introduced by Möbius in his *Barycentrische Calcul* [5]. If we place weights w_i at the points $\{O,A,B,C\}$, then P is the center of mass of the whole system, and:

$$x = \frac{w_A}{w_O+w_A+w_B+w_C} \qquad y = \frac{w_B}{w_O+w_A+w_B+w_C} \qquad z = \frac{w_C}{w_O+w_A+w_B+w_C} \tag{5}$$

$$1-x-y-z = \frac{w_O}{w_O+w_A+w_B+w_C}$$

Barycentric coordinates are the ratios of volumes of the tetrahedrons that point P forms with the faces of the affine frame $\{OABC\}$ with regard to the volume of the tetrahedron they form:

$$x = \frac{V_{OPBC}}{V_{OABC}}, \qquad y = \frac{V_{OAPC}}{V_{OABPC}} \qquad z = \frac{V_{OABP}}{V_{OABC}} \qquad 1-x-y-z = \frac{V_{PABC}}{V_{OABC}} \tag{6}$$

Let us prove, for instance, the first equality:

$$V_{OPBC} = \frac{1}{6}OP \wedge OB \wedge OC = \frac{1}{6}x\;OA \wedge OB \wedge OC \tag{7}$$

In the same way:

$$V_{PABC} = \frac{1}{6}PA \wedge PB \wedge PC = \frac{1}{6}(OA-OP) \wedge (OB-OP) \wedge (OC-OP) \tag{8}$$

By means of the distributive property, and since terms having repeated factors vanish we obtain:

$$V_{PABC} = \frac{1}{6}(1-x-y-z)OA \wedge OB \wedge OC \tag{9}$$

Since the volume of the tetrahedron $OABC$ is:

$$V_{OABC} = \frac{1}{6}OA \wedge OB \wedge OC \tag{10}$$

the equalities (6) follow.

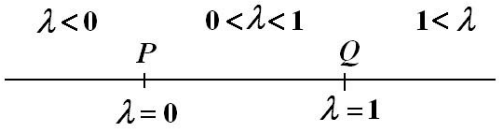

FIGURE 2. Values of a barycentric coordinate for a line described by the barycentric linear combination (11) of two points P and Q. A generic point X on the line lies between P and Q if $0 < \lambda < 1$.

The advantage of barycentric coordinates is that the four points of the affine frame are treated on an equal footing, which gives a more symmetrical description of some geometric problems.

4. EQUATIONS OF LINES AND PLANES IN BARYCENTRIC COORDINATES

The barycentric equation of a general point X on the line passing through the points P and Q is (figure 2):

$$X = (1-\lambda)P + \lambda Q \qquad \lambda \in \mathbb{R} \tag{11}$$

Therefore, three points are collinear if their product is null:

$$P,Q,R \quad \text{collinear} \quad \Longleftrightarrow \quad PQR = 0 \tag{12}$$

$$\begin{aligned} PQR = &\begin{vmatrix} 1-x_P-y_P & x_P & y_P \\ 1-x_Q-y_Q & x_Q & y_Q \\ 1-x_R-y_R & x_R & y_R \end{vmatrix} OAB + \begin{vmatrix} 1-x_P-z_P & x_P & z_P \\ 1-x_Q-z_Q & x_Q & z_Q \\ 1-x_R-z_R & x_R & z_R \end{vmatrix} OAC \\ &+ \begin{vmatrix} 1-y_P-z_P & y_P & z_P \\ 1-y_Q-z_Q & y_Q & z_Q \\ 1-y_R-z_R & y_R & z_R \end{vmatrix} OBC + \begin{vmatrix} x_P & y_P & z_P \\ x_Q & y_Q & z_Q \\ x_R & y_R & z_R \end{vmatrix} ABC = 0 \end{aligned}$$

In the same way, every point X in a plane containing the points P, Q and R is given by a barycentric linear combination of them:

$$X = (1-\lambda-\mu)P + \lambda Q + \mu R \qquad \lambda,\mu \in \mathbb{R} \tag{13}$$

where $(1-\lambda,\lambda,\mu)$ are the barycentric coordinates of frame $\{P,Q,R\}$ for the plane they form. In fact, four points are coplanar if and only if they are linearly dependent, that is, their product is null:

$$P,Q,R,S \quad \text{coplanar} \iff \quad PQRS = \begin{vmatrix} 1-x_P-y_P-z_P & x_P & y_P & z_P \\ 1-x_Q-y_Q-z_Q & x_Q & y_Q & z_Q \\ 1-x_R-y_R-z_R & x_R & y_R & z_R \\ 1-x_S-y_S-z_S & x_S & y_S & z_S \end{vmatrix} OABC = 0 \tag{14}$$

The equation of a plane with affine coordinates is:

$$a\,x+b\,y+c\,z+d=0 \tag{15}$$

which becomes homogeneous by changing to barycentric coordinates:

$$d(1-x-y-z)+(a+d)x+(b+d)y+(c+d)z=0 \tag{16}$$

Indicating by $e=a+d$, $f=b+d$, $g=c+d$ we have:

$$d\,(1-x-y-z)+e\,x+f\,y+g\,z=\begin{bmatrix}d & e & f & g\end{bmatrix}\begin{pmatrix}1-x-y-z\\ x\\ y\\ z\end{pmatrix}=0 \tag{17}$$

The equation of every plane is a linear combination of the equations of the faces $\{1-x-y-z=0,\ x=0,\ y=0,\ z=0\}$ of the reference tetrahedron (affine frame) *OABC* (figure 1). From now on, the first barycentric coordinate will be indicated as $t=1-x-y-z$ in order of brevity. In this way, the equation of a plane becomes:

$$d\,t+e\,x+f\,y+g\,z=\begin{bmatrix}d & e & f & g\end{bmatrix}\begin{pmatrix}t\\ x\\ y\\ z\end{pmatrix}=0 \tag{18}$$

Every plane is determined uniquely (except for a multiplicative constant) by the coefficients $[d,e,f,g]$ that will be called *dual coordinates* and indicated with brackets in order to differentiate them from point coordinates (t,x,y,z) that will be indicated with parentheses. Fishback [14] already used this bracket notation for his *natural homogeneous plane coordinates*, which are dual of Plücker coordinates and also represent planes. All the possible values of the dual coordinates form a geometric space to be called the *dual space*. Every point of the dual space represents a plane in the point space. In the same way, every plane of the dual space expresses a point in the point space.

Barycentric coordinates are not only applied to points. They can also be applied to planes. For instance, the sheaf of planes of the line intersection of the planes $\pi_1: a_1x+b_1y+c_1z+d_1=0$ and $\pi_2: a_2+b_2+c_2+d_2=0$ is expressed as (figure 3):

$$(1-\lambda)(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0 \qquad \lambda\in\mathbb{R} \tag{19}$$

In barycentric coordinates:

$$(1-\lambda)(d_1t+e_1x+f_1y+g_1)+\lambda(d_2t+e_2x+f_2y+g_2z)=0 \qquad \lambda\in\mathbb{R} \tag{20}$$

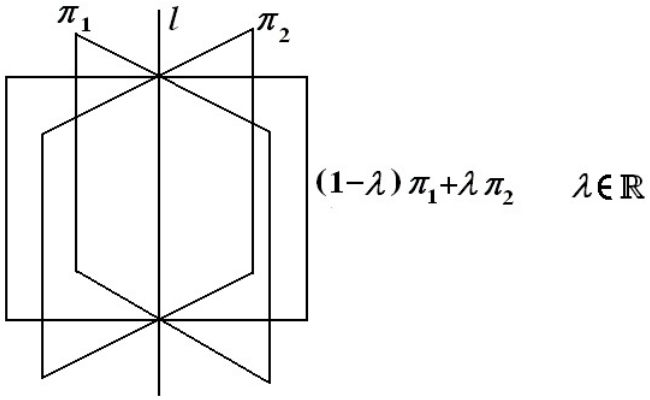

FIGURE 3. The sheaf of planes being linear combination of two given planes intersect in a line.

or simply:

$$(1-\lambda)\pi_1+\lambda\pi_2=(1-\lambda)\begin{bmatrix}d_1\\e_1\\f_1\\g_1\end{bmatrix}+\lambda\begin{bmatrix}d_2\\e_2\\f_2\\g_2\end{bmatrix} \tag{21}$$

that is, a sheaf of planes is the same as a line in the dual space, or equivalently a line in the point space is also a line in the dual space. The equation (21) represents all the planes containing the line except for the plane $\pi_1-\pi_2$. But this is only an arithmetic trouble because if we change $[d_1,e_1,f_1,g_1]$ for $[-d_1,-e_1,-f_1,-g_1]$ that expresses the same plane, and take barycentric linear combinations, then the former plane will be included in the sheaf while another plane will be excluded. Then, geometrically there is no exclusion although there is arithmetic trouble with infinity. In relation to this, let us revise the equation of a line in barycentric coordinates. The equations of a line as the intersection of two planes are:

$$r:\begin{cases}d_1t+e_1x+f_1y+g_1z=0\\ \\ d_2t+e_2x+f_2y+g_2z=0\end{cases} \tag{22}$$

Since any two points (t_1,x_1,y_1,z_1) and (t_2,x_2,y_2,z_2) on the line satisfy these equations, we can write the matrix equation of a line:

$$\begin{bmatrix}d_1 & e_1 & f_1 & g_1\\ d_2 & e_2 & f_2 & g_2\end{bmatrix}\begin{pmatrix}t_1 & t_2\\ x_1 & x_2\\ y_1 & y_2\\ z_1 & z_2\end{pmatrix}=0 \tag{23}$$

Now, each row of the left matrix (dual coordinates) can be exchanged for a linear combination of both rows without changing the line, which means that a plane defining the line can be exchanged for another plane of the sheaf of planes of the line. In the same way, each column of the right matrix (point coordinates) can be exchanged for a linear combination of both columns without changing the line, which means that a point defining the line can be exchanged for another point on the line. A limitation is still present: the addition of the four point coordinates

must be the unity, but we will get rid of it. This limitation does not apply to the left matrix of dual coordinates, and it looks strange that it only applies to point coordinates in the right matrix.

5. QUADRICS

The equation of a quadric is a quadratic equation of the affine coordinates:

$$a\,x^2+b\,y^2+c\,z^2+d\,x\,y+e\,x\,z+f\,y\,z+g\,x+h\,y+i\,z+j=0 \tag{24}$$

When changing to barycentric coordinates, the equation of a quadric becomes homogeneous:

$$\begin{aligned} &m_{11}(1-x-y-z)^2+m_{22}x^2+m_{33}y^2+m_{44}z^2+2m_{12}(1-x-y-z)x \\ &+2m_{13}(1-x-y-z)y+2m_{14}(1-x-y-z)z+2m_{23}x\,y+2m_{24}x\,z+2m_{34}y\,z=0 \end{aligned} \tag{25}$$

$$\begin{pmatrix} t & x & y & z \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{22} & m_{34} & m_{44} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t & x & y & z \end{pmatrix} \mathbf{M} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = 0 \tag{26}$$

where $t=1-x-y-z$. The matrix $\mathbf{M}$ of a quadric is therefore symmetric. Notice that barycentric coordinates yield a more symmetrical equation for quadrics than affine coordinates.

5.1. **Tangential quadric of a proper quadric.** The tangential or dual quadric of a proper quadric ($\det\mathbf{M}\neq 0$) is the surface whose points are the dual of the planes tangent to the given quadric. The dual coordinates allow us to compute easily the tangential quadric. Let us differentiate the quadric equation (26) (δ represents the ordinary differential):

$$\begin{pmatrix} \delta t & \delta x & \delta y & \delta z \end{pmatrix} \mathbf{M} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t & x & y & z \end{pmatrix} \mathbf{M} \begin{pmatrix} \delta t \\ \delta x \\ \delta y \\ \delta z \end{pmatrix} = 0 \tag{27}$$

Since $\mathbf{M}^T=\mathbf{M}$ we can take the transpose of the second term to have:

$$\begin{pmatrix} \delta t & \delta x & \delta y & \delta z \end{pmatrix} \mathbf{M} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = 0 \tag{28}$$

This means that the plane tangent to the quadric at the point $P_0(t_0,x_0,y_0,z_0)$ is:

$$\begin{pmatrix} t-t_0 & x-x_0 & y-y_0 & z-z_0 \end{pmatrix} \mathbf{M} \begin{pmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} t & x & y & z \end{pmatrix} \mathbf{M} \begin{pmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0 \tag{29}$$

because P_0 satisfies the quadric equation (26). This means that the dual coordinates of the tangent plane that will be denoted as $[t',x'y'z']$ are:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \mathbf{M} \begin{pmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} \tag{30}$$

Let X be the column matrix of the coordinates of any point in the quadric and U the matrix of the dual coordinates of the tangent plane at this point. Then we have:

$$U = \mathbf{M}\,X \qquad \Rightarrow \qquad \mathbf{M}^{-1}U = X \qquad \Rightarrow \qquad X^T = U^T\mathbf{M^{-1}} \tag{31}$$

because $\mathbf{M}^{-1}$ is symmetrical as well as $\mathbf{M}$. Substitution in the equation of the quadric (26) yields:

$$X^T\mathbf{M}\,X = U^T\mathbf{M}^{-\mathbf{1}}U = 0 \tag{32}$$

that is, the matrix of the tangential quadric is the inverse of the matrix of the proper quadric. The same statement was already proven for a conic in the plane ([10], p. 134).

5.2. **Eigenpoints of a quadric.** The matrix $\mathbf{M}$ of a quadric is symmetric and it can always be reduced to diagonal form:

$$\mathbf{M} = \mathbf{B}\,\mathbf{D}\,\mathbf{B}^{-1} \qquad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{33}$$

where $\mathbf{B}$ is the matrix of change of basis, and λ_i are the eigenvalues of the quadric matrix. Since the equation of a quadric is homogeneous, the eigenvalues are determined except for a factor $k \neq 0$, that is, $\mu_i = k\lambda_i$ can be eigenvalues of another matrix of the same quadric as well. Then, what is characteristic of the quadric is the ratio of eigenvalues λ_i/λ_j. Let X' be any *eigenpoint* of the quadric, that is, the algebraic eigenvectors of $\mathbf{M}$ whose components are barycentric coordinates of a point of the three-dimensional affine space.

$$X' = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \mathbf{B} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{34}$$

Then, the equation of the quadric becomes:

$$X'^T D\,X' = \lambda_1(t')^2 + \lambda_2(x')^2 + \lambda_3(y')^2 + \lambda_4(z')^2 = 0 \tag{35}$$

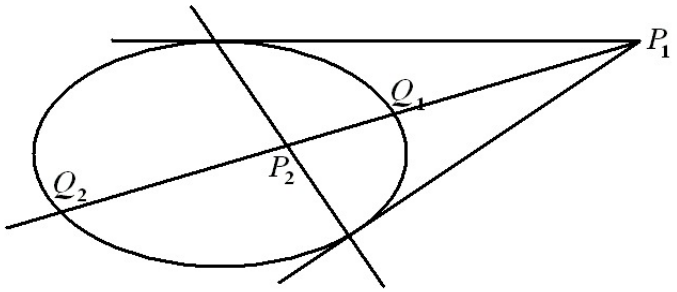

FIGURE 4. Intersection of a line with a quadric surface, here represented by a conic.

The quadric will only exist if there are eigenvalues with different sign. The eigenpoints of a quadric are not points of the quadric surface because they do not satisfy the quadric equation. They form a frame where the equation of the quadric becomes the quadratic form (35) without any crossed terms. The eigenpoints belonging to different eigenvalues are always algebraically orthogonal:

$$\lambda_i \neq \lambda_j \qquad \Rightarrow \qquad P_i^T P_j = 0 \tag{36}$$

Eigenpoints found for the same eigenvalue are not usually algebraically orthogonal. However, in the subspace they form we can take linear combinations of them being orthogonal to each other. If we have already done this, then the four eigenpoints of the quadric are algebraically orthogonal to each other. Let us prove that every pair of eigenpoints whose line intersects the quadric are harmonic conjugate of these intersections, that is, they form a harmonic range. The intersections Q_1 and Q_2 of the line passing through the eigenpoints P_1 and P_2 with the quadric (figure 4) are obtained from the equation system:

$$\begin{cases} Q^T\ \mathbf{M}\ Q = 0 \\ Q = \mu P_1 + (1-\mu)P_2 \qquad \mu \in \mathbb{R} \qquad P_1 \neq P_2 \end{cases} \tag{37}$$

Substitution of the second equation in the first one yields:

$$[\mu P_1^T + (1-\mu)P_2^T]\ \mathbf{M}\ [\mu P_1 + (1-\mu)P_2] = 0 \tag{38}$$

$$[\mu P_1^T + (1-\mu)P_2^T]\ [\mu\lambda_1 P_1 + (1-\mu)\lambda_2 P_2] = 0 \tag{39}$$

because $\mathbf{M}\ P_i = \lambda_i\ P_i$. After applying the distributive property we find:

$$\begin{aligned} &\mu^2\lambda_1 P_1^T\ P_1 + (1-\mu)^2\lambda_2 P_2^T\ P_2 \\ &\quad = \mu^2 - 2\mu\frac{\lambda_2 P_2^T P_2}{\lambda_1 P_1^T P_1 + \lambda_2 P_2^T P_2} + \frac{\lambda_2 P_2^T P_2}{\lambda_1 P_1^T P_1 + \lambda_2 P_2^T P_2} = 0 \end{aligned} \tag{40}$$

where the algebraic orthogonality of P_1 and P_2, that is $P_1^T P_2 = 0$, has been taken into account. Then, the addition and product of the two solutions μ_1 and μ_2 to this second-degree equation are:

$$\mu_1 + \mu_2 = 2\mu_1\mu_2 = \frac{2\lambda_2 P_2^T P_2}{\lambda_1 P_1^T P_1 + \lambda_2 P_2^T P_2} \qquad \Rightarrow \qquad \frac{1}{\mu_1} + \frac{1}{\mu_2} = 2 \tag{41}$$

Now, let us see that the cross ratio $(P_1 Q_1 P_2 Q_2) = 2$. By taking $Q_i = \mu_i P_1 + (1-\mu_i)P_2$ then it is easy to obtain:

$$(P_1 Q_1 P_2 Q_2) = \frac{\overline{P_1P_2}\;\overline{Q_1Q_2}}{\overline{P_1Q_2}\;\overline{Q_1P_2}} = \frac{\mu_1 - \mu_2}{\mu_1(1-\mu_2)} = 2 \tag{42}$$

Therefore $P_1Q_1P_2Q_2$ is a harmonic range. This means that each eigenpoint is located in a polar plane of each other. In fact, since three eigenpoints lie in the polar plane of the fourth with respect to the quadric, and this holds for each eigenpoint, the eigenpoints P_1, P_2, P_3, P_4 form an autopolar tetrahedron of the quadric: each face is the polar plane of each vertex, and each vertex is the pole of each face with respect to the quadric.[1]

5.3. **Polarities.** The polar plane of a given point P with respect to a quadric is the plane containing the points which form a harmonic range with the two intersections with the quadric of every line drawn from P. When both intersection points are coincident, they become tangency points of lines drawn from P to the quadric. These tangency points, when they exist, are a conic section that is the intersection of the polar plane with the quadric. The question arising now is how to calculate the polar plane of a point with respect to a quadric.

Theorem 5.1. *The equation of the polar plane of a point P with respect to the quadric with matrix* $\mathbf{M}$ *is* $X^T\mathbf{M}P=0$.

Proof. Look at figure 4. The intersection points Q_1 and Q_2 satisfy the equations (37) and (38). Since P_2 is in the polar plane of P_1, it satisfies its equation, and we have instead of (40):

$$P_2^T\mathbf{M}P_1 = 0 \qquad \Rightarrow \qquad \mu^2 - 2\mu\frac{P_2^T\mathbf{M}P_2}{P_1^T\mathbf{M}P_1 + P_2^T\mathbf{M}P_2} + \frac{P_2^T\mathbf{M}P_2}{P_1^T\mathbf{M}P_1 + P_2^T\mathbf{M}P_2} = 0 \tag{43}$$

whence:

$$\mu_1 + \mu_2 = 2\mu_1\mu_2 = \frac{2P_2^T\mathbf{M}P_2}{P_1^T\mathbf{M}P_1 + P_2^T\mathbf{M}P_2} \tag{44}$$

and then in the same way as (42):

$$(P_1 Q_1 P_2 Q_2) = \frac{\mu_1 - \mu_2}{\mu_1(1-\mu_2)} = 2 \tag{45}$$

[1]The quadratic equation (40) can have complex instead of real solutions, but equation (41) is still satisfied. In this case the points of intersection will be imaginary, as usual in complex geometry. Typically, an ellipsoid has an eigenpoint inside and three eigenpoints outside, and they form an autopolar tetrahedron.

□

Therefore, the dual coordinates D of the polar plane of a point P having as eqaution $X^T D = 0$ are:

$$D = \mathbf{M}P \tag{46}$$

This linear transformation induced by every quadric with matrix $\mathbf{M}$ that maps points into dual points is called a *polarity*.

6. Exterior algebra of points

While Grassmann's exterior product of vectors is widely used nowadays in all branches of mathematics, Grassmann's exterior algebra of points has been forgotten. However, Grassmann defined very clearly the exterior algebra of points ("combinatorial multiplication of points") ([4], p. 138). Peano reviewed Grassmann's *Ausdehnungslehre* [7], and more concretely Grassmann's exterior algebra of points [8]. Peano made an important distinction between *forms* and *multivectors*, and he introduced the operator ω, which applied to a form F yields a multivector v:

$$\omega(F) = v \qquad F \in \wedge^{n+1}(P) \qquad v \in \wedge^{n}(E) \tag{47}$$

where P is the geometric space of points with algebraic dimension $n+1$, and E its n-dimensional vector space of translations in P. The operator ω has been reviewed more extensively by Greco et al. [11].
The algebra of points is always exterior. No interior product of points is considered at all. Then, the exterior product of points $\{P_i\}$ will be indicated by adjunction $P_1P_2\cdots P_i$ and simply called *product* without the adjective *exterior*. It corresponds to what Grassmann called *combinatorial* or *progressive product*. The exterior algebra allows us to increase dimensions by successive multiplications. This is an intrinsec property of exterior algebra that applies to every kind of elements (points, geometric vectors, differentials or others) in an algebraic vector space owing to the property that the exterior product of linearly dependent elements is null. Then, we have the following statements:
1) The product of a point by itself is null.
2) The product of two diferent points A and B is the line $\overline{AB}$.
3) The product of three non-collinear points A, B and C is the plane they form. Otherwise the product is zero.
4) The product of four non-coplanar points is the three-dimensional space where they live. Otherwise it is zero.
5) And so on, until completing the dimension of the hyperspace of points.

It is clear that a non-vanishing product of points corresponds to the operator *join* of the projective geometry, and owing to this reason Grassmann called it *progressive*. All these products can be computed by means of a basis of points. For instance, let A, B, C and D be four points in the three-dimensional space. Then, if we express them in barycentric coordinates with respect to the basis $\{P_1,P_2,P_3,P_4\}$ we have:

$$
\begin{aligned}
A &= a_1\,P_1 + a_2\,P_2 + a_3\,P_3 + a_4\,P_4 \qquad & B &= b_1\,P_1 + b_2\,P_2 + b_3\,P_3 + b_4\,P_4 \\
C &= c_1\,P_1 + c_2\,P_2 + c_3\,P_3 + c_4\,P_4 \qquad & D &= d_1\,P_1 + d_2\,P_2 + d_3\,P_3 + d_4\,P_4
\end{aligned} \tag{48}
$$

In matrix form:

$$
\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \tag{49}
$$

Then, we can make the exterior products:

$$
\begin{aligned}
AB = {} & P_1P_2 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + P_1P_3 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + P_1P_4 \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \\
& + P_2P_3 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + P_2P_4 \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + P_3P_4 \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}
\end{aligned} \tag{50}
$$

In the same way:

$$
\begin{aligned}
ABC = {} & P_1P_2P_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + P_1P_2P_4 \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} \\
& + P_1P_3P_4 \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} + P_2P_3P_4 \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix}
\end{aligned} \tag{51}
$$

If the form AB is understood as the line passing through the points A and B, then one sees that the algebraic space of lines has dimension 6, and the lines $\{P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4\}$, which are edges of the reference tetrahedron $P_1P_2P_3P_4$, form a basis of this space. In the same way, if ABC is interpreted as the plane containing these three points, then the planes $\{P_1P_2P_3, P_1P_2P_4, P_1P_3P_4, P_2P_3P_4\}$ form a basis of the algebraic space of planes in the space.

The exterior product can be applied to points or dual points on an equal footing. Then, in the former equalities A, B, C and D can be dual points representing planes. then we have:
1) The product of two different dual points A and B, which represent the planes π_A and π_B, is the line intersection of both planes: $\pi_A \cap \pi_B$.
3) The product of three dual points A, B and C representing the planes π_A, π_B and π_C is their point of interesection: $\pi_A \cap \pi_B \cap \pi_C$.
4) The product of four dual points is a real number if they are not coplanar in the dual space. Otherwise, the product is zero, which means that the four planes they represent have a common intersection (point, line or plane depending on the rank).

It is then clear that the product of dual points corresponds to what Grassmann called the *regressive product*, and that the same exterior algebra but now applied to dual points corresponds to the *meet* operator of the projective geometry if the product is not null. Barnabei, Brini and Rota [12] said:

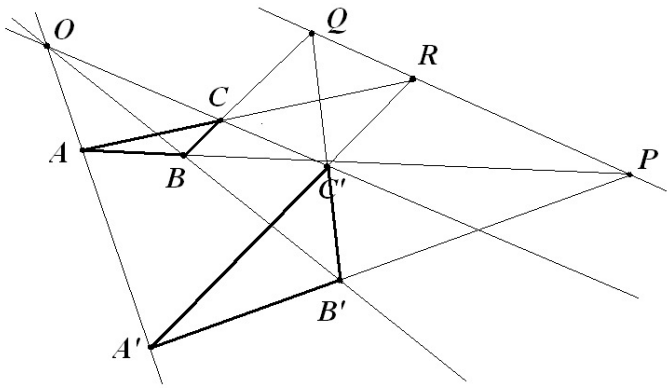

FIGURE 5. Desargues' theorem.

> Elie Cartan found the regressive product to be superfluous and awkward. By vector space duality, a pairing of two exterior algebras of V and V^* could easily be made with one kind of product, the one that came to be called the wedge.
> ...
> Grassmann's idea was to develop a calculus for the join and meet of linear varieties in projective space, a calculus that is actually realized by the progressive and regressive products. It has been amply demonstrated that this calculus furnishes the definitive notation for such computations.

However, they did not treat affine and projective geometries in the way Cartan indicated. We agree with Cartan in the fact that the exterior algebra applied to both point and dual spaces suffices for describing and computing both join and meet operators, because the join and meet operators in the point space are respectively the meet and join operators in the dual space. The only precaution is to provide consistent rules of computations for them. For instance, a line can be determined by two points as well as by the intersection of two planes, like the matrix equation (23) of a line shows. Then, joining points A and B is expressed by the product AB. The meet of the planes π_C and π_D is expressed by their product expressed in dual coordinates. Both computations must provide the same result, as proven ([13], p. 72).

7. ENTERING INTO PROJECTIVE GEOMETRY

7.1. **Projective theorems.** Surprisingly, barycentric coordinates allow us to prove theorems of projective geometry. Let us see, for instance, three main theorems of the plane projective geometry: Desargues', Pappus' and Pascal's theorems.

Theorem 7.1 (Desargues)**.** *Let ΔABC and $\Delta A'B'C'$ be two triangles. Let P be the intersection of the prolongations of sides AB and $A'B'$, Q the intersection of the prolongations of BC and $B'C'$, and R the intersection of the prolongations of CA and $C'A'$. Points P, Q and R are collinear if and only if the lines $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$ meet at the same point O (figure 5).*

Proof. [10], p. 48. The hypothesis states that $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$ intersect at the same point O:

$$O = a\,A + (1-a)A' = b\,B + (1-b)B' = c\,C + (1-c)C' \qquad a,b,c \in \mathbb{R} \tag{52}$$

From the first equality we obtain:

$$a\,A - b\,B = -(1-a)A' + (1-b)B' \tag{53}$$

Dividing by $a-b$, the addition of coefficients becomes the unity, and then the equality represents the interesection of lines $\overline{AB}$ and $\overline{A'B'}$, which is the point P:

$$P = \frac{a}{a-b}A - \frac{b}{a-b}B = \frac{a-1}{a-b}A' - \frac{b-1}{a-b}B' \tag{54}$$

Likewise, by taking the second and third terms or the first and third terms in (52) we also obtain:

$$Q = \frac{b}{b-c}B - \frac{c}{b-c}C = \frac{b-1}{b-c}B' - \frac{c-1}{b-c}C' \tag{55}$$

$$R = \frac{c}{c-a}C - \frac{a}{c-a}A = \frac{c-1}{c-a}C' - \frac{a-1}{c-a}A' \tag{56}$$

Let us check that the determinant of the barycentric coordinates of P, Q and R with respect to the basis ABC is null in order to prove that they are collinear.

$$\det(PQR) = \begin{vmatrix} \frac{a}{a-b} & -\frac{b}{a-b} & 0 \\ 0 & \frac{b}{b-c} & -\frac{c}{b-c} \\ -\frac{a}{c-a} & 0 & \frac{c}{c-a} \end{vmatrix} \propto \begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ -a & 0 & c \end{vmatrix} = 0 \tag{57}$$

This ends the proof in one direction. The proof in the other direction is algebraically the same after applying duality and changing points by lines. □

Theorem 7.2 (Pappus)**.** *If A, B, and C lie on a line and D, E, F lie on another line in the plane, then the intersections points of $\overline{AE}$ with $\overline{BF}$, $\overline{AD}$ with $\overline{CF}$ and $\overline{BD}$ with $\overline{CE}$ are collinear (figure 6).*

Proof. [10], p. 240. Let us take $\{A,B,D\}$ as a basis for points in the plane. Since P belongs to lines $\overline{AE}$ and $\overline{BF}$, the points A, P and E are collinear as well as B, P and F, so that the determinants of their barycentric coordinates are null:

$$\begin{vmatrix} 1 & 0 & 0 \\ p_A & p_B & p_D \\ e_A & e_B & e_D \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 1 & 0 \\ p_A & p_B & p_D \\ f_A & f_B & f_D \end{vmatrix} = 0 \tag{58}$$

whence we have:

$$(p_A, p_B, p_D) = \frac{(f_A e_D, e_B f_D, eDf_D)}{f_A e_D + e_B f_D + e_D f_D} \tag{59}$$

Since Q belongs to lines $\overline{AD}$ and $\overline{CF}$ we have likewise:

$$\begin{vmatrix} 1 & 0 & 0 \\ q_A & q_B & q_D \\ 0 & 0 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} c_A & c_B & 0 \\ q_A & q_B & q_D \\ f_A & f_B & f_D \end{vmatrix} = 0 \tag{60}$$

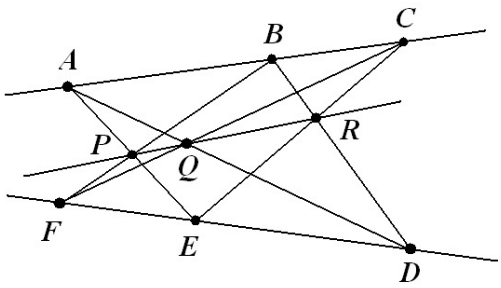

FIGURE 6. Pappus' theorem.

whence we obtain:

$$(q_A, q_B, q_D) = \frac{(c_A f_B - c_B f_A, 0, -c_B f_D)}{c_A f_B - c_B f_A - c_B f_D} \tag{61}$$

Note that $c_D = 0$ because C lies on line $\overline{AB}$ and is a linear combination of only these points. Since R belongs to lines $\overline{CE}$ and $\overline{BD}$ we have likewise:

$$\begin{vmatrix} c_A & c_B & 0 \\ r_A & r_B & r_D \\ e_A & e_B & e_D \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 1 & 0 \\ r_A & r_B & r_D \\ 0 & 0 & 1 \end{vmatrix} = 0 \tag{62}$$

whence we obtain:

$$(r_A, r_B, r_D) = \frac{(0, c_A e_B - c_B e_A, c_A e_D)}{c_A e_B - c_B e_A + c_A e_D} \tag{63}$$

Let us see that the determinant of points P, Q and R vanishes:

$$\det(P,Q,R) \propto \begin{vmatrix} e_D f_A & e_B f_D & e_D f_D \\ c_A f_B - c_B f_A & 0 & -c_B f_D \\ 0 & c_A e_B - e_A c_B & c_A e_D \end{vmatrix} \propto \begin{vmatrix} \frac{f_A}{f_D} & \frac{e_B}{e_D} & 1 \\ \frac{c_A}{c_B} - \frac{f_A}{f_B} & 0 & -\frac{f_D}{f_B} \\ 0 & \frac{e_B}{e_A} - \frac{c_B}{c_A} & \frac{e_D}{e_A} \end{vmatrix} \tag{64}$$

$$= \frac{-e_A f_B + f_A e_B}{e_A f_B} = 0$$

because points D, E and F are collinear:

$$\begin{vmatrix} 0 & 0 & 1 \\ e_A & e_B & e_D \\ f_A & f_B & f_D \end{vmatrix} = e_A f_B - e_B f_A = 0 \tag{65}$$

Therefore, P, Q and R are collinear. □

Theorem 7.3 (Pascal's hexagram)**.** *Let A, B, C, D, E and F be six distinct points on a proper conic. Let P be the intersection of the line $\overline{AE}$ with $\overline{BF}$, Q the intersection of $\overline{AD}$ with $\overline{CF}$, and R be the intersection of $\overline{BD}$ with $\overline{CE}$. Then P, Q and R are collinear (figure 7).*

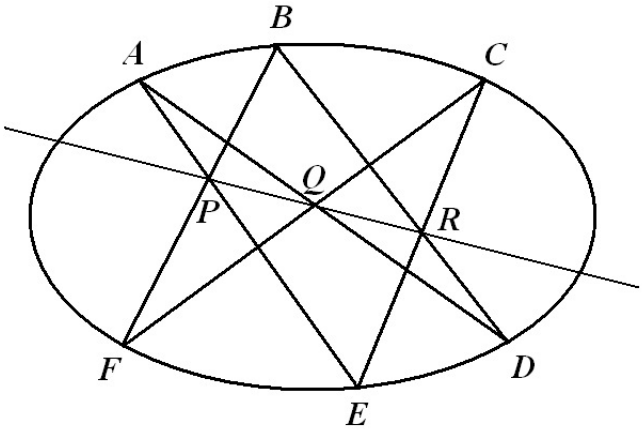

FIGURE 7. Pascal's hexagram theorem.

Proof. [10], p. 270. Five points determine a conic. Let us take $\{A,B,C\}$ as an affine frame of the plane and express D and X (the general point of the conic) with barycentric coordinates in this frame:

$$D = d_A A + d_B B + d_C C \qquad \text{with} \quad d_A + d_B + d_C = 1 \tag{66}$$

$$X = x_A A + x_B B + x_C C \qquad \text{with} \quad x_A + x_B + x_C = 1 \tag{67}$$

The projective cross ratio r is ([10], p. 131):

$$r = (X,ABCD) = \frac{XA \wedge XC \; XB \wedge XD}{XA \wedge XD \; XB \; \wedge XC} = \frac{x_B(x_A d_C - x_C d_A)}{(x_B d_C - x_C d_B)x_A} \tag{68}$$

which yields the following equation:

$$0 = r\, x_C x_A d_B + (1-r)x_A x_B d_c - x_B x_C d_A \tag{69}$$

The fifth point E also satisfies this equation:

$$0 = r\, e_C e_A d_B + (1-r)e_A e_B d_C - e_B e_C d_A \tag{70}$$

which results in:

$$r = \frac{\frac{d_A}{e_A} - \frac{d_C}{e_C}}{\frac{d_B}{e_B} - \frac{d_C}{e_C}} \tag{71}$$

The substitution of r into the equation (69) gives the equation of the conic:

$$\begin{vmatrix} d_A e_A & d_B e_B & d_C e_C \\ d_A x_A & d_B x_B & d_C x_C \\ e_A x_A & e_B x_B & e_C x_C \end{vmatrix} = 0 \tag{72}$$

Since P is the intersection of lines $\overline{AE}$ and $\overline{BF}$, A, P and E are collinear as well as B, P and F:

$$\begin{vmatrix} 1 & 0 & 0 \\ p_A & p_B & p_C \\ e_A & e_B & e_C \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 1 & 0 \\ p_A & p_B & p_C \\ f_A & f_B & f_C \end{vmatrix} = 0 \quad \Rightarrow (p_A, p_B, p_C) = \frac{(e_C f_A,\ e_B f_C,\ e_C f_C)}{e_C f_A + e_B f_C + e_C f_C} \tag{73}$$

Since Q is the intersection of lines $\overline{AD}$ and $\overline{CF}$, A, Q and D are collinear as well as C, Q and F:

$$\begin{vmatrix} 1 & 0 & 0 \\ q_A & q_B & q_C \\ d_A & d_B & d_C \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 0 & 1 \\ q_A & q_B & q_C \\ f_A & f_B & f_C \end{vmatrix} = 0 \quad \Rightarrow (q_A, q_B, q_C) = \frac{(d_B f_A,\ d_B f_B,\ d_C f_B)}{d_B f_A + d_B f_B + d_C f_B} \tag{74}$$

Since R is the intersection of lines $\overline{BD}$ and $\overline{CE}$, B, R and D are collinear as well as C, R and E:

$$\begin{vmatrix} 0 & 1 & 0 \\ r_A & r_B & r_C \\ d_A & d_B & d_C \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 0 & 1 \\ r_A & r_B & r_C \\ e_A & e_B & e_C \end{vmatrix} = 0 \quad \Rightarrow (r_A, r_B, r_C) = \frac{(d_A e_A,\ d_A e_B,\ d_C e_A)}{d_A e_A + d_A e_B + d_C e_A} \tag{75}$$

The points P, Q and R will be collinear if and only if their determinant is null:

$$\begin{vmatrix} p_A & p_B & p_C \\ q_A & q_B & q_C \\ r_A & r_B & r_C \end{vmatrix} \propto \begin{vmatrix} e_C f_A & e_B f_C & e_C f_C \\ d_B f_A & d_B f_B & d_C f_B \\ d_A e_A & d_A e_B & d_C e_A \end{vmatrix} \propto \begin{vmatrix} \frac{f_A}{f_C} & \frac{e_B}{e_C} & 1 \\ \frac{f_A}{f_B} & 1 & \frac{d_C}{d_B} \\ 1 & \frac{e_B}{e_A} & \frac{d_C}{d_A} \end{vmatrix} \propto \begin{vmatrix} \frac{1}{f_C} & \frac{1}{e_C} & \frac{1}{d_C} \\ \frac{1}{f_B} & \frac{1}{e_B} & \frac{1}{d_B} \\ \frac{1}{f_A} & \frac{1}{e_A} & \frac{1}{d_A} \end{vmatrix}$$
$$\propto \begin{vmatrix} d_A e_A & d_B e_B & d_C e_C \\ d_A f_A & d_B f_B & d_C f_C \\ e_A f_A & e_B f_B & e_C f_C \end{vmatrix} = 0 \tag{76}$$

where in the last step there is a transposition. The determinant is equal to zero because F fits the equation (72) of a conic passing through five points A, B, C, D and E. Therefore P, Q and R are collinear. □

Remark. By duality, Brianchon's theorem is proven in the same algebraic way ([10], p. 271).

7.2. **Points at infinity.** The algebraic tools provided by affine frames and barycentric coordinates are enough to prove projective theorems. However, the affine geometric space lacks the points at infinity, necessary to give completeness to projective geometry. In order to include points at infinity, Fishback ([14], p. 115) and Coxeter ([15], p. 271) took homogeneous coordinates proportional to barycentric coordinates they called (and we will call) *projective coordinates*. Now, the addition of coordinates does not need to be the unity. Then, every point X in the geometric space is written as a linear combination of the points of the affine frame $\{O,A,B,C\}$, and it is the same point although we multiply all the coordinates by a constant $k \neq 0$:

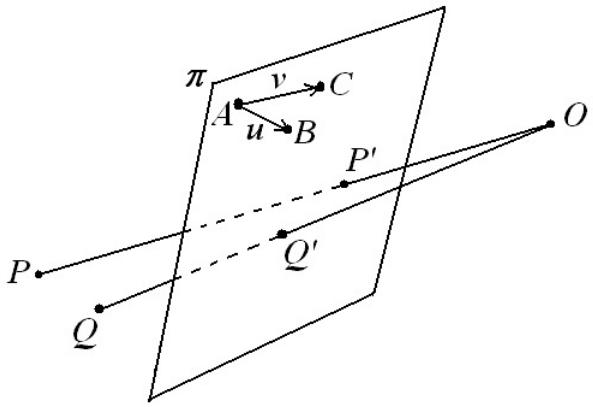

FIGURE 8. Projection onto a plane.

$$X = t\,O + x\,A + y\,B + z\,C = k\,t\,O + k\,x\,A + k\,y\,B + k\,z\,C \qquad k \in \mathbb{R} - \{0\} \tag{77}$$

In fact, in the definition itself, barycentric coordinates are already homogeneous. Consider t, x, y, z as weights placed at the points O, A, B, C, and let X be the center of mass of the system. Now, increase all the weights in the same ratio k. It is obvious that the centre of mass is the same point X. Then, barycentric coordinates do not need themselves to have a sum equal to the unity. This "normalization" is only necessary when we wish to transform barycentric into affine coordinates. As we have seen, the equation of a plane (18), a line (23), a conic (72) or a quadric (26) are already homogeneous. Therefore, using homogeneous coordinates makes no difference. Dual coordinates are homogeneous as well, and it makes no sense that point coordinates are not homogeneous.

Homogeneous coordinates allow us to handle points at infinity because the addition of their coordinates is zero, which means that their affine coordinates are infinite:

$$X_\infty = (-x-y-z)O + x\,A + y\,B + z\,C = x\,OA + y\,OB + z\,OC \tag{78}$$

By coupling points, we can see that a point at infinity is equivalent to a direction indicated by a vector. Proportional vectors indicate the same point at infinity. In this algebraic way, points at infinity are incorporated to the equations deduced by means of barycentric coordinates without any change to them. These equations simply apply to all configurations without exceptions. Look for a moment at the proofs of Desargues', Pappus' and Pascal's theorems, and notice that denominators in the fractions could be null for a certain point configurations, which means that these proofs would no longer be valide. However, homogeneous coordinates avoid this trouble: denominators can simply be erased, and configurations with points at infinity can also be included. This is a fundamental question when dealing with projections.

7.3. **Projection onto a plane.** Let us study the projection with centre O of all the points in the space onto a given plane (figure 8). Every projected point P' satisfies:

$$\begin{cases} P' = (1-\lambda)P + \lambda O \\ P' = (1-\mu-\nu)A + \mu B + \nu C \end{cases} \qquad \lambda, \mu, \nu \in \mathbb{R} \tag{79}$$

By equating both equations and isolating P we obtain:

$$P = \frac{(1-\mu-\nu)A + \mu B + \nu C - \lambda O}{1-\lambda} \tag{80}$$

Since barycentric coordinates are a quotient of volumes we have:

$$-\frac{\lambda}{1-\lambda} = \frac{PABC}{OABC} = \frac{PA \wedge PB \wedge PC}{OA \wedge OB \wedge OC} = \frac{PA \wedge AB \wedge AC}{OA \wedge OB \wedge OC} = \frac{PA \wedge u \wedge v}{OA \wedge u \wedge v} \tag{81}$$

whence:

$$\lambda = \frac{PA \wedge u \wedge v}{PO \wedge u \wedge v} \qquad 1-\lambda = \frac{AO \wedge u \wedge v}{PO \wedge u \wedge v} \tag{82}$$

Therefore, the projected point P' is:

$$P' = \frac{AO \wedge u \wedge v}{PO \wedge u \wedge v} P + \frac{PA \wedge u \wedge v}{PO \wedge u \wedge v} O \tag{83}$$

If the plane has the equation $a\,x + b\,y + c\,z + d = 0$, we have:

$$P' = \frac{a(x_O - x_A) + b(y_O - y_A) + c(z_O - z_A)}{a(x_O - x_P) + b(y_O - y_P) + c(z_O - z_P)} P + \frac{a(x_A - x_P) + b(y_A - y_P) + c(z_A - z_P)}{a(x_O - x_P) + b(y_O - y_P) + c(z_O - z_P)} O \tag{84}$$

but A is a point in the plane satisfying its equation. Therefore:

$$P' = \frac{(a\,x_O + b\,y_O + c\,z_O + d)P - (a\,x_P + b\,y_P + c\,z_P + d)O}{a(x_O - x_P) + b(y_O - y_P) + c(z_O - z_P)} \tag{85}$$

Writing the equation of the plane with barycentric coordinates $d\,t + f\,x + g\,y + h\,z = 0$:

$$P' = \frac{(d\,t_O + f\,x_O + g\,y_O + h\,z_O)P - (d\,t_P + f\,x_P + g\,y_P + h\,z_P)O}{d(t_O - t_P) + f(x_O - x_P) + g(y_O - y_P) + h(z_O - z_P)} \tag{86}$$

If we take homogeneous coordinates, then the denominator is no longer necessary. Then, the former equality is written in matrix form:

$$P' = P\,O^T\,D - O\,P^T\,D \tag{87}$$

where all the points are taken as columns of homogeneous coordinates, and $D^T = [d,\ f,\ g,\ h]$ are the dual coordinates of the plane. The product of the two matrices on the right of each term is a real number, which is a coefficient of the linear combination of points. Since the product of matrices has the associative and distributive properties, we can write:

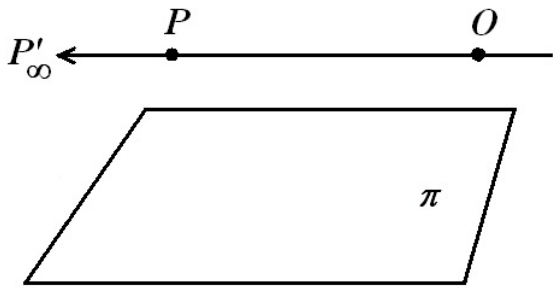

FIGURE 9. Projection yielding a point at infinity.

$$P' = (P\,O^T - O\,P^T)D = \mathbf{M}\,D \tag{88}$$

where $\mathbf{M} = P\,O^T - O\,P^T$ is a skew-symmetric matrix of order 4 with null determinant because the degrees of freedom are less for P' than for D. This equation means that the projected point P' is a linear skew-symmetric mapping of the dual coordinates of the plane of projection. If we take a plane of projection parallel to the line OP, the projected point P' will be a point at infinity (figure 9). The equation (88) for homogeneous coordinates will yield this point without making any exception. Of course, we can also take the centre of projection O at the infinity to have a parallel projection. This case is also included in the equation (88), which can be rearranged taking into account that $O^T D$ and $P^T D$ are real numbers:

$$P' = (O^T D\,I - O\,D^T)P = \mathbf{N}\,P \qquad \text{rank}(\mathbf{N}) = 3 \qquad \det(\mathbf{N}) = 0 \tag{89}$$

where I is the identity of order 4. Now, every point P' projected onto plane D is a linear function of a point P in the space, and therefore the matrix $\mathbf{N}$ has rank 3.

8. PROJECTIVITIES AND PROJECTIVE COORDINATES

Since barycentric and homogeneous coordinates are enough to prove projective theorems, it is obvious that they already are projective coordinates, that is, coordinates adapted to the description of the projective problems and properties of the three-dimensional space. Therefore, what are the differences between and affine frame and a projective frame? Notice that one element is implicit and not displayed in the affine frame shown in figure 1: the coordinate lines are parallel. If we draw them, we will obtain figure 10. There, the so called *unit point U* with affine coordinates $x = 1$, $y = 1$ and $z = 1$ is indicated. The unit point is an additional information necessary to define projective grids and coordinates. In the case of affine coordinates, the unit point lies at the vertex of the parallelepiped generated by the affine frame $\{O,A,B,C\}$.

8.1. **Geometric definition of projective coordinates.** If we deformate the affine grid of figure 10 we obtain the projective grid of figure 11. The new projective grid is defined by the image points of five points in the three-dimensional space, no four of which are coplanar. Since the reference tetrahedron has four vertices, an additional point is needed. One usually takes the unit point U, but any other noncoplanar point can be taken. In this way the mapping $\{O,A,B,C,U\} \to \{O',A',B',C',U'\}$ defines a *projectivity* (also called *homography*), that is, a projective transformation of the three-dimensional space into itself.

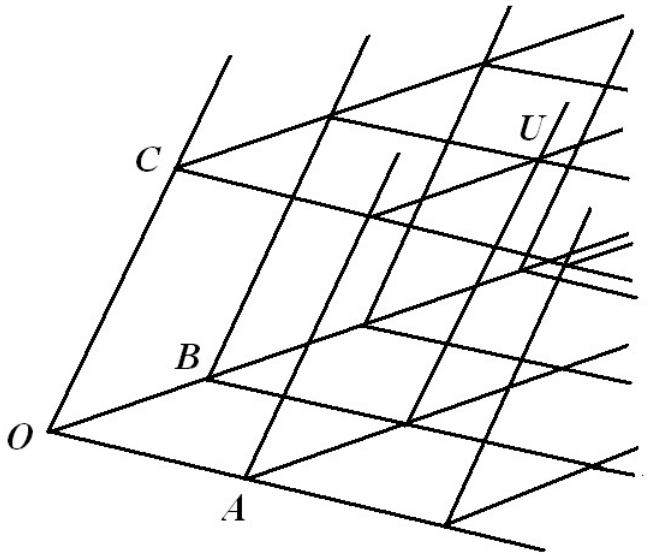

FIGURE 10. An affine frame $\{O,A,B,C\}$ always has parallel line coordinates. It is a particular case of projective frames. The unit point U is the vertex of the parallelepiped generated by $\{O,A,B,C\}$

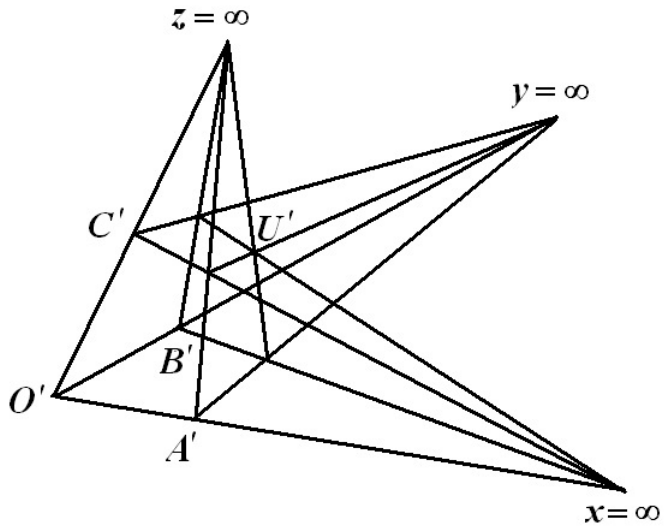

FIGURE 11. A projective frame $\{O',A',B',C',U'\}$ is defined by one point more than an affine frame $\{O',A',B',C'\}$. The unit point U' is additionally taken although any other point suffices. Line coordinates are not parallel in general but they are concurrent at points having the corresponding coordinate equal to infinity.

An example of complete projective grid in the plane is displayed in figure 12. Four non-coplanar points $\{A,B,C,D\}$ are enough to build a full projective grid in a plane. It contains an infinite number of complete quadrilaterals that indicate respectively harmonic ranges[2] of points such as:

[2]Depending on how the four points are ordered on a line, the cross ratio of a harmonic range can be 2, -1 or 1/2. From the distinct values the cross ratio can take for permutations of points, we defined the *harmonic characteristic* which is null for harmonic ranges, and equal to infinity when any pair of points are coincident([[10], p. 94).

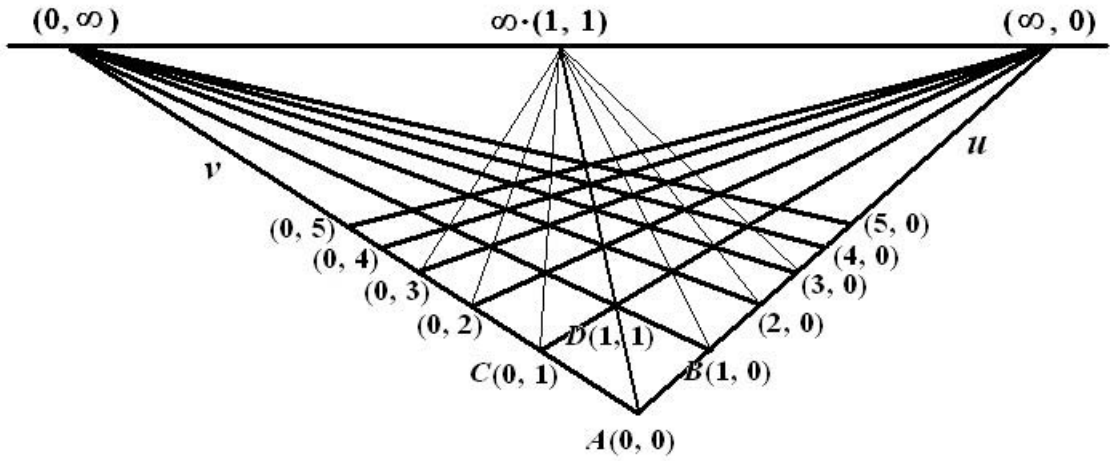

FIGURE 12. Example of a projective grid in the plane. The projective frame is $\{A,B,C,D\}$ where D is the unit point. Four points, no three of which are collinear, define a projective grid if we assign coordinates to them. Points with integer values of coordinates are displayed. The line joining points at infinity is called the line at infinity.

$$(ABU_2U_\infty) = \frac{AU_2 \;\; BU_\infty}{AU_\infty \;\; BU_2} = 2 \qquad U_2 = (2,0) \quad U_\infty = (\infty,0) \tag{90}$$

This means that the projective length of AU_2 doubles that of BU_2, which is also displayed by coordinates. Therefore, the measuring of distances d between points and comparison of segments on a line in a projective grid is carried out by means of the cross ratio with respect to the point at infinity on the same line:

$$\frac{d(X_1X_3)}{d(X_2X_3)} = \frac{(X_1X_3) \;\; (X_2X_\infty)}{(X_2X_3) \;\; (X_1X_\infty)} \qquad \text{or} \qquad \frac{d(X_1X_3)}{d(X_1X_2)} = \frac{(X_1X_3) \;\; (X_2X_\infty)}{(X_1X_2) \;\; (X_3X_\infty)} \tag{91}$$

This measure is invariant under projectivities. For an affine grid, $X_1X_\infty = X_2X_\infty = X_3X_\infty = \infty$ and (91) reduces to the usual comparison of segments on a line by means of simple ratio:

$$\frac{d(X_1X_3)}{d(X_2X_3)} = \frac{X_1X_3}{X_2X_3} \qquad \text{or} \qquad \frac{d(X_1X_3)}{d(X_1X_2)} = \frac{X_1X_3}{X_1X_2} \tag{92}$$

Another example of plane projective coordinates are trilinear coordinates, for which the point $I(1,1,1)$ is taken as the incentre of the reference triangle ABC. Homogeneous barycentric coordinates are then obtained from multiplication of the trilinear coordinates by the lengths of the sides $\|BC\|$, $\|CA\|$ and $\|AB\|$ respectively. For homogeneous barycentric coordinates the point $G(1,1,1)$ is the centroid of the reference triangle ABC.

8.2. **Algebraic definition of projective coordinates.** This geometric definition of projective grids and coordinates must have its counterpart in a suitable algebraic definition.

Theorem 8.1. *Taking a fixed projective frame, every non-singular linear transformation of the projective coordinates* (77) *obtained from barycentric coordinates is a projectivity (homography); and conversely, every projectivity is a non-singular linear transformation of projective coordinates.*

This theorem for Plücker coordinates ([16], p. 29) is already known. However its validity for projective coordinates obtained from barycentric coordinates must be proven.

Proof. [17]. A non-singular linear transformation of projective coordinates (77) can be written by means of a non-singular matrix $\mathbf{M}_{4\times 4}$:

$$P' = \mathbf{M}\,P \qquad \det\mathbf{M} \neq 0 \tag{93}$$

where P and P' are given as column matrices of projective coordinates:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{94}$$

Firstly, let us prove that this linear transformation maps collinear points into collinear points. If P, Q and R are three collinear points then:

$$R = \lambda P + \mu Q \qquad \lambda, \mu \in \mathbb{R} \tag{95}$$

Notice that $\lambda + \mu = 1$ is not necessary because projective coordinates are homogeneous. Applying the linear map $\mathbf{M}$ we have:

$$R' = \mathbf{M}R = \lambda\mathbf{M}P + \mu\mathbf{M}Q = \lambda P' + \mu Q' \tag{96}$$

owing to the distributive property of matrix product. Therefore, the images P', Q' and R' are collinear as well.
Secondly, let us prove that this linear transformation preserves the cross ratio. Let A, B, C and D be four distinct collinear points. Then we can write C and D as barycentric linear combinations of A and B (barycentric coordinates are necessary to measure distances on the line):

$$C = (1-\lambda)A + \lambda B \qquad D = (1-\mu)A + \mu B \qquad \lambda, \mu \in \mathbb{R} - \{0,1\} \tag{97}$$

Then we have:

$$AC = \lambda AB \qquad AD = \mu AB \qquad BC = (\lambda - 1)AB \qquad BD = (\mu - 1)AB \tag{98}$$

The cross ratio is then:

(99) $$\frac{AC\ BD}{AD\ BC} = \frac{\lambda(1-\mu)}{\mu(1-\lambda)}$$

The linear transformation maps points A and B into A' and B' respectively:

(100) $$A' = k\,\mathbf{M}\,A \qquad B' = l\,\mathbf{M}\,B \qquad k,l \in \mathbb{R} - \{0\}$$

where k and l are numeric factors necessary to pass from the projective coordinates of A' and B' to their barycentric coordinates. In the same way, the points C and D are transformed into C' and D':

(101) $$C' = m\,\mathbf{M}\,C \qquad D' = n\,\mathbf{M}\,D \qquad m,n \in \mathbb{R} - \{0\}$$

The coefficients m and n are related to k and l as we see by substitution of C and D for their barycentric expression:

(102) $$C' = m\,\mathbf{M}[(1-\lambda)A + \lambda B] = \frac{m(1-\lambda)}{k}A' + \frac{m\lambda}{l}B'$$

With barycentric coordinates, the addition of the coefficients of linear combination must be the unity:

(103) $$\frac{m(1-\lambda)}{k} + \frac{m\lambda}{l} = 1 \qquad \Rightarrow \qquad m = \frac{k\,l}{k\lambda + l(1-\lambda)}$$

Therefore:

(104) $$C' = \frac{l(1-\lambda)}{k\lambda + l(1-\lambda)}A' + \frac{k\lambda}{k\lambda + l(1-\lambda)}B'$$

Analogously:

(105) $$D' = \frac{l(1-\mu)}{k\mu + l(1-\mu)}A' + \frac{k\mu}{k\mu + l(1-\mu)}B'$$

The vectors are therefore:

(106) $$A'C' = \frac{k\lambda}{k\lambda + l(1-\lambda)}A'B' \qquad A'D' = \frac{k\mu}{k\mu + l(1-\mu)}A'B'$$

$$B'C' = \frac{l(\lambda-1)}{k\lambda + l(1-\lambda)}A'B' \qquad B'D' = \frac{l(\mu-1)}{k\mu + l(1-\mu)}A'B'$$

The cross ratio of the transformed points is now equal to that of the initial points:

$$\frac{A'C'\ \ B'D'}{A'D'\ \ B'C'} = \frac{\lambda(1-\mu)}{\mu(1-\lambda)} = \frac{AC\ \ BD}{AD\ \ BC} \tag{107}$$

Since the cross ratio of collinear points is preserved, this linear transformation is a projectivity. Since a projectivity transforms lines into lines, that is, collinear points into collinear points, it must be a linear transformation of the projective coordinates, whose more general expression is the multiplication by a non-singular square matrix. □

Theorem 8.2. *Five points of the three-dimensional space, no four of which are coplanar and their images, no four of which are coplanar, determine a projectivity in a unique way.*

Proof. To determine a projectivity means to determine all the entries of the matrix $\mathbf{M}$ except for an arbitrary factor k. Let us suppose that the images of the projective frame $\{O,A,B,C,U\}$ are $\{O',A',B',C',U'\}$, where O is the origin of coordinates and U is the unity point. Taking $\{O,A,B,C,U\}$ as the reference frame of all coordinates, we have $O(1,0,0,0)$, $A(0,1,0,0)$, $B(0,0,1,0)$, $C(0,0,0,1)$ and $U(-2,1,1,1)$, $O'(o_1,o_2,o_3,o_4)$, $A'(a_1,a_2,a_3,a_4)$, $B'(b_1,b_2,b_3,b_4)$, $C'(c_1,c_2,c_3,c_4)$ and $U'(u_1,u_2,u_3,u_4)$. Then:

$$k\,O' = \mathbf{M}\,O \quad \Rightarrow \quad k\begin{pmatrix} o_1\\ o_2\\ o_3\\ o_4\end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{pmatrix}\begin{pmatrix} 1\\ 0\\ 0\\ 0\end{pmatrix} \qquad k\neq 0 \tag{108}$$

$$l\,A' = \mathbf{M}\,A \quad \Rightarrow \quad l\begin{pmatrix} a_1\\ a_2\\ a_3\\ a_4\end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{pmatrix}\begin{pmatrix} 0\\ 1\\ 0\\ 0\end{pmatrix} \qquad l\neq 0$$

$$n\,B' = \mathbf{M}\,B \quad \Rightarrow \quad n\begin{pmatrix} b_1\\ b_2\\ b_3\\ b_4\end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{pmatrix}\begin{pmatrix} 0\\ 0\\ 1\\ 0\end{pmatrix} \qquad n\neq 0$$

$$p\,C' = \mathbf{M}\,C \quad \Rightarrow \quad p\begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ 1\end{pmatrix} \qquad p\neq 0$$

Therefore the matrix $\mathbf{M}$ has the form:

$$\mathbf{M} = \begin{pmatrix} k\,o_1 & l\,a_1 & n\,b_1 & p\,c_1\\ k\,o_2 & l\,a_2 & n\,b_2 & p\,c_2\\ k\,o_3 & l\,a_3 & n\,b_3 & p\,c_3\\ k\,o_4 & l\,a_4 & n\,b_4 & p\,c_4\end{pmatrix} \tag{109}$$

In order to determine the factors k, l, n and p we apply the projectivity to the unit point U:

$$U' = \mathbf{M}\, U \qquad \Rightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} k\, o_1 & l\, a_1 & n\, b_1 & p\, c_1 \\ k\, o_2 & l\, a_2 & n\, b_2 & p\, c_2 \\ k\, o_3 & l\, a_3 & n\, b_3 & p\, c_3 \\ k\, o_4 & l\, a_4 & n\, b_4 & p\, c_4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{110}$$

which yields the equation system:

$$\begin{cases} -2k\, o_1 + l\, a_1 + n\, b_1 + p\, c_1 = u_1 \\ -2k\, o_2 + l\, a_2 + n\, b_2 + p\, c_2 = u_2 \\ -2k\, o_3 + l\, a_3 + n\, b_3 + p\, c_3 = u_3 \\ -2k\, o_4 + l\, a_4 + n\, b_4 + p\, c_4 = u_4 \end{cases} \tag{111}$$

The solution to this equation system provides the values for the unknown coefficients k, l, n and p. The projective coordinates of $U'(u_1, u_2, u_3, u_4)$ are homogeneous and other proportional values can be taken. In this case, the values of k, l, n and p rise proportionally and therefore $\mathbf{M}$ also does, that is, the matrix $\mathbf{M}$ is determined except for a constant, which means that their entries have 15 degress of freedom. □

Theorem 8.3. *Every transformation of the three-dimensional space that preserves projective coordinates and maps a projective frame $\{ABCDE\}$ into another projective frame $\{A'B'C'D'E'\}$ is a projectivity:*

$$E = a\, A + b\, B + c\, C + d\, D \quad \Rightarrow \quad E' = a\, A' + b\, B' + c\, C' + d\, D' \tag{112}$$

Proof. By taking the initial frame as the reference frame for all coordinates we have:

$$\begin{aligned} A' &= m_{11}A + m_{21}B + m_{31}C + m_{41}D \qquad & B' &= m_{12}A + m_{22}B + m_{32}C + m_{42}D \\ C' &= m_{13}A + m_{23}B + m_{33}C + m_{43}D \qquad & D' &= m_{14}A + m_{24}B + m_{34}C + m_{44}D \end{aligned} \tag{113}$$

where the indexes of the coefficients m_{ij} of linear combination have been taken in a suitable way. Then we have:

$$\begin{aligned} E' = A(m_{11}a + m_{12}b + m_{13}c + m_{14}d) + B(m_{21}a + m_{22}b + m_{23}c + m_{24}d) \\ + C(m_{31}a + m_{32}b + m_{33}c + m_{34}d) + D(m_{41}a + m_{42}b + m_{43}c + m_{44}d) \end{aligned} \tag{114}$$

If we denote with primes the coordinates of E' in the old basis, $E' = a'A + b'B + c'C + d'D$ and we can then write the former equality in matrix form:

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{115}$$

which is just a projectivity $E' = \mathbf{M}\, E$ with $\det(\mathbf{M}) \neq 0$ because there are not four coplanar points in any frame. □

As there happens for tranformations of vectors, a projectivity can be defined in two equivalent ways: as a linear transformation of projective coordinates keeping the projective frame fixed (passive transformation) or as a change of the projective frame keeping the coordinates fixed (active transformation).

9. Comments on the bibliography

Barnabei, Brini and Rota [12], in one of the first attempts to treat projective geometry following the ideas of Peano, wrote a very cumbersome formalism that can be got rid of in projective geometry as we have shown. Following this line of research, Hawrylycz [18] proved Bricard's theorem. In our opinion, they were not able to grasp Peano's ideas mainly developed in [8]. In fact, Peano deals with finite extensions and one must go to Grassmann [4] in order to find the infinite extensions of projective spaces.

Hestenes and Ziegler [19] identified the *meet* (Grassmann's regressive) product $A \vee B$ of two blades A and B with the dual of the exterior product of their duals $\tilde{A}$ and $\tilde{B}$:

$$A \vee B = (\tilde{A} \wedge \tilde{B})\, e_{1\cdots n} \qquad \tilde{A} = A\, e^{-1}_{1\cdots n} \tag{116}$$

They already realized that this definition only works if the addition of grades of A and B is higher than the dimension n of the generator vector space of the Clifford algebra because the addition of grades of $\tilde{A}$ and $\tilde{B}$ is then lower than n. Otherwise $\tilde{A} \wedge \tilde{B} = 0$. This shows the insufficiency of their definition. Then they went to projective geometry by means of the nowadays standard definition of projective spaces $\mathbb{PR}_{n-1}$ as the set of points projected from the vector space V_n. We have already seen that this projection is not necessary at all to deal with projective geometry of a given space. With this step, they converted the products of vectors into products of points in the projective space. They proved Desargues' and Pappus' theorems with this algebra of points through nonintuive and longer proofs than those given here. They also proved Bricard's theorem, whose proof with our easier formalism is left to readers as an exercise.

Oliver Conradt states in [20, 21] a dual geometric algebra obtained by multiplication of all the elements by the pseudoscalar, and defines there a scalar product $\circ$ and an exterior product $\vee$. From this double Clifford algebra, he obtains the double exterior algebra where two multivectors are considered equivalent if they differ by a non null factor. In this way, he defines the double projective algebra as the double exterior algebra plus this equivalence relation of homogeneity. He says that two elements A and B are incident only if $A \wedge B = 0$ and $A \vee B = 0$. He also states that connection (*join*) and intersection (*meet*) correspond to the exterior product $\wedge$ and dual exterior product $\vee$. The symmetric outlining of exterior and dual exterior products by Conradt is close to the one here explained. However, he does not realize the importance of barycentric coordinates when dealing with projective geometry. On the other hand, he still needed geometric algebra and Hodge dual to state the dual exterior algebra, when both resources are not really necessary: the linear relation (18) between dual and point coordinates that defines a plane is enough to substantiate duality with no assumption about the metric properties or the scalar product of the three-dimensional space. Now, in the Alterman conference Conradt has already forsaked the geometric product from his projective algebra like we have been doing from the beginning.

Charles Gunn [22] also states two complementary exterior algebras for projective geometry, but still resorting to the definition of projective spaces $\mathbb{RP}^n$, which we have shown to be superfluous. He bases the algebraic treatment of projective geometry on the geometric product

and an arbitrary and nonintuitive metric of points that depends on the Euclidean or hyperbolic character of the space, while we have shown that projective properties are independent of the metric. He gives well the reference tetrahedron as basis of both exterior algebras, but he has not grasped that the barycentric coordinates are the coefficients of linear combination for this basis. Disregarding any principle of symmetry, his metric has different values of the squares of the basis points of the reference tetrahedron, sometimes 1 and other times 0, which makes a big mess of the algebraic treatment. Anyway, he is not aware of the existence of the papers of Oliver Conradt on the same topic.

Although these authors have glimpsed that duality can be treated with an operation similar to the exterior product but with dual elements, they have not yet grasped that this exterior product needs to have well-defined dual coordinates and basis. On the other hand, they have never outlined that the *join* operator must be identified with the exterior product of points given with barycentric coordinates, in the way Coxeter did [15]. They have not understood that metric must be absent in a purely projective geometry implying that duality must simply be defined as a linear relation among coordinates, which is independent of any Clifford algebra. Finally, they have not seen that a frame of points is not only a basis for points, but also that its edges are a basis for lines and its faces are a basis for planes. With these bases, the exterior product becomes a practical tool for computing projective properties of geometric elements.

10. CONCLUSIONS

Möbius' barycentric coordinates suffices for working with projective geometry and proving geometric theorems, because they are already homogeneous by definition and can describe points at infinity not included in the affine space. Cartesian coordinates are obtained from barycentric coordinates from division by their addition. Therefore, the barycentric coordinates are themselves projective coordinates. Then, projectivities are defined as their linear transformations. The usefulness and easiness of barycentric coordinates for dealing with projections, quadrics, polarities, projective theorems, etc. is obvious. Grassmann's exterior algebra of points is enough to build geometric elements of higher dimensions. In this way, their non-null exterior product corresponds to the *join* operator. Likewise, a non-null exterior product of dual points expressed with dual coordinates corresponds to the *meet* operator. In the three-dimensional space, an affine frame of four non-coplanar points defines an affine grid, which has parallel coordinate lines. However, a projective frame needs five points, one point more than an affine frame, in order to fix a projective grid, whose coordinate lines are no longer parallel. Finally, notice that projective coordinates and Grassmann's exterior algebra of points can be generalized to any dimension. They are not exclusive of the three-dimensional space.

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GRASSMANNIAN ALGEBRAS AND THE ERLANGEN PROGRAM WITH EMPHASIS ON PROJECTIVE GEOMETRY

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Abstract. Grassmann's legacy is certainly constituted by his many revolutionary concepts and by exterior algebra, rightly attributed to him and which sometimes bears his name. But he also put a foot in the door of Cliifford algebra and, to quote É. Cartan, he also created *a very fruitful geometric calculus* —specially for projective geometry— where both points and vectors pertain to the *first or primitive class*[1]. Grassmann did his work[2] during the golden age of synthetic geometry, which also was the stone age of the algebraic foundations of projective geometry. As we shall show, these foundations are subordinate to those of affine geometry, which is the reason why É. Cartan developed his general theory of connections starting not with the Euclidean or projective ones, but with affine connections. The same will be the case here for the corresponding elementary or Klein geometries, which the theory of the different types of connections generalizes[3].

The use of algebra that respects the equivalence of all points in affine geometry —thus the absence of a "zero point"— leads to the concept of canonical affine frame bundle, where the frames are constituted by a point and a vector basis. But bundles of frames made of points or of lines or, in dimension n, of linear varieties of dimension $(n-1)$, may also be used in affine geometry[4]. This leads us to consider the relation of frame bundles to Klein geometries.

The representation up to a proportionality constant of projective transformations as homographies, which constitute the projective group of matrices, almost fits the Erlangen program. But the subgroup that leaves a point unchanged —essential in Klein geometries— and the matrix representation of the affine group are typically overlooked. So has been, therefore, the issue of what synthetic projective transformations are directly related to the post-affine entries in the homographies. We exhibit the subgroup of such transformations and show that the proper homologies —i.e. not involving elements at infinity— are directly related to those entries.

We re-interpret from the canonical frame bundle González's version of Möbius-Grassmann-Peano theory, the usefulness of that bundle being enriched in the process. Thus, his special barycentric coordinates now also belong to a theory of moving frames where one includes "frames that do not move". Improper elements, arising from the use of homogeneous coordinates, are not needed if duality is not taken too far, as when one replaces the statement that "parallel lines do not intersect" with the statement that

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[2]Herman Grassmann, *A New Branch of Mathematics: The Ausdenungslehre of 1844, and Other Works*. Open Court, Chicago, 1995.

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"they intersect at a point at infinity". It is worth noting that the line at infinity is dual to the centroid of a triangle, which is not a special point. So, duality is a very important correspondence, but does not respect the equivalence of all points (unless, of course, we were to create an unnecessary superstructure that mimicked the bundles of frames). Thus González's treatment of Grassmann's system for projective geometry takes it closer to the theory of the moving frame. Of course, there is nothing moving in this case, since nothing needs to do so in the Klein geometries; only their Cartanian generalizations need that the frames "move".

We proceed to briefly summarize Cartan's derivation of the equations of structure of projective connections[5]

Finally, the Kähler calculus[6] can claim to have Grassmann in its ascendancy. We shall illustrate how it blends Clifford algebra with exterior calculus.

1. Introduction

1.1. **Of algebra and geometry.** This introduction deals at an elementary level with the three concepts in the title of the paper. Clifford algebra is not mentioned because it has very little to do with the Erlangen program and projective geometry. In algebra, there are not points. Hence, projective geometry cannot be a by-product of Clifford algebra. In contrast, Grassmann's progressive-regressive system has points and vectors together in the first or fundamental class.

In Grassmann's time, there was not a concept of geometry like in the Erlangen program. The latter was born in 1872, almost as late as his death. Of the same decade is the birth of Clifford algebra, which supersedes the algebraic part of Grassmann's system, not its geometric part, with which it is entangled. If not Clifford algebra, what supersedes the geometric part?

Progress on a better algebraic approach to geometry through algebra required an understanding of the essential difference between algebra and geometry. As we shall explain, É. Cartan must have known this very well, as his authorship of the theory of connections indicates. The key concept for progress was "frame bundles", which are principal fiber spaces. More on this is to be found further below.

As J. Dieudonné wrote on Cartan in context of Riemannian spaces ("here" in the quotation that follows"):

> "Finally, it is fitting to mention the most unexpected extension of Klein's ideas in differential geometry... By an extremely original generalization, É. Cartan was able to show here as well that the idea of "operation" still plays a fundamental role; but it is necessary to replace to replace the group with a more complex object, called the "principal fiber space"; one can roughly represent it with a family of isomorphic groups, parametrized by the different points under consideration; the action of each of these groups attaches objects of an "infinitesimal nature" (tangent vectors, tensors, differential forms) at the same point; and it is by "pulling up to the principal fiber" that É. Cartan was able to inaugurate a new era in the study (local and global) of Riemannian spaces and their generalizations.

[5]É. Cartan, "Sur les varietées à connexion projective", *Bull. Soc. math* **52** (1924) 205-241.

[6]E. Kähler, "Der innere Differentialkalkül", *Rendiconti di Matematica e delle sue Applicazioni* **XXI** (1962) 425-523.

We have got this quotation from the introduction to the book "The Method of Equivalence" by R. Gardner, who credits his assistant Adam Falk with the translation from Dieudonné's introduction to *The Erlangen Program* by Felix Klein.

We return to the relation between Grassmann's system and the modern version of the Erlangen program. In his 1908 paper, *Complex Numbers,* Cartan used 17 of its 146 pages to describe Grassmann's work, mostly the latter's progressive-regressive system. As soon as two years later, two of this paper's sections carried "moving trihedron" in their titles. He might instead have used the retrospectively obvious term "moving frames". He used frames and their bundles to introduce algebraic structure in geometries.

Cartan chose affine geometry as the most relevant one for his first generalization of any Klein geometry. He did so in a way consistent with is view that, for each qualified Lie group, there is a geometry which is to it what Euclidean geometry is to the Euclidean group. He did not choose the more general projective geometry, or the more pertinent Lorentz-Minkowski geometry, as he could have done given that he presented general relativity as if it were the motivation of his work on a general theory of connections.

Affine spaces are the most general manifolds that, without improper elements, are globally associated with vector spaces. These are a matter of algebra and have a special element, the zero. Affine spaces are a matter of geometry and do not have a special element, no special point, no "zero". These few statement is all that one needs to understand Cartan's generalization of Klein geometries, and to fill gaps in the study of projective geometry from the perspective of the Erlangen program.

1.2. **Of Grassmannian algebras and geometry.** Exterior algebra is the name of the algebra for which Grassmann is best known by the general mathematical public. But this recognition is a meager favor to him since it distracts from his many significant mathematical contributions. Speaking specifically of algebras, his formulation of a multitude of new products virtually amounts to his introduction of informal quotient algebras (of equally informal tensor algebras) by binary relations. But his most important algebraic work was his exterior-interior system.

A comment by Dieudonné in his paper *The Tragedy of Grassmann* seems pertinent here: "Grassmann is primarily interested in n-dimensional geometry, and not in algebra...". Here, we are not interested in whether this quoted statement is correct or not, but in that it implicitly states the need to distinguish between these two branches of mathematics. The distinction appears to have been ignored by many modern algebraists.

Grassmann's system certainly has much to do with geometry, as can be inferred from Dieudonné's comment. By far, the biggest name in geometry since the Ausdehnungslehre is É. Cartan. he was able to distinguish very clearly between algebra (under the name of calculus) and geometry in Grassmann's work. In the aforementioned 17 pages, he had this to say about Grassmann's system: "If one applies to geometry the extensive calculus ... of which H. Grassmann's has developed the laws, one obtains a very fruitful *geometric analysis.* (Emphasis added).

Fifty four pages later Cartan devoted five pages to "The systems of complex numbers and the groups of transformations". The title "Complex Numbers" of the paper could have been "Algebras that Generalize the Complex Numbers Systems". And given his view —which is the modern one— on the relation of groups of transformations to the concept of geometry, he might have titled those five papers as "The relation of algebra to geometry."

Of special interest here is that paper's section "The systems of Clifford and of Lipschitz". Cartan viewed their work as purely algebraic, which we mention here to insist on the difference between algebra and geometry. This difference is most easily mixed in Euclidean geometry because the closest link to an algebraic representation of geometry that there was in Grassmann's times was analytic geometry in 3-D Euclidean space. The latter is too often misidentified with 3-D Euclidean vector space, the base space of Clifford algebra for that dimension and signature.

In his monumental geometric work, Cartan used exterior algebra and the dot product of vectors. He did not need more than those products for developing the theory of connections.

1.3. **The Erlangen program.** In the original Erlangen program, a geometry was conceived as the study of anything that is left invariant under the transformations of a Lie group. Retrospectively, this was too lose a concept. As per Cartan's reformulation and extension of that program, one has to distinguish between elementary geometries, nowadays called Klein geometries, and their generalizations, which go by the name of the theory of connections.

A Klein geometry is a pair of group, G, and subgroup, G_o, and a property involving them that we shall discuss later on. In Cartan's generalization of Klein geometry, they are retained but only in differential form (Also for later discussion is the presence of G_o as common group of all the fibers, though the elements of any two different fibers cannot be identified).

Both, Klein geometries and their generalizations share the form of a system of equations known as equations of structure. In the case of Klein geometries, they constitute integrability conditions for another differential system (of connection equations). Upon integration, one obtains the group G. This is not the case for the generalization. Suffice to remember what Dieudonné said of the replacement of the group (G) by an action on a frame bundle. That revolution by Cartan contained another conceptual revolution, namely the following. In the original Erlangen program, Riemannian geometry was the geometry of an infinite Lie group. But Cartan pointed out that not only are such groups not part of what defines a geometry, but they actually mask out what is geometric in geometry. Their presence is not denied; it is just a matter of not assigning them a relevance that they lack.

It follows from these considerations on the superseding by Cartan of the original Erlangen program that he also superseded and made clear the relationship between geometry and algebra present in very entangled form in Grassmann's work.

1.4. **Of projective geometry and Clifford algebra.** In their paper "Projective Geometry with Clifford Algebra", Hestenes and Ziegler make statements such as:

> "... projective geometry has not been fully integrated into modern mathematics. The reason ... is to be found in incompatibilities of method".

In order to fully integrate projective geometry into modern mathematics, one has to show how it fits into the modern concept of geometry. There is nothing of this sort in their paper, which is not surprising in any case; it is difficult, if not impossible, to find in the literature statements about some simple transformations that complement the affine transformations to yield the projective group for the same dimension. In the process, one should go beyond defining the projective G_o as the subgroup of the projective group that leaves a point unchanged. And that is only a beginning. Later on they state:

> " ... we seek an efficient formulation of projective geometry with a coherent mathematical system which provides equally efficient formulations of the full range of geometric concepts ..."

The authors should also be specific with regards to "efficiency for what", certainly not for proving theorems as those by Desargues and Pappus. In his *Treatise on Plane Geometry through Geometric Algebra"*, R. González has already shown how to prove those theorems with just a little exterior algebra, which certainly is a Clifford solution where only its exterior contents is used. The proofs then can be done in the back of an envelope if one first reorganizes his material by dealing with the relation of his frames to those of the canonical frame bundle, and thus to the Erlangen program. The algebraic treatment of those theorems then fit in the back of an envelope.

We celebrate the spirit of both of those quotations, but suggest that proponents of the Hestenes-Ziegler approach to projective geometry go back to the drawing board and tell us what their full paraphernalia of concepts is needed for, after R. González has shown what they are not needed for. On the other hand, I agree with those authors that, in my view, their system has greater advantages than the system of Rota and his followers, though we would certainly wish that these made their counter argument, not only to the Hestenes-Ziegler claim, but also to any claims to be found in this paper and that we have just announced.

Here is the specific claim to be rebuffed by those who may still have a better approach: The Erlangen program is first and foremost what brings order to the rich but disorganized body of projective geometry.

2. Bundles, Geometries

2.1. **Concept of canonical frame bundles for affine and Euclidean spaces.** As we are about to argue, geometry is much simpler than it looks if one understand spaces of the same dimension, n. It is simply a matter of whether there is or not a special or null element, the zero in the case of vector spaces. Let us start by recalling some basic concepts.

An affine space is a set of points (they could be elements of some general type, but this could create confusion here; so our "points" are going to be what non-mathematicians call points) such that the following conditions are satisfied. To each ordered pair (P, Q) of points one can make correspond a vector, denoted $\overrightarrow{PQ}$, of an n-dimensional vector space, V_n, in such a way that

$$\overrightarrow{PQ} = -\overrightarrow{QP}, \tag{1}$$

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} \tag{2}$$

and that, given an arbitrary but explicitly chosen point O and vector $\boldsymbol{v}$, there is a unique point P such that

$$\overrightarrow{OP} = \boldsymbol{v}. \tag{3}$$

If V_n is endowed with a dot product, it is called Euclidean vector space, E_n. An affine space that is associated with a Euclidean vector space is called a Euclidean space.

A frame of an affine (respectively Euclidean) space is a pair of a point together with a basis of the associated vector space (respectively Euclidean vector space).

There is something about the concept of affine or Euclidean space that is subtly inconsistent, at least when compared with the idea we have of these spaces, say the plane, ordinary 3-space, etc. They do not have a special point and yet, in order to make use of algebra in them, we have to choose a point Ω as zero. We thus brake the symmetry of all points. The oddity is not nper se in making the choice, but the fact that choice makes part of the definition.

Rather than escape the critique just make, it is better to accept it and restore the symmetry of all points, as follows. We consider the set of all pairs of point and vector bases for a given, affine space. This set of all frames constitutes the affine frame bundle. The set is called bundle because it has a fibrated structure, the fiber at a point being constituted by all the bases at the point, each one of them being accompanied by the point itself.

In the affine frame bundles, the group acting on the fibers obviously is the corresponding (i.e. for same n) linear group. In the fibers of the Euclidean frame bundles, the groups of notations plays, that role. Dieudonné had in mind the bundles so constructed when he referred to the "unexpected extension of Klein's ideas" with which Cartan generalized the concept of Riemannian manifolds, among other.

The reason to refer to these bundles as canonical will soon become evident. The alternative ones are not directly related to the absence of a special element, but they have interest for the treatment of some specific problems.

2.2. **Elementary geometries and frame bundles.** Connections are properly defined on frame bundles, and are usually introduced in a context of generalized geometries, as if the concept did not pertain to the Klein geometries. Hence, a unified version of geometry requires considerations of the frame bundle of the elementary geometries, which we just did in the affine and Euclidean cases. The concepts of elementary geometry and frame bundle are very close to each other. Let me proceed to be a little bit more formal regarding these concepts.

A Klein geometry is a pair (G, G_0), where G is a Lie group and G_0 is a closed subgroup of G having the property that no subgroup of G_0 other than the identity is normal in G. Recall that a subgroup is said to be closed if it is closed as a subset, i.e. its complement is open. G_0 is said to be normal (or invariant) in G if, for all $g \in G$ we have $gG_0 = G_0g$ (right and left cosets coincide). There is then a quotient group G/G_0, which means that we eliminate the distinction between elements of G_0. Thus, if we ignore the difference between basses in the affine frame bundle, the frames are distinguished only by the frames where they are. They then constitute the affine space. We have just reverse. This is of interest if we start with a Lie group rather than with a specific elementary geometry. Then the quotient group G/G_0 allows for the "fibrated structure" by creating a base space of cosets. Readers who do not quite follow, should not bother too much, at least not yet; later we shall speak again of these concepts from a different and possibly more familiar angle.

Under the name of elementary geometry, the definition just given is in the paper "On the Completeness of Cartan Connections" by Yeaton H. Clifton, which was communicated by S.S. Cheen and published in Journal of Mathematics and Mechanics, vol. 16, no. 6 (1966). And under the name Klein geometry, one can find in Wikipedia a largely comparable definition. The difference lies in whether one requires or not that G_0 be normal in G. This issue is entangled with the issue of whether G_0 itself must be normal in G, which in turn is related to the far more important issue of whether definitions

of elementary geometries through groups of matrices faithfully represent the geometric contents.

We shall deal with this "new issue" later in the paper, asking ourselves the extent to which a matrix representation in dimension $n+1$ of the affine transformations for dimension n is a faithful one in a new sense of the term faithful representation. It would seem that the standard answer should be in the affirmative. But let us rather respond that, to say the least, subtleties are usually ignored, which may cause surprises. Important to us in this respect is that affine and Euclidean geometries are to be considered as elementary or Klein geometries ab initio. Also, with a new to generalized geometries, we must have in mind that differential geometry deals with differential forms, where there is no room for rasing these issues in the first place.

Let us now look formally at the concept of frame bundles. These are principal fiber bundles of frames. A principal fiber bundle is a fiber bundle in which the group in the fibers, G_0, and the set on which G_0 acts by left translation are isomorphic. Knowing where we want to get, ;et us go to the opposite end of the argument, to the general concept of bundle and proceed to specialized until we reach frame bundles.

A bundle is a triple (B, S, π), where B and S are topological spaces and π is a surjective map $\pi : B \to S$. For example, B could be the set of affine frames for a given dimension. Then S would be the affine space itself and π would be the projections of frames, yielding the point where each frame is at. Let F_x be the fiber at x, i.e. the set of points $\pi^{-1}(x)$. Let ua assume that F_x is independent of x. Denoted as F, it is called the typical fiber. In the affine case, F_x is the set of all the bases for dimensional n at the point x, and F would be simply the set of all bases for the given dimension without reference to where the bases are at.

The most interesting bundles are the fiber bundles, (B, S, π, G_0). This term is used when F has a topological group of homeomorphisms acting on it (there is also a requirement about a covering of S, but we shall skip it for present peer poses). In the example under consideration, affine geometry, G_0 is the linear group for dimension n.

A fiber bundle is said to be principal if there is an isomorphism between F and G_0, this is certainly the case between the sets of all vector bases and the linear groups.

2.3. **Different frame bundles for the same elementary geometry.** In (B, S, π, G_0), we have a group G_0, but not a group G of which G_0 would be an invariant subgroup. This has to do with the fact that the bundle of frames of a generalized geometry is not isomorphic to a group; but it is for elementary geometries. In these cases, and for the same reason why does not need to add F to (B, S, π, G_0), we could replace B with G and write instead (G, S, π, G_0). Then S is the quotient space G/G_0, and π is implicit. So the concept of frame bundle is, in general, more general than the concept of elementary geometry. But, when these geometries are concerned, they are more general than specific frame bundles, since they do not require specification of the frames. There may be different frame bundles on the same elementary geometry.

Bundles alternative to the canonical frame bundle of affine frames would be constituted by what we shall call frames of points, or by sets of lines on planes, and of planes in 3-space, etc.

This idea of alternative bundles is already present in the description that Cartan made of Grassmann's system in those 17 papers that we mentioned before. While speaking of barycentric baes in Grassmann's systems, he said:

> "Instead of four points to which one has assigned mass one, one can take as primitive units
>
> $$e_1, e_2, e_3, e_4$$
>
> be they four points affected by arbitrary masses, be they three points and a free vector, be they two points and two free vectors, be they one point and three free vectors".

Of course, those are examples pertaining to the Grassmann system, not to Cartan's own system, of which we already started to speak above. Out motivation to consider frame bundles of points and of linear varieties (straight lines, planes and hyperplanes) has to to with the fact that incidence and collineation are relevant issues in projective geometry. From now on, *line* will mean *straight line*. For those issues, frames of lines (in the plane, and of planes in 3-space) and of points will be highly useful, specially in dealing with traditional theorems of essentially synthetic nature. Projective geometry in higher dimensions will be automatically present in algebraically dealing with its foundations.

3. Alternative, non canonical frame bundles

Starting about the end of this page, all the material is totally from R. González Calvet's "*Treatise of Plane Geometry Through Geometric Algebra*". We have simply reorganized material. Even the figures are lifted from his book, but with a point Ω added.

3.1. **Frames of points.** A point has no plane in algebra because it does not have dimension and yet t is not a zero. But its representation by a vector fits in algebra. In affine space, we choose a point Ω and assign a vector to each point. Let O be a point to which we want we want to assign a vector basis. The canonical frame will be written as $(\overrightarrow{\Omega O}, a_1, \ldots, a_n)$. An arbitrary point R in the n-dimensional affine plane can be expressed as

$$\overrightarrow{\Omega R} = \overrightarrow{\Omega O} + l^i \boldsymbol{a}_i, \tag{4}$$

with summation over repeated indices. Now $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ can be viewed as vectors representative of points $P, Q, \ldots$ etc such that

$$\overrightarrow{OP} = \boldsymbol{a}, \;\; \overrightarrow{OQ} = \boldsymbol{a}_2, \quad etc. \tag{5}$$

Equation (1) can then be written as

$$\overrightarrow{\Omega R} = \overrightarrow{\Omega O} + l^1 \overrightarrow{OP} + l^2 \overrightarrow{OQ} + \ldots \tag{6}$$

The first vector on the right of (3) has its origin at Ω. All the other have origins at O.

A frame bundle of points will be defined as the set of vectors $(\overrightarrow{\Omega O}, \overrightarrow{\Omega P}, \overrightarrow{\Omega Q}, \ldots)$. Thus all vectors have the same origin. The l^i's are the linear coordinates of $\overrightarrow{OR}$ relative to the basis $(\boldsymbol{a}_i)$ at O. Relative to $(\overrightarrow{\Omega O}, \overrightarrow{\Omega P}, \overrightarrow{\Omega Q}, \ldots)$ we have

$$\begin{aligned} \overrightarrow{\Omega R} &= \overrightarrow{\Omega O} + l^1(\overrightarrow{\Omega P} - \overrightarrow{\Omega O}) + l^2(\overrightarrow{\Omega Q} - \overrightarrow{\Omega O}) + \ldots = \\ &= \overrightarrow{\Omega O}(1 - l^1 - l^2 - \ldots) + l^1 \overrightarrow{\Omega P} + l^2 \overrightarrow{\Omega Q} + \ldots \end{aligned} \tag{7}$$

We shall refer to

$$1 - \sum_{i=1}^{n} l^i, l^1, \ldots, l^n \tag{8}$$

as the point coordinates of R, and also as its non-homogeneous redundant coordinates. In the plane, these are $(1-a-b,a,b)$ in terms of the frame $(\overrightarrow{\Omega O},\overrightarrow{\Omega P},\overrightarrow{\Omega Q})$ for an arbitrary point. For a line, α, in the plane, we have:

$$\alpha:\overrightarrow{\Omega R}=\overrightarrow{\Omega A'}+a\overrightarrow{A'A}+a(\overrightarrow{\Omega A}-\overrightarrow{\Omega A'})=a\overrightarrow{\Omega A}+(1-a)\overrightarrow{\Omega A'}. \tag{9}$$

Or, we could have written, similarly,

$$\alpha:\overrightarrow{\Omega R}=a\overrightarrow{\Omega A}+(1-a)\overrightarrow{\Omega A'} \tag{10}$$

In (9), we would speak of coordinates $(O,a,1-a)$ relatively to the basis $(\overrightarrow{\Omega O},\overrightarrow{\Omega A},\overrightarrow{\Omega A'})$ and equivalently $(a,1-a)$ relatively to the basis $(\overrightarrow{\Omega A},\overrightarrow{\Omega A'})$. These last coordinates are adapted to the line. If we had three points, (A,B,C), given in such a way that we do not know whether they are colinear, we shall not use (9) and (10). Because six vectors would be involved: $\overrightarrow{\Omega A}$, $\overrightarrow{OA'}$, $\overrightarrow{OB}$, $\overrightarrow{OB'}$, $\overrightarrow{OC}$ and $\overrightarrow{OC'}$. We would take a genera; basis, the same one for each of the three lines determined by the three pairs of points.

We now proceed to study collineation of point. Three points (A,B,C) are on the same line if the exterior product $\overrightarrow{AB}\wedge\overrightarrow{AC}$ is zero. In terms of these "point frame" coordinates,

$$\overrightarrow{\Omega A}=(1-a-a')\overrightarrow{\Omega O}+a'\overrightarrow{\Omega Q}, \tag{11}$$

and similarly for $\overrightarrow{\Omega B}$ and $\overrightarrow{\Omega C}$. The same condition can be written as

$$\begin{vmatrix}1-a-a' & a & a'\\ 1-b-b' & b & b'\\ 1-c-c' & c & c'\end{vmatrix}=0. \tag{12}$$

This should be clear since we can further write this determinant as

$$\ldots=\begin{vmatrix}1 & a & a'\\ 1 & b & b'\\ 1 & c & c'\end{vmatrix}=\begin{vmatrix}1 & a & a'\\ 0 & b-a & b'-a'\\ \alpha & c-a' & c'-a'\end{vmatrix}=\begin{vmatrix}b-a & b'-a'\\ c-a & c'-a'\end{vmatrix} \tag{13}$$

But

$$\overrightarrow{AB}-\overrightarrow{\Omega B}-\overrightarrow{\Omega A}=(b-a)\overrightarrow{\Omega P}+(b'-a')\overrightarrow{\Omega Q}$$
$$\overrightarrow{AC}=\overrightarrow{\Omega C}-\overrightarrow{\Omega A}=(c-a)\overrightarrow{\Omega P}+(c'-a')\overrightarrow{\Omega Q}. \tag{14}$$

Hence, if and only is a,b and c are collinera, we have

$$O=\overrightarrow{AB}\wedge\overrightarrow{AC}=\begin{vmatrix}b-a & b'-a'\\ c-a & c'-a'\end{vmatrix}\overrightarrow{\Omega P}\wedge\overrightarrow{\Omega Q}. \tag{15}$$

The determination in (15) must then be zero and, through (10) (12) must follows

()

The intersection of two lines, when given in the form (9), must satisfy

$$a\overrightarrow{\Omega A}+(1-a)\overrightarrow{\Omega A'}=b\overrightarrow{\Omega B}+(1-b)\overrightarrow{\Omega B'} \tag{16}$$

This involves four vectors, one for each point. In order to compute the point of intersection of the two lines, we would have to refer all four vectors to a common basis. But there are many problems in projective geometry where one has relations among lines determined by points such that the specific location of the points is irrelevant. A pair of points determines a line but the role of those points in a theorem can also he played by any other two points in the same line. It is for this reason that (16) is important. For example,let A,A',B',B be four points in a quadrilateral. For simplicity, we shall assume

that there are not any parallel sides. Let B' be the diagonally opposite vertex to A. The intersecting point of the prolongation of the opposite sides AA' and $\overrightarrow{BB'}$ will satisfy (16). This equation could be rearranged as

$$a\overrightarrow{\Omega A} - b\overrightarrow{\Omega B} = -(1-a)\overrightarrow{\Omega A'} + (1-b)\overrightarrow{\Omega B'}, \tag{17}$$

and further written as

$$\frac{a}{a-b}\overrightarrow{\Omega A} - \frac{b}{a-b}\overrightarrow{\Omega B} = -\frac{1-a}{a-b}\overrightarrow{\Omega A'} + \frac{1-b}{a-b}\overrightarrow{\Omega B'}. \tag{18}$$

3.2. **Frames of lines and pencils of lines.** For expediency purpose; we appeal to the reader's knowledge of analytic geometry to understand that, in the plane, any line, δ, can be expressed as a linear combination of not more than three other intersecting lines. We shall name them as α, β and γ. Then

$$\delta = a\alpha + b\alpha + c\gamma. \tag{19}$$

But $\lambda\delta$ represents the same line. Hence (a, b, c) constitutes homogeneous coordinates of δ. We can introduce non-homogeneous coordinates by imposing the condition $a + b + c1$, which we choose in order to establish some parallelism with bases of points. Just think of the fact that the points to which a basis of points (which actually is a basis of three vectors with origin at the same point) generate a triangle. The lines on which those sides sit may be taken as a basis of lines constitutes a line in the pencil of lines through the intersection. The two line correspond to when one of the two coefficients is zero.

Let α, β and γ be the lines opposite to the points Q, P and O. The pencils through Q, P and O. The pencils through O, P and Q will be written respectively as

$$c\alpha + (1-c)\beta, \quad b\alpha + (1-\beta)\gamma, \quad a\beta + (1-a)\gamma.$$

These expressions are useful because, if we want to consider a line which is known to go through a point, we may not need to first compute the point but simply that the line belongs to a goven pencil of lines.

The expression for a pencil of lines in the plane uses a specific pair of lines, say α and β. But one could have chosen any other pair line through the same point by adding and subtracting the null quantities $a'\alpha - a'\alpha$ and/or $b'\beta - b'\beta$. Thus

$$a\alpha + b\alpha = [a'\alpha + b'\beta] + [(a-a')\alpha + (b-b')\beta]. \tag{20}$$

The contents of each bracket represents a line of the pencil $a\alpha + b\alpha$.

We now consider a situation that is reminiscent of the intersection of opposite lines of a quadrilateral. Let (α, α') and (β, β') be two pairs of lines. They determine respective pencils $a\alpha + (1-a)\alpha'$ and $b\beta + (1-b)\beta'$. The line that joints the "centers" of the two pencils belongs to both pencils. Hence it must satisfy

$$a\alpha + (1-a)\alpha' = b\beta + (1-b)b'. \tag{21}$$

We assume for simplicity that there are no parallel lines between these four. We rewrite this expression as

$$a\alpha - b\beta = -(1-a)\alpha' + (1-b)\beta'. \tag{22}$$

We have rewritten the original condition as one to be satisfied by the line that joins the pencils through the intersection of α and β, and through the intersection of α' and β'. Notice the similarity of the pair of equations (16)-(17) with the pair (21)-(22).

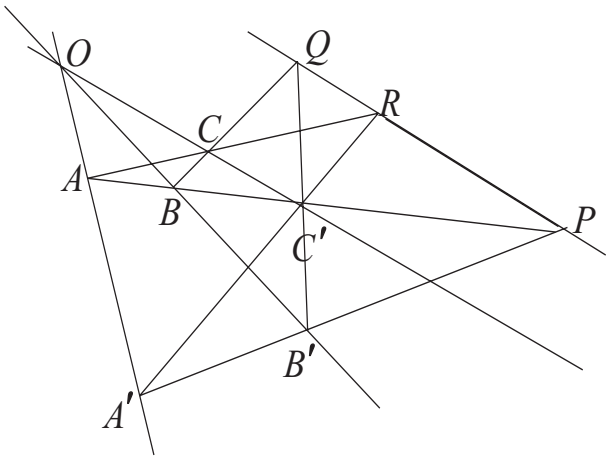

Fig. 1

3.3. **Theorem of Desargues.** Desargues theorem reads: Given triangles ABC and $A'B'C'$, let P, Q and R be the respective intersection points of the line of AB with that of $A'P'$, of the line of BC with $B'C'$, and of the line of AC with that of $A'C'$. These three points are colinear if and only the lines of AA', BB' and CC' meet at the same point.

Proof of Desargues direct theorem. Eqs. (16) and (18) give ΩD and ΩP. Hence

$$\overrightarrow{\Omega P} = \frac{a}{a-b}\overrightarrow{\Omega A} - \frac{b}{a-b}\overrightarrow{\Omega B}. \tag{23}$$

Similarly,

$$\overrightarrow{\Omega Q} = \frac{b}{b-c}\overrightarrow{\Omega B} - \frac{c}{b-c}\overrightarrow{\Omega C}, \tag{24}$$

and

$$\overrightarrow{\Omega R} = \frac{c}{c-a}\overrightarrow{\Omega C} - \frac{a}{c-a}\overrightarrow{\Omega A}. \tag{25}$$

We form the determinant of the coefficients of $\overrightarrow{\Omega P}$, $\overrightarrow{\Omega Q}$ and $\overrightarrow{\Omega R}$ in terms of the basis $(\overrightarrow{\Omega A}, \overrightarrow{\Omega B}, \overrightarrow{\Omega C})$. Since, we wish to find out whether the determinant is zero, it suffices to check the equivalent determinant

$$\begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ -a & 0 & c \end{vmatrix}$$

which is clearly null.

Proof of Desargues inverse theorem

We bow assume that P, Q and R are collinear. Consider the figure formed by the four lines PA, PA', RA and RA'. Let us denote those lines with Greek letters as in the figure. The line containing P and R belongs to the pencils at P and at R. Hence (21) applies. For the pencils at A and A', Eq. (22) applies. We rewrite it as

$$\frac{a}{a-b}\alpha - \frac{b}{a-b}\beta = -\frac{1-a}{a-b}\alpha' + (1-b)\beta'.$$

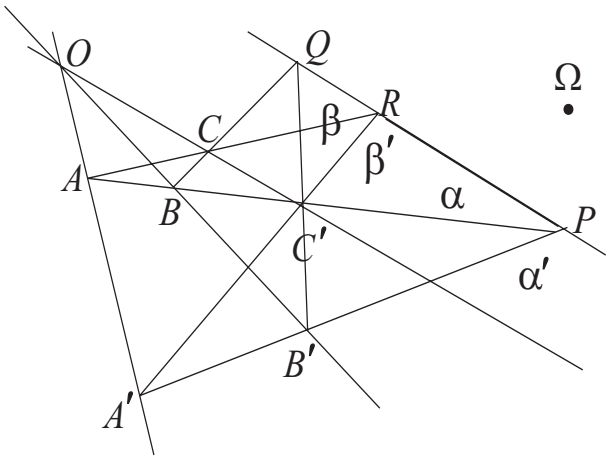

Fig. 2

We take the left hand side to represent the line through A and A'. We would name the lines QB and QB' as γ and γ',with β, β', γ and γ' playing the role of α, α', β and β' above, The line through C and C' will be given by the linear combination

$$\frac{b}{b-c}\beta - \frac{c}{b-c}\gamma.$$

Finally, the line through B and B' will be given by

$$\frac{c}{c-a}\gamma - \frac{a}{c-a}\alpha.$$

For these three lines to intersect they must belong to the same pencil. Whether they do or not depends on whether the determinant of the components of the three lines in terms of the basis (α, β, γ) of lines is zero or not. For this, we simply need to compute whether the determinant

$$\begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ -a & 0 & c \end{vmatrix}$$

is zero, which it certainly is. The full proof of Desargues theorems now complete.

Pappus theorem

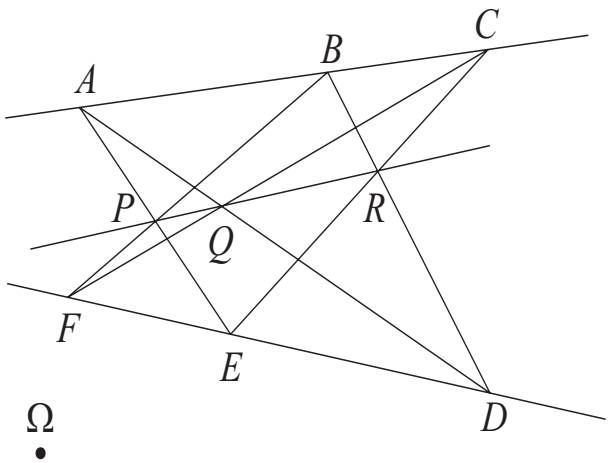

Fig. 3

If A, B and C lie on a line and F, E and D lie on another line, the intersection points of AE with BF, of AD with CF, and of BD with CE lie on the same line.

We shall use the basis $(\overrightarrow{\Omega A}, \overrightarrow{\Omega B}, \overrightarrow{\Omega D})$. The respective coordinates of the points A, B and D are $(1,0,0,(0,1,0))$ and $(0,0,1)$. For all other points, X, the coordinates will be names as x_A, x_B and x_D with $x_A + x_B + x_D = 1$.

Point P is collinear with A and E, on the one hand, and with B and F on the other. Hence we must have

$$\begin{vmatrix} 1 & 0 & 0 \\ p_A & p_B & p_D \\ e_A & e_B & e_D \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 1 & 0 \\ p_A & p_B & p_D \\ f_A & f_B & f_D \end{vmatrix} = 0$$

together with

$$1 = p_A + p_B + p_D.$$

We do not need p_A, p_B and p_D, but just three number proportional to them, as we shall later see. So using Rouche's theorem, we get $(p'_A, p'_B, p'_C) = (f_A e_D, f_D e_B, f_D e_D) = f_D e_D(\frac{f_A}{f_D}, \frac{e_B}{e_D}, 1)$. From the collinearity of A, Q and D, we get $q_B = 0$. Using then the collinearity of C, Q and F and $1 = q_A + q_D$, we get

$$(q'_A, q'_B, q'_D) = (c_A f_B - c_B f_A, O, -c_B f_D) = c_B f_B(\frac{c_A}{c_B} - \frac{f_A}{f_B}, 0, -\frac{f_D}{f_B}).$$

Finally, proceeding similarly for the point R

$$(r'_A, r'_B, r'_D) = (0, c_A e_B - e_A c_B, c_A e_D) = e_A c_A(0, \frac{e_B}{e_A} - \frac{c_B}{c_A}, \frac{e_D}{e_A}).$$

We finally, proceed to compute the determinant formed with the components in the last parentheses of the last three lines of equations and develop it by the third line:

$$\begin{vmatrix} \frac{f_A}{f_D} & \frac{e_B}{e_D} & 1 \\ \frac{c_A}{c_B} - \frac{f_A}{f_B} & 0 & -\frac{f_D}{f_B} \\ 0 & \frac{e_B}{e_A} - \frac{c_B}{c_A} & \frac{e_D}{e_D} \end{vmatrix} = -(e_{B/A} - c_{B/A}) \begin{vmatrix} f_{A/D} & 1 \\ c_{A/B} - f_{A/B} & -f_{D/B} \end{vmatrix}$$
$$- e_{D/A} e_{B/D}(c_{A/B} - f_{A/B})$$

where the notation should be obvious. The determinant of the 2×2 matrix is $-f_{A/B} - c_{A/B} + f_{A/B} = -c_{A/B}$. So finally,

$$\ldots = (e_{B/A} - c_{B/A})c_{A/B} - e_{B/A}(c_{A/B} - f_{A/B})) = -1 + e_{B/A} f_{A/B}$$
$$= \frac{-e_A f_B + e_B f_A}{e_A f_B} = 0,$$

where we have used that, since E and F are colinear with D, we have

$$0 = \begin{vmatrix} 0 & 0 & 1 \\ e_A & e_B & e_D \\ f_A & f_B & f_D \end{vmatrix} = e_A f_B - e_B f_A.$$

Note: This paper is unfinished.

Alterman Conference

on Geometric Algebra

General section

ON CLIFFORD ALGEBRAS AND RELATED FINITE GROUPS AND GROUP ALGEBRAS

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Abstract. Albuquerque and Majid [8] have shown how to view Clifford algebras $C\ell_{p,q}$ as twisted group rings whereas Chernov has observed [13] that Clifford algebras can be viewed as images of group algebras of certain 2-groups modulo an ideal generated by a nontrivial central idempotent. Abłamowicz and Fauser [3–5] have introduced a special transposition anti-automorphism of $C\ell_{p,q}$, which they called a "transposition", which reduces to reversion in algebras $C\ell_{p,0}$ and to conjugation in algebras $C\ell_{0,q}$. The purpose of this talk is to bring these concepts together in an attempt to investigate how the algebraic properties of real Clifford algebras, including their periodicity of eight, are a direct consequence of the central product structure of Salingaros vee groups viewed as 2-groups.

Keywords. central product, dihedral group, elementary abelian group, extra-special group, Clifford algebra, Gray code, group algebra, Hopf algebra, quaternionic group, Salingaros vee group, twisted group algebra, Walsh function

Mathematics Subject Classification (2010). Primary: 15A66, 16S35, 20B05, 20C05, 68W30

Contents

1. Introduction

The main goal of this survey paper is to show how certain finite groups, in particular, Salingaros vee groups [29–31], and elementary abelian group $(\mathbb{Z}_2)^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n-times), and their group algebras and twisted groups algebras, arise in the context of Clifford algebras $C\ell_{p,q}$.

Chernov's observation [13] that Clifford algebras $C\ell_{p,q}$ can be viewed as images of (non-twisted) group algebras of suitable 2-groups, conjectured to be Salingaros vee groups [34], allows one to gain a new viewpoint on these algebras and to relate classical group-theoretical results [15, 17, 24], in particular, on finite 2-groups, to the theory of Clifford algebras. Salingaros classified the groups $G_{p,q}$ – referred to as *Salingaros vee groups* – into five non-isomorphic classes N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k}, and S_k. These groups, according to the theory of finite 2-groups [15, 24], are central products of extra special groups D_8 – the dihedral group and Q_8 – the quaternionic group, both of order 8, and their centers $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4$. Thus, the properties of these groups and the fact that they fall into the five classes, is reflected by the fact that Clifford algebras $C\ell_{p,q}$ also fall into five isomorphism classes which is well known [14, 20, 21] and references therein. The structure theorem on these algebras is recalled in Appendix A. Furthermore, the "periodicity of eight" of Clifford algebras viewed as the images of Salingaros vee groups, seems to be related to, if not predicted by, the structure of these groups and their group algebras. Thus, Section 2 is devoted to this approach to Clifford algebras.

Section 3 is devoted to a review of the basic properties of Salingaros vee groups $G_{p,q}$ appearing as finite subgroups of the group of units $C\ell^{\times}_{p,q}$. Furthermore, we will review certain important subgroups of these groups appearing in the context of certain stabilizer groups of primitive idempotents in $C\ell_{p,q}$ [4, 5].

Section 4 is devoted to a review of the central product structure of Salingaros vee groups.

In Section 5, we recall how the elementary abelian group $(\mathbb{Z}_2)^n$ appears in the context of defining Clifford product on the set of monomials $\mathbf{e}_{\underline{a}}$ indexed by binary n-tuples $\underline{a}$ from $(\mathbb{Z}_2)^n$. In this first context, Walsh functions – essentially, irreducible characters of $(\mathbb{Z}_2)^n$ – and Gray code – as a certain isomorphism of $(\mathbb{Z}_2)^n$ – are used to define the $C\ell_{p,q}$ algebra product [21, Page 284] and references therein. In particular, a formula given by Lounesto dates back to 1935 and is being attributed to Brauer and Weyl [10]. It will be shown how this formula, applicable only to real Clifford algebras $C\ell_{p,q}$ over quadratic vector spaces (V, Q) with a non-degenerate quadratic form Q of signature (p, q), and for an orthonormal set of basis elements (group generators), can be easily extended to Clifford algebras $C\ell_{p,q,r}$ for degenerate quadratic form Q with $\dim V^{\perp} = r$.

Finally, in Section 6, we briefly recall the group $(\mathbb{Z}_2)^n$ as it appears again in the context of the Clifford algebra $C\ell_{p,q}$ as a twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ viewed as a Hopf algebra with a certain quasi-triangular structure [8, 16]. This structure is needed to twist the commutative product in the group algebra $\mathbb{R}[(\mathbb{Z}_2)^n]$ in a manner similar to the Brauer and Weyl formula, so that the twisted product is the Clifford product in $C\ell_{p,q}$. It is recalled that the "transposition" anti-involution of $C\ell_{p,q}$ introduced in [3–5] is actually the antipode in the Hopf algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$.[1]

Our standard references on the group theory are [15, 17, 27]; in particular, for the theory of p-groups we rely on [24]; for Clifford algebras we use [14, 20, 21] and references

[1] We remark that twisted group rings can also be described as certain special Ore extensions known as skew polynomial rings [12].

therein; on representation theory we refer to [19]; and for the theory of Hopf algebras we refer to [25].

2. Clifford Algebras as Images of Group Algebras

Using Chernov's idea [13], in this section we want to show how Clifford algebras $C\ell_{p,q}$ can be viewed as images of group algebras $\mathbb{R}[G]$ of certain 2-groups. It is conjectured [34] that the group G, up to an isomorphism, is the Salingaros vee group $G_{p,q}$ [29–31]. These groups, and their subgroups, have been recently discussed in [4, 5, 11, 22, 23].

Definition 1. Let G be a finite group and let $\mathbb{F}$ be a field[2]. Then the *group algebra* $\mathbb{F}[G]$ is the vector space

$$\mathbb{F}[G] = \left\{ \sum_{g\in G} \lambda_g g,\ \lambda_g \in \mathbb{F} \right\} \tag{1}$$

with multiplication defined as

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h (gh) = \sum_{g\in G}\sum_{h\in G} \lambda_h \mu_{h^{-1}g} g \tag{2}$$

where all $\lambda_g, \mu_h \in \mathbb{F}$. [19]

Thus, group algebras are associative unital algebras with the group identity element playing the role of the algebra identity. In the theory of representations of finite groups, all irreducible inequivalent representations are related to a complete decomposition of the group algebra over $\mathbb{C}$ viewed as a *regular* $\mathbb{C}$-module (cf. [19, Maschke Theorem]). The theory is rich on its own. The theory of group characters can then be derived from the representation theory [19], or, as it is often done, from the combinatorial arguments and the theory of characters of the symmetric group [28]. Since in this survey we are only interested in finite groups, we just recall for completeness that every finite group is isomorphic to a subgroup of a symmetric group [27].

We begin by recalling a definition of a p-group.

Definition 2. Let p be a prime. A group G is a *p-group* if every element in G is of order p^k for some $k \geq 1$.

Note that any finite group G of order p^n is a p-group. A classical result states that a center of any p-group is nontrivial, and, by Cauchy's theorem we know that every finite p-group has an element of order p. Thus, in particular, the center of any finite p-group has an element of order p [15, 17, 27]. In the following, we will be working only with finite 2-groups such as, for example, the group $(\mathbb{Z}_2)^n$ and Salingaros vee groups $G_{p,q}$ of order 2^{1+p+q}.

Two important groups in the theory of finite 2-groups and hence in this paper, are the *quaternionic group* Q_8 and the *dihedral group* D_8 (the symmetry group of a square under rotations and reflections), both of order $|Q_8| = |D_8| = 8$. These groups have the following presentations:

Definition 3. The *quaternionic group* Q_8 has the following two presentations:

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle \tag{3a}$$

$$= \langle I, J, \tau \mid \tau^2 = 1, I^2 = J^2 = \tau, IJ = \tau JI \rangle \tag{3b}$$

[2]Usually, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ although finite fields are also allowed. In this paper, we will be looking at the real Clifford algebras $C\ell_{p,q}$ as images of real group algebras or as real twisted group algebras.

Thus, $Q_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where the group elements have orders as follows: $|a^2| = 2$, $|a| = |a^3| = |b| = |ab| = |a^2b| = |a^3b| = 4$, so the order structure of Q_8 is $(1, 1, 6)$,[3] and the center $Z(Q_8) = \{1, a^2\} \cong \mathbb{Z}_2$. Here, we can choose $\tau = a^2$. While the presentation (3a) uses only two generators, for convenience and future use, we prefer presentation (3b) which explicitly uses a central element τ of order 2.

Definition 4. The *dihedral group* D_8 (the symmetry group of a square) has the following two presentations:

$$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle \tag{4a}$$

$$= \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \tag{4b}$$

Thus, $D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $|a^2| = |b| = |ab| = |a^2b| = |a^3b| = 2$, $|a| = |a^3| = 4$, the order structure of D_8 is $(1, 5, 2)$, and $Z(D_8) = \{1, a^2\} \cong \mathbb{Z}_2$. Here, we can choose $\tau = b$, $\sigma = a$, hence, $\sigma^2 \in Z(D_8)$. That is, σ^2 is our central element of order 2, and our preferred presentation of D_8 is (4b).

In the following two examples, we show how one can construct the Clifford algebra $C\ell_{0,2} \cong \mathbb{H}$ (resp. $C\ell_{1,1}$) as an image of the group algebra of Q_8 (resp. D_8).

Example 1. (Constructing $\mathbb{H} \cong C\ell_{0,2}$ as $\mathbb{R}[Q_8]/\mathcal{J}$)
Define an algebra map ψ from the group algebra $\mathbb{R}[Q_8] \to \mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$ as follows:

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad I \mapsto \mathbf{i}, \quad J \mapsto \mathbf{j}, \tag{5}$$

Then, $\mathcal{J} = \ker\psi = (1 + \tau)$ for the central element τ of order 2 in Q_8[4], so $\dim_{\mathbb{R}} \mathcal{J} = 4$ and ψ is surjective. Let $\pi : \mathbb{R}[Q_8] \to \mathbb{R}[Q_8]/\mathcal{J}$ be the natural map $u \mapsto u + \mathcal{J}$. There exists an isomorphism $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to \mathbb{H}$ such that $\varphi \circ \pi = \psi$ and

$$\pi(I^2) = I^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(I^2)) = \psi(\tau) = -1 = (\psi(I))^2 = \mathbf{i}^2,$$
$$\pi(J^2) = J^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(J^2)) = \psi(\tau) = -1 = (\psi(J))^2 = \mathbf{j}^2,$$
$$\pi(IJ + JI) = IJ + JI + \mathcal{J} = (1 + \tau)JI + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(IJ + JI)) = \psi(0) = 0 = \psi(I)\psi(J) + \psi(J)\psi(I) = \mathbf{ij} + \mathbf{ji}.$$

Thus, $\mathbb{R}[Q_8]/\mathcal{J} \cong \psi(\mathbb{R}[Q_8]) = \mathbb{H} \cong C\ell_{0,2}$ provided the central element τ is mapped to -1 (see also [13]).

Example 2. (Constructing $C\ell_{1,1}$ as $\mathbb{R}[D_8]/\mathcal{J}$)
Define an algebra map ψ from the group algebra $\mathbb{R}[D_8] \to C\ell_{1,1}$ such that:

$$1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_2, \tag{6}$$

where $C\ell_{1,1} = \mathrm{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$. Then, $\ker\psi = (1+\sigma^2)$ where σ^2 is a central element of order 2 in D_8. Let $\mathcal{J} = (1 + \sigma^2)$. Thus, $\dim_{\mathbb{R}} \mathcal{J} = 4$ and ψ is surjective. Let $\pi : \mathbb{R}[D_8] \to \mathbb{R}[D_8]/\mathcal{J}$ be the natural map $u \mapsto u + \mathcal{J}$. There exists an isomorphism $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{1,1}$ such that $\varphi \circ \pi = \psi$ and

$$\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \text{ and } \varphi(\pi(\tau^2)) = \psi(1) = 1 = \psi(\tau^2) = (\mathbf{e}_1)^2,$$
$$\pi(\sigma^2) = \sigma^2 + \mathcal{J} \text{ and } \varphi(\pi(\sigma^2)) = \psi(\sigma^2) = -1 = (\mathbf{e}_2)^2,$$
$$\pi(\tau\sigma + \sigma\tau) = \tau\sigma + \sigma\tau + \mathcal{J} = \sigma\tau(1 + \sigma^2) + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(\tau\sigma + \sigma\tau) = \psi(0) = 0 = \psi(\tau)\psi(\sigma) + \psi(\sigma)\psi(\tau) = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1.$$

Thus, $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ provided the central element σ^2 is mapped to -1.

[3] That is, Q_8 has one element of order 1; one element of order 2; and six elements of order 4.

[4] Here, $(1 + \tau)$ denotes an ideal in $\mathbb{R}[Q_8]$ generated by $1 + \tau$. Note that the two elements $\frac{1}{2}(1 \pm \tau)$ are idempotents which provide an *orthogonal decomposition* of the unity in $\mathbb{R}[Q_8]$.

It is not difficult to modify Example 2 and construct $C\ell_{2,0}$ as the quotient algebra $\mathbb{R}[D_8]/\mathcal{J}$ by changing only the definition of the algebra map ψ given in (6) to

$$1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_1\mathbf{e}_2, \tag{7}$$

Then, the rest of Example 2 follows except that of course now $(\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = 1$. Thus, one can construct $C\ell_{2,0}$ as $\mathbb{R}[D_8]/\mathcal{J}$ with again $\mathcal{J} = (1 + \sigma^2)$.

We remark that the fact that can use the group D_8 twice should not come as surprise since $C\ell_{1,1} \cong C\ell_{2,0}$ (as real Clifford algebras) due to one of the isomorphism theorems stating that $C\ell_{p,q} \cong C\ell_{q+1,p-1}$, [21, Page 215] (see also [6, 14, 20]) and that we only have, up to an isomorphism, two non-abelian groups of order eight, namely, Q_8 and D_8.

We summarize our two examples as follows. In preparation for Chernov's theorem [13], notice that elements in each group Q_8 and D_8 can be written as follows:

- The quaternionic group Q_8:

  $$Q_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0, 1\},\, k = 0, 1, 2\}$$

  where $\tau = a^2$ is the central element of order 2 in Q_8, $g_1 = a$, and $g_2 = b$. Thus,

  $$(g_1)^2 = a^2 = \tau, \quad (g_2)^2 = b^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

  Observe that $|g_1| = |g_2| = 4$ and $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ where $\mathcal{J} = (1 + \tau)$.
- The dihedral group D_8:

  $$D_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0, 1\},\, k = 0, 1, 2\}$$

  where $\tau = a^2$ is the central element of order 2 in D_8, $g_1 = b$, and $g_2 = a$. Thus,

  $$(g_1)^2 = b^2 = 1, \quad (g_2)^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

  Observe that $|g_1| = 2$, $|g_2| = 4$ and $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ where $\mathcal{J} = (1 + \tau)$.

Chernov's theorem states the following.

Theorem 1 (Chernov)**.** *Let G be a finite 2-group of order 2^{1+n} generated by a central element τ of order 2 and additional elements $g_1, \ldots, g_n$, which satisfy the following relations:*

$$\tau^2 = 1, \quad (g_1)^2 = \cdots = (g_p)^2 = 1, \quad (g_{p+1})^2 = \cdots = (g_{p+q})^2 = \tau, \tag{8a}$$

$$\tau g_j = g_j \tau, \quad g_i g_j = \tau g_j g_i, \quad i, j = 1, \ldots, n = p + q, \tag{8b}$$

so that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \mid \alpha_k \in \{0, 1\},\, k = 0, 1, \ldots, n\}$. Let $\mathcal{J} = (1 + \tau)$ be an ideal in the group algebra $\mathbb{R}[G]$ and let $C\ell_{p,q}$ be the universal real Clifford algebra generated by $\{\mathbf{e}_k\}, k = 1, \ldots, n = p + q$, where

$$\mathbf{e}_i^2 = Q(\mathbf{e}_i) \cdot 1 = \varepsilon_i \cdot 1 = \begin{cases} 1 & \text{for } 1 \le i \le p; \\ -1 & \text{for } p + 1 \le i \le p + q; \end{cases} \tag{9a}$$

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 0, \quad i \ne j, \quad 1 \le i, j \le n. \tag{9b}$$

Then, (a) $\dim_{\mathbb{R}} \mathcal{J} = 2^n$; (b) There exists a surjective algebra homomorphism ψ from the group algebra $\mathbb{R}[G]$ to $C\ell_{p,q}$ so that $\ker \psi = \mathcal{J}$ and $\mathbb{R}[G]/\mathcal{J} \cong C\ell_{p,q}$.

Remark 1. Chernov's theorem does not give the existence of the group G. It only states that should such group exist whose generators satisfy relations (8), the result follows. It is not difficult to conjecture that the group G in that theorem is in fact the Salingaros vee group $G_{p,q}$, that is, $\mathbb{R}[G_{p,q}]/\mathcal{J} \cong C\ell_{p,q}$ (see [34]). In fact, we have seen it in Examples 1 and 2 above.

Chernov's theorem. Observe that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n}\} \mid \alpha_k \in \{0,1\}, k = 0,1,\ldots,n\}$. The existence of a central element τ of order 2 is guaranteed by a well-known fact that the center of any p-group is nontrivial, and by Cauchy Theorem. [27] Define an algebra homomorphism $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ such that

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad g_j \mapsto \mathbf{e}_j, \quad j = 1,\ldots,n. \tag{10}$$

Clearly, $\mathcal{J} \subset \ker\psi$. Let $u \in \mathbb{R}[G]$. Then,

$$u = \sum_{\alpha} \lambda_\alpha \tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} = u_1 + \tau u_2 \tag{11}$$

where

$$u_i = \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(i)} g_1^{\alpha_1} \cdots g_n^{\alpha_n}, \quad i = 1,2, \tag{12a}$$

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \quad \text{and} \quad \widetilde{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n. \tag{12b}$$

Thus, if $u \in \ker\psi$, then

$$\psi(u) = \sum_{\widetilde{\alpha}} (\lambda_{\widetilde{\alpha}}^{(1)} - \lambda_{\widetilde{\alpha}}^{(2)}) \mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n} = 0 \tag{13}$$

implies $\lambda_{\widetilde{\alpha}}^{(1)} = \lambda_{\widetilde{\alpha}}^{(2)}$ since $\{\mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n}\}$ is a basis in $C\ell_{p,q}$. Hence,

$$u = (1+\tau) \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(1)} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \in \mathcal{J}. \tag{14}$$

Thus, $\dim_{\mathbb{R}} \ker\psi = 2^n$, $\ker\psi = \mathcal{J}$, $\dim_{\mathbb{R}} \mathbb{R}[G]/\mathcal{J} = 2^{1+n} - 2^n = 2^n$, so ψ is surjective. Let $\varphi : \mathbb{R}[G]/\mathcal{J} \to C\ell_{p,q}$ be such that $\varphi \circ \pi = \psi$ where $\pi : \mathbb{R}[G] \to \mathbb{R}[G]/\mathcal{J}$ is the natural map. Then, since $\psi(g_j) = \mathbf{e}_j$, $\pi(g_j) = g_j + \mathcal{J}$, we have $\varphi(\pi(g_j)) = \varphi(g_j + \mathcal{J}) = \psi(g_j) = \mathbf{e}_j$ and

$$\begin{aligned} \pi(g_j)\pi(g_i) + \pi(g_i)\pi(g_j) &= (g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J}) \\ &= (g_j g_i + g_i g_j) + \mathcal{J} = (1+\tau) g_j g_i + \mathcal{J} = \mathcal{J} \end{aligned} \tag{15}$$

because $g_i g_j = \tau g_j g_i$ in $\mathbb{R}[G]$, τ is central, and $\mathcal{J} = (1+\tau)$. Thus, $g_j + \mathcal{J}, g_i + \mathcal{J}$ anticommute in $\mathbb{R}[G]/\mathcal{J}$ when $i \neq j$. Also,

$$\pi(g_i)\pi(g_i) = (g_i + \mathcal{J})(g_i + \mathcal{J}) = (g_i)^2 + \mathcal{J} = \begin{cases} 1 + \mathcal{J}, & 1 \le i \le p; \\ \tau + \mathcal{J}, & p+1 \le i \le n; \end{cases} \tag{16}$$

due to the relations (8a) on g_i in G. Observe, that

$$\tau + \mathcal{J} = (-1) + (1+\tau) + \mathcal{J} = (-1) + \mathcal{J} \text{ in } \mathbb{R}[G]/\mathcal{J}. \tag{17}$$

To summarize, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is generated by the cosets $g_i + \mathcal{J}$ which satisfy these relations:

$$(g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J}) = \mathcal{J}, \tag{18a}$$

$$(g_i)^2 + \mathcal{J} = \begin{cases} 1 + \mathcal{J}, & 1 \le i \le p; \\ (-1) + \mathcal{J}, & p+1 \le i \le n; \end{cases} \tag{18b}$$

Thus, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is a Clifford algebra isomorphic to $C\ell_{p,q}$ provided $\mathcal{J} = (1+\tau)$ for the central element τ of order 2 in G. □

3. Salingaros Vee Groups $G_{p,q} \subset C\ell^{\times}_{p,q}$

Let $\dim_{\mathbb{R}} V = n$ and Q be a non-degenerate quadratic form on V:

$$Q(\mathbf{x}) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_n x_n^2, \tag{19}$$

$\varepsilon_i = \pm 1$ and $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in V$ for an orthonormal basis $\mathcal{B}_1 = \{\mathbf{e}_i, 1 \leq i \leq n\}$. Q has an arbitrary signature $-n \leq p - q \leq n$ where p (resp. q) denotes the number of $+1$'s (resp. -1's) in (19), and $p + q = n$. Let $C\ell_{p,q}$ be the universal Clifford algebra of (V, Q) obtained, for example, via Chevalley's construction [21, Chapter 22].

Then, let $\mathcal{B}$ be the canonical basis of $\bigwedge V$ generated by $\mathcal{B}_1$, $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$, by underlined Roman characters. The basis elements of $\bigwedge V$, or, of $C\ell_{p,q}$ due to the linear space isomorphism $\bigwedge V \to C\ell_{p,q}$, can be indexed by these finite ordered subsets as $\mathbf{e}_{\underline{i}} = \wedge_{i\in\underline{i}}\, \mathbf{e}_i$.

Now, let $G_{p,q}$ be a finite group in any real Clifford algebra $C\ell_{p,q}$ (simple or semisimple) with a with a binary operation being just the Clifford product, namely:

$$G_{p,q} = \{\pm\mathbf{e}_{\underline{i}} \mid \mathbf{e}_{\underline{i}} \in \mathcal{B} \text{ with } \mathbf{e}_{\underline{i}}\mathbf{e}_{\underline{j}} \text{ denoting the Clifford product}\}. \tag{20}$$

Thus, $G_{p,q}$ may be presented as follows:

$$G_{p,q} = \langle -1, \mathbf{e}_1, \ldots, \mathbf{e}_n \mid \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \text{ for } i \neq j \text{ and } \mathbf{e}_i^2 = \pm 1\rangle, \tag{21}$$

where $\mathbf{e}_i^2 = 1$ for $1 \leq i \leq p$ and $\mathbf{e}_i^2 = -1$ for $p + 1 \leq i \leq n = p + q$. In the following, the elements $\mathbf{e}_{\underline{i}} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}$ will be denoted for short as $\mathbf{e}_{i_1 i_2 \cdots i_k}$ for $k \geq 1$ while $\mathbf{e}_\emptyset$ will be denoted as 1, the identity element of $G_{p,q}$ (and $C\ell_{p,q}$).

This 2-group of order $2 \cdot 2^{p+q} = 2^{n+1}$ is known as *Salingaros vee group* and it has been discussed, for example, by Salingaros [29–31], Varlamov [32, 33], Helmstetter [18], Abłamowicz and Fauser [4, 5], Maduranga and Abłamowicz [23], and most recently by Brown [11]. We should recall here that $G_{p,q}$ is a discrete subgroup of $\mathbf{Pin}(p, q) \subset \mathbf{\Gamma}_{p,q}$ (Lipschitz group) (Lounesto [21]).

In preparation for discussing properties of the groups $G_{p,q}$ and related to them subgroups, we recall a definition of the derived subgroup $G' \subset G$ and a proposition that gives some of its properties [27].

Definition 5. If G is a group and $x, y \in G$, then their *commutator* $[x, y]$ is the element $xyx^{-1}y^{-1}$. If X and Y are subgroups of G, then the *commutator subgroup* $[X, Y]$ of G is defined by

$$[X, Y] = \langle [x, y] \mid x \in X, y \in Y\rangle, \tag{22}$$

that is, the group $[X, Y]$ is generated by all the commutators $[x, y]$. In particular, the *derived subgroup* G' of G is defined as $G' = [G, G]$.

Proposition 1. *Let G be a group.*

(i) *G' is a normal subgroup of G, and G/G' is abelian.*
(ii) *If H is a normal subgroup of G and G/H is abelian, then $G' \subseteq H$.*

3.1. **Transposition Anti-Involution in $C\ell_{p,q}$.** Let us now recall a definition and some of its basic properties of a special anti-involution $T_{\varepsilon}\tilde{}$ in a Clifford algebra $C\ell_{p,q}$ referred to as "transposition". This anti-involution was introduced in [3–5] where its properties were investigated at length. In particular, it allowed for an introduction of a reciprocal basis in a Clifford algebra $C\ell_{p,q}$ and, subsequently, a new spinor product on spinor spaces, and a classification of its (infinite) groups of invariance. In the following, we limit ourselves only to reviewing certain finite groups appearing in this context.

Definition 6. The *transposition* $T_\varepsilon^{\tilde{}}$ of $C\ell_{p,q}$ is defined as:

$$T_\varepsilon^{\tilde{}} : C\ell_{p,q} \to C\ell_{p,q}, \qquad \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\mathbf{e}_{\underline{i}} \mapsto \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}(\mathbf{e}_{\underline{i}})^{-1} \tag{23}$$

It is the *antipode* map S known from the theory of group algebras $\mathbb{F}[G]$

$$\mathbb{F}[G] \to \mathbb{F}[G], \qquad \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in G} \lambda_g g^{-1} \tag{24}$$

viewed as Hopf algebras [25]. Here are a few of its properties and a few finite related groups. For more details, see [3–5].

- $T_\varepsilon^{\tilde{}}$ is an anti-involution of $C\ell_{p,q}$ which reduces to reversion in $C\ell_{p,0}$ and to conjugation in $C\ell_{0,q}$.
- Depending on the value of $(p-q)$ mod 8, where (p,q) is the signature of Q, $T_\varepsilon^{\tilde{}}$ gives rise to transposition, Hermitian complex, and Hermitian quaternionic conjugation of spinor representation matrices.
- $T_\varepsilon^{\tilde{}}(\mathbf{e}_{\underline{i}}) = \mathbf{e}_{\underline{i}}^{-1}$ hence $T_\varepsilon^{\tilde{}}(\mathbf{e}_{\underline{i}}) = \mathbf{e}_{\underline{i}}$ (resp. $T_\varepsilon^{\tilde{}}(\mathbf{e}_{\underline{i}}) = -\mathbf{e}_{\underline{i}}$) when $(\mathbf{e}_{\underline{i}})^2 = 1$ (resp. $(\mathbf{e}_{\underline{i}})^2 = -1$) (elements of order 2 and 4, respectively, in $G_{p,q}$).
- $T_\varepsilon^{\tilde{}}(f) = f$ for any primitive idempotent f.
- Let $S = C\ell_{p,q}f$ be a spinor (minimal left) ideal in a simple algebra $C\ell_{p,q}$ generated by a primitive idempotent f. Then, $T_\varepsilon^{\tilde{}}$ defines a dual spinor space $S^* = T_\varepsilon^{\tilde{}}(S)$ and a $\mathbb{K}$-valued, where $\mathbb{K} = fC\ell_{p,q}f$, spinor norm $(\psi,\phi) = T_\varepsilon^{\tilde{}}(\psi)\phi$ on S invariant under (infinite) group $G_{p,q}^\varepsilon$ (with $G_{p,q} < G_{p,q}^\varepsilon$) different, in general, from spinor norms related to reversion and conjugation in $C\ell_{p,q}$.
- $G_{p,q}$ act transitively on a complete set $\mathcal{F}$, $|\mathcal{F}| = 2^{q-r_{q-p}}$, of mutually annihilating primitive idempotents where r_i is the Radon-Hurwitz number. See a footnote in Appendix A for a definition of r_i.
- The normal stabilizer subgroup $G_{p,q}(f) \lhd G_{p,q}$ of f is of order $2^{1+p+r_{q-p}}$ and monomials m_i in its (non-canonical) left transversal together with f determine a spinor basis in S.
- The stabilizer groups $G_{p,q}(f)$ and the invariance groups $G_{p,q}^\varepsilon$ of the spinor norm have been classified according to the signature (p,q) for $(p+q) \le 9$ in simple and semisimple algebras $C\ell_{p,q}$.
- $G_{p,q}$ permutes the spinor basis elements modulo the commutator subgroup $G'_{p,q}$ by left multiplication.
- The ring $\mathbb{K} = fC\ell_{p,q}f$ is $G_{p,q}$-invariant.

3.2. **Important Finite Subgroups of $C\ell_{p,q}^\times$.** In this section, we summarize properties and definitions of some finite subgroups of the group of invertible elements $C\ell_{p,q}^\times$ in the Clifford algebra $C\ell_{p,q}$. These groups were defined in [3–5].

- $G_{p,q}$ – Salingaros vee group of order $|G_{p,q}| = 2^{1+p+q}$,
- $G'_{p,q} = \{1,-1\}$ – the commutator subgroup of $G_{p,q}$,
- Let $\mathcal{O}(f)$ be the orbit of f under the conjugate action of $G_{p,q}$, and let $G_{p,q}(f)$ be the stabilizer of f. Let

$$N = |\mathcal{F}| = [G_{p,q} : G_{p,q}(f)] = |\mathcal{O}(f)| = |G_{p,q}|/|G_{p,q}(f)| = 2\cdot 2^{p+q}/|G_{p,q}(f)| \tag{25}$$

  then $N = 2^k$ (resp. $N = 2^{k-1}$) for simple (resp. semisimple) $C\ell_{p,q}$ where $k = q - r_{q-p}$ and $[G_{p,q} : G_{p,q}(f)]$ is the index of $G_{p,q}(f)$ in $G_{p,q}$.
- $G_{p,q}(f) \lhd G_{p,q}$ and $|G_{p,q}(f)| = 2^{1+p+r_{q-p}}$ (resp. $|G_{p,q}(f)| = 2^{2+p+r_{q-p}}$) for simple (resp. semisimple) $C\ell_{p,q}$.

- The set of commuting monomials $\mathcal{T} = \{\mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k}\}$ (squaring to 1) in the primitive idempotent $f = \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_1}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_k})$ is point-wise stabilized by $G_{p,q}(f)$.
- $T_{p,q}(f) := \langle \pm 1, \mathcal{T} \rangle \cong G'_{p,q} \times \langle \mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k} \rangle \cong G'_{p,q} \times (\mathbb{Z}_2)^k$, the *idempotent group* of f with $|T_{p,q}(f)| = 2^{1+k}$,
- $K_{p,q}(f) = \langle \pm 1, m \mid m \in \mathcal{K} \rangle < G_{p,q}(f)$ – the *field group* of where f is a primitive idempotent in $C\ell_{p,q}$, $\mathbb{K} = fC\ell_{p,q}f$, and $\mathcal{K}$ is a set of monomials (a transversal) in $\mathcal{B}$ which span $\mathbb{K}$ as a real algebra. Thus,

$$|K_{p,q}(f)| = \begin{cases} 2, & p - q = 0, 1, 2 \bmod 8; \\ 4, & p - q = 3, 7 \bmod 8; \\ 8, & p - q = 4, 5, 6 \bmod 8. \end{cases} \tag{26}$$

- $G^{\varepsilon}_{p,q} = \{g \in C\ell_{p,q} \mid T_{\varepsilon}\tilde{}(g)g = 1\}$ (infinite group)

Before we state the main theorem from [5] that relates the above finite groups to the Salingaros vee groups, we recall the definition of a *transversal*.

Definition 7. Let K be a subgroup of a group G. A *transversal* ℓ of K in G is a subset of G consisting of exactly one element $\ell(bK)$ from every (left) coset bK, and with $\ell(K) = 1$.

Theorem 2 (Main Theorem)**.** *Let f be a primitive idempotent in $C\ell_{p,q}$ and let $G_{p,q}$, $G_{p,q}(f)$, $T_{p,q}(f)$, $K_{p,q}(f)$, and $G'_{p,q}$ be the groups defined above. Let $S = C\ell_{p,q}f$ and $\mathbb{K} = fC\ell_{p,q}f$.*

(i) *Elements of $T_{p,q}(f)$ and $K_{p,q}(f)$ commute.*
(ii) $T_{p,q}(f) \cap K_{p,q}(f) = G'_{p,q} = \{\pm 1\}$.
(iii) $G_{p,q}(f) = T_{p,q}(f)K_{p,q}(f) = K_{p,q}(f)T_{p,q}(f)$.
(iv) $|G_{p,q}(f)| = |T_{p,q}(f)K_{p,q}(f)| = \frac{1}{2}|T_{p,q}(f)||K_{p,q}(f)|$.
(v) $G_{p,q}(f) \lhd G_{p,q}$, $T_{p,q}(f) \lhd G_{p,q}$, *and* $K_{p,q}(f) \lhd G_{p,q}$. *In particular, $T_{p,q}(f)$ and $K_{p,q}(f)$ are normal subgroups of $G_{p,q}(f)$.*
(vi) *We have:*

$$G_{p,q}(f)/K_{p,q}(f) \cong T_{p,q}(f)/G'_{p,q}, \tag{27}$$

$$G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/G'_{p,q}. \tag{28}$$

(vii) *We have:*

$$(G_{p,q}(f)/G'_{p,q})/(T_{p,q}(f)/G'_{p,q}) \cong G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/\{\pm 1\} \tag{29}$$

and the transversal of $T_{p,q}(f)$ in $G_{p,q}(f)$ spans $\mathbb{K}$ over $\mathbb{R}$ modulo f.
(viii) *The transversal of $G_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{K}$ modulo f.*
(ix) *We have $(G_{p,q}(f)/T_{p,q}(f)) \lhd (G_{p,q}/T_{p,q}(f))$ and*

$$(G_{p,q}/T_{p,q}(f))/(G_{p,q}(f)/T_{p,q}(f)) \cong G_{p,q}/G_{p,q}(f) \tag{30}$$

and the transversal of $T_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{R}$ modulo f.
(x) *The stabilizer $G_{p,q}(f)$ can be viewed as*

$$G_{p,q}(f) = \bigcap_{x \in T_{p,q}(f)} C_{G_{p,q}}(x) = C_{G_{p,q}}(T_{p,q}(f)) \tag{31}$$

where $C_{G_{p,q}}(x)$ is the centralizer of x in $G_{p,q}$ and $C_{G_{p,q}}(T_{p,q}(f))$ is the centralizer of $T_{p,q}(f)$ in $G_{p,q}$.

3.3. **Summary of Some Basic Properties of Salingaros Vee Groups $G_{p,q}$.** In the following, we summarize some basic properties of Salingaros vee groups $G_{p,q}$.

- $|G_{p,q}| = 2^{1+p+q}$, $|G'_{p,q}| = 2$ because $G'_{p,q} = \{\pm 1\}$,
- When $p + q \geq 1$, $G_{p,q}$ is not simple as it has a nontrivial normal subgroup of order 2^m for every $m < 1 + p + q$ (because every p-group of order p^n has a normal subgroup of order p^m for every $m \neq n$).
- When $p + q \geq 1$, the center of any group $G_{p,q}$ is non-trivial since $2 \mid |Z(G_{p,q})|$ and so every group $G_{p,q}$ has a central element τ of order 2. It is well-known that for any prime p and a finite p-group $G \neq \{1\}$, the center of G is non-trivial (Rotman [27]).
- Every element of $G_{p,q}$ is of order 1, 2, or 4.
- Since $[G_{p,q} : G'_{p,q}] = |G_{p,q}|/|G'_{p,q}| = 2^{p+q}$, each $G_{p,q}$ has 2^{p+q} linear characters (James and Liebeck [19]).
- The number N of conjugacy classes in $G_{p,q}$, hence, the number of irreducible inequivalent representations of $G_{p,q}$, is $1 + 2^{p+q}$ (resp. $2 + 2^{p+q}$) when $p + q$ is even (resp. odd) (Maduranga [22]).
- We have the following result (see also Varlamov [33]):

Theorem 3. *Let $G_{p,q} \subset C\ell_{p,q}$. Then,*

$$Z(G_{p,q}) = \begin{cases} \{\pm 1\} \cong \mathbb{Z}_2 & \text{if } p - q \equiv 0, 2, 4, 6 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p - q \equiv 1, 5 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong \mathbb{Z}_4 & \text{if } p - q \equiv 3, 7 \pmod 8. \end{cases} \tag{32}$$

as a consequence of $Z(C\ell_{p,q}) = \{1\}$ (resp. $\{1, \beta\}$) when $p + q$ is even resp. odd) where $\beta = \mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_n$, $n = p + q$, is the unit pseudoscalar in $C\ell_{p,q}$.

- In Salingaros' notation, the five isomorphism classes denoted as $N_{2k-1}, N_{2k}, \Omega_{2k-1}, \Omega_{2k}, S_k$ correspond to our notation $G_{p,q}$ as follows:

TABLE 1. Vee groups $G_{p,q}$ in Clifford algebras $C\ell_{p,q}$

Group	Center	Group order	$\dim_{\mathbb{R}} C\ell_{p,q}$
N_{2k-1}	$\mathbb{Z}_2$	2^{2k+1}	2^{2k}
N_{2k}	$\mathbb{Z}_2$	2^{2k+1}	2^{2k}
Ω_{2k-1}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{2k+2}	2^{2k+1}
Ω_{2k}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{2k+2}	2^{2k+1}
S_k	$\mathbb{Z}_4$	2^{2k+2}	2^{2k+1}

$$\begin{aligned} N_{2k-1} &\leftrightarrow G_{p,q} \subset C\ell_{p,q},\ p - q \equiv 0, 2 \pmod 8,\ \mathbb{K} \cong \mathbb{R}; \\ N_{2k} &\leftrightarrow G_{p,q} \subset C\ell_{p,q},\ p - q \equiv 4, 6 \pmod 8,\ \mathbb{K} \cong \mathbb{H}; \\ \Omega_{2k-1} &\leftrightarrow G_{p,q} \subset C\ell_{p,q},\ p - q \equiv 1 \pmod 8,\ \mathbb{K} \cong \mathbb{R} \oplus \mathbb{R}; \\ \Omega_{2k} &\leftrightarrow G_{p,q} \subset C\ell_{p,q},\ p - q \equiv 5 \pmod 8,\ \mathbb{K} \cong \mathbb{H} \oplus \mathbb{H}; \\ S_k &\leftrightarrow G_{p,q} \subset C\ell_{p,q},\ p - q \equiv 3, 7 \pmod 8,\ \mathbb{K} \cong \mathbb{C}. \end{aligned}$$

(Salingaros [29–31], Brown [11], Varlamov [33])

The first few vee groups $G_{p,q}$ of low orders $4, 8, 16$ corresponding to Clifford algebras $C\ell_{p,q}$ in dimensions $p+q = 1, 2, 3$, are:

$$\begin{aligned}
\text{Groups of order 4:} \quad & G_{1,0} = D_4, \quad G_{0,1} = \mathbb{Z}_4, \\
\text{Groups of order 8:} \quad & G_{2,0} = D_8 = N_1, \quad G_{1,1} = D_8 = N_1, \quad G_{0,2} = Q_8 = N_2, \\
\text{Groups of order 16:} \quad & G_{3,0} = S_1, \quad G_{2,1} = \Omega_1, \quad G_{1,2} = S_1, \quad G_{0,3} = \Omega_2.
\end{aligned}$$

where D_8 is the dihedral group of a square, Q_8 is the quaternionic group, and $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. For a construction of inequivalent irreducible representations and characters of these groups see Maduranga and Abłamowicz [23], and Maduranga [22].

4. Central Product Structure of $G_{p,q}$

We recall first a few definitions and results pertaining to finite p-groups that will be needed in the sequel.

Definition 8 (Gorenstein [17])**.** A finite abelian p-group is *elementary abelian* if every nontrivial element has order p.

Example 3. ($D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ is elementary abelian)
$(\mathbb{Z}_p)^k = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (k-times), in particular, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, etc, are elementary abelian.

Definition 9 (Dornhoff [15])**.** A finite p-group P is *extra-special* if (i) $P' = Z(P)$, (ii) $|P'| = p$, and (iii) P/P' is elementary abelian.

Example 4. (D_8 is extra-special)
$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ is extra-special because:

- $Z(D_8) = D_8' = [D_8, D_8] = \langle a^2\rangle$, $|Z(D_8)| = 2$,
- $D_8/D_8' = D_8/Z(D_8) = \langle\langle a^2\rangle, a\langle a^2\rangle, b\langle a^2\rangle, ab\langle a^2\rangle\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 5. (Q_8 is extra-special)
$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\rangle$ is extra-special because:

- $Z(Q_8) = Q_8' = [Q_8, Q_8] = \langle a^2\rangle$, $|Z(Q_8)| = 2$,
- $Q_8/Q_8' = Q_8/Z(Q_8) = \langle\langle a^2\rangle, a\langle a^2\rangle, b\langle a^2\rangle, ab\langle a^2\rangle\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let us recall now definitions of internal and external central products of groups.

Definition 10 (Gorenstein [17])**.**

(1) A group G is an *internal central product* of two subgroups H and K if:
   (a) $[H, K] = \langle 1\rangle$;
   (b) $G = HK$;

(2) A group G is an *external central product* $H \circ K$ of two groups H and K with $H_1 \leq Z(H)$ and $K_1 \leq Z(K)$ if there exists an isomorphism $\theta : H_1 \to K_1$ such that G is $(H \times K)/N$ where

$$N = \{(h, \theta(h^{-1})) \mid h \in H_1\}.$$

Clearly: $N \lhd (H \times K)$ and $|H \circ K| = |H||K|/|N| \leq |H \times K| = |H||K|$.

Here we recall an important result on extra-special p-groups as central products.

Lemma 1 (Leedham-Green and McKay [24])**.** *An extra-special p-group has order p^{2n+1} for some positive integer n, and is the iterated central product of non-abelian groups of order p^3.*

As a consequence, we have the following lemma and a corollary. For their proofs, see [11].

Lemma 2. *$Q_8 \circ Q_8 \cong D_8 \circ D_8 \ncong D_8 \circ Q_8$, where D_8 is the dihedral group of order 8 and Q_8 is the quaternion group.*

Corollary 1.

- $G_{3,1} \cong D_8 \circ D_8 \cong Q_8 \circ Q_8$,
- $G_{4,0} \cong D_8 \circ Q_8 \cong Q_8 \circ D_8$.

The following theorem is of critical importance for understanding the central product structure of Salingaros vee groups.

Theorem 4 (Leedham-Green and McKay [24])**.** *There are exactly two isomorphism classes of extra-special groups of order 2^{2n+1} for positive integer n. One isomorphism type arises as the iterated central product of n copies of D_8; the other as the iterated central product of n groups isomorphic to D_8 and Q_8, including at least one copy of Q_8. That is,*

1: $D_8 \circ D_8 \circ \cdots \circ D_8 \circ D_8$, *or,*
2: $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$.

where it is understood that these are iterated central products; that is, $D_8 \circ D_8 \circ D_8$ is really $(D_8 \circ D_8) \circ D_8$ and so on.

Thus, the above theorem now explains the following theorem due to Salingaros regarding the iterative central product structure of the finite 2-groups named after him.

Theorem 5 (Salingaros Theorem [31])**.** *Let $N_1 = D_8$, $N_2 = Q_8$, and $(G)^{\circ k}$ be the iterative central product $G \circ G \circ \cdots \circ G$ (k times) of G. Then, for $k \geq 1$:*

(1) $N_{2k-1} \cong (N_1)^{\circ k} = (D_8)^{\circ k}$,
(2) $N_{2k} \cong (N_1)^{\circ k} \circ N_2 = (D_8)^{\circ (k-1)} \circ Q_8$,
(3) $\Omega_{2k-1} \cong N_{2k-1} \circ (\mathbb{Z}_2 \times \mathbb{Z}_2) = (D_8)^{\circ k} \circ (\mathbb{Z}_2 \times \mathbb{Z}_2)$,
(4) $\Omega_{2k} \cong N_{2k} \circ (\mathbb{Z}_2 \times \mathbb{Z}_2) = (D_8)^{\circ (k-1)} \circ Q_8 \circ (\mathbb{Z}_2 \times \mathbb{Z}_2)$,
(5) $S_k \cong N_{2k-1} \circ \mathbb{Z}_4 \cong N_{2k} \circ \mathbb{Z}_4 = (D_8)^{\circ k} \circ \mathbb{Z}_4 \cong (D_8)^{\circ (k-1)} \circ Q_8 \circ \mathbb{Z}_4$.

In the above theorem:

- $\mathbb{Z}_2$, $\mathbb{Z}_4$ are cyclic groups of order 2 and 4, respectively;
- D_8 and Q_8 are the dihedral group of a square and the quaternionic group;
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ is elementary abelian of order 4;
- N_{2k-1} and N_{2k} are extra-special of order 2^{2k+1}; e.g., $N_1 = D_8$ and $N_2 = Q_8$;
- $\Omega_{2k-1}, \Omega_{2k}, S_k$ are of order 2^{2k+2}.
- $\circ$ denotes the iterative central product of groups with, e.g., $(D_8)^{\circ k}$ denotes the iterative central product of k-copies of D_8, etc.,

We can tabulate the above results for Salingaros vee groups $G_{p,q}$ of orders ≤ 256, ($p+q \leq 7$) (Brown [11]) in the following table:

5. Clifford Algebras Modeled with Walsh Functions

Until now, the finite 2-groups such as the Salingaros vee groups $G_{p,q}$ have appeared either as finite subgroups of the group of units $C\ell_{p,q}^{\times}$ in the Clifford algebra, or, as groups whose group algebra modulo a certain ideal generated by $1+\tau$ for some central element τ

TABLE 2. Salingaros Vee Groups $|G_{p,q}| \leq 256$

Isomorphism Class	Salingaros Vee Groups
N_{2k}	$N_0 \cong G_{0,0},\ N_2 \cong Q_8 \cong G_{0,2},\ N_4 \cong G_{4,0},\ N_6 \cong G_{6,0}$
N_{2k-1}	$N_1 \cong D_8 \cong G_{2,0},\ N_3 \cong G_{3,1},\ N_5 \cong G_{0,6}$
Ω_{2k}	$\Omega_0 \cong G_{1,0},\ \Omega_2 \cong G_{0,3},\ \Omega_4 \cong G_{5,0},\ \Omega_6 \cong G_{6,1}$
Ω_{2k-1}	$\Omega_1 \cong G_{2,1},\ \Omega_3 \cong G_{3,2},\ \Omega_5 \cong G_{0,7}$
S_k	$S_0 \cong G_{0,1},\ S_1 \cong G_{3,0},\ S_2 \cong G_{4,1},\ S_3 \cong G_{7,0}$

of order 2 was isomorphic to the given Clifford algebra $C\ell_{p,q}$. In these last two sections, we recall how the (elementary abelian) group $(\mathbb{Z}_2)^n$ can be used to define a Clifford product on a suitable vector space.

In this section, we recall the well-known construction of the Clifford product on the set of monomial terms $\mathbf{e}_{\underline{a}}$ indexed by binary n-tuples $\underline{a} \in (\mathbb{Z}_2)^n$, which, when extended by linearity, endows the set with the structure of the Clifford algebra $C\ell_{p,q}$. This approach is described in Lounesto [21, Chapter 21]. We will show how it can be extended to Clifford algebras $C\ell_{p,q,r}$ over (real) quadratic vector spaces with degenerate quadratic forms.

In the last section we will briefly mention the approach of Albuquerque and Majid [25] in which the Clifford algebra structure is introduced in a suitably twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ using Hopf algebraic methods.

Let $\mathcal{B}^n = \{\underline{a} = a_1 a_2 \dots a_n \mid a_i = 0, 1,\ \underline{a} \oplus \underline{b} = \underline{c} \text{ as } c_i = a_i + b_i \bmod 2\}$ be a group of binary n-tuples with addition $\oplus$, that is, $\mathcal{B}^n \cong (\mathbb{Z}_2)^n$.

Definition 11 (Walsh function)**.** A *Walsh function* $w_{\underline{a}}$ indexed by $\underline{a} \in \mathcal{B}^n$ is a function from $\mathcal{B}^n$ to the multiplicative group $\{\pm 1\}$ defined as

$$w_{\underline{a}}(\underline{b}) = (-1)^{\sum_{i=1}^n a_i b_i} = \pm 1, \quad \underline{a}, \underline{b} \in \mathcal{B}^n, \tag{33}$$

which satisfies $w_{\underline{k}}(\underline{a} \oplus \underline{b}) = w_{\underline{k}}(\underline{a}) w_{\underline{k}}(\underline{b})$ and $w_{\underline{a}}(\underline{b}) = w_{\underline{b}}(\underline{a})$.

Observe that the first condition on $w_{\underline{a}}$ simply states that each $w_{\underline{a}}$ is a group homomorphism from $\mathcal{B}^n$ to the group $\{\pm 1\}$.

Definition 12 (Gray code)**.** A *Gray code* $g : \mathcal{B}^n \to \mathcal{B}^n$ with the property $g(\underline{a} \oplus \underline{b}) = g(\underline{a}) \oplus g(\underline{b})$ is defined as

$$g(\underline{k})_1 = k_1, \quad g(\underline{k})_i = k_{i-1} + k_i \bmod 2, \quad i = 2, \dots, n. \tag{34}$$

Thus, g is a group isomorphism which reorders Walsh functions into a *sequency order* with a *single digit change code* [21, Section 21.2, page 281].

Given that the Gray code g is an isomorphism, Lounesto defines its inverse $h : \mathcal{B}^n \to \mathcal{B}^n$ as

$$h(\underline{a})_i = \sum_{j=1}^{i} a_j \bmod 2. \tag{35}$$

Now, take an $\mathbb{R}$-vector space $\mathcal{A}$ with a basis consisting of 2^n elements $\mathbf{e}_{\underline{a}}$ labeled by the binary n-tuples $\underline{a} = a_1 a_2 \dots a_n$ as

$$\mathbf{e}_{\underline{a}} = \mathbf{e}_1^{a_1} \mathbf{e}_2^{a_2} \cdots \mathbf{e}_n^{a_n}, \quad a_i = 0, 1; \tag{36}$$

for some n symbols $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and define an algebra product on $\mathcal{A}$ which on the basis elements $\mathbf{e}_{\underline{a}}$ is computed as follows:

$$\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = (-1)^{\sum_{i=1}^{p} a_i b_i} w_{\underline{a}}(h(\underline{b}))\mathbf{e}_{\underline{a}\oplus\underline{b}}, \tag{37}$$

for some $1 \leq p \leq n$. Then, together with this product, $\mathcal{A}$ becomes the Clifford algebra $C\ell_{p,q}$, where $q = n - p$, of a non-degenerate quadratic form Q of signature (p, q). See Lounesto [21, Page 284] and his reference to (37) as the formula of Brauer and Weyl from 1935 [10].

Remark 2. Observe that if the scalar factor in front of $\mathbf{e}_{\underline{a}\oplus\underline{b}}$ in (37) were set to be identically equal to 1, then we would have $\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = \mathbf{e}_{\underline{b}}\mathbf{e}_{\underline{a}}$ for any $\mathbf{e}_{\underline{a}}, \mathbf{e}_{\underline{b}} \in \mathcal{A}$. Thus, the algebra $\mathcal{A}$ would be isomorphic to the (abelian) group algebra $\mathbb{R}[G]$ where $G \cong (\mathbb{Z}_2)^n$. That is, the scalar factor introduces a twist in the algebra product in $\mathcal{A}$ and so it makes $\mathcal{A}$, hence the Clifford algebra $C\ell_{p,q}$, isomorphic to the twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$.

Formula (37) is encoded as a procedure `cmulWalsh3` in `CLIFFORD`, a Maple package for computations with Clifford algebras [2, 7]. It has the following pseudo-code.

```
cmulWalsh3:=proc(eI::clibasmon,eJ::clibasmon,B1::{matrix,list(nonnegint)})
local a,b,ab,monab,Bsig,flag,i,dim_V_loc,ploc,qloc,_BSIGNATUREloc;
global dim_V,_BSIGNATURE,p,q;
options `Copyright (c) 2015-2016 by Rafal Ablamowicz and Bertfried Fauser.
        All rights reserved.`;
if type(B1,list) then
   ploc,qloc:=op(B1);
   dim_V_loc:=ploc+qloc:
   _BSIGNATUREloc:=[ploc,qloc]:
   else
   ploc,qloc:=p,q; <<<-- this reads global p, q
   dim_V_loc:=dim_V: <<<-- this reads global dim_V
   _BSIGNATUREloc:=[ploc,qloc]:
   if not _BSIGNATURE=[ploc,qloc] then _BSIGNATURE:=[p,q] end if:
end if:
a:=convert(eI,clibasmon_to_binarytuple,dim_V_loc);
b:=convert(eJ,clibasmon_to_binarytuple,dim_V_loc);
ab:=oplus(a,b);
monab:=convert(ab,binarytuple_to_clibasmon);
return twist(a,b,_BSIGNATUREloc)*Walsh(a,hinversegGrayCode(b))*monab;
end proc:
```

Now let us take a real quadratic vector space (V, Q) with a degenerate quadratic form Q such that $\dim V^{\perp} = r$, while Q restricted to the orthogonal complement of $V^{\perp}$ in V has signature (p, q), $(\dim V = n = p + q + r)$), and we let a basis $\mathbf{e}_i, 1 \leq i \leq n$ be such that $Q(\mathbf{e}_i) = 1$ resp. $Q(\mathbf{e}_i) = -1$, resp. $Q(\mathbf{e}_i) = 0$, for $0 \leq i \leq p$, resp. $p + 1 \leq i \leq p + q$, resp. $p + q + 1 \leq i \leq p + q + r$. We can now generate a universal Clifford algebra as the graded tensor product $C\ell_{p,q}r \cong C\ell_{p,q} \hat{\otimes} \bigwedge V^{\perp}$ with a Clifford product obtained by modifying the above formula (37) as follows: we introduce an extra scalar factor in front of $\mathbf{e}_{\underline{a}\oplus\underline{b}}$. This factor equals 1 or, resp. 0, depending whether the monomial elements $\mathbf{e}_{\underline{a}}$ and $\mathbf{e}_{\underline{b}}$ do not share, resp. do share, a common basis element $\mathbf{e}_i$ which squares to 0 in $C\ell_{p,q,r}$, that is, such that $Q(\mathbf{e}_i) = 0$.

A modified pseudo-code of such procedure called `cmulWalsh3pqr` has been encoded in a new experimental package `eClifford` for computations in $C\ell_{p,q,r}$ [1].

```
cmulWalsh3pqr:=proc(eI::eclibasmon,eJ::eclibasmon,B1::list(nonnegint))
local ploc,qloc,rloc,dim_V_loc,_BSIGNATUREloc,a,b,ab,monab,maxmaxindex,r_factor;
global twist,Walsh,hinverseGrayCode,oplus;
options `Copyright (c) 2015-2016 by Rafal Ablamowicz and Bertfried Fauser.
        All rights reserved.`;
if nops(B1)=2 then
   ploc,qloc:=op(B1);
   rloc:=0;
elif nops(B1)=3 then
   ploc,qloc,rloc:=op(B1);
else
   error `three non-negative integers p,q,r are needed in the list entered as
          the last argument but received \%1 instead`,B1
end if;
```

```
15 dim_V_loc:=ploc+qloc+rloc:
16 maxmaxindex:=max(op(eClifford:-eextract(eI)),op(eClifford:-eextract(eJ)));
17 if evalb(maxmaxindex>dim_V_loc) then
18    error `maximum index \%1 found in the arguments \%2 and \%3 is larger
19          then dim_V = \%4 derived from the last argument \%5`,
20          maxmaxindex,eI,eJ,dim_V_loc,B1
21 end if;
22 _BSIGNATUREloc:=[ploc,qloc]:
23 a:=convert(eI,eclibasmon_to_binarytuple,dim_V_loc);
24 b:=convert(eJ,eclibasmon_to_binarytuple,dim_V_loc);
25 if rloc=0 then
26    r_factor:=1
27 else
28    r_factor:=mul((1+(-1)^(a[i]*b[i]))/2,i=ploc+qloc+1..(ploc+qloc+rloc));
29 end if;
30 if r_factor=0 then return 0 else
31    ab:=oplus(a,b);
32    monab:=convert(ab,binarytuple_to_eclibasmon);
33    return twist(a,b,_BSIGNATUREloc)*Walsh(a,hinversegGrayCode(b))*monab;
34 end if;
35 end proc:
```

In the above, the code lines 25-33 accommodate the additional factor called `r_factor` which equals 1 or 0 as indicated above[5]. In particular, the Clifford algebra $C\ell_{0,0,n} \cong \bigwedge V$, the exterior (Grassmann) algebra.

6. Clifford Algebras $C\ell_{p,q}$ as Twisted Group Algebras

In this last section we give a formal definition of a *twisted group ring* (algebra) following Passman [26, Section 2], and briefly refer to the paper by Albuquerque and Majid [25] in which the authors discuss twisting of a real group algebra of $(\mathbb{Z}_2)^n$ by using Hopf algebraic methods.

Definition 13 (Passman [26])**.** The *twisted group ring* $k^t[G]$ [26, Sect. 2] is an associative k-algebra, k is a field, with a basis $\{\bar{x} \mid x \in G\}$ and multiplication defined distributively for all $x, y \in G$ as

$$\bar{x}\bar{y} = \gamma(x,y)\,\overline{xy}, \qquad \gamma(x,y) \in k^\times = k \setminus \{0\}. \tag{38}$$

where the function $\gamma : G \times G \to k^\times$ satisfies

$$\gamma(x,y)\gamma(xy,z) = \gamma(y,z)\gamma(x,yz), \quad \forall z,y,z \in (\mathbb{Z}_2)^n \quad \text{(cocycle condition)} \tag{39}$$

to assure associativity $(\bar{x}\bar{y})\,\bar{z} = \bar{x}\,(\bar{y}\,\bar{z})$ in $k^t[G]$ for any $x, y, z \in G$.

Lemma 3 (Passman [26])**.** *The following relations hold in* $k^t[G]$.

(i) $\gamma(1,1)^{-1}\overline{1}$ *is the identity in* $k^t[G]$*;*
(ii) $\bar{x}^{-1} = \gamma(x,x^{-1})\gamma(1,1)^{-1}\overline{x^{-1}} = \gamma(x^{-1},x)\gamma(1,1)^{-1}\overline{x^{-1}}$, $\forall x \in G$*;*
(iii) $(\bar{x}\bar{y})^{-1} = \bar{y}^{-1}\bar{x}^{-1}$, $\forall x, y \in G$.

If $\gamma(1,1) = 1$ in part (i) of the above lemma, then we call γ *normalized*, which can always be achieved by scaling. In part (ii), the inverse $\bar{x}^{-1}$ is the result of the action of the antipode on $\bar{x}$ in the Hopf algebra sense, or, it can be viewed as the (un-normalized) action of the transposition map $T_\varepsilon^{\sim}$ introduced in [3–5] and mentioned in Section 3.1.

For a Hopf algebraic discussion of Clifford algebras $C\ell_{p,q}$ as twisted group algebras $\mathbb{R}^t[(\mathbb{Z}_2)^n]$, where the twisting is accomplished via a 2-cocycle F which twists the group algebra $k[(\mathbb{Z}_2)^n]$ into a cotriangular Hopf algebra with a suitable cotriangular structure $\mathcal{R}$, see [8,16] and references therein. Note that if γ is trivial, then the twist is trivial and the twisted group algebra is just the group algebra $k[G]$; if it is given by the `XOR` function on binary tuples, we get the Grassmann product (including a graded tensor product, or a

[5]Note that such factor can also be computed by an `XOR` operation [9].

graded switch; if γ is the choice described by Lounesto in (37), we get the Clifford algebra $C\ell_{p,q}$ [9].

7. Conclusions

As stated in the Introduction, the main goal of this survey paper has been to collect and summarize properties of certain finite 2-groups which appear in Clifford algebras $C\ell_{p,q}$. On one hand, these Salingaros-defined groups $G_{p,q}$ appear as subgroups of the group of invertible elements. These subgroups play an important role in relation to the set of orthogonal primitive idempotents, with the help of which one defines spinorial representations. It has been observed by Salingaros, that these groups belong to five non-isomorphic families. On the other hand, one knows that all Clifford algebras $C\ell_{p,q}$ are classified into five different families of simple and semisimple algebras depending on the values of (p, q) and $p + q$ (the Periodicity of Eight). Another connection with finite Salingaros groups appears via Chernov's observation that the algebras $C\ell_{p,q}$ can be viewed as images of group algebras, most likely of the groups $G_{p,q}$ modulo a suitable ideal generated by a central nontrivial idempotent in the group algebra. This shows that the theory of extra-special 2-groups has a direct bearing on the structure of the Clifford algebras $C\ell_{p,q}$. Finally, we have observed how Clifford algebras can be obtained by twisting a group algebra of $(\mathbb{Z}_2)^n$, either by using the Walsh functions, or equivalently but in a more sound mathematical way, by using a 2-cocycle and the formalism of cotriangular Hopf algebras [16].

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Appendix A. The Structure Theorem on Clifford Algebras

In this appendix we list the main structure theorem for real Clifford algebras $C\ell_{p,q}$. For more information on Clifford algebras, see [14, 20, 21].

Structure Theorem. *Let $C\ell_{p,q}$ be the universal Clifford algebra over (V, Q), Q is non-degenerate of signature (p, q).*

(a) *When $p-q \neq 1 \bmod 4$ then $C\ell_{p,q}$ is a simple algebra of dimension 2^{p+q} isomorphic with a full matrix algebra $\mathrm{Mat}(2^k, \mathbb{K})$ over a division ring $\mathbb{K}$ where $k = q-r_{q-p}$ and r_i is the Radon-Hurwitz number.*[6] *Here $\mathbb{K}$ is one of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ when $(p-q) \bmod 8$ is $0, 2$, or $3, 7$, or $4, 6$.*

(b) *When $p - q = 1 \bmod 4$ then $C\ell_{p,q}$ is a semisimple algebra of dimension 2^{p+q} isomorphic to $\mathrm{Mat}(2^{k-1}, \mathbb{K}) \oplus \mathrm{Mat}(2^{k-1}, \mathbb{K})$, $k = q - r_{q-p}$, and $\mathbb{K}$ is isomorphic to $\mathbb{R}$ or $\mathbb{H}$ depending whether $(p - q) \bmod 8$ is 1 or 5. Each of the two simple direct components of $C\ell_{p,q}$ is projected out by one of the two central idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots n})$.*

(c) *Any element f in $C\ell_{p,q}$ expressible as a product*

$$f = \frac{1}{2}(1 \pm \mathbf{e}_{i_1})\frac{1}{2}(1 \pm \mathbf{e}_{i_2}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{i_k}) \tag{40}$$

[6] The Radon-Hurwitz number is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0$, $r_1 = 1$, $r_2 = r_3 = 2, r_4 = r_5 = r_6 = r_7 = 3$.

where $\mathbf{e}_{i_j}$, $j = 1, \ldots, k$, *are commuting basis monomials in* $\mathcal{B}$ *with square* 1 *and* $k = q - r_{q-p}$ *generating a group of order* 2^k, *is a primitive idempotent in* $C\ell_{p,q}$. *Furthermore,* $C\ell_{p,q}$ *has a complete set of* 2^k *such primitive mutually annihilating idempotents which add up to the unity* 1 *of* $C\ell_{p,q}$.

(d) *When* $(p-q) \bmod 8$ *is* $0, 1, 2$, *or* $3, 7$, *or* $4, 5, 6$, *then the division ring* $\mathbb{K} = fC\ell_{p,q}f$ *is isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$ *or* $\mathbb{H}$, *and the map* $S \times \mathbb{K} \to S$, $(\psi, \lambda) \mapsto \psi\lambda$ *defines a right* $\mathbb{K}$*-module structure on the minimal left ideal* $S = C\ell_{p,q}f$.

(e) *When* $C\ell_{p,q}$ *is simple, then the map*

$$C\ell_{p,q} \xrightarrow{\gamma} \mathrm{End}_{\mathbb{K}}(S), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi \tag{41}$$

gives an irreducible and faithful representation of $C\ell_{p,q}$ *in* S.

(f) *When* $C\ell_{p,q}$ *is semisimple, then the map*

$$C\ell_{p,q} \xrightarrow{\gamma} \mathrm{End}_{\mathbb{K}\oplus\hat{\mathbb{K}}}(S \oplus \hat{S}), \quad u \mapsto \gamma(u), \quad \gamma(u)\psi = u\psi \tag{42}$$

gives a faithful but reducible representation of $C\ell_{p,q}$ *in the double spinor space* $S \oplus \hat{S}$ *where* $S = \{uf \,|\, u \in C\ell_{p,q}\}$, $\hat{S} = \{u\hat{f} \,|\, u \in C\ell_{p,q}\}$ *and* $\hat{\ }$ *stands for the grade-involution in* $C\ell_{p,q}$. *In this case, the ideal* $S \oplus \hat{S}$ *is a right* $\mathbb{K} \oplus \hat{\mathbb{K}}$*-module structure,* $\hat{\mathbb{K}} = \{\hat{\lambda} \,|\, \lambda \in \mathbb{K}\}$, *and* $\mathbb{K}\oplus\hat{\mathbb{K}}$ *is isomorphic to* $\mathbb{R}\oplus\mathbb{R}$ *when* $p-q = 1 \bmod 8$ *or to* $\mathbb{H} \oplus \hat{\mathbb{H}}$ *when* $p - q = 5 \bmod 8$.

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talk

POLYNOMIAL PERMUTATIONS OF FINITE RINGS AND FORMATION OF LATIN SQUARES

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Combinatorial designs have wide applications in various fields, including coding theory and cryptography. Many examples of combinatorial designs can be listed like linked design, balanced design, one-factorization etc. Latin square is one such combinatorial concept. In this talk, we have considered different types of permutation polynomials over some finite rings. Over finite rings, we have observed that univariate permutation polynomials permute the ring elements whereas bivariate permutation polynomials form Latin squares. The Latin squares formed thus by permutation polynomials over finite rings are discussed with respect to various Latin square properties.

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SCALED PLANAR NEARRINGS

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Abstract: A nearring $(N,+,\cdot)$ is a structure similar to a ring, but without the request to be additively commutative and with just one of the distributive laws. In this work, we deal with a special nearring structure called planar nearring and introduce a new structure called scaled planar nearring. We prove that every scaled planar nearring is zero symmetric and deduce some structure theorems. We illustrate that the scaling factor of the scaled planar nearring can be used to understand ideas from projective geometry. Let $(F,+,\cdot)$ be a finite nearfield and $N = F \times F$. It is well known that if F is a field then the affine plane $(N, \boldsymbol{L}, \varepsilon)$ is desarguesian and that finite nearfields (which are not fields) can be used to construct non-desarguesian planes. We demonstrate that a suitable choice of scaling factor can be made to construct desarguesian planes or non-desarguesian planes.

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HIGHER-DIMENSIONAL REPRESENTATIONS OF SL_2 AND ITS REAL FORMS VIA PLÜCKER EMBEDDING

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ABSTRACT. The paper provides a geometric perspective on the inclusion map $\mathfrak{sl}_2 \cong \mathfrak{so}_3 \subset \mathfrak{so}_n$ realized via the Plücker embedding. It relates higher-dimensional representations of the complex Lie group SO_3 and its real forms in terms of $\mathsf{SO}(n)$ and $\mathsf{SO}(p,q)$ transformations to different subspaces described by means of the well-known double fibrations used in twistor theory. Moreover, incidence criteria, explicit matrix realizations and various factorization techniques are constructed (e.g. Euler and Wigner decompositions) in this generalized setting. Examples are provided for $n = 3$, 4 and 5 in the context of special relativity, classical and quantum mechanics.

PRELIMINARIES

Let us consider the action of $\mathsf{SO}(3)$ on $\mathbb{R}^n$ by means of plane rotations restricted to some three-dimensional irreducible subspace $V \subset \mathbb{R}^n$ (with trivial continuation on its orthogonal complement $V^\perp \cong \mathbb{R}^n/V$). Each plane $\{\theta\} \subset V$ may be identified with a projective bi-vector (simple two-form) $[\theta] \in \mathrm{P}\Lambda^2(\mathbb{R}^n)$, and thus, with a generator $\hat{\Theta}$ in $\mathfrak{so}(n)$. The latter correspondence is ensured by the Euclidean metric in $\mathbb{R}^n$ and the former one - by the Plücker embedding, briefly revised below in the more general complex setting. In this context it is convenient to operate with the geometric algebra $C\ell(\mathbb{C}^n)$ spanned by the canonical basis $\{1, \hat{\mathbf{e}}_{i_1}\hat{\mathbf{e}}_{i_2}, \ldots, \hat{\mathbf{e}}_{i_1}\hat{\mathbf{e}}_{i_2}\ldots\hat{\mathbf{e}}_{i_k}\}$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $1 \leq k \leq n$ where the vectors $\hat{\mathbf{e}}_i$ constitute an orthonormal basis of $\mathbb{C}^n$ and the geometric product yields $\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}$ for each pair $\mathbf{u},\mathbf{v} \in \mathbb{C}^n$. The non-degenerate quadratic form $\mathbf{u}^2 = \mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + \ldots u_n^2$ identifies exterior k-forms in $\Lambda^k(\mathbb{C}^n)$ with homogeneous elements of *grade k* denoted here as $\mathscr{V}_k^n \subset C\ell(\mathbb{C}^n)$ and in particular, decomposable forms - with monomials in $C\ell(\mathbb{C}^n)$, usually referred to as *k-blades* $\mathscr{B}_k^n \subset \mathscr{V}_k^n$, i.e., $\theta \in \mathscr{B}_k^n \Leftrightarrow \exists\, \{\mathbf{u}_i\}_{i=1}^k \in \mathbb{C}^n,\ \theta = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \ldots \mathbf{u}_k$. Note also that the Hodge operator $*$ providing an explicit realization of the isomorphism $\mathscr{V}_k^n \cong \mathscr{V}_{n-k}^n$ is constructed in $C\ell(\mathbb{C}^n)$ with the aid of the *pseudoscalar* $\mathrm{I} = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\ldots\hat{\mathbf{e}}_n$. Its restriction on blades yields a duality $\mathscr{B}_k^n \cong \mathscr{B}_{n-k}^n$ interpreted geometrically as an orthogonal complement. Finally, one has the Plücker embedding

$$\mathrm{G}_k^n \;\cong\; \mathscr{B}_k^n/_{\mathbb{C}^*} \quad \xrightarrow{\text{pl}} \quad \mathrm{P}\Lambda^k(\mathbb{C}^n) \cong \mathscr{V}_k^n/_{\mathbb{C}^*} \tag{1}$$

where G_k^n stands for the complex Grassmannian whose points are k-dimensional subspaces of $\mathbb{C}^n$ and we identify the equivalence class $[\theta] = \{\lambda\theta\,|\,\lambda \in \mathbb{C}^*\}$ of a k-blade $\theta = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \ldots \mathbf{u}_k$ with the linear subspace $\{\theta\} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_k\} \subset \mathbb{C}^n$. In particular, for $k = 2$ the inclusion $\mathscr{B}_2^n \subset \mathscr{V}_2^n$ (from which we later derive the inclusions $\mathfrak{so}_3 \subset \mathfrak{so}_n$) is given by the Plücker relations

$$\theta \wedge \theta = 0 \quad \rightarrow \quad \theta^{[ij}\theta^{k]l} = 0.$$

Note that for $n = 4$, this system reduces to a single equation describing the famous *Klein quadric*

$$\theta^{12}\theta^{34} + \theta^{23}\theta^{14} + \theta^{31}\theta^{24} = 0 \tag{2}$$

while in the general case it may be interpreted as an intersection of $\binom{n}{k} - k(n-k) - 1$ quadrics.

1. The Inclusion $\mathsf{SL}_2 \subset \mathsf{SO}_4$ via Twistor Correspondence

In twistor theory one considers a flag manifold[1] $\mathscr{F}^4_{1,2}$ referred to as the *correspondence space* between the *projective twistor space* $\mathbb{P}^3 = \mathrm{G}^4_1$ and the complexified compactified Minkowski space $\mathbf{M} \cong \mathrm{G}^4_2$ (embedded in $\mathbb{P}^5$ via the Plücker map) realizing the double fibration

$$\mathrm{G}^4_1 \quad \xleftarrow{\mu} \quad \mathscr{F}^4_{1,2} \quad \xrightarrow{\nu} \quad \mathrm{G}^4_2.$$

Here μ and ν are natural projections, which yield the *twistor correspondence* $\rho = \nu \circ \mu^{-1}$ between $\mathbb{P}^3$ and $\mathbf{M}$ in terms of *incidence relations* (see [1] for details). In particular, it associates each line $\ell \subset \mathbb{C}^4$ through the origin with the subset of planes intersecting at ℓ and conversely, each plane $\{\theta\} \subset \mathbb{C}^4$ through the origin is mapped via ρ^{-1} into the set of its one-dimensional subspaces. On the other hand, the image of the Plücker map (1) contains a family of subsets $\{\mathscr{P}\}$, on which the exterior product vanishes, i.e., for each pair of 2-blades $\theta_{i,j} \in \mathscr{P}$, we have $\theta_i \wedge \theta_j = 0$. Geometrically, this means that the planes $\{\theta_i\}$ and $\{\theta_j\}$ intersect along a line, just as in $\mathbb{C}^3$. There are two essentially distinct families of 2-parameter subsets of G^4_2. The first type, referred to as α-planes, consist of planes through a given line ℓ, i.e., the image of the twistor correspondence ρ for fixed $\ell \in \mathbb{P}^3$. One also has β-families consisting of planes that intersect along different lines but are still restricted to a three-dimensional subspace $V_\beta \in \mathrm{G}^4_2$. Their normal complements form an α-set with $V_\beta \perp \ell = \cap_{\{\theta_i\} \in \mathscr{P}_\beta} \{\theta_i\}^\perp$. Similarly, the union of the normal complements in an α-family forms a β-plane $V_\beta = \cup_{\{\sigma_i\} \in \mathscr{P}_\alpha} \{\sigma_i\}^\perp$. One may construct this duality using the Hodge operator or equivalently, consider the dual twistor correspondence

$$\mathrm{G}^4_3 \quad \xleftarrow{\mu_*} \quad \mathscr{F}^4_{2,3} \quad \xrightarrow{\nu_*} \quad \mathrm{G}^4_2.$$

Note that the incidence relations are now reversed: each hyperplane $V_\beta \subset \mathbb{C}^4$ is mapped via $\rho_* = \nu_* \circ \mu_*^{-1}$ to the set of planes contained in it, while ρ_*^{-1} sends each plane to a family of hyper-planes containing it as their common intersection. It is not hard to see that both α-planes (ρ-images of lines) and β-planes (ρ_*-images of hyper-planes) $\mathscr{P} \subset \mathrm{G}^4_2$ are isomorphic to $\mathbb{P}^2$ and may be labeled with the pre-image in the corresponding projective twistor space: $\mathbb{P}^3$ and G^4_3, respectively. In view of the duality between α and β planes, given by

$$\mathscr{P}_\alpha \quad \xrightarrow{\rho^{-1}} \quad \ell \quad \xrightarrow{\perp} \quad V_\beta \quad \xrightarrow{\rho_*} \quad \mathscr{P}_\beta \tag{3}$$

one may always parametrize with $\ell \in \mathbb{P}^3$ indicating either the common intersection in the α-set, or the common normal in the β-family. Thus, we see that only β-planes correspond to irreducible subspaces[2] $V_\beta \in \mathbb{C}^4$ realizing the corresponding representation of SO_3 in $\mathbb{C}^4$. One may consider the example of a β-plane spanned by the basic bi-vectors in $C\ell(\mathbb{C}^3)$, i.e., $\mathscr{P}_\beta = \{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_2\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1\} \;\to\; V_\beta = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ with $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$. This yields, on the one hand, the standard embedding of $\mathbb{C}^3 \subset \mathbb{C}^4$ as the subspace normal to the fourth direction $\hat{\mathbf{e}}_4$, while on the other, the above bi-vectors generate the quaternion basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, so it also gives the embedding $\mathfrak{sl}_2 \cong \mathfrak{so}_3 \subset C\ell(\mathbb{C}^4)$. Similarly, a typical α-plane has the form $\mathscr{P}_\alpha \cong \{\hat{\mathbf{e}}_1\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_2\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_3\hat{\mathbf{e}}_4\}$ and is obviously not closed under the geometric product, e.g. $\hat{\mathbf{e}}_1\hat{\mathbf{e}}_4\hat{\mathbf{e}}_2\hat{\mathbf{e}}_4 = \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1$. Thus, it does not provide a representation of SL_2, while each β-plane does, as it generates (via ρ_*^{-1}) an irreducible subspace for SL_2. Note that the above constructions may be restricted to the real case in different ways, depending on the choice of signature for the scalar product $\mathbf{u} \cdot \mathbf{v}$. Moreover, they could easily be extended to the more general setting of a smooth manifold X, whose tangent bundle admits a natural action of $\mathsf{SO}(n)$, $\mathsf{SO}(p,q)$ or (in the complex case) SO_n. Then, all definitions need to be made local (fiber-wise) and the parallel transport of $V_\beta \in \mathrm{TX}$ would depend on the bundle holonomy. We leave the detailed study of this matter for further research.

[1] $\mathscr{F}^n_{k,m}(\mathbb{C})$ consists of k-dimensional subspaces embedded in m-dimensional subspaces of $\mathbb{C}^n$ with $k < m < n$.

[2] As we see below, in the case of isotropic ℓ the irreducible subspace is given by the overlap of the α and β sets.

Before we proceed with real forms, let us point out that the splitting of the semi-simple Lie algebra $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$ allows for expressing a generic SO_4 transformation in a matrix form as

$$\mathscr{R}(\mathbf{c}\otimes\tilde{\mathbf{c}}) = \frac{1}{\sqrt{(1+\mathbf{c}^2)(1+\tilde{\mathbf{c}}^2)}} \begin{pmatrix} 1-\mathbf{c}\cdot\tilde{\mathbf{c}}+\mathbf{c}\tilde{\mathbf{c}}^t+\tilde{\mathbf{c}}\mathbf{c}^t+(\mathbf{c}+\tilde{\mathbf{c}})^\times & \mathbf{c}-\tilde{\mathbf{c}}+\tilde{\mathbf{c}}\times\mathbf{c} \\ (\tilde{\mathbf{c}}-\mathbf{c}+\tilde{\mathbf{c}}\times\mathbf{c})^t & 1+\mathbf{c}\cdot\tilde{\mathbf{c}} \end{pmatrix}$$

where the two complex vector-parameters (Rodrigues' vectors) $\mathbf{c},\tilde{\mathbf{c}} \in \mathbb{P}^3$ are obtained by projecting the corresponding quaternions $\zeta = (\zeta_0, \boldsymbol{\zeta}),\ \tilde{\zeta} = (\tilde{\zeta}_0, \tilde{\boldsymbol{\zeta}}) \in \mathbb{H}^*$, i.e., $\mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0}$ and $\tilde{\mathbf{c}} = \dfrac{\tilde{\boldsymbol{\zeta}}}{\tilde{\zeta}_0}$. Multiplication in $\mathbb{H}^*$ then yields for SO_3 the composition law

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - \mathbf{c}_2 \cdot \mathbf{c}_1} \tag{4}$$

which extends to a composition in SO_4 due to the local product structure. Let us also denote

$$\mathbf{c} = \boldsymbol{\alpha} + \boldsymbol{\beta}, \qquad \tilde{\mathbf{c}} = \boldsymbol{\alpha} - \boldsymbol{\beta}$$

along with $\tilde{\mathscr{R}} = \mathscr{R} - \mathscr{R}^t$ and derive from the matrix representation (cf. [2])

$$\boldsymbol{\alpha} = \frac{1}{\mathrm{tr}\mathscr{R}} \left(\tilde{\mathscr{R}}_{32}, \tilde{\mathscr{R}}_{13}, \tilde{\mathscr{R}}_{21}\right)^t, \qquad \boldsymbol{\beta} = \frac{1}{\mathrm{tr}\mathscr{R}} \left(\tilde{\mathscr{R}}_{14}, \tilde{\mathscr{R}}_{24}, \tilde{\mathscr{R}}_{34}\right)^t.$$

One may also introduce the so-called *tensor-parameter* Θ in a block-matrix form as

$$\Theta = \begin{pmatrix} \boldsymbol{\alpha}^\times & \beta \\ -\boldsymbol{\beta}^t & 0 \end{pmatrix} \in \mathfrak{so}_4 \tag{5}$$

and express group elements either via the exponential map or by means of the Cayley transform

$$\mathrm{Cay}(\Theta) = \frac{\mathscr{I} + \Theta}{\mathscr{I} - \Theta} \tag{6}$$

where $\mathscr{I}$ denotes the identity in every dimension and since $\mathscr{I}+\Theta$ and $(\mathscr{I}-\Theta)^{-1}$ commute, we have written their product as a fraction. Let $\Theta_{1,2}$ denote the tensor-parameters (5) of two SO_3 transformations and $\theta_{1,2}$ be the associated 2-vectors (also given in terms of $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}_k$) corresponding (via the metric η) to $\mathfrak{so}_3$ elements acting in the same irreducible subspace V_β if

$$\theta_1 \wedge \theta_1 = \theta_2 \wedge \theta_2 = \theta_1 \wedge \theta_2 = 0, \qquad \theta_i = \eta(\Theta_i). \tag{7}$$

The latter may be written for $\boldsymbol{\alpha}_{1,2}$ and $\boldsymbol{\beta}_{1,2}$ as a set of orthogonality conditions in the form

$$\boldsymbol{\alpha}_i \perp \boldsymbol{\beta}_j, \qquad i,j = 1,2$$

which guarantee that both representations coincide, i.e., $\mathscr{R}(\mathbf{c}\otimes\tilde{\mathbf{c}}) = \mathrm{Cay}(\Theta)$. For example, $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = 0$ yields a restriction of the vectorial parameterizations $\mathbf{c}\otimes\tilde{\mathbf{c}}$ to the diagonal, corresponding to a rotation in the hyperplane normal to $\hat{\mathbf{e}}_4$. In the general setting, however, one needs to determine the common invariant axis ℓ as an intersection of the orthogonal complements

$$\ell = \Sigma_1 \cap \Sigma_2, \qquad \Sigma_{1,2} = \{\ker\Theta_{1,2}\} = \{\Theta_{1,2}\}^\perp.$$

In particular, each plane rotation $\mathrm{Cay}(\Theta) \in \mathsf{S}O_4$, i.e., one that satisfies $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$, leaves the plane $\Sigma = \{(\boldsymbol{\alpha}, 0)^t, (\boldsymbol{\alpha}\times\boldsymbol{\beta}, \boldsymbol{\alpha}^2)^t\}$ invariant assuming $\boldsymbol{\alpha} \neq 0$. A vanishing $\boldsymbol{\alpha}$, on the other hand, yields $\Sigma = \{(\boldsymbol{\beta}^\perp, 0)^t\}$ as formula (5) indicates. Then, using the relation (3) one easily obtains the corresponding β-set realizing the SL_2 representation. Moreover, $\mathscr{P}_\beta$ is obviously closed under the composition law (4) that may be written for the tensor-parameters as

$$\langle \Theta_2, \Theta_1 \rangle = \frac{\Theta_1 + \Theta_2 + [\Theta_2, \Theta_1]}{1 - (\Theta_1, \Theta_2)} \tag{8}$$

where $[\cdot,\cdot]$ is the matrix commutator and $(\Theta_1,\Theta_2) = -\dfrac{1}{2}\mathrm{Tr}\,(\Theta_1\Theta_2)$ is minus the Killing form. Note also that each pair of distinct 2-blades $\Theta_{1,2}$ satisfying (7) defines an irreducible space via

$$\{\theta_{1,2}\} \xrightarrow{\eta^{-1}\circ\mathrm{pl}} \pm\hat{\Theta}_{1,2}, \qquad \mathscr{P}_\beta = \left\{\mathrm{pl}^{-1}\circ\eta\,\langle\tau_2\hat{\Theta}_2, \tau_1\hat{\Theta}_1\rangle,\ \tau_{1,2}\in\mathbb{P}^1\right\} \xrightarrow{\rho_*^{-1}} V_\beta.$$

2. Higher-Dimensional Extension

For $n > 3$ we have a family of inclusions $\mathfrak{sl}_2 \subset \mathfrak{so}_n$ generalizing the above construction for $\mathfrak{so}_4$. The formal definition of α and β planes here remains the same: consider the double fibrations

$$\mathrm{G}_1^n \xleftarrow{\mu} \mathscr{F}_{1,2}^n \xrightarrow{\nu} \mathrm{G}_2^n, \qquad \mathrm{G}_3^n \xleftarrow{\mu^*} \mathscr{F}_{2,3}^n \xrightarrow{\nu^*} \mathrm{G}_2^n$$

and define the α-planes as the ρ images of lines $\ell \subset \mathbb{C}^n$ through the origin, while the β-planes are obtained similarly as ρ_* images of three-dimensional subspaces of $\mathbb{C}^n$. Despite the close analogy with the case $n = 4$, the duality (3) between these objects is lost due to the increased codimension. In particular, one can no longer label the irreducible subspaces $\mathscr{P}_\beta$ with a single direction ℓ - the invariant subspace $V_\beta^\perp$ appears as the $n-3$ dimensional intersection of all $n-2$ dimensional orthogonal complements of planes in $\mathscr{P}_\beta$, so (3) remains valid with $V_\beta^\perp$ instead of ℓ and $\rho(V_\beta^\perp)$ replacing $\mathscr{P}_\alpha$. Furthermore, there is no quaternion or vector-parameter construction to aid our calculations for $n > 4$, although tensor-parameters and the Cayley representation (6) are always available and the composition law (8) holds in each V_β-plane $\mathscr{P}_\beta$. Note that in the three-dimensional case the tensor-parameter in formula (6) may be replaced with the Hodge dual of the vector-parameter $\Theta = \mathbf{c}^\times$, which allows us to write the rotation matrix explicitly as

$$\mathscr{R}(\mathbf{c}) = \frac{1 - \mathbf{c}^2 + 2\,\mathbf{c}\,\mathbf{c}^t + 2\,\mathbf{c}^\times}{1 + \mathbf{c}^2}. \tag{9}$$

Now, let us obtain a similar expression for the restriction to a particular $\mathscr{P}_\beta$ in terms of $\Theta \in \mathfrak{so}_3$. Firstly, we note that if Θ is decomposable, i.e., $\theta = \eta(\Theta)$ satisfies $\theta \wedge \theta = 0$, one has (see [3])

$$\Theta^{k+2} = -|\Theta|^2 \Theta^k, \qquad |\Theta|^2 = (\Theta, \Theta).$$

Next, we expand formula (6) in a geometric (Neumann) series, thus obtaining

$$\mathscr{R} = \mathrm{Cay}(\Theta) = (1 + \Theta) \sum_{k=0}^{\infty} \Theta^k$$

to which we apply the above recurrent relation and finally end up with

$$\mathscr{R}(\Theta) = \mathscr{I} + 2\,\frac{\Theta + \Theta^2}{1 + |\Theta|^2}. \tag{10}$$

Certainly, each plane rotation in a given β-inclusion has an invariant vector $\mathbf{n}$ within V_β, which may be extracted from $\{\theta\}^\perp = \ker\Theta$ via projection onto V_β, i.e., using the V_β-restriction of the Hodge duality. This, however, is not necessary for our construction as we may use the adjoint action of SO_3 on projective 2-blades $[\hat{\Theta}]$ realized as $\mathfrak{so}_n$ generators and interpreted as planes[3]

$$\mathscr{R}\,[\hat{\Theta}] = [\mathrm{Ad}_{\mathscr{R}}\,\hat{\Theta}] = [\mathscr{R}\,\hat{\Theta}\,\mathscr{R}^{-1}], \qquad \hat{\Theta} \in \mathfrak{so}_n.$$

In particular, $\mathscr{R}$ leaves invariant both the rotation plane $\{\theta\}$ and its orthogonal complement $\{*\theta\} = \{\theta\}^\perp$. On the other hand, in the V_β-restriction the angle between two planes equals the angle between their normal directions, so we may also express matrix entries in a basis of planes $\{\theta_i\} \in \mathrm{G}_2^4 \leftrightarrow [\Theta_i] = \pm\hat{\Theta}_i \in \mathfrak{so}_n$ rather than vectors, with the aid of the Killing form, e.g.

$$g_{ij} = (\hat{\Theta}_i, \hat{\Theta}_j), \qquad r_{ij} = (\hat{\Theta}_i, \mathrm{Ad}_{\mathscr{R}}\,\hat{\Theta}_j), \qquad \omega = (\hat{\Theta}_1, [\hat{\Theta}_2, \hat{\Theta}_3]). \tag{11}$$

[3]We identify 2-vectors $\mathscr{V}_2^n$ with 2-forms $\Lambda^2(\mathbb{C}^n)$ and elements of $\mathfrak{so}_n$ using the machinery of geometric algebra.

3. Real Forms and Wigner Little Groups

The real forms $\mathsf{SO}(3)$ and $\mathsf{SO}(2,1)$ are realized as subgroups of $\mathsf{SO}(n)$, $\mathsf{SO}(p,q)$ and $\mathsf{SO}^*(n)$. Obviously, only the first type appears in $\mathsf{SO}(n)$ and only the second one in $\mathsf{SO}(2,2)$. In all other cases both the compact and non-compact three-dimensional subgroups are included. In particular, it is straightforward to obtain the embedding of the real form $\mathsf{SO}(3) \subset \mathsf{SO}_3$ by restricting the above twistor construction to $\mathbb{R}$. It is also far more intuitive, e.g. the projective twistor space consisting of different copies of V_β reduces to $\mathbb{RP}^3 \cong \mathsf{SO}(3)$, which yields locally $\mathsf{SO}(4) \cong \mathsf{SO}(3) \times \mathsf{SO}(3)$. This is quite similar to the axis-angle decomposition of $\mathsf{SO}(3)$ or $\mathsf{SO}(2,1)$ seen as a manifestation of the first Hopf fibration or its dual, respectively (cf. [4])

$$\mathbb{S}^1 \;\to\; \mathbb{S}^3 \;\to\; \mathbb{S}^2, \qquad \mathbb{S}^1 \;\to\; \mathsf{AdS}_3 \;\to\; \mathbb{D}. \tag{12}$$

This technique applies to the higher-dimensional case as well. For example, one may construct the irreducible subspace V_β by pointing its $n-3$ dimensional orthogonal complement $V_\beta^\perp$ obtained as $V_\beta^\perp = \cap\{\theta_i\}^\perp$, for which it suffices to have only two planes $\{\theta_{1,2}\}$ satisfying (7). The situation does not change much if we allow (ultra-)hyperbolic signatures. However, there is one important peculiarity inherited from the complex case: if $V_\beta^\perp$ contains an isotropic direction ℓ, the non-trivial overlap of the corresponding α and β plane makes V_β tangent to the light cone at ℓ and thus, yields a representation of a parabolic subgroup of (pseudo-)Euclidean motions. In [5, 6] this has been referred to as *light cone singularity* in the specific cases of $\mathbb{R}^{2,1}$ and $\mathbb{R}^{3,1}$.

Let us now consider the proper Lorentz group. The local isomorphism $\mathsf{SO}_3 \cong \mathsf{SO}^+(3,1)$ allows for using either a complex vector-parameter or a tensor-parameter in the $\mathbb{R}^{3,1}$ representation[4]

$$\mathbf{c} = \boldsymbol{\alpha} + i\boldsymbol{\beta} \quad\longleftrightarrow\quad \Theta = \begin{pmatrix} \boldsymbol{\alpha}^\times & \boldsymbol{\beta} \\ \boldsymbol{\beta}^t & 0 \end{pmatrix} \quad\longrightarrow\quad \mathrm{Cay}(\Theta) \tag{13}$$

where $\mathbf{c} \in \mathbb{P}^3$ yields also the block-matrix form (see [7])

$$\Lambda(\mathbf{c}) = \frac{1}{|1+\mathbf{c}^2|} \begin{pmatrix} 1-|\,\mathbf{c}\,|^2 + \mathbf{c}\bar{\mathbf{c}}^t + \bar{\mathbf{c}}\mathbf{c}^t + (\mathbf{c}+\bar{\mathbf{c}})^\times & i(\bar{\mathbf{c}} - \mathbf{c} + \bar{\mathbf{c}} \times \mathbf{c}) \\ i(\bar{\mathbf{c}} - \mathbf{c} - \bar{\mathbf{c}} \times \mathbf{c})^t & 1+|\,\mathbf{c}\,|^2 \end{pmatrix} \in \mathsf{SO}^+(3,1). \tag{14}$$

Note that (6) and (14) coincide (up to a sign) only in the decomposable case $\Im\mathbf{c}^2 = 0$. Then the intersection relations (7) for the set of planes $\{\theta_i\}$ associated with $\{\mathbf{c}_i\}$ via (13) are written as

$$\Im\, \mathbf{c}_i \cdot \mathbf{c}_j = 0. \tag{15}$$

Demanding additionally orthochronicity for Λ yields $\mathrm{Cay}(\Theta) = \pm\Lambda(\mathbf{c})$, thus allowing vectors in $\ker\Theta$ to be associated with an eigenvalue $\lambda = -1$ of $\Lambda(\mathbf{c})$, still leaving $\ker\Theta$ invariant. In the decomposable setting each $V_\beta \in \mathbb{R}^{3,1}$ yields a representation of a Wigner little group if $\mathbf{p} \in \ell$ is interpreted as a momentum of a quantum particle. For example, $\mathbf{p}^2 > 0$ (space-like) corresponds to the embedding $\mathsf{SO}(2,1) \subset \mathsf{SO}(3,1)$, while $\mathbf{p}^2 < 0$ (time-like) yields $\mathsf{SO}(3) \subset \mathsf{SO}(3,1)$ and finally, for a light-like (isotropic) momentum $\mathbf{p}^2 = 0$ we have a realization of the Euclidean group in the plane $E(2)$. The first two are obvious, while the latter needs further explanation. Note that in this case $V_\alpha \cap V_\beta = \{\ell\}^\perp$ yields the tangent to the light cone at ℓ. In the complex representation it corresponds to a projective line $\mathbb{P}^1$ with homogeneous coordinates set by the choice of a basis, i.e., $\{\mathbf{c}_\circ, \boldsymbol{\kappa}\}$ where $\mathbf{c}_\circ$ determines the isotropic directions and $\boldsymbol{\kappa} \in \{\mathbf{c}_\circ\}^\perp / \{\mathbf{c}_\circ\}$. Then, SL_2 acts on $\mathbb{P}^1$ via linear-fractional transformations and this action determines a stationary subgroup of a point (e.g. infinity) isomorphic to $E(2)$. We show this result below using a spinor construction, which is very close to Wigner's original proof [8]. Note also that for each pair of vectors $\mathbf{c}_{i,j} \in \mathbf{c}_\circ^\perp$ one has $\mathbf{c}_i \times \mathbf{c}_j \in \{\mathbf{c}_\circ\}$ and thus, $(\mathbf{c}_i \cdot \mathbf{c}_j)^2 = \mathbf{c}_i^2\,\mathbf{c}_j^2$ so $\mathbf{c}_\circ^\perp$ is closed under the composition (4). Moreover, the isotropic vector $\mathbf{c}_\circ = \boldsymbol{\alpha}_\circ + i\boldsymbol{\beta}_\circ$ determines the null direction $\ell = \{(\boldsymbol{\alpha}_\circ \times \boldsymbol{\beta}_\circ, \boldsymbol{\alpha}_\circ^2)^t\}$ as an intersection of the orthogonal complements $\Sigma_i = \{\theta_i\}^\perp = \ker\hat{\Theta}_i$.

[4] Interpreted as the Maxwell tensor of the EM field with electric component given by β and magnetic one by α.

Next, let us consider the other real forms of SO_4 and note the splitting of their Lie algebras

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3), \qquad \mathfrak{so}(2,2) \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1), \qquad \mathfrak{so}^*(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2,1)$$

which allows for introducing a (split) quaternion or vector-parameter description in the form of a tensor product $\mathbf{c} \otimes \tilde{\mathbf{c}}$. Obviously, all little groups in the first case are isomorphic to $\mathsf{SO}(3)$ due to the spherical symmetry. As for $\mathsf{SO}(2,2)$, it is not hard to see that it contains only copies of $\mathsf{SO}(2,1)$ and the parabolic subgroup $E(1,1)$, the Poincaré group of $1+1$ dimensional space-time. Finally, $\mathsf{SO}^*(4)$ clearly contains both $\mathsf{SO}(3)$ and $\mathsf{SO}(2,1)$, but its natural realization is not as an isometry group of a pseudo-Euclidean space. Thus, for now we ignore the $\mathsf{SO}^*(n)$ real forms and concentrate only on $\mathsf{SO}(n)$ and $\mathsf{SO}(p,q)$. Note also that for $\mathsf{SO}(3,1)$ and $\mathsf{SO}(2,2)$ we have a convenient representation in terms of spinors. In the former case for example, following the twistor approach once more, we may associate to each projective null direction a two-sphere

$$\mathscr{L}(\mathbb{R}^{3,1}) \ni \mathbf{x} \quad \longrightarrow \quad \Psi = \begin{pmatrix} x_4 + x_1 & x_3 - ix_2 \\ x_3 + ix_2 & x_4 - x_1 \end{pmatrix}, \qquad |\Psi|^2 = -\det \Psi = 0$$

and use a pair of homogeneous coordinates $z, w \in \mathbb{C}$ to express

$$\mathbf{x} = \big(|z|^2 - |w|^2,\, 2\Im(z\bar{w}),\, 2\Re(z\bar{w}),\, |z|^2 + |w|^2 \big)^t$$

which can be written in spinor form as

$$\Psi = \begin{pmatrix} |z|^2 & \bar{z}w \\ z\bar{w} & |w|^2 \end{pmatrix} = \begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} \big(z,\, w \big) = \psi\psi^\dagger.$$

The parameter $\tau = z/w \in \mathbb{P}^1$ describes the projective null direction $\{\Psi\}$ and the group PSL_2 acts on $\mathbb{P}^1$ via linear-fractional transformations. Choosing for example $(z,w) = (1,0)$, i.e., $\tau = \infty$, it is easy to see that the parabolic subgroup preserving this point is isomorphic to $E(2)$.

Similarly, in the case of $\mathsf{SO}(2,2)$ we may express each null vector in the form

$$\mathscr{L}(\mathbb{R}^{2,2}) \ni \mathbf{x} \quad \longrightarrow \quad \Psi = \begin{pmatrix} x_4 + ix_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_4 - ix_3 \end{pmatrix}, \qquad |\Psi|^2 = -\det \Psi = 0$$

and again use a pair of complex parameters $z, w \in \mathbb{C}$ for its entries

$$\mathbf{x} = \big(\Re(z^2 - w^2),\, \Im(z^2 - w^2),\, 2\Im(\bar{z}w),\, |z|^2 - |w|^2 \big)^t$$

which yields with the notation $\xi_\pm = z \pm w$ the spinor representation

$$\Psi = \begin{pmatrix} z^2 - w^2 & (z+w)(\bar{z} - \bar{w}) \\ (\bar{z} + \bar{w})(z - w) & \bar{z}^2 - \bar{w}^2 \end{pmatrix} = \begin{pmatrix} \xi_+ \\ \bar{\xi}_+ \end{pmatrix} \big(\xi_-,\, \bar{\xi}_- \big) = \psi_+ \psi_-^t.$$

Setting $\xi_+ = \xi_- = 1$ we obtain a two-dimensional abelian stationary subgroup generated by

$$T(\nu_\pm) = \begin{pmatrix} 1 + i\nu_\pm & -i\nu_\pm \\ i\nu_\pm & 1 - i\nu_\pm \end{pmatrix}, \qquad \nu_\pm \in \mathbb{R}. \tag{16}$$

Moreover, the action of $\mathbb{R}^* \otimes \mathbb{R}^*$ on $\xi_+ \otimes \xi_-$ clearly preserves the null direction $\{\Psi\}$ and as it is not hard to see, one may equivalently work with the invariant hyperbolic subgroup

$$S(\lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \in \mathsf{SO}(1,1) \cong \mathbb{R} \tag{17}$$

acting on $\Psi = \psi_+ \psi_-^t$ via $S(\lambda) \otimes S^t(-\lambda)$, which corresponds to the off-diagonal inclusion $\mathsf{SO}(2,1) \subset \mathsf{SO}(2,2)$. Transition to the real basis $\mathfrak{su}(1,1) \to \mathfrak{sl}(2,\mathbb{R})$ yields a composition of (coupled) dilatations and (upper-triangular) parabolic transformations in the two copies of $\mathsf{SL}(2,\mathbb{R})$. The compound action is then isomorphic to a pseudo-Euclidean planar motion via

$$\tau_\pm = \cot(\arg \xi_\pm) \quad \longrightarrow \quad \tau'_\pm = \mathrm{e}^{\pm 2\lambda} \tau_\pm - 2\nu_\pm. \tag{18}$$

Thomas Precession and Holonomy. The Wigner rotation, together with its infinitesimal version referred to as Thomas precession [9], is a well-known relativistic effect usually attributed to Einstein's non-commutative summation rule for velocities. The rotation phase generated by the interaction of two non-parallel pure boosts given in vector-parameter form as $\mathbf{c}_{1,2} = i\boldsymbol{\beta}_{1,2}$ may easily be obtained from the non-commutative part of the composition (4) as

$$\Re\langle i\boldsymbol{\beta}_2, i\boldsymbol{\beta}_1\rangle = \frac{\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2}{1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}.$$

In the case of a continuously deformed boost it is convenient to express $i\boldsymbol{\beta}$ as a purely imaginary quaternion $\boldsymbol{\zeta} \in \mathfrak{su}(2)$ and thus obtain the infinitesimal precession in the non-inertial (Thomas) frame in the form of a gauge potential

$$\mathscr{A} = \Im \frac{\boldsymbol{\zeta}^* \mathrm{d}\boldsymbol{\zeta}}{1-|\boldsymbol{\zeta}|^2}. \tag{19}$$

The latter may be associated via differentiation with a skew-symmetric Hannay form (cf. [10]) interpreted as the curvature of a principle $\mathsf{SO}(3)$ bundle over the unit ball $\mathbb{B}_3$ (rapidity space). Thomas precession may then be seen as a consequence of the non-trivial holonomy and thus related a geometric phase. In particular, for $\mathbb{R}^4$ there is no precession as the fibration $\mathsf{SO}(3) \to \mathsf{SO}(4) \to \mathbb{S}^3$ is globally trivial. A parallel is often made between the precession of Foucault's pendulum associated with the first Hopf map (12) and the Thomas effect in the $2+1$ dimensional case related to its non-compact version. As it has been pointed out in [11], this construction allows for higher-dimensional generalization related to the geometry of spheres $\mathbb{S}^n$ and Lobachevsky spaces $\mathbb{L}_n$. Namely, if $\Theta_{1,2} = \eta^{-1}(\theta_{1,2})$ are the tensor-parameters of two non-parallel plane (pseudo-)rotations with the intersection condition $\{\theta_1\} \cap \{\theta_2\} = \ell \in \mathbb{P}^1$ satisfied, they determine both an α-set (labeled by ℓ) and a β-set V_β via the composition law (8). Moreover, we clearly have $\ell \in V_\beta$ and one may factorize into a longitudinal and transversal with respect to ℓ components: $\langle\Theta_2,\Theta_1\rangle = \langle\Theta^\circ,\tilde{\Theta}\rangle$, where we refer to the ℓ-preserving part Θ° as the generalized Wigner rotation (or in the infinitesimal case, Thomas precession). With the aid of formula (8) one easily retrieves the corresponding explicit expressions respectively as[5]

$$\Theta^\circ = \frac{[\Theta_1,\Theta_2]}{1-(\Theta_1,\Theta_2)}, \qquad \mathrm{d}\Theta^\circ = \frac{[\Theta,\mathrm{d}\Theta]}{1-|\Theta|^2} \tag{20}$$

assuming in the latter case that Θ and $\mathrm{d}\Theta$ intersect at ℓ as well. Moreover, the associated geometric phase in the compact setting may be associated with an $\mathsf{SO}(3)$ sub-bundle and the precession angle is given as $\vartheta^\circ = 2\arctan|\Theta^\circ|$. In the non-compact case, however, the line of intersection ℓ may happen to be space-like, time-like or null, which yields different types of homogeneous spaces, e.g. in Minkowski space-time $\mathbb{R}^{3,1}$ those are given by the familiar cosets

$$\mathbb{B}_3 \cong \mathsf{SO}^+(3,1)/\mathsf{SO}(3), \qquad \mathrm{dS}_3 \cong \mathsf{SO}^+(3,1)/\mathsf{SO}(2,1), \qquad \mathscr{L}(\mathbb{R}^{3,1}) \cong \mathsf{SO}^+(3,1)/E(2)$$

while in the ultra-hyperbolic case $\mathbb{R}^{2,2}$ one has (the first bundle being obviously globally trivial)

$$\mathrm{AdS}_3 \cong \mathsf{SO}(2,2)/\mathsf{SO}(2,1), \qquad \mathscr{L}(\mathbb{R}^{2,2}) \cong \mathsf{SO}(2,2)/E(1,1).$$

Thus, in the generic case of $\mathsf{SO}(p,q)$ with $p+q>4$ one always has the classical elliptic phases, while for $pq \neq 0$ hyperbolic and parabolic ones appear as well, the latter being of two possible types if $p,q>1$. Adding the trivial case of $\mathsf{SO}(4)$, we obtain all possible inclusions of three-dimensional sub-bundles, which generalize the geometry of the Thomas precession. For example, the embeddings of the above homogeneous spaces in $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$, respectively, yield $\ell \in V_\beta$ as a normal direction at each point of the base. However, this is not the flat $\mathsf{U}(1)$ bundle as the plane $\{\theta\}^\circ = V_\beta \cap \ell^\perp$ varies from fiber to fiber due to the non-commutativity (holonomy) and increasing the codimension certainly allows more freedom. Note also that one may consider a dynamical picture, in which the "normal" direction ℓ evolves through various transitions of the geometric type. This, however, goes far beyond the scope of the present text.

[5] These expressions apply to the Thomas frame, while in the laboratory one the signs are reversed.

4. The Factorization Problem

The Two-Factors Setting. Consider first the decomposition of a SO_3 transformation into a pair of complex rotations. The Euler invariant axis theorem yields the corresponding necessary and sufficient condition as an equality of matrix entries, namely $r_{21} = g_{21}$, written explicitly as

$$\mathscr{R} = \mathscr{R}_2\mathscr{R}_1 \quad \Leftrightarrow \quad \hat{\mathbf{c}}_2 \cdot \mathscr{R}\,\hat{\mathbf{c}}_1 = \hat{\mathbf{c}}_2 \cdot \hat{\mathbf{c}}_1 \tag{21}$$

where $\hat{\mathbf{c}}_i$ is the invariant vector of $\mathscr{R}_i$. The above may be expressed also in the form

$$\mathbf{c} \cdot \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2 = \mathbf{c} \times \hat{\mathbf{c}}_1 \cdot \mathbf{c} \times \hat{\mathbf{c}}_2 \tag{22}$$

with $\mathbf{c}$ denoting the vector-parameter of the compound transformation. Hence, if one of the axes is set (for a fixed compound complex rotation $\mathscr{R}$), the other is restricted to a plane, namely

$$\hat{\mathbf{c}}_1 \;\perp\; \mathbf{c} \times \hat{\mathbf{c}}_2 - \mathbf{c} \times (\mathbf{c} \times \hat{\mathbf{c}}_2) \qquad \hat{\mathbf{c}}_2 \;\perp\; \mathbf{c} \times \hat{\mathbf{c}}_1 + \mathbf{c} \times (\mathbf{c} \times \hat{\mathbf{c}}_1)$$

and the solutions for the angles are readily given by formula (4) as

$$\tau_1 = \frac{\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2 \cdot \mathbf{c}}{(\hat{\mathbf{c}}_1 \cdot \hat{\mathbf{c}}_{[2}})(\hat{\mathbf{c}}_{1]} \cdot \mathbf{c})}, \qquad \tau_2 = \frac{\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2 \cdot \mathbf{c}}{(\hat{\mathbf{c}}_2 \cdot \hat{\mathbf{c}}_{[1}})(\hat{\mathbf{c}}_{2]} \cdot \mathbf{c})} \tag{23}$$

where $[\cdot,\cdot]$ denotes anti-symmetrization of indices, i.e., $a_{[i}b_{j]} = a_ib_j - a_jb_i$ and τ_i are the *scalar parameters* determined from $\mathbf{c}_i = \tau_i\hat{\mathbf{c}}_i$. Note that in the compact (real) case as long as the vectors $\hat{\mathbf{c}}_i$ are unit[6] one has for the two rotation angles in the decomposition $\phi_i = 2\arctan\tau_i$. The above expressions, on the other hand, hold in the general complex setting $\mathsf{SO}_3 \cong \mathsf{SO}^+(3,1)$ and its non-compact real form $\mathsf{SO}(2,1)$ in particular (cf. [12]). In all those cases one has a simple way to choose the second axis $\hat{\mathbf{c}}_2$ depending on both $\hat{\mathbf{c}}_1$ and $\mathbf{c}$, namely as $\hat{\mathbf{c}}_2 \parallel \hat{\mathbf{c}}_1 \times \mathscr{R}\,\hat{\mathbf{c}}_1$, which reduces the equality (21) to $0 = 0$ (see [6] for details). In the n-dimensional setting, however, it is convenient to work with 2-blades realized in terms of $\mathfrak{so}_n$ matrices, as already discussed. Then, we use formula (10) instead of (9), replace the usual dot product with $(A,B) = -\frac{1}{2}\mathrm{Tr}(AB)$ and the cross product with the matrix commutator. This allows for expressing (22) in the form

$$([\hat{\Theta}_1, \hat{\Theta}_2], \Theta) = ([\hat{\Theta}_1, \Theta], [\hat{\Theta}_2, \Theta]) \tag{24}$$

where $\hat{\Theta}_i \in \mathfrak{so}_n$ are the generators corresponding to the rotation planes $\{\theta_i\}$ in (8) with $\Theta_i = \tau_i\hat{\Theta}_i$, while Θ is the tensor-parameter of the compound transformation. Note that (24) is homogeneous in $\hat{\Theta}_{1,2}$ as one should expect. Thus, in the compact case it is convenient to normalize $\hat{\Theta}_i = |\Theta_i|^{-1}\,\Theta_i$ in order to preserve the trigonometric interpretation of the scalar parameters τ_i, while in the non-compact setting such procedure is not justified due to the presence of a light cone structure. Next, from equation (24) one easily derives the orthogonality conditions

$$\hat{\Theta}_1 \;\perp\; \mathrm{ad}_\Theta(1 - \mathrm{ad}_\Theta)\hat{\Theta}_2, \qquad \hat{\Theta}_2 \;\perp\; \mathrm{ad}_\Theta(1 + \mathrm{ad}_\Theta)\hat{\Theta}_1$$

with $\mathrm{ad}_X Y = [X,Y]$, which may be used for adjusting an arbitrary system $\{\Theta\}, \{\hat{\Theta}_{1,2}\} \in \mathscr{P}_\beta$ to one that allows for a decomposition via appropriately constructed projectors. Then, using formula (8), one easily derives the solutions for the scalar parameters in the form

$$\tau_1 = -\frac{([\hat{\Theta}_1, \hat{\Theta}_2], \Theta)}{([\hat{\Theta}_1, \hat{\Theta}_2], [\hat{\Theta}_1, \Theta])}, \qquad \tau_2 = \frac{([\hat{\Theta}_1, \hat{\Theta}_2], \Theta)}{([\hat{\Theta}_1, \hat{\Theta}_2], [\hat{\Theta}_2, \Theta])}. \tag{25}$$

Note also that in the regular setting we may safely choose $\hat{\Theta}_2 = [\hat{\Theta}_1, \mathrm{Ad}_{\mathscr{R}}\hat{\Theta}_1]$ and thus, obtain[7]

$$\tau_1 = |\hat{\Theta}_1|^{-2}(\hat{\Theta}_1, \Theta), \qquad \tau_2 = \left[|\hat{\Theta}_1|^2 + (\hat{\Theta}_1, \mathrm{Ad}_{\mathscr{R}}\hat{\Theta}_1)\right]^{-1}. \tag{26}$$

[6] Such normalization is unnecessary due to the homogeneity of (21) and in the non-compact case we avoid it.

[7] See [6] for a derivation and a detailed treatment of both the gimbal lock and isotropic singular cases.

Generic (Pseudo-)Rotations. Let us note that when $\mathscr{R}$ is not a plane (pseudo-)rotation, most of the above expressions lose their validity. However, as pointed out in [3], it is always possible to decompose a generic SO_n transformation into a product of $\left\lceil \frac{n}{2} \right\rceil$ plane factors. Then, invariance arguments yield the geometric necessary and sufficient condition in the form of $(n-2)^2$ equalities of the type (21) - one for each basic pair of vectors $\hat{\mathbf{c}}_1 \in \Sigma_1$ and $\hat{\mathbf{c}}_2 \in \Sigma_2$, where $\Sigma_i = \{\theta_i\}^\perp$ denote the orthogonal complements of the rotation planes in the decomposition. Only for $n = 3$ or in the case $\hat{\Theta}_1 \perp \hat{\Theta}_2$ this reduces to a single equality of matrix entries in the form $r_{21} = g_{21}$, which is guaranteed if we adjust one of the planes with respect to the other as

$$\hat{\Theta}_1 \perp (1 - \mathrm{Ad}_{\mathscr{R}})\hat{\Theta}_2, \qquad \hat{\Theta}_2 \perp (1 - \mathrm{Ad}_{\mathscr{R}}^{-1})\hat{\Theta}_1.$$

Similarly, $n = 4$ yields four equalities of the above type to be satisfied and thus we are left with two free parameters fixed by the choice of $\tau_{1,2}$, while for $n = 5$, the dimension of the group is ten and one has nine orthogonality conditions, which leaves only one free parameter, so in this case the values of τ_1 and τ_2 are dependant, similarly to the gimbal lock setting. This technique may be seen as an extension to the well-known Wigner decomposition initially introduced for $\mathsf{SO}^+(3,1)$ in [8], where it has been pointed out that each proper Lorentz transformation may be written as a product $\mathscr{R}_2\Lambda_z\mathscr{R}_1$ with $\mathscr{R}_{1,2}$ being pure rotations and Λ_z - a boost in the z,t-plane. Wigner's procedure may be realized by composing a conjugation with $\mathscr{R}_1^{-1}$ that brings the z-axis in the desired position and a boost followed by another rotation with $\mathscr{R}_2\mathscr{R}_1$. Certainly, if the compound transformation is a plane boost itself, it is simply conjugated to Λ_z with a properly chosen rotation. This construction, however, is not available for $n > 5$ as one needs more plane factors in the decomposition. For more clarity here we focus on the case $n = 4$ only. The vectorial parametrization of $\mathsf{SO}^+(3,1)$ shows for instance that the boost-rotation decomposition $\mathbf{c} = \langle i\tau_2\hat{\mathbf{u}}_2, \tau_1\hat{\mathbf{u}}_1\rangle$ with $\hat{\mathbf{u}}_{1,2} \in \mathbb{S}^2$ leads to a system of six coupled quadratic equations for $\tau_{1,2}$ that is inconsistent unless the four conditions (21) for the invariant planes determined by $\hat{\mathbf{u}}_{1,2}$ are satisfied, leaving us with two parameters $\tau_{1,2}$ determined by (23). Moreover, these conditions are quite restrictive on $\hat{\mathbf{u}}_{1,2}$, e.g. $\hat{\mathbf{u}}_1 \perp \hat{\mathbf{u}}_2$ turns out to be possible only if $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$, while $\hat{\mathbf{u}}_1 \parallel \hat{\mathbf{u}}_2$ implies also $\boldsymbol{\alpha} \parallel \boldsymbol{\beta}$. Both these cases are considered in [6] and explicit solutions are provided too. Here we discuss one more specific setting related to the isotropic singularity. Firstly, recall that the subspace $\mathbf{c}_\circ^\perp$ (with $\mathbf{c}_\circ^2 = 0$) is closed under the composition law (4). Since our group has complex dimension three, this space is spanned by two vectors $\mathbf{c}_\circ^\perp = \{\mathbf{c}_\circ, \boldsymbol{\kappa}\}$, the latter of which may be chosen real and unit, e.g. $\boldsymbol{\kappa} = |\boldsymbol{\alpha}_\circ|^{-2}\boldsymbol{\alpha}_\circ \times \boldsymbol{\beta}_\circ$. Then, each $\mathbf{c} \in \mathbf{c}_\circ^\perp$ has the representation $\mathbf{c} = \mu\mathbf{c}_\circ + \nu\boldsymbol{\kappa}$. As it is not hard to see, $\boldsymbol{\kappa}^\times$ acts on vectors in the isotropic direction $\mathbf{c}_\circ$ as multiplication with $-i$ and from the composition law (4) one has

$$\mathbf{c} = \mu\mathbf{c}_\circ + \nu\boldsymbol{\kappa} = \langle \tilde{\mu}_+\mathbf{c}_\circ, \nu\boldsymbol{\kappa}\rangle = \langle \nu\boldsymbol{\kappa}, \tilde{\mu}_-\mathbf{c}_\circ\rangle, \qquad \tilde{\mu}_\pm = \frac{\mu}{1 \pm i\nu}. \tag{27}$$

Since $\mathbf{c}_\circ$ is isotropic, $\tilde{\mu}_\pm\mathbf{c}_\circ$ still determines a plane transformation, which is not true for $\nu\boldsymbol{\kappa}$. However, as it has been shown in [6] based on formula (4), for each $\boldsymbol{\kappa} \in \mathbb{R}^3$ and $\nu \in \mathbb{C}$ one has

$$\nu\boldsymbol{\kappa} = \langle i\lambda_\pm\boldsymbol{\kappa}, \rho_\pm\boldsymbol{\kappa}\rangle, \qquad \rho_\pm = \frac{|\nu|^2 - 1 \pm \sqrt{\Delta}}{2\Re\nu}, \qquad \lambda_\pm = \frac{2\Im\nu}{|\nu|^2 + 1 \pm \sqrt{\Delta}} \tag{28}$$

with $\Delta = (|\nu|^2 - 1)^2 + (2\Re\nu)^2$. Moreover, each vector $\mathbf{c} \in \mathbb{C}^3$ belongs to some plane $\mathbf{c}_\circ^\perp$ determined by a null vector $\mathbf{c}_\circ \in \ker\left(\mathbf{c}^\times \pm i\sqrt{\mathbf{c}^2}\right)$. Thus, we may factorize each SO_3 transformation into a pair of coaxial boost and rotation, and a parabolic transformation in the complex plane $\{\mathbf{c}_\circ\}$ fixed by $\boldsymbol{\kappa}^\perp$. Then, the six parameters of $\mathsf{SO}^+(3,1)$ may be encoded in the rotation angle, boost rapidity, the coordinates of $\boldsymbol{\kappa} \in \mathbb{S}^2$ plus the "intensity and polarization" of $\tilde{\mu}_\pm\mathbf{c}_\circ$. Consider for example the vector-parameter $\mathbf{c} = (3+2i, 3i-2, 2+i)^t$ and use the above technique to determine $\mathbf{c}_\circ = (1, i, 0)^t$ and $\boldsymbol{\kappa} = (0, 0, 1)^t$ that yields $\mu = 3+2i$ and $\nu = 2+i$. Then, with the aid of (27) and (28), one obtains the factorizations $\mathbf{c} = \langle (1 - 3i/2)\,\mathbf{c}_\circ, i(3 \mp 2\sqrt{2})\,\boldsymbol{\kappa}, (1 \pm \sqrt{2})\,\boldsymbol{\kappa}\rangle$ and $\mathbf{c} = \langle (1 \pm \sqrt{2})\,\boldsymbol{\kappa}, i(3 \mp 2\sqrt{2})\,\boldsymbol{\kappa}, (1/4 + 5i/4)\,\mathbf{c}_\circ\rangle$ with commutative boost and rotation parts.

Euler-Type Decompositions. The covariant technique developed in [5] allows for a straightforward extension to higher-dimensional plane (pseudo-)rotations. Here we shall consider the complex version of the construction as it unifies the two cases. The vector algebra in $\mathscr{P}_\beta$ may be obtained with the aid of the commutator and the Killing form in the standard matrix representation of $\mathfrak{so}_n$, in which the compound rotation $\mathscr{R}(\Theta)$ generated by $\Theta \in \mathfrak{so}_3$ via the Cayley transform acts by conjugations $\mathrm{Ad}_{\mathscr{R}} X = \mathscr{R} X \mathscr{R}^{-1}$. Then, having chosen a set of decomposition planes $\{\theta_k\} \leftrightarrow \pm\hat{\Theta}_k$, it is straightforward to determine the matrix entries (11) along with

$$\omega_1 = \left([\hat{\Theta}_1, \hat{\Theta}_2], \mathrm{Ad}^{-1}_{\mathscr{R}} \hat{\Theta}_3\right), \qquad \omega_3 = \left(\mathrm{Ad}_{\mathscr{R}} \hat{\Theta}_1, [\hat{\Theta}_2, \hat{\Theta}_3]\right).$$

Denoting for convenience also $\omega_2 = \omega$, we express the unknown scalar parameters in the form

$$\tau_i^{\pm} = \frac{\sigma_i}{\omega_i \pm \sqrt{\Delta}}, \qquad \sigma_i = \varepsilon_{ijk}(g_{jk} - r_{jk}), \quad j > k \tag{29}$$

where ε_{ijk} stands for the Levi-Civita symbol and the discriminant is given explicitly as

$$\Delta = \begin{vmatrix} g_{11} & g_{12} & r_{31} \\ g_{21} & g_{22} & g_{23} \\ r_{31} & g_{32} & g_{33} \end{vmatrix}$$

that needs to be non-negative on the real forms. Moreover, in the compact case the generalized Euler angles are expressed as $\phi_k^{\pm} = 2\arctan\tau_k^{\pm}$ (provided the $\hat{\mathbf{c}}_i$'s are unit). For example, in the classical Bryan setting of three distinct mutually normal planes one has the solution in the form

$$\phi_1^{\pm} = 2\arctan\frac{r_{32}}{r_{33} \pm \sqrt{1-r_{31}^2}}, \quad \phi_2^{\pm} = -2\arctan\frac{r_{31}}{1 \pm \sqrt{1-r_{31}^2}}, \quad \phi_3^{\pm} = 2\arctan\frac{r_{21}}{r_{11} \pm \sqrt{1-r_{31}^2}}.$$

Note, however, that formula (29) does not cover the cases of gimbal lock and light-cone singularities. The former one appears whenever the following condition is satisfied

$$\hat{\Theta}_3 = \pm\mathrm{Ad}_{\mathscr{R}} \hat{\Theta}_1 \tag{30}$$

which corresponds to a topological singularity leading to the degenerate solution

$$\tau' = \frac{r_{22} - g_{22}}{([\hat{\Theta}_1, \hat{\Theta}_2], \mathrm{Ad}^{-1}_{\mathscr{R}} \hat{\Theta}_2)}, \qquad \tau_2 = \frac{r_{11} - g_{11}}{([\hat{\Theta}_1, \hat{\Theta}_2], \mathrm{Ad}_{\mathscr{R}} \hat{\Theta}_1)} \tag{31}$$

that is given equivalently by formula (23). Here τ' is the scalar parameter of $\langle \tau_1 \hat{\Theta}_1, \pm\tau_3 \hat{\Theta}_1 \rangle$, e.g. in the compact case we have $\phi_1 \pm \phi_3 = 2\tan\tau'$. In the non-compact setting one needs to consider also the isotropic singularity that has been studied in [5, 6] for $\mathsf{SO}_3 \cong \mathsf{SO}^+(3,1)$ and its real form $\mathsf{SO}(2,1)$. In order to regularize in the former case one takes scalar products with $\bar{\mathbf{c}}_\circ$ instead of $\mathbf{c}_\circ$ in the isotropic direction (cf. [6]). As for the n-dimensional embedding, we let $\bar{\Theta}_\circ$ be the $\mathfrak{so}_n$-representation of $\bar{\mathbf{c}}_\circ^{\times}$ (the complex conjugate of the corresponding null vector). Then, denoting also $\Psi^\circ = (\Psi, \bar{\Theta}_\circ)$ for an arbitrary $\Psi \in \mathfrak{so}_n$, it is not hard to obtain $\tau_3 = 0$ and

$$\tau_1 = -\frac{[\hat{\Theta}_2, \Theta]^\circ}{[\hat{\Theta}_1, \hat{\Theta}_2]^\circ + (\hat{\Theta}_2, \Theta)\hat{\Theta}_1^\circ - (\hat{\Theta}_1, \hat{\Theta}_2)\Theta^\circ}, \qquad \tau_2 = \frac{[\hat{\Theta}_1, \Theta]^\circ}{[\hat{\Theta}_1, \hat{\Theta}_2]^\circ + (\hat{\Theta}_1, \Theta)\hat{\Theta}_2^\circ - (\hat{\Theta}_1, \hat{\Theta}_2)\Theta^\circ}.$$

Note, however, that although the orthogonality conditions (15) hold in $\mathbf{c}_\circ^{\perp}$, one needs also (21) to be satisfied for each pair of vectors $\hat{\mathbf{c}}_1 \in \{\hat{\Theta}_1\}^{\perp}$ and $\hat{\mathbf{c}}_2 \in \{\hat{\Theta}_2\}^{\perp}$ to guarantee real solutions. Then, the above result may be interpreted, based on formula (13), as a $\mathsf{SO}(3,1)$ decomposition. In order to avoid both types of singularities it is sufficient (although not necessary) to make sure there is no common eigenvector for the $\hat{\mathbf{c}}_i^{\times}$'s and either $\hat{\mathbf{c}}_1$ or $\hat{\mathbf{c}}_3$ is null, like in the Iwasawa decomposition of $\mathsf{SL}(2,\mathbb{R})$. Another option is to enforce the isotropic setting via (27) and (28). Various other factorization techniques have been proposed lately, mostly in terms of real forms (see [6] and the reference therein), each one allowing for a convenient n-dimensional extension.

NUMERICAL EXAMPLES

Consider a proper Lorentz transformation generated via formula (14) by the vector-parameter $\mathbf{c} = (1+2i, 2i-1, -1)^t$ and let us choose $\hat{\mathbf{c}}_1 = (1-2i, 1+2i, 1)^t$. This choice clearly satisfies

$$\Im\,\mathbf{c}^2 = \Im\,\hat{\mathbf{c}}_1^2 = \Im\,\mathbf{c}\cdot\hat{\mathbf{c}}_1 = 0.$$

Therefore, formula (26) yields real solutions for the scalar parameters $\tau_1 = \frac{1}{5}$, $\tau_2 = \frac{1}{2}$ and with

$$\hat{\mathbf{c}}_2 = \hat{\mathbf{c}}_1 \times \mathscr{R}(\mathbf{c})\,\hat{\mathbf{c}}_1 = (2+4i, 4i-2, 0)^t$$

where $\mathscr{R}(\mathbf{c})$ is obtained via formula (9), we easily factorize $\Lambda = \Lambda_2\Lambda_1$ with

$$\Lambda = \begin{pmatrix} 0 & 2 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -2 & 3 \end{pmatrix}, \qquad \Lambda_1 = \frac{1}{5}\begin{pmatrix} 6 & -4 & 3 & -6 \\ 1 & 6 & -2 & 4 \\ -2 & 3 & 4 & 2 \\ -4 & 6 & -2 & 9 \end{pmatrix}, \qquad \Lambda_2 = \frac{1}{5}\begin{pmatrix} 1 & 6 & -2 & 4 \\ 6 & 1 & -2 & 4 \\ 2 & 2 & -9 & 8 \\ 4 & 4 & -8 & 11 \end{pmatrix}.$$

The normal planes are given as $\Sigma = \{(1,-1,-1,0)^t, (2,-2,4,3)^t\}$ for Λ and for $\Lambda_{1,2}$ we have

$$\Sigma_1 = \{(1,1,1,0)^t, (-2,-2,4,3)^t\}, \qquad \Sigma_2 = \{(1,-1,0,0)^t, (0,0,2,1)^t\}.$$

The common normal direction ℓ that determines the embedding $\mathscr{P}_\beta$ of the corresponding little group may be obtained as an intersection of the above planes: $\ell = \Sigma_1 \cap \Sigma_2 = \{(0,0,2,1)^t\}$ that is space-like, hence, in this particular case we have a representation of the little group $\mathsf{SO}(2,1)$. Note also that the condition (24) is satisfied for $\hat{\Theta}_{1,2}$ and Θ corresponding respectively to $\hat{\mathbf{c}}_{1,2}$ and $\mathbf{c}$ via (13), so the solutions for $\tau_{1,2}$ may be obtained equivalently by means of formula (25).

Our next example concerns the group $\mathsf{SO}^+(4,1)$ playing a central role in various physical theories. Its Lie algebra is isomorphic to $\mathfrak{sp}_{1,1}$ and may thus be realized in terms of 2×2 quaternion matrices. Now, let us choose a pseudo-rotation in the plane $\{\theta\}$ that is given in matrix terms as

$$\Theta = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \xrightarrow{\text{Cay}} \quad \Lambda(\Theta) = \frac{1}{7}\begin{pmatrix} 1 & -8 & -2 & 4 & -6 \\ -4 & -3 & -6 & -2 & -4 \\ 2 & -2 & 3 & -6 & 2 \\ 8 & 6 & -2 & -3 & 8 \\ 6 & 8 & 2 & -4 & 13 \end{pmatrix} \in \mathsf{SO}(4,1)$$

and pick another plane, e.g. $\{\theta'\} = \{(1,1,0,-1,1)^t, (0,2,1,2,0)^t\}$ intersecting $\{\theta\}$ along a line. Then, the common subspace normal to both $\{\theta\}$ and $\{\theta'\}$ is spanned by the vectors $\mathbf{u} = (1,0,0,0,-1)^t$ and $\mathbf{v} = (2,-1,0,1,0)^t$. Its orthogonal complement in $\mathbb{R}^{4,1}$ defines the irreducible space V_β of the corresponding three-dimensional subgroup, in which we can choose a basis $\mathbf{a}_1 = \hat{\mathbf{e}}_3$, $\mathbf{a}_2 = \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_4$ and $\mathbf{a}_3 = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_5$ that yields the sub-basis of $\mathfrak{so}_{4,1}$ generators

$$\hat{\Theta}_1 = \eta^{-1}(\mathbf{a}_2 \wedge \mathbf{a}_3), \quad \hat{\Theta}_2 = \eta^{-1}(\mathbf{a}_3 \wedge \mathbf{a}_1), \quad \hat{\Theta}_3 = \eta^{-1}(\mathbf{a}_1 \wedge \mathbf{a}_2), \quad \eta = \mathrm{diag}(1,1,1,1,-1)$$

one may use in the factorization $\Lambda(\Theta) = \Lambda(\tau_3\hat{\Theta}_3)\Lambda(\tau_2\hat{\Theta}_2)\Lambda(\tau_1\hat{\Theta}_1)$. Then, formula (29) gives

$$\tau_1^{\pm} = \frac{4}{9 \pm \sqrt{33}}, \qquad \tau_2^{\pm} = \frac{2}{-7 \pm \sqrt{33}}, \qquad \tau_1^{\pm} = \frac{2}{3 \mp \sqrt{33}}.$$

One may also consider the classical Euler decomposition with $\hat{\Theta}_3 = \hat{\Theta}_1$, in which case

$$\tau_1^{\pm} = \frac{1 \mp \sqrt{10}}{6}, \qquad \tau_1^{\pm} = \pm\frac{1}{\sqrt{10}}, \qquad \tau_3^{\pm} = -\frac{3 \pm \sqrt{10}}{2}.$$

Note that this method provides exact solutions (in algebraic functions) for any finite dimension.

FINAL REMARKS

In the present text most of the constructions are given in terms of matrix representations due to their overwhelming popularity among the scientific community allowing the above results to be available for testing and straightforward implementation. However, it is far more natural to use geometric algebra instead. For example, the vector-parameter originates in the set $\mathbb{H}^*$ of invertible even grade elements of the Clifford algebra of $\mathbb{R}^3$ and may thus be derived explicitly

$$q \in \mathbb{H}^* \longrightarrow \mathbf{c} = \frac{\langle q \rangle_2}{\langle q \rangle_0} \tag{32}$$

with the aid of grade projectors $\langle \cdot \rangle_k$. Then, the composition law (4) is given in this context as

$$\langle \mathbf{c}_k, \mathbf{c}_{k-1}, \dots \mathbf{c}_1 \rangle = \frac{\langle q_k q_{k-1} \cdots q_1 \rangle_2}{\langle q_k q_{k-1} \cdots q_1 \rangle_0} \tag{33}$$

for an arbitrary number of factors. This construction appears naturally superposing the geometric product in $\mathbb{H}$ with spin projection. Furthermore, it makes associativity obvious, which is not the case with the usual vectorial description. Another advantage is that the above expressions remain valid if we increase the dimension of the ambient space with no need of artificial extensions as the transition from (4) to (8). The only restriction is that the elements q_i close a multiplicative subgroup isomorphic to $\mathbb{H}^*$. In this setting it is obviously more natural to interpret vector-parameters as (weighted) planes rather than lines and make use of (8) and (10). Basic operations such as projection, reflection, intersection, rejection or rotation of subspaces are expressed quite naturally in the language of geometric algebra as well (see [13] for details).

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A SYSTEMATIC CONSTRUCTION OF REPRESENTATIONS OF QUATERNIONIC TYPE

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The appearance of quaternions in representation theory is usually by accident, both poorly understood yet thought to be deeply meaningful [5, 6]. Examples are the representations of root systems in 3D and 4D in terms of (pure) quaternions, as well as representations of quaternionic type of the polyhedral and other groups.

I have demystified the former in previous work, showing that 4D root systems are induced from 3D root systems in complete generality; in particular, the 3D root systems can only be realised in terms of pure quaternions when the corresponding reflection group contains the inversion [1, 2, 3]. The emergence of the 4D root systems hinges on the Clifford algebra of 3D, or rather its even subalgebra. The spinors describing rotations in 3D (from even products of the reflection generating root vectors) can be endowed with a well-known 4D Euclidean distance. The axioms of a root system are then easily satisfied: firstly, via the Euclidean metric a 3D spinor group can be treated as a collection of vectors in 4D; secondly, Clifford spinorial methods provide a double cover of rotations such that the 4D collection of vectors contains the negatives of those vectors, and thirdly with respect to the 4D Euclidean distance and using some other properties of spinors, the collection of 4D vectors is closed under reflections amongst themselves. These representations of root systems in terms of quaternions therefore systematically hinge purely on the geometry of 3D and the accident that the even subalgebra, i.e. the spinors, is quaternionic.

Here I discuss systematically the representation theory of the intimately related polyhedral groups [4, 2]. The 8D Clifford algebra of 3D space allows one to easily define various representations: the trivial one, parity, the usual 3×3 rotation matrix representation acting on a 3D vector achieved by sandwiching a vector with the corresponding versor, or the 8×8 representation of the group elements as reshuffling the multivector components in the whole 8D algebra under multivector multiplication. The representations we will focus on, however, are those defined by acting with any spinor on another general spinor. This reshuffles the components of the general spinor, which can also be expressed as a 4×4 matrix acting on the spinor in column format. It is not surprising that again because of the quaternionic nature of the even subalgebra the representations of quaternionic type of the polyhedral groups arise naturally and geometrically in a systematic way. Both observations therefore demystify quaternionic phenomena as consequences of 3D geometry, and in particular the 'mysterious deep significance' is simply provided by their spinorial nature – both simple yet underappreciated.

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A CONFORMAL GEOMETRIC ALGEBRA CONSTRUCTION OF THE MODULAR GROUP

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I will discuss a new construction of the modular group [1]. My interest in the modular group stems from recent Moonshine observations, relating string theory, finite simple groups and (mock) modular forms [2, 3]. The modular group is a subgroup of the 2D conformal group and I use the conformal model in Geometric Algebra with the corresponding Clifford realisations of the conformal group [4, 5] to construct a new realisation of the modular group. The double cover of the modular group is the braid group; of course this Clifford construction in fact provides a double cover of the conformal and thus modular groups, and we will discuss the relation with the braid group.

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THE SIMPLEST NONASSOCIATIVE GENERALIZATION OF SUPERSYMMETRY

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ABSTRACT. Nonassociative generalization of supersymmetry is suggested. 3- and 4-point associators for supersymmetric generators are considered. On the basis of zero Jacobiators for three supersymmetric generators, we have obtained the simplest form of 3-point associators. The connection between 3- and 4-point associators is considered. On the basis of this connection, 4-point associators are obtained. The Jacobiators for the product of four supersymmetric generators are calculated. We discuss the possible physical meaning of numerical coefficients presented on the right-hand sides of associators. The possible connection between supersymmetry, hidden variables, and nonassociativity is discussed.

1. INTRODUCTION

Supersymmetry is a well-defined mathematical theory that probably has the application in physics: it is a branch of particle physics that, using a proposed type of spacetime symmetry, relates two basic classes of elementary particles – bosons and fermions. In the standard approach supersymmetric generators are associative and anticommutative. Here we want to consider a nonassociative generalization of supersymmetry. We offer some 3-point associators for supersymmetric generators Q_a and $Q_{\dot{a}}$. Using some relation between 3- and 4-point associators, we obtain some limitations on the possible form of 4-point associators. On the basis of these limitations, we offer 4-point associators for supersymmetric generators.

Nonassociative structures appear in: (a) quantum chromodynamics [1]; (b) Maxwell and Dirac equations [2, 3]; (c) string theory [4]; (d) nonassociative quantum mechanics [5, 6]. For other ways of introducing nonassociative structures into physics, see the monographs [7, 8].

Here we would like to introduce nonassociative structures into supersymmetry and to discuss the physical consequences of such a procedure.

2. THE SIMPLEST 3-POINT ASSOCIATORS

Recall the definition of associator

$$[A,B,C] = (AB)C - A(BC), \tag{1}$$

where A,B,C are nonassociative quantities. Now we want to demonstrate that it is possible to introduce the generalization of supersymmetry other than that given in Refs. [9, 10]. Let us define the following 3-point associators:

$$[Q_a, Q_b, Q_c] = \alpha_1 Q_a \varepsilon_{bc} + \alpha_2 Q_b \varepsilon_{ac} + \alpha_3 Q_c \varepsilon_{ab}, \tag{2}$$

$$[Q_{\dot{a}}, Q_b, Q_c] = \beta_1 Q_{\dot{a}} \varepsilon_{bc}, \tag{3}$$

$$\left[Q_a, Q_{\dot{b}}, Q_c\right] = \beta_2 Q_{\dot{b}} \varepsilon_{ac}, \tag{4}$$

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$$[Q_a, Q_b, Q_{\dot c}] = \beta_3 Q_{\dot c}\varepsilon_{ab}, \tag{5}$$

$$[Q_a, Q_{\dot b}, Q_{\dot c}] = \gamma_1 Q_a \varepsilon_{\dot b\dot c}, \tag{6}$$

$$[Q_{\dot a}, Q_b, Q_{\dot c}] = \gamma_2 Q_b \varepsilon_{\dot a\dot c}, \tag{7}$$

$$[Q_{\dot a}, Q_{\dot b}, Q_c] = \gamma_3 Q_c \varepsilon_{\dot a\dot b}, \tag{8}$$

$$[Q_{\dot a}, Q_{\dot b}, Q_{\dot c}] = \delta_1 Q_{\dot a}\varepsilon_{\dot b\dot c} + \delta_2 Q_{\dot b}\varepsilon_{\dot a\dot c} + \delta_3 Q_{\dot c}\varepsilon_{\dot a\dot b}, \tag{9}$$

where

$$\varepsilon_{\dot a\dot b} = \varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{10}$$

$$\varepsilon^{ab} = \varepsilon^{\dot a\dot b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

The dotted indices are lowered and raised by using $\varepsilon_{\dot a\dot b}, \varepsilon^{\dot a\dot b}$, and the undotted – by $\varepsilon_{ab}, \varepsilon^{ab}$.

In order to introduce some limitations on the nonassociative algebra, we want to calculate the Jacobiator

$$J(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y] = [x,y,z] + [y,z,x] + [z,x,y] - [x,z,y] - [y,x,z] - [z,y,x], \tag{12}$$

where x, y, z are either $Q_{a,\dot a}$ or their product. Let us calculate Jacobiators

$$J(Q_a, Q_b, Q_c) = 2(\alpha_1 - \alpha_2 + \alpha_3) Q_a \varepsilon_{bc}, \tag{13}$$

$$J(Q_{\dot a}, Q_b, Q_c) = 2(\beta_1 - \beta_2 + \beta_3) Q_{\dot a}\varepsilon_{bc}, \tag{14}$$

$$J(Q_a, Q_{\dot b}, Q_c) = 2(\beta_1 - \beta_2 + \beta_3) Q_{\dot b}\varepsilon_{ca}, \tag{15}$$

$$J(Q_a, Q_b, Q_{\dot c}) = 2(\beta_1 - \beta_2 + \beta_3) Q_{\dot c}\varepsilon_{ab}, \tag{16}$$

$$J(Q_a, Q_{\dot b}, Q_{\dot c}) = 2(\gamma_1 - \gamma_2 + \gamma_3) Q_a \varepsilon_{\dot b\dot c}, \tag{17}$$

$$J(Q_{\dot a}, Q_b, Q_{\dot c}) = 2(\gamma_1 - \gamma_2 + \gamma_3) Q_b \varepsilon_{\dot c\dot a}, \tag{18}$$

$$J(Q_{\dot a}, Q_{\dot b}, Q_c) = 2(\gamma_1 - \gamma_2 + \gamma_3) Q_c \varepsilon_{\dot a\dot b}, \tag{19}$$

$$J(Q_{\dot a}, Q_{\dot b}, Q_{\dot c}) = 2(\delta_1 - \delta_2 + \delta_3) Q_{\dot a}\varepsilon_{\dot b\dot c}. \tag{20}$$

To have zero Jacobiators, we have to have the following equations for the parameters $\alpha, \beta, \gamma, \delta$:

$$\alpha_1 - \alpha_2 + \alpha_3 = \beta_1 - \beta_2 + \beta_3 = \gamma_1 - \gamma_2 + \gamma_3 = \delta_1 - \delta_2 + \delta_3 = 0. \tag{21}$$

Thus, we see that perhaps the most natural and the simplest choice of the parameters $\alpha, \beta, \gamma, \delta$ is

$$\alpha_2 = \beta_2 = \gamma_2 = \delta_2 = 0, \tag{22}$$

$$|\alpha_1| = |\alpha_3| = |\beta_1| = |\beta_3| = |\gamma_1| = |\gamma_3| = |\delta_1| = |\delta_3| = \frac{\hbar}{\ell_0}, \tag{23}$$

where the factor $\hbar/\ell_0$ is introduced for equalizing the dimensionality of the left-hand sides and right-hand sides of equations (2)-(9); ℓ_0 is some characteristic length. Thus

$$\alpha_1 = -\alpha_3 = \frac{\hbar}{\ell_0}\zeta_1, \tag{24}$$

$$\beta_1 = -\beta_3 = \frac{\hbar}{\ell_0}\zeta_2, \tag{25}$$

$$\gamma_1 = -\gamma_3 = \frac{\hbar}{\ell_0}\zeta_3, \tag{26}$$

$$\delta_1 = -\delta_3 = \frac{\hbar}{\ell_0}\zeta_4, \tag{27}$$

(28)

where

$$\zeta_{1,2,3,4} = \begin{cases} \text{either} & \pm 1 \\ \text{or} & \pm i \end{cases}. \tag{29}$$

Thus we have the following 3-point associators

$$[Q_a, Q_b, Q_c] = \frac{\hbar}{\ell_0}\zeta_1 \left(Q_a \varepsilon_{bc} - Q_c \varepsilon_{ab}\right), \tag{30}$$

$$[Q_{\dot a}, Q_b, Q_c] = \frac{\hbar}{\ell_0}\zeta_2 Q_{\dot a}\varepsilon_{bc}, \tag{31}$$

$$\left[Q_a, Q_{\dot b}, Q_c\right] = 0, \tag{32}$$

$$[Q_a, Q_b, Q_{\dot c}] = -\frac{\hbar}{\ell_0}\zeta_2 Q_{\dot c}\varepsilon_{ab}, \tag{33}$$

$$\left[Q_a, Q_{\dot b}, Q_{\dot c}\right] = \frac{\hbar}{\ell_0}\zeta_3 Q_a \varepsilon_{\dot b \dot c}, \tag{34}$$

$$[Q_{\dot a}, Q_b, Q_{\dot c}] = 0, \tag{35}$$

$$\left[Q_{\dot a}, Q_{\dot b}, Q_c\right] = -\frac{\hbar}{\ell_0}\zeta_3 Q_c \varepsilon_{\dot a \dot b}, \tag{36}$$

$$\left[Q_{\dot a}, Q_{\dot b}, Q_{\dot c}\right] = \frac{\hbar}{\ell_0}\zeta_4 \left(Q_{\dot a}\varepsilon_{\dot b \dot c} - Q_{\dot c}\varepsilon_{\dot a \dot b}\right) \tag{37}$$

3. 4-POINT ASSOCIATORS

First, we want to consider the connection between 3- and 4-point associators. For example, the 4-point associator $[Q_x Q_y, Q_z, Q_w]$ is

$$[Q_x Q_y, Q_z, Q_w] = \left(\left(Q_x Q_y\right) Q_z\right) Q_w - \left(Q_x Q_y\right)\left(Q_z Q_w\right), \tag{38}$$

where x, y, z, w are any combinations of dotted and undotted indices. The last term on the right-hand side of equation (38) is $\left(Q_x Q_y\right)\left(Q_z Q_w\right)$ and it cannot be obtained by multiplying of any 3-point associator from the left-hand sides of (2)-(9) neither by Q_a nor $Q_{\dot a}$. Nevertheless, there is the relation between the 3- and 4-point associators:

$$[Q_x Q_y, Q_z, Q_w] - [Q_x, Q_y Q_z, Q_w] + [Q_x, Q_y, Q_z Q_w] = [Q_x, Q_y, Q_z]\, Q_w - Q_x\, [Q_y, Q_z, Q_w]. \tag{39}$$

3.1. **4-point associators without dots.**

Let us assume the following 4-point associators

$$\begin{aligned}[Q_a Q_b, Q_c, Q_d] = {} & \rho_{1,1} Q_a Q_b \varepsilon_{cd} + \rho_{2,1} Q_a Q_c \varepsilon_{bd} + \rho_{3,1} Q_a Q_d \varepsilon_{bc} + \rho_{4,1} Q_b Q_c \varepsilon_{ad} + \\ & \rho_{5,1} Q_b Q_d \varepsilon_{ac} + \rho_{6,1} Q_c Q_d \varepsilon_{ab},\end{aligned} \tag{40}$$

$$\begin{aligned}[Q_a, Q_b Q_c, Q_d] = {} & \mu_{1,1} Q_a Q_b \varepsilon_{cd} + \mu_{2,1} Q_a Q_c \varepsilon_{bd} + \mu_{3,1} Q_a Q_d \varepsilon_{bc} + \mu_{4,1} Q_b Q_c \varepsilon_{ad} + \\ & \mu_{5,1} Q_b Q_d \varepsilon_{ac} + \mu_{6,1} Q_c Q_d \varepsilon_{ab},\end{aligned} \tag{41}$$

$$\begin{aligned}[Q_a, Q_b, Q_c Q_d] = {} & \nu_{1,1} Q_a Q_b \varepsilon_{cd} + \nu_{2,1} Q_a Q_c \varepsilon_{bd} + \nu_{3,1} Q_a Q_d \varepsilon_{bc} + \nu_{4,1} Q_b Q_c \varepsilon_{ad} + \\ & \nu_{5,1} Q_b Q_d \varepsilon_{ac} + \nu_{6,1} Q_c Q_d \varepsilon_{ab}.\end{aligned} \tag{42}$$

Then the relation (39) gives us the following relations between 3- and 4-point nonassociative structure constants

$$\rho_{1,1} - \mu_{1,1} + \nu_{1,1} = \alpha_1, \tag{43}$$

$$\rho_{2,1} - \mu_{2,1} + \nu_{2,1} = \alpha_2, \tag{44}$$

$$\rho_{3,1} - \mu_{3,1} + \nu_{3,1} = \alpha_1 + \alpha_3, \tag{45}$$

$$\rho_{4,1} - \mu_{4,1} + \nu_{4,1} = 0, \tag{46}$$

$$\rho_{5,1} - \mu_{5,1} + \nu_{5,1} = \alpha_2, \tag{47}$$

$$\rho_{6,1}-\mu_{6,1}+\nu_{6,1} = \alpha_3. \tag{48}$$

Perhaps the simplest limitations on the nonassociative structure constants $\rho_{i,1},\mu_{i,1},\nu_{i,1}$ are as follows:

$$\rho_{2,1} = \rho_{5,1}, \mu_{2,1}=\mu_{51}, \nu_{2,1}=\nu_{5,1}; \tag{49}$$

$$\rho_{3,1} = \rho_{1,1}+\rho_{6,1}, \mu_{3,1}=\mu_{1,1}+\mu_{6,1}, \nu_{3,1}=\nu_{1,1}+\nu_{6,1}; \tag{50}$$

$$\rho_{4,1}-\mu_{4,1}+\nu_{4,1} = 0. \tag{51}$$

Then the following solution of equations (49)-(51) that is compatible with (22) and (24) can be found:

$$\rho_{1,1} = -\rho_{6,1}=\nu_{1,1}=-\nu_{6,1}=\frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1; \tag{52}$$

$$\mu_{i,1} = 0, i=1,2,\dots 6, \tag{53}$$

$$\rho_{2,1} = \rho_{3,1}=\rho_{4,1}=\rho_{5,1}=\nu_{2,1}=\nu_{3,1}=\nu_{4,1}=\nu_{5,1}=0. \tag{54}$$

Finally, 4-point associators are

$$[Q_aQ_b,Q_c,Q_d] = \frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\left(Q_aQ_b\varepsilon_{cd}-Q_cQ_d\varepsilon_{ab}\right), \tag{55}$$

$$[Q_a,Q_bQ_c,Q_d] = 0, \tag{56}$$

$$[Q_a,Q_b,Q_cQ_d] = \frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\left(Q_aQ_b\varepsilon_{cd}-Q_cQ_d\varepsilon_{ab}\right). \tag{57}$$

One can immediately check that the Jacobiator

$$J\left(Q_aQ_b,Q_c,Q_d\right)=0. \tag{58}$$

3.2. **4-point associators with one dot.** In this section we consider 4-point associators with one dot moving from the left on the right side of associator.

3.2.1. *1-st case.* We seek 4-point associators with one dot as follows:

$$\begin{aligned}[Q_{\dot a}Q_b,Q_c,Q_d] &= \rho_{1,2}\varepsilon_{cd}Q_{\dot a}Q_b+\rho_{2,2}\varepsilon_{bd}Q_{\dot a}Q_c+\rho_{3,2}\varepsilon_{bc}Q_{\dot a}Q_d+\\ &\rho_{4,2}\left(Q_bQ_c,x_{d\dot a}\right)+\rho_{5,2}\left(Q_bQ_d,x_{c\dot a}\right)+\rho_{6,2}\left(Q_cQ_d,x_{b\dot a}\right),\end{aligned} \tag{59}$$

$$\begin{aligned}[Q_{\dot a},Q_bQ_c,Q_d] &= \mu_{1,2}\varepsilon_{cd}Q_{\dot a}Q_b+\mu_{2,2}\varepsilon_{bd}Q_{\dot a}Q_c+\mu_{3,2}\varepsilon_{bc}Q_{\dot a}Q_d+\\ &\mu_{4,2}\left(Q_bQ_c,x_{d\dot a}\right)+\mu_{5,2}\left(Q_bQ_d,x_{c\dot a}\right)+\mu_{6,2}\left(Q_cQ_d,x_{b\dot a}\right),\end{aligned} \tag{60}$$

$$\begin{aligned}[Q_{\dot a},Q_b,Q_cQ_d] &= \nu_{1,2}\varepsilon_{cd}Q_{\dot a}Q_b+\nu_{2,2}\varepsilon_{bd}Q_{\dot a}Q_c+\nu_{3,2}\varepsilon_{bc}Q_{\dot a}Q_d+\\ &\nu_{4,2}\left(Q_bQ_c,x_{d\dot a}\right)+\nu_{5,2}\left(Q_bQ_d,x_{c\dot a}\right)+\nu_{6,2}\left(Q_cQ_d,x_{b\dot a}\right).\end{aligned} \tag{61}$$

3.2.2. *2-nd case.* We seek 4-point associators with one dot as follows:

$$\begin{aligned}\left[Q_aQ_{\dot b},Q_c,Q_d\right] &= \rho_{1,3}\varepsilon_{cd}Q_aQ_{\dot b}+\rho_{2,3}\left(Q_aQ_c,x_{d\dot b}\right)+\rho_{3,3}\left(Q_aQ_d,x_{c\dot b}\right)+\\ &\rho_{4,3}\varepsilon_{ad}Q_{\dot b}Q_c+\rho_{5,3}\varepsilon_{ac}Q_{\dot b}Q_d+\rho_{6,3}\left(Q_cQ_d,x_{a\dot b}\right),\end{aligned} \tag{62}$$

$$\begin{aligned}\left[Q_a,Q_{\dot b}Q_c,Q_d\right] &= \mu_{1,3}\varepsilon_{cd}Q_aQ_{\dot b}+\mu_{2,3}\left(Q_aQ_c,x_{d\dot b}\right)+\mu_{3,3}\left(Q_aQ_d,x_{c\dot b}\right)+\\ &\mu_{4,3}\varepsilon_{ad}Q_{\dot b}Q_c+\mu_{5,3}\varepsilon_{ac}Q_{\dot b}Q_d+\mu_{6,3}\left(Q_cQ_d,x_{a\dot b}\right),\end{aligned} \tag{63}$$

$$\begin{aligned}\left[Q_a,Q_{\dot b},Q_cQ_d\right] &= \nu_{1,3}\varepsilon_{cd}Q_aQ_{\dot b}+\nu_{2,3}\left(Q_aQ_c,x_{d\dot b}\right)+\nu_{3,3}\left(Q_aQ_d,x_{c\dot b}\right)+\\ &\nu_{4,3}\varepsilon_{ad}Q_{\dot b}Q_c+\nu_{5,3}\varepsilon_{ac}Q_{\dot b}Q_d+\nu_{6,3}\left(Q_cQ_d,x_{a\dot b}\right).\end{aligned} \tag{64}$$

3.2.3. *3-rd case.* We seek 4-point associators with one dot as follows:

$$
\begin{aligned}
[Q_aQ_b,Q_{\dot c},Q_d] &= \rho_{1,4}\,(Q_aQ_b,x_{d\dot c})+\rho_{2,4}\varepsilon_{bd}Q_aQ_{\dot c}+\rho_{3,4}\,(Q_aQ_d,x_{b\dot c})+ \\
&\quad \rho_{4,4}\varepsilon_{ad}Q_bQ_{\dot c}+\rho_{5,4}\,(Q_bQ_d,x_{a\dot c})+\rho_{6,4}\varepsilon_{ab}Q_{\dot c}Q_d, \qquad (65)\\
[Q_a,Q_bQ_{\dot c},Q_d] &= \mu_{1,4}\,(Q_aQ_b,x_{d\dot c})+\mu_{2,4}\varepsilon_{bd}Q_aQ_{\dot c}+\mu_{3,4}\,(Q_aQ_d,x_{b\dot c})+ \\
&\quad \mu_{4,4}\varepsilon_{ad}Q_bQ_{\dot c}+\mu_{5,4}\,(Q_bQ_d,x_{a\dot c})+\mu_{6,4}\varepsilon_{ab}Q_{\dot c}Q_d, \qquad (66)\\
[Q_a,Q_b,Q_{\dot c}Q_d] &= \nu_{1,4}\,(Q_aQ_b,x_{d\dot c})+\nu_{2,4}\varepsilon_{bd}Q_aQ_{\dot c}+\nu_{3,4}\,(Q_aQ_d,x_{b\dot c})+ \\
&\quad \nu_{4,4}\varepsilon_{ad}Q_bQ_{\dot c}+\nu_{5,4}\,(Q_bQ_d,x_{a\dot c})+\nu_{6,4}\varepsilon_{ab}Q_{\dot c}Q_d. \qquad (67)
\end{aligned}
$$

3.2.4. *4-th case.* We seek 4-point associators with one dot as follows:

$$
\begin{aligned}
\left[Q_aQ_b,Q_cQ_{\dot d}\right] &= \rho_{1,5}\left(Q_aQ_b,x_{c\dot d}\right)+\rho_{2,5}\left(Q_aQ_c,x_{b\dot d}\right)+\rho_{3,5}\varepsilon_{bc}Q_aQ_{\dot d}+ \\
&\quad \rho_{4,5}\left(Q_bQ_c,x_{a\dot d}\right)+\rho_{5,5}\varepsilon_{ac}Q_bQ_{\dot d}+\rho_{6,5}\varepsilon_{ab}Q_cQ_{\dot d}, \qquad (68)\\
\left[Q_a,Q_bQ_c,Q_{\dot d}\right] &= \mu_{1,5}\left(Q_aQ_b,x_{c\dot d}\right)+\mu_{2,5}\left(Q_aQ_c,x_{b\dot d}\right)+\mu_{3,5}\varepsilon_{bc}Q_aQ_{\dot d}+ \\
&\quad \mu_{4,5}\left(Q_bQ_c,x_{a\dot d}\right)+\mu_{5,5}\varepsilon_{ac}Q_bQ_{\dot d}+\mu_{6,5}\varepsilon_{ab}Q_cQ_{\dot d}, \qquad (69)\\
\left[Q_a,Q_b,Q_cQ_{\dot d}\right] &= \nu_{1,5}\left(Q_aQ_b,x_{c\dot d}\right)+\nu_{2,5}\left(Q_aQ_c,x_{b\dot d}\right)+\nu_{3,5}\varepsilon_{bc}Q_aQ_{\dot d}+ \\
&\quad \nu_{4,5}\left(Q_bQ_c,x_{a\dot d}\right)+\nu_{5,5}\varepsilon_{ac}Q_bQ_{\dot d}+\nu_{6,5}\varepsilon_{ab}Q_cQ_{\dot d}. \qquad (70)
\end{aligned}
$$

3.2.5. *Final form of the associators. Jacobiators.* Taking into account the relation (39) (as in Section 3.1), we obtain the following 4-point associators with one dot

$$
\begin{aligned}
[Q_{\dot a}Q_b,Q_c,Q_d] &= -\frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{cd}Q_{\dot a}Q_b+\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{6,2}\,(x_{b\dot a},Q_cQ_d)\,, \qquad (71)\\
[Q_{\dot a},Q_bQ_c,Q_d] &= 0, \qquad (72)\\
[Q_{\dot a},Q_b,Q_cQ_d] &= -\frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{cd}Q_{\dot a}Q_b-\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{6,2}\,(x_{b\dot a},Q_cQ_d)\,, \qquad (73)\\
\left[Q_aQ_{\dot b},Q_c,Q_d\right] &= \frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{cd}Q_aQ_{\dot b}+\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{6,3}\left(x_{a\dot b},Q_cQ_d\right), \qquad (74)\\
\left[Q_a,Q_{\dot b}Q_c,Q_d\right] &= 0, \qquad (75)\\
\left[Q_a,Q_{\dot b},Q_cQ_d\right] &= \frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{cd}Q_aQ_{\dot b}-\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{6,3}\left(x_{a\dot b},Q_cQ_d\right), \qquad (76)\\
[Q_aQ_b,Q_{\dot c},Q_d] &= \frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{ab}Q_{\dot c}Q_d+\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{1,4}\,(x_{d\dot c},Q_aQ_b)\,, \qquad (77)\\
[Q_a,Q_bQ_{\dot c},Q_d] &= 0, \qquad (78)\\
[Q_a,Q_b,Q_{\dot c}Q_d] &= \frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{ab}Q_{\dot c}Q_d-\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{1,4}\,(x_{d\dot c},Q_aQ_b)\,, \qquad (79)\\
\left[Q_aQ_b,Q_c,Q_{\dot d}\right] &= -\frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{ab}Q_cQ_{\dot d}+\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{4,5}\left(x_{c\dot d},Q_aQ_b\right), \qquad (80)\\
\left[Q_a,Q_bQ_c,Q_{\dot d}\right] &= 0, \qquad (81)\\
\left[Q_a,Q_b,Q_cQ_{\dot d}\right] &= -\frac{1}{2}\frac{\hbar}{\ell_0}\zeta_1\varepsilon_{ab}Q_cQ_{\dot d}-\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{4,5}\left(x_{c\dot d},Q_aQ_b\right). \qquad (82)
\end{aligned}
$$

Considering the relation (39), we have found that

$$\zeta_2=-\zeta_1, \qquad (83)$$

and we set

$$\left|\tilde\rho_{6,2}\right|=\left|\tilde\rho_{6,3}\right|=\left|\tilde\rho_{1,4}\right|=\left|\tilde\rho_{4,5}\right|=1. \qquad (84)$$

One can check that the Jacobiators are

$$J(Q_{\dot a}Q_b, Q_c, Q_d) = 0 \tag{85}$$

$$J\left(Q_aQ_{\dot b}, Q_c, Q_d\right) = 0 \tag{86}$$

$$J(Q_aQ_b, Q_{\dot c}, Q_d) = 0 \tag{87}$$

$$J\left(Q_aQ_b, Q_c, Q_{\dot d}\right) = 0 \tag{88}$$

if

$$\tilde\rho_{1,4} = \tilde\rho_{6,2}, \tag{89}$$

$$\tilde\rho_{4,5} = \tilde\rho_{6,3}. \tag{90}$$

3.3. **4-point associators with two dots.** In this section we consider the case with two dots and with the different locations.

3.3.1. *1-st case.* We seek 4-point associators with two dots as follows:

$$\begin{aligned}\left[Q_{\dot a}Q_{\dot b}, Q_c, Q_d\right] &= \rho_{1,6}Q_{\dot a}Q_{\dot b}\varepsilon_{cd} + \rho_{2,6}\left(Q_{\dot a}Q_c, y_{d\dot b}\right) + \rho_{3,6}\left(Q_{\dot a}Q_d, y_{c\dot b}\right) + \\ &\rho_{4,6}\left(Q_{\dot b}Q_c, y_{d\dot a}\right) + \rho_{5,6}\left(Q_{\dot b}Q_d, y_{c\dot a}\right) + \rho_{6,6}Q_cQ_d\varepsilon_{\dot a\dot b} + \\ &\rho_{7,6}M_{\left(c\dot a,d\dot b\right)} + \rho_{8,6}M_{\left(c\dot b,d\dot a\right)},\end{aligned} \tag{91}$$

$$\begin{aligned}\left[Q_{\dot a}, Q_{\dot b}Q_c, Q_d\right] &= \mu_{1,6}Q_{\dot a}Q_{\dot b}\varepsilon_{cd} + \mu_{2,6}\left(Q_{\dot a}Q_c, y_{d\dot b}\right) + \mu_{3,6}\left(Q_{\dot a}Q_d, y_{c\dot b}\right) + \\ &\mu_{4,6}\left(Q_{\dot b}Q_c, y_{d\dot a}\right) + \mu_{5,6}\left(Q_{\dot b}Q_d, y_{c\dot a}\right) + \mu_{6,6}Q_cQ_d\varepsilon_{\dot a\dot b} + \\ &\mu_{7,6}M_{\left(c\dot a,d\dot b\right)} + \mu_{8,6}M_{\left(c\dot b,d\dot a\right)},\end{aligned} \tag{92}$$

$$\begin{aligned}\left[Q_{\dot a}, Q_{\dot b}, Q_cQ_d\right] &= \nu_{1,6}Q_{\dot a}Q_{\dot b}\varepsilon_{cd} + \nu_{2,6}\left(Q_{\dot a}Q_c, y_{d\dot b}\right) + \nu_{3,6}\left(Q_{\dot a}Q_d, y_{c\dot b}\right) + \\ &\nu_{4,6}\left(Q_{\dot b}Q_c, y_{d\dot a}\right) + \nu_{5,6}\left(Q_{\dot b}Q_d, y_{c\dot a}\right) + \nu_{6,6}Q_cQ_d\varepsilon_{\dot a\dot b} + \\ &\nu_{7,6}M_{\left(c\dot a,d\dot b\right)} + \nu_{8,6}M_{\left(c\dot b,d\dot a\right)}.\end{aligned} \tag{93}$$

3.3.2. *2-nd case.* We seek 4-point associators with two dots as follows:

$$\begin{aligned}\left[Q_{\dot a}Q_b, Q_{\dot c}, Q_d\right] &= \rho_{1,7}\left(Q_{\dot a}Q_b, x_{d\dot c}\right) + \rho_{2,7}Q_{\dot a}Q_{\dot c}\varepsilon_{bd} + \rho_{3,7}\left(Q_{\dot a}Q_d, x_{b\dot c}\right) + \\ &\rho_{4,7}\left(Q_bQ_{\dot c}, x_{d\dot a}\right) + \rho_{5,7}Q_bQ_d\varepsilon_{\dot a\dot c} + \rho_{6,7}\left(Q_{\dot c}Q_d, x_{b\dot a}\right) + \\ &\rho_{7,7}M_{(b\dot a,d\dot c)} + \rho_{8,7}M_{(b\dot c,d\dot a)},\end{aligned} \tag{94}$$

$$\begin{aligned}\left[Q_{\dot a}, Q_bQ_{\dot c}, Q_d\right] &= \mu_{1,7}\left(Q_{\dot a}Q_b, x_{d\dot c}\right) + \mu_{2,7}Q_{\dot a}Q_{\dot c}\varepsilon_{bd} + \mu_{3,7}\left(Q_{\dot a}Q_d, x_{b\dot c}\right) + \\ &\mu_{4,7}\left(Q_bQ_{\dot c}, x_{d\dot a}\right) + \mu_{5,7}Q_bQ_d\varepsilon_{\dot a\dot c} + \mu_{6,7}\left(Q_{\dot c}Q_d, x_{b\dot a}\right) + \\ &\mu_{7,7}M_{(b\dot a,d\dot c)} + \mu_{8,7}M_{(b\dot c,d\dot a)},\end{aligned} \tag{95}$$

$$\begin{aligned}\left[Q_{\dot a}, Q_b, Q_{\dot c}Q_d\right] &= \nu_{1,7}\left(Q_{\dot a}Q_b, x_{d\dot c}\right) + \nu_{2,7}Q_{\dot a}Q_{\dot c}\varepsilon_{bd} + \nu_{3,7}\left(Q_{\dot a}Q_d, x_{b\dot c}\right) + \\ &\nu_{4,7}\left(Q_bQ_{\dot c}, x_{d\dot a}\right) + \nu_{5,7}Q_bQ_d\varepsilon_{\dot a\dot c} + \nu_{6,7}\left(Q_{\dot c}Q_d, x_{b\dot a}\right) + \\ &\nu_{7,7}M_{(b\dot a,d\dot c)} + \nu_{8,7}M_{(b\dot c,d\dot a)}.\end{aligned} \tag{96}$$

3.3.3. *3-rd case.* We seek 4-point associators with two dots as follows:

$$\begin{aligned}\left[Q_{\dot a}Q_b, Q_c, Q_{\dot d}\right] &= \rho_{1,8}\left(Q_{\dot a}Q_b, x_{c\dot d}\right) + \rho_{2,8}\left(Q_{\dot a}Q_c, x_{b\dot d}\right) + \rho_{3,8}Q_{\dot a}Q_{\dot d}\varepsilon_{bc} + \\ &\rho_{4,8}\left(Q_bQ_c\varepsilon_{\dot a\dot d}\right) + \rho_{5,8}\left(Q_bQ_{\dot d}, x_{c\dot a}\right) + \rho_{6,8}\left(Q_cQ_{\dot d}, x_{b\dot a}\right) + \rho_{7,8}M_{\left(b\dot a,c\dot d\right)} + \\ &\rho_{8,8}M_{\left(b\dot d,c\dot a\right)},\end{aligned} \tag{97}$$

$$\begin{aligned}\left[Q_{\dot a}, Q_bQ_c, Q_{\dot d}\right] &= \mu_{1,8}\left(Q_{\dot a}Q_b, x_{c\dot d}\right) + \mu_{2,8}\left(Q_{\dot a}Q_c, x_{b\dot d}\right) + \mu_{3,8}Q_{\dot a}Q_{\dot d}\varepsilon_{bc} + \\ &\mu_{4,8}\left(Q_bQ_c\varepsilon_{\dot a\dot d}\right) + \mu_{5,8}\left(Q_bQ_{\dot d}, x_{c\dot a}\right) + \mu_{6,8}\left(Q_cQ_{\dot d}, x_{b\dot a}\right) +\end{aligned}$$

$$\mu_{7,8}M_{(b\dot{a},c\dot{d})}+\mu_{8,8}M_{(b\dot{d},c\dot{a})}, \tag{98}$$

$$\begin{aligned}\left[Q_{\dot{a}},Q_b,Q_cQ_{\dot{d}}\right] &= \nu_{1,8}\left(Q_{\dot{a}}Q_b,x_{c\dot{d}}\right)+\nu_{2,8}\left(Q_{\dot{a}}Q_c,x_{b\dot{d}}\right)+\nu_{3,8}Q_{\dot{a}}Q_d\varepsilon_{bc}+\\ &\nu_{4,8}\left(Q_bQ_c\varepsilon_{\dot{a}\dot{d}}\right)+\nu_{5,8}\left(Q_bQ_{\dot{d}},x_{c\dot{a}}\right)+\nu_{6,8}\left(Q_cQ_{\dot{d}},x_{b\dot{a}}\right)+\\ &\nu_{7,8}M_{(b\dot{a},c\dot{d})}+\nu_{8,8}M_{(b\dot{d},c\dot{a})}.\end{aligned} \tag{99}$$

3.3.4. *4-th case.* We seek 4-point associators with two dots as follows:

$$\begin{aligned}\left[Q_aQ_{\dot{b}},Q_{\dot{c}},Q_d\right] &= \rho_{1,9}\left(Q_aQ_{\dot{b}},x_{d\dot{c}}\right)+\rho_{2,9}\left(Q_aQ_{\dot{c}},x_{d\dot{b}}\right)+\rho_{3,9}\varepsilon_{\dot{b}\dot{c}}Q_aQ_d+\\ &\rho_{4,9}\varepsilon_{ad}Q_{\dot{b}}Q_{\dot{c}}+\rho_{5,9}\left(Q_{\dot{b}}Q_d,x_{a\dot{c}}\right)+\rho_{6,9}\left(Q_{\dot{c}}Q_d,x_{a\dot{b}}\right)+\\ &\rho_{7,9}M_{(a\dot{b},d\dot{c})}+\rho_{8,9}M_{(a\dot{c},d\dot{b})},\end{aligned} \tag{100}$$

$$\begin{aligned}\left[Q_a,Q_{\dot{b}}Q_{\dot{c}},Q_d\right] &= \mu_{1,9}\left(Q_aQ_{\dot{b}},x_{d\dot{c}}\right)+\mu_{2,9}\left(Q_aQ_{\dot{c}},x_{d\dot{b}}\right)+\mu_{3,9}\varepsilon_{\dot{b}\dot{c}}Q_aQ_d+\\ &\mu_{4,9}\varepsilon_{ad}Q_{\dot{b}}Q_{\dot{c}}+\mu_{5,9}\left(Q_{\dot{b}}Q_d,x_{a\dot{c}}\right)+\mu_{6,9}\left(Q_{\dot{c}}Q_d,x_{a\dot{b}}\right)+\\ &\mu_{7,9}M_{(a\dot{b},d\dot{c})}+\mu_{8,9}M_{(a\dot{c},d\dot{b})},\end{aligned} \tag{101}$$

$$\begin{aligned}\left[Q_a,Q_{\dot{b}},Q_{\dot{c}}Q_d\right] &= \nu_{1,9}\left(Q_aQ_{\dot{b}},x_{d\dot{c}}\right)+\nu_{2,9}\left(Q_aQ_{\dot{c}},x_{d\dot{b}}\right)+\nu_{3,9}\varepsilon_{\dot{b}\dot{c}}Q_aQ_d+\\ &\nu_{4,9}\varepsilon_{ad}Q_{\dot{b}}Q_{\dot{c}}+\nu_{5,9}\left(Q_{\dot{b}}Q_d,x_{a\dot{c}}\right)+\nu_{6,9}\left(Q_{\dot{c}}Q_d,x_{a\dot{b}}\right)+\\ &\nu_{7,9}M_{(a\dot{b},d\dot{c})}+\nu_{8,9}M_{(a\dot{c},d\dot{b})}.\end{aligned} \tag{102}$$

3.3.5. *5-th case.* We seek 4-point associators with two dots as follows:

$$\begin{aligned}\left[Q_aQ_{\dot{b}},Q_c,Q_{\dot{d}}\right] &= \rho_{1,10}\left(Q_aQ_{\dot{b}},x_{c\dot{d}}\right)+\rho_{2,10}\varepsilon_{b\dot{d}}Q_aQ_c+\rho_{3,10}\left(Q_aQ_{\dot{d}},x_{c\dot{b}}\right)+\\ &\rho_{4,10}\left(Q_{\dot{b}}Q_c,x_{a\dot{d}}\right)+\rho_{5,10}\varepsilon_{ac}Q_{\dot{b}}Q_{\dot{d}}+\rho_{6,10}\left(Q_cQ_{\dot{d}},x_{a\dot{b}}\right)+\\ &\rho_{7,10}M_{(a\dot{b},c\dot{d})}+\rho_{8,10}M_{(a\dot{d},c\dot{b})},\end{aligned} \tag{103}$$

$$\begin{aligned}\left[Q_a,Q_{\dot{b}}Q_c,Q_{\dot{d}}\right] &= \mu_{1,10}\left(Q_aQ_{\dot{b}},x_{c\dot{d}}\right)+\mu_{2,10}\varepsilon_{b\dot{d}}Q_aQ_c+\mu_{3,10}\left(Q_aQ_{\dot{d}},x_{c\dot{b}}\right)+\\ &\mu_{4,10}\left(Q_{\dot{b}}Q_c,x_{a\dot{d}}\right)+\mu_{5,10}\varepsilon_{ac}Q_{\dot{b}}Q_{\dot{d}}+\mu_{6,10}\left(Q_cQ_{\dot{d}},x_{a\dot{b}}\right)+\\ &\mu_{7,10}M_{(a\dot{b},c\dot{d})}+\mu_{8,10}M_{(a\dot{d},c\dot{b})},\end{aligned} \tag{104}$$

$$\begin{aligned}\left[Q_a,Q_{\dot{b}},Q_cQ_{\dot{d}}\right] &= \nu_{1,10}\left(Q_aQ_{\dot{b}},x_{c\dot{d}}\right)+\nu_{2,10}\varepsilon_{b\dot{d}}Q_aQ_c+\nu_{3,10}\left(Q_aQ_{\dot{d}},x_{c\dot{b}}\right)+\\ &\nu_{4,10}\left(Q_{\dot{b}}Q_c,x_{a\dot{d}}\right)+\nu_{5,10}\varepsilon_{ac}Q_{\dot{b}}Q_{\dot{d}}+\nu_{6,10}\left(Q_cQ_{\dot{d}},x_{a\dot{b}}\right)+\\ &\nu_{7,10}M_{(a\dot{b},c\dot{d})}+\nu_{8,10}M_{(a\dot{d},c\dot{b})}.\end{aligned} \tag{105}$$

3.3.6. *6-th case.* We seek 4-point associators with two dots as follows:

$$\begin{aligned}\left[Q_aQ_b,Q_{\dot{c}},Q_{\dot{d}}\right] &= \rho_{1,11}\varepsilon_{\dot{c}\dot{d}}Q_aQ_b+\rho_{2,11}\left(Q_aQ_{\dot{c}},x_{b\dot{d}}\right)+\rho_{3,11}\left(Q_aQ_{\dot{d}},x_{b\dot{c}}\right)+\\ &\rho_{4,11}\left(Q_bQ_{\dot{c}},x_{a\dot{d}}\right)+\rho_{5,11}\left(Q_bQ_{\dot{d}},x_{a\dot{c}}\right)+\rho_{6,11}\varepsilon_{ab}Q_{\dot{c}}Q_{\dot{d}}+\\ &\rho_{7,11}M_{(a\dot{c},b\dot{d})}+\rho_{8,11}M_{(a\dot{d},b\dot{c})},\end{aligned} \tag{106}$$

$$\begin{aligned}\left[Q_a,Q_bQ_{\dot{c}},Q_{\dot{d}}\right] &= \mu_{1,11}\varepsilon_{\dot{c}\dot{d}}Q_aQ_b+\mu_{2,11}\left(Q_aQ_{\dot{c}},x_{b\dot{d}}\right)+\mu_{3,11}\left(Q_aQ_{\dot{d}},x_{b\dot{c}}\right)+\\ &\mu_{4,11}\left(Q_bQ_{\dot{c}},x_{a\dot{d}}\right)+\mu_{5,11}\left(Q_bQ_{\dot{d}},x_{a\dot{c}}\right)+\mu_{6,11}\varepsilon_{ab}Q_{\dot{c}}Q_{\dot{d}}+\\ &\mu_{7,11}M_{(a\dot{c},b\dot{d})}+\mu_{8,11}M_{(a\dot{d},b\dot{c})},\end{aligned} \tag{107}$$

$$\begin{aligned}\left[Q_a,Q_b,Q_{\dot{c}}Q_{\dot{d}}\right] &= \nu_{1,11}\varepsilon_{\dot{c}\dot{d}}Q_aQ_b+\nu_{2,11}\left(Q_aQ_{\dot{c}},x_{b\dot{d}}\right)+\nu_{3,11}\left(Q_aQ_{\dot{d}},x_{b\dot{c}}\right)+\\ &\nu_{4,11}\left(Q_bQ_{\dot{c}},x_{a\dot{d}}\right)+\nu_{5,11}\left(Q_bQ_{\dot{d}},x_{a\dot{c}}\right)+\nu_{6,11}\varepsilon_{ab}Q_{\dot{c}}Q_{\dot{d}}+\\ &\nu_{7,11}M_{(a\dot{c},b\dot{d})}+\nu_{8,11}M_{(a\dot{d},b\dot{c})}.\end{aligned} \tag{108}$$

3.3.7. *Final form of the associators. Jacobiators.* Using the relation (39) for the connection between 3- and 4-point associators and the simplifications similar to those of used in section 3.1, we obtain

$$[Q_{\dot a}Q_{\dot b},Q_c,Q_d] = -\frac{\hbar}{\ell_0}\zeta_2 Q_{\dot a}Q_{\dot b}\varepsilon_{cd}+\frac{\hbar}{\ell_0}\tilde\rho_{7,6}M_{(c\dot a,d\dot b)}+\frac{\hbar}{\ell_0}\tilde\rho_{8,6}M_{(c\dot b,d\dot a)}, \tag{109}$$

$$[Q_{\dot a},Q_{\dot b}Q_c,Q_d] = \frac{\hbar}{\ell_0}\tilde\mu_{7,6}M_{(c\dot a,d\dot b)}+\frac{\hbar}{\ell_0}\tilde\mu_{8,6}M_{(c\dot b,d\dot a)}, \tag{110}$$

$$[Q_{\dot a},Q_{\dot b},Q_cQ_d] = -\frac{\hbar}{\ell_0}\zeta_3 Q_cQ_d\varepsilon_{\dot a\dot b}+\frac{\hbar}{\ell_0}\tilde\nu_{7,6}M_{(c\dot a,d\dot b)}+\frac{\hbar}{\ell_0}\tilde\nu_{8,6}M_{(c\dot b,d\dot a)}, \tag{111}$$

$$\begin{aligned}[Q_{\dot a}Q_b,Q_{\dot c},Q_d] &= \frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{1,7}\left(Q_{\dot a}Q_b,x_{d\dot c}\right)+\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{6,7}\left(x_{b\dot a},Q_{\dot c}Q_d\right)+\\ &\frac{\hbar}{\ell_0}\tilde\rho_{7,7}M_{(b\dot a,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\rho_{8,7}M_{(b\dot c,d\dot a)},\end{aligned} \tag{112}$$

$$[Q_{\dot a},Q_bQ_{\dot c},Q_d] = \frac{\hbar}{\ell_0}\tilde\mu_{7,7}M_{(b\dot a,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\mu_{8,7}M_{(b\dot c,d\dot a)}, \tag{113}$$

$$\begin{aligned}[Q_{\dot a},Q_b,Q_{\dot c}Q_d] &= -\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{1,7}\left(Q_{\dot a}Q_b,x_{d\dot c}\right)-\frac{1}{2}\frac{\hbar}{\ell_0}\tilde\rho_{6,7}\left(x_{b\dot a},Q_{\dot c}Q_d\right)+\\ &\frac{\hbar}{\ell_0}\tilde\nu_{7,7}M_{(b\dot a,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\nu_{7,7}M_{(b\dot a,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\nu_{7,7}M_{(b\dot a,d\dot c)}+\\ &\frac{\hbar}{\ell_0}\tilde\nu_{8,7}M_{(b\dot c,d\dot a)},\end{aligned} \tag{114, 115}$$

$$\begin{aligned}[Q_{\dot a}Q_b,Q_c,Q_{\dot d}] &= \frac{\hbar}{\ell_0}\tilde\rho_{1,8}\left(Q_{\dot a}Q_b,x_{c\dot d}\right)+\frac{\hbar}{\ell_0}\zeta_2 Q_{\dot a}Q_{\dot d}\varepsilon_{bc}+\frac{\hbar}{\ell_0}\tilde\rho_{4,8}\varepsilon_{\dot a\dot d}Q_bQ_c+\\ &\frac{\hbar}{\ell_0}\tilde\rho_{6,8}\left(Q_cQ_{\dot d},x_{b\dot a}\right)+\frac{\hbar}{\ell_0}\tilde\rho_{7,8}M_{(b\dot a,c\dot d)}+\frac{\hbar}{\ell_0}\tilde\rho_{8,8}M_{(b\dot d,c\dot a)},\end{aligned} \tag{116}$$

$$[Q_{\dot a},Q_bQ_c,Q_{\dot d}] = -2\frac{\hbar}{\ell_0}\tilde\rho_{4,8}\varepsilon_{\dot a\dot d}Q_bQ_c+\tilde\mu_{7,8}M_{(b\dot a,c\dot d)}+\tilde\mu_{8,8}M_{(b\dot d,c\dot a)}, \tag{117}$$

$$\begin{aligned}[Q_{\dot a},Q_b,Q_cQ_{\dot d}] &= -\frac{\hbar}{\ell_0}\tilde\rho_{1,8}\left(Q_{\dot a}Q_b,x_{c\dot d}\right)+\frac{\hbar}{\ell_0}\zeta_2 Q_{\dot a}Q_{\dot d}\varepsilon_{bc}+\frac{\hbar}{\ell_0}\tilde\rho_{4,8}\varepsilon_{\dot a\dot d}Q_bQ_c-\\ &\frac{\hbar}{\ell_0}\tilde\rho_{6,8}\left(Q_cQ_{\dot d},x_{b\dot a}\right)+\frac{\hbar}{\ell_0}\tilde\nu_{7,8}M_{(b\dot a,c\dot d)}+\frac{\hbar}{\ell_0}\tilde\nu_{8,8}M_{(b\dot d,c\dot a)},\end{aligned} \tag{118}$$

$$\begin{aligned}[Q_aQ_{\dot b},Q_{\dot c},Q_d] &= \frac{\hbar}{\ell_0}\tilde\rho_{1,9}\left(Q_aQ_{\dot b},x_{d\dot c}\right)+\frac{\hbar}{\ell_0}\zeta_3\varepsilon_{b\dot c}Q_aQ_d+\frac{\hbar}{\ell_0}\tilde\rho_{4,9}\varepsilon_{ad}Q_{\dot b}Q_{\dot c}+\\ &\frac{\hbar}{\ell_0}\tilde\rho_{6,9}\left(Q_{\dot c}Q_d,x_{a\dot b}\right)+\frac{\hbar}{\ell_0}\tilde\rho_{7,9}M_{(a\dot b,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\rho_{8,9}M_{(a\dot c,d\dot b)},\end{aligned} \tag{119}$$

$$[Q_a,Q_{\dot b}Q_{\dot c},Q_d] = -2\frac{\hbar}{\ell_0}\tilde\rho_{4,9}\varepsilon_{ad}Q_{\dot b}Q_{\dot c}+\frac{\hbar}{\ell_0}\tilde\mu_{7,9}M_{(a\dot b,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\mu_{8,9}M_{(a\dot c,d\dot b)}, \tag{120}$$

$$\begin{aligned}[Q_a,Q_{\dot b},Q_{\dot c}Q_d] &= -\frac{\hbar}{\ell_0}\tilde\rho_{1,9}\left(Q_aQ_{\dot b},x_{d\dot c}\right)+\frac{\hbar}{\ell_0}\zeta_3\varepsilon_{b\dot c}Q_aQ_d+\frac{\hbar}{\ell_0}\tilde\rho_{4,9}\varepsilon_{ad}Q_{\dot b}Q_{\dot c}-\\ &\frac{\hbar}{\ell_0}\tilde\rho_{6,9}\left(Q_{\dot c}Q_d,x_{a\dot b}\right)+\frac{\hbar}{\ell_0}\tilde\nu_{7,9}M_{(a\dot b,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\nu_{8,9}M_{(a\dot c,d\dot b)},\end{aligned} \tag{121}$$

$$\begin{aligned}[Q_aQ_{\dot b},Q_c,Q_{\dot d}] &= \frac{\hbar}{\ell_0}\tilde\rho_{1,10}\left(Q_aQ_{\dot b},x_{c\dot d}\right)+\frac{\hbar}{\ell_0}\tilde\rho_{6,10}\left(x_{a\dot b},Q_cQ_{\dot d}\right)+\\ &\frac{\hbar}{\ell_0}\tilde\rho_{7,10}M_{(a\dot b,d\dot c)}+\frac{\hbar}{\ell_0}\tilde\rho_{8,10}M_{(a\dot c,d\dot b)},\end{aligned} \tag{122}$$

$$[Q_a, Q_{\dot b} Q_c, Q_{\dot d}] = \frac{\hbar}{\ell_0}\tilde\mu_{7,10} M_{(a\dot b, c\dot d)} + \frac{\hbar}{\ell_0}\tilde\mu_{8,10} M_{(a\dot d, c\dot b)}, \tag{123}$$

$$[Q_a, Q_{\dot b}, Q_c Q_{\dot d}] = -\frac{\hbar}{\ell_0}\tilde\rho_{1,10}\left(Q_a Q_{\dot b}, x_{c\dot d}\right) - \frac{\hbar}{\ell_0}\tilde\rho_{6,10}\left(x_{a\dot b}, Q_c Q_{\dot d}\right) +$$

$$\frac{\hbar}{\ell_0}\tilde\nu_{7,10} M_{(a\dot b, d\dot c)} + \frac{\hbar}{\ell_0}\tilde\nu_{8,10} M_{(a\dot c, d\dot b)}, \tag{124}$$

$$[Q_a Q_b, Q_{\dot c}, Q_{\dot d}] = -\frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot c\dot d} Q_a Q_b + \frac{\hbar}{\ell_0}\tilde\rho_{7,11} M_{(a\dot c, b\dot d)} + \frac{\hbar}{\ell_0}\tilde\rho_{8,11} M_{(a\dot d, b\dot c)}, \tag{125}$$

$$[Q_a, Q_b Q_{\dot c}, Q_{\dot d}] = \frac{\hbar}{\ell_0}\tilde\mu_{7,11} M_{(a\dot c, b\dot d)} + \frac{\hbar}{\ell_0}\tilde\mu_{8,11} M_{(a\dot d, b\dot c)}, \tag{126}$$

$$[Q_a, Q_b, Q_{\dot c} Q_{\dot d}] = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ab} Q_{\dot c} Q_{\dot d} + \frac{\hbar}{\ell_0}\tilde\nu_{7,11} M_{(a\dot c, b\dot d)} + \frac{\hbar}{\ell_0}\tilde\nu_{8,11} M_{(a\dot d, b\dot c)}, \tag{127}$$

where $(\ldots,\ldots)$ is either commutator or anticommutator, and with the limitations

$$\tilde\rho_{7,6} - \tilde\mu_{7,6} + \tilde\nu_{7,6} = 0, \tag{128}$$

$$\tilde\rho_{8,6} - \tilde\mu_{8,6} + \tilde\nu_{8,6} = 0, \tag{129}$$

$$\tilde\rho_{7,7} - \tilde\mu_{7,7} + \tilde\nu_{7,7} = 0, \tag{130}$$

$$\tilde\rho_{8,7} - \tilde\mu_{8,7} + \tilde\nu_{8,7} = 0, \tag{131}$$

$$\tilde\rho_{7,8} - \tilde\mu_{7,8} + \tilde\nu_{7,8} = 0, \tag{132}$$

$$\tilde\rho_{8,8} - \tilde\mu_{8,8} + \tilde\nu_{8,8} = 0, \tag{133}$$

$$\tilde\rho_{7,9} - \tilde\mu_{7,9} + \tilde\nu_{7,9} = 0, \tag{134}$$

$$\tilde\rho_{8,9} - \tilde\mu_{8,9} + \tilde\nu_{8,9} = 0, \tag{135}$$

$$\tilde\rho_{7,10} - \tilde\mu_{7,10} + \tilde\nu_{7,10} = 0, \tag{136}$$

$$\tilde\rho_{8,10} - \tilde\mu_{8,10} + \tilde\nu_{8,10} = 0, \tag{137}$$

$$\tilde\rho_{7,11} - \tilde\mu_{7,11} + \tilde\nu_{7,11} = 0, \tag{138}$$

$$\tilde\rho_{8,11} - \tilde\mu_{8,11} + \tilde\nu_{8,11} = 0. \tag{139}$$

One can check that the Jacobiators are (here we consider the simplest case $x_{a,\dot b} = M_{(a\dot b, c\dot d)} = 0$)

$$J\left(Q_{\dot a} Q_{\dot b}, Q_c, Q_d\right) = 0, \tag{140}$$

$$J\left(Q_{\dot a}, Q_{\dot b} Q_c, Q_d\right) = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{cd}\left[Q_{\dot a}, Q_{\dot b}\right] - \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot a\dot b}\left[Q_c, Q_d\right], \tag{141}$$

$$J\left(Q_{\dot a}, Q_{\dot b}, Q_c Q_d\right) = 0, \tag{142}$$

$$J\left(Q_{\dot a} Q_b, Q_{\dot c}, Q_d\right) = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{bd}\left[Q_{\dot a}, Q_{\dot c}\right] - \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot a\dot c}\left[Q_b, Q_d\right], \tag{143}$$

$$J\left(Q_{\dot a}, Q_b Q_{\dot c}, Q_d\right) = \frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{bd}\left[Q_{\dot a}, Q_{\dot c}\right] + \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot a\dot c}\left[Q_b, Q_d\right], \tag{144}$$

$$J\left(Q_{\dot a}, Q_b, Q_{\dot c} Q_d\right) = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{bd}\left[Q_{\dot a}, Q_{\dot c}\right] - \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot a\dot c}\left[Q_b, Q_d\right], \tag{145}$$

$$J\left(Q_{\dot a} Q_b, Q_c Q_{\dot d}\right) = \frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{bc}\left[Q_{\dot a}, Q_{\dot d}\right] + \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot a\dot d}\left[Q_b, Q_c\right], \tag{146}$$

$$J\left(Q_{\dot a}, Q_b Q_c, Q_{\dot d}\right) = 0, \tag{147}$$

$$J\left(Q_{\dot a}, Q_b, Q_c Q_{\dot d}\right) = \frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{bc}\left[Q_{\dot a}, Q_{\dot d}\right] + \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot a\dot d}\left[Q_b, Q_c\right], \tag{148}$$

$$J\left(Q_a Q_{\dot b}, Q_{\dot c}, Q_d\right) = \frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ad}\left[Q_{\dot b}, Q_{\dot c}\right] + \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot b\dot c}\left[Q_a, Q_d\right], \tag{149}$$

$$J\left(Q_a, Q_{\dot{b}} Q_{\dot{c}}, Q_d\right) = 0, \tag{150}$$

$$J\left(Q_a, Q_{\dot{b}}, Q_{\dot{c}} Q_d\right) = \frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ad}\left[Q_{\dot{b}}, Q_{\dot{c}}\right] + \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot{b}\dot{c}}\left[Q_a, Q_d\right], \tag{151}$$

$$J\left(Q_a Q_{\dot{b}}, Q_c, Q_{\dot{d}}\right) = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ac}\left[Q_{\dot{b}}, Q_{\dot{d}}\right] - \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot{b}\dot{d}}\left[Q_a, Q_c\right], \tag{152}$$

$$J\left(Q_a, Q_{\dot{b}} Q_c, Q_{\dot{d}}\right) = \frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ac}\left[Q_{\dot{b}}, Q_{\dot{d}}\right] + \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot{b}\dot{d}}\left[Q_a, Q_c\right], \tag{153}$$

$$J\left(Q_a, Q_{\dot{b}}, Q_c Q_{\dot{d}}\right) = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ac}\left[Q_{\dot{b}}, Q_{\dot{d}}\right] - \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot{b}\dot{d}}\left[Q_a, Q_c\right], \tag{154}$$

$$J\left(Q_a Q_b, Q_{\dot{c}} Q_{\dot{d}}\right) = 0, \tag{155}$$

$$J\left(Q_a, Q_b Q_{\dot{c}}, Q_{\dot{d}}\right) = -\frac{\hbar}{\ell_0}\zeta_2 \varepsilon_{ab}\left[Q_{\dot{c}}, Q_{\dot{d}}\right] - \frac{\hbar}{\ell_0}\zeta_3 \varepsilon_{\dot{c}\dot{d}}\left[Q_a, Q_b\right], \tag{156}$$

$$J\left(Q_a, Q_b, Q_{\dot{c}} Q_{\dot{d}}\right) = 0, \tag{157}$$

(158)

if

$$\tilde{\rho}_{4,8} = \zeta_3, \tag{159}$$

$$\tilde{\rho}_{4,9} = \zeta_2, \tag{160}$$

$$\left\{Q_{\dot{a}}, Q_{\dot{b}}\right\} = \{Q_a, Q_b\} = 0. \tag{161}$$

3.4. **4-point associators with two dots.** This case is identical to subsection 3.2 after replacing $\dot{a} \leftrightarrow a$.

4. The connection with supersymmetry

Now we want to pounce on supersymmetry. In this case the operators $Q_{a,\dot{b}}$ obey the following anticommutators

$$\{Q_a, Q_b\} = \left\{Q_{\dot{a}}, Q_{\dot{b}}\right\} = 0, \tag{162}$$

$$\{Q_a, Q_{\dot{a}}\} = Q_a Q_{\dot{a}} + Q_{\dot{a}} Q_a = 2\sigma^{\mu}_{a\dot{a}} P_{\mu}, \tag{163}$$

where the operator P can be a nonassociative generalization of standard accosiative operator $-i\hbar\partial_{\mu}$; the Pauli matrices $\sigma^{\mu}_{a\dot{a}}, \sigma_{\mu}^{a\dot{a}}$ are defined in the standard way

$$\sigma^{\mu}_{a\dot{a}} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \tag{164}$$

$$\sigma_{\mu}^{a\dot{a}} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \tag{165}$$

with the orthogonality relations for the Pauli matrices

$$\sigma_{\mu}^{a\dot{a}} \sigma^{\nu}_{a\dot{a}} = 2\delta^{\nu}_{\mu}, \quad \sigma_{\mu}^{a\dot{a}} \sigma^{\mu}_{b\dot{b}} = 2\delta^{a}_{b}\delta^{\dot{a}}_{\dot{b}}. \tag{166}$$

Let us note the following interesting relation which follows from (30)

$$\left(Q_a Q_b\right) Q_b = -\frac{\hbar}{\ell_0}\zeta_1 Q_b \varepsilon_{ab}. \tag{167}$$

Roughly speaking, one can say that this expression "destroys" sometimes $Q_a^2 = 0$ property of Grassmann numbers. This results in distinctions between supersymmetries based on associative and nonassociative generators. But this difference will be (in the dimensionless form) of the order of l_{Pl}/ℓ_0, where ℓ_0 is some characteristic length. For example, if $\ell_0 = \Lambda^{-1/2}$ (where Λ is the cosmological constant) then this difference will be $\approx 10^{-120}$.

5. SUPERSYMMETRY, HIDDEN VARIABLES, AND NONASSOCIATIVITY

In this section we want to consider a possible connection between supersymmetry, hidden variables, and nonassociativity.

First of all we want to remind what is the hidden variables theory. In Wiki [11] one can find the following definition of hidden variables theories "... hidden variable theories were espoused by some physicists who argued that the state of a physical system, as formulated by quantum mechanics, does not give a complete description for the system; i.e., that quantum mechanics is ultimately incomplete, and that a complete theory would provide descriptive categories to account for all observable behavior and thus avoid any indeterminism. ...

... In 1964, John Bell showed that if local hidden variables exist, certain experiments could be performed involving quantum entanglement where the result would satisfy a Bell inequality. ...

Physicists such as Alain Aspect [12] and Paul Kwiat [13] have performed experiments that have found violations of these inequalities up to 242 standard deviations[14] (excellent scientific certainty). This rules out local hidden variable theories.

... Gerard 't Hooft [14, 15] has disputed the validity of Bell's theorem on the basis of the superdeterminism loophole and proposed some ideas to construct local deterministic models."

We want to pay attention to what we talked about the associative observables. That is natural for physical quantities in quantum mechanics. But in Ref. [16] the possibility of consideration of nonassociative hidden variables is discussed. In this case these quantities are unobservable ones.

Let us consider what happens in our case. We have supersymmetric decomposition of (probably generalized) momentum operator (163). The constituents $Q_{a,\dot{a}}$ are unobservable according to the nonassociative properties (30)-(37) and (55)-(57). Following this way, we can say that we have unobservable nonassociative operators $Q_{a,\dot{a}}$ that are similar to hidden variables. The main difference compared with the standard hidden variables is unobservability of the nonassociative hidden-like variables.

6. DISCUSSION AND CONCLUSIONS

Thus we have considered a nonassociative generalization of supersymmetry. We have shown that: (a) one can choose such a form of 3-point associators that the corresponding Jacobiators are zero; (b) there is the relation between 3- and 4-point associators; (c) using these expressions, one can find 4-point associators.

We have seen that in all definitions of associators there is the Planck constant and some characteristic length ℓ_0. The presence of the Planck constant permits us to make natural assumptions that these associators can be regarded as a nonassociative generalization of the Heisenberg uncertainty principle. In this case the characteristic length ℓ_0 will be a new fundamental constant, and the corrections arising in this case have the order of $\hbar/\ell_0$. For example, if $\ell_0 \approx \Lambda^{-1/2}$ (where Λ is the cosmological constant) then the dimensionless corrections $\approx 10^{-120}$, i.e., are negligible. Instead of introducing a fundamental length ℓ_0, we can introduce a fundamental momentum $\mathscr{P}_0 = \hbar/\ell_0$. Physical consequences of introducing new fundamental quantities ℓ_0 or $\mathscr{P}_0$ (that are consequences of nonassociativity) are:

- There appears a minimum momentum $\mathscr{P}_0$.
- There appears a maximum length ℓ_0.

- The appearance of the maximum length ℓ_0 leads to the fact that the curvature is bounded below: $R_{min} \approx 1/\ell_0^2$.
- The minimum momentum $\mathscr{P}_0$ and the maximum length ℓ_0 are connected by the Heisenberg uncertainty principle: $\mathscr{P}_0\ell_0 \approx \hbar$.
- The experimental manifestation of possible nonaccosiativity can arise only for a physical phenomenon when either the momentum $\approx \mathscr{P}_0 \approx 10^{-60}\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-1}$ or on the scales $l \approx \ell_0 \approx \Lambda^{-1/2} \approx 10^{26}\mathrm{m}$ (if the fundamental length $\ell_0 \approx \Lambda^{-1/2}$).

We have also discussed a possible interpretation of nonassociative supersymmetric generators $Q_{a,\dot{a}}$ as hidden-like variables in quantum theory. The main idea here is that the nonassociativity leads to unobservability of these variables.

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GA AND GC APPLIED TO PRE-SPATIAL ARITHMETIC SCHEME TO ENHANCE MODELLING EFFECTIVENESS IN BIOPHYSICAL APPLICATIONS

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ABSTRACT. The classic vector analysis is unable to describe the Quantum Mechanics, Relativity and Quantum Field Theory so that an increasing attention to the geometric algebra (GA) and geometric calculus (GC) has been paid. In arbitrary multiscale complex system modelling the bottom-up discrete approach under the "discreteness hypothesis" is mandatory to achieve effective operative solutions. We present an exponential, pre-spatial arithmetic scheme ("all-powerful scheme") to overcome the limitation of the traditional top-down probabilistic modelling veil opacity. *CICT* (computational information conservation theory) new awareness of a discrete hyperbolic geometry subspace (reciprocal-space, RS) of coded heterogeneous hyperbolic structures, underlying the familiar **Q** Euclidean direct-space (DS) surface representation, shows that any natural number n in **N** has associated a specific, non-arbitrary exterior phase relationship that we have to take into account to full conserve overall system component information content by computation in DS. GA and GC unified mathematical language with *CICT* can offer a competitive and effective "Science 2.0" universal arbitrary multiscale computational framework for biophysical and advanced applications.

1. INTRODUCTION

In his classical book on quantum mechanical principles [1], Heisenberg delineates with his usual clarity two possible ways to build quantum mechanics (QM), both of which were coherent with the uncertainty principle:

- (a)- a space-time description tending to save the classical concepts by means of a statistical connection;
- (b)- adopting a pre-spatial mathematical scheme and a different concept of causality.

Quantum events only happen in interactions between systems, and the fact that a quantum event has happened is only true with respect to the systems involved in the interaction. The unique account of the state of the world of the classical theory is thus fractured into a multiplicity of accounts, one for each possible "observing" physical system. In ordinary QM we assume that the entropy is a vector that represents superposition state of the pure states. Any observer by a measure can see only one of the possible values of the superpose vector of entropies. Therefore, we can consider the entropy of a quantum system as a vector of the superposition of many different entropies whose values are conditioned by the observer. Any observer makes the system in superposition collapse to a classical value of entropy, i.e. it fixes classical information by a measurement. When the observer does not interact with a quantum system, the composed system given by the quantum "microsystem+observer" (and thus, somewhat the universe) is in an entangled state. When the observer interacts with the "microsystem" under consideration thus becoming an active element, the composed system given by the "quantum microsystem+observer" is in a pure state and follows the classical thermodynamics defined by one specific entropy and, in correspondence, the measured variable of the system under consideration takes the measured value. The superpose vector of the quantum entropies (whose

values are conditioned by the observer) is the fundamental entity that explains in what sense, when a quantum event happens, it happens only with respect a peculiar observing physical system. In quantum mechanics, one can say that a variable of a quantum system takes the value q because the vector given by a superposition of different Boltzmann entropies collapses to one specific entropy which corresponds to the value q of the variable under consideration: an interaction between the quantum system and an observer happens which produces the collapse of the vector of the superposed entropies into one specific entropy and the observer involved in this interaction measures just the value q of the variable under consideration. As concerning point (a), according to personal experience, geometric algebra (GA), geometric calculus (GC) and spacetime algebra (STA) [2, 3] are the current best tools to satisfy those requirements by reliable computational point of view. In the 1930s there was no cogent reason to choose the second way (b). Nowadays, we know that incredibly small groups of atoms, much too small to display exact statistical laws, do play a dominating role in the very orderly and lawful events within a living organism, so approach (a) cannot be followed. They have control of the observable large-scale characteristics of its functioning; and in all this very sharp and very strict biological laws are displayed. The great revelation of quantum theory, discovered by Max Planck in 1900, is that features of a discreteness were discovered in the Book of Nature at system microscale (nanoscale) level, in context in which anything other than continuity seemed to be absurd according to the views held until then at macroscale level. Quantum nonlocality offers a valid reason to explore a different approach where the nonlocal features of QM are not necessarily restricted to the entangled amplitudes, but are instead connected to a time-reversal approach, or some timeless background [4]. In fact, according to the principle of complementarity, there is also another representation of the uncertainty relation where the phase assumes a precise value; this representation which focuses on the wave-like features of the system cannot be assumed simultaneously with the particle representation [5, 6]. The relation between these two representations is expressed by the uncertainty relation, similar to the Heisenberg relation between position and momentum:

$$\Delta N \Delta \Phi \geq 1/2 \tag{1}$$

connecting the uncertainty of the number of quanta (particle structure of the system) ΔN and the uncertainty of the phase (which describes the rhythm of fluctuation of the system) $\Delta\Phi$. For point (a) approach, the difference between classical and quantum information is similar to the difference between Euclidean and non-Euclidean geometry in the parameter space determined by the quantum entropy. Traditionally, superposition states characteristic of quantum mechanics are determined by the deformation of the geometry of the background and are associated with a vector of superposed entropies. Here, we like to present a new approach where we suggest a non-Euclidean geometry emerging out of optimally combined phased generators as the source of a quantum information. As a sound example, in this paper, first we focus our attention on the relativistic electrodynamics inertial observer (IO) space-time split (STS) case, supported by spacetime algebra (STA) as spacetime representation system, to show how the full vector information content of rational correspondence at higher representation level (macroscale) for IO STS cannot be conserved by traditional approach (point (a)). Then, we present and discuss *CICT* (Computational Information Conservation Theory) solution by a pre-spatial arithmetic scheme to overcome and minimize the probabilistic modeling veil opacity (point (b)). We like to show how STS invariant physical quantities can be related to the variables employed by a specific interacting inertial observer (IO), with large information dissipation unavoidably, if specific computational awareness and care are not taken into account. To gain a better understanding about the operational compromise offered by the inertial observer spacetime splitting and the Space-Time Algebra (STA) main properties, Subsection 1.1 and Subsection 1.2 are propedeutic to later sections.

1.1. **Inertial Observer (IO) Spacetime Splitting.** Here we follow the line of thought reported in Hestenes' STA [3, 7], emphasizing the fundamental role of the IO. STA is built up from combinations of one time-like basis vector γ_0 and three orthogonal space-like vectors $\{\gamma_1, \gamma_2, \gamma_3\}$, under the multiplication rule [3]:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma\mu = 2\eta_{\mu\nu}, \tag{2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric with signature (+ - - -). We choose the signature here that will produce the STA that correctly contains the relative Euclidean 3-space (direct space, DS) as a proper subalgebra. Since there are one positive sign and three negative signs in this dot product, we say that it has a mixed signature (+ - - -) and denote the vector space of 4-vectors as $\mathscr{M}_{1,3}$ to make this signature explicit. So, all physical vector quantities in special relativity are postulated to be 4-vectors in $\mathscr{M}_{1,3}$ that satisfy the Minkowski metric. These vectors are geometric quantities that do not depend on the choice of reference frame, so they shall be called proper relativistic objects in what follows. Importantly, the Minkowski metric is qualitatively different from the standard Euclidean dot product since it does not produce a positive length. Indeed, the mixture of positive and negative signs can make the length of a vector positive, negative, or zero, which produces three qualitatively different classes of vectors. We call these classes of vectors timelike, spacelike, and lightlike, respectively. Physically speaking, STA is a complete and natural algebraic language for compactly describing physical quantities that satisfy the postulates of special relativity. Mathematically speaking, it is the largest associative algebra that can be constructed with the vector space of spacetime equipped with the Minkowski metric. It is an orthogonal Clifford algebra (CA) [8], which is a powerful tool that enables manifestly frame-independent and coordinate free manipulations of geometrically significant objects. By STA, Maxwell's equations reduce to a terse single equation:

$$\nabla F = \mu_0 c J, \tag{3}$$

where c is the speed of light, μ_0 is the parameter that international Standards Organizations call the "magnetic constant," commonly called "vacuum permeability," with the "defined" value $\mu_0 = 4\pi \cdot 10^{-7} H \cdot m^{-1}$ [9], with usual definition of variables [3]. The classical approach starts taking into consideration a generic electromagnetic field $\bar{F}$, described by Riemann–Silberstein vector [10]. The field $\bar{F}$ is defined as a complexified 3-dimensional vector field [11]. The value of $\bar{F}$ at an event is a bivector according to GA [2]. The field bivector $\bar{F}$ is the same for all observers; there is no question about how it transforms under a change of reference system. However, it is easily related to a representation of coupled electric and magnetic fields in a given inertial system (inertial observator, IO). GA can show how spacetime invariant physical quantities can be related to the variables employed by IO quite easily. An inertial system is determined by a single unit time-like vector γ_0, which can be regarded as tangent (with exact application point P unknown) to the worldline of an observer at rest in the system. According to STA this vector determines a "split" of ST into Space and Time, in other words a projection from 4-dimensional space into (3+1)-dimensional space, with a chosen reference frame by means of the following two operations [3]:

- a collapse of the chosen time axis, yielding a 3D space spanned by bivectors;
- a projection of the 4D space onto the chosen time axis, yielding a 1D space of scalars;

which are most simply expressed by the equation:

$$x\gamma_0 = t + \mathbf{x}, \tag{4}$$

where $t = x \cdot \gamma_0$ and $\mathbf{x} = x \wedge \gamma_0$. This is just an example of a projective transformation in CA. It is a linear mapping of each spacetime point $x(\gamma_1, \gamma_2, \gamma_3)$ into a scalar t designating a time and a vector $\mathbf{x}$ designating a position. The position vectors for all spacetime points compose a 3-dimensional Euclidean vector space $\mathbf{R^3}$. We denote the vectors in $\mathbf{R^3}$ with boldface type to

distinguish them from vectors in $R^{1,3}$. Let $R^{p,q}$ denote a vector space of dimension $n = p+q$ over the reals consisting of a p-dimensional subspace of vectors with positive signature orthogonal to a q-dimensional subspace of vectors with negative signature [3]. We argue that it is exactly at this point that "both/and" becomes "either/or" representation by usual Science 1.0 approach. In other words, quantum system representation is mapped to classical system additive representation with loss of overall system fundamental coherence information (the field $\bar{F}$ information phase connecting time component to space component coherently into their primitive spacetime continuum, at their original gauge level). Classically, the position vectors for all spacetime points compose a 3-dimensional Euclidean vector space $\mathbf{R^3}$. In this context, "orthogonality" of vectors means a vanishing inner product. Let $R_{p,q}$ denote the CA generated by the vectors of $R^{p,q}$ and let $R^k_{p,q}$ be the $\binom{n}{k}$-dimensional subspace of k-vectors in $R_{p,q}$. Elements of the $R^k_{p,q}$ 1-dimensional subspace are called "pseudoscalars" [2] of $R^{p,q}$ or $R_{p,q}$. An arbitrary element of $R_{p,q}$ is called a "multivector" or Clifford number [2]. The primary reference-frame-independent objects of special relativity (i.e., scalars, 4-vectors, bivectors/anti-symmetric rank-2 tensors, pseudo-4-vectors, and pseudoscalars) are unified as distinct grades of a single object known as a "multivector." An element of grade k geometrically corresponds to an oriented surface of dimension k (e.g., 4-vectors are line segments, while bivectors are oriented plane segments) [2]. For the important case of a vector space with Euclidean (positive) signature, it is convenient to introduce the abbreviations $R^n \equiv R^{n,0}$, $R_n \equiv R_{n,0}$ and $R^k_n \equiv R^k_{n,0}$. The equation $\mathbf{x} = x \wedge \gamma_0$ tells us that a vector in $\mathbf{R^3}$ is actually a bivector in $R^2_{1,3}$. In fact, $\mathbf{R^3}$ consists of the set of all bivectors in $R^2_{1,3}$ which have the vector γ_0 as a factor. Algebraically, this can be characterized as the set of all bivectors in $R^2_{1,3}$ which anticommute with γ_0. This determines a unique mapping of the electromagnetic bivector $\bar{F}$ into the GA R_3 of the given inertial system. In fact, the space-time split of $\bar{F}$ by γ_0 is obtained by decomposing $\bar{F}$ into a part:

$$\bar{E} = 1/2(\bar{F} - \gamma_0 \bar{F} \gamma_0), \tag{5}$$

which anticommutes with γ_0 and a part:

$$\bar{B}cI = 1/2(\bar{F} + \gamma_0 \bar{F} \gamma_0), \tag{6}$$

which commutes with γ_0, so we have:

$$\bar{F} = (\bar{E} + \bar{B}cI), \tag{7}$$

where $I = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_{0123}$ is the unit pseudoscalar. Some authors prefer to multiply the right hand side by an overall constant where $\sqrt{\varepsilon_0}$ is the permittivity of free space (vacuum):

$$\bar{F} = (\bar{E} + \bar{B}ci)\sqrt{\varepsilon_0} \tag{8}$$

or equivalently, by an algebraic point of view for representation systems [12], and more precisely, by a computational point of view for description systems [13]:

$$\bar{F} = (\bar{E} + \bar{B}ci)/\sqrt{\mu_0}. \tag{9}$$

$\bar{F}$ is defined as a complexified 3-dimensional vector field. The value of $\bar{F}$ at an event is a bivector (a "bivector" is the vector part of a biquaternion; the biquaternions are just the complexification of the (real) quaternions). Incidentally, by QM point of view, the quantum mechanical derivation that produces eq.9 also produces Maxwell's vacuum equations in an intriguing form that resembles the Dirac equation for massless spin-1 particles [14, 15, 16, 17, 18, 19, 20]. Treating the complex vector in eq.9 as a single-particle wave function in the first-quantized quantum mechanical sense produces consistent results, and suggests that this vector should be a more natural representation for classical e.m. fields as well. Remarkably, performing the standard second-quantization procedure using the complex Riemann—Silberstein wave function in eq.9 identically reproduces the correctly quantized e.m. field [16, 17, 18] in QFT (Quantum

Field Theory). Although $\bar{B}I$ commutes with γ_0, $\bar{B}$ must anticommute since I does. Therefore, we are right to denote $\bar{B}$ as a vector in $\mathbf{R^3}$. By iteratively appending all objects generated by the wedge product to the initial vector space $\mathscr{M}_{1,3}$, we construct the full Clifford STA $\mathscr{C}l_{1,3}$. This notation indicates that the STA is a CA generated from the metric signature (+ - - -). Importantly, all components in this CA are purely real. In this case, the GA generated by $\mathbf{R^3}$ is identical with the even subalgebra of $R_{1,3}$, so we write:

$$R_3 = R^+_{1,3}. \tag{10}$$

Moreover, eq.4 determines a split of the bivectors in $R^2_{1,3}$ into vectors and bivectors of R_3, as expressed by writing:

$$R^2_{1,3} = R^1_3 + R^2_3. \tag{11}$$

This split is not unique, however, as it depends on the choice of the vector γ_0. A complete discussion of space-time splits is given in [2, 3]. Finally, we should point out that the purpose of STS is merely to relate invariant physical quantities to the system macroscale variables employed by specific IO.

1.2. **STA Fundamental Properties.** STA is the Hestenes' name for the CA $\mathscr{C}l_{1,3}(R)$, or equivalently the geometric algebra $G_4 \equiv G(\mathscr{M}_{1,3})$. Of course, $\bar{E}$ and $\bar{B}$ in eq.7 are just the electric and magnetic fields in the γ_0-system, and the split of $\bar{F}$ into electric and magnetic fields will be different for different inertial systems (different IOs). We argue that it is exactly at this point that quantum system representation is mapped to classical system additive representation with loss of overall system fundamental coherence information (the field $\bar{F}$ information phase connecting time component to space components coherently into their primitive space-time continuum, at their original gauge level) by traditional Physics approach. As we briefly indicated, the embedding of CAs $\mathscr{C}l_{3,0} \subset \mathscr{C}l_{1,3}$ motivated our choice of metric signature for spacetime to be (+ - - -). The new basis vectors share the algebra of the gamma matrices but are usually not equated with them. This embedding is part of a larger sequence of CA embeddings starting from $\mathscr{C}l_{0,0} \subset \mathscr{C}l_{0,1} \subset \mathscr{C}l_{0,2} \subset \mathscr{C}l_{3,0} \subset \mathscr{C}l_{1,3} \subset \mathscr{C}l_{4,1} \subset \mathscr{C}l_{2,4}$ which corresponds to, respectively, the real numbers, complex numbers (Schrödinger spinors), quaternions (Hamilton spinors), relative 3-space (Pauli spinors), spacetime/electromagnetism (Maxwell spinors), relativistic electron (Dirac spinors), and conformal space (Penrose twistors) [21]. Here, we like to focus our attention to the fact that the classic vector space $\mathscr{M}_{1,3}$ does not contain the complete physical picture implied by special relativity, since it must be augmented by quantities like $F^{\mu\nu}$ to obtain the complete picture of special relativity in a systematic and principled way. Indeed, standard treatments of electromagnetism, involving relative 3-vectors, use both the symmetric dot product and the antisymmetric vector cross product to properly discuss the physical implications of the theory. The classic vector space $\mathscr{M}_{1,3}$ only specifies the relativistic version of the dot product in the form of the Minkowski metric. As you can see, without introducing the proper relativistic notion of the wedge product (proper generalization of the vector cross product to relativistic 4-vectors) the physical picture of spacetime is incomplete. Mathematically, the introduction by STA of a product on a vector space creates an algebra and the wedge product is precisely Grassmann's exterior wedge product, adopted by Cartan to define differential forms [22]. The wedge product enlarges the mathematical space in order to properly accommodate quantities like the electromagnetic field tensor $F^{\mu\nu}$. Evidently, the intrinsic complex structure inherent to the geometry of spacetime has deep and perhaps under-appreciated consequences for even our classical field theories. The complex vector from eq.7 will appear naturally as a relative 3-space expansion of a bivector field, which is a proper geometric (and thus frame-independent) object in spacetime. Moreover, this bivector field will have a crucial fundamental difference from Riemann–Silberstein vector [12]: the scalar imaginary i in Riemann–Silberstein vector will be replaced by an algebraic unit pseudoscalar

I (also satisfying $I^2 = -1$) that is intrinsically meaningful to and required by the geometric structure of spacetime. This replacement makes the complex form of $\bar{F}$ reference-frame independent, which is impossible when the mathematical representation is restricted to the usual scalar imaginary i only. The electromagnetic field is an irreducibly complex object with an intrinsic phase. This phase necessarily involves the intrinsic pseudoscalar (unit 4-volume) I of spacetime, and it is intimately related to the appearance of electromagnetic waves and circular polarizations, with no need for any ad-hoc addition of a complex scalar field. As a matter of fact, this replacement of the scalar i with I has to be systematic throughout the electromagnetic theory, where it appears in electromagnetic waves, elliptical polarization, the normal variables for canonical field mode quantization, and the dual symmetric internal rotations of the vacuum field, to arrive to full ST system macroscale representation information conservation. From a multiscale complex system computational perspective (see [23] for a definition of multiscale complex system), all approaches that use a top-down scale-free approach allow for starting from an exact global solution panorama of analytic solution families, which offers a shallow local solution computational precision to real specific needs (in other words, from global to local point of view overall system information is not conserved, as misplaced precision leads to information dissipation). In fact, usually further analysis and validation (by probabilistic and stochastic methods) is necessary to get localized computational solution of any practical value, in real application. A local discrete solution is worked out and computationally approximated as the last step in their line of reasoning, that started from an overall continuous system approach (from continuum to discreteness hypothesis approach). In arbitrary multiscale complex system modelling the bottom-up discrete approach under the "discreteness hypothesis" is mandatory to achieve effective solutions [23]. To design and develop more robust, resilient and antifragile cyber-physical multiscale complex system modelling, we need novel tools to combine effectively and efficiently analytical asymptotic exact global solution panoramas to deep local computational precision achievement, to overcome the limitation of the traditional top-down probabilistic modelling veil opacity. We can achieve that goal taking into consideration the *CICT* computational system pre-spatial arithmetic scheme.

2. *CICT* Pre-spatial Arithmetic Scheme

It is possible to show a new point of view for computational space representation, where the complex upper half-plane can naturally emerge by the interplay of the coupled inversion of two counter oriented upper quarter planes. That representation can be even read as the self-reflexion of a reciprocal conformal relation of an "Outer Symbolic Representation" (OSR) to its corresponding reflected fundamental "Inner OpeRational Representation" (IOR) and vice-versa, where D in Z, $A = 1/D$ and $D = 1/A$, and $DA = AD = \overline{0}1.\overline{0}$, for positive oriented upper quarter-space (right upper quarter plane) and $DA = AD = -\overline{0}1.\overline{0}$, for negative oriented upper quarter-space (left upper quarter plane). The relationship of these two counter-oriented upper quarter-spaces can be thought as a reciprocally projective conformal correspondence to be the first operative example in literature for explicit self-reflexive oriented numeric representation space [24], that can be used as powerful reference framework to develop "OpeRational" Representation (OR)[25] applications for maximum information conservation (i.e. minimum information dissipation). For the sake of simplicity, we show main properties of one oriented upper quarter-space only, because the second one is just its reflection with respect a vertical axis, so LTR (Left-To-Right) representation can turn into RTL (Right-To-Left) representation and vice-versa. In virtue of this relationship, a direct representation turns into additive complemented representation and vice-versa, when needed. The same information content can get two different numerical representations which can resonate to each other freely, at the same time, with respect to an inversive reference boundary element (i.e., centered unitary segment

in 1-d, unitary circumference in 2-d, unitary sphere in 3-d, unitary iperspherе in 4-d, etc.) between an n^{th}-dimensional OSR (Outer Space Representation) and a correspondent IOR (Inner Space Representation), according to the Kelvin transform [26]: an n^{th}-dimensional OSR to its corresponding IOR and vice-versa. Reversing a convergent sequence into a divergent one and vice-versa is the fundamental property to reach information conservation, i.e. information reversibility. Traditional Number Theory and modern Numeric Analysis use mono-directional interpretation (LTR) for **Q** Arithmetic single numeric group generator, so information entropy generation cannot be avoided in contemporary computational algorithm and application. On the contrary, according to *CICT*, it is quite simple to show information conservation and generator reversibility, by using basic considerations only. We like to show our new point of view for number interpretation and processing discussing two fundamental topics in order to achieve more convenient and effective numeric algorithm development: Arithmetic Closure and Information Conservation in Section 2.1., and Exponential Periodic Closure and OECS (Optimized Exponential Cyclic Sequence) in Section 2.2.

2.1. **Arithmetic Closure and Information Conservation.** Let us consider fraction $1/D$, where D in **Z**, or Egyptian fraction, with no loss of generality for common fraction. Common fraction, given by Egyptian fraction multiplied by $a \in \mathbf{Z}$, is considered a simple integer division usually. The first repetition of basic period for reptend corresponds to the first full scale interval where number information can be conserved: we call it "Representation Fundamental Domain" (RFD) [25]. In general, D, the denominator of the considered OSR, is given by a finite decimal word of length W digits. From IOR X, the related RFD_L can be obtained, by a numeric word length of L_X digits. Elementary number theory considerations give us worst case word length L_X, with no loss of information, by:

$$L_X = D - 1 \tag{12}$$

digits, if and only if 10 is a primitive root modulo D. Otherwise L_X is a factor of $D-1$. If the period of the corresponding repeating decimal to $1/D$ for prime D is equal $D-1$, then the repeating decimal part is called "cyclic number" and D can be referred as "primitive number" or "solid number" (SN) of order 1 by *CICT* or "full reptend prime" elsewhere [27]. Thus a SN is necessarily prime. It is a sufficient qualification. Conversely a prime number may not be a SN of order 1. So, the usual worst case word length L_X for X, given by eq.12, can get RFD_L with no loss of information, related to D, just in case D is SN of order 1 or SN_1 for short. In this way, an OSR $1/D$ can be associated to a corresponding IOR and vice-versa, and an IOR RFD_L to a corresponding OSR and vice-versa, which are able to resonate freely to each other (we can say we got "resonant representation correspondence"). In classic arithmetic long division algorithm (the one you learn to divide at elementary school), usual dominant result (quotient, Q) is important, and traditionally minority components (remainders, R) are always discarded. So, with no loss of generality, let us consider, the first integer to show SN_1 property manifestly, that is number "7." In this case $D = \overline{0}7.\overline{0}$, so that worst case length analysis gives $L_X = D - 1 = 6$ digits. Recording of quotient Q and remainder R evolutive information generation is needed. The full-information content of long division $1/7$ would be stored into two decimal coupled words $< Q, R >$ of length $M_{QR} = (6+6)$ digits in total, for exact arbitrary

precision computation. In fact, for long division 1 by 7, we have:

$$
\begin{aligned}
&L_1 = 1, && Q_1 = 1/10^1, && R_1 = 3/10^1\\
&L_2 = 2, && Q_2 = 14/10^2, && R_2 = 02/10^2\\
&L_3 = 3, && Q_3 = 142/10^3, && R_3 = 006/10^3\\
&L_4 = 4, && Q_4 = 1428/10^4, && R_4 = 0004/10^4\\
&L_5 = 5, && Q_5 = 14285/10^5, && R_5 = 00005/10^5\\
&L_6 = 6, && Q_6 = 142857/10^6, && R_6 = 000001/10^6\\
&L_7 = 7, && Q_7 = 1428571/10^7, && R_7 = 0000003/10^7
\end{aligned}
\tag{13}
$$

and $L_X = 6$, and $T_p = 6$. Therefore, 7 is just a SN_1 and its full information content is captured by a six-digit word length RFD_6, for Q_6 and the same for R_6. Eqs. 13 sum up the first stages of elementary long division of 1 by 7. But from eqs. 13 it is straightforward to note too:

$$
\frac{1}{7} = \frac{1}{10}\sum_{k=0}^{\infty}\left(\frac{3}{10}\right)^k = \frac{14}{10^2}\sum_{k=0}^{\infty}\left(\frac{02}{10^2}\right)^k = \frac{142}{10^3}\sum_{k=0}^{\infty}\left(\frac{006}{10^3}\right)^k = \dots \tag{14}
$$

The Remainder R_L, at any computation evolutive stage, is the fixed multiplicative ratio of a formal power series associated to optimal decimal representations of $1/7$, at increasing arbitrary accuracy levels. Thus $1/7$ associated information content can be lossless compressed down to minimal $M_{QR} = (1+1)$ digits in total. As a matter of fact, any word couple $< Q_L, R_L >$ can be thought to be equivalent to and can represent a different real measurement instrument class, defined by RFD word length L_X. It is easy to see that, in general, but not always, the greater D, the lengthier L_X, and greater the achievable compression ratio E_w. In this simple case $E_w = (2*6)/2 = 6:1$, for number 7. Furthermore, by realizing that the remainder R_1 is the fixed multiplicative ratio of a power series, the computation of 3^n (mod 7) for $n = 1,2,3,\dots$, till its exponential closure, gives the "Fundamental Cyclic Remainder Sequence" ($FCRS$):

$$
R_1 = 3, \quad R_2 = 2, \quad R_3 = 6, \quad R_4 = 4, \quad R_5 = 5, \quad R_6 = 1 \tag{15}
$$

from which the "Fundamental Cyclic Quotient Sequence" ($FCQS$) can be readily regenerated by $7 * R_n$ (mod 10):

$$
Q_1 = 1, \quad Q_2 = 4, \quad Q_3 = 2, \quad Q_4 = 8, \quad Q_5 = 5, \quad Q_6 = 7. \tag{16}
$$

So, quotient and remainder information can always be regenerated anew by remainder information only, but not vice-versa. In fact, starting from $Q_1 = 1, R_1 = 3$, we get: $Q_2 = Q_1/10 + Q_1R_1/100 = 13/100$ and raw $r_2 = 09/100$. Now r_2(mod07) gives $q_2 = 1$ and $R_2 = 02/00$. We can use this information to obtain the final exact value of Q_2 and R_2 at $A = 10^{-2}$ accuracy level, in the following way:

$$
Q_2 = Q_1/10 + Q_1R_1/100 + q_2/100 = 14/100 \quad \textit{and} \quad R_2 = 02/100 \tag{17}
$$

Again, starting from $Q_2 = 14/100$, $R_2 = 02/100$, we get:

$$
Q_4 = 14/100 + Q_2R_2/10000 = 1428/10000 \quad \textit{and} \quad R_4 = 0004/10000, \tag{18}
$$

and so on. In this way, with remainder knowledge only, it is always possible to regenerate exact quotient and new remainder information at any arbitrary accuracy, with full information conservation. It is like to process tail information to regenerate the associated body information. Thanks to the above properties, the division algorithm can become free from trial and error like in Finite Segment P-adic representation systems, but with no usually associated coding burden [28]. For every OSR $D \geq \overline{D}$, where $\overline{D}$ is the additive 10^W complement of D, i.e. $\overline{D} = (10^W - D)$,

the fundamental related IOR, for LTR sequences, it is immediately given, for all values of D in Z, by the following formal power series:

$$\frac{1}{D} = \sum_{k=0}^{\infty} \frac{1}{10^W} \left(\frac{\overline{D}}{10^W} \right)^k, \tag{19}$$

where W is the word representation precision length of the denominator D. When $\overline{D} > D$ the formal power series of eq.19 can be rescaled mod$D$, to give multiple convergence paths to $1/7$, but with different "convergence speeds." The total number of allowed convergent paths, as monotonic power series, is given by the corresponding Q_L value, at the considered accuracy level. For instance, in the case of:

$$\frac{1}{07} = \sum_{k=0}^{\infty} \frac{1}{10^2} \left(\frac{93}{10^2} \right)^k, \tag{20}$$

we see that $D = 07$, $W = 2$, $A = 10^{-2}$ and $\overline{D} = 93 > D = 07$. Therefore, re-scaling the fixed multiplicative ratio mod$D$, we have:

$$\frac{1}{07} = \sum_{k=0}^{\infty} \frac{2}{10^2} \left(\frac{86}{10^2} \right)^k, \tag{21}$$

$$\frac{1}{07} = \sum_{k=0}^{\infty} \frac{3}{10^2} \left(\frac{79}{10^2} \right)^k, \tag{22}$$

$$\vdots$$

$$\frac{1}{07} = \sum_{k=0}^{\infty} \frac{14}{10^2} \left(\frac{02}{10^2} \right)^k, \tag{23}$$

for a total of 14 conservative paths converging to the same limit value $1/07$, with different convergence velocities till a maximum value ($Q_2 = 14/10^2$). Eq.23 corresponds to the maximum convergence velocity to $1/07$ for $A = 10^{-2}$. Working at accuracy level $A = 10^{-3}$, the total number of allowed convergent paths to $1/007$, as monotonic power series (as allowed conservative paths), are 142, at accuracy level $A = 10^{-4}$, they are 1428, at accuracy level $A = 10^{-5}$, they are 14285, at accuracy level $A = 10^{-6}$, they are 142857 exactly, and so on, till maximum machine word length and beyond: like discrete quantum paths denser and denser to one another, towards a never ending "blending quantum continuum," by a top-down point of view. Rational IORs are able to capture two different type of information related to an OSR: modulo (usual quotient information) and associated outer or extrinsic period (EP, exterior or external or extrinsic phase), which phased generator inner or intrinsic period (IP, interior or internal or intrinsic phase) can be computed from. So, rational information can be better thought to be isomorphic to vector information rather than to usual scalar one, at least. Furthermore, our knowledge of $RFDQ_L$ and corresponding $RFDR_L$ can allow reversing numeric power convergent sequence to its corresponding numeric power divergent sequence uniquely (reversing a convergence to a divergence and vice-versa is the basic property to allow information conservation; i.e. information reversibility). In fact, inversion operation always maps OSR D ending with digit 3 into $RFDQ$ ending with digit 3, D ending with digit 7 into $RFDQ$ ending with digit 7, and D ending with digit 1 into $RFDQ$ ending with digit 9 and vice-versa. So, it is possible to verify our previous statement immediately. As an example, by considering eq.15 and eq.16 as $RFDR_6$ and $RFDQ_6$ of OSR $1/7$ RTL respectively, we can recover number information by a simple

arithmetic point of view. We start with $R_6 = 1$, $R_5 = 5$ and $Q_6 = 7$. So:

$$
\begin{aligned}
R_6 &= 1*10^0, \quad Q_6 = 7*1 \\
R_5 &= 5*10^1, \quad R_5 - R_6 = 5*10^1 - 1*10^0 = 49 = 7*7*10^0 \\
R_4 &= 4*10^2, \quad R_4 - R_5 = 4*10^2 - 5*10^1 = 350 = 7*5*10^1 \\
R_3 &= 6*10^3, \quad R_3 - R_4 = 6*10^3 - 4*10^2 = 5,600 = 7*8*10^2 \\
R_2 &= 2*10^4, \quad R_2 - R_3 = 2*10^4 - 6*10^3 = 14,000 = 7*2*10^3 \\
R_1 &= 3*10^5, \quad R_1 - R_2 = 3*10^5 - 2*10^4 = 280,000 = 7*4*10^4 \\
R_0 &= 1*10^6, \quad R_0 - R_1 = 1*10^6 - 3*10^5 = 700,000 = 7*1*10^5.
\end{aligned} \tag{24}
$$

Adjacent weighted remainder difference gives correspondent quotient digit multiplied by p. It is even possible to show body and head regeneration by tail modular power interplay from a geometric (power series) point of view. We start with $Q_6 = 7$, $R_6 = 1$ and $R_5 = 5$. So:

$$
\begin{aligned}
q_1 &= 7 = Q_6 \; and \; r_1 = 1 = R_6; \\
q_2 &= Q_6 + 10R_5 = 57, \\
&\quad and \; r_2 = 5 = R_5; \\
q_3 &= Q_6 + 10Q_6R_5 = 357, \\
&\quad raw \; r_3 = 25 \; and \; r_3 \, mod \, D = 4 = R_4; \\
q_4 &= Q_6 + 10Q_6R_5 + 50(10Q_6R_5) = 17857, \\
&\quad raw \; r_4 = 125, and \; 4r_4, mod \, D = 6 = R_3; \\
q_5 &= Q_6 + 10Q_6R_5 + 50(10Q_6R_5) + 2500(10Q_6R_5) = 892857, \\
&\quad raw \; r_5 = 625 \; and \; r_5 \, mod \, D = 2 = R_2; \\
q_6 &= Q_6 + 10Q_6R_5 + 50(10Q_6R_5) + 2500(10Q_6R_5) + \\
&\quad + 125000(10Q_6R_5) = 44642857, \; raw \; r_6 = 3125 \; and \; r_6 \, mod \, D = 3 = R_1; \\
q_7 &= Q_6 + 10Q_6R_5 + 50(10Q_6R_5) + 2500(10Q_6R_5) + 125000(10Q_6R_5) + \\
&\quad + 6250000(10Q_6R_5) = 2232142857, \; raw \; r_7 = 15625 \; and \; r_7 \, mod \, D = 1 = R_0.
\end{aligned}
$$

By looking at partial sums $q_n, n = 1,2,\ldots,7$, it is like to process tail information to make emerge its encoded body information RTL, to generate the associated divergence head, computing associated remainder r_n at each step. You get a numerical power divergent sequence that is uniquely related to OSR $1/7$, which numeric body information can be synthetically RTL generated by:

$$
Div\left(\frac{1}{07}\right) = \sum_{k=0}^{\infty} 7(5\cdot 10)^k = 7\sum_{k=0}^{\infty}(5\cdot 10)^k = 7Div\left(\frac{1}{49}\right), \tag{25}
$$

where "*Div*" means "Divergence of." The period of the repeating decimal of $1/p^2$ is usually "pT_p," where T_p is the period of the repeating decimal corresponding to $1/p$. In our example, $p = 7$, a *SN*, therefore RFD period is $T_p = 6$, and $1/49 = 1/p^2$. It is immediate to verify that RFD period of $1/49$ is given just by $pT_p = 7\text{x}6 = 42$ digits. At present, there are three known primes only for which this is not true apparently, and for which the period of $1/p^2$ is the same as the period of $1/p$ because p^2 divides "$10^{p-1} - 1$." They are $3,487,56598313$ [29]. Similarly, the period of the repeating decimal of $1/p^k$ is usually $p^{k-1}T_p$. If p and q are primes other than 2 or 5, the decimal representation of the fraction $1/pq$ has a specific period. An example is $1/119$, where $119 = 7 \times 17$:

$$
\begin{aligned}
\lambda(7\times 17) &= LCM(\lambda(7), \lambda(17)) \\
&= LCM(6,16) = 48,
\end{aligned} \tag{26}
$$

where $\lambda(7\times 17)$ is the "Carmichael function" [30] and *LCM* denotes the "Least Common Multiple." The period T of $1/pq$ is a factor of $\lambda(pq)$ and it happens to be 48 in this case. So, the information content half of RFD(119) is just given by the following Optimized Exponential Cyclic Sequence (OECS, [40]) structured by 48 digits:

$$00840336134453781512605042016806722689075630252 1. \tag{27}$$

The other information half of RFD(119) is given by the sequence of corresponding 48 remainders. Therefore, the period T of the repeating decimal of $1/pq$ is $LCM(T_p,T_q)$ where T_p is the period of the repeating decimal of $1/p$ and T_q is the period of the repeating decimal of $1/q$. If $1/m$ is a repeating decimal and $1/n$ is a terminating decimal, them $1/(mn)$ has a nonperiodic part whose length is that of $1/n$ and a repeating part whose length is that of $1/m$ [31].

2.2. **Exponential Closure and OECS.** We explicitly computed $RFDQ_6$ and $RFDR_6$ corresponding to OSR $1/7$ and we found the total minimal word length allowed $L_1 = L_{Q_1} + L_{R_1} = 2$ digits, to preserve full information content associated to OSR $1/7$. Furthermore, digit period T_p is equal to $6 = p-1$, and therefore $D = 7$ is a full reptend prime or better *SN*. Now, all the other members of the "Word Family Group" (WFG) RFD($1/7$) or $SN = 7$ can be derived immediately from "ancestor sequence", by cyclic permutation only, obtaining:

TABLE 1. Q_L and R_L values for each component of Word Family Group $SN = 7$

$1/7$	0.	$Q_1=1$	$Q_2=4$	$Q_3=2$	$Q_4=8$	$Q_5=5$	$Q_6=7$
		$R_1=3$	$R_2=2$	$R_3=6$	$R_4=4$	$R_5=5$	$R_6=1$
$2/7$	0.	$Q_1=2$	$Q_2=8$	$Q_3=5$	$Q_4=7$	$Q_5=1$	$Q_6=4$
		$R_1=6$	$R_2=4$	$R_3=5$	$R_4=1$	$R_5=3$	$R_6=2$
$3/7$	0.	$Q_1=4$	$Q_2=2$	$Q_3=8$	$Q_4=5$	$Q_5=7$	$Q_6=1$
		$R_1=2$	$R_2=6$	$R_3=4$	$R_4=5$	$R_5=1$	$R_6=3$
$4/7$	0.	$Q_1=5$	$Q_2=7$	$Q_3=1$	$Q_4=4$	$Q_5=2$	$Q_6=8$
		$R_1=5$	$R_2=1$	$R_3=3$	$R_4=2$	$R_5=6$	$R_6=4$
$5/7$	0.	$Q_1=7$	$Q_2=1$	$Q_3=4$	$Q_4=2$	$Q_5=8$	$Q_6=5$
		$R_1=1$	$R_2=3$	$R_3=2$	$R_4=6$	$R_5=4$	$R_6=5$
$6/7$	0.	$Q_1=8$	$Q_2=5$	$Q_3=7$	$Q_4=1$	$Q_5=4$	$Q_6=2$
		$R_1=4$	$R_2=5$	$R_3=1$	$R_4=3$	$R_5=2$	$R_6=6$
$7/7$	0.	$Q_1=9$	$Q_2=9$	$Q_3=9$	$Q_4=9$	$Q_5=9$	$Q_6=9$
		$R_1=7$	$R_2=7$	$R_3=7$	$R_4=7$	$R_5=7$	$R_6=7$.

Please, from Table 1, note that remainder $R_L = a = 1,2,\ldots,p-1$, from "ancestor sequence", can be thought as pointer to Q_{L+1} to get the corresponding a/D quotient cyclical numeric string immediately, with no computation at all. This is a unique property shared by all *SN* numbers. So, the final overall "Family Group" (FG) information compression ratio is given by $E_f = (2*6*6)/2 = 36:1$. The $1/7$ traditional Number Theory approach tells us that this FG has six group generators. We have just verified the general rule for which the generators of $\mathbf{Z}/\mathbf{nZ}$ are the residue classes of integers which are co-prime to $n = D$ (the number of generators is given by $\varphi(n)$, where φ is the Euler's totient function [32]. Traditional Number Theory and modern Numeric Analysis are just a mono-directional interpretation (LTR) for $\mathbf{Q}$ Arithmetic group generators, so information entropy cannot be avoided by traditional computational applications. On the contrary, according to our new *CICT* approach, presented in previous section, it is quite simple to show information conservation and generator reversibility. In fact, from the first one of eqs.13 and eqs.24, comparing to Table 2 (column $a = 3$ and $a = 5$), it is possible to verify that numeric body information for $1/7$ is generated LTR by generator 3^k (mod7) and RTL by 5^k (mod7) conservatively, with OECSs (periodic closures) at $a^{k(p-1)}$ for $k = 1,2,\ldots$, thanks to Fermat's little theorem [33]. Furthermore, LTR generator 2^k (mod7)

corresponds to LTR generator 4^k (mod7), with half-cycle period, and generator 6^k (mod7) is self-corresponding alternately to unity. FG $1/7$ contains the smallest cyclic-numbers of order one, shown manifestly. Next larger prime number p to show *SN* property might be a good

TABLE 2. Modular Power a (mod7)

Number a	01	02	03	04	05	06	07	08
Power Order								
01	01	02	03	04	05	06	00	01
02	01	04	02	02	04	01	00	01
03	01	01	06	01	06	06	00	01
04	01	02	04	04	02	01	00	01
05	01	04	05	02	03	06	00	01
06	01	01	01	01	01	01	00	01
SubCycle ♯	06	02	01	02	01	03	00	06

candidate for exhibiting cyclic-number property on an even larger scale. Unfortunately there does not seem to be any simple rule dictating which prime number p will have *SN* property and which will not, so that one just has to check each prime out by long division to see. It turns out that the next higher prime number with the desired *SN* property is $p = D = 17$ where $T_p = 16$ digits. A search for still larger prime numbers reveals that they are not at all rare. In fact no less than seven more prime numbers smaller than 100 generate cyclic numbers of order one or *SN*. They are: $19, 23, 29, 47, 59, 61$ and 97. So, what about OSR when D is not *SN*? In that case L_X is the result of the exponential modular interplay with D and 10^k (mod$D$). Let us see a simple example. Let us consider a prime p , after 7. In this case $p = D = 13$, with $T_p = 6$, and 10^k (mod13) gives the following repeating sequence $(10, 09, 12, 03, 04, 01)$ with cycle six for remainders. Therefore, even in this case $L_X = 6$ digits like for $1/7$, but 10^k (mod13) it is not a full sequence generator, this time (see Table 3, column $a = 10$). One cycle six is sufficient to characterize half "Family Members" (FMs) only, this time. In fact, in the obtained sequence only six terms for the counting sequence from 01 to $(D-1) = 12$ are present. What about the other half FMs? It is immediate to find that they are given by $2 * 10^k$ (mod13) with the following repeating sequence $(07, 05, 11, 06, 08, 02)$ for remainders, with cycle six again. Finally we got all the terms! So $1/13$ WFG presents a RFD structure split into two prime subdomains ($1/13$ and $2/13$) both with $T_p = 6$. Therefore $D = 13$ is a SN of order 2, or SN_2 for short. Now, from Table 3, it is manifest that inner RFDs can be thought as structured combinatorical

TABLE 3. Modular Power a (mod13)

Number a	01	02	03	04	05	06	07	08	09	10	11	12	13	14
Power Order														
01	01	02	03	04	05	06	07	08	09	10	11	12	00	01
02	01	04	09	03	12	10	10	12	03	09	04	01	00	01
03	01	08	01	12	08	08	05	05	01	12	05	12	00	01
04	01	03	03	09	01	09	09	01	09	03	03	01	00	01
05	01	06	09	10	05	02	11	08	03	04	07	12	00	01
06	01	12	01	01	12	12	12	12	01	01	12	01	00	01
07	01	11	03	04	08	07	06	05	09	10	02	12	00	01
08	01	09	09	03	01	03	03	01	03	09	09	01	00	01
09	01	05	01	12	05	05	08	08	01	12	03	12	00	01
10	01	10	03	09	12	04	04	12	09	03	10	01	00	01
11	01	07	09	10	08	11	02	05	03	04	06	12	00	01
12	01	01	01	01	01	01	01	01	01	01	01	01	00	01
SubCycle ♯	12	01	04	02	03	01	01	03	04	02	01	06	00	12

optimizations of their FG generators to achieve full information content conservation by power

modular reversibility (LTR and RTL). Again, from Table 3, it is possible to verify immediately that numeric body information for family subgroup $1/13$ is generated LTR by generator $10^k = 62^k = 72^k$ (mod13) and RTL by $4^k = 22^k = 112^k$ (mod13) conservatively, with exponential periodic closure at $a^{k(p-1)/2}$ for $k = 1,2,\ldots$. Furthermore, LTR generator 9^k (mod13) corresponds to LTR generator 3^k (mod13), with four subcycles, and generator 12^k (mod13) is self-corresponding alternately to unity (to get a periodic word with one mapped "fixed point" at 12). Numeric body information for family subgroup 2/13 is generated LTR by generator 7^k (mod13) and RTL by 2^k (mod13) conservatively, with exponential periodic closure at $a^{k(p-1)}$ for $k = 1,2,\ldots$. Furthermore, LTR generator 5^k (mod13) corresponds to RTL generator 8^k (mod13), with three subcycles, and LTR generator 11^k (mod13) corresponds to LTR generator 6^k (mod13), (to get a periodic word with a mapped "pairing"). Actually, since space is limited, the discussion here will not be extended to the subgroup interplay of the family group and polycyclic groups. Good general references on polycyclic groups are [34, 35]. Word inner generator combinatorical structure can be arranged for "pairing" and "fixed point" properties for digit group with the same word length. As a matter of fact, those properties ("pairing" and "fixed point") are just the operational manifestation of universal categorical dichotomy hardwired into integer digit and digit group themselves (evenness and oddness, correspondence and incidence fundamental discrete relations). So, FG $1/13$ contains the smallest numeric cyclic-words of order two. What about more second order *SN* FGs? The basic ones are also generated by a subset of prime fractions $1/p$. The other prime numbers smaller than 100 which generate second order *SN* FG are $31,43,67,71,83$ and 89. In the same manner one can now go on to define, and find, *SN* FG of order three, four, five, and higher and higher, according to computational needs. The smallest prime number producing third order *SN* FG is 103. In this case, $T_p = 34$ digits (instead of 102 digits a first order *SN* FG would have). The seven smallest primes generating *SN* FGs of order four through ten are respectively $53,11,79,211,41,73$, and 281, but you must reach prime number 353 to get the first *SN* FG of order 11. All the prime numbers less than 100 have now been covered with the exception of $p = 37$, with $T_p = 3$ digits. Therefore 37 is a *SN* FG of order 12. Till now we discussed peculiar number property focusing our attention, for simplicity of presentation, on prime numbers only, but we do not wish to imply that a number has to be a prime in order to generate intriguing cycles. We already saw, briefly, that multiplication composition of prime numbers can generate an entire universe of new number-cycles. The decimal forms of quite arbitrary fractions nearly always have some unexpected fascination in their cycling behaviour. For example, consider FG $(1/21)$ with $T_p = 6$, or FG $(1/39)$ with $T_p = 6$, or FG $(1/63)$ with $T_p = 6$, or FG $(1/91)$ with $T_p = 6$ again!

3. *CICT* PG ALL-POWERFUL FRAMEWORK

CICT [36] new awareness of a discrete HG (hyperbolic geometry) subspace ("Reciprocal-Space", RS) of coded heterogeneous hyperbolic structures [43], underlying the familiar $\mathbb{Q}$ Euclidean "Direct-Space" (DS) surface representation, shows that any natural number n in $\mathbb{N}$ has associated a specific, non-arbitrary extrinsic or external phase relationship that we have to take into account to full conserve overall system component information content by computation in DS [37]. Recently, *CICT* has shown that, by Shannon entropy approach only, even the current, most sophisticated instrumentation system is completely unable to reliably discriminate so called "Random Noise" (RN) from any combinatorically optimized encoded message, which *CICT* called "Deterministic Noise" (DN) [13]. Stochastic vs. combinatorically optimized noise generation ambiguity emphasises the major "Information Double-Bind" (IDB) problem even in current most advanced research laboratory and instrumentation system, just at the inner core of human knowledge extraction by experimentation in contemporary science [13]. *CICT* solved the IDB problem by "Phased Generator" (PG) approach. *CICT* is a natural

framework for arbitrary multi-scale (AMS) computer science, biophysical and systems biology modelling in the current landscape of modern QFT [38, 39], created to overcome the computational limitations focused in previous sections. Thanks to this line of generative thinking, it is possible to realize that traditional rational number system can be even regarded as a highly sophisticated open logic, powerful and flexible LTR and RTL formal language of languages, with self-defining consistent words and rules, starting from elementary generators and relations [13]. Further, *CICT* ODR (Observation, Description, Representation) approach [40] can take advantage immediately from those properties to develop system computational functional closures to achieve information conservation countermeasure at each operative step automatically. Then, all computational information usually lost by classic information approach, based on the traditional noise-affected data stochastic model only, can be captured and fully recovered to arbitrary precision by a corresponding complementary codomain, step-by-step. Theoretically, codomain information can be used to correct any computed result, achieving computational information conservation (virtually noise-free data), according to *CICT* Infocentric Worldview [13]. In this way, overall system resilience and antifragility can be developed quite easily [41]. In fact, traditional digital computational resources are unable to capture and to manage not only the full information content of a single Real Number **R**, but even Rational Number **Q** is managed by information dissipation. In numeric representation of Rational Number **Q**, rational proper quotient is represented by infinite repetition of a basic digit cycle, called "reptend" (the repeating decimal part) [27]. According to *CICT*, the first repetition of basic digit cycle corresponds to the first full scale interval where number information can be conserved completely, and we call it *RFD*. In this case the new computational representation of rational number $\mathbb{Q}$, is called "OpeRational Representation" (OR), just to remember that *CICT* is able to conserve Rational Number **Q** full information content, much better than previous computational approaches. *CICT* sees natural integers **N** as specific numeric resonances emerging out of the OECS (optimized exponential cyclic sequence) [40] manifold of rational values **Q**. In turn, OECS can be thought emerging out from the peculiar numeric resonances of irrational numeric sequences generated by DS-RS coherent cross-interaction with their duals, according to the following two algebraic identities:

$$\frac{1}{D} = \frac{1}{(\sqrt[M]{D})^P} \cdot \frac{1}{(\sqrt[M]{D})^{(M-P)}} = \frac{1}{D^{\frac{P}{M}}} \cdot D^{\frac{P-M}{M}}, \tag{28}$$

for $P > 0$ and $M \geq P = 1, 2, \ldots, \infty$, where $D \in \mathbb{Q}$; and

$$\frac{1}{D} = (\sqrt[M]{D})^P \cdot \frac{1}{(\sqrt[M]{D})^{(M+P)}} = D^{\frac{P}{M}} \cdot \frac{1}{D^{\frac{M+P}{M}}}, \tag{29}$$

for $P < 0$. DS-RS cross-interaction with their duals is assumed to be our spacetime representation fundamental property. Eventually, *CICT* OECSs have strong connection even to classic DFT algorithmic structure for discrete data, Number-Theoretic Transform (NTT), Laplace and Mellin Transforms [13]. Coherent precision correspondence (numeric resonance) leads to transparency, ordering, reversibility, cosmos, simplicity, clarity, and to algorithmic quantum incomputability on real macroscopic machines [37]. *CICT* natural framework is quite flexible and can be used under two major operational representation schemes: "Formal Recurrence Sequence (FRS) in Section 3.1, and "Formal Power Series" (FPS) (Section 3.2) presentation, respectively.

3.1. *CICT* **FRS Presentation.** *CICT* can be presented and used in term of recurrence relation. In this case, rational geometric series can be seen as simple recurrence sequences in a wider recurrence operative framework where all algebraic recurrence sequences of any countable higher order include all the lower order ones and they can be optimally mapped to rational number system **Q** ORs and generating functions to OECS [40]. For instance, arithmetic progression and

Lucas sequences are recurrence sequences of the second order. Lucas sequences are certain integer sequences that satisfy Lucas recurrence relation defined by polynomials $U_n(P,Q)$ and $V_n(P,Q)$, where U_n, V_n are specific polynomials and P, Q are fixed integer coefficients.[23] Any other sequence satisfying this recurrence relation can be represented as a linear combination of the Lucas sequences $U_n(P,Q)$ and $V_n(P,Q)$. Famous examples of Lucas sequences include the Fibonacci numbers, Mersenne numbers, Pell numbers, Lucas numbers, Jacobsthal numbers, and a superset of Fermat numbers. *CICT* is able to fold any recursion sequence of the first order into one digit number D, any recursion sequence of second order into a two digit number D, any recursion sequence of the third order into a three digit number D and so on to higher orders. Then, you can compute their asymptotic convergence velocities as related roots from corresponding first, second, third, etc..., order equations respectively. In papers presented elsewhere, the interested reader can go deeper into *CICT* FRS presentation main properties [23, 43].

3.2. *CICT* **FPS Presentation.** Basic *CICT* result can be even presented in term of classic monopolar power series, or in general by formal power series, to show the close relationships to classic and modern control theory approaches for causal continuous-time and discrete-time linear systems. The *CICT* fundamental relationship that ties together numeric body information of RTL to RTL monotonic power series in any base (in this case decimal, with no loss of generality) with D ending by digit 9 is given by the following equation:

$$\frac{1}{D} = \sum_{k=0}^{\infty} \frac{1}{10^W} \left(\frac{\overline{D}}{10^W} \right)^k \Rightarrow \cdots \Leftarrow Div \left(\frac{1}{D} \right) = \sum_{k=0}^{\infty} (D+1)^k, \tag{30}$$

where $\overline{D}$ is the additive 10^W complement of D, i.e. $\overline{D} = (10^W - D)$, W is the word representation precision length of the denominator D and "*Div*" means "Divergence of". When $\overline{D} > D$ the formal power series on the left of eq.30 can be re-scaled mod$D$, to give multiple convergence paths to $1/D$, but with different "convergence speeds." The total number of allowed convergent paths, as monotonic power series, is given by the corresponding Q_L value, at the considered accuracy level. So, increasing the level of representation accuracy, the total number of allowed convergent paths to 1/D, as monotonic power series (as allowed conservative paths), increases accordingly and can be counted exactly, and so on, till maximum machine word length and beyond: like discrete quantum paths denser and denser to one another, towards a never ending "blending quantum continuum," by a top-down perspective. By using this approach, it is possible to generate LTR and RTL remainder sequences that show same quotient body information (multi-scale periodic) and specific quotient head and tail information to compute deterministic boundary values, to sustain body periodicity with no information dissipation (full information conservation and reversibility) [25]. Further generalizations of right term of eq.30 related to D ending by digit 1 or 3 or 7 are straightforward. In fact, we can re-write the right term of eq.30 as:

$$Div \left(\frac{1}{D} \right) = \sum_{k=0}^{\infty} G^k, \tag{31}$$

where G is a specific RTL generator. Then, it is immediate to verify that for $G = (9D+1)$, $G = (3D+1)$ and $G = (7D+1)$ the RTL periodic string will match the LTR one generated by:

$$\frac{1}{D} = \sum_{k=0}^{\infty} \frac{1}{10^W} \left(\frac{\overline{D}}{10^W} \right)^k. \tag{32}$$

It will match exactly the sequences generated by number word D of length W, right ending by digit 1 or 3 or 7 respectively. As monotonic power series (as allowed subspace paths), by increasing the subspace representation accuracy, the total number of allowed convergent paths increases accordingly till maximum machine word length and beyond; like discrete quantum

paths denser and denser to one another, towards never ending "blending quantum continuum," called "quantum mixture" by current FMM (Finite Mixture Model) top-down statistical point of view for composite AMS system, as we discussed in previous section. On the contrary, for *CICT* the finer geometry of subspace itself becomes scale dependent. While differentiable trajectories found in standard mathematical physics are automatically scale invariant, it is the main insight of the *CICT* theory that also certain non-differentiable paths (resultant paths, emerging from lower scales combined quantum trajectory interactions, which explicitly depend on the scale and accuracy of the observer) can be scale invariant. Rational representations are able to capture two different types of information at the same time, modulus (usual quotient information) and associated outer or information period (extrinsic phase, represented by RFD [25]) which an inner, or intrinsic phase can be computed from. So, rational information can be better thought to be isomorphic to vector information rather than to usual scalar one, at least. According to Hestenes [8], we can take the mathematical concepts of modular magnitude and direction as basic, and introduce the concept of vector as the basic kind of directed number, with an additional, associated, specific phasing relation. Directed numbers are defined implicitly by specifying rules for adding and multiplying vectors. According to *CICT* vector representation precision, directed numbers (modular magnitude and direction) can be related uniquely to long division rational remainder sequences to identify convenient "quantum support field" relations, which subspace inner phased generators can be computed from. Furthermore, our knowledge of $RFDQ_L$ and corresponding $RFDR_L$ can allow reversing numeric power convergent sequence to its corresponding numeric power divergent sequence uniquely [25]. Reversing a convergence sequence to a divergence one and vice-versa is the basic property to reach information conservation, i.e. information reversibility, as from eq.30. For future reference, it is convenient to use a compact notation for LTR geometric series as follows:

$$\frac{1}{D} = \sum_{k=0}^{\infty} \frac{N}{10^W} \left(\frac{CR}{10^W} \right)^k \equiv N_W(CR_W), \tag{33}$$

where D is denominator, N numerator, CR power series constant ratio, and W the length of their digit strings. Then, it is immediate to verify that at $W = 1$ (accuracy = 10^{-1}), unity can emerge out of a fundamental symmetrical multiplicity of different monopolar countable paths, but with different "convergence speeds" (in this case 9):

$1(9) \equiv 1/1$, $(8) \equiv 1/1$, $3(7) \equiv 1/1$, $4(6) \equiv 1/1$, $5(5) \equiv 1/1$, $6(4) \equiv 1/1$, $7(3) \equiv 1/1$, $8(2) \equiv 1/1$, $9(1) \equiv 1/1$.

At $W = 2$ (accuracy = 10^{-2}, therefore 99 different monopolar countable paths), we have: $01(99) \equiv 1/1$, $02(98) \equiv 1/1$, $03(97) \equiv 1/1$, $04(96) \equiv 1/1$, $05(95) \equiv 1/1$, …, $96(04) \equiv 1/1$, $97(03) \equiv 1/1$, $98(02) \equiv 1/1$, $99(01) \equiv 1/1$.

At $W = 3$ (accuracy = 10^{-3}, therefore 999 different monopolar countable paths), we have: $001(999) \equiv 1/1$, $002(998) \equiv 1/1$, $003(997) \equiv 1/1$, $004(996) \equiv 1/1$, $005(995) \equiv 1/1$, …, $996(004) \equiv 1/1$, $997(003) \equiv 1/1$, $998(002) \equiv 1/1$, $999(001) \equiv 1/1$, and so on for $W = 4$, 5, …, $W \in \mathbb{N}$. The interested reader to go deeper into *CICT* FPS presentation is referred to [13, 25, 40, 42]

4. Recovering the Exterior Phase Relationship

To conserve the full ST information content, we need to avoid the loss of overall system fundamental information (the pre-splitting information phase loss, connecting time component to space component coherently into their primitive spacetime continuum field $\bar{F}$, at their original gauge level), which is manifestly an ill-posed problem by classic scientific approach (Science 1.0). *CICT* new awareness of the discrete HG (hyperbolic geometry) based

RS of coded heterogeneous hyperbolic structures [36], underlying the familiar **Q** Euclidean DS surface representation can open the way to AMS information conservation by the *CICT* PG approach [13, 25]. First, let us introduce the LTR "Symbolic Compression Operator" SCO $\equiv < M \mid DS >$, where DS is a finite digit string of length W and M is the number of times DS is repeated to get the final unfolded digit string in full (e.g. $< 4 \mid 1 > \equiv 1111$ or $< 2 \mid 123 > \equiv 123123$). Usual symbolic string operations can be applied to SCO. Then, we can write usual rational number OR corresponding to their symbolic representation as [25]:

$$\begin{array}{rl} Q_1 = \frac{1}{D_1} = \frac{1}{9} & = \sum_{k=0}^{\infty} \frac{1}{10^1} \left(\frac{1}{10^1}\right)^k = 0.1111111111111111\ldots \\ Q_2 = \frac{1}{D_2} = \frac{1}{99} & = \sum_{k=0}^{\infty} \frac{01}{10^2} \left(\frac{1}{10^2}\right)^k = 0.01010101010101010\ldots \\ Q_3 = \frac{1}{D_3} = \frac{1}{999} & = \sum_{k=0}^{\infty} \frac{001}{10^3} \left(\frac{1}{10^3}\right)^k = 0.00100100100100100\ldots \\ \ldots & \ldots, \end{array} \tag{34}$$

in a more compact RFD, [25] Q_W format as:

$$\begin{array}{rl} Q_1 = \frac{1}{D_1} = \frac{1}{9} & \equiv 0.{<}\infty|1> = 0.{<}\infty|(<0 \mid 0><1 \mid 1>)> \\ Q_2 = \frac{1}{D_2} = \frac{1}{99} & \equiv 0.{<}\infty|01> = 0.{<}\infty|(<1 \mid 0><1 \mid 1>)> \\ Q_3 = \frac{1}{D_3} = \frac{1}{999} & \equiv 0.{<}\infty|001> = 0.{<}\infty|(<2 \mid 0><1 \mid 1>)> \\ \ldots & \ldots \\ Q_n = \frac{1}{D_n} \equiv \frac{1}{<n|9>} & = 0.{<}\infty|(<n \mid 0><1 \mid 1>)> \\ \ldots & \ldots \end{array} \tag{35}$$

In the same way, we can write:

$$\begin{array}{rl} P_1 = \frac{1}{DD_1} = \frac{1}{1} & = \sum_{k=0}^{\infty} \frac{9}{10^1} \left(\frac{1}{10^1}\right)^k = 0.99999999999999999\ldots \\ P_2 = \frac{1}{DD_2} = \frac{1}{11} & = \sum_{k=0}^{\infty} \frac{09}{10^2} \left(\frac{1}{10^2}\right)^k = 0.09090909090909090\ldots \\ P_3 = \frac{1}{DD_3} = \frac{1}{111} & = \sum_{k=0}^{\infty} \frac{009}{10^3} \left(\frac{1}{10^3}\right)^k = 0.00900900900900900\ldots \\ \ldots & \ldots \\ P_1 = \frac{1}{DD_1} = \frac{1}{1} & \equiv 0.{<}\infty|9> = 0.{<}\infty|(<0 \mid 0><1 \mid 9>)> \\ P_2 = \frac{1}{DD_2} = \frac{1}{11} & \equiv 0.{<}\infty|09> = 0.{<}\infty|(<1 \mid 0><1 \mid 9>)> \\ P_3 = \frac{1}{DD_3} = \frac{1}{111} & \equiv 0.{<}\infty|009> = 0.{<}\infty|(<2 \mid 0><1 \mid 9>)> \\ \ldots & \ldots \\ P_n = \frac{1}{DD_n} \equiv \frac{1}{<n|1>} & = 0.{<}\infty|(<n \mid 0><1 \mid 9>)> \\ \ldots & \ldots \end{array} \tag{36}$$

Now, we can realize that P_1 RFD is related by Q_1 RFD, P_2 RFD is related by Q_2 RFD, P_3 RFD is related by Q_3 RFD,... and vice-versa by periodic scale relativity (precision length) W = 1, 2, 3, ..., respectively. So, to conserve the full information content of rational correspondence between Q_1 and P_1, Q_2 and P_2, Q_3 and P_3, ..., we realize that we have to take into account not only the usual Q_1 and P_1, etc., modulus information, but even their related periodic precision length information W = 1, 2, 3, ..., respectively (external or extrinsic phase representation). Furthermore we see that:

$$\begin{array}{rl} O_1 = \frac{1}{DDD_1} = \frac{1}{9} & \equiv 0.{<}\infty|1> = 0.{<}\infty|(<1 \mid 1>)> \\ O_2 = \frac{1}{DDD_2} = \frac{1}{09} & \equiv 0.{<}\infty|11> = 0.{<}\infty|(<2 \mid 1>)> \\ O_3 = \frac{1}{DDD_3} = \frac{1}{009} & \equiv 0.{<}\infty|111> = 0.{<}\infty|(<3 \mid 1>)> \\ \ldots & \ldots \\ O_n = \frac{1}{DDD_n} \equiv \frac{1}{(<n|0><1|9>)} & = 0.{<}\infty|(<n \mid 1>)> \\ \ldots & \ldots \end{array} \tag{37}$$

The coherent representations DDD_1, DDD_2, DDD_3, etc. emerge out of a LTR infinity of symbolic structured infinite length sequences as in eq.37. So, DDD_n is the coherent relation representation of traditional scalar modulus D_1 in eq.34 as denominator at precision W, while scalar modulus D_1 in eq.34 has to be interpreted as the decoherenced relation representation of DDD_n denominators in eq.37. In general, for any Natural number $D \in \mathbf{N}$ we can write:

$$\frac{1}{< n \mid RFD(D) >} \equiv \tag{38}$$
$$\equiv < (< k + W(n-1) \mid 0 >< 1 \mid D >) > .$$
$$< (< k + W(n-1) \mid 0 >< 1 \mid D >) >,$$

where RFD is at precision W and k is the number of leading zeros in RFD (if any) at precision W. Now, for $n \to \infty$, we get:

$$\frac{1}{< n \mid RFD(D) >} \equiv \tag{39}$$
$$\equiv < (< k + W(\infty - 1) \mid 0 >< 1 \mid D >) > .$$
$$< (< k + W(\infty - 1) \mid 0 >< 1 \mid D >) > = \bar{0}D.\bar{0}.$$

Then, Natural numbers $\mathbf{N}$ emerge from eq.39 as a conceptual abstraction, just as the rightmost approximated part of those LTR structured sequences when $n \to \infty$, for $n \in \mathbf{N}$, with left truncated decimal part (represented by leading zeroes on any finite computational machine). In this way we loose the basic coherence relationship with its fundamental string generators (i.e. information dissipation and system decoherence) completely. Leading zeros in positional notation representation system for *CICT* $\mathbf{Q}$ Arithmetic do count effectively, and can model the system quantum-classical transition quite effectively [23]. If we do not take into account leading zeros information, we lose the correct rational RFD correspondence information (coherence) which an outer or extrinsic phase for each RTL (right-to-left) string generator can be computed from (i.e. from their optimized exponential cyclic sequences (OECS) of R_W [37]) first. Therefore, Rational numeric representations are able to capture two different types of information at the same time, modulus (usual quotient information) and RFD associated information period (exterior or external or extrinsic phase, EP), which inner generator intrinsic period (interior or internal or intrinsic generator phase, IP) can be computed from. So, rational information can be better thought to be isomorphic to vector information rather than to usual scalar one, at least. As an example, according to our SCO approach, the correct coherent relation representation of traditional scalar modulus $D = \bar{0}7.\bar{0}$ as denominator of Egyptian fraction, is given by:

$$CQ_1 = \frac{1}{CD_1} \equiv \frac{1}{< \infty \mid (< \infty \mid 0 >< 1 \mid 7 >) >} \equiv \tag{40}$$
$$\equiv 0. < \infty | RFD(7) > \equiv 0. < \infty | 142857 > .$$

Therefore, to conserve the full information content of rational correspondence at higher level, we realize that we have to take into account not only the usual modulus information, but even the related periodic precision length information $W = 6$ (numeric period or external phase representation) in this case (i.e. $CD_1 = 000007$ as base RFD). We can use Euler's formula to establish the usual fundamental relationship between trigonometric functions and the complex exponential function:

$$e^{ix} = cosx + isinx, \tag{41}$$

where e is the natural mathematical constant and $i = \sqrt{-1}$. The final result is:

$$CQQ_1 = \frac{1}{CDD_1} = \frac{1}{7} e^{i\frac{\pi(2n+1)}{3}} = \frac{1}{7}(cos(\frac{2\pi(n+1)}{6}) + isin(\frac{2\pi(n+1)}{6})) \tag{42}$$

and

$$CDD_1 = \frac{1}{CQQ_1} = 7(e^{-i\frac{\pi(2n+1)}{3}}) = 7(cos(\frac{\pi(2n-1)}{3}) + isin(\frac{\pi(2n-1)}{3})) = \\ = 7(\frac{1}{2} - i\frac{\sqrt{3}}{2}) \quad (p.v.) \tag{43}$$

for n = 0, 1, 2, 3, ...in **N**, where (p.v.) means "principal value". *CICT* shows that any natural number $D \in \mathbf{Q}$ has associated a specific, non-arbitrary external phase relationship (Optimized Exponential Cyclic Sequence, OECS, coherence information, [40]) that we must take into account to conserve its full information content by computation in Euclidean space [36]. The careful reader will have already guessed the relationship of our result to de Moivre number or root of unity (i.e. any complex number that gives 1.0 when raised to some integer power of n). In this way, we can exploit rational numbers **Q** full information content to get stronger solutions to current AMS system modeling problems [12], by keeping their information coherence. Our RFD period knowledge associated to specific rational number D reveals its associated EP information immediately. From EP, *CICT* allows to compute each RFD inner generator IP to go through the statistical modeling veil opacity, by phased generator approach, as you already saw in Section 2. This result, offered by $1/D$ rational remainder sequence, can be interpreted as the optimized combination of elementary quantum phase irrational fluctuations.

5. Conclusion

In arbitrary multiscale complex system modelling the bottom-up discrete approach under the "discreteness hypothesis" is mandatory to achieve effective solutions. In fact, *CICT* shows that any natural number $D \in \mathbf{N}$ has associated a specific, non-arbitrary external phase relationship (OECS coherence information, [40]) that we must take into account to full conserve its information content by computation in Euclidean DS [36]. RFD period knowledge associated to specific rational number D reveals its EP information immediately. From EP, *CICT* allows to compute each RFD inner generator IP to tear aside or minimize the traditional probabilistic modeling veil opacity. *CICT* achieves this result by $1/D$ rational remainder sequence, interpreted as optimized combination of elementary quantum phase generators. Therefore, to conserve the full information content of rational correspondence at higher level, we realize that we have to take into account not only the usual modulus information, but even the related RFD periodic precision length information. According to QFT, the gauge invariance in quantum physics becomes the natural partner of the phase invariance to produce the representation and description of our world. Quantum fluctuations give rise to electromagnetic potentials which spread the phase fluctuations beyond the system at the phase velocity. This gives an intrinsic nonlocalizability to the system and prevents a direct observation of quantum fluctuations. Through the electromagnetic potential, the system gets a chance to communicate with other systems and subsystems [42]. QFT has emerged from a major ontological paradigm shift with respect to Classical Physics which still provides the traditional framework of the vision of nature of most "Science 1.0" current scientists. This change of paradigm has not yet been completely grasped at operative level in full by many contemporary scientists and researchers in different disciplined areas, so that not all the implications of this change have been realized hitherto, even less their related applications. So, the discreteness approach, developed under the "discreteness hypothesis" assumption, in specific scientific disciplines, has been considered in peculiar application areas only. It has been further slowly developed by

a few specialists and less understood by a wider audience. Unfortunately, over the centuries, the two large scientific research areas (continuum based and discreteness based) have followed separate mathematical development paths with no or quite little, inconsistent synergic coupling. That is the main reason why QFT is still mostly overlooked by traditional scientific and engineering researchers for arbitrary multi-scale system modelling, from system nano-microscale to macroscale. Reversing a convergent sequence into a divergent one and vice-versa is the fundamental property to reach information conservation, i.e. information reversibility. Eventually, *CICT* OECS have strong connection even to classic DFT algorithmic structure for discrete data, Number-Theoretic Transform (NTT), Laplace and Mellin Transforms [12]. As a matter of fact, to grasp a more reliable representation of reality, researchers and scientists need two intelligently articulated hands; both stochastic and combinatorical approach synergistically articulated by natural coupling; let's say we need a fresh "Science 2.0" approach [42]. *CICT* combined to GA and GC unified mathematical language can offer an effective and convenient "Science 2.0" universal framework, by considering information not only on the statistical manifold of model states but also on the combinatorical manifold of low-level discrete, phased generators and empirical measures of noise sources, related to experimental high-level overall perturbation. It is the fresh QFT approach. *CICT* coherent precision correspondence allows us to tear aside or minimize the probabilistic veil reaching transparency, ordering, reversibility, cosmos, simplicity and clarity, by a simple arithmetic scheme strongly coupled to GA and GC representations [44]. This paper is a relevant contribution towards an effective and convenient "Science 2.0" universal framework to develop more effective biophysics modelling and simulation for competitive application and beyond [45].

ABBREVIATIONS

AMS: Arbitrary Multi-Scale
CA: Clifford Algebra
CICT: Computational Information Conservation Theory
DN: Deterministic Noise
DS: Direct-Space
EP: Exterior or External or Extrinsic Phase
FCRS: Fundamental Cyclic Remainder Sequence
FCQS: Fundamental Cyclic Quotient Sequence
FG: Family Froup
FM: Family Member
FPS: Formal Power Series
FRS: Formal Recurrence Sequence
GA: Geometric Algebra
GC: Geometric Calculus
IDB: Information Double-Bind
IO: Inertial Observer
IOR: Inner OpeRational Representatio
IP: Interior or Internal or Intrinsic Phase
LCM: Least Common Multiple
LTR: Left-To-Right
NTT: Number-Theoretic Transform
ODR: Observation, Description, Representation
OECS: Optimized Exponential Cyclic Sequence
OR: OpeRational
OSR: Outer Symbolic Representation
PG: Phased Generator

QFT: Quantum Field Theory
QM: Quantum Mechanics
RFD: Representation Fundamental Domain
RN: Random Noise
RS: Reciprocal Space
RTL: Right-To-Left
SN: Solid Number
ST: Spacetime
STA: Space-Time Algebra
WFG: Word Family Group

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ON MATRIX REPRESENTATIONS OF GEOMETRIC (CLIFFORD) ALGEBRAS

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ABSTRACT. The representations of geometric (Clifford) algebras with real square matrices are reviewed in order to see whether some advantage can be gained when considering them from an arithmetic point of view, meaning without resorting to algebraic structure. Many isometries such as rotations and Lorentz transformations are written as similarity transformation of matrices, which will be chosen as the general definition of direct isometry. Then, it will be deduced in which geometric algebras all the known isometries as well as the transformations that could be considered isometries can be written as similarity transformations or need additionally reversion. Since similar matrices have the same characteristic polynomial, the norm of every element of a geometric algebra should be defined from some combination of its coefficients. We propose to define the norm of every element of a geometric algebra as the n^{th}-root of the absolute value of the determinant of its matrix representation with dimension $n \times n$, which is the independent term of the characteristic polynomial. Some examples of the usefulness of this definition will be provided in order to confirm its consistency and generality.

1. INTRODUCTION

The matrix representations of associative algebras have played a very important role in software programming, and this is not an exception for geometric algebras. Since the best way to compute rotations and other isometries in the Euclidean space is by means of geometric algebra, the technological applications of matrix representations are obvious for programmers. Otherwise, the appearance of quantum mechanics and the discovery of the relativistic wave equation for the electron by Paul A. M. Dirac, brought forth the matrix representation of the space-time geometric algebra to the foreground of the physics, as stressed by Hestenes in his *Space-Time Algebra* recently reedited [1]. The development of particle physics has also enhanced modeling the fundamental interactions by means of groups of matrices, which are representations of geometric algebras many times.

2. FUNDAMENTALS OF MATRIX REPRESENTATIONS

Since those attending this Summer School come from different disciplines, some of them coming from the mathematical world and others coming from physics or the engineering world, in this section we recall some fundamental facts about matrix representations that can be trivial for those with good mathematical training.

A matrix representation $M(A)$ of an associative algebra $A(\mathbb{F})$ over a division algebra[1] $\mathbb{F}$ is a square matrix algebra or subalgebra with entries in $\mathbb{F}$ that is homomorphic to $A(\mathbb{F})$, that is, for every pair of elements $a, b \in A(\mathbb{F})$ there are two corresponding matrices $M(a), M(b)$ for which their linear combination and multiplication have the corresponding representations:

$$\forall a,b \in A(\mathbb{F}) \qquad M(\lambda a + \mu b) = \lambda M(a) + \mu M(b) \qquad \lambda,\mu \in \mathbb{F} \tag{1}$$

[1] According to Frobenius' theorem, the division associative algebras are only real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ or other algebras isomorphic to them.

$$\forall a,b \in A(\mathbb{F}) \qquad M(ab) = M(a)M(b) \tag{2}$$

Let us recall that a similarity transformation of matrices is the transformation given by:

$$M' = S^{-1}MS \qquad \det M \neq 0 \tag{3}$$

Then, a matrix representation is determined up to a similarity transformation of matrices because:

$$S^{-1}M(\lambda a + \mu b)S = S^{-1}(\lambda M(a) + \mu M(b))S = \lambda S^{-1}M(a)S + \mu S^{-1}M(b)S \tag{4}$$

According to eq. (3), it can be written as:

$$M'(\lambda a + \mu b) = \lambda M'(a) + \mu M'(b) \tag{5}$$

In the same way:

$$S^{-1}M(ab)S = S^{-1}M(a)S\, S^{-1}M(b)S \tag{6}$$

which can be written as:

$$M'(ab) = M'(a)M'(b) \tag{7}$$

Then, if a given set of matrices represents a set of elements of the algebra $A(\mathbb{F})$, another set of matrices obtained from a similarity transformation represents them as well, and they are said to be the same matrix representation. Therefore, what defines a matrix representation is not the specific matrices but their invariants under a similarity transformation. Now we already see that similarity transformations (3) play a very important role in the theory of matrix representations, which is enhanced by some new geometric considerations.

A matrix representation is said to be *faithful* if different elements are always represented by different matrices:

$$a \neq b \qquad \Rightarrow \qquad M(a) \neq M(b) \tag{8}$$

The faithful representations are the only ones being of interest since the others collapse different elements into the same matrices, which is not suitable for any kind of technical application. The homomorphism becomes an isomorphism for faithful representations. A matrix representation needs to have enough dimension to be faithful. In this way, the following inequality is obvious:

$$M(A) \quad \text{faithful} \qquad \Rightarrow \qquad \dim A(\mathbb{F}) \leq \dim M(A) \tag{9}$$

since $M(A)$ are matrices with entries in $\mathbb{F}$. The most interesting representations are those being faithful and having the minimal dimension. The representation obtained from the multiplication rule of the elements of the algebra $A(\mathbb{F})$ is called *regular*. For instance, let us deduce the regular

representation of quaternions $q = a + b\,i + c\,j + d\,k\ \ (a,b,c,d, \in \mathbb{R})$ from multiplication by the four units:

$$\begin{array}{l} 1\,q = \quad a + b\,i + c\,j + d\,k \\ i\,q = -b + a\,i - d\,j + c\,k \\ j\,q = -c + d\,i + a\,j - b\,k \\ k\,q = -d - c\,i + b\,j + a\,k \end{array} \tag{10}$$

Therefore, the regular representation of quaternions is [2]:

$$a + b\,i + c\,j + d\,k = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \tag{11}$$

Sometimes the regular representation is not faithful, but if the algebra has the unity element, then its regular representation is always faithful. Since every associative algebra can be embedded in an associative algebra with unity element, every associative algebra has a faithful matrix representation [2].

3. Representations of geometric algebras with real square matrices

From now on, matrix representations of geometric algebras with entries in $\mathbb{R}$ will only be considered, although we will also make use of some results about representations over complex numbers and quaternions.
As you have seen in the former lectures, a geometric (Clifford) algebra $Cl(E_n)$ of a geometric vector space E_n with dimension n has dimension 2^n:

$$\dim(E_n) = n \qquad \Rightarrow \qquad \dim Cl(E_n) = 2^n \tag{12}$$

independently of its signature ($n = p + q$). Let us indicate $Cl_n = Cl(E_n)$. Every faithful representation of a Clifford algebra with real square matrices of order k must satisfy:

$$M_{k\times k}(Cl_n) \quad \text{faithful} \qquad \Rightarrow \qquad 2^n \leq k^2 \tag{13}$$

On the other hand, the matrix representation of geometric algebras always have order $k = 2^l$ (see theorem (1) below) so that we have:

$$M_{k\times k}(Cl_n) \quad \text{faithful} \qquad \Rightarrow \qquad 2^n \leq 2^{2l} \tag{14}$$

Therefore, the following inequality is satisfied:

$$n \leq 2l \tag{15}$$

For instance, we have $l = 2$ for the geometric algebra $Cl_{3,0}$, that is, the faithful matrix representation with minimal order has order $2^2 = 4$. Quaternions are the geometric algebra $Cl_{0,2}$ and a subalgebra of $Cl_{3,0}$. Their regular representation (11) has also order 4.
Since the regular representations are obtained from multiplication by the elements of the basis

of the algebra, and $\dim(Cl_n) = 2^n$, the regular representation will be formed by matrices with this order:

$$M(Cl_n) \quad \text{regular} \qquad \Rightarrow \qquad \text{order} \quad M(Cl_n) = 2^n \tag{16}$$

while a faithful representation only needs that:

$$M(Cl_n) \quad \text{faithful} \qquad \Rightarrow \qquad \text{order} \quad M(Cl_n) \geq 2^{n/2} \tag{17}$$

Therefore, faithful representations with order lower than that of regular representation are usually taken.

4. Representations of geometric algebras of lower dimension

Let us see the matrix representations of the smallest geometric algebra.

4.1. **Complex numbers ($\mathbb{C} = Cl_{0,1}$).**

Complex numbers are given by the rule of multiplication:

$$(a+b\,i)(c+d\,i) = a\,c - b\,d + i\,(a\,d + b\,c) \qquad \forall a,b,c,d \in \mathbb{R} \tag{18}$$

The regular representation is obtained from:

$$\begin{aligned} 1\,(a+b\,i) &= a+b\,i \\ i\,(a+b\,i) &= -b+a\,i \end{aligned} \tag{19}$$

whence:

$$a+b\,i = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{20}$$

4.2. **Hyperbolic numbers ($\mathscr{H} = Cl_{1,0}$).**

Hyperbolic numbers are defined by a unit e_1 with positive square. Then their multiplication rule is:

$$(a+b\,e_1)(c+d\,e_1) = a\,c + b\,d + e_1\,(a\,d + b\,c) \qquad \forall a,b,c,d \in \mathbb{R} \tag{21}$$

The regular representation is obtained from:

$$\begin{aligned} 1\,(a+b\,e_1) &= a+b\,e_1 \\ e_1\,(a+b\,e_1) &= b+a\,e_1 \end{aligned} \tag{22}$$

whence:

$$a+b\,e_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{23}$$

4.3. **The geometric algebra of the Euclidean plane $Cl_{2,0}$.** This algebra is defined by two unitary orthogonal vectors in the plane having positive square:

$$e_1^2 = e_2^2 = 1 \qquad e_1 e_2 = -e_2 e_1 = e_{12} \qquad Cl_{2,0} = \langle 1, e_1, e_2, e_{12} \rangle \tag{24}$$

Its minimal faithful representation is:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{25}$$

Every multivector of $Cl_{2,0}$ is then represented by the following matrix:

$$a + b\, e_1 + c\, e_2 + d\, e_{12} = \begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix} \tag{26}$$

$Cl_{2,0}$ has two subalgebras: the hyperbolic numbers and the complex numbers. The complex numbers are given by:

$$z \in \mathbb{C} \qquad \Longleftrightarrow \qquad z = a + b e_{12} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{27}$$

The hyperbolic numbers are given by:

$$t \in \mathscr{H} \qquad \Longleftrightarrow \qquad t = a + b e_1 = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \tag{28}$$

As said above, a matrix representation is defined up to a similarity transformation. Therefore, the representation (28) is as good as (23) and is considered the same matrix representation, since it is only defined by its invariants under similarity transformation of matrices, trace and determinant:

$$\mathrm{tr}(t) = 2a \qquad \det(t) = a^2 - b^2 \tag{29}$$

The advantage of using (28) instead of (23) lies on the fact that it is a diagonal matrix, which makes computations easier.

4.4. **The geometric algebra of the hyperbolic plane ($Cl_{1,1}$).** It is defined by two unit vectors with distinct square:

$$e_1^2 = 1 \qquad e_2^2 = -1 \qquad e_1 e_2 = -e_2 e_1 = e_{12} \qquad Cl_{1,1} = \langle 1, e_1, e_2, e_{12} \rangle \tag{30}$$

Its minimal faithful representation is:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{31}$$

Notice that the difference between (25) and (31) is only an interchange of matrices between e_2 and e_{12}. Therefore $Cl_{2,0}$ and $Cl_{1,1}$ have the same representation and are isomorphic:

$$Cl_{2,0} \simeq Cl_{1,1} \tag{32}$$

The elements of the generator vector space $E = \langle e_1, e_2 \rangle$ are called hyperbolic vectors because of their Lorentzian (hyperbolic) norm:

$$\|a\, e_1 + b\, e_2\|^2 = (a\, e_1 + b\, e_2)^2 = a^2 - b^2 \tag{33}$$

Notice that the square of their norm and their determinant are opposite:

$$\det(a\, e_1 + b\, e_2) = \begin{vmatrix} a & b \\ -b & -a \end{vmatrix} = -a^2 + b^2 \tag{34}$$

This is not a casual coincidence, which will be explained later on. On the other hand, elements of the form $a + b\, e_{12}$ are the even subalgebra of the hyperbolic numbers already found in (23). Summarizing, the matrix algebra $M_{2\times 2}$ is the representation of both geometric algebras of the Euclidean and hyperbolic planes, which are isomorphic $Cl_{2,0} \simeq Cl_{1,1}$. The only difference between $Cl_{2,0}$ and $Cl_{1,1}$ is which vector space is considered as generator of the geometric algebra.

4.5. **Quaternions ($\mathbb{H} = Cl_{0,2}$).** It is the algebra generated by two units having a negative square (imaginary units):

$$e_1^2 = e_2^2 = -1 \qquad e_1 e_2 = -e_2 e_1 \tag{35}$$

Then, their product is also an imaginary unit:

$$e_{12}^2 = -e_1^2 e_2^2 = -1 \tag{36}$$

and we cannot distinguish one unit from the others. Since e_{12} anticommutes with e_1 and e_2 and:

$$e_1 e_{12} = -e_2 \qquad e_2 e_{12} = e_1 \tag{37}$$

we can rename the units as $i = e_1$, $j = e_2$ and $k = e_{12}$ to see that $Cl_{0,2}$ is the algebra of quaternions discovered by Hamilton [3] on October 16$^{\text{th}}$, 1843:

$$i^2 = j^2 = k^2 = -1 \tag{38}$$

$$i\, j = -j\, i = k \qquad j\, k = -k\, j = i \qquad k\, i = -i\, k = j \tag{39}$$

$$Cl_{0,2} = \mathbb{H} = \langle 1, i, j, k \rangle \tag{40}$$

Hamilton deduced that the norm of a quaternion $\|q\|$ is a generalization of the Euclidean norm of complex numbers:

$$\|a + b\, i + c\, j + d\, k\|^2 = a^2 + b^2 + c^2 + d^2 \tag{41}$$

Gibbs pointed out [7] that the norm of a quaternion can be obtained as the square root of the determinant of its matrix representation (11), although it is really its 4$^{\text{th}}$ root:

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = \left(a^2+b^2+c^2+d^2\right)^2 \tag{42}$$

This is not a simple coincidence, which caught my attention and led me to outline a new definition of norm.

4.6. **The geometric algebra of the three-dimensional Euclidean space ($Cl_{3,0}$).** Its geometric algebra $Cl_{3,0}$ is defined by:

$$e_1^2 = e_2^2 = e_3^2 = 1 \qquad e_i\, e_j = -e_j\, e_i \qquad \forall i \neq j \tag{43}$$

Then $Cl_{3,0} = \langle 1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}\rangle$. It is easy to see that the subalgebra of the elements of even grade (called even subalgebra) are quaternions $\mathbb{H} = \langle 1, e_{12}, e_{23}, e_{31}\rangle$. In fact, Hamilton himself [3] deduced quaternions as quotients of two vectors $q = v/w$, meaning $v/w = v\, w^{-1}$. However, a small detail must be taken into account when dealing with quaternions. Hamilton defined quaternions through the anticommutation rule (39). However, the multiplication of the anticommuting bivector units e_{ij} yields a somewhat different result:

$$e_{23}e_{31} = -e_{12} \qquad e_{31}e_{12} = -e_{23} \qquad e_{12}e_{23} = -e_{31} \tag{44}$$

Then, one must take into account a change of sign between Hamilton's and geometric algebra units:

$$i = -e_{23} \qquad j = -e_{31} \qquad k = -e_{12} \tag{45}$$

Therefore, a rotation through an angle θ in the plane of the unitary bivector n is written with Hamilton units as [4]

$$q' = \left(\cos\frac{\theta}{2} + n\sin\frac{\theta}{2}\right)\, q\, \left(\cos\frac{\theta}{2} - n\sin\frac{\theta}{2}\right) \qquad \text{for} \qquad i,j,k \qquad \|n\| = 1 \tag{46}$$

while in geometric algebra:

$$q' = \left(\cos\frac{\theta}{2} - n\sin\frac{\theta}{2}\right)\, q\, \left(\cos\frac{\theta}{2} + n\sin\frac{\theta}{2}\right) \qquad \text{for} \qquad e_{ij} \qquad \|n\| = 1 \tag{47}$$

If we write $r = \cos\frac{\theta}{2} + n\sin\frac{\theta}{2}$, the rotation can be written as:

$$q' = r^{-1}\, q\, r \tag{48}$$

which resembles a similarity transformation of matrices (3). This is not by chance again as we will see later.

4.7. **Pauli matrices.** Pauli matrices were introduced to explain the spin of the electron. Let us recall them:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{49}$$

Notice that:

$$\sigma_i^2 = 1 \qquad \text{and} \qquad \sigma_i\,\sigma_j = -\sigma_j\,\sigma_i \qquad \forall i \neq j \tag{50}$$

which shows that they are a complex representation of $Cl_{3,0}$:

$$\sigma_x = e_1 \qquad \sigma_y = e_2 \qquad \sigma_z = e_3 \tag{51}$$

The direct product by the real representation of complex numbers (27) yields:

$$e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{52}$$

Through the products of these units we obtain the quaternion units:

(53)

$$e_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad e_{23} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad e_{31} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Comparison with eq. (11) leads to the identification $e_{12} = i$, $e_{23} = k$ and $e_{31} = j$. Of course, it can also be used $e_{12} = -k$, $e_{23} = -i$ and $e_{31} = -j$, which means a conjugation plus a rotation of the coordinate axes. Anyway, the matrix representation is the same and the precaution consists of taking a set of matrices that are consistent with the rule of geometric product.

4.8. **Dirac matrices and spacetime geometric algebra.** Dirac matrices γ_i were introduced by P.A.M. Dirac in 1928 [8] when dealing with the relativistic wave equation of the electron. They are:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Anbother matrix sometimes called γ_4 or γ_5 is introduced:

$$\gamma_4 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{55}$$

Dirac matrices form a set of anticommuting units:

$$\gamma_0^2 = \gamma_4^2 = 1 \qquad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \qquad \gamma_i\gamma_j = -\gamma_j\gamma_1 \qquad \forall i \neq j \tag{56}$$

$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ generate the geometric algebra $Cl_{1,3}$ and have the Lorentzian metric of the Minkowski spacetime. Ettore Majorana discovered that, after a suitable choice of the matrices, the Dirac equation for the electron becomes real [9]:

$$\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \nabla) + \beta'\mu\right]\psi = 0 \tag{57}$$

where $\mu = 2\pi mc/h$, and α_i and β' are:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{58}$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \beta' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Multiplication by β' on the left yields:

$$\left[\beta'\frac{1}{c}\frac{\partial}{\partial t} - (\beta'\alpha, \nabla) - \mu\right]\psi = 0 \tag{59}$$

Since the correspondence between energy-momentum operators and partial derivatives is:

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \qquad \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i} \tag{60}$$

we have:

$$e_0 = \beta' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad e_1 = \beta'\alpha_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{61}$$

$$e_2 = \beta'\alpha_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad e_3 = \beta'\alpha_z = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

They are a set of anticommuting matrices:

$$e_i e_j = -e_j e_1 \qquad \text{for } i \neq j \tag{62}$$

which have the metric of Minkowski spacetime:

$$e_0^2 = -1 \qquad e_1^2 = e_2^2 = e_3^2 = 1 \tag{63}$$

and are generators of $Cl_{3,1}$. The faithful matrix representation of $Cl_{3,1}$ with lower dimension is the algebra of all real 4×4 matrices:

$$Cl_{3,1} \simeq M_{4\times 4}(\mathbb{R}) \tag{64}$$

Therefore, the matrix algebra $M_{4\times 4}(\mathbb{R})$ is the real geometric algebra of spacetime.

5. REPRESENTATION OF GEOMETRIC ALGEBRAS OF ANY DIMENSION

What the representations of geometric algebras with square matrices are is a known result stated in the followimg theorem that we recall:

Theorem 1 (Square matrix representations of Clifford algebras [15])**.** *Let $Cl_{p,q}$ be a geometric algebra then:*
1) If $p-q \neq 1 \mod 4$ then $Cl_{p,q}$ is a simple algebra of dimension 2^n, $n = p+q$, isomorphic with a full matrix algebra $M_{2^k\times 2^k}[\mathbb{K}]$ where $k = q - r_{q-p}$ and $\mathbb{K}$ is a division (associative) algebra ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), and r_i is the Radon-Hurwitz number.
2) If $p-q = 1 \mod 4$ then $Cl_{p,q}$ is a semisimple algebra of dimension 2^n, $n = p+q$, isomorphic to $M_{2^{k-1}\times 2^{k-1}}[\mathbb{K}] \oplus M_{2^{k-1}\times 2^{k-1}}[\mathbb{K}]$ where $k = q - r_{q-p}$ and $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{H}$ depending whether $p-q = 1 \mod 8$ or $p-q = 5 \mod 8$. The Radon-Hurwitz number is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0$, $r_1 = 1$, $r_2 = r_3 = 2$, $r_4 = r_5 = r_6 = r_7 = 3$.

Corollary 1 (Real square matrix representations)**.** *The reduced faithful representation of a geometric algebra $Cl_{p,q}$ with real square matrices is a full matrix algebra $M_{2^k\times 2^k}$ or a subalgebra of this matrix algebra such that $n = p+q \leq 2k$.*

Corollary 2 (Central simple algebra)**.** *If $p-q$ is even, $Cl_{p,q}$ is a central simple algebra. If $p-q = 3 \mod 4$ then $Cl_{p,q}$ is a simple algebra but it is no longer central.*

Proof. For $p-q$ (or $n = p+q$) even the pseudoscalar $e_{1\ldots n}$ anticommutes with all 1-vectors, whence it cannot belong to the centre of $Cl_{p,q}$, and the centre is only $\mathbb{R}$. Then, $Cl_{p,q}$ is a central simple algebra. For $p-q$ odd, the pseudoscalar $e_{1\ldots n}$ commutes with all the elements of the geometric algebra, that is, it belongs to the centre of the $Cl_{p,q}$, an algebra over $\mathbb{R}$. Therefore $Cl_{p,q}$ is no longer central. □

Therefore, all the representations of geometric algebras are matrices of order 2^n with entries in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), ${}^2\mathbb{R}$ (hyperbolic numbers) or ${}^2\mathbb{H}$ [5, 6], yielding the classification of geometric algebras given in table 1 for $p+q < 8$. Representations of geometric algebras of higher dimension are obtained from the periodicity theorem [5, 6]:

$$Cl_{p+8,q} \simeq Cl_{p,q+8} \simeq Cl_{p,q} \otimes \mathbb{R}(16) \tag{65}$$

where $\mathbb{R}(16)$ represents the real square matrices of order 16.

Matrix representations of Clifford algebras $Cl_{p,q}$ [6]

$p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$p+q$															
0								$\mathbb{R}$							
1							$\mathbb{C}$		$^2\mathbb{R}$						
2						$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$					
3					$^2\mathbb{H}$		$\mathbb{C}(2)$		$^2\mathbb{R}(2)$		$\mathbb{C}(2)$				
4				$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$			
5			$\mathbb{C}(4)$		$^2\mathbb{H}(2)$		$\mathbb{C}(4)$		$^2\mathbb{R}(4)$		$\mathbb{C}(4)$		$^2\mathbb{H}(2)$		
6		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$	
7	$^2\mathbb{R}(8)$		$\mathbb{C}(8)$		$^2\mathbb{H}(4)$		$\mathbb{C}(8)$		$^2\mathbb{R}(8)$		$\mathbb{C}(8)$		$^2\mathbb{H}(4)$		$\mathbb{C}(8)$

TABLE 1. $A(d)$ means the real algebra of $d \times d$ matrices $M_{d\times d}$ with entries in the ring $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $^2\mathbb{R}$, $^2\mathbb{H}$. $^2\mathbb{R}$ are diagonal matrices with real entries, which are the matrix representation of hyperbolic numbers, and $^2\mathbb{H}$ are diagonal matrices having quaternions as entries.

From table 1, it is easy to obtain the real matrix representation of any geometric algebra by doing the corresponding direct product with the representation of $\mathbb{C}$ (see eq. (27)) or $\mathbb{H}$ (see eq. (11)) if needed.

5.1. **Isomorphisms of Clifford algebras.** As table (5) shows, some geometric algebras have the same matrix representation because they are isomorphic. The isomorphisms of Clifford algebras were studied by Élie Cartan ([12], p. 464), and P. Lounesto reviewed them in [6]. They are $Cl_{p,q} = Cl_{q+1,p-1}$ if $p \geq 1$, $Cl_{p,q} = Cl_{p-4,q+4}$ if $p \geq 4$ and $Cl_{p,q} \otimes Cl_{1,1} = Cl_{p+1,q+1}$.

6. NEW DEFINITIONS AND THEOREMS ABOUT ISOMETRIES

A main question in the theory of transformations among elements of geometric algebra is what we understand by *isometry*. An intuitive definition is that an *isometry* is a transformation preserving the norm (lenght) of 1-vectors. However, if we regard complex numbers or quaternions we will say that an *isometry* is a transformation preserving the norm, but then we will define their norm as the square root of the product by its conjugate. Then, the concept of isometry is linked to the concept of *norm*, which is not trivial except for some particular cases. A point of view from matrix representation can be an advantage since it allows us to give general definitions of all these concepts. Let us see some of them:

Definition 1 (Complete geometric algebra)**.** *A complete geometric algebra is a geometric algebra whose representation is the full real matrix algebra of a given order.*

Since the dimension of a geometric algebra $Cl_{p,q}$ is 2^n, where $n = p+q$, and its real matrix representation is the matrix algebra of order 2^k or some of its subalgebras, if $Cl_{p,q}$ is a complete geometric algebra then:

$$2^n = (2^k)^2 \qquad \Rightarrow \qquad k = \frac{n}{2} \tag{66}$$

Therefore, complete geometric algebras are represented by real matrix algebras of order $2^{n/2}$. Examples of complete geometric algebras are $Cl_{2,0} \simeq Cl_{1,1} \simeq M_{2\times 2}(\mathbb{R})$ or $Cl_{3,1} \simeq Cl_{2,2} \simeq M_{4\times 4}(\mathbb{R})$.

Definition 2 (Norm)**.** *The norm $\|a\|$ of every element a of a geometric algebra is the n^{th}-root of the absolute value of the determinant of its matrix representation $M(a)$ of order n.*

$$\|a\| = \sqrt[n]{|\det(M_{n\times n}(a))|} \tag{67}$$

This extends what was appraised by Gibbs [7] for quaternions (42). As an example of the usefulness of this definition, let us calculate the norm of a bivector w of the space-time algebra $Cl_{3,1}$ from Majorana's representation (61) ([13] p. 13):

$$w = a\, e_{01} + b\, e_{02} + c\, e_{03} + d\, e_{23} + f\, e_{31} + g\, e_{12}$$

$$\det(w) = (a^2 + b^2 + c^2 - d^2 - e^2 - f^2)^2 + 4(a\, f + b\, g + c\, h)^2 \tag{68}$$

$$\|w\| = \sqrt[4]{(a^2 + b^2 + c^2 - d^2 - e^2 - f^2)^2 + 4(a\, f + b\, g + c\, h)^2}$$

If w is the electromagnetic field:

$$w = E_x e_{01} + E_y e_{02} + E_z e_{03} + c\, B_x e_{23} + B_y e_{31} + B_z e_{12} \tag{69}$$

see that

$$a^2 + b^2 + c^2 - d^2 - e^2 - f^2 = E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2) \tag{70}$$

is the true invariant of the electromagnetic field and

$$a\, f + b\, g + c\, h = c(E_x B_x + E_y B_y + E_z B_z) \tag{71}$$

is its pseudoinvariant, which is preserved under Lorentz transformations but changes the sign under inversion of the coordinate axes [14].

Corollary 3. *The norm of a product of elements of a geometric algebra is equal to the product of the norms of each one of them.*

$$\|a\ b\ c\ ...\| = \|a\|\|b\|\|c\|... \qquad a,b,c \in Cl_{p,q} \tag{72}$$

and therefore $\|a^n\| = \|a\|^n$. Since an isometry must preserve the norm, a temptative definition of isometry can be the following.

Definition 3 (Isometries)**.** *Isometries are defined as those transformations T that are linear and commute with the power of every element of the geometric algebra, that is:*

$$T(\lambda a + \mu b) = \lambda T(a) + \mu T(b) \qquad T(a^n) = [T(a)]^n \qquad \forall a,b \in Cl_{p,q} \quad \lambda,\mu \in \mathbb{R} \tag{73}$$

Theorem 2. *Isometries preserve the characteristic polynomial of the matrix representation of every element of geometric algebra.*

Proof. Let $p(x)$ be the characteristic polynomial of a matrix A:

$$p(x) = \det(A - x\,I) = \sum_{i=1}^{k} a_i\, x^i \tag{74}$$

According to the Hamilton-Cayley theorem, the substitution of the matrix A into its own characteristic polynomial $p(x)$ yields zero:

$$p(A) = \det(A - A\,I) = \sum_{i=1}^{k} a_i\, A^i = 0 \qquad a_i \in \mathbb{R} \tag{75}$$

If we apply an isometry to this equality we have:

$$T(p(A)) = T\left(\sum_{i=1}^{k} a_i\, A^i\right) = \sum_{i=1}^{k} a_i\, [T(A)]^i = 0 \tag{76}$$

that is, $T(A)$ is also a root of $p(x)$, or equivalently $p(x)$ is also the characteristic polynomial of $T(A)$. □

Corollary 4. *Isometries preserve the norm of all the elements of a geometric algebra.*

Proof. Since isometries preserve the characteristic polynomial of their matrix representation, and the determinant is its independent term, it follows that the determinant and the norm are preserved. □

Corollary 5. *If $T(w)$ is an isometry of an element w of a geometric algebra and $w\,T(w) \in \mathbb{R}$ then $\|w\| = \sqrt{|w\,T(w)|}$.*

This is an immediate consequence of the corollary (4). Examples are the norm of a complex number or a quaternion. Since the conjugate $q*$ is an isometry of q and $q\,q* > 0$ then we have:

$$q \in \mathbb{H} \qquad \|q*\| = \|q\| \qquad \Rightarrow \qquad \|q\| = \sqrt{q\,q*} \tag{77}$$

Another example is the norm of a space-time bivector (68). By doing the square of the bivector w we obtain:

$$w^2 = a^2 + b^2 + c^2 - d^2 - e^2 - f^2 + 2(a\,d + b\,f + c\,g)e_{0123} \tag{78}$$

where it has been taking into account that $e_{01}^2 = e_{02}^2 = e_{03}^2 = 1$ and $e_{23}^2 = e_{31}^2 = e_{12}^2 = -1$. Since $e_{0123}^2 = -1$, w^2 is a complex number whose norm is well known. Then, according to (27), $\det(w^2)$ is just given by the determinant (68), whence $\|w\|$ follows.

Corollary 6. *Isometries leave invariant the identity and real numbers.*

Proof. It is a consequence of the application of the definition of isometry (3) to a power of the identity I:

$$I^2 = I \quad \Rightarrow \quad T(I^2) = T(I) \quad \Longleftrightarrow \quad [T(I)]^2 = T(I)I \quad \Longleftrightarrow \quad T(I) = I \tag{79}$$

Notice that the last simplification needs the existence of $[T(I)]^{-1}$, that is, it needs that $\det[T(I)] \neq 0$, which is warranted because $\det[T(I)] = \det(I) = 1$. The last simplification could not be carried on for idempotents, whose determinant is null. Then idempotents can be changed by isometries. By linearity we see that isometries also preserve real numbers:

$$T(\lambda I) = \lambda T(I) = \lambda I \qquad \lambda \in \mathbb{R} \tag{80}$$

□

Definition (3) implies that $T(A^2) = [T(A)]^2$, but what about $T(A\ B)$? We have two options giving two definitions.

Definition 4 (Direct isometries)**.** *Direct isometries T of a geometric algebra $Cl_{p,q}$ are defined as its automorphisms:*

$$T(\lambda a + \mu b) = \lambda T(a) + \mu T(b) \qquad T(a\,b) = T(a)T(b) \qquad \forall a,b \in Cl_{p,q} \quad \lambda,\mu \in \mathbb{R} \tag{81}$$

Definition 5 (Indirect isometries)**.** *Indirect isometries T of a geometric algebra $Cl_{p,q}$ are defined as its antiautomorphisms:*

$$T(\lambda a + \mu b) = \lambda T(a) + \mu T(b) \qquad T(a\,b) = T(b)T(a) \qquad \forall a,b \in Cl_{p,q} \quad \lambda,\mu \in \mathbb{R} \tag{82}$$

Definition 6 (Units)**.** *A unit of a geometric algebra is a matrix with square equal to $\pm I$ (the identity) obtained from tensor products of the four units of $Cl_{2,0} \simeq Cl_{1,1}$ or from isometries of these tensor products.*

Let us recall the four units (25) or (31):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{83}$$

For instance, a unit of the geometric algebra $Cl_{3,1} = Cl_{2,0} \otimes Cl_{2,0}$ would be:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{84}$$

However, the following matrix, despite having as square $\pm I$, will not be considered a unit:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{85}$$

because it cannot be obtained as tensor products of (83). How units look like are clearly shown in all the matrices (not only γ_i) considered by Dirac [8]. Despite being complex, they also fit the definition (6). Here we only consider real matrices. On the other hand, it is not needed to have only 0 and 1 as entries. For instance, an isometric matrix of the second unit of (83) must have the same trace TrM=0 and determinant $\det M = -1$ and can be:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{86}$$

It is, in fact, a linear combination of the second and four units, which are isometric.

Theorem 3 (Pseudo-Pythagorean)**.** *Let $\{u_i\}$ be a set of anticommuting units. Then they satisfy:*

$$\|\sum_i a_i u_i\| = \sqrt{|\sum_i a_i^2 \chi_i|} \qquad a_i \in \mathbb{R} \qquad \chi_i = u_i^2 = \pm 1 \tag{87}$$

If all the units have the same square, it becomes the Pythagorean theorem.

Proof. Let us calculate the square of a linear combination of anticommuting units $w = \sum_i a_i e_i$:

$$w^2 = \left(\sum_i a_i u_i\right)^2 = \sum_i a_i^2 u_i^2 = \sum_i a_i^2 \chi_i I \tag{88}$$

since the crossed terms cancel out owing to the anticommutativity. Then the determinant is:

$$\det\left(w^2\right) = (\det(w))^2 = \left(\sum_i a_i^2 \chi_i\right)^n \qquad \Rightarrow \qquad \|w\| = \sqrt{|\sum_i a_i^2 \chi_i|} \tag{89}$$

The absolute value arises because n (the order of the matrix representation of a geometric algebra) is always an even number. □

Notice that the only supposition is that they are anticommuting units, but there are no considerations about whether they are generators or have the same grade.

Theorem 4. *Isometries transform a set of anticommuting units into another set of anticommuting units.*

Proof. Let $\{u_k\}$ be a set of anticommuting units. Then, for each pair u_i and u_j of them, we have:

$$i \neq j \quad u_i\, u_j + u_j\, u_i = 0 \quad \Longleftrightarrow T(u_i u_j + u_j u_i) = T(u_i)T(u_j) + T(u_j)T(u_i) = 0 \tag{90}$$

This equality is satisfied for $T(a\ b) = T(a)T(b)$ as well as for $T(a\ b) = T(b)T(a)$. Then $\{T(u_i)\}$ is a set of orthogonal units. □

Notice that no consideration about grades is present, and only anticommutativity is considered.

Corollary 7. *Those isometries preserving grades transform orthogonal vectors into orthogonal vectors, since they are anticommuting.*

Definition 7. *Two elements are said to be isometric if one can be obtained from the other through any isometry.*

Theorem 5. *Isometric units always have the same square.*

Proof. Let us denote by $\chi_i = u_i^2 = \pm 1$. Then we have:

$$u_j = T(u_i) \quad \Rightarrow \quad u_j^2 = \chi_j = T(u_i^2) = \chi_i \tag{91}$$

since isometries preserve real numbers according to the corollary (6). □

Theorem 6 (Equivalence)**.** *To be isometric is an equivalence relation (indicated with $\sim$), and two isometric elements are equivalent.*

Proof. Let us check the three properties of equivalence relations:
1. Reflexive property. Each element is isometric to itself by the identity.
2. Symmetric property. If $b = T(a)$ then $a = T^{-1}(b)$. Let us verify that T^{-1} is also an isometry.

$$[T^{-1}(T(A))]^n = A^n = T^{-1}(T(A^n)) = T^{-1}([T(A)]^n) \tag{92}$$

3. Transitive property. If $b = T(a)$ and $c = S(b)$ then $c = S(T(b))$. Let us check that $S \circ T$ is also an isometry:

$$(S \circ T)(A^n) = S(T(A^n)) = S([T(A)]^n) = [S(T(A))]^n = [(S \circ T)(A)]^n \qquad A \in Cl_{p,q} \tag{93}$$

In properties 2 and 3 linearity is also easily checked. □

Theorem 7. *A set of anticommuting units that have the same square are equivalent.*

Proof. Let u_1 and u_2 be two anticommuting units that have the same square. Then:

$$u_2^2 = u_1^2 \qquad \Rightarrow \qquad u_2 = u_1\, u_1\, u_2^{-1} = u_1\, u_{12}\chi \tag{94}$$

where $\chi = u_1^2 = u_2^2 = \pm 1$. The fact that both units have the same square implies that $u_{12}^2 = -1$, that is, u_{12} behaves as the imaginary unit. Let us separate two cases depending on the value of χ:

1) If $\chi = +1$ then we have:

$$\begin{aligned} u_2 = u_1\, u_{12} &= u_1 \left(\cos\frac{\pi}{4} + u_{12}\sin\frac{\pi}{4}\right)^2 \\ &= \left(\cos\frac{\pi}{4} - u_{12}\sin\frac{\pi}{4}\right) u_1 \left(\cos\frac{\pi}{4} + u_{12}\sin\frac{\pi}{4}\right) = t^{-1}\, u_1\, t \end{aligned} \tag{95}$$

2) If $\chi = -1$ then we have:

$$\begin{aligned} u_2 = -u_1\, u_{12} &= u_1 \left(\cos\frac{3\pi}{4} + u_{12}\sin\frac{3\pi}{4}\right)^2 \\ &= \left(\cos\frac{3\pi}{4} - u_{12}\sin\frac{3\pi}{4}\right) u_1 \left(\cos\frac{3\pi}{4} + u_{12}\sin\frac{3\pi}{4}\right) = t'^{-1}\, u_1\, t' \end{aligned} \tag{96}$$

In both cases, each unit is obtained from the other by means of a similarity transformation, which is a direct isometry. In fact, these isometries are rotations in the Euclidean plane u_{12} through an angle $\pm\pi/2$. Owing to the transitivity of the equivalence relation, this result applies

also to a set of any number of anticommuting units whenever they have the same square. Then, in $Cl_{p,q}$, all the p units of square $+1$ are equivalent among them, and so are the q units of square -1. □

7. Direct isometries

Many direct isometries such as rotations or axial symmetries can be written as similarity transformation of matrices:

$$T(v) = t^{-1} v\, t \qquad \det t \neq 0 \tag{97}$$

This operator is general and can be applied to every element of a geometric algebra. But, can all direct isometries be written as similarity transformation of matrices? The Skolem-Noether theorem gives us the answer.

Theorem 8 (Skolem-Noether)**.** *Let R, S be finite dimensional algebras, R simple and S central simple. If $f,\ g\ :\ R \to S$ are homomorphisms then there is an element $s \in S$ such that, for all $r \in R$, $g(r) = s^{-1} f(r)\ s$.* [16].

Corollary 8. *Every automorphism of a central simple algebra is an inner automorphism.*

Proof. If $R = S$, then f, g are inner automorphisms. Take f as the identity, then for every automorphism $g(r)$, there exists $s \in S$ such that $g(r) = s^{-1} r\ s$. □

Theorem 9. *In the geometric algebras $Cl_{p,q}$ with $n = p + q$ even (which include complete geometric algebras), every direct isometry can be written as a similarity transformation of matrices:*

$$\forall v, w \in Cl_{p,q} \quad \left.\begin{array}{c} T(v\ w) = T(v)T(w) \\ T(\lambda v + \mu w) = \lambda T(v) + \mu T(w) \end{array}\right\} \Rightarrow T(v) = t^{-1} v\, t \quad \textit{if} \quad p - q \neq 1 \tag{98}$$

Proof. According to corollary (2) geometric algebras $Cl_{p,q}$ with $p + q$ even are central simple algebras, where the Skolem-Noether theorem (8) applies to. Therefore, every direct isometry (automorphism) is an inner automorphism, that is, a similarity transformation of matrices. □

7.1. **Axial symmetries.** In axial symmetries the component of a vector with the direction of a given axis is preserved, while the components orthogonal to the axis are reversed. The operator is simply the direction vector d of the axis:

$$v' = d^{-1} v\, d = \frac{d\ v\ d}{d^2} \tag{99}$$

In the particular case that v is a 1-vector, it can be resolved into proportional and perpendicular components, and then:

$$v' = d^{-1} v\, d = \frac{d\ (v_{\parallel} + v_{\perp})\ d}{d^2} = v_{\parallel} - v_{\perp} \tag{100}$$

7.2. **Reflections.** Reflections in a $n-1$-hyperplane of $Cl_{p,q}$ ($n = p+q$) preserve all the components except that being perpendicular to the hyperplane. Their operator is $e_{1\ldots n-1}$, where n is the direction perpendicular to the $n-1$-hyperplane:

$$v' = e_{1\ldots n-1}^{-1} v\, e_{1\ldots n-1} \tag{101}$$

If v is a 1-vector, we can resolve it into coplanar and perpendicular components $v_\parallel$ and $v_\perp$ respectively:

$$v' = e_{1\cdots n-1}^{-1}(v_\parallel + v_\perp)e_{1\cdots n-1} \tag{102}$$

Now we make use of the fact that $p+q = 2k$ is an even number for the complete geometric algebras. Therefore $n-1 = 2k-1$ is an odd number, and there is an odd number of permutations in order to change the order of $v_\perp$ and $e_{1\cdots n-1}$:

$$v_\perp e_{1\cdots n-1} = -e_{1\cdots n-1} v_\perp \tag{103}$$

The coplanar component $v_\parallel$ can be resolved into components of the vectors of the plane:

$$v_\parallel = \sum_{i=1}^{n-1} \lambda_i e_i \qquad \lambda_i \in (R) \tag{104}$$

which commute with $e_{1\cdots n-1}$:

$$v_\parallel e_{1\cdots n-1} = \sum_{i=1}^{n-1} \lambda_i e_i\, e_{1\cdots n-1} = e_{1\cdots n-1} \sum_{i=1}^{n-1} \lambda_i e_i = e_{1\cdots n-1} v_\parallel \tag{105}$$

since one e_i is the same, and there are only $n-2$ vectors in $e_{1\cdots n-1}$ with which e_i anticommutes. Gathering both results we have:

$$v' = e_{1\cdots n-1}^{-1}(v_\parallel + v_\perp)e_{1\cdots n-1} = v_\parallel - v_\perp \tag{106}$$

which is the sought result for a reflection in a hyperplane perpendicular to the e_n direction. If the hyperplane is perpendicular to any given direction d, then the reflection operator is $e_{1\cdots n}d$ and:

$$v' = (e_{1\cdots n}d)^{-1} v\, e_{1\cdots n}d \tag{107}$$

Therefore, reflections can always be written as similary transformations according to (101) in geometric algebras having an even number of generators, including complete geometric algebras. Reflections cannot be written as similarity transformations in a geometric algebra with an odd number of generators, which includes the important case of Cl_3, the geometric algebra of the Euclidean space. This means that, in order to describe reflections, different operators must be applied to elements of different grade raising the trouble of knowing which is the reflection operator for elements with mixed grade, such as spinors. This clearly shows the incompleteness of some geometric algebras such as Cl_3. Moreover, the general reflection operator which allows to write reflections in a plane of the Euclidean space as a similarity transformation is a 4×4 real matrix with the same order as the matrix representation of Cl_3, but it is simply not

included in Cl_3. Programmers have an immediate application here. They are using the same matrix algebra. and must only take the suitable matrix for the reflection operator. They are not exiting the matrix algebra although they are going out of Cl_3, but this is no trouble for the program, which is really working in $M_{4\times4}(\mathbb{R}) \simeq Cl_{3,1}$.

7.3. **Rotations.** Rotations are isometries that change the components in a plane of the geometric algebra, not necessarily of two vectors. Let u_1 and u_2 be two anticommuting units of the geometric algebra. Then, rotations in the plane u_{12} through an angle ξ are given by:

$$v' = \exp\left(-\frac{\xi u_{12}}{2}\right) v \, \exp\frac{\xi u_{12}}{2} \qquad \xi \in \mathbb{R} \tag{108}$$

The operator $\exp(\xi u_{12}/2)$ is usually called a *rotor*. Now, we have two cases:
1) $u_1^2 = u_2^2$. Since $u_{12}^2 = -1$, the exponential is complex and the plane is Euclidean:

$$v' = \left(\cos\frac{\xi}{2} - u_{12}\sin\frac{\xi}{2}\right) v \left(\cos\frac{\xi}{2} + u_{12}\sin\frac{\xi}{2}\right) \tag{109}$$

v' is a periodic function of the circular angle ξ with period 4π instead of 2π! This is a very known fact in quantum mechanics [10, 11]. The true identity is only achieved after a rotation through 4π. Although an exact mathematical identity is obtained for $\xi = 2\pi$, this is not the case for neighbouring values and therefore there is not identity in quantum mechanics owing to the uncertainty principle.

2) $u_1^2 = -u_2^2$. Since $u_{12}^2 = 1$, the exponential and the plane are hyperbolic:

$$v' = \left(\cosh\frac{\psi}{2} - u_{12}\sinh\frac{\psi}{2}\right) v \left(\cosh\frac{\psi}{2} + u_{12}\sinh\frac{\psi}{2}\right) \tag{110}$$

If space-time is considered, the hyperbolic rotation is called *Lorentz transformation*, and the rotation operator is called a *boost*. The relation between the argument of the hyperbolic rotation ψ and the relative velocity V of two inertial frames is given by:

$$\tanh\psi = \frac{V}{c} \tag{111}$$

where c is the light speed.
The rotation (108) leaves invariant all the components orthogonal to the plane u_{12}: if the unit u_3 anticommutes with u_1 and u_2 then commutes with u_{12} and does not change under rotation: $u_3' = u_3$. This also applies to every linear combination of units orthogonal to the plane u_{12}.

7.4. **Duality.** Exterior and geometric algebras are graded algebras, whose elements of grade k called k-multivectors are obtained from successive products of elements of the generator space E, called *geometric vectors* or simply 1-vectors. Duality is the inversion of grades. The dual of a k-multivector is defined as the $(n-k)$-multivector obtained from multiplication by the pseudoscalar $e_{1\cdots n}$:

$$v' = v\, e_{1\cdots n} \tag{112}$$

This is the so called *Hodge duality*. However, this expresion is not a similarity transformation, and then another expression must be sought. In order to split the Hodge duality operator

into two parts like (95), one needs the pseudoscalar $e_{1\cdots n}$ to anticommute with all the e_i, the generator units of the geometric algebra:

$$e_1\, e_{1\cdots n} = -e_{1\cdots n} e_i \tag{113}$$

and to have negative square:

$$e_{1\cdots n}^2 = -1 \tag{114}$$

When is this last condition satisfied for $Cl_{p,q}$? Let us calculate the square of the pseudoscalar:

$$e_{12\cdots n}^2 = (-1)^{n-1+n-2+\cdots+1} e_1^2 e_2^2 \cdots e_n^2 = (-1)^{\frac{n(n-1)}{2}+q} = (-1)^{\frac{n^2+q-p}{2}} \tag{115}$$

where $n = p+q$ as usual. The anticommutation (113) of the pseudoscalar with 1-vectors only takes place for geometric algebras with an even number of generators $n = 2k$ because then there is an odd number $n-1$ of permutations (e_i repeated once). In this case:

$$n = 2k \qquad \Rightarrow \qquad e_{12\cdots n}^2 = (-1)^{\frac{q-p}{2}} \tag{116}$$

Then, the pseudoscalar has negative square if $p-q = 2 \mod 4$ [17]. Complete geometric algebras always have an even number of generators, but they do not sometimes satisfy the condition $p-q = 2 \mod 4$. In this way let us see the following isomorphism theorem:

Theorem 10 (Binary isomorphism)**.** *$Cl_{p,q} \simeq Cl_{p+1,q-1}$ if $p-q = 0 \mod 4$ and $q \geq 1$.*

Proof. If $q-p = 0 \mod 4$, $e_{12\cdots n}^2 = 1$ and $n = p+q$ is even. Since the pseudoscalar $e_{12\cdots n}$ anticommutes with e_i for n even, it can be renamed as $e'_n = e_{12\cdots n}$ and taken as generator. Then $\{e_1, e_2, \cdots, e'_n\}$ generate $Cl_{p+1,q-1}$, and $e'^2_{12\cdots n} = (-1)^q e_n$ is the new pseudoscalar with negative square. □

Therefore, each Clifford algebra with $p-q = 0 \mod 4$ is isomorphic to another Clifford algebra with $p-q = 2 \mod 4$ and vice versa. This isomorphism applies to all the Clifford algebras with $n = p+q$ even. Some examples are $Cl_{1,1} \simeq Cl_{2,0}$, $Cl_{2,2} \simeq Cl_{3,1}$, $Cl_{3,3} \simeq Cl_{4,2}$, $Cl_{0,4} \simeq Cl_{1,3}$, $Cl_{1,5} \simeq Cl_{2,4}$, etc. An example are Dirac's matrices γ_i, generators of $Cl_{1,3} \simeq Cl_{0,4}$. Now, the role γ_4 (55) plays with regard to the set of generators $\{\gamma_0, \gamma_1, \gamma_2, \gamma 3\}$ becomes clear. It is the pseudoscalar of the algebra multiplied by i that can exchange its role and also act as generator. Complete geometric algebras always have an even number of generator units, and the binary isomorphism (theorem (10)) assures us that they have a Clifford structure with $p-q = 2 \mod 4$ in which the pseudoscalar is an imaginary unit. Let us then suppose that this is the case. Then, we can write Hodge duality for a vector v as:

$$v' = v\, e_{1\cdots n} = v\, \frac{(1+e_{1\cdots n})^2}{2} = \frac{1-e_{1\cdots n}}{\sqrt{2}}\, v\, \frac{1+e_{1\cdots n}}{\sqrt{2}} \tag{117}$$

where we have used De Moivre's identity $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2$ considering $e_{1\cdots n} = i$ as the imaginary unit. Therefore, duality can be written as a similarity transformation:

$$v' = t^{-1} v\, t \qquad t = \frac{1 + e_{1\cdots n}}{\sqrt{2}} \qquad \text{or simply} \qquad t = 1 + e_{1\cdots n} \tag{118}$$

for Clifford structures $p - q = 2 \mod 4$, which include all the complete geometric algebras if a suitable election of generators is taken. This case includes the important space-time algebra $Cl_{3,1}$.

Warning. The operator (118) acts exactly as Hodge duality for the grades $1, \cdots, n-1$ but it leaves scalars and pseudoscalars invariant. For instance, we have:

$$e'_1 = e_{2\cdots n} \tag{119}$$

Then we expect that:

$$1' = (e_1^2)' = (e'_1)^2 = {e_{2\cdots n}}^2 = 1 \tag{120}$$

This is ensured by a similarity transformation. Then it is not right and makes no sense to say that the dual of scalars are pseudoscalars and vice versa. Notice that if instead of $\{e_i\}$ you take as generators of the complete geometric algebra their dual units $\{e_{1\cdots i-1,i+1,\cdots n}\}$, their product generates the same pseudoscalar. Then, duality does not change pseudoscalars as commonly argued in some textbooks, and its right operator is (118).

Let us now see whether the three main involutions can be written as similarity transformations of matrices.

7.5. **Grade involution.** It consist of changing the direction of 1-vectors. For geometric algebras with an even number of generator units, the pseudoscalar anticommutes with vectors, which suffices to state the grade involution operator:

$$v' = (e_{1\cdots n})^{-1}\, v\, e_{1\cdots n} = -v \qquad \text{if} \qquad n = 2k \tag{121}$$

As usual, the similarity transformation operator applies to all the elements of geometric algebra. Then 2-vectors do not change, 3-vector changes the sign and so on. The grade involution is clearly a direct isometry.

8. INDIRECT ISOMETRIES

Let us recall the definition (5) of indirect isometries. Indirect isometries are the antiautomorphisms of a geometric algebra, which change the order of the factors when applied to a product. An example of antiautomorphism (and involution) of geometric algebras is reversion.

8.1. **Reversion.** Reversion R is the transformation that changes the order of the basis vectors in any given multivector [18]:

$$R(e_{i\cdots k}) = e_{k\cdots i} \tag{122}$$

The number of necessary permutations of adjacent vectors in order to reverse the unit $e_{1\cdots k}$ is $k(k-1)/2$:

$$e_{1\cdots k} = (-1)^{k-1+k-2+\cdots+1} e_{k\cdots i} = (-1)^{\frac{k(k-1)}{2}} e_{k\cdots i} \tag{123}$$

Depending on the value of $k \mod 4$ we have the following cases:

$$k \mod 4 = \begin{cases} 0,\ 1 & \Rightarrow & \frac{k(k-1)}{2} & \text{even} & \Rightarrow & R\langle e\rangle_k = \langle e\rangle_k \\ 2,\ 3 & \Rightarrow & \frac{k(k-1)}{2} & \text{odd} & \Rightarrow & R\langle e\rangle_k = -\langle e\rangle_k \end{cases} \tag{124}$$

where $\langle e\rangle_k$ indicates not only $e_{1\cdots k}$ but any other unit of grade k. Then scalars and 1-vectors do not change the sign, 2-vectors and 3-vectors change the sign and so on.
Reversion is an involutory antiautomorphism:

$$\forall\, u_i \in Cl_{p,q} \qquad R(u_1 \cdots\ u_n) = R(u_n)\cdots R(u_1) \tag{125}$$

$$R(R(u_1 \cdots\ u_n)) = u_1 \cdots\ u_n \tag{126}$$

If u_i are vectors then reversion simply changes their order:

$$(u_1 \cdots\ u_n)' = u_n \cdots u_1 \qquad \forall\, u_i \in \left\langle Cl_{p,q}\right\rangle_1 \tag{127}$$

which is a consequence of (124) owing to the distributive property of the geometric product.
Let us see the following theorem about indirect isometries.

Theorem 11 (Inner antiautomorphism)**.** *Any indirect isometry of a geometric algebra $Cl_{p,q}$ with $n = p+q$ even (including complete geometric algebras) is the composition of reversion with a similarity transformation of matrices.*

Proof. Let S be an indirect isometry, R the reversion and $T = S \circ R$ a direct isometry. Then we have:

$$\begin{aligned} S(u\ v) = S(v)S(u) \qquad \Rightarrow \qquad & T(u\ v) = S(R(u\ v)) = S(R(v)R(u)) \\ & = S(R(u))S(R(v)) = T(u)T(v) \end{aligned} \tag{128}$$

On the other hand, S and R are linear and therefore T. In consequence T is an automorphism and a direct isometry. Since every direct isometry T can be written as a similarity transformation of matrices for $n = p+q$ even, we finally have:

$$S(u\ v) = S(v)S(u) \quad \Rightarrow \quad S(u) = t^{-1}R(u)\ t \qquad \forall u, v \in Cl_{p,q} \qquad p+q = 2k \tag{129}$$

□

Let us now review some indirect isometries:

8.2. **Clifford conjugation.** Clifford conjugation $\sim$ is the composition of reversion and grade involution. Therefore, Clifford conjugation is an involutory antiautomorphism and an indirect isometry:

$$u' = (e_{1\cdots n})^{-1}R(u)\ e_{1\cdots n} \tag{130}$$

Chisholm and Farwell [18] gave how the signs of multivectors change for reversion, grade involution and Clifford conjugation. Their table 27.1.3 can be summarized in the following table:

Sign changes of $\langle e \rangle_k$ under main involutions of Clifford algebras $Cl_{p,q}$

$0 \le k \le n = p+q$	Reversion	Grade involution	Clifford conjugation
$k = 0 \mod 4$	+	+	+
$k = 1 \mod 4$	+	−	−
$k = 2 \mod 4$	−	+	−
$k = 3 \mod 4$	−	−	+

TABLE 2. + means that $\langle e \rangle_k$ is preserved, and − means that its sign is changed under the involution: $\langle e \rangle'_k = -\langle e \rangle_k$.

8.3. **Transposition or Hermitian conjugation.** Transposition (usually denoted with ^{T}) is the transformation consisting of the transposition of the real matrix representation of every element in a geometric algebra. Transposition is an antiautomorphism:

$$(u\,v)^T = v^T u^T \qquad (\lambda u + \mu v)^T = \lambda u^T + \mu v^T \qquad \forall u, v \in Cl_{p,q} \quad \lambda, \mu \in \mathbb{R} \tag{131}$$

Let us see that complex numbers become conjugate under transposition:

$$(a + b\,i)^T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a - b\,i = (a + b\,i)^* \tag{132}$$

The Hermitian conjugate (usually represented with †) of a matrix with complex entries is defined as the transpose matrix with complex conjugate entries. If a complex matrix is expanded by means of direct product with the real representation of complex numbers given above, the Hermitian conjugate corresponds to the transposition of the expanded real matrix. Let us see, how units change under transposition.

Theorem 12 (Symmetry of unit matrices)**.** *The real representations of units with positive or negative square are respectively symmetric or skew-symmetric matrices.*

Proof. Let u be a unit having positive square, and let v be an eigenvector (column matrix) of u with eigenvalue λ. Then

$$u^2 = I, \quad u\,v = \lambda\,v \quad \Rightarrow \quad u^2\,v = v = \lambda^2 v \quad \Rightarrow \quad \lambda^2 = 1 \tag{133}$$

where I is the identity matrix. Multiplying $u\,v$ by the transposed matrices we have:

$$v^T u^T u\,v = v^T \lambda^2 v = v^T v \quad \Rightarrow \quad u^T u = I \tag{134}$$

But u is a unit such that $u^2 = I$. Then:

$$u^T u = I = u^2 \quad \Rightarrow \quad u^T = u^2 u^{-1} = u \tag{135}$$

That is, the real representation of a unit having positive square is a symmetric matrix:

$$u^2 = I \qquad \Rightarrow \qquad u^T = u \tag{136}$$

Let us now suppose that the unit u has negative square, and let v be an eigenvector (column matrix) of u with eigenvalue λ:

$$u^2 = -I, \quad u\,v = \lambda\,v \quad \Rightarrow \quad u^2\,v = -v = \lambda^2 v \quad \Rightarrow \quad \lambda^2 = -1 \tag{137}$$

Since the eigenvalues are imaginary, there are invariant subspaces for v with dimension 2, and λ must be treated as a matrix:

$$\lambda = \pm i = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{138}$$

Multiplying $u\,v$ by the transposed matrices we have:

$$v^T u^T u\,v = v^T \lambda^T \lambda v = v^T v \quad \Rightarrow \quad u^T u = I \tag{139}$$

But u is a unit such that $u^2 = -I$. Then:

$$u^T u = I = -u^2 \quad \Rightarrow \quad u^T = -u^2 u^{-1} = -u \tag{140}$$

That is, the real representation of units with negative square are skew-symmetric matrices:

$$u^2 = -I \qquad \Rightarrow \qquad u^T = -u \tag{141}$$

□

In fact, Hermitian conjugation has been already defined from its action under the units of the geometric algebra[18]:

$$\begin{array}{lll} e^\dagger_{i\cdots k} = e_{i\cdots k} & \Longleftrightarrow & e^2_{i\cdots k} = I \\ e^\dagger_{i\cdots k} = -e_{i\cdots k} & \Longleftrightarrow & e^2_{i\cdots k} = -I \end{array} \tag{142}$$

The operator of Hermitian conjugation must discriminate units with positive square from units with negative square, and it seems that in $Cl_{p,q}$ the unit $e_{1\cdots p}$ could be the suitable operator. Let us see how it is exchanged with vector units:

$$\begin{array}{lll} p \quad \text{odd} \quad \Rightarrow & \left\{ \begin{array}{ll} e_i e_{1\cdots p} = e_{1\cdots p} e_i & \text{if} \quad i \leq p \\ e_i e_{1\cdots p} = -e_{1\cdots p} e_i & \text{if} \quad i > p \end{array} \right. \\ p \quad \text{even} \quad \Rightarrow & \left\{ \begin{array}{ll} e_i e_{1\cdots p} = -e_{1\cdots p} e_i & \text{if} \quad i \leq p \\ e_i e_{1\cdots p} = e_{1\cdots p} e_i & \text{if} \quad i > p \end{array} \right. \end{array} \tag{143}$$

where $1 \leq i \leq n = p+q$. On the other hand, we also have:

$$
\begin{array}{lll}
q \quad \text{odd} \quad \Rightarrow & \left\{ \begin{array}{ll} e_i e_{p+1\cdots p+q} = -e_{p+1\cdots p+q} e_i & \text{if} \quad i \leq p \\ e_i e_{p+1\cdots p+q} = e_{p+1\cdots p+q} e_i & \text{if} \quad i > p \end{array} \right. \\
q \quad \text{even} \quad \Rightarrow & \left\{ \begin{array}{ll} e_i e_{p+1\cdots p+q} = e_{p+1\cdots p+q} e_i & \text{if} \quad i \leq p \\ e_i e_{p+1\cdots p+q} = -e_{p+1\cdots p+q} e_i & \text{if} \quad i > p \end{array} \right.
\end{array}
\tag{144}
$$

The suitable anticommutation with unit vectors having negative square (Hermitian conjugation) is only possible for p odd or q even. All geometric algebras admit one of both possibilities. If p is even and q is odd, then the isomorphism $Cl_{p,q} \simeq Cl_{q+1,p-1}$ (valid for all geometric algebras[6]) changes the parity of p and q. Then, the general operator of Hermitian conjugation is:

$$
\forall v \in Cl_{p,q} \left\{ \begin{array}{lll} p \quad \text{odd} & v^\dagger = \left(e_{1\cdots p}\right)^{-1} R(v) e_{1\cdots p} \\ \\ q \quad \text{even} & v^\dagger = \left(e_{p+1\cdots p+q}\right)^{-1} R(v) e_{p+1\cdots p+q} \end{array} \right.
\tag{145}
$$

Of course, reversion must also be applied since Hermitian conjugation is the same as transposition of real matrices, which is an antiautomorphism. In the important case of space-time geometric algebra $Cl_{3,1}$, Hermitian conjugation is obtained through the operator e_{123}.

$$
v^\dagger = (e_{123})^{-1} R(v) e_{123} \qquad \forall v \in Cl_{3,1}
\tag{146}
$$

For instance:

$$
e_0^\dagger = (e_{123})^{-1} e_0 e_{123} = -(e_{123})^{-1} e_{123} e_0 = -e_0
\tag{147}
$$

$$
e_{123}^\dagger = (e_{123})^{-1} R(e_{123}) e_{123} = (e_{123})^{-1} e_{321} e_{123} = (e_{123})^{-1} = -e_{123}
\tag{148}
$$

since $e_{123}^2 = -1$.
From the Hermitian conjugation (145), we can isolate reversion to get a useful formula for reversion:

$$
\forall v \in Cl_{p,q} \left\{ \begin{array}{lll} p \quad \text{odd} & R(v) = e_{1\cdots p}\, v^\dagger \left(e_{1\cdots p}\right)^{-1} \\ \\ q \quad \text{even} & R(v) = e_{p+1\cdots p+q}\, v^\dagger \left(e_{p+1\cdots p+q}\right)^{-1} \end{array} \right.
\tag{149}
$$

The equalities (124) yield a fast algorithm for computing reversion with matrix representation. If the matrices are real, Hermitian conjugation consists of the very easy algorithm of taking the transpose matrix. Then, after multiplication by the matrix representation of the product of the units with positive or negative square depending on the case, we have reversed the matrix very fastly. For instance, in $Cl_{3,1} \simeq M_{4\times 4}$, using the Majorana representation (61) we have for every element represented by a matrix A:

$$
R(A) = e_{123}\, A^T \left(e_{123}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A^T \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\tag{150}
$$

without resolving A in multivectors.

Since every square matrix can be resolved into an addition of a symmetric matrix and a skew-symmetric matrix in a unique way, the linear space of square matrices of order m is direct sum of the linear space of skew-symmetric matrices of the same order, which has dimension $m(m-1)/2$, and the linear space of symmetric matrices, which has dimension $m(m+1)/2$. Therefore, the total number w of units with positive and negative square in the basis of a complete geometric algebra $Cl_{p,q}$ is:

$$w_+ = 2^{\frac{n}{2}-1}(2^{\frac{n}{2}}+1) \qquad w_- = 2^{\frac{n}{2}-1}(2^{\frac{n}{2}}-1) \qquad n = p+q \tag{151}$$

because $m = 2^{\frac{n}{2}}$. For instance $Cl_{3,1} \simeq M_{4\times 4}(\mathbb{R})$ has $n = 4$, $w_+ = 10$ and $w_- = 6$.

9. Conclusions

The norm of every element v of a geometric algebra is defined as the k^{th}-root of the absolute value of the determinant of its real square matrix representation $M_{k\times k}(v)$, and isometries are defined as those transformations that commute with the power of every element of the algebra $T(v^n) = [T(v)]^n$. Therefore isometries preserve the characteristic polynomial of the matrix representation of every element. Then, direct isometries of a geometric algebra are defined as its automorphisms, while indirect isometries are defined as its antiautomorphisms. For the geometric algebras $Cl_{p,q}$ with $n = p+q$ even, which are central simple algebras, direct isometries can be written as similarity transformations of matrices $T(u) = t^{-1}u\,t$, while indirect isometries can be written as the composition of reversion R with similarity transformations of matrices. In this way, rotations, reflections, axial symmetries, duality, Clifford conjugation, transposition and Hermitian conjugation are reviewed and written in canonical form (in the two ways indicated above). Canonical forms $T(u) = t^{-1}u\,t$ or $S(u) = t^{-1}R(u)\,t$ for isometries can be applied to every element of the geometric algebra, homogeneous or heterogeneous in grade, which has not only theoretical importance but also practical usefulness since it simplifies algorithms for programmers.

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BINOCULAR COMPUTER VISION BASED ON CONFORMAL GEOMETRIC ALGEBRA

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ABSTRACT. We solve a generalized binocular vision problem by means of conformal geometric algebra. We consider two cameras whose position is determined by an arbitrary Euclidean transformation which can depend on an arbitrary number of parameters. We represent these transformations by algebra elements which allows us to derive general equations of 3D object reconstruction from two camera projection planes by formal manipulation in conformal geometric algebra. Such equation can be solved w.r.t. either motor or projection unknown parameters, the others can be determined consequently. We present two specific examples, show the explicit form of two particular motors and solve the appropriate equations completely.

1. INTRODUCTION

Binocular vision problem can be understood as two particular objectives, i.e. 3D object reconstruction from the data obtained by a pair of cameras and the converse decomposition of a 3D object into two projection planes. Our task is to define the camera position transformation to achieve the prescribed shape of an object in question whose position is given as part of the input data. Typically, this is solved to centre the observed object on one camera. Furthermore, we assume that the cameras are in arbitrary mutual position whose parameters are the only data needed. In classical literature, e.g. [5, 1], the coordinate system is attached to one of the cameras which will be referred to as the first one. In our case, we choose the coordinate system connected to the base of the camera stand to simplify the correspondence of the rotation axes. We consider both cameras to be the pin–hole model and introduce the description of each of them as a pair of algebra elements, more precisely as a point F (focus) and a point pair $K = P \wedge Q$ from the camera projection plane such that the lines $F \wedge P \wedge e_\infty$ and $K \wedge e_\infty$ are orthogonal.

In this setting, the problem is solved by symbolic computations with conformal geometric algebra (CGA) objects in general, but we discuss the projective geometric algebra (PGA) setting simultaneously. Indeed, the last Section shows one of possible directions that we intend to explore in the future, particularly the human vision where PGA is not sufficient tool. Another direction to be considered is e.g. the image analysis from the omnidirectional camera.

The reason to use the GA (CGA) structures to solve such topics lies in the fact that the properties are invariant of the particular mechanism. In paricular, we derive formulas (3) and (4) which hold in general for any system. In Section 5 we show on two elementary examples the application of our computations onto two particular mechanisms and we present the numeric results for several specific settings. In the sense of an object oriented approach, we divide the binocular vision problem into three parts. The most abstract one contains the elaboration with the GA elements by means of standard operations and exploits their natural properties as the OPNS and IPNS representation of the Euclidean object. Our aim is to achieve a mechanism independent geometric solution. Afterwards, we choose a particular GA (despite the fact that

in this text we assume to work within CGA and use several arguments valid for CGA only, from the remarks in the text it is obvious that e.g. in the case of a pin–hole camera the choice of CGA is not completely necessary). Finally, we introduce the equations of the particular mechanism which follows from the forward kinematic chain. Thus in general, the motors transforming the camera position from the initial ideal state to the actual one are obtained. Note that the computations are performed in Maple, package `CLIFFORD` [7], and the final set of equations, whose form does not demand deeper knowledge of CGA, can be used in engineering applications.

2. Conformal geometric algebra – CGA

We recall some elementary facts about CGA and specify our particular setting. Note that the properties and definitions of conformal geometric algebras can be found in e.g. [2, 5]. Classically, for modelling a 3D robot, the CGA is the Clifford algebra $\mathscr{C}l(4,1)$ and an embedding

$$c : \mathbb{R}^3 \to \mathbb{K}^4 \subset \mathbb{R}^{4,1}$$

of the 3D Euclidean space is considered, where $\mathbb{K}^4$ is a null cone in the Minkowski space $\mathbb{R}^{4,1}$. We describe the embedding explicitly in a suitably chosen basis which is denoted by $e_0, e_1, e_2, e_3, e_\infty \in \mathbb{R}^{4,1}$ usually. Namely, the basis is chosen such that the appropriate quadratic form is

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1_{3\times 3} & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where $1_{3\times 3}$ is an identity 3×3 matrix. Consequently, the embedding is of the form

$$c(x) = x + \frac{1}{2}x^2 e_\infty + e_0.$$

Note that e_0 and e_∞ play the role of the origin and the infinity, respectively. Note also that x^2 is a scalar which is equal to $\|x\|^2$, the square of usual Euclidean norm. We recall that the algebra operation is called the *geometric (Clifford) product* and two further elementary operations on CGA are derived; the *inner product (left contraction)* and *outer (wedge) product*. We recall just the basic properties used within the following text. On $\mathbb{R}^{4,1}$, the inner product and outer product correspond to the symmetric and antisymmetric part of the geometric product, respectively. Generally, the wedge product of two basis blades E_k and E_l of grades k and l, respectively, is defined as

$$E_k \wedge E_l := \langle E_k E_l \rangle_{k+l}$$

and the left contraction is defined as

$$E_k \cdot E_l := \langle E_k E_l \rangle_{l-k} \text{ if } l \geq k \text{ and } 0 \text{ otherwise},$$

where $\langle\ \rangle_k$ is the grade projection into grade k. Consequently, the inverse mapping $c^{-1} : \mathbb{K}^4 \to \mathbb{R}^3$ is defined by

$$c^{-1}(x) = P^{\perp}_{e_\infty \wedge e_0}\left(\frac{x}{-x \cdot e_\infty}\right) = x + \frac{x \cdot (e_\infty \wedge e_0)^{-1}}{x \cdot e_\infty}(e_\infty \wedge e_0),$$

where $P^{\perp}$ denotes the orthogonal complement to the projection onto $e_\infty \wedge e_0$. The embedding c has the fundamental property that the inner product of two conformal points is, up to the factor

$-1/2$, the square of the Euclidean distance. Indeed,

$$c(x)\cdot c(y) = \left(x+\tfrac{1}{2}x^2e_\infty+e_0\right)\cdot\left(y+\tfrac{1}{2}y^2e_\infty+e_0\right)$$
$$= \left\langle xy+\tfrac{1}{2}(y^2x-x^2y)e_\infty+(x-y)e_0+\tfrac{1}{2}x^2e_\infty e_0+\tfrac{1}{2}y^2e_0e_\infty\right\rangle_0$$
$$= xy-\tfrac{1}{2}x^2-\tfrac{1}{2}y^2 = -\tfrac{1}{2}(x-y)^2 = -\tfrac{1}{2}\|x-y\|^2.$$

Thus the embedding c, together with the inner product, gives a linearization of a squared distance. As a consequence, the round objects which are given by quadratic expressions (spheres) are represented by linear objects and the Euclidean motions are represented by orthogonal transformations.

Algebraically, we consider the geometric algebra on $\mathbb{R}^{4,1}$, i.e. the algebra denoted as $\mathbb{G}_{4,1}$ which is the Clifford algebra $\mathscr{C}l(4,1)$. In terms of the basis e_0,e_1,e_2,e_3,e_∞ of $\mathbb{R}^{4,1}$, a basis of $\mathbb{G}_{4,1}$ is given by Grassmann monomials (blades) as displayed in tabular 1.

scalars	1
vectors	e_1, e_2, e_3, e_0, e_∞
2–blades	$e_1\wedge e_2$, $e_1\wedge e_3$, $e_1\wedge e_0$, $e_1\wedge e_\infty$, $e_2\wedge e_3$, $e_2\wedge e_0$, $e_2\wedge e_\infty$, $e_3\wedge e_0$, $e_3\wedge e_\infty$, $e_0\wedge e_\infty$
3–blades	$e_1\wedge e_2\wedge e_3$, $e_1\wedge e_2\wedge e_0$, $e_1\wedge e_2\wedge e_\infty$, $e_1\wedge e_3\wedge e_0$, $e_1\wedge e_3\wedge e_\infty$, $e_1\wedge e_0\wedge e_\infty$, $e_2\wedge e_3\wedge e_0$, $e_2\wedge e_3\wedge e_\infty$, $e_2\wedge e_0\wedge e_\infty$, $e_3\wedge e_0\wedge e_\infty$
4–blades	$e_1\wedge e_2\wedge e_3\wedge e_0$, $e_1\wedge e_2\wedge e_3\wedge e_\infty$, $e_1\wedge e_2\wedge e_0\wedge e_\infty$, $e_1\wedge e_3\wedge e_0\wedge e_\infty$, $e_2\wedge e_3\wedge e_0\wedge e_\infty$
pseudoscalar	$I = e_1\wedge e_2\wedge e_3\wedge e_0\wedge e_\infty$

TABLE 1. Wedge basis of $\mathscr{C}l(4,1)$

The Euclidean objects which can be represented by elements in this algebra are spheres of any dimension in particular, i.e. point pairs, circles, spheres, points (spheres of zero radius), linear objects of any dimension, i.e. flat points, lines, planes, and also Euclidean directional and tangential elements. They are represented in the following way. Let $\mathscr{S}\subseteq\mathbb{R}^3$ is one of the Euclidean elements listed above. Then it is viewed as an element $S\in\mathbb{G}_{4,1}$ such that

$$x\in\mathscr{S}\Leftrightarrow c(x)\wedge S=0.$$

Note that this is so–called outer product representation (OPNS). Dually, i.e. in IPNS representation, $\mathscr{S}$ is represented by an element S^* and the condition reads

$$x\in\mathscr{S}\Leftrightarrow c(x)\cdot S^*=0.$$

Note that on the algebra level the duality is obtained by dividing an algebra element S by the pseudoscalar I, i.e. $S^*=S/I$.

In the OPNS representation, the outer product $\wedge$ indicates the construction of a geometric object with the help of points P_i that lie on it. A point pair (0D sphere) is defined by two points $P_1\wedge P_2$. A circle (1D sphere) is defined by three points $P_1\wedge P_2\wedge P_3$ or a point pair and a point. Finally, a sphere (2D sphere) is defined by four points $P_1\wedge P_2\wedge P_3\wedge P_4$ or two point pairs, etc. A plane and line can also be defined by points that lie on it and by the point at infinity, i.e. a line is represented by $P_1\wedge P_2\wedge e_\infty$ and a plane by $P_1\wedge P_2\wedge P_3\wedge e_\infty$. In IPNS, a sphere can be represented by its center P and its radius r as $P-\frac{1}{2}r^2e_\infty$. A plane is defined as $n+de_\infty$, where n is the normal vector of the plane and d is the distance to the origin.

In this sense, the wedge product is a constructive operator, i.e. $A \wedge B$ is an object spanned by A and B etc. The duality operator allows for a definition of the dual to wedge product, so called *meet*,

$$A \vee B = (A^* \wedge B^*)^*.$$

Geometrically, this gives a CGA representative of the intersection of objects A and B.

The next advantage of CGA is that the Euclidean motions are also represented by certain algebra elements. Namely, an Euclidean transformation of an element S is in CGA realized by conjugation with a unique orthogonal invertible element $T \in \mathbb{G}_{4,1}$, i.e.

$$S \mapsto TST^{-1}.$$

Note that the inverse T^{-1} can be replaced by reverse $\tilde{T}$ since for orthogonal transformations $T\tilde{T} = 1$, [5]. For instance, the translation in the direction $t = t_1e_1 + t_2e_2 + t_3e_3$ is realized by the multivector (translator)

$$T = 1 - \frac{1}{2} te_\infty$$

and the rotation around the origin and the normalized axis $\vec{L} = (L_1, L_2, L_3)$ by an angle ϕ is realized by the multivector (rotor)

$$R = \mathrm{e}^{-\frac{1}{2}L\phi} = \cos\frac{\phi}{2} - L\sin\frac{\phi}{2},$$

where $L = L_1(e_2 \wedge e_3) + L_2(e_3 \wedge e_1) + L_3(e_1 \wedge e_2)$. The rotation around a general point and axis is then a composition $TR\tilde{T}$ of the translation to the origin, rotation R and reverse translation. A general composition of a translator with a rotor is called a motor.

Remark 2.1. For some aplications it is sufficient to consider an embedding in a 4D projective space only. To generate three dimensional projective geometric algebra (PGA), Euclidean vectors are embedded in an four dimensional affine space:

$$p(x) = xe_1 + ye_2 + ze_3 + e_4$$

where the e_i are the canonical basis of $\mathbb{R}^4$. If $a \in \mathbb{R}^3$, then the corresponding homogenized vector $p(a)$ is denoted as A and given vector $A \in \mathbb{G}_4$, the corresponding Euclidean vector is

$$a = \frac{A}{A \cdot e_4} - e_4.$$

Using the concept OPNS the bivector $A \wedge B$ represents the line trough a and b (after homogenisation), where $A = p(a), B = p(b)$ are the points. Finally, OPNS of the outer product of three homogeneous vectors in $\mathbb{G}_3$ represents a plane. Note that this PGA approach is implicitly contained in the CGA approach.

3. Camera position

First of all, we need to describe efectively a position of a camera in space. We represent the camera by two CGA elements

- conformal point F representing the camera center (focus),
- point pair $P \wedge Q$ defining the camera image plane π,

as illustrated in figure 1. We can obtain the classical camera descriptions by straightforward computations as

- focal distance $f = -2\sqrt{F \cdot P}$,
- camera direction $(F - P) \wedge e_\infty$,
- camera plane $\pi = P \wedge Q \wedge (F \wedge P \wedge e_\infty)^*$.

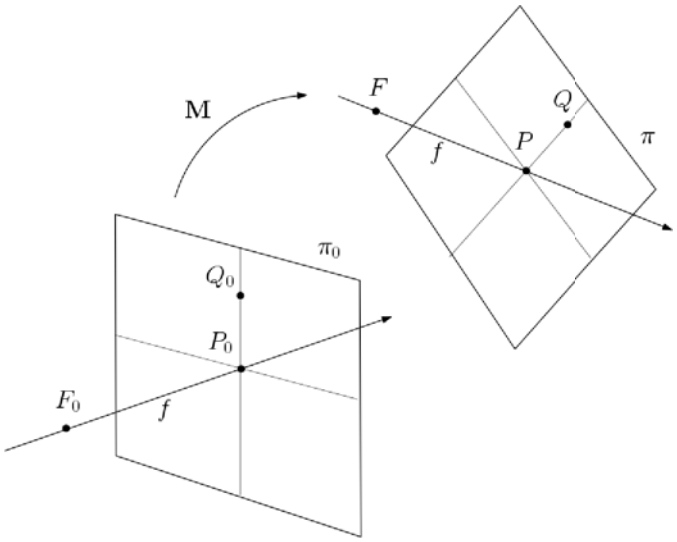

FIGURE 1. Description of camera position

Note that the line $P \wedge Q \wedge e_\infty$ has to be always orthogonal to line $F \wedge P \wedge e_\infty$, i.e.

$$(F \wedge P \wedge e_\infty) \cdot (P \wedge Q \wedge e_\infty) = 0.$$

Given an initial position F_0, P_0, Q_0, the actual position in the space is obtained by an Euclidean transformation which in CGA is given by a conjugation with a motor M (see figure 1 again). The advantage of CGA is that we can transform the whole geometric objects, i.e. any geometric entity constructed from F, P, Q. For instance, the actual position of the camera center is

$$F = MF_0\tilde{M}, \tag{1}$$

and the actual position of the image plane is given by

$$\pi = M\pi_0\tilde{M}. \tag{2}$$

In the same way, one can obtain also the actual camera direction $M((F_0 - P_0) \wedge e_\infty)\tilde{M}$ etc. For a concrete application one only needs to specify the initial position and the motor M.

Remark 3.1. In PGA, the camera position is also represented by two objects; the projective point F representing the camera center and line $P \wedge Q$ which lies in camera plane such that the line $P \wedge Q$ is orthogonal to line $F \wedge P$, i.e. $(F \wedge P) \cdot (P \wedge Q) = 0$. Similarly to CGA, the camera image plane is obtained as $\pi = P \wedge Q \wedge (F \wedge P)^*$.

4. POSE ESTIMATION

Let us have two cameras in a general position, given by two arbitrary motors M_1, M_2, and a line symmetry L which projects to conjugate lines L_1, L_2 in the two camera image planes π_1, π_2 (see figure 2). We show how simple is to compute the images of L or to reconstruct L from its images by CGA approach. Indeed, given a line L in 3D, its image on a camera plane is given as the intersection of these plane with a plane spanned by L and the corresponding camera focus. In terms of the wedge product and the meet we thus have

$$L_k = (L \wedge F_k) \vee \pi_k, \quad k = 1, 2, \tag{3}$$

where the current positions F_k and π_k are computed from their initial position by (1) and (2), respectively, with M being either M_1 or M_2. Concerning the inverse problem, the real line is reconstructed from the given images as the intersection of the plane spanned by L_1 and F_1 with the plane spanned by L_2 and F_2. Thus its representative in CGA reads

$$L = (F_1 \wedge L_1) \vee (F_2 \wedge L_2). \tag{4}$$

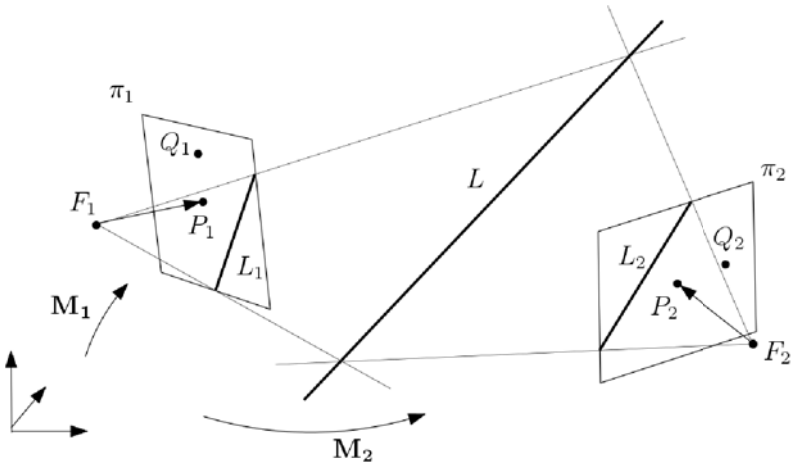

FIGURE 2. A line symmetry projected to two cameras

Let us emphasize that formulas (3), (4) are valid for cameras in general positions based on arbitrary motors M_1 and M_2 and that the computation above are valid for conformal geometric algebra and also for the projective geometric algebra.

Note that the conjugate lines L_1 and L_2 are considered as lines in 3D space. For applications we would rather need to express them in the 2D image coordinates. Therefore, we define maps $\iota_k : \mathbb{R}^2 \to \mathbb{R}^3$ which transform the xy–plane into π_k, $k = 1,2$. Then a formula for L_k in the planar image coordinates is given by $\iota_k^{-1}(L_k)$. Hence

$$L_k^{2D} := \iota_k^{-1}(L_k) = \delta_k e_1 \wedge e_2 \wedge e_\infty + \cos(\alpha_k) e_1 \wedge e_\infty \wedge e_0 + \sin(\alpha_k) e_2 \wedge e_\infty \wedge e_0,$$

for a suitable planar Plückner coordinates based on δ_k and α_k. These can be extracted from L_k as follows

$$\vec{L}_k = (e_0 \wedge e_\infty) \cdot L_k^{2D},$$
$$(\delta_k)^* = (L_k^{2D} - \vec{L}_k \wedge e_0 \wedge e_\infty) \wedge e_3 \wedge e_0,$$

where $\vec{L}_k \equiv \cos(\alpha_k) e_1 + \sin(\alpha_k) e_2$. Indeed, compute $(e_0 \wedge e_\infty) \cdot L_k^{2D}$ as

$$\begin{aligned}
&\langle (e_0 \wedge e_\infty) \cdot \iota_k^{-1}(L_k) \rangle_{3-2} \\
&= \langle \delta_k (e_0 \wedge e_\infty)(e_1 \wedge e_2 \wedge e_\infty) + \cos(\alpha_k)(e_0 \wedge e_\infty)(e_1 \wedge e_\infty \wedge e_0) + \sin(\alpha_k)(e_0 \wedge e_\infty)(e_2 \wedge e_\infty \wedge e_0) \rangle_1 \\
&= \langle \delta_k (-1 - e_0 e_\infty)(e_1 e_2 e_\infty) + \cos(\alpha_k)(-1 - e_0 e_\infty)(e_1(1 + e_\infty e_0)) + \sin(\alpha_k)(-1 - e_0 e_\infty)(e_2(1 + e_\infty e_0)) \rangle_1 \\
&= \langle -\delta_k (e_1 e_2 e_\infty) + \cos(\alpha_k)(-e_1(1 + e_\infty e_0) - e_0 e_\infty e_1) + \sin(\alpha_k)(-e_2(1 + e_\infty e_0) - e_0 e_\infty e_2) \rangle_1 \\
&= \langle -\delta_k (e_1 e_2 e_\infty) + \cos(\alpha_k)(-e_1 - e_1 e_\infty e_0 - e_0 e_\infty e_1) + \sin(\alpha_k)(-e_2 - e_2 e_\infty e_0 - e_0 e_\infty e_2) \rangle_1 \\
&= -\cos(\alpha_k) e_1 - \sin(\alpha_k) e_2.
\end{aligned}$$

In the very similar way compute $(L_k^{2D} - \vec{L}_k \wedge e_0 \wedge e_\infty)$ as

$$\begin{aligned}
&\delta_k e_1 \wedge e_2 \wedge e_\infty + \cos(\alpha_k) e_1 \wedge e_\infty \wedge e_0 + \sin(\alpha_k) e_2 \wedge e_\infty \wedge e_0 + (\cos(\alpha_k) e_1 + \sin(\alpha_k) e_2) \wedge e_0 \wedge e_\infty \\
&= \delta_k e_1 \wedge e_2 \wedge e_\infty
\end{aligned}$$

and finally,

$$(L_k^{2D} - \vec{L}_k \wedge e_0 \wedge e_\infty) \wedge e_3 \wedge e_0 = \delta_k e_1 \wedge e_2 \wedge e_\infty \wedge e_3 \wedge e_0 = (\delta_k)^*.$$

Let us remark that, in the inverse problem, we do not need to read off the Plückner coordinates directly. If we read off two points $[x_k^1, y_k^1]$, $[x_k^2, y_k^2]$ of the line L_k instead, then we get

$$\begin{aligned}
\iota_k^{-1}(L_k) = {} & (x_k^1 y_k^2 - x_k^2 y_k^1) e_1 \wedge e_2 \wedge e_\infty + (x_k^2 - x_k^1) e_1 \wedge e_\infty \wedge e_0 \\
& + (y_k^2 - y_k^1) e_2 \wedge e_\infty \wedge e_0.
\end{aligned}$$

Remark 4.1. In PGA, the geometric elements $F_1, F_2, L_1, L_2, L, \pi_1, \pi_2$ are represented differently (in a 4D space) but the fundamental formulae (3) and (4) have exactly the same form.

5. MODELS AND THE PROBLEM

We will demonstrate our apparatus on two models; each of them constitutes of two pinhole cameras but their relative position and freedome is different. The problem is to adapt the given system by changing free parameters (angles of rotations) such that the image of a given line in one camera or in both cameras (if possible) is in a specific position. Concretly, in the examples below we want the image in one of the camera to be centered exactly to the x–axis of the image coordinates. This problem in general is solved completely by formulas (3) and (4). An explicit solution is solved by the following algorithm based on these formulas. We consider that the configuration of a given system depends on a set of parameters $\phi = (\phi_1, \phi_2, \cdots)$ and our aim is to find ϕ such that the image of a line in the k–th camera coincides with a given reference line L_k^{ref} (x–axis in the examples below). Note that the algorithm has two parts. In the first one, we reconstruct the world line L from its images captured by the camera system in a configuration ϕ^0. The second part computes images of L in a general configuration ϕ.

- specify the system by specifying two motors M_1, M_2 and the map ι_k and their dependance on parameters ϕ,
- an input: two 2D images L_1^{2D}, L_2^{2D} of a real line L captured on cameras in a positions described by parameters ϕ^0,
- compute $L_k(\phi^0) = \iota_k(L_k^{2D})$ for $k = 1, 2$,
- compute positions of camera centers for $k = 1, 2$ according to (1)

$$F_k(\phi^0) = M_k(\phi^0) F_0 \tilde{M}_k(\phi^0),$$

- compute the real line L according to (4)

$$L = (F_1(\phi^0) \wedge L_1(\phi^0)) \vee (F_2(\phi^0) \wedge L_2(\phi^0))$$

- compute camera centers and image planes in a position ϕ according to (1) and (2)

$$F_k(\phi) = M_k(\phi) F_0 \tilde{M}_k(\phi),$$
$$\pi_k(\phi) = M_k(\phi) \pi_0 \tilde{M}_k(\phi),$$

- compute its images in cameras in a position ϕ according to (3)

$$L_k(\phi) = (L \wedge F_k(\phi)) \vee \pi_k(\phi),$$

- solve equation $\iota_k^{-1}(L_k(\phi)) = L_k^{ref}$ for ϕ.

For both models we consider that the camera initial position is placed in the inertial coordinate frame such that

$$F_0 = c(0,0,0) = e_0,$$
$$P_0 = c(0,0,f) = f e_3 + \frac{1}{2} f^2 e_\infty + e_0,$$
$$Q_0 = c(0,1,f) = e_2 + f e_3 + \frac{1}{2}(f^2+1) e_\infty + e_0.$$

Consequently, the initial camera axis $F_0 \wedge P_0 \wedge e_\infty = f e_3 \wedge e_\infty \wedge e_0$ is the z–axis and the initial image plane $\pi_0 = P_0 \wedge Q_0 \wedge (F_0 \wedge P_0 \wedge e_\infty)^*$ is given by

$$\pi_0 = -\frac{1}{2} f^2 e_1 \wedge e_2 \wedge e_3 \wedge e_\infty + \frac{1}{2} f e_1 \wedge e_2 \wedge e_\infty \wedge e_0.$$

The map $\iota_k : \mathbb{R}^2 \to \mathbb{R}^3$ maps xy–plane to π_0 (the translation about the focal distance f in the z–direction) and then to π_k (transformation M_k) and thus is given by conjugation by the motor $M_k exp(-1/2 f e_3 \wedge e_\infty)$, $k = 1,2$, i.e.

$$\iota_k(L_k^{2D}) = M_k(1 - \frac{f}{2} e_3 \wedge e_\infty) L_k^{2D} (1 + \frac{f}{2} e_3 \wedge e_\infty) \tilde{M}_k.$$

Since it is an orthogonal transformation in CGA, its inverse is given simply by a conjugation by the reversed motor. Now we have all what we need to solve the problem in CGA, except the specification of M_1 and M_2.

5.1. **First model.** Let us consider a system of two cameras attached to one rotational axis such that the first of them can rotate also around a perpendicular axis. A precise scheme with an attached coordinate frame is displayed in Figure 3. The formulae for the motors can be read

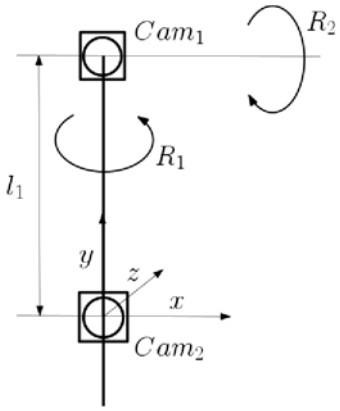

FIGURE 3. Scheme of the first model

off the figure directly. Namely, we have

$$M_1 = R_2 R_1 T_1,$$
$$M_2 = R_1,$$

where the translation T_1 and the rotations R_1, R_2 are given by

$$T_1 = 1 - \frac{1}{2} l_1 e_2 \wedge e_\infty,$$
$$R_1 = \cos(\frac{\phi_1}{2}) + \sin(\frac{\phi_1}{2})(e_3 \wedge e_1),$$
$$R_2 = \cos(\frac{\phi_2}{2}) + \sin(\frac{\phi_2}{2}) \ell_2$$

and where the axis ℓ_2 of the second rotation is

$$\ell_2 = R_1 T_1 (e_2 \wedge e_3) \tilde{T}_1 \tilde{R}_1.$$

Thus the appropriate motors are determined and thus a solution can be found according to the algorithm given above. In this particular example, it is possible to find a complete solution in symbolic form using `CLIFFORD` [7]. Namely, given Plückner coordinates

$(m_1, m_2, m_3, d_1, d_2, d_3)$ of the world line L, its images in cameras are

$$\begin{aligned}
\vec{L}_1 &= \Big(m_2\cos\phi_2 - (m_1 - l_1 d_3)\sin\phi_1\sin\phi_2 + (m_3 + l_1 d_1)\cos\phi_1\sin\phi_2\Big)e_1 \\
&\quad + \Big((m_1 - l_1 d_3)\cos\phi_1 + (m_3 + l_1 d_1)\sin\phi_1\Big)e_2 \\
\delta_1 &= f(m_1 - l_1 d_3)\sin\phi_1\cos\phi_2 - f(m_3 + l_1 d_1)\cos\phi_1\cos\phi_2 \\
\vec{L}_2 &= m_2 e_1 + (m_1\cos\phi_1 + m_3\cos\phi_1)e_2 \\
\delta_2 &= f m_1\sin\phi_1 - f m_3\cos\phi_1
\end{aligned}$$

In the opposite way, given Plückner coordinates (x_1, y_1, δ_1) (x_2, y_2, δ_2) of L_1 and L_2, respectively, the real line L is reconstructed as

$$\begin{aligned}
m_1 &= l_1 x_2 y_1 f^2\sin\phi_1\cos\phi_2 - l_1 x_2\delta_2 f\cos\phi_1\cos\phi_2 + l_1 y_1\delta_1 f\sin\phi_1\sin\phi_2 - l_1\delta_1\delta_2\cos\phi_1\sin\phi_2, \\
m_2 &= l_1 x_1 x_2 f^2\cos\phi_2 + l_1 x_1\delta_1 f\sin\phi_2, \\
m_3 &= l_1 x_2 y_1 f^2\cos\phi_1\cos\phi_2 + l_1 x_2\delta_2 f\sin\phi_1\cos\phi_2 + l_1 y_1\delta_1 f\cos\phi_1\sin\phi_2 + l_1\delta_1\delta_2\sin\phi_1\sin\phi_2, \\
d_1 &= -y_2\delta_1 f\sin\phi_1\sin\phi_2 + x_2 y_1 f^2\sin\phi_1\cos\phi_2, \\
&\quad -(\delta_1\delta_2 + x_2 y_1 f^2)\cos\phi_1\sin\phi_2 + (x_1\delta_1 f - x_2\delta_2 f)\cos\phi_1\cos\phi_2, \\
d_2 &= -y_2\delta_2 f - x_2 y_1 f^2\sin\phi_2 + y_1\delta_1 f\cos\phi_2, \\
d_3 &= (\delta_1\delta_2 + x_1 x_2)\sin\phi_1\sin\phi_2 + (x_1\delta_1 f - x_2\delta_2 f)\sin\phi_1\cos\phi_2, \\
&\quad - y_1\delta_1 f\cos\phi_1\sin\phi_2 - x_2 y_1 f^2\cos\phi_1\cos\phi_2.
\end{aligned}$$

Now we can solve our main equation

$$L_k^{2D}(\phi) = L_k^{ref}$$

for ϕ explicitly. To present a set of results we choose the kinematic parameters $\ell_1 = 1$ and $f = 0.045$. Furthermore, we assume that the initial camera positions are determined by the angles $\phi_1^0 = 0$ and $\phi_2^0 = -\frac{\pi}{8}$. If the Plückner coordinates coincide within this setting we conclude that the visualized line is placed in front of the system. We set the angle to be $\frac{\pi}{10}$, $\frac{\pi}{20}$ and $\frac{\pi}{100}$, respectively, and the appropriate distance to be 0.1 and 0.01. The numeric results are shown in columns three to four of Table 2.

The image within camera number one can be placed into Plückner plane coordinates $(1,0,0)$ by a combination of both rotation changes. The larger the object distance, the rotation changes become more minor. Clearly, the data in the above table correspond with the fact that if $d_{1,2}$ is decreased the visualized object is becoming more remote.

5.2. **Second model.** The second model (see figure 4) is also based on two revolute joints, represented by angles rotations R_1, R_2 but the cameras do not lie one on one axis. In this case, the system can be described by the following set of motors.

$$\begin{aligned}
M_1 &= R_1 T_1, \\
M_2 &= R_2 R_1 T_2,
\end{aligned}$$

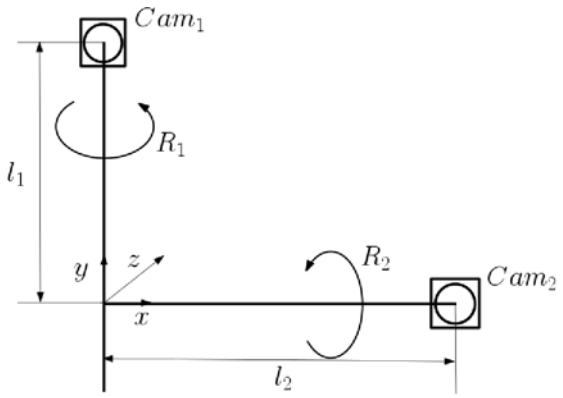

FIGURE 4. Scheme of the second model

where the translations T_1, T_2 and the rotations R_1, R_2 are given by

$$T_1 = 1 - \frac{1}{2} l_1 e_2 \wedge e_\infty,$$
$$T_2 = 1 - \frac{1}{2} l_2 e_1 \wedge e_\infty,$$
$$R_1 = \cos(\frac{\phi_1}{2}) + \sin(\frac{\phi_1}{2})(e_3 \wedge e_1),$$
$$R_2 = \cos(\frac{\phi_2}{2}) + \sin(\frac{\phi_2}{2})\ell_2$$

and where the axis ℓ_2 of the second rotation is

$$\ell_2 = R_1(e_2 \wedge e_3)\tilde{R}_1.$$

$\phi_{1,2}$	$d_{1,2}$	First model: r_1	r_2	Second model: r_1	r_2
$\pi/10$	0.1	-3.01	1.56	0.14	4.31
	0.01	-2.64	0.68	0.95	0.38
$\pi/20$	0.1	-3.08	1.55	0.07	1.15
	0.01	-2.88	0.63	0.61	0.27
$\pi/100$	0.1	-3.13	1.54	0.01	1.15
	0.01	-3.09	0.61	0.14	0.22

TABLE 2. The results where the image of a line in the camera coincides with x–axis

Finally, we present an example given by the kinematics setting $\ell_1 = 1, \ell_2 = 0.5$ and $f = 0.045$. The initial configuration is given by $\phi_1^0 = \phi_2^0 = 0$ and the Plückner coordinates are the same as in the first model. The numeric results are shown in columns five to six of Table 2. again, the data in the above table correspond with the fact that if $d_{1,2}$ is decreased the visualized object is becoming more remote.

6. HUMAN–LIKE VISION

The models described above can be solved alternatively in PGA since all objects which occured were flat, i.e. points, lines and planes, as remarked in 3.1 and 4.1. However, the CGA approach allows for generalizations like human–like vision. By this expression, we mean a binocular vision as above but with cameras which do not project on plane but rather on a sphere. Thus it is similar to human vision but the relative position of "eyes" can vary and may depend on several parameters. For instance, the "eyes" can rotate as in the models above. In

CGA, sphere is in principle the same as plane and thus we can use the theory from sections 3, 4 and also the algorithm from section 5 with minor changes. Instead of a camera plane π, we have a camera sphere σ but the transformation is the same as in (2), i.e.

$$\sigma = M\sigma_0\tilde{M}.$$

The role of F is the same and P defines the direction of an "eye". Instead of Q one needs more data to recover the whole image sphere σ. The basic formulas (3) and (4) are also still valid but as images of L we get conjugate circles c_1, c_2 instead of conjugate lines L_1, L_2, i.e.

$$c_k = (L \wedge F_k) \vee \sigma_k, \quad k = 1,2,$$
$$L = (F_1 \wedge c_1) \vee (F_2 \wedge c_2).$$

Note that the images c_1, c_2, even in their initial position, are 3D objects. A possibility how to get an equivalent 2D information is to take their stereographic projections which is easy in CGA. Using CGA one can also easily recover classical notions as the point of fixation or the horopter. Of course, the fixation does not need to exist in our general setting. But if it exists, it is obviously given by

$$Fix = (F_1 \wedge P_1 \wedge e_\infty) \vee (F_2 \wedge P_2 \wedge e_\infty)$$

and the horopter circle reads

$$Hor = F_1 \wedge F_2 \wedge Fix.$$

7. Conclusion

The motivation for this paper is given by a specific engineering application of binocular vision. More precisely, we consider two cameras attached to a mechanic manipulator on different axes. The whole construction provides several degrees of freedom. The initial setting reads the position of both cameras and the reference line projection of the observed object. The goal is to identify the reference line and centre it on an arbitrary camera consequently. Note that without the reference line exact 3D position it is not possible to solve the problem, although this was not part of the setting.

We solved this problem for arbitrary manipulator kinematics by means of CGA geometric objects and transformations. In Section 5 we introduced the algorithm for a particular kinematics choice and derived the explicit equation $\iota_k^{-1}(L_k(\phi)) = L_k^{ref}$ the solution of which is the desired configuration. We solved the problem for two specific configurations including the presentation of several outputs. Final equations of the first example (Section 5.1) are presented in the form without the CGA symbolic which we find suitable for applications. Conformal algebra thus provides an effective system description. Its object oriented nature then allows to solve several problems simultaneously. The last step demands to solve the system of non linear equations. This should be done in special software that needs the classical form of equations. Note that even the system transformation into the explicit form is realized by CGA elements manipulation.

Within the process we needed to describe the inner configuration of a pin–hole camera which was done in Section 3. Furthermore, some unknown identities transforming the CGA line description into the Plückner coordinates were introduced, see Section 4.

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talk

IDEALS IN MATRIX NEARRINGS AND GROUP NEARRINGS

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Let N be a zero symmetric right nearring with identity 1 and N^n denote the direct sum of n copies $(n \geq 2)$ of the underlying group $(N,+)$. We consider the $n \times n$ matrix nearring over N, denoted by $M_n(N)$, generated by the set of functions $\{[a;i,j] : 1 \leq i,j \leq n, a \in N\}$. It is well known that ideals in the base nearring N and related ideals in the corresponding matrix nearring $M_n(N)$ have been extensively studied in $[5,11,13]$. In this talk, some observations have been made on idempotent elements, nilpotent elements in the nearring and the corresponding matrix nearring. In case of a finite group G, with $|G|=n$, the notion of group nearring, defined in [3], is closely related to $M_n(N)$. Few analogue relationships between the ideals of nearring and that of group nearring are presented. For preliminary definitions and results on nearrings, we refer [6, 12].

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CONSTANT SOLUTIONS OF YANG-MILLS EQUATIONS AND GENERALIZED PROCA EQUATIONS

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ABSTRACT. In this paper we present some new equations which we call Yang-Mills-Proca equations (or generalized Proca equations). This system of equations is a generalization of Proca equation and Yang-Mills equations and it is not gauge invariant. We present a number of constant solutions of this system of equations in the case of arbitrary Lie algebra. In details we consider the case when this Lie algebra is Clifford algebra or Grassmann algebra. We consider solutions of Yang-Mills equations in the form of perturbation theory series near the constant solution.

INTRODUCTION

In this paper we present some new equations which we call Yang-Mills-Proca equations. This system of equations is a generalization of Proca equation and Yang-Mills equations. In Sections 1-3 we present some well-known facts about Maxwell's equations, Proca equation, and Yang-Mills equations and give citations to the literature.

A. Proca introduce his equation as a generalization of Maxwell's equations. It later emerged that Proca equation describes massive particle of spin 1. In a similar manner we generalize Yang-Mills equations. New equations can be considered as the partial case of Yang-Mills equations with some current that satisfies an additional condition. Yang-Mills theory provides models for all fundamental types of interactions.

The considered system of Yang-Mills-Proca equations is not gauge invariant, but it is invariant with respect to a global transformation which depends on some Lie group (see section 4). We are interested in constant solutions of this system of equations. We study the corresponding algebraic system of cubic equations in the case of arbitrary Lie algebra L and present some solutions to this system (sections 5 and 6). In details we consider the case when this Lie algebra is Clifford algebra with respect to the commutator (see sections 8 and 9). In the section 7 we consider solutions of Yang-Mills equations in the form of perturbation theory series near constant solutions.

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1. Relativistic form of Maxwell's equations

Let $\mathbb{R}^{1,3}$ be the Minkowski space with Cartesian coordinates x^μ, $\mu = 0,1,2,3$ and let $\partial_\mu = \partial/\partial x^\mu$ be partial derivatives. The metric tensor of the Minkowski space is given by the diagonal matrix

$$\eta = \|\eta_{\mu\nu}\| = \|\eta^{\mu\nu}\| = \mathrm{diag}(1,-1,-1,-1). \tag{1}$$

Components of tensors (tensor fields) are enumerated by small Greek letters. If we consider a tensor field of type (r,s) and of rank $r+s$ with components $u^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} = u^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}(x)$, $x \in \mathbb{R}^{1,3}$, then we write $u^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} \in \mathrm{T}^r_s$ or $u \in \mathrm{T}^r_s$. With the aid of metric tensor we can raise or lower indices of components of tensor fields. For example, $f^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} f_{\alpha\beta}$.

We use natural system of units where the speed of light and the positron charge are equal to one.

Let us write down Maxwell's equations (1862) in relativistic form [21][1]

$$\begin{aligned} \partial_\mu a_\nu - \partial_\nu a_\mu &= f_{\mu\nu}, \\ \partial_\mu f^{\mu\nu} &= j^\nu, \end{aligned} \tag{2}$$

where $a_\mu \in \mathrm{T}_1$ is a potential of electromagnetic field, $f_{\mu\nu} = -f_{\nu\mu} \in \mathrm{T}_2$ is a strength of electromagnetic field, and $j^\nu \in \mathrm{T}^1$ is a 4-vector of current. It follows from (2) that $\partial_\nu j^\nu = 0$. In the first equation in (2) there are two free (not contracted) indices μ,ν and in the second equation there is one free index ν. Therefore, the first equation is satisfied for all $\mu,\nu = 0,1,2,3$ and the second equation is satisfied for all $\nu = 0,1,2,3$.

If we substitute $f_{\mu\nu}$ from the first equation of (2) into the second equation, then we get an equation of second order for electromagnetic potential

$$\partial_\mu \partial^\mu a^\nu - \partial^\nu(\partial_\mu a^\mu) = j^\nu. \tag{3}$$

Systems of equations (2) and (3) are invariant w.r.t. a gauge transformation

$$\begin{aligned} a_\mu &\to \acute{a}_\mu = a_\mu + \partial_\mu \sigma, \\ f_{\mu\nu} &\to \acute{f}_{\mu\nu} = f_{\mu\nu}, \\ j^\nu &\to \acute{j}^\nu = j^\nu, \end{aligned}$$

where $\sigma = \sigma(x)$ is a twice differentiable function $\sigma : \mathbb{R}^{1,3} \to \mathbb{R}$.

Pseudo-Euclidean space $\mathbb{R}^{p,q}$. The relativistic form of Maxwell's equation gives us the possibility of considering these equations in arbitrary n dimensional pseudo-Euclidean space $\mathbb{R}^{p,q}$, (p,q are nonnegative integer numbers and $p+q=n$) with Cartesian coordinates x^μ, $\mu = 1,\dots,n$ and with a metric tensor given by the diagonal $n \times n$- matrix

$$\eta = \|\eta_{\mu\nu}\| = \|\eta^{\mu\nu}\| = \mathrm{diag}(1,\dots,1,-1,\dots,-1).$$

with p pieces of 1 and q pieces of -1 on the diagonal.

In the sequel we consider Maxwell's equations (2) and other systems of equations in pseudo-Euclidean space $\mathbb{R}^{p,q}$.

[1] We use Einstein convention about summation w.r.t. replicated indices.

2. PROCA EQUATIONS

In 1936 [17] the Romanian physicist Alexandru Proca invented the following modification of the relativistic Maxwell equations:

$$\begin{aligned} \partial_\mu a_\nu - \partial_\nu a_\mu &= f_{\mu\nu}, \\ \partial_\mu f^{\mu\nu} + m^2 a^\nu &= 0, \end{aligned} \tag{4}$$

where m is a real constant (mass of a particle of spin 1). From the equations (4) for $m \neq 0$ it follows the condition (Lorentz gauge)

$$\partial_\mu a^\mu = 0. \tag{5}$$

Hence, it follows from (3) that the system of equations (4) can be reduced to Klein-Gordon-Fock equation for each component of a^ν

$$\partial_\mu \partial^\mu a^\nu + m^2 a^\nu = 0. \tag{6}$$

Consider Maxwell's equations (2) with vector of current j^ν that satisfies the condition $\partial_\nu j^\nu = 0$ and the conditions

$$\partial_\mu \partial^\mu j^\nu + m^2 j^\nu = 0. \tag{7}$$

In that case, Maxwell's equations have a solution

$$a_\mu = -\frac{1}{m^2} j_\mu.$$

This solution is not gauge invariant.

Consequently solutions of Proca equations can be considered as partial subclass of solutions of Maxwell's equations with right hand side (current j^ν) that satisfies additional conditions (7).

3. YANG-MILLS EQUATIONS

Let K be a semisimple Lie group; L be the real Lie algebra of the Lie group K. A Lie algebra L is a real vector space of dimension N with basis $t^1, \ldots, t^N$. Multiplication of elements of L is given by Lie bracket $[A,B] = -[B,A]$, which satisfies Jacobi's identity. Multiplication of basis elements is given with the aid of real structural constants $c_l^{rs} = -c_l^{sr}$ $(r,s,l = 1,\ldots,N)$ of the Lie algebra L

$$[t^r, t^s] = c_l^{rs} t^l. \tag{8}$$

In this work we represent elements of the Lie algebra L and the Lie group K by square matrices of respective dimension or by elements of Clifford algebra $C\ell(p,q)$. In both cases Lie bracket is given by commutator $[A,B] = AB - BA$, where on right hand side we use matrix multiplication of matrices or Clifford multiplication of Clifford algebra elements.

By $L\mathrm{T}_b^a$ we denote a set of tensor fields of the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of type (a,b) and of rank $a+b$ with values in the Lie algebra L.

Consider the following equations in pseudo-Euclidean space $\mathbb{R}^{p,q}$:

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu] &= F_{\mu\nu}, \\ \partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}] &= J^\nu \end{aligned} \tag{9}$$

where $A_\mu \in L\mathrm{T}_1$, $J^\nu \in L\mathrm{T}^1$, $F_{\mu\nu} = -F_{\nu\mu} \in L\mathrm{T}_2$, ρ is a real constant (interaction constant). These equations are called *Yang-Mills equations* (system of Yang-Mills equations). One suggests that $A_\mu, F_{\mu\nu}$ are unknown and J^ν is known vector with values in Lie algebra L. One says that equations (9) define *Yang-Mills field* $(A_\mu, F_{\mu\nu})$, where A_μ is *potential* and $F_{\mu\nu}$ is *strength* of

Yang-Mills field. A vector J^ν is called *non-Abelian current* (in the case of Abelian group K vector J^ν is called *current*).

The components of the skew-symmetric tensor $F_{\mu\nu}$ from the first equation of (9) can be substituted into the second equation to get one equation of second order for the potential of Yang-Mills field

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]) - \rho[A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]] = J^\nu. \tag{10}$$

Let us consider equation (9) from another point of view. Let $A_\mu \in L\mathrm{T}_1$ be arbitrary covector with values in L, which smoothly depends on $x \in \mathbb{R}^{p,q}$. By $F_{\mu\nu}$ denote the expression

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu], \tag{11}$$

and by J^ν denote the expression

$$J^\nu := \partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}].$$

Now we can consider the expression $\partial_\nu J^\nu - \rho[A_\nu, J^\nu]$ and, with the aid of simple calculations, we may verify that

$$\partial_\nu J^\nu - \rho[A_\nu, J^\nu] = 0. \tag{12}$$

This identity is called *non-Abelian conservation law* (in case of Abelian Lie group K we have $\partial_\nu J^\nu = 0$, i.e., divergence of the vector J^ν is equal to zero).

Therefore non-Abelian conservation law (12) is a consequence of Yang-Mills equations (9).

Consider tensor fields $A_\mu, F_{\mu\nu}, J^\nu$ that satisfy Yang-Mills equations (9). Let us take a scalar field with values in Lie group $S = S(x) \in K$ and consider transformed tensor fields

$$\begin{aligned} \acute{A}_\mu &= S^{-1}A_\mu S - S^{-1}\partial_\mu S, \\ \acute{F}_{\mu\nu} &= S^{-1}F_{\mu\nu}S, \\ \acute{J}^\nu &= S^{-1}J^\nu S. \end{aligned} \tag{13}$$

These tensor fields satisfy the same Yang-Mills equations

$$\partial_\mu \acute{A}_\nu - \partial_\nu \acute{A}_\mu - \rho[\acute{A}_\mu, \acute{A}_\nu] = \acute{F}_{\mu\nu},$$
$$\partial_\mu \acute{F}^{\mu\nu} - \rho[\acute{A}_\mu, \acute{F}^{\mu\nu}] = \acute{J}^\nu,$$

i.e., equations (9) are invariant w.r.t. transformations (13). Transformation (13) is called *gauge transformation* (or *gauge symmetry*), and the Lie group K is called *gauge group* of Yang-Mills equations (9).

Partial solutions of Yang-Mills equations. During the last 60 years several classes of solutions of Yang-Mills equations were discovered. Namely, monopoles (Wu, Yang, 1968 [22]), instantons (Belavin, Polyakov, Schwartz, Tyupkin, 1975 [5]), merons (de Alfaro, Fubini, Furlan, 1976 [4]) and so on.[2]

4. Yang-Mills-Proca equations

Let K be semisimple Lie group; L be the real Lie algebra of the Lie group K. Consider equations in pseudo-Euclidean space $\mathbb{R}^{p,q}$

$$\partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu] = F_{\mu\nu},$$
$$\partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}] + m^2 A^\nu = 0, \tag{14}$$

[2]See review of Actor, 1979 [3] and review of Zhdanov and Lagno, 2001 [23].

where $A_\mu \in L\mathrm{T}_1$, $F_{\mu\nu} = -F_{\nu\mu} \in L\mathrm{T}_2$; m, ρ are real constants and $[A,B] = -[B,A]$ is a Lie bracket. We call these equations *Yang-Mills-Proca equations (YMP)*.

Let us discuss some properties of this system of equations.

If $m \neq 0$ then YMP system of equations (14) implies the identity (generalized Lorentz gauge)

$$\partial_\mu A^\mu = 0. \tag{15}$$

System of equation (14) is not gauge invariant, but it is invariant w.r.t. a global (not dependent on $x \in \mathbb{R}^{p,q}$) transformation

$$A_\mu \to \acute{A}_\mu = S^{-1} A_\mu S, \quad F_{\mu\nu} \to \acute{F}_{\mu\nu} = S^{-1} F_{\mu\nu} S,$$

where S is an element of a Lie group K and S is independent on x.

System of equations (14) can be reduced to the following equation (system of equations) of second order for A_μ:

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]) - \rho[A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]] + m^2 A^\nu = 0. \tag{16}$$

For $m = 0$ this equation coincides with the Yang-Mills equation (10) with the trivial right hand side ($J^\nu = 0$).

Using the condition (15) and the formula

$$\partial_\mu [A,B] = [\partial_\mu A, B] + [A, \partial_\mu B],$$

the equation (16) can be rewritten in the form

$$\partial_\mu \partial^\mu A^\nu - 2\rho[A^\mu, \partial_\mu A^\nu] + \rho[A_\mu, \partial^\nu A^\mu] + \rho^2 [A_\mu, [A^\mu, A^\nu]] + m^2 A^\nu = 0. \tag{17}$$

5. Constant solutions of Yang-Mills-Proca equations

We are looking for constant (not dependent on $x \in \mathbb{R}^{p,q}$) solutions $A_\mu \in L\mathrm{T}_1$ of Yang-Mills-Proca equations (16) for fixed constant m. For this solutions

$$\partial_\mu A_\nu = 0.$$

Therefore, the system of nonlinear differential equations (16) reduces to the system of algebraic (cubic) equations

$$[A_\mu, [A^\mu, A^\nu]] = \lambda A^\nu, \tag{18}$$

where $\lambda = -m^2/\rho^2$. Constant solutions of Yang-Mills equations (when $\lambda = 0$) are discussed in [18] and [19].

Let us write down components of vector A^μ with values in Lie algebra L in the form of decomposition w.r.t. a basis of the Lie algebra

$$A^\mu = a^\mu_r t^r,$$

where real coefficients a^μ_r ($\mu = 1, \ldots, n$; $r = 1, \ldots, N$) define N vectors. Substituting these decompositions into the equations (18) and using the relations (8), we get nN algebraic cubic equations for nN unknown coefficients a^μ_r. A resulting system of cubic equations contains a real parameter λ and structure constants c^{rs}_l of the Lie algebra[3].

We cannot give any standard method to solve a system of cubic equations (if we have one cubic equation with one unknown, then we can use Cardano's formula). Nevertheless, we

[3] In the case of the Minkowski space $\mathbb{R}^{1,3}$ and (three dimensional) Lie algebra $\mathrm{su}(2)$ of special unitary Lie group $\mathrm{SU}(2)$ we have a system of 12 equations with 12 unknowns a^μ_r ($\mu = 1,2,3,4$; $r = 1,2,3$).

have found (guess) several classes of solutions of the system of equations (18) (for $\lambda > 0$, for $\lambda < 0$, and for $\lambda = 0$).

Commuting solutions of the system of equations (18). Any set of n mutually commuting elements (matrices) A_μ of the Lie algebra L

$$[A_\mu, A_\nu] = 0$$

is a solution of the system of equations (18) with $\lambda = 0$ (i.e. $m = 0$). Such solutions of Yang-Mills equations were considered by M. Ikeda and Y. Miyachi (1962, [10]).

6. Anti-commuting solutions of Yang-Mills-Proca equations

Let us remind that we represent elements of the Lie group L in the form of square matrices of some size or in the form of elements of Clifford algebra $C\ell(p,q)$ and in both cases the Lie bracket is expressed by the commutator $[A,B] = AB - BA$. By $\mathbf{1}$ we denote the identity matrix of corresponding size or the identity element of Clifford algebra.

Theorem 6.1. *Consider the pseudo-Euclidian space $\mathbb{R}^{p,q}$ of dimension $n = p+q \geq 2$. Let us take a parameter $\theta = 1$ or $\theta = -1$. If the Lie algebra L contains n elements A_μ such that*

$$A_\mu A_\nu + A_\nu A_\mu = 2\theta\eta_{\mu\nu}\mathbf{1}, \tag{19}$$

then these elements A_μ satisfy the system of equations (18) with $\lambda = 4\theta(n-1)$.

Proof. Let $\theta = 1$ and the Lie algebra L contains n elements (components of a covector) A_μ that satisfy relations (19). In other words, A_μ are such that

$$A_\mu A_\nu = -A_\nu A_\mu, \quad \mu \neq \nu$$

and

$$(A_\mu)^2 = \eta_{\mu\mu}\mathbf{1}.$$

For such set of elements A_μ we have relations

$$\begin{aligned} &A_\mu A^\nu A^\mu = A^\mu A^\nu A_\mu = (2-n)A^\nu, \\ &A^\mu A_\mu = n\mathbf{1}, \end{aligned} \tag{20}$$

which follow from the theorem about generators contractions (see [15], page 242).

Let us calculate the left hand side of the equation (18). Replacing Lie brackets by commutators and using formulas (20) we get

$$\begin{aligned} [A_\mu, [A^\mu, A^\nu]] &= A_\mu A^\mu A^\nu - A_\mu A^\nu A^\mu - A^\mu A^\nu A_\mu + A^\nu A^\mu A_\mu \\ &= 2nA^\nu - 2A_\mu A^\nu A^\mu \\ &= 4(n-1)A^\nu. \end{aligned}$$

The proposition is proved. For $\theta = -1$ a proof is similar. ■

The set of elements A_μ (19) generates Cliford algebra of dimension n or, in some cases, of dimension $n-1$ (see, for more details, Section 8). For example, in the case of real Clifford algebra of odd dimension $n = p+q$ and signatures $p-q = 1 \mod 4$, we have such solution (19) that elements A_μ are dependent. For example, if $n = 3$, then for each of the signatures $(2,1)$, $(1,2)$, $(0,3)$ there exists a solution to the system of equations (19) such that A_1, A_2, A_3 are arbitrary variables that satisfy the condition $\mathrm{tr}(A_1A_2A_3) = 0$. But for signatures $(2,1)$ and $(0,3)$ there exists an additional solution to the system of equations (19) of the form A_1, A_2, A_3, where $A_3 = A_1A_2$. In this case $\mathrm{tr}(A_1A_2A_3) \neq 0$.

Theorem 6.2. *Consider the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of dimension $n = p+q \geq 2$. Let us take a parameter $\theta = 1$ or $\theta = -1$. Suppose that a set of n elements (covector components) A_μ satisfy identities (19) and the Lie algebra L contains a set of n elements $\acute{A}_\mu$ such that this set is obtained from the set A_μ by taking r ($1 \leq r \leq n-2$) elements of this set equal to zero. Then the set $\acute{A}_\mu$ satisfies the system of equations (18) with $\lambda = 4\theta(\acute{n}-1)$, where $\acute{n} = n-r \geq 2$.*

Proof. A proof follows from the proof of Theorem 6.1. ■

Multiplication of a solution by a constant. Suppose that a covector $A_\mu \in L\mathrm{T}_1$ is independent of x and satisfies the system of equations (18), and $\kappa \in \mathbb{R}$ is a nonzero constant; then the covector $\breve{A}_\mu = \kappa A_\mu$ also satisfies the system of equations (18) but with the parameter

$$\breve{\lambda} = \kappa^2 \lambda.$$

Let us summarize our reasoning. Suppose that a pair of nonnegative integer numbers (p,q) defines a signature of the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of dimension $n = p+q \geq 2$; then the number of pairs $(\acute{p},\acute{q})$ that satisfy the conditions

$$\acute{p} \leq p, \quad \acute{q} \leq q, \quad \acute{p}+\acute{q} \geq 2$$

are equal to $(p+1)(q+1)-3$. For "appropriate" Lie algebras L, any of these pair $(\acute{p},\acute{q})$ is connected to the pair of constant solutions $A_1,\ldots,A_n$ (for $\theta = \pm 1$) of the system of equations (18) in the pseudo-Euclidean space $\mathbb{R}^{p,q}$ with constant $\lambda = 4\theta(\acute{p}+\acute{q}-1)$. These solutions are defined up to multiplication by a real nonzero constant κ (in this case the constant λ is multipied by κ^2).

What are "appropriate" Lie algebras L. "Appropriate" Lie algebras L must contain a subalgebra that is isomorphic to Clifford algebra $C\ell^{\mathbb{R}}(p,q)$. Otherwise a number of considered constant anti-commutative solutions of Yang-Mills-Proca system of equations is decreased.

Constant solutions of Yang-Mills-Proca system of equations in Minkowski space. As an example, consider constant solutions of Yang-Mills-Proca system of equations in Minkowski space $\mathbb{R}^{1,3}$ with Cartesian coordinates x^μ, $\mu = 0,1,2,3$ and with the diagonal metric tensor (1). We need four vectors (tetrad) y_a^μ, $\mu = 0,1,2,3$, $a = 0,1,2,3$, which are numbered by Latin index a and satisfy relations

$$y_a^\mu y_b^\nu \eta^{ab} = \eta^{\mu\nu}. \tag{21}$$

By Theorem 6.1 we must take a covector A_μ with values in some real Lie algebra L and components of this covector satisfy relations (19) with $\theta = 1$, or $\theta = -1$. From the theory of Dirac equations we know that the set of four matrices γ^a, $a = 0,1,2,3$ in Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

satisfies relations (19) with $\theta = 1$ and the matrices $i\gamma^a$ satisfy conditions (19) with $\theta = -1$. The Hermitian conjugated matrices satisfy conditions

$$(\gamma^a)^\dagger = \gamma^0\gamma^a\gamma^0, \quad (i\gamma^a)^\dagger = -\gamma^0 i\gamma^a\gamma^0. \tag{23}$$

Let us remind definitions of Lie group of special pseudo-unitary matrices $\mathrm{SU}(2,2)$ and its real Lie algebra $\mathrm{su}(2,2)$

$$\begin{aligned} \mathrm{SU}(2,2) &= \{S \in \mathrm{Mat}(4,\mathbb{C}) : S^\dagger \beta S = \beta,\ \det S = 1\}, \\ \mathrm{su}(2,2) &= \{s \in \mathrm{Mat}(4,\mathbb{C}) : \beta s^\dagger \beta = -s,\ \mathrm{tr}\, s = 1\}, \end{aligned}$$

where $\beta = \mathrm{diag}(1,1,-1,-1)$. From these definitions and from formulas (23) we see that $i\gamma^a \in \mathrm{su}(2,2)$ ($\beta = \gamma^0$).

Whence if we take a Lie algebra $L = \mathrm{su}(2,2)$ then the following vector with values in L

$$A^\mu = \kappa y^\mu_a \gamma^a \tag{24}$$

satisfies conditions (19) for $\theta = -1$ and by Theorem 6.1 this vector is a solution to the system of equations (18) with constant $\lambda = -12\kappa^2$, (κ is real parameter). So, for the real Lie algebra $L = \mathrm{su}(2,2)$, we get a constant solution of the Yang-Mills-Proca system of equations (14) in Minkowski space $\mathbb{R}^{1,3}$ with real constants that are connected by the relation

$$\frac{m^2}{\rho^2} = 12\kappa^2.$$

Constant solutions of Yang-Mills-Proca system of equations in Euclidean space $\mathbb{R}^3$. As a second example let us consider constant solutions of Yang-Mills-Proca system of equations in Euclidean space $\mathbb{R}^3$ with Cartesian coordinates x^μ, $\mu = 1,2,3$ and with the diagonal metric tensor given by 3×3 identity matrix $\eta = \mathrm{diag}(1,1,1)$. We need three independent of x vectors y^μ_a, $\mu = 1,2,3$, $a = 1,2,3$, which are numbered by Latin index a and satisfy relations

$$y^\mu_a y^\nu_b \eta^{ab} = \eta^{\mu\nu}. \tag{25}$$

Let us remind definitions of Lie group of special unitary matrices $\mathrm{SU}(2)$ and its real Lie algebra $\mathrm{su}(2)$

$$\begin{aligned} \mathrm{SU}(2) &= \{S \in \mathrm{Mat}(2,\mathbb{C}) : S^\dagger = S^{-1},\ \det S = 1\}, \\ \mathrm{su}(2) &= \{s \in \mathrm{Mat}(2,\mathbb{C}) : s^\dagger = -s,\ \mathrm{tr}\, s = 1\}. \end{aligned}$$

Let τ^a be the Pauli matrices multiplied by imaginary unit i

$$\tau^1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{26}$$

We see that $\tau^a \in \mathrm{su}(2)$. And these matrices satisfy conditions (18) with $\theta = -1$. Therefore, if we take a Lie algebra $L = \mathrm{su}(2)$, then the following vector with values in L

$$A^\mu = \kappa y^\mu_a \tau^a \tag{27}$$

satisfies conditions (19) with $\theta = -1$ and by Theorem 6.1 this vector is a solution to the system of equations (18) with constant $\lambda = -8\kappa^2$. So, in the Euclidean space $\mathbb{R}^3$ we get a constant solution of Yang-Mills-Proca system of equations with Lie algebra $L = \mathrm{su}(2)$ and with real constants that are connected by the relation

$$\frac{m^2}{\rho^2} = 8\kappa^2.$$

Note that this example deals with a class of additional solutions (with $\mathrm{tr} A_1 A_2 A_3 \neq 0$) which were considered after the proof of theorem 6.1.

7. Solutions of Yang-Mills equations in the form of perturbation theory series

Consider the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of dimension $n = p+q \geq 2$ with Cartesian coordinates x^μ and with the metric tensor $\eta_{\mu\nu}$. Let γ^μ be constant (independent of x) vector field (with values in matrix algebra or in Clifford algebra) such that components γ^μ satisfy the relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\theta\eta^{\mu\nu}\mathbf{1}$, where the parameter $\theta = 1$ or $\theta = -1$. A real Lie algebra L is such that $\gamma^\mu \in L\mathrm{T}^1$. Now we consider the system of Yang-Mills equations (10) with the Lie algebra L, with the parameter $\rho = 1$, and with right hand side

$$J^\nu = 4\theta(n-1)\gamma^\nu. \tag{28}$$

By Theorem 6.1 this system of Yang-Mills equations, in particular, has constant solution $A_\mu = \gamma_\mu$ and $[\gamma_\mu, [\gamma^\mu, \gamma^\nu]] = 4\theta(n-1)\gamma^\nu$.

Our aim is to consider solutions of Yang-Mills equations in the form of perturbation theory series near the constant solution $A_\mu = \gamma_\mu$. We take a small parameter $\varepsilon < 1$ and substitute the expression

$$A_\mu = \sum_{k=0}^{\infty} \varepsilon^k \overset{k}{A}_\mu$$

into the left hand side of equation (10). Let us write the result in the form of power series w.r.t. ε

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]) - [A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]] = \sum_{k=0}^{\infty} \varepsilon^k Q_k^\nu = 4\theta(n-1)\gamma^\nu,$$

where

$$\begin{aligned} Q_k^\nu \;=\; & \partial_\mu\partial^\mu \overset{k}{A}{}^\nu - \partial^\nu\partial_\mu \overset{k}{A}{}^\mu - \sum_{l=0}^{k}([\partial_\mu \overset{l}{A}{}^\mu, \overset{k-l}{A}{}^\nu] + [\overset{l}{A}{}^\mu, \partial_\mu \overset{k-l}{A}{}^\nu]) \\ & - \sum_{s=0}^{k}[\overset{k-s}{A}_\mu, \partial^\mu \overset{s}{A}{}^\nu - \partial^\nu \overset{s}{A}{}^\mu - \sum_{r=0}^{s}[\overset{r}{A}{}^\mu, \overset{s-r}{A}{}^\nu]]. \end{aligned} \tag{29}$$

For every integer $k \geq 0$, the components of vector Q_k^ν depend on $\overset{0}{A}_\mu, \ldots, \overset{k}{A}_\mu$. Therefore some approximate solutions of Yang-Mills equations (10) with the right hand side (28) can be found with the aid of the following procedure. Let us take $\overset{0}{A}_\mu = \gamma_\mu$. Then we get

$$Q_0^\nu = [\gamma_\mu, [\gamma^\mu, \gamma^\nu]] = 4\theta(n-1)\gamma^\nu.$$

Substitute $\overset{0}{A}_\mu = \gamma_\mu$ into the expression (29) for $k = 1$ and take

$$Q_1^\nu = 0. \tag{30}$$

As a result, we get a system of linear partial differential equations with constant coefficients and with variables $\overset{1}{A}_\mu$. Let us take any solution of this system of equations (for example, a plane wave solution) and let us substitute this solution (together with $\overset{0}{A}_\mu = \gamma_\mu$) into the expressions

$$Q_2^\nu = 0.$$

Now we get a system of linear partial differential equations with variable coefficients (dependent on $x \in \mathbb{R}^{p,q}$) for variables $\overset{2}{A}_\mu$. Again we take any solution of this system of equations and substitute this solution into the expression $Q_3^\nu = 0$. Continuing this procedure, we get $\overset{k}{A}_\mu$ for any integer $k \geq 0$. So we get an approximate solution (up to terms of order ε^k) of Yang-Mills equations with the right hand side (28).

Let us summarize. If we look for approximate solutions of Yang-Mills system of equations with the right hand side (28) near the constant solution $A_\mu = \gamma_\mu$, then we arrive at one linear system of partial differential equations with constant coefficients (for $\overset{1}{A}_\mu$) and a sequence of linear systems of partial differential equations with variable coefficients (for $\overset{2}{A}_\mu, \dots, \overset{k}{A}_\mu$).

Let us consider in more details the system of equations (30) for $\overset{1}{A}_\mu \equiv B_\mu$

$$\begin{aligned} Q_1^\nu \equiv\ & \partial_\mu \partial^\mu B^\nu - \partial^\nu \partial_\mu B^\mu + [\gamma^\nu, \partial_\mu B^\mu] - 2[\gamma^\mu, \partial_\mu B^\nu] + [\gamma_\mu, \partial^\nu B^\mu] \\ & + [\gamma_\mu, [\gamma^\mu, B^\nu]] + [\gamma_\mu, [B^\mu, \gamma^\nu]] + [B_\mu, [\gamma^\mu, \gamma^\nu]] = 0, \end{aligned} \tag{31}$$

which are the linearization of Yang-Mills system of equations (10) (with parameter $\rho = 1$ and with the right hand side $J^\nu = 4\theta(n-1)\gamma^\nu$) near the constant solution $A_\mu = \gamma_\mu$. For this system of equations (31) one can easily find a class of simple solutions. Namely, let B_μ be a vector such that every component of this vector commute with all γ^ν. For even n we have $B_\mu = b_\mu \mathbf{1}$, where $b_\mu = b_\mu(x)$ is a covector that satisfies Maxwell's equations with zero right hand side

$$\partial_\mu \partial^\mu b^\nu - \partial^\nu \partial_\mu b^\mu = 0.$$

For odd n we have $B_\mu = b_\mu \mathbf{1} + \hat{b}_\mu \gamma^1 \dots \gamma^n$, where covectors $b_\mu = b_\mu(x)$, $\hat{b}_\mu = \hat{b}_\mu(x)$ satisfy Maxwell's equations with zero right hand side ($j^\nu = 0$). Evidently such B_μ satisfy the equations (31).

8. Yang-Mills-Proca equations in Clifford algebra

Let us recall the basic notation. We consider real $C\ell^{\mathbb{R}}(p,q)$ [8] and complexified $C\ell^{\mathbb{C}}(p,q) = \mathbb{C} \otimes C\ell^{\mathbb{R}}(p,q)$ [12] Clifford algebra, $p+q=n$. In the general case, we write $C\ell^{\mathbb{F}}(p,q)$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$. The construction of real and complexified Clifford algebras is discussed in details in [12], [14] and [15].

Let e be the identity element and e^a, $a = 1, \dots, n$ [6] be generators of the Clifford algebra $C\ell^{\mathbb{R}}(p,q)$. Generators satisfy conditions $e^a e^b + e^b e^a = 2\eta^{ab} e$, where $\eta = ||\eta^{ab}||$ is the diagonal matrix with p pieces of $+1$ and q pieces of -1 on the diagonal. Elements $e^{a_1 \dots a_k} = e^{a_1} \cdots e^{a_k}$, $a_1 < \cdots < a_k$, $k = 1, \dots, n$, together with the identity element e, form the basis of Clifford algebra.

We denote by $C\ell_k^{\mathbb{R}}(p,q)$ the vector space spanned by the basis elements $e^{a_1 \dots a_k}$. Elements of $C\ell_k^{\mathbb{R}}(p,q)$ are said to be elements of grade k. We have $C\ell^{\mathbb{R}}(p,q) = \bigoplus_{k=0}^n C\ell_k^{\mathbb{R}}(p,q)$.

Clifford algebra can be considered as a Lie algebra with respect to the commutator $[U,V] = UV - VU$, $U, V \in C\ell^{\mathbb{F}}(p,q)$. It is well-known that the following set is a center of Clifford algebra

$$\mathrm{Cen}(C\ell^{\mathbb{F}}(p,q)) = \begin{cases} C\ell_0^{\mathbb{F}}(p,q), & \text{if } n \text{ is even;} \\ C\ell_0^{\mathbb{F}}(p,q) \oplus C\ell_n^{\mathbb{F}}(p,q) & \text{if } n \text{ is odd.} \end{cases}$$

The following set

$$C\ell_{\text{Ⓢ}}^{\mathbb{F}}(p,q) = C\ell^{\mathbb{F}}(p,q) \setminus \mathrm{Cen}(C\ell^{\mathbb{F}}(p,q))$$

is a Lie subalgebra of Clifford algebra (see [13]).

Now we are looking for the constant solutions of Yang-Mills-Proca equations in the case of Lie algebra $L = C\ell_{\text{Ⓢ}}^{\mathbb{F}}(p,q)$. We have

$$[A_\mu, [A^\mu, A^\nu]] = \lambda A^\nu, \tag{32}$$

where $A^\mu \in L = C\ell_{\text{Ⓢ}}^{\mathbb{F}}(p,q)$.

We have

$$[A_\mu,[A^\mu,A^\nu]] = A_\mu A^\mu A^\nu - A_\mu A^\nu A^\mu - A^\mu A^\nu A_\mu + A^\nu A^\mu A_\mu = \{A^\nu, A^\mu A_\mu\} - 2A_\mu A^\nu A^\mu,$$

where $\{U,V\} = UV + VU$ is anticommutator. So, equations (32) can be rewritten in the following form

$$\{A^\nu, A^\mu A_\mu\} - 2A_\mu A^\nu A^\mu = \lambda A^\nu. \tag{33}$$

It is easy to see (see also section 6), that there is the following class of solutions of these equations:

$$(A^\mu)^2 = \frac{\lambda \eta^{\mu\mu} e}{4(n-1)}, \quad \mu = 1,2,\dots,n; \qquad \{A^\mu, A^\nu\} = 0, \quad \mu \neq \nu. \tag{34}$$

Really, we have

$$\{e^\nu, e^\mu e_\mu\} - 2e_\mu e^\nu e^\mu = 2ne^\nu - 2(2-n)e^\mu = (2n-4+2n)e^\nu = 4(n-1)e^\nu,$$

because of the property $e_a e^b e^a = (2-n)e^b$ of Clifford algebra generators (see [15]).

Note, that after normalization such elements A^μ (34) will be generators: 1) of Clifford algebra $C\ell^{\mathbb{F}}(p,q)$ or $C\ell^{\mathbb{F}}(q,p)$, $p+q=n$ (in the case of real Clifford algebra $\mathbb{F}=\mathbb{R}$ there are 2 cases of signatures: (p,q) and (q,p); complex Clifford algebra $\mathbb{F}=\mathbb{C}$ does not depend on the signature); 2) of Clifford algebra of smaller dimension $n-1$ (for $p-q=1 \mod 4$ in the case of real Clifford algebra and for $p-q=1,3 \mod 4$ in the case of complex Clifford algebra, see [16] and [20]); 3) of Grassmann algebra (for $\lambda = 0$, see Section 9).

Also, there are proportional (they commute, see Section 5) solutions A_μ of equations (33) with $\lambda = 0$ because of the form (32). Also, there are such solutions A_μ that some of them equal zero and the remaining ones generate a basis of Clifford algebra of smaller dimension (see Theorem 6.2).

Let us consider some examples in the cases of small dimensions $n = 2,3$.

$n = 2$**.** In this case we have $C\ell^{\mathbb{F}}_{\mathbb{S}}(p,q) = C\ell^{\mathbb{F}}_1(p,q) \oplus C\ell^{\mathbb{F}}_2(p,q)$. From (33) we get

$$(A^2)^2A^1 + A^1(A^2)^2 - 2A^2A^1A^2 = \lambda\eta^{22}A^1,$$
$$(A^1)^2A^2 + A^2(A^1)^2 - 2A^1A^2A^1 = \lambda\eta^{11}A^2.$$

Using $(A^1)^2$, $(A^2)^2$, $\{A^1,A^2\} \in C\ell^{\mathbb{F}}_0(p,q) = \mathrm{Cen}(C\ell^{\mathbb{F}}(p,q))$ we obtain

$$(A^2)^2A^1 - A^2A^1A^2 = \lambda\frac{\eta^{22}}{2}A^1,$$
$$(A^1)^2A^2 - A^1A^2A^1 = \lambda\frac{\eta^{11}}{2}A^2.$$

Further,

$$2(A^2)^2A^1 - A^2\{A^1,A^2\} = \lambda\frac{\eta^{22}}{2}A^1,$$
$$2(A^1)^2A^2 - A^1\{A^1,A^2\} = \lambda\frac{\eta^{11}}{2}A^2$$

and

$$2((A^2)^2e - \lambda\frac{\eta^{22}}{4}e)A^1 - A^2\{A^1,A^2\} = 0, \tag{35}$$
$$2((A^1)^2e - \lambda\frac{\eta^{11}}{4}e)A^2 - A^1\{A^1,A^2\} = 0.$$

It is easy to see that the following expressions are solution to this system of equations:

$$(A^1)^2 = \frac{\lambda\eta^{11}}{4}e, \qquad (A^2)^2 = \frac{\lambda\eta^{22}}{4}e, \qquad \{A_1,A_2\} = 0.$$

If one of 4 scalar expressions in (35) does not equal to zero, then we obtain proportional solutions

$$A_1 = \mu A_2, \qquad \mu = \frac{\{A^1,A^2\}}{2(A^2)^2} = \frac{2(A^1)^2}{\{A^1,A^2\}}$$

(or analogously $A_2 = \mu A_1$), or one of A_1, A_2 equals to zero (we have $\lambda = 0$ in these cases). So, we obtain all solutions of the system of equations (32) in the case $n = 2$ for $L = C\ell^{\mathbb{F}}_{\mathbb{S}}(p,q)$.

$n = 3$**.** In this case we have $C\ell^{\mathbb{F}}_{\mathbb{S}}(p,q) = C\ell^{\mathbb{F}}_1(p,q) \oplus C\ell^{\mathbb{F}}_2(p,q)$. System of 3 equations (33) for A^1,A^2,A^3 can be rewritten in the following form

$$\eta^{22}((A^2)^2A^1 + A^1(A^2)^2 - 2A^2A^1A^2) + \eta^{33}((A^3)^2A^1 + A^1(A^3)^2 - 2A^3A^1A^3) = \lambda A^1,$$
$$\eta^{33}((A^3)^2A^2 + A^2(A^3)^2 - 2A^3A^2A^3) + \eta^{11}((A^1)^2A^2 + A^2(A^1)^2 - 2A^1A^2A^1) = \lambda A^2,$$
$$\eta^{11}((A^1)^2A^3 + A^3(A^1)^2 - 2A^1A^3A^1) + \eta^{22}((A^2)^2A^3 + A^3(A^2)^2 - 2A^2A^3A^2) = \lambda A^3.$$

Using $(A^i)^2 \in C\ell^{\mathbb{F}}_0(p,q) \oplus C\ell^{\mathbb{F}}_3(p,q)$ and $\{A_i,A_j\} \in C\ell^{\mathbb{F}}_0(p,q) \oplus C\ell^{\mathbb{F}}_3(p,q) = \mathrm{Cen}(C\ell^{\mathbb{F}}(p,q))$, we obtain

$$\eta^{22}((A^2)^2A^1 - A^2A^1A^2) + \eta^{33}((A^3)^2A^1 - A^3A^1A^3) = \frac{\lambda}{2}A^1,$$
$$\eta^{33}((A^3)^2A^2 - A^3A^2A^3) + \eta^{11}((A^1)^2A^2 - A^1A^2A^1) = \frac{\lambda}{2}A^2,$$
$$\eta^{11}((A^1)^2A^3 - A^1A^3A^1) + \eta^{22}((A^2)^2A^3 - A^2A^3A^2) = \frac{\lambda}{2}A^3,$$

and

$$\eta^{22}(2(A^2)^2A^1 - A^2\{A^1,A^2\}) + \eta^{33}(2(A^3)^2A^1 - A^3\{A^1,A^3\}) = \frac{\lambda}{2}A^1,$$
$$\eta^{33}(2(A^3)^2A^2 - A^3\{A^2,A^3\}) + \eta^{11}(2(A^1)^2A^2 - A^1\{A^2A^1\}) = \frac{\lambda}{2}A^2,$$
$$\eta^{11}(2(A^1)^2A^3 - A^1\{A^3,A^1\}) + \eta^{22}(2(A^2)^2A^3 - A^2\{A^3,A^2\}) = \frac{\lambda}{2}A^3,$$

and

$$A^1(2\eta^{22}(A^2)^2 + 2\eta^{33}(A^3)^2 - \frac{\lambda}{2}e) + A^2(-\eta^{22}\{A^1,A^2\}) + A^3(-\eta^{33}\{A^1,A^3\}) = 0,$$
$$A^1(-\eta^{11}\{A^2,A^1\}) + A^2(2\eta^{33}(A^3)^2 + 2\eta^{11}(A^1)^2 - \frac{\lambda}{2}e) + A^3(-\eta^{33}\{A^2,A^3\}) = 0,$$
$$A^1(-\eta^{11}\{A^3,A^1\}) + A^2(-\eta^{22}\{A^3,A^2\}) + A^3(2\eta^{11}(A^1)^2 + 2\eta^{22}(A^2)^2 - \frac{\lambda}{2}e) = 0.$$

Elements in round brackets are elements of the center of Clifford algebra. If they equal to zero, then we obtain the following solution of the system of equations:

$$\eta^{11}(A^1)^2 = \eta^{22}(A^2)^2 = \eta^{33}(A^3)^2 = \frac{\lambda}{8}e, \qquad \{A_i,A_j\} = 0.$$

To obtain other solutions we must consider all the remaining cases (if at least one of expressions in round brackets does not equal to zero). It is easy to see, that among solutions there will be proportional solutions A_μ with $\lambda = 0$; commuting solutions with $\lambda = 0$; solutions like: $A_1 = 0$ and 2 elements A_2, A_3 generate basis of Clifford algebra of dimension $n = 2$, and similar others.

9. Grassmann numbers as solutions of Yang-Mills-Proca equations in Clifford algebra

Now we want to discuss one another class of solutions of Yang-Mills-Proca equations (32) with $\lambda = 0$. It is easy to see that Grassmann numbers [9] are solutions of these equations. If we take Lie algebra $L = C\ell^{\mathbb{C}}_{(\mathbb{S})}(p,q)$ (let us consider only complex case), then we must realize Grassmann algebra as a subalgebra of Clifford algebra. We denote complexified Grassmann algebra of dimension n by $\Lambda^{\mathbb{C}}(n)$. We can also consider degenerate Clifford algebra $C\ell^{\mathbb{C}}(p,q,r)$ in more generale case.

We have the following well-known construction (see Clifford-Jordan-Wigner representation [11], [7]). Let us consider complex Clifford algebra $C\ell^{\mathbb{C}}(n) = C\ell^{\mathbb{C}}(n,0)$ of even dimension $n = p+q = 2N$ (or odd dimension $n = 2N+1$). With the use of generators e^a we can construct the following elements

$$\theta^k = \frac{1}{2}(e^k + ie^{N+k}), \qquad k = 1,\dots,N,$$
$$\pi^k = \frac{1}{2}(e^k - ie^{N+k}), \qquad k = 1,\dots,N.$$

Note, that in the opposite way we have:

$$e^k = \theta^k + \pi^k, \qquad e^{k+N} = i(\theta^k - \pi^k).$$

It is easy to verify that these elements satisfy conditions

$$\theta^k\pi^l + \pi^l\theta^k = \delta^{kl}, \qquad \theta^k\theta^l + \theta^l\theta^k = 0, \qquad \pi^k\pi^l + \pi^l\pi^k = 0.$$

So, we have 2 sets θ^k and π^k of Grassmann numbers with some connections between each other.

Now let us consider degenerate Clifford algebras $C\ell^{\mathbb{C}}(p,q,r)$ (see [1], [2]) with generators

$$e^1,\dots,e^p, \quad \varepsilon^1,\dots,\varepsilon^q, \quad \theta^1,\dots,\theta^r,$$

where $(e^k)^2 = 1$, $(\varepsilon^l)^2 = 1$, $(\theta^m)^2 = 0$ for $k = 1,\dots,p$, $l = 1,\dots,q$, $m = 1,\dots,r$.

Jacobson radical (intersection of all maximal ideals) consists of elements

$$I = \sum_A a_A\theta^A + \sum_{A,B} b_{AB}e^A\theta^B + \sum_{A,B} c_{AB}\varepsilon^A\theta^B + \sum_{A,B,C} d_{ABC}e^A\varepsilon^B\theta^C,$$

and it is nilpotent. Algebra $C\ell^{\mathbb{C}}(p,q,r)$ is not semi-simple. But it is well known that we can realize it in matrix algebra in the following way. Consider $\psi : C\ell^{\mathbb{C}}(p,q,r) \to C\ell^{\mathbb{C}}(p+r,q+r)$

$$\begin{aligned} e^k &\to e^k, \qquad k = 1,\dots,p,\\ \varepsilon^l &\to \varepsilon^l, \qquad l = 1,\dots,q,\\ \theta^m &\to e^{p+m} + \varepsilon^{q+m}, \qquad m = 1,\dots,r. \end{aligned}$$

For example, $C\ell^{\mathbb{C}}(0,0,2) = \Lambda^{\mathbb{C}}(2) \to C\ell^{\mathbb{C}}(2,2)$, with $\theta^1 \to e^1 + \varepsilon^1$ and $\theta^2 \to e^2 + \varepsilon^2$. We can consider standard matrix representation of square complex matrices of order 4. Degenerate Clifford algebra is a subalgebra of this algebra of matrices.

So, in Clifford algebra $C\ell^{\mathbb{C}}(p,q)$ we can realize the following algebras

$$C\ell^{\mathbb{C}}(p-1,q-1,1), \qquad C\ell^{\mathbb{C}}(p-2,q-2,2), \qquad \dots, \qquad C\ell^{\mathbb{C}}(p-m,q-m,m),$$

where $m = min(p,q)$.

Grassmann algebra of even dimension n can be represented using square complex matrices of order 2^n (it is isomorphic to a subalgebra of the algebra of such matrices), whenever complex Clifford algebra is isomorphic to an algebra of square complex matrices of order

$2^{\frac{n}{2}}$. That is why we can always realize $\frac{n}{2}$ Grassmann numbers in Clifford algebra $C\ell^{\mathbb{C}}(p,q)$, $p+q=n$.

Let us give one example. In the case of signature $(p,q)=(1,3)$ we have the following solution of (32):

$$A_1 = T^{-1}a(ie^{23}-e^{13})T, \qquad A_2 = T^{-1}b(e^{03}-e^{3})T, \qquad A_3 = 0, \qquad A_4 = 0; \qquad \lambda = 0$$

for any invertible element $T \in C\ell^{\mathbb{C}}(p,q)$ and $a,b \in \mathbb{C}$. Really, these Clifford algebra elements satisfy conditions of Grassmann algebra: $A_1^2 = A_2^2 = 0, \quad A_1A_2 = -A_2A_1$. In this case we can use the following 2 matrices:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

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talk

COHERENT STATES AND BEREZIN TRANSFORMS ATTACHED TO LANDAU LEVELS

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In general, coherent states $(|x\rangle)_{x\in X}$ are a specific overcomplete family of normalized vectors in the Hilbert space $\mathcal{H}$ of the problem that describes the quantum phenomena and solves the identity of $\mathcal{H}$ as

$$1_{\mathcal{H}} = \int_X |x\rangle \langle x| \, d\mu(x) .$$

These states have long been known for the harmonic oscillator and their properties have frequently been taken as models for defining this notion for other models. We review the definition and properties of coherent states with examples. We construct coherent states attached to Landau levels (discrete energies of a uniform magnetic field) on three known examples of Kähler manifolds X: the Poincaré disk $\mathbb{D}$, the Euclidean plane $\mathbb{C}$ and the Riemann sphere $C\mathbb{P}^1$. After defining their corresponding integral transforms, we obtain characterization theorems for spaces of bound states of the particle. Generalization to $\mathbb{C}^n$ and to the complex unit ball $\mathbb{B}^n$ and $C\mathbb{P}^n$ are also discussed. In these cases, we apply a coherent states quantization method to recover the corresponding Berezin transforms and we give formulae representing these transforms as functions of Laplace-Beltrami operators.

talk

LINEAR 2-NORMED SPACES AND 2-BANACH ALGEBRAS

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The concept of linear 2- normed spaces was introduced by Siegfried Gahler in 1963 [6], which is nothing but a two dimensional analogue of a normed space. This concept had received the attention of a wider audience after the publication of a paper by A. G. White in 1969 entitled 2-Banach spaces [7]. In this talk we would like to present the recent developments in Linear 2-normed spaces and 2-Banach algebras. We introduce the idea of expansive, non-expansive and contraction mappings in linear 2-normed spaces eventually some of its properties are established. The analogous of Banach fixed point theorem for contraction mappings in linear 2- normed spaces is obtained, which leads to the existence of the solution of strong accretive operator equation in linear 2-normed space. Some more analogues results in Linear 2-normed spaces and 2-Banach algebras are obtained.

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FACIAL RECOGNITION USING MODERN ALGEBRA AND MACHINE LEARNING

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Abstract: Facial Recognition Systems have been garnering attention from various researchers and enthusiasts in recent times, they are being deployed for numerous applications like those in identifying faces from a mass of subjects, noise removal from captured images, forensic applications, etc. The principle idea is to identify and extract individual features from the image of an individual's face. The RGB image is first converted to grayscale and then in reference to a threshold intensity, are transformed into a binary matrix. The end-point demarcations of individual features, called feature points are identified from the image and the relative distances between relevant points are calculated using a wide range of algebraic functions like Euclid distance, eigenvectors etc. The concept used here is of reservation tables. These distances are stored in the form of vectors and are then transformed as required. The images are classified on the basis of similar distances between concurrent features, and are grouped together under one class. The principle avoids, similar faces getting misclassified. An accuracy of 88 percent is obtained, using this method. Similar faces come in a diagonal in the reservation table, and non similar faces are the vice versa, in this paper. This is a novel approach, as face matching is done according to position in reservation table. The different facial features are shown as a set of similar faces, which are a plot on the reservation table, in any given direction.

Keywords: face recognition, machine learning, principal component analysis.

1. INTRODUCTION

Due to increase threat perceptions, face recognition systems have become very handy, in finding out the real identification[1] of the person. The different features, are the different parts of the face. The features are eye brows, moustache, nostrils, etc. In order to find these features, algebra and machine learning is used in varying degrees of automation in different angles. But, none of these methods offer complete autonomy of movement of the face; but static faces work fine, with all existing methods.

Another problem with faces[3] by existing methods is that, they are more feature concentric, and also features are many a times dislocated, as shown in Fig 1.1.[1] The algorithms used are more towards registered faces.

[1]Figures and Tables have been listed at the end of the paper.

Even in situations where clear faces are available the accuracy is hampered by a number of factors. The determination of an entity's distance from another is done on the basis of propagation time of image signal, and to be able to determine the same, the facial messages from the human being are often clear, are taken. This however requires that the faces are taken at the right angle, and also faces of the same person are taken over a period of time.

In reality, acquiring frontal faces is difficult . Due to varying angles, of the image features, some times, feature extraction becomes difficult.

Objectives:

The purpose of our research is to get faces of individuals, and then extract their features, to classify them into a proper database.

• Get the Image from a camera. Register the images.

• Find the features, using algebra and Machine Learning.

• Classify the images.

Methods:

The faces will function in a predefined linear motion at a set altitude, modelled after pipelined instructions. They are designed to avoid physical impacts among them using collision free scheduling, implemented using the reservation table.

The cameras used for the faces would be enabled with heat sensing technology and night vision; and would trigger an alert in case any untoward activity of wrong faces detected.

There would be beacons placed at appropriate place of the landscape at optimal distances, which would be used to determine the location of each face via triangulation - trigonometry triangle solution (based on the same principle as GNSS systems); which in turn would be fed to the algorithm running at the servers.

Once the faces are detected, they are classified, using appropriate algorithm.

A relation R between the sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \ldots A_n$. This relation R is called an n-ary relation . A relation R on S is said to be equivalence if R is (ii) reflexive that is, $(a,a) \in R$ for all $a \in S$; (iii) symmetric if $(a,b) \in R$ implies $(b,a) \in R$; (iii) transitive, that is, $(a,b) \in R, (b,c) \in R$ implies $(a,c) \in R$.

The equivalence classes are considered for obtaining cycles.

Triangulation Theory (Algebra): Geometry [12] on Face recognition, is to find features, which follow dimension reduction [13]. This has been achieved using Principal Component Analysis in this paper. Also, Reservation tables have been used to show the two D map of the faces, across time. We use the term Triangulation[5] here loosely. Triangulation is a process by which the location of a facial features can be determined by measuring either the radial distance, or the direction, of the received face from two or three different points. This is shown in Fig 1.2.

We set up three transmitters at the corners of the area under question. Timers on-board all devices, transmitters and faces alike, will be synchronized and maintained. Messages containing the transmission time will be received by the features of the faces on three different instances. Now, the radial distance from the three will be cross-referenced and the exact location of the face with respect to the already known positions of the transmitters will be calculated and fed to the algorithm.

This is implemented by the Reservation Table Algorithm. Here, we assume a virtual grid of equally sized squares, spread between the three transmitters. According to the positions received by the triangulation solution, the reservation table is set up.

Reservation Table:

A reservation table, also known as the Gantt chart, is a 2D chart[6], a way of representing the task flow pattern of a pipelined system. The table is shown in Fig 1.3. Before applying the reservation tables, PCA [6], Principal Component Analysis, is applied to get the features. The method contains the following steps:

• Get Data

• Subtract Mean

• Calculate Covariance Matrix

• Calculate Eigen Values and Eigen Vectors of Matrix

• Form the feature Vector.

In this particular embodiment for the purpose of surveillance, all the faces are at the same height in order to keep a uniform field of view and to avoid intersections of visual jurisdiction. This transforms the three dimensional space to a two dimensional plane and makes it an appropriate situation to successfully implement the reservation table model. The positions of the faces taken by triangulation are represented as ‘X’s across a virtual grid of equally sized squares, the size of whom depends upon the implementation, as shown in Fig 1.3.

Now, we calculate the Forbidden Latencies which are the set of distances which should not be moved in order to avoid collision between successive faces. This is to find similar faces of a person. Forbidden Latencies as equivalence classes points: 2, 4, 5, 7

The Permissible Latencies are taken from the universal set and are mutually exclusive from the set of Forbidden Latencies. The last non-forbidden (8) can be ignored in the collision vector. Permissible Latencies as equivalence classes points: 1, 3, 6

The collision vector is made by representing the elements present in the universal set in descending order– ‘1’ for the forbidden and ‘0’ for the permissible set. Collision Vector: 1011010

Now, we use the collision vector to form the state diagram, as shown in Fig 1.4 which is a graphical representation of all achievable vectors in the algorithm.

Algorithm:

1. current state := collision vector

2. repeat

3. repeat

4. determine the rightmost unvisited 0 in the current state

5. right shift the current state by the calculated spaces

6. perform bitwise-OR on the shifted current state and the collision vector

7. if the derived state is new add the state to the diagram

8. else

make the appropriate transition links

9. until no more zeroes are left in the current state

10. current state := next state in list

11. until no states are left in the list

The simple cycles from the graph are used to calculate the Minimum Average Latency (MAL) and the greedy cycles are used to calculate the Minimum Greedy Cycle Average Latency (MGCAL). A Latency Cycle is a latency sequence which repeats the same subsequence or cycle indefinitely. By convention, we label the minimum outgoing latency from a state with a '*' and the last available latency is labelled with a '+' and a cycle labelled with that latency is formed for each state.The cycles which don't consist of even '*' labels are called Simple Cycles and in which each latency is encountered exactly once; while the rest, the ones with all '*'s are called Greedy Cycles. The Average Latency of a latency cycle is obtained by dividing the sum of all latencies by the number of latencies along the cycle. Latency cycles are extensively studied in [10] and [11].

Latency Cycles: (equivalence classes)

One Edged: (3*), (6), (8)

Two Edged: (1*, 8*), (3, 8), (6, 8)

Three Edged: (1*, 8, 8), (3, 6, 8), (6, 6, 8), (3, 3*, 8), (6, 3*, 8)

Greedy Cycles: (3*), (1*, 8*)

Now, we consider one and two edged cycles, since they give the optimum result, as shown in Table 1.1.

Now, from the aforementioned table, we get the Minimum Average Latency (MAL) and Minimum Greedy Cycle Average Latency (MGCAL).

MAL: 3.

MGCAL: 3.

For further proceedings, we consider only the greedy cycles which have the average latency equal to the MAL.

Now, Frame Size = MAL*Steps in Horizontal Grid = 3*3 = 9.

According to the figure shown above, the faces will now move in a linear fashion as dictated by the MAL calculated. Similar faces are across the diagonal, and non similar faces are vice versa.

Once repetition starts, we mark the first marker as the set-up time. Efficiency of this system = No. of filled spaces when repetition starts/ Frame size *100 = 8/9*100 =88 percent.

This efficiency signifies the utilization aerial space as done by the faces.

Once they cross the border specified by the table, they will be turned around and the function for collision free scheduling will be called again, and the reservation table will be reset in the opposite direction. This would ensure that there would be constant feeds from all areas and no spot would be left unsupervised.

When the rightmost face crosses the right boundary, the reservation table is turned around. This new configuration is formulated in the same fashion as done for left to right movement. It yields us this chart: The repetition starts at the sixth stage, the identical boxes are highlighted in fig. 1.5.

The efficiency for the system is calculated as: 8/9*100 = 88 percent.

This proves that the efficiency of the system remains same throughout the operation and such a high static efficiency is desirable for a reliable surveillance operation.

Additional Features for Functionality:

A thermographic camera is a device that forms an image using infrared radiation, similar to a common camera that forms an image using visible light. Instead of the 400 to 700 nanometer range of the visible light camera, infrared cameras operate in wavelengths as long as 14,000 nm.We set up a wireless network by using a router as an access point and configuring each face to the required bandwidth while implementing Bandwidth Control, to ensure appropriate resources for each face's stream. The data streams from each face are set to be viewable at the control station. Since, the entire project is set up on the premise of security, it is essential that the faces themselves be equipped for the same. These features can be applied by the discretion of that particular implementation instance.

Result: All the results, as shown in Fig 1.8, have been summarized, using reservation tables, and the concept of abstract algebra, to get the correct matches. The matching accuracy has been found out to be 88 percent. The intermediate faces also have been shown.

Conclusion: The above method, removes duplicates, and matches faces of the same person taken over a period of time. As a part of future work, wrong features, getting classified can be avoided.

.

Fig 1.1: Image of a person.

Fig 1.2: Triangulation Diagram

Collision Free Scheduling:

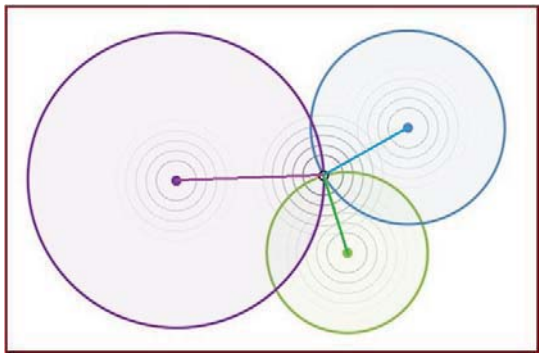
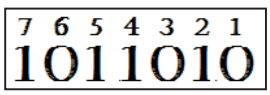
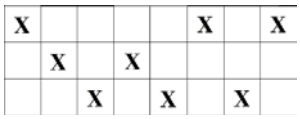


Fig 1.3: Reservation Table.

Collision Vector: 1011010





State Diagram: Fig 1.4

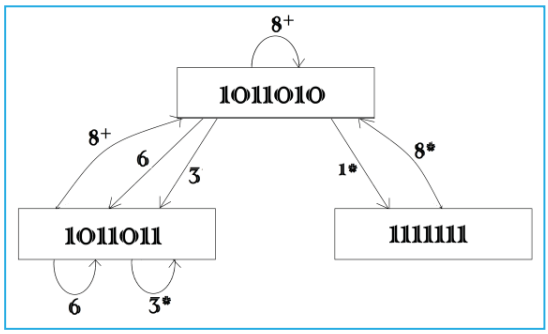


Table 1.1

Cycle	Average Latency
(3*)	3
(6)	6
(8)	8
(1*, 8*)	4.5
(6, 8)	7
(3, 8)	5.5

Fig 1.5: Detailed Reservation Table .

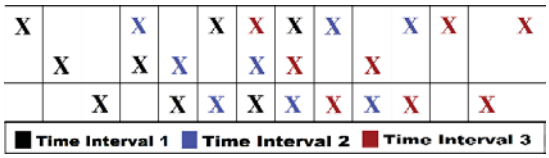


Fig 1.6: Similar Patterns.

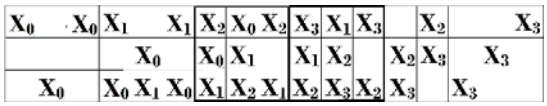
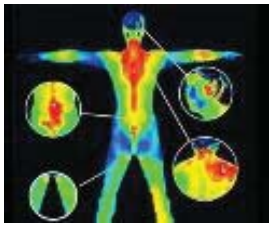


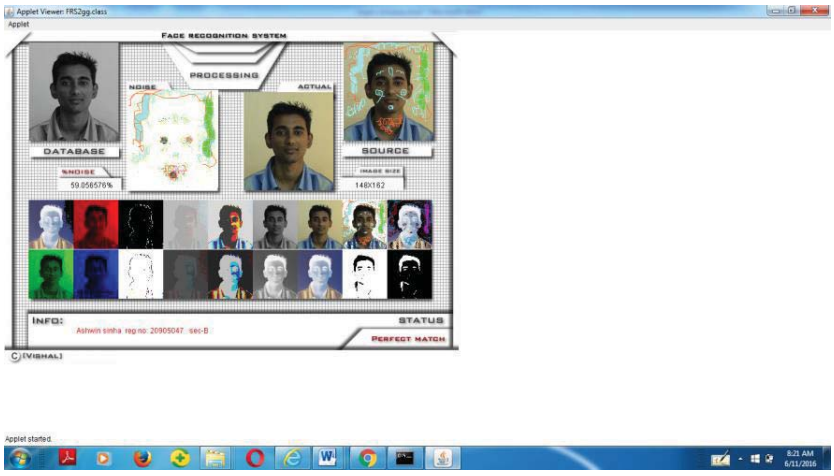
Fig 1.7: Camera to capture an image.



Results: Fig 1.8: Normal Faces.

Fig 1.9: Intermediate Faces.

Fig 1.10: Face Matching.



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CLIFFORD ALGEBRA IMPLEMENTATIONS IN MAXIMA

ABSTRACT. The contribution focuses on the packages *clifford* and *cliffordan* for the computer algebra system *Maxima* [13]. *Maxima* is the open source descendant of the first computer algebra system and features a rich functionality from a large number of shared packages. The Maxima language is based on the ideas of functional programming, which is particularly well suited for transformations of formal mathematical expressions. While *clifford* implements Clifford algebras $C\ell_{p,q,r}$ of arbitrary signatures and order based on the elementary construction of Macdonald, *cliffordan* features geometric calculus functionality. Using *clifford* expressions containing geometric, outer and inner products can be simplified. Applications of *clifford* and *cliffordan* in linear algebra and calculus will be demonstrated.

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1. INTRODUCTION

Proponents of Geometric algebra and Geometric calculus promote the view that these approaches unify, simplify, and generalize vast areas of mathematics that involve geometric ideas. Clifford algebras provide natural generalizations of complex, dual and double numbers into the concept of *Clifford numbers*. It is then natural to use existing *Computer Algebra Systems* (CAS) to implement various geometric algebra instances and provide them as tools for science and engineering. Several Clifford and Geometric algebra packages have been developed for different types of CAS and engineering numerical suites. For Maple since the late 1990's, Ablamowicz and Fauser develop the package `CLIFFORD` [1]. Computations in `CLIFFORD` are based on Chevalley's definition of Clifford algebra as a subalgebra of the algebra of endomorphisms of Grassmann algebra [3]. Another package, `BIGEBRA` [2], builds on Maple's `CLIFFORD`, with the aim to explore Hopf algebras and provide a useful tool for experimental mathematics. Finally, the package called `eCLIFFORD` computes the Clifford product in $C\ell_{p,q,r}$ using Walsh functions.

For Mathematica, the `clifford.m` package [4] introduces the concept of Clifford and Grassmann algebras, multivectors, and the geometric product. Blades are represented by tuples of numbers. Although not in symbolic computer algebra, the recent work of Sangwine and Hitzer is also noteworthy [15]. The package is the first comprehensive work for Matlab. It features operations with matrices of Clifford multivectors, including LU decomposition.

For Maxima, `atensor`, since 2004, and the `clifford`-based packages, since 2015, are designed to be a symbolic computational tools for applied mathematicians and physicists. The package `atensor` partially implements generalized (tensor) algebras [17]. The packages *clifford* and *cliffordan* authored by the presenter, implement Clifford algebras $C\ell_{p,q,r}$ of arbitrary signatures and orders. `clifford` emphasizes simplification, including the ability to treat multivectors as "sparse" purely symbolical objects [14].

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There is an elementary construction of Clifford algebras given by Macdonald [9]. This construction of $\mathbb{G}^n$ is suitable for direct implementation in a computer algebra system supporting symbolical transformations of expressions. From design perspective it was the preferred choice in `clifford`, while `atensor` implements the inner-product quadratic form- based approach. Before proceeding further with the construction, we give some remarks for computer algebra systems, which are given little attention by pure mathematicians but are rather important in computer science.

The relation "=" is considered a symmetric equivalence relation, while the syntactic equality "$\cong$" pertains only to the symbolical structure of the expression. In the computer algebra systems, there is a clear distinction between these two relations with the symbol "=" corresponding only to syntactic equality. However, here I will comply to usual mathematical notational convention and use "=" in both ways. In such way, expressions (or their parts) can be compared syntactically and matched against existing transformation rules, without the need of evaluating them first. In addition, there is an another important concept. Maxima distinguishes between two forms of applications of operators – forms which are `nouns` and forms which are `verbs`. The difference is that the `verb` form of an operator evaluates its arguments and produces an output result, while the `noun` form appears as an inert composite symbol in an expression, without being executed. A `verb` form can be mutated into a `noun` form and vice-versa. This allows for context-dependent evaluation, which is especially suited for symbolic processing.

1.1. **Expression representation and transformation in Maxima.** *Maxima* is the open source descendant of the first ever computer algebra system and features a rich functionality from a large number of shared packages. While written in Lisp, *Maxima* has its own programming language. The system also offers the possibility of running batch unit tests. Maxima supports several primitive data types [10]: *numbers* (rational, float and arbitrary precision); *strings* and *symbols*. In addition there are compound data types, such as *lists*, *arrays*, *matrices* and *structs*. There are also special symbolic constants, such as the Boolean constants `true` and `false` or the complex imaginary unit `%i`.

Several types of operators can be defined in Maxima. An operator is a defined symbol that may be *unary* `prefix`, *unary* `postfix`, *binary* `infix`, *n-ary* `matchfix`, or `nofix` types. For example, the inner and outer products defined in `clifford` are of the `binary infix` type. The scripting language allows for defining new operators with specified precedence or redefining the precedence of existing operators.

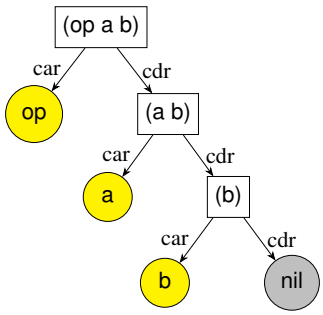


FIGURE 1. Expression representation in Lisp. A general expression can be represented as a list with the first element being the operator `op` and the rest of the elements representing operator arguments. For instance, the expression `op(a,b)` will be represented by the list `(op a b)`, which is the ordered pair of the atom `op`, and another list, `(a b)`, which, in turn, is represented by another ordered pair.

The Maxima language is based on the ideas of functional programming, which is particularly well suited for transformations of formal mathematical expressions. Maxima programs can be automatically translated and compiled to Lisp within the program environment itself. Third-party Lisp programs can be also loaded and accessed from within the system. The manner in which Maxima represents expressions, function calls and index expressions using the Lisp language is particularly relevant for the design of the `clifford`-based packages. In the underlying Lisp representation a Maxima *expression* is a tree containing sequences of operators, numbers and symbols. Every Maxima expression is simultaneously also a λ-construct and its value is the value of the last assigned member. This is a design feature inherited from Lisp. Maxima expressions are represented by underlying Lisp constructs. The core concept of the Lisp language is the idea of a *list* representation of the language constructs. The list is represented recursively by an ordered pair, the first element of which is the head (or `car`), the second element the tail (or `cdr`) of the list, which is also a list (see. Fig. 1). List elements are themselves either *lists* or *atoms*: e.g., a number, a symbol, or the empty list (`nil`). This representation enables the possibility to define transformation rules. In such way a part of an expression can be matched against a pattern and rewritten (see Listing 1).

A very powerful feature of the system is the ability to define custom transformation rules. Various transformation rules can be associated with any given operator in Maxima. Maxima has an advanced pattern matching mechanism, which supports nesting of operators and simplification. User-defined rules can be added to the built-in simplifier using one of two commands: `tellsimp` or `tellsimpafter`. Rules in both sets are identified by the main operator of the expression. Rules specified using `tellsimp` are applied before the built-in simplification, while `tellsimpafter` rules are applied after the built-in simplification. The augmented simplification is then treated as built-in, so subsequent tellsimp rules are applied before those defined previously. An example is given in Listing 1 used in the implementation of `clifford`.

2. Properties of Clifford algebras in view of Maxima implementation

I will give a brief exposition on the properties of Clifford algebras over the reals following [9]. Any Clifford algebra $\mathbb{G}^n$ is an associative unitary algebra, which is generated by a vector space V spanned by the orthonormal basis $\{e_1 \dots e_n\}$, over a field $\mathbb{K}$ of characteristic different from 2. The unit is usually skipped in notation (assuming implicit conversion between the scalar unit of $\mathbb{K}$ and the vector unit of V wherever necessary) and the square of the vector v is denoted conveniently by v^2.

2.1. Construction of the algebra $\mathbb{G}^n$.

Definition 1. *We define a generator symbol* **e** *and adjoin a natural number index $k \leq n$ to the symbol $e \mapsto e_k$ producing a set of n* ***basis vectors*** *$E := \{e_1 \dots e_n\}$ (also called* ***generators*** *of the algebra). Next, we define a canonical lexicographic order $\prec$ over E, such that $i < j \Longrightarrow e_i \prec e_j$.*

Finally, we define the (non-commutative) associative Clifford product *or geometric multiplication operation with properties:*

Closure: *The algebra is closed with regard to the geometric and scalar multiplication. For k basis vectors $e_1, \dots, e_k \in \mathbb{G}^n$ and the scalar λ the product*

$$\lambda e_1 \dots e_k \in \mathbb{G}^n \tag{C}$$

where we assume implicit conversion between the Clifford product and the multiplication operation of the field $\mathbb{K}$ whenever necessary.

Reducibility: *For all basis vectors*

(R) $$e_k e_k = \sigma_k$$

where $\sigma \in \{1,-1,0\}$ *are scalars of the field* $\mathbb{K}$*. The notation* $C\ell_{p,q,r}(\mathbb{K})$ $(p+q+r=n)$ *is interpreted as the convention that* p *elements of the orthonormal basis square to* 1*,* q *elements square to* -1 *and* r *(degenerate) elements square to* 0*.*

Anti-Commutativity: *For every two basis vectors, such that* $e_i \prec e_j$

(A-C) $$e_i e_j = -e_j e_i$$

Scalar Commutativity: *For all basis vectors* e_i *and scalars* λ

(S-C) $$e_i \lambda = \lambda e_i$$

The construction is well-defined since the existence of $\mathbb{G}^n$ implies the existence of $\mathbb{G}^{n-1}$ while $\mathbb{G}^1$ represent the double, complex or dual numbers respectively. Note also that the closure and scalar commutativity properties are not included in the construction of Macdonald [9]. This construction can be carried out without modifications in *Maxima* using the built-in associative non-commutative multiplication dot "." operator (Listing 1). Note that the closure property is implicit for any operator definition in *Maxima*.

Definition 2. *A multivector of the Clifford algebra is a linear combination of elements over the* 2^n*-dimensional vector space spanned by the power-set*

$$P(E) := \{1, e_1, \ldots, e_n, e_1e_2, e_1e_3, \ldots, e_1e_2 \ldots e_n\}.$$

LISTING 1. Clifford algebra construction in `clifford`

```
/*
Abstract Cliford algebra construction
*/
matchdeclare([aa, ee], lambda([u], not freeof(asymbol,u) and freeof ("+", u
    ) and not scalarp(u) ), [bb,cc], true,
[kk, mm, nn], lambda([z],  integerp(z) and z>0) );

if get('clifford,'version)=false then (
    tellsimp(aa[kk].aa[kk], signature[kk] ),
    tellsimpafter(aa[kk].aa[mm], dotsimp2(aa[kk].aa[mm])),
    tellsimpafter(bb.ee.cc, dotsimpc(bb.ee.cc)),
    tellsimp(bb^nn, bb^^nn)
);
```

Here `signature` corresponds to the variable σ. While the reordering of products is executed by the function `dotsimp2`.

The constitutive equations define a canonical representation of a multivector expression, which allows for automatic simplification. The main lemma is given in [9] which is repeated here for convenience:

Lemma 1 (Permutation equivalence)**.** *Let* $B = e_{k_1} \ldots e_{k_i}$ *be an arbitrary Clifford product, where the* i *basis vectors are not necessarily different. Then*

$$B = sP_{\prec}\left\{e_{k_1} \ldots e_{k_i}\right\}$$

where $s = \pm 1$ *is the sign of permutation of* B *and* $P_{\prec}\left\{e_{k_1} \ldots e_{k_i}\right\}$ *is the product permutation according to the ordering* $\prec$*.*

Proof. The proof follows directly from the anti-commutativity of multiplication for any two basis vectors A-C, observing that the sign of a permutation of S can be defined from its decomposition into the product of transpositions as $sgn(B) = (-1)^m$, where m is the number of transpositions in the decomposition. □

LISTING 2. Parity of permutation function

```
permsign(arr):=block([k:0, len, ret:0 ] ,
    mode_declare([k, len], fixnum),
    if not listp(arr) then return (false),
    len:length(arr),
    for i:1 thru len do (
        if not mapatom(arr[i]) then ret:nil,
        for j:i+1 thru len do
            if ordergreatp(arr[i], arr[j]) then k:k+1
    ),
    if ret#nil then
        if evenp(k)  then 1 else -1
    else 0
);
```

Further we can define a *simplified form* according to the action of eq. R. By means of this distinction it is convenient to define *blade* objects.

Definition 3. *A blade of grade k is a product of k basis elements in simplified form.*

Conventionally, 0-blades are scalars, 1-blades are vectors etc. A general multivector M can be decomposed by the grade projection operators $\langle\ \rangle_k$ into a direct sum of different subspaces:

$$M = \sum_{k=0}^{n} \langle M \rangle_k . \tag{1}$$

Defining simplified forms allows for an implementation of an efficient Clifford product simplification algorithm given in Listing 3.

LISTING 3. Clifford product simplification in `clifford`

```
dotsimpc(ab):=block([ba, c:1, v, w:1, q, r, l, sop],
 ....
        sop:inop(ab),
        if mapatom(ab) or freeof(".", ab) or
            sop='nil or sop="^"  or
                sop="^^" then return(ab),
        if sop="+" then
            map(dotsimpc, ab)
        else if sop="*" then (
        [r,l]: oppart(ab, lambda([u], freeof(".", u))),
        if _debug=true then display(sop, r, l),
            r:subst(nil=1, r),
            l:subst(".","*",l),
            r*dotsimpc(l)
            ) else (
                ba:copy(ab),
                v:inargs(ba),
                if _debug=true then display(sop, v),
                    w:sublist( v, lambda ([z], not freeof(asymbol,z) and
                        mapatom(z))),
                    w:permsign(w),
                if w#0 then (
                    v:sort(v),
                    for q in v do c:c.q,
                    if _debug=true then  display(w, v),
                    w*c
            ) else ab
        )
    );
```

2.2. **Involutions, inner and outer product decompositions.** There are three important involutions which change signs of blades – the reflection $A^\bullet$, the order reversion $A^\sim$ and the Clifford conjugation $A^\star$. The Clifford conjugation is the composition of reversion and reflection. The sign mutations for the different involutions are shown in Table 1.

TABLE 1. Sign mutation table for Clifford algebras $C\ell_{p,q,r}$.

	k mod 4			
	0	1	2	3
$A^\bullet$	+	−	+	−
$A^\sim$	+	+	−	−
$A^\star = A^{\bullet\sim}$	+	−	−	+

The Clifford product of **vectors** decomposes into a sum of *inner* and *outer* products according to

$$ab = a\cdot b + a\wedge b = \langle ab\rangle_0 + \langle ab\rangle_2 \tag{2}$$

Hestenes further identifies the outer product of vectors with the Grassmann's outer product [7][Ch. 1] and defines the outer product by extension for blades (not for scalars!) as

$$a \wedge B := \frac{1}{2}(aB + (-1)^r Ba) \tag{3}$$

$$a \cdot B := \frac{1}{2}(aB - (-1)^r Ba) \tag{4}$$

where r is the grade of B and a is a vector. In the general case for a grade k blade A_k and grade l blade A_l

$$A_k \wedge B_l = \begin{cases} \langle A_k B_l \rangle_{k+l} & k+l \leq n \\ 0 & k+l > n \end{cases}$$

The inner product, on the other hand, can be extended to blades in several different ways depending on the sign of $k-l$ resulting in *left*, *right* or *symmetric contractions*. For the latter we have

$$A_k \cdot B_l := \langle A_k B_l \rangle_{|k-l|}$$

In addition, the *scalar product* is defined as

$$A_k \star B_l := \langle A_k B_l \rangle_0$$

giving rise to the symmetric contraction decomposition [6]:

$$A_k \cdot B_l = A_k \rfloor B_l + A_k \lfloor B_l - A_k \star B_l$$

2.3. **Inverse elements.** The Clifford product provides a definite advantage compared to the Gibbs-Heaviside dot and cross products by its invertibility. One way of constructing an inverse of a vector is as follows. We can interpret eq. R as defining a quadratic form as follows. For a general vector

$$v = \sum_{k=1}^{n} v_k e_k$$

we define the quadratic form

$$Q(v) := vv = v \star v \tag{5}$$

For a given Clifford algebra $C\ell_{p,q,r}(\mathbb{R})$ over the real numbers. Then elementary calculations show that

$$Q(v) = \sum_{k=1}^{n} \sigma_k v_k^2 = v_1^2 + \ldots + v_p^2 - v_{p+1}^2 - \ldots - v_{p+q}^2.$$

which can be used in other equivalent definitions of the Cliford algebras, for example as given in [11]. For the elements for which $Q(v) \neq 0$ the inverse element can be reduced into the canonical form

$$v^{-1} = \frac{v}{vv} = \frac{v}{Q(v)}. \tag{6}$$

For order $n \leq 5$ closed form-expressions have been identified [16]. It is still an open question what the general formula for a multivector inverse can be in the abstract approach for orders higher than 5.

3. CLIFFORD ALGEBRAS IN MAXIMA

The initial Clifford support in Maxima was offered by the `atnesor` package, which is distributed as an add-on to the main system. At present the `clifford` package implements more extensive functionality related to Clifford algebras [14]. The code is distributed under GNU Lesser General Public License. A permanent repository of the presented version of the package is available through Zenodo repository [13]. A development version is hosted at GitHub (`http://dprodanov.github.io/clifford/`). The package includes also a large number of unit tests for the core functionality and offers several interactive demonstrations. My intention is to have *clifford* serving as core engine of any type of Clifford algebra computations possible in Maxima. Since there are multiple possible directions of development every new functionality set (for example geometric calculus or visualizations) is spun off from the core *clifford* engine in a specialized package. To fully support this strategy, I have endorsed minimalistic design approach, which surprisingly coincided with the Clifford algebra construction offered by [9], about which I was not aware until recently. The package relies extensively on the Maxima simplification functionality, and its features are fully integrated into the Maxima simplifier. The `clifford` package defines multiple rules for pre- and post-simplification of Clifford products, outer products, scalar products, inverses and powers of Clifford vectors. Using this functionality, any combination of products can be simplified into the canonical representation. The main features of the package are summarized in Table 2.

TABLE 2. Main functions in the `clifford` package.

function name	functionality
simplification	
`cliffsimpall` (expr)	full simplification of expressions
`dotsimp1` (ab)	canonic reordering of dot products
`dotsimpc` (ab)	simplification of dot products
`dotinvsimp` (ab)	simplification of inverses
`powsimp` (ab)	simplification of exponents
involutions	
`dotreverse` (ab)	Clifford reverse of product
`cinvolve` (expr)	Clifford involution of expression
`dotconjugate` (expr)	Clifford conjugate of expression
grade functions	
`grade` (expr)	grade decomposition of expression
`scalarpart` (expr)	$< expr >_0$
`vectorpart` (expr)	$< expr >_1$
`grpart` (expr,k)	$< expr >_k$
`mvectorpart` (expr)	$< expr >_{2+}$
`bdecompose` (expr)	blade decomposition of expression

Example 1 (Quaternions)**.** *The quaternion algebra $C\ell_{0,2}$ can be initialized by issuing the command*

```
clifford(e,0,2);
```

The function `mtable1` *computes and simplifies the geometric products of a list of elements and returns the multiplication matrix.*

```
mtable1([1, e[1],e[2], e[1] . e[2]]);
```

$$\begin{pmatrix} 1 & e_1 & e_2 & e_1.e_2 \\ e_1 & -1 & e_1.e_2 & -e_2 \\ e_2 & -e_1.e_2 & -1 & e_1 \\ e_1.e_2 & e_2 & -e_1 & -1 \end{pmatrix}$$

Here we give some examples in $\mathbb{G}^3$:

Example 2 (Outer product)**.** *Outer product evaluation:*

```
e[1]&e[2] & e[3];
```

$$e_1 e_2 e_3$$

```
(1+e[1])&(1+e[1]);
```

$$1+2e_1$$

```
(1+e[1])&(1-e[1]);
```

$$1$$

In a similar way the outer product table can be computed:

```
mtable2o();
```

$$\begin{pmatrix} 1 & e_1 & e_2 & e_3 & e_1\cdot e_2 & e_1\cdot e_3 & e_2\cdot e_3 & e_1\cdot e_2\cdot e_3 \\ e_1 & 0 & e_1\cdot e_2 & e_1\cdot e_3 & 0 & 0 & e_1\cdot e_2\cdot e_3 & 0 \\ e_2 & -e_1\cdot e_2 & 0 & e_2\cdot e_3 & 0 & -e_1\cdot e_2\cdot e_3 & 0 & 0 \\ e_3 & -e_1\cdot e_3 & -e_2\cdot e_3 & 0 & e_1\cdot e_2\cdot e_3 & 0 & 0 & 0 \\ e_1\cdot e_2 & 0 & 0 & e_1\cdot e_2\cdot e_3 & 0 & 0 & 0 & 0 \\ e_1\cdot e_3 & 0 & -e_1\cdot e_2\cdot e_3 & 0 & 0 & 0 & 0 & 0 \\ e_2\cdot e_3 & e_1\cdot e_2\cdot e_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_1\cdot e_2\cdot e_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3 (Associativity of outer product)**.** *We create 3 vectors with scalar components:*

```
a1:cvect(a),b1:cvect(b), c1:cvect(c);
```

$$a_1 e_1 + a_2 e_2 + a_3 e_3$$

And then test for the associativity of the vector product:

```
(a1 & b1 )& c1,factor;
```

$$(-c_1 b_2 a_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + a_1 b_2 c_3)\,(e_1\cdot e_2\cdot e_3)$$

```
a1 & (b1 & c1),factor
```

$$(-c_1 b_2 a_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + a_1 b_2 c_3)\,(e_1\cdot e_2\cdot e_3)$$

The modulus of the triple product can be identified as the determinant of the coordinate matrix as will be shown further. Moreover, we can verify an the mixed product identity from vector algebra:

$$c\cdot(a\times b) = det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

```
c1| (-%iv. (a1 & b1)),dotsimpc;
```

$$-c_1b_2a_3+b_1c_2a_3+c_1a_2b_3-a_1c_2b_3-b_1a_2c_3+a_1b_2c_3$$

```
M1: matrix(
[a[1],a[2],a[3]],
[b[1],b[2],b[3]],
[c[1],c[2],c[3]]
)$
D2:determinant(M1);
```

$$(-c_1b_2+b_1c_2)a_3-a_2(-c_1b_3+b_1c_3)+a_1(-c_2b_3+b_2c_3)$$

Example 4 (Inner products)**.**
Inner product evaluation :

```
((1+e[1])/2)|( (1+e[1])/2),factor;
```

$$\frac{1+e_1}{2}$$

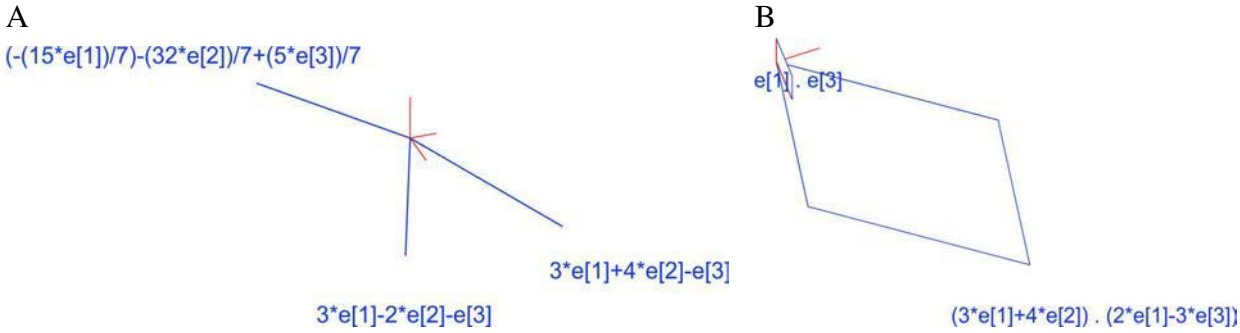
In `clifford` *the left and right contractions and the symmetrical product are supported by the command line switches* `lc` *for left contraction,* `rc` *for right contraction and* `sym` *for the symmetric product. The example below calculates left contraction in* $\mathbb{G}^3$.

```
inprotype:lc;
mtable2i();
```

$$\begin{pmatrix}
1 & e_1 & e_2 & e_3 & e_1.e_2 & e_1.e_3 & e_2.e_3 & e_1.e_2.e_3 \\
0 & 1 & 0 & 0 & e_2 & e_3 & 0 & e_2.e_3 \\
0 & 0 & 1 & 0 & -e_1 & 0 & e_3 & -e_1.e_3 \\
0 & 0 & 0 & 1 & 0 & -e_1 & -e_2 & e_1.e_2 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -e_3 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & e_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -e_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

3.1. **Visualization of blades.** The `clidraw` package offers elementary functionality for visualization of multivector operations. Projections of multivectors on 2 and 3 dimensional subspaces can be drawn using the underlying Maxima graphical functionality. Reflection of the vector $a=3e_1+4e_2-e_3$ by $b=3e_1-2e_2-e_3$ is plotted in Fig. 2A. Bi-vectors spanned by the vectors $\{e_1,e_2\}$ and $\{3e_1+4e_2,2e_1-3e_3\}$ are plotted in Fig. 2B.

FIGURE 2. Vector and bi-vector display in the `clidraw` package.



A – plotting of vector operations; B – plotting of bi-vectors.

4. GENERALIZED DERIVATIVES IN CLIFFORD ALGEBRAS

4.1. **Multi-vector derivatives.** Vector derivative of a function is well-defined in the case a Clifford algebra is non-degenerate ($r = 0$), which will be assumed further on.

Definition 4 (Reciprocal frames)**.** *Consider the multivector $R = r_k e_k$, in the subspace spanned by $e_{i_1} \wedge \dots e_{i_m}$. The frame e_k can be extracted by partial differentiation with respect to coordinates:*

$$e_k = \frac{\partial R}{\partial r_k}$$

Then for every fame we define the reciprocal frame as $e^k = e_k^{-1}$.

Definition 5 (Geometric derivative)**.** *Consider the vector r spanned by the subspace $e_1 \wedge \dots e_m$, $m \leq n$. Following* [7] *the vector derivative ∇_r of the multi-vector valued function $f(x)$ is defined as*

$$\nabla_r f(x) = \sum_{i=1}^{m} e^i \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon} = e^i \partial_i f(x) \tag{7}$$

where $\varepsilon > 0$ is a scalar vanishing in limit and the limit procedure is understood in norm.

the definition can be readily extended to multi-vectors. The vector (but not multivector) derivative can be decomposed into inner and exterior components

$$\nabla_r F = \nabla_r \cdot F + \nabla_r \wedge F$$

in a way similar to the decomposition of the geometric product.

A local fractional variant of the derivative [18] called *fractional velocity* [5, 18, 12] can also be constructed

Definition 6.

$$\nabla^\beta_\pm f(x) = \sum_{i=1}^{n} \pm e^i \lim_{\varepsilon \to 0} \frac{f(x \pm \varepsilon e_i) - f(x)}{\varepsilon^\beta} = e^i \partial^\beta_{\pm i} f(x) \tag{8}$$

for the exponent $0 < \beta \leq 1$.

Then for (locally) differentiable functions [12]:

$$\partial^\beta_{\pm k} f(x) = \lim_{\varepsilon \to 0} \frac{1}{\beta} \varepsilon^{1-\beta} \partial_k f(x \pm \varepsilon e_i)$$

5. GEOMETRIC CALCULUS FUNCTIONALITY IN MAXIMA

Maxima supports symbolic differentiation and integration in the real and complex domains. This functionality can be extended also to Clifford numbers. The `cliffordan` package, which is based on `clifford` implements symbolical differentiation based on the vector derivative. The main building block is the total derivative function shown in Listing 4.

LISTING 4. Clifford-valued total differentiation of an expression in *cliffordan*

```
/*
Clifford-valued
total differentiation; */

ctotdiff(f,x):=block( [ret:0, lv],
    if mapatom(x) then
        ret: diff( f, x )
    else (
        lv: sublist( listofvars(x), lambda ([z], freeof(asymbol,z))),
        for u in lv do
            ret: ret + cinv(diff(x, u)). subst(".", "*", diff( f, u ))
    ),
ret
);
```

The main functions of the packages are listed in Table 3

name	functionality
`ctotdiff`(f, x)	total derivative w.r.t. multivector x
`ctotintdiff`(f, x)	inner total derivative w.r.t. x
`ctotextdiff`(f, x)	outer total derivative w.r.t. x
`vectdiff`(f, ee, k)	vector derivative of order k w.r.t. basis vector list *ee*
`mvectdiff`(f, x, k)	multivector derivative of order k w.r.t. multivector x
`parmvectdiff`(f, x, k)	partial multivector derivative of order k w.r.t. multivector x
`convderiv`(f, t, xx, [vs])	convective derivative w.r.t. multivector x
`coordsubst`(x, eqs)	substitutes coordinates in multivector x w.r.t. new variables in the list *eqs*
`clivolel`(x, eqs)	computes volume element of $Span\{x\}$ w.r.t. new variables in the list *eqs*

TABLE 3. Main functions in `cliffordan`

5.1. **Potential problems in $C\ell_{3,0}$.** We shall give a presentation of the potential problem in the geometric algebra $\mathbb{G}^3 = C\ell_{3,0}$. In electrostatic or magnetostatic setting the Green's function of the system

$$\nabla_r G = \delta(r)$$

where $\delta(x)$ is the Dirac's Delta function is

$$G(x,y,z) = \frac{e_1 x + e_2 y + e_3 z}{4\pi \sqrt{(x^2+y^2+z^2)^3}} \tag{9}$$

Direct calculation issuing the commands

```
xx:e[1]*x+e[2]*y+e[3]*z$
G:xx/sqrt(-cnorm(xx))^3$
mvectdiff(GG,xx)
```

evaluates to 0 for $x \neq 0, y \neq 0, z \neq 0$. In the last calculation the factor is skipped for simplicity. $G(x,y,z)$ can be derived from the following scalar potential

$$V(x,u,z) = -\frac{C}{\sqrt{x^2+y^2+z^2}} \tag{10}$$

where C is an arbitrary constant matching the initial or boundary conditions.

```
mvectdiff(-1/sqrt(x^2+y^2+z^2),xx);
```

yielding

$$\frac{e_1 x + e_2 y + e_3 z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

5.2. **Coordinate transformations.** Geometric algebra implementation in `clifford` allows for transparent coordinate substitutions.

Example 5. *In the following example we verify the properties of the Green's function in cylindrical coordinates.*

```
GG_c:coordsubst(G, cyl_eq),factor;
```

$$\frac{e_1 \rho \cos\phi + e_2 \rho \sin\phi + e_3 z}{(\rho^2+z^2)^{\frac{3}{2}}}$$

```
rc:coordsubst(r, cyl_eq);
```

$$(e_1 \cos\phi + e_2 \sin\phi)\,\rho + e_3 z$$

```
V:coordsubst(-1/sqrt(-cnorm(r)),cyl_eq);
```

$$-\frac{1}{\sqrt{\rho^2+z^2}}$$

```
rc:coordsubst(r, cyl_eq);
```

$$(e_1 \cos\phi + e_2 \sin\phi)\,\rho + e_3 z$$

```
mvectdiff(V,rc);
```

yielding

$$\frac{e_1 \rho \cos\phi + e_2 \rho \sin\phi + e_3 z}{(\rho^2+z^2)^{\frac{3}{2}}}$$

as expected.

```
mvectdiff(GG_c,rc);
```

which yields 0 as expected.

5.3. **Homogeneous d'Alembert equation in** `clifford`**.** Clifford algebras offer a convenient way of combining objects of different grades in the form of inhomogeneous sums. For example, the sum of a scalar and a 3-vector in $\mathbb{G}^3$ (called a *paravector*) is a well-defined inhomogeneous object that can be used in calculations. The Euler-Lagrange field equations corresponding to the scalar Lagrangian density $\mathscr{L}(q, \partial_x q)$ involving a field q and its derivatives $\partial_x q$ with respect to the coordinates x are derived in [8].

$$p = \hat{\nabla}_x \, q, \quad x = x_i e_i \tag{11}$$

$$\hat{\nabla}_q \mathscr{L} = \left(\hat{\nabla}_p \mathscr{L}\right) \nabla_x \tag{12}$$

where r is a radius-vector and the hat denotes partial derivation.

Example 6. *The wave equation was derived in 1747 by Jean-Baptiste le Rond d'Alembert in the analysis of the problem of vibrating strings. The following application derives this equation using the Euler-Lagrange framework. The Euler-Lagrange equation for the mixed object* $p = \nabla_r A$ *can be represented as:*

$$p^\star = \hat{\nabla}_{x^\star} \, q$$

$$\hat{\nabla}_{q^\star} \mathscr{L} = \nabla_x \left(\hat{\nabla}_{p^\star} \mathscr{L}\right)$$

Here the derivation procedure is replicated using the `clifford` *package. Let A be the paravector potential given by*

$$A = A_t + e_1 A_x + e_2 A_y + e_3 A_z, \tag{13}$$

which can be constructed by the command

```
AA: celem(A,[t,x,y,z])$
```

Then the geometric derivative object is given by applying geometric derivative using a paravector $x = t - r$*:*

$$F = \nabla_{t-r} A, \tag{14}$$

given by the command

```
F:mvectdiff(AA,t-r)$
```

resulting in a mixture of scalar, vector and bi-vector components:

$$F = \langle F \rangle_0 + \langle F \rangle_1 + \langle F \rangle_2 \tag{15}$$

The wave equation for the paravector potential can be derived from a purely quadratic Lagrangian composed from the components of the geometric derivative of F

$$\mathscr{L}_a = \frac{1}{2} \langle F^2 \rangle_0, \tag{16}$$

```
L:lambda([x],1/2*scalarpart(cliffsimpall(x.x)))(F);
```

$$\begin{aligned}
&((A_{tt})^2+(A_{tx})^2+(A_{ty})^2+(A_{tz})^2-2\,(A_{tx})\,(A_{xt})+(A_{xt})^2-2\,(A_{tt})\,(A_{xx})\\
&+(A_{xx})^2-(A_{xy})^2-(A_{xz})^2-2\,(A_{ty})\,(A_{y_t})+(A_{y_t})^2\\
&+2\,(A_{xy})\,(A_{y_x})-(A_{y_x})^2-2\,(A_{tt})\,(A_{y_y})\\
&+2\,(A_{xx})\,(A_{y_y})+(A_{y_y})^2-(A_{y_z})^2-2\,(A_{tz})\,(A_{zt})+(A_{zt})^2\\
&+2\,(A_{xz})\,(A_{zx})-(A_{zx})^2+2\,(A_{y_z})\,(A_{zy})-(A_{zy})^2-2\,(A_{tt})\,(A_{zz})\\
&+2\,(A_{xx})\,(A_{zz})+2\,(A_{y_y})\,(A_{zz})+(A_{zz})^2)/2\,,
\end{aligned}$$

where we can recognize the identity

$$\mathscr{L}_a=\frac{1}{2}\left(\langle F\rangle_0^2+\langle F\rangle_1^2+\langle F\rangle_2^2\right) \tag{17}$$

```
S:scalarpart(F)$
V:vectorpart(F)$
Q:grpart(F,2)$
L-1/2*(S.S+V.V+Q.Q),cliffsimpall;
```

yielding 0.

Finally, applying the functional derivative, that is the Euler-Lagrange functional yields:

```
dA:mvectdiff(AA,r);
EuLagEq2(L, t+r,[AA,dA]);
```

$$\begin{aligned}
&\left(-A_{ttt}+A_{txx}+A_{tyy}+A_{tzz}\right)+e_1\left(-A_{xtt}+A_{xxx}+A_{xyy}+A_{xzz}\right)\\
&+e_2\left(-A_{y_{tt}}+A_{y_{xx}}+A_{y_{yy}}+A_{y_{zz}}\right)+e_3\left(-A_{ztt}+A_{zxx}+A_{zyy}+A_{zzz}\right),
\end{aligned}$$

which can be recognized as the D'Alembertian for the components of the paravector potential:

$$\nabla_{t+r}\nabla_{t-r}A=0. \tag{18}$$

The Maxima expression can be decomposed in a matrix form as

```
(%i51)  bdecompose(%);
```

$$\begin{aligned}
&[[[1],\left(-A_{ttt}+A_{txx}+A_{tyy}+A_{tzz}\right)],\\
&[[e_1,e_2,e_3],\begin{pmatrix}-A_{xtt}+A_{xxx}+A_{xyy}+A_{xzz}\\-A_{y_{tt}}+A_{y_{xx}}+A_{y_{yy}}+A_{y_{zz}}\\-A_{ztt}+A_{zxx}+A_{zyy}+A_{zzz}\end{pmatrix}],\\
&[[0],(0)],[[0],(0)]]
\end{aligned}$$

if one wishes to solve for individual components.

6. Outlook

This paper demonstrates applications of Clifford algebra in several areas of applied sciences, notably: visualizations of geometric objects, coordinate transformations, derivation of the Green function of the Poisson's equation and variational problems. In summary, Clifford algebra packages of Maxima can be considered as sufficiently mature for use as research tools. Further development of the Clifford algebra tools will be directed towards more extensive linear algebra functionality and fractional calculus.

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The author wishes to acknowledge conference organizers for pointing out subtleties in definitions of inner and outer products which led improvements of the add-on functionality of `clifford`.

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Appendix: Example Code listings

Listing 5. Inner and outer products example

```
1          /* Initialization */
```

```
load('clifford);
clifford(e,0,2);
mtable1([1, e[1],e[2], e[1] . e[2]]);
clifford(e, 3);
e[1] &e[2] &e[3];
(1+e[1])&(1+e[1]);
(1+e[1])&(1-e[1]);
mtable2o();
inprotype:lc;
((1+e[1])/2)|( (1+e[1])/2),factor;
mtable2i();
```

LISTING 6. Associatvity of outer products example

```
/* Initialization */
load('clifford);
clifford(e, 3);
a1:cvect(a),b1:cvect(b), c1:cvect(c);
b1:cvect(b);
c1:cvect(c);
(a1 & b1 )& c1,factor;
a1 & (b1 & c1),factor;
L1:a1 | (b1 & c1);
L2:(a1|b1).c1 - (a1 | c1) .b1,expand;
L1-L2;
D1:c1| (-%iv. (a1 & b1)),dotsimpc;

M1: matrix(
[a[1],a[2],a[3]],
[b[1],b[2],b[3]],
[c[1],c[2],c[3]]
);
D2:determinant(M1);

equal(D1,D2),pred;
```

LISTING 7. Potential theory example

```
/* Initialization */
load('clifford);
load ('cliffordan);
clifford(e,3);

r:cvect([x,y,z]);
G:r/sqrt(-cnorm(r))^3;
mvectdiff(G,r);
mvectdiff(-1/sqrt(-cnorm(r)),r);
mvectdiff(-1/sqrt(-cnorm(r)),r,2);


cyl_eq:[x=rho*cos(phi), y=rho*sin(phi)];
declare( [rho, phi], scalar);
GG_c:coordsubst(G, cyl_eq),factor;
rc:coordsubst(r, cyl_eq);
mvectdiff(GG_c,rc);
V:coordsubst(-1/sqrt(-cnorm(r)),cyl_eq);
mvectdiff(V,rc);
```

LISTING 8. Lagrangian example

```
/* Initialization */
load('clifford);
load ('cliffordan);
clifford(e,3);

derivabbrev:true;
AA: celem(A,[t,x,y,z]);
dependsv(A,[t,x,y,z]);
r:cvect([x,y,z]);
F:mvectdiff(AA,t-r);
L:lambda([x],1/2*scalarpart(cliffsimpall(x.x)))(F);
S:scalarpart(F);
V:vectorpart(F);
Q:grpart(F,2);
L-1/2*(S.S+V.V+Q.Q),cliffsimpall;
dA:mvectdiff(AA,r);
EuLagEq2(L, t+r,[AA,dA]);
bdecompose(%);
```

FOCK REPRESENTATIONS AND DEFORMATION QUANTIZATION OF KÄHLER MANIFOLDS

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ABSTRACT. The goal of this paper is to construct the Fock representation of noncommutative Kähler manifolds. Noncommutative Kähler manifolds studied here are constructed by deformation quantization with separation of variables, which was given by Karabegov. The algebra of the noncommutative Kähler manifolds contains the Heisenberg-like algebras. Local complex coordinates and partial derivatives of a Kähler potential satisfy the commutation relations between creation and annihilation operators. A Fock space is constituted by states obtained by acting creation operators on a vacuum which is annihilated by all annihilation operators. The algebras on noncommutative Kähler manifolds are represented as those of linear operators acting on the Fock space. In representations studied here, creation operators and annihilation operators are not Hermitian conjugate with each other, in general. Therefore, the bases of the Fock space are not the Hermitian conjugates of those of the dual vector space. In this sense, we call the representation the twisted Fock representation. In this presentation, we construct the twisted Fock representations for arbitrary noncommutative Kähler manifolds given by deformation quantization with separation of variables, and give a dictionary to translate between the twisted Fock representations and functions on noncommutative Kähler manifolds concretely.

1. INTRODUCTION

Why is it worth investigating noncommutative geometry? One of the big reasons is found in physics. We come across field theories on noncommutative spaces in various situations. Many theoretical physicists consider that string theory is a promising candidates for fundamental theory including quantum gravity. In string theories, there exist boundaries on which the end points of open strings can be attached, and the boundaries are called D-branes. The effective theories of the D-branes with some background fields are given as gauge theories on noncommutative manifolds [39]. One of other examples of field theories on noncommutative manifolds is given in matrix models [4, 15] which are candidates for non-perturbative definitions of string theories. In the models, some classical solutions of their equations of motion correspond to gauge theories on noncommutative space. These facts imply that the spacetime of our universe is not a usual manifold but a noncommutative manifold. Therefore, many physicists have tried to construct field theories on noncommutative manifolds. (See, for example, review papers [27, 40, 32].)

So far, various formulations have been developed to construct noncommutative space; Connes' noncommutative geometry, deformation quantization, geometric quantization, fuzzy

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space, q-deformation and so on. They are partially related with each other, but not equivalent. In this article, we only consider deformation quantization to construct noncommutative manifolds.

The origin of deformation quantization would be seen in the equivalence between a differentiable manifold and the algebra of C^∞ functions on it, which is shown by Pursell and Shanks [30]. By extending this equivalence, a deformation of a commutative algebra of C^∞ functions on a differentiable manifold to a noncommutative algebra gives a noncommutative manifold corresponding to it. This is the basic concept of deformation quantization.

Deformation quantization was introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [5]. After [5], many methods for deformation quantization were proposed [12, 28, 13, 22]. In particular, deformation quantizations of Kähler manifolds were provided in [25, 26, 9, 10]. (For a recent review, see [37].) In this paper, we consider deformation quantization with separation of variables, which was introduced by Karabegov to construct noncommutative Kähler manifolds [17, 18, 20].

Deformation quantization of some manifold is usually given by a noncommutative product which is defined as a product in a form of a formal power series of deformation parameter. This product is called a star product. In this paper, $\hbar$ is used for the deformation parameter. Basically, a formal power series defining a star product is obtained as a solution of an infinite system of differential equations, and it is proved that there exists a unique deformation quantization as the solution of the system. Though the existence of solutions is proved for a wide class of manifolds, explicit expressions of star products are not obtained without few kinds of manifolds. For example, the star products on noncommutative Euclidean spaces are given by a famous star product that is called the Moyal product. On manifolds with spherically symmetric metrics, explicit star products are also given in the context of the Fedosov's deformation quantization [13].

It is known that explicit formulas of star products are given for locally symmetric Kähler manifolds [14]. One of the aim of this article is to review how to obtain the explicit expressions of deformation quantizations with separation of variables for $\mathbb{C}P^N$ and $\mathbb{C}H^N$.[1] To make star products, we have to solve the infinite system of differential equations. In these cases, systems of differential equations are translated into algebraic recursion relations, and thus we can solve them. Then, expressions of star products are explicitly given in all order of $\hbar$.

As mentioned above, deformation quantization is given by an associative algebra on a set of formal power series of C^∞ functions with a star product. An advantage of deformation quantization is that usual analytical techniques are available on noncommutative manifolds. On the other hand, when we study physics on noncommutative manifolds given by deformation quantization, there is a difficulty. In this formulation, physical quantities are also given as formal power series of a noncommutative parameter. It is difficult to interpret formal power series as observable quantities. In addition, a formal power series prevents from applying several useful methods like the principle of least action. A typical way to solve the difficulties is to make a representation of the noncommutative algebra.

A main purpose of this article is to describe the Fock representation of noncommutative Kähler manifolds. The algebras on noncommutative Kähler manifolds constructed by deformation quantization with separation of variables contain the Heisenberg algebras. Local complex coordinates and partial derivatives of a Kähler potential satisfy the commutation relations between creation and annihilation operators. A Fock space is spanned by a vacuum state, which is annihilated by all annihilation operators, and states given by acting creation operators on

[1] Star products on the fuzzy $\mathbb{C}P^N$ are investigated in [3, 21, 16]. A deformation quantization of the hyperbolic plane was provided in [8].

this vacuum. The algebras on noncommutative Kähler manifolds are represented as those of linear operators acting on the Fock space. We call the representation of the algebra the Fock representation. In representations studied in this article, creation operators and annihilation operators are not Hermitian conjugate with each other, in general. Therefore, the bases of the Fock space are not the Hermitian conjugates of those of the dual vector space. In this case, the representation is called the twisted Fock representation. [2] We here describe the twisted Fock representation for an arbitrary noncommutative Kähler manifold where the noncommutative Kähler manifold are constructed by using deformation quantization with separation of variables [17, 18, 20].

The results of the twisted Fock representation of the noncommutative Kähler manifolds are summarized as the following dictionary, Table 1. In this dictionary, $z^i, \overline{z}^i$ $(i = 1, \cdots N)$ are

TABLE 1. Functions - Fock operators Dictionary

Functions	Fock operators
$e^{-\Phi/\hbar}$	$\|\vec{0}\rangle\langle\vec{0}\|$
z_i	$a_i^\dagger$
$\frac{1}{\hbar}\partial_i\Phi$	$\underline{a}_i$
$\overline{z}^i$	$a_i = \sum\sqrt{\frac{\vec{m}!}{\vec{n}!}}H_{\vec{m},\vec{k}}H^{-1}_{\vec{k}+\vec{e}_i,\vec{n}}\|\vec{m}\rangle\langle\vec{n}\|$
$\frac{1}{\hbar}\partial_{\overline{i}}\Phi$	$\underline{a}_i^\dagger = \sum\sqrt{\frac{\vec{m}!}{\vec{n}!}}(k_i+1)H_{\vec{m},\vec{k}+\vec{e}_i}H^{-1}_{\vec{k},\vec{n}}\|\vec{m}\rangle\langle\vec{n}\|$

local complex coordinates of some open subset of an N dimensional Kähler manifold. Φ is its Kähler potential and H is defined by $e^{\Phi/\hbar} = \sum H_{\vec{m},\vec{n}}z^{\vec{m}}\overline{z}^{\vec{n}}$, where $z^{\vec{m}} = z_1^{m_1}z_2^{m_2}\cdots z_N^{m_N}$ for $\vec{m} = (m_1, m_2, \cdots, m_N)$, and $\overline{z}^{\vec{n}}$ is similarly defined. $a_i^\dagger$ and $\underline{a}_i$ are essentially a creation operator and an annihilation operator, respectively. a_i and $\underline{a}_i^\dagger$ are Hermitian conjugate with $a_i^\dagger$ and $\underline{a}_i$, respectively. Note that $a_i^\dagger$ is not a Hermitian conjugate of $\underline{a}_i$, in general. The strict definitions of the above quantities are given in Section 2 and 4.

The twisted Fock algebra is defined on a local coordinate chart. The star product with separation variables are glued between charts with nonempty overlap. Therefore, transition functions between the twisted Fock algebras on two charts having an overlapping region are also constructed. But, the detail of the transition between the Fock algebras are skipped due to a limit to the pages of this proceedings. We observe the twisted Fock representations of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ as examples.

The organization of this article is as follows. In Section 2, we review deformation quantization with separation of variables which is the way to make noncommutative Kähler manifolds. In Section 3, star products for noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are explicitly constructed as examples of deformation quantization with separation of variables. In Section 4, a twisted Fock representation is constructed on a chart of a general Kähler manifold. In Section 5, the Fock representations of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are given as examples. We summarize our results in Section 6.

[2] Historically, Berezin constructed a kind of the Fock representations of some noncommutative Kähler manifolds[6, 7]. After them, many studies of this subject are reported [31, 37, 38, 29].

2. Review of Kalabegov's deformation quantization

In this section, we review how to construct noncommutative Kähler manifolds by deformation quantization with separation of variables.

An N-dimensional complex Kähler manifolds is described by using a Kähler potential. Let Φ be a Kähler potential and ω be a Kähler 2-form:

$$\begin{aligned} \omega &:= ig_{k\bar{l}}dz^k \wedge d\bar{z^l}, \\ g_{k\bar{l}} &:= \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z^l}}. \end{aligned} \tag{1}$$

In this paper, we use the Einstein summation convention over repeated indices. The $g^{\bar{k}l}$ is the inverse of the metric $g_{k\bar{l}}$:

$$g^{\bar{k}l}g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}. \tag{2}$$

In the following, we denote

$$\partial_k = \frac{\partial}{\partial z^k}, \qquad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z^k}}. \tag{3}$$

Definition 1 (Deformation quantization (weak sense))**.** Deformation quantization is defined as follows. $\mathscr{F}$ is defined as a set of formal power series:

$$\mathscr{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k,\ f_k \in C^\infty \right\}. \tag{4}$$

A star product is defined as

$$f * g = \sum_k C_k(f,g)\hbar^k \tag{5}$$

such that the product satisfies the following conditions.

(I) $*$ is associative product.
(II) C_k is a bidifferential operator.
(III) C_0 and C_1 are defined as
$C_0(f,g) = fg$,
$C_1(f,g) - C_1(g,f) = i\{f,g\}$,
where $\{f,g\}$ is the Poisson bracket.
(IV) $f * 1 = 1 * f = f$.

Note that this definition of the deformation quantization is weaker than the usual definition of deformation quantization. The difference between them is seen in $C_1(f,g) - C_1(g,f) = i\{f,g\}$. In the strong sense of deformation quantization the condition $C_1(f,g) = \frac{i}{2}\{f,g\}$ is required. For example, the Moyal product satisfies this condition. But deformation quantizations with the separation of variables do not satisfy this condition. In the following, "deformation quantization" is used in the weak sense.

Definition 2 (A star product with separation of variables)**.** $*$ is called a star product with separation of variables when

$$a * f = af \tag{6}$$

for a holomorphic function a and

$$f * b = fb \tag{7}$$

for an anti-holomorphic function b.

We use

$$D^{\bar{l}} = g^{\bar{l}k}\partial_k = i\{\overline{z}^l, \cdot\}$$

and

$$\mathscr{S} := \left\{A | A = \sum_{\alpha} a_\alpha D^\alpha, \ \ a_\alpha \in C^\infty\right\},$$

where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. There are some useful formulas which $D^{\bar{l}}$ satisfies,

$$[D^{\bar{l}}, D^{\bar{m}}] = 0 \ , \ \forall l, m \tag{8}$$

$$[D^{\bar{l}}, \partial_{\bar{m}}\Phi] = \delta^{\bar{l}}{}_{\bar{m}} \tag{9}$$

$$\partial_k = g_{k\bar{l}} D^{\bar{l}}. \tag{10}$$

Using them, one can construct a star product as a differential operator L_f such that $f * g = L_f g$.

Theorem 2.1. *For arbitrary ω, there exist a star product with separation of variables $*$ and it is constructed as follows. Let f be an element of $\mathscr{F}$ and $A_n \in \mathscr{S}$ be a differential operator whose coefficients depend on f i.e.*

$$A_n = a_{n,\alpha}(f) D^\alpha, \qquad D^\alpha = \prod_{i=1}^{n} (D^{\bar{i}})^{\alpha_i}, \tag{11}$$

where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then,

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n \tag{12}$$

is uniquely determined such that it satisfies the following conditions,

$$\left[L_f, \ R_{\partial_{\bar{l}}\Phi}\right] = 0, \tag{13}$$

$$L_f 1 = f * 1 = f, \tag{14}$$

where $R_{\partial_{\bar{l}}\Phi} = \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}$. Then, a star product defined by $f * g := L_f g$ is associative,

$$L_h(L_g f) = h * (g * f) = (h * g) * f = L_{L_h g} f. \tag{15}$$

Recall that each two of $D^{\bar{i}}$ commute each other, so if multi index α is fixed then the A_n is uniquely determined. Similarly, a differential operator R_g which define the right $*$-multiplication, $f * g = R_g f$ is determined by

$$\left[R_g, \ L_{\partial_l\Phi}\right] = 0, \tag{16}$$

$$R_g 1 = 1 * g = g, \tag{17}$$

where $L_{\partial_l\Phi} = \partial_l\Phi + \hbar\partial_l$.

The following proposition is also used in the following sections.

Proposition 2.2. *We denote the left $*$-multiplication operation for a generic function f as L_f i.e. $L_f g = f * g$. The right $*$-multiplication operation for f is defined similarly by $R_f g := g * f$. L_f (R_f) is obtained by using $L_{\overline{z}^l}$ (R_{z^l}) where $L_{\overline{z}^l}$ (R_{z^l}) is defined by $L_{\overline{z}^l} g = \overline{z}^l * g$ ($R_{z^l} g = g * z^l$):*

$$L_f = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial \overline{z}}\right)^\alpha f \ (L_{\overline{z}} - \overline{z})^\alpha, \tag{18}$$

$$R_f = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial z}\right)^\alpha f \ (R_z - z)^\alpha. \tag{19}$$

From the definition of the star product, we easily find

$$[\frac{1}{\hbar}\partial_i\Phi,\ z^j]_* = \delta_{ij}, \qquad [z^i,\ z^j]_* = 0, \qquad [\partial_i\Phi,\ \partial_j\Phi]_* = 0, \tag{20}$$

$$[\bar{z^i},\ \frac{1}{\hbar}\partial_{\bar{j}}\Phi]_* = \delta_{ij}, \qquad [\bar{z^i},\ \bar{z^j}]_* = 0, \qquad [\partial_{\bar{i}}\Phi,\ \partial_{\bar{j}}\Phi]_* = 0, \tag{21}$$

where $[A,\ B]_* = A*B - B*A$. Hence, $\{z^i, \partial_j\Phi \mid i,j = 1,2,\cdots,N\}$ and $\{\bar{z^i}, \partial_{\bar{j}}\Phi \mid i,j = 1,2,\cdots,N\}$ constitute $2N$ sets of the creation and annihilation operators under the star product. However, operators in $\{z^i, \partial_j\Phi\}$ do not commute with ones in $\{\bar{z^i}, \partial_{\bar{j}}\Phi\}$, e.g., $z^i * \bar{z^j} - \bar{z^j} * z^i \neq 0$.

3. Examples of star products

In this section, as examples of the deformation quantization of separation variables, we study noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$. Both manifolds are typical Kähler manifolds, and they are also locally symmetric manifolds. This section is based on the studies in [33, 34, 24].

Example 1. Noncommutative $\mathbb{C}P^N$

In the inhomogeneous coordinates z^i $(i = 1,2,\cdots,N)$, the Kähler potential of $\mathbb{C}P^N$ is given by

$$\Phi = \ln\left(1 + |z|^2\right), \tag{22}$$

where $|z|^2 = \sum_{k=1}^N z^k\bar{z^k}$. The metric $(g_{i\bar{j}})$ is

$$ds^2 = 2g_{i\bar{j}}dz^i d\bar{z^j}, \tag{23}$$

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}\Phi = \frac{(1+|z|^2)\delta_{ij} - z^j\bar{z^i}}{(1+|z|^2)^2}, \tag{24}$$

and the inverse of the metric $(g^{\bar{i}j})$ is

$$g^{\bar{i}j} = (1+|z|^2)\left(\delta_{ij} + z^j\bar{z^i}\right). \tag{25}$$

To derive explicit forms of a star product on $\mathbb{C}P^N$, the following relations play an important role,

$$\partial_{\bar{i_1}}\partial_{\bar{i_2}}\cdots\partial_{\bar{i_n}}\Phi = (-1)^{n-1}(n-1)!\ \partial_{\bar{i_1}}\Phi\partial_{\bar{i_2}}\Phi\cdots\partial_{\bar{i_n}}\Phi, \tag{26}$$

$$\left[\partial_{\bar{i}}, D^{\bar{j}}\right] = \partial_{\bar{i}}\Phi D^{\bar{j}} + \delta_{ij}\partial_{\bar{k}}\Phi D^{\bar{k}}, \tag{27}$$

$$\begin{aligned}\left[\partial_{\bar{i}},\ c_{\bar{j_1}\bar{j_2}\cdots\bar{j_n}}D^{\bar{j_1}}D^{\bar{j_2}}\cdots D^{\bar{j_n}}\right] &= \partial_{\bar{i}}c_{\bar{j_1}\bar{j_2}\cdots\bar{j_n}}D^{\bar{j_1}}D^{\bar{j_2}}\cdots D^{\bar{j_n}} + nc_{\bar{j_1}\cdots\bar{j_n}}\partial_{\bar{i}}\Phi D^{\bar{j_1}}\cdots D^{\bar{j_n}} \\ &+ nc_{\overline{ij_1}\cdots\bar{j_{n-1}}}\partial_{\bar{k}}\Phi D^{\bar{k}}D^{\bar{j_1}}\cdots D^{\bar{j_{n-1}}} + n(n-1)c_{\overline{ij_1}\cdots\bar{j_{n-1}}}D^{\bar{j_1}}\cdots D^{\bar{j_{n-1}}},\end{aligned} \tag{28}$$

where the coefficients $c_{\bar{j_1}\bar{j_2}\cdots\bar{j_n}}$ are totally symmetric under the permutations of the indices.

We construct the operator $L_{\bar{z^l}}$, which is corresponding to the left star product by $\bar{z^l}$. $L_{\bar{z^l}}$ is expanded as a power series of $\hbar$,

$$L_{\bar{z^l}} = \bar{z^l} + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty}\hbar^n A_n, \tag{29}$$

where A_n $(n \geq 2)$ is a formal series of the differential operators $D^{\bar{k}}$. We assume that A_n has the following form,

$$A_n = \sum_{m=2}^{n} a_m^{(n)}\partial_{\bar{j_1}}\Phi\cdots\partial_{\bar{j_{m-1}}}\Phi D^{\bar{j_1}}\cdots D^{\bar{j_{m-1}}}D^{\bar{l}}, \tag{30}$$

where the coefficients $a_m^{(n)}$ do not depend on z^i and $\bar{z}^{\bar{l}}$.

From the condition $\left[L_{\bar{z}^{\bar{l}}}, \partial_{\bar{i}}\Phi + \bar{h}\partial_{\bar{i}}\right] = 0$, the operators A_n are required to satisfy the following recursion relations,

$$[A_n, \partial_{\bar{i}}\Phi] = [\partial_{\bar{i}}, A_{n-1}], \qquad (n \geq 2) \tag{31}$$

where $A_1 = D^{\bar{l}}$. $A_2 = \partial_{\bar{j}}\Phi D^{\bar{j}} D^{\bar{l}}$ is easily obtained from the above equation. Using the expression (30), the left hand side of the recursion relation (31) becomes

$$\begin{aligned}
[A_n, \partial_{\bar{i}}\Phi] &= \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi \left[D^{\bar{j}_1} \cdots D^{\bar{j}_m} D^{\bar{l}}, \partial_{\bar{i}}\Phi\right] \\
&= \sum_{m=2}^{n-1} a_{m+1}^{(n)} \left\{ m\partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi \partial_{\bar{i}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} + \delta_{il}\partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_m}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_m} \right\} \\
&\quad + a_2^{(n)} \left(\partial_{\bar{i}}\Phi D^{\bar{l}} + \delta_{il}\partial_{\bar{j}}\Phi D^{\bar{j}} \right).
\end{aligned}$$

On the other hand, the right hand side of (31) is calculated as

$$\begin{aligned}
[\partial_{\bar{i}}, A_{n-1}] &= \sum_{m=2}^{n-1} a_m^{(n-1)} \left[\partial_{\bar{i}},\ \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_m} D^{\bar{l}} \right] \\
&= \sum_{m=2}^{n-1} \left(a_m^{(n-1)} + m a_{m+1}^{(n-1)} \right) \\
&\quad \times \left(m\partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi \partial_{\bar{i}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} + \delta_{il}\partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_m}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_m} \right) \\
&\quad + a_2^{(n-1)} \left(\partial_{\bar{i}}\Phi D^{\bar{l}} + \delta_{il}\partial_{\bar{j}} D^{\bar{j}} \right).
\end{aligned}$$

Thus, from (31) we find

$$a_2^{(n)} = a_2^{(n-1)} = \cdots = a_2^{(2)} = 1, \tag{32}$$

and the following recursion relation

$$a_m^{(n)} = a_{m-1}^{(n-1)} + (m-1) a_m^{(n-1)}. \tag{33}$$

To solve this equation, we introduce a generating function

$$\alpha_m(t) \equiv \sum_{n=m}^{\infty} t^n a_m^{(n)}, \tag{34}$$

for $m \geq 2$. Then the relation (33) is written as

$$\alpha_m(t) = t\left[\alpha_{m-1}(t) + (m-1)\alpha_m(t)\right], \tag{35}$$

and $\alpha_m(t)$ is solved as

$$\begin{aligned}
\alpha_m(t) &= \frac{t}{1-(m-1)t}\alpha_{m-1}(t) \\
&= t^{m-2} \prod_{n=2}^{m-1} \frac{1}{1-nt} \times \alpha_2(t).
\end{aligned} \tag{36}$$

Since $\alpha_2(t)$ is easily calculated from (32) as

$$\alpha_2(t) = \sum_{n=2}^{\infty} t^n a_2^{(n)} = \sum_{n=2}^{\infty} t^n = \frac{t^2}{1-t}, \tag{37}$$

$\alpha_m(t)$ is finally determined as

$$\alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1-nt} = \frac{\Gamma(1-m+\frac{1}{t})}{\Gamma(1+\frac{1}{t})}, \qquad (m \geq 2). \tag{38}$$

We mention that the function $\alpha_m(t)$ coincides with the generating function for the Stirling numbers of the second kind $S(n,k)$, and $a_m^{(n)}$ is related to $S(n,k)$ as

$$a_m^{(n)} = S(n-1, m-1). \tag{39}$$

Summarizing the above calculations, the explicit formula of $L_{\bar{z}^l}$ is derived as

$$\begin{aligned} L_{\bar{z}^l} &= \bar{z}^l + \bar{h}D^{\bar{l}} + \sum_{n=2}^{\infty} \bar{h}^n \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} \\ &= \bar{z}^l + \bar{h}D^{\bar{l}} + \sum_{m=2}^{\infty} \left(\sum_{n=m}^{\infty} \bar{h}^n a_m^{(n)} \right) \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} \\ &= \bar{z}^l + \sum_{m=1}^{\infty} \alpha_m(\bar{h}) \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}. \end{aligned} \tag{40}$$

Here we defined $\alpha_1(t) = t$. Similarly, it can be shown that the right star product by z^l, $R_{z^l} f = f * z^l$ is expressed as

$$\begin{aligned} R_{z^l} &= z^l + \bar{h}D^l + \sum_{n=2}^{\infty} \bar{h}^n \sum_{m=2}^{n} a_m^{(n)} \partial_{j_1}\Phi \cdots \partial_{j_{m-1}}\Phi D^{j_1} \cdots D^{j_{m-1}} D^l \\ &= z^l + \sum_{m=1}^{\infty} \alpha_m(\bar{h}) \partial_{j_1}\Phi \cdots \partial_{j_{m-1}}\Phi D^{j_1} \cdots D^{j_{m-1}} D^l, \end{aligned} \tag{41}$$

where $D^i = g^{i\bar{j}}\partial_{\bar{j}}$.

From the theorem 2.1, proposition 2.2, (40) and (41), we obtain the following theorem.

Theorem 3.1. *A star product with separation of variables for $\mathbb{C}P^N$ with the Kähler potential $\Phi = \ln\left(1+|z|^2\right)$ is given by*

$$f * g = L_f g = R_g f. \tag{42}$$

Here differential operators L_f and R_g are determined by the differential operators $L_{\bar{z}}$ and R_z whose expressions are given in (40) and (41) through the relation (18) and (19), respectively.

We can now calculate the star products among z^i and $\bar{z}^i$, for instance,

$$z^i * z^j = z^i z^j, \tag{43}$$

$$z^i * \bar{z}^j = z^i \bar{z}^j, \tag{44}$$

$$\bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \tag{45}$$

$$\begin{aligned} \bar{z}^i * z^j &= \bar{z}^i z^j + \bar{h}\delta_{ij}(1+|z|^2)_2F_1\left(1,1;1-1/\bar{h};-|z|^2\right) \\ &\quad + \frac{\bar{h}}{1-\bar{h}} \bar{z}^i z^j (1+|z|^2)_2F_1\left(1,2;2-1/\bar{h};-|z|^2\right), \end{aligned} \tag{46}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Example 2. Noncommutative $\mathbb{C}H^N$

Since a star product with separation of variables on $\mathbb{C}H^N$ can be also derived by a similar procedure to the case of $\mathbb{C}P^N$, we briefly summarize it here.

The Kähler potential of $\mathbb{C}H^N$ is given by

$$\Phi = -\ln\left(1-|z|^2\right). \tag{47}$$

The metric $g_{i\bar{j}}$ and the inverse metric $g^{\bar{i}j}$ are defined as

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}\Phi = \frac{(1-|z|^2)\delta_{ij}+\bar{z}^i z^j}{(1-|z|^2)^2}, \tag{48}$$

$$g^{\bar{i}j} = (1-|z|^2)\left(\delta_{ij}-\bar{z}^i z^j\right). \tag{49}$$

Then we find the following relations similar to (26)-(28);

$$\partial_{\bar{i}_1}\partial_{\bar{i}_2}\cdots\partial_{\bar{i}_n}\Phi = (n-1)!\,\partial_{\bar{i}_1}\Phi\partial_{\bar{i}_2}\Phi\cdots\partial_{\bar{i}_n}\Phi, \tag{50}$$

$$\left[\partial_{\bar{i}}, D^{\bar{j}}\right] = -\partial_{\bar{i}}\Phi D^{\bar{j}} - \delta_{ij}\partial_{\bar{k}}\Phi D^{\bar{k}}, \tag{51}$$

$$\begin{aligned}\left[\partial_{\bar{i}},\ c_{\bar{j}_1\bar{j}_2\cdots\bar{j}_n}D^{\bar{j}_1}D^{\bar{j}_2}\cdots D^{\bar{j}_n}\right] &= \partial_{\bar{i}}c_{\bar{j}_1\bar{j}_2\cdots\bar{j}_n}D^{\bar{j}_1}D^{\bar{j}_2}\cdots D^{\bar{j}_n} - nc_{\bar{j}_1\cdots\bar{j}_n}\partial_{\bar{i}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_n}\\ &- nc_{\overline{ij}_1\cdots\bar{j}_{n-1}}\partial_{\bar{k}}\Phi D^{\bar{k}}D^{\bar{j}_1}\cdots D^{\bar{j}_{n-1}} - n(n-1)c_{\overline{ij}_1\cdots\bar{j}_{n-1}}D^{\bar{j}_1}\cdots D^{\bar{j}_{n-1}},\end{aligned} \tag{52}$$

The operator $L_{\bar{z}^l}$ is expanded as a power series of the noncommutative parameter $\hbar$,

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty}\hbar^n B_n. \tag{53}$$

We assume that B_n has the following form,

$$B_n = \sum_{m=2}^{n}(-1)^{n-1}b_m^{(n)}\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}}. \tag{54}$$

The factor $(-1)^{n-1}$ in the front of the coefficient $b_m^{(n)}$ is introduced for convenience.

Requiring $\left[L_{\bar{z}^l}, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}\right] = 0$, the following conditions for the coefficients $b_m^{(n)}$ similar to (32) and (33) are found,

$$\begin{aligned} b_2^{(n)} &= b_2^{(n-1)} = \cdots = b_2^{(2)} = 1,\\ b_m^{(n)} &= b_{m-1}^{(n-1)} + (m-1)b_m^{(n-1)}.\end{aligned} \tag{55}$$

Thus $b_m^{(n)}$ coincides with $a_m^{(n)}$, and we obtain the explicit representation of the star product with separation of variables on $\mathbb{C}H^N$,

$$\begin{aligned} L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty}\hbar^n\sum_{m=2}^{n}(-1)^{n-1}b_m^{(n)}\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}}\\ &= \bar{z}^l + \sum_{m=1}^{\infty}(-1)^{m-1}\beta_m(\hbar)\partial_{\bar{j}_1}\Phi\cdots\partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1}\cdots D^{\bar{j}_{m-1}}D^{\bar{l}},\end{aligned} \tag{56}$$

$$\begin{aligned} R_{z^l} &= z^l + \hbar D^l + \sum_{n=2}^{\infty}\hbar^n\sum_{m=2}^{n}(-1)^{n-1}b_m^{(n)}\partial_{j_1}\Phi\cdots\partial_{j_{m-1}}\Phi D^{j_1}\cdots D^{j_{m-1}}D^{l}\\ &= z^l + \sum_{m=1}^{\infty}(-1)^{m-1}\beta_m(\hbar)\partial_{j_1}\Phi\cdots\partial_{j_{m-1}}\Phi D^{j_1}\cdots D^{j_{m-1}}D^{l},\end{aligned} \tag{57}$$

with

$$\beta_n(t) = (-1)^n\alpha_n(-t) = \frac{\Gamma(1/t)}{\Gamma(n+1/t)}. \tag{58}$$

From the theorem 2.1 and proposition 2.2, we obtain the following theorem.

Theorem 3.2. *A star product with separation of variables for $\mathbb{C}H^N$ with the Kähler potential $\Phi = -\ln\left(1-|z|^2\right)$ is given by*

$$f * g = L_f g = R_g f. \tag{59}$$

Here differential operators L_f and R_g are determined through the relation (18) and (19), respectively, by the differential operators $L_{\bar{z}}$ and R_z whose expressions are given in (56) and (57).

Using the representations of the star product for $\mathbb{C}H^N$, we can calculate the star products among z^i and $\bar{z}^i$,

$$z^i * z^j = z^i z^j, \tag{60}$$

$$z^i * \bar{z}^j = z^i \bar{z}^j, \tag{61}$$

$$\bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \tag{62}$$

$$\begin{aligned} \bar{z}^i * z^j =& \bar{z}^i z^j + \hbar\delta_{ij}(1-|z|^2)_2F_1\left(1,1;1+1/\hbar;|z|^2\right) \\ &- \frac{\hbar}{1+\hbar}\bar{z}^i z^j (1-|z|^2)_2F_1\left(1,2;2+1/\hbar;|z|^2\right). \end{aligned} \tag{63}$$

Remarks

Before closing this section, we make some remarks.

The above star products of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ can be also represented in some other expressions. We show that L_f on these manifolds has the following form,

$$L_f g = f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1\bar{k}_1}\cdots g_{j_n\bar{k}_n}\left(D^{j_1}\cdots D^{j_n} f\right)\left(D^{\bar{k}_1}\cdots D^{\bar{k}_n} g\right). \tag{64}$$

The coefficient $c_n(\hbar)$ is determined by the condition $[L_f, \hbar\partial_{\bar{i}} + \partial_{\bar{i}}\Phi] = 0$. For the case of $\mathbb{C}P^N$, this condition becomes

$$\begin{aligned} [L_f, \hbar\partial_{\bar{i}} + \partial_{\bar{i}}\Phi] =& \sum_{n=1}^{\infty}\left[n(1-\hbar(n-1))c_n(\hbar) - \hbar c_{n-1}(\hbar)\right] \\ &\times g_{l\bar{i}} g_{j_1\bar{k}_1}\cdots g_{j_{n-1}\bar{k}_{n-1}}\left(D^l D^{j_1}\cdots D^{j_{n-1}} f\right) D^{\bar{k}_1}\cdots D^{\bar{k}_{n-1}} = 0. \end{aligned} \tag{65}$$

By solving the recursion relation, $n(1-\hbar(n-1))c_n(\hbar) - \hbar c_{n-1}(\hbar) = 0$, under the initial condition $c_0 = 1$, $c_n(\hbar)$ is obtained as

$$c_n(\hbar) = \frac{\Gamma(1-n+1/\hbar)}{n!\Gamma(1+1/\hbar)} = \frac{\alpha_n(\hbar)}{n!}, \tag{66}$$

where $\alpha_n(\hbar)$ is given in (38). Similarly, the operator L_f on $\mathbb{C}H^N$ can be represented in the form of (64) with $c_n(\hbar) = \beta_n(\hbar)/n!$ where $\beta_n(\hbar)$ is defined in (58).

The expression of L_f (64) can be recast in another form by the use of the covariant derivatives on the manifolds. Non-vanishing components of the Christoffel symbols on a Kähler manifolds are only Γ^i_{jk} and $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$. Therefore, for scalars f and g

$$\begin{aligned} g^{j_1\bar{k}_1}\cdots g^{j_n\bar{k}_n}\nabla_{\bar{k}_1}\cdots\nabla_{\bar{k}_n} f &= g^{j_1\bar{k}_1}\nabla_{\bar{k}_1}\left(g^{j_2\bar{k}_2}\cdots g^{j_n\bar{k}_n}\nabla_{\bar{k}_2}\cdots\nabla_{\bar{k}_n} f\right) \\ &= g^{j_1\bar{k}_1}\partial_{\bar{k}_1}\left(g^{j_2\bar{k}_2}\cdots g^{j_n\bar{k}_n}\nabla_{\bar{k}_2}\cdots\nabla_{\bar{k}_n} f\right) \\ &= D^{j_1}\left(g^{j_2\bar{k}_2}\cdots g^{j_n\bar{k}_n}\nabla_{\bar{k}_2}\cdots\nabla_{\bar{k}_n} f\right) \\ &= D^{j_1}\cdots D^{j_n} f, \end{aligned} \tag{67}$$

$$g^{\bar{j}_1 k_1}\cdots g^{\bar{j}_n k_n}\nabla_{k_1}\cdots\nabla_{k_n} g = D^{\bar{j}_1}\cdots D^{\bar{j}_n} g. \tag{68}$$

Using these relations, $L_f g$ for $\mathbb{C}P^N$ and $\mathbb{C}H^N$, eq. (64), is represented as

$$L_f g = f * g = \sum_{n=0}^{\infty} c_n(\hbar) g^{\bar{j}_1 k_1} \cdots g^{\bar{j}_n k_n} \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f\right) \left(\nabla_{k_1} \cdots \nabla_{k_n} g\right). \tag{69}$$

4. The Fock representation of noncommutative Kähler manifolds

In this section, we study the Fock space on an open subset U in a general Kähler manifold M which is diffeomorphic to a connected open subset of $\mathbb{C}^N$ and an algebra given by a set of linear operators acting on the Fock space. This section is based on the work in [36].

As mentioned in Section 2, from the (20) and (21) $\{z^i, \partial_j \Phi \mid i,j = 1,2,\cdots,N\}$ and $\{\bar{z}^i, \partial_{\bar{j}} \Phi \mid i,j = 1,2,\cdots,N\}$ are candidates for the creation and annihilation operators under the star product $*$. We introduce $a_i^\dagger, a_i, \underline{a}_i^\dagger$ and $\underline{a}_i$ $(i = 1,2,\ldots,N)$ by

$$a_i^\dagger = z^i, \quad \underline{a}_i = \frac{1}{\hbar}\partial_i \Phi, \quad a_i = \bar{z}^i, \quad \underline{a}_i^\dagger = \frac{1}{\hbar}\partial_{\bar{i}} \Phi. \tag{70}$$

Then they satisfy the following commutation relations which are similar to the usual commutation relations for the creation and annihilation operators but slightly different,

$$[\underline{a}_i,\, a_j^\dagger]_* = \delta_{ij}, \qquad [a_i^\dagger,\, a_j^\dagger]_* = 0, \qquad [\underline{a}_i,\, \underline{a}_j]_* = 0, \tag{71}$$

$$[a_i,\, \underline{a}_j^\dagger]_* = \delta_{ij}, \qquad [\underline{a}_i^\dagger,\, \underline{a}_j^\dagger]_* = 0, \qquad [a_i,\, a_j]_* = 0. \tag{72}$$

These are different from the ordinary creation and annihilation operators, because these two sets of creation and annihilation operators are not given as direct sum, in other words,

$$[a_i, a_i^\dagger]_* \text{ and } [\underline{a}_i,\, \underline{a}_j^\dagger]_* \tag{73}$$

do not vanish in general.

The star product with separation of variables has the following property under the complex conjugation.

Proposition 4.1.

$$\overline{f * g} = \overline{L_f g} = \bar{g} * \bar{f} \tag{74}$$

Proof. L_f is written by using $L_{\bar{z}}$ as (18). $L_{\bar{z}}$ is determined by (13) and (14). Similarly, R_z is determined by (16) and (17). Recall that these conditions uniquely determine $L_{\bar{z}}$ and R_z, respectively. This means that $L_{\bar{z}}$ is complex conjugate with R_z, $\overline{L_{\bar{z}}} = R_z$, and leads to

$$\begin{aligned} \overline{L_f g} &= \sum_\alpha \frac{1}{\alpha!} \left(\frac{\partial}{\partial z}\right)^\alpha \bar{f}\, \overline{(L_{\bar{z}} - \bar{z})^\alpha}\, \bar{g} \\ &= \sum_\alpha \frac{1}{\alpha!} \left(\frac{\partial}{\partial z}\right)^\alpha \bar{f} (R_z - z)^\alpha \bar{g} \\ &= R_{\bar{f}} \bar{g} = \bar{g} * \bar{f}. \end{aligned} \tag{75}$$

□

The Fock space is defined by a vector space spanned by the bases which is generated by acting $a_i^\dagger$ on $|\vec{0}\rangle$,

$$\begin{aligned} |\vec{n}\rangle &= |n_1, \cdots, n_N\rangle \\ &= c_1(\vec{n})(a_1^\dagger)_*^{n_1} * \cdots * (a_N^\dagger)_*^{n_N} * |\vec{0}\rangle, \end{aligned} \tag{76}$$

where $|\vec{0}\rangle = |0, \cdots, 0\rangle$ satisfies

$$\underline{a}_i * |\vec{0}\rangle = 0, \qquad (i = 1, \cdots, N), \tag{77}$$

and $(A)_*^n$ stands for $\overbrace{A * \cdots * A}^{n}$. $c_1(\vec{n})$ is a normalization coefficient which does not depend on z^i and $\bar{z}^{\bar{i}}$. Here, we define the basis of a dual vector space by acting $\underline{a}_i$ on $\langle\vec{0}|$,

$$\begin{aligned} \underline{\langle\vec{m}|} &= \underline{\langle m_1, \cdots, m_N|} \\ &= \langle\vec{0}| * (\underline{a}_1)_*^{m_1} * \cdots * (\underline{a}_N)_*^{m_N} c_2(\vec{m}), \end{aligned} \tag{78}$$

and

$$\langle\vec{0}| * a_i^\dagger = 0, \qquad (i = 1, \cdots, N), \tag{79}$$

where $c_2(\vec{m})$ is also a normalization constant. The underlines are attached to the bra vectors in order to emphasize that $\underline{\langle\vec{m}|}$ is not Hermitian conjugate to $|\vec{m}\rangle$. In this article, we set the normalization constants as

$$c_1(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \quad c_2(\vec{n}) = \frac{1}{\sqrt{\vec{n}!}}, \tag{80}$$

where $\vec{n}! = n_1! n_2! \cdots n_N!$.

Definition 3. The local twisted Fock algebra (representation) F_U is defined as an algebra given by a set of linear operators acting on the Fock space defined on U:

$$F_U := \{\sum_{\vec{n},\vec{m}} A_{\vec{n}\vec{m}} |\vec{n}\rangle \underline{\langle\vec{m}|} \mid A_{\vec{n}\vec{m}} \in \mathbb{C}\}. \tag{81}$$

and multiplication between its elements is given by the star product $*$.

In the remaining part of this section, we construct concrete expressions of functions which are elements of the local twisted Fock algebra.

Lemma 4.2 (Berezin)**.** *For arbitrary Kähler manifolds (M, ω), there exists a Kähler potential $\Phi(z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N)$ such that*

$$\Phi(0, \ldots, 0, \bar{z}^1, \ldots, \bar{z}^N) = 0, \ \Phi(z^1, \ldots, z^N, 0, \ldots, 0) = 0. \tag{82}$$

This is easily shown as follow. If a Kähler potential Φ satisfying $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi$ does not satisfy (82), then we redefine a new Kähler potential Φ' as

$$\begin{aligned} &\Phi'(z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N) \\ &:= \Phi(z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N) - \Phi(0, \ldots, 0, \bar{z}^1, \ldots, \bar{z}^N) - \Phi(z^1, \ldots, z^N, 0, \ldots, 0). \end{aligned} \tag{83}$$

$\Phi(z^1, \ldots, z^N, 0, \ldots, 0)$ is a holomorphic function and $\Phi(0, \ldots, 0, \bar{z}^1, \ldots, \bar{z}^N)$ is an anti-holomorphic function. Kähler potentials have ambiguities of adding holomorphic and anti-holomorphic functions. This Φ' satisfies the condition (82). In the following, we abbreviate $\Phi(z^1, \ldots, z^N, \bar{z}^1, \ldots, \bar{z}^N)$ to $\Phi(z, \bar{z})$ for convenience.

In [33], it is shown that $e^{-\Phi/\hbar}$ corresponds to a vacuum projection operator $|\vec{0}\rangle\langle\vec{0}|$ for the noncommutative $\mathbb{C}P^N$. We extend this statement for general Kähler manifolds.

Proposition 4.3. *Let (M,ω) be a Kähler manifold, Φ be its Kähler potential with the property (82), and $*$ be a star product with separation of variables given in the previous section. Then the function*

$$|\vec{0}\rangle\langle\vec{0}| := e^{-\Phi/\hbar}, \tag{84}$$

satisfies

$$\underline{a}_i * |\vec{0}\rangle\langle\vec{0}| = 0, \qquad |\vec{0}\rangle\langle\vec{0}| * a_i^\dagger = 0, \tag{85}$$

$$\Big(|\vec{0}\rangle\langle\vec{0}|\Big) * \Big(|\vec{0}\rangle\langle\vec{0}|\Big) = e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar} = |\vec{0}\rangle\langle\vec{0}|. \tag{86}$$

Overview of proof (A more detailed proof is given in [36].)
We define the following normal ordered quantity,

$$:e^{-\Sigma_i a_i^\dagger \underline{a}_i}: := \prod_{i=1*}^{N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_i^\dagger)_*^n * (\underline{a}_i)_*^n. \tag{87}$$

It is easy to show that $:e^{-\Sigma_i a_i^\dagger \underline{a}_i} := |\vec{0}\rangle\langle\vec{0}|$ in the same way as in the case of the ordinary harmonic oscillator, Therefore, all we have to do is to show

$$:e^{-\Sigma_i a_i^\dagger \underline{a}_i} := e^{-\Phi/\hbar}. \tag{88}$$

This can be done as follows:

$$\begin{aligned} :e^{-\Sigma_i a_i^\dagger \underline{a}_i}: &= \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}!} (a^\dagger)_*^{\vec{n}} * (\underline{a})_*^{\vec{n}} \\ &= \sum_{\vec{n}} \frac{(-1)^{|n|}}{\vec{n}!\hbar^{|n|}} (z)_*^{\vec{n}} * (\partial\Phi)_*^{\vec{n}}. \end{aligned} \tag{89}$$

In this paper, we use the following notation: for an N-tuple A_i $(i = 1,2,\cdots,N)$ and an N-vector $\vec{n} = (n_1, n_2, \cdots, n_N)$,

$$(A)_*^{\vec{n}} = (A_1)_*^{n_1} * (A_2)_*^{n_2} * \cdots * (A_N)_*^{n_N}, \tag{90}$$

$$\vec{n}! = n_1! n_2! \cdots n_N!, \qquad |n| = \sum_{i=1}^{N} n_i. \tag{91}$$

By using $(z)_*^{\vec{n}} = (z)^{\vec{n}} = (z^1)^{n_1} \cdots (z^N)^{n_N}$, the above is recast as

$$\begin{aligned} &\sum_{n_1,n_2,\ldots,n_N=0}^{\infty} \frac{1}{n_1! n_2! \cdots n_N!} (-z^1)^{n_1} \cdots (-z^N)^{n_N} e^{-\frac{\Phi(z,\bar{z})}{\hbar}} \partial_1^{n_1} \cdots \partial_N^{n_N} e^{\frac{\Phi(z,\bar{z})}{\hbar}} \\ &= e^{-\frac{\Phi(z,\bar{z})}{\hbar}} e^{\frac{\Phi(0,\bar{z})}{\hbar}} \\ &= e^{-\frac{\Phi(z,\bar{z})}{\hbar}}. \end{aligned} \tag{92}$$

Here, the final equality follows from the condition (82). □

From a similar calculation to the above proof, we can also show the following relations with respect to a_i and $\underline{a}_i^\dagger$,

$$|\vec{0}\rangle\langle\vec{0}| = e^{-\Phi/\hbar} =: e^{-\Sigma_i \underline{a}_i^\dagger a_i} := \prod_{i=1*}^{N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\underline{a}_i^\dagger)_*^n * (a_i)_*^n, \tag{93}$$

$$a_i * |\vec{0}\rangle\langle\vec{0}| = 0, \qquad |\vec{0}\rangle\langle\vec{0}| * \underline{a}_i^\dagger = 0. \tag{94}$$

Lemma 4.4 (Sako, Suzuki, Umetsu [33]). *$e^{-\Phi/\hbar} = |0\rangle\langle 0|$ satisfies the relation*

$$|0\rangle\langle 0| * f(z,\overline{z}) = e^{-\Phi/\hbar} * f(z,\overline{z}) = e^{-\Phi/\hbar} f(0,\overline{z}) = |0\rangle\langle 0| f(0,\overline{z}), \tag{95}$$

$$f(z,\overline{z}) * |0\rangle\langle 0| = f(z,\overline{z}) * e^{-\Phi/\hbar} = f(z,0) e^{-\Phi/\hbar} = f(z,0)|0\rangle\langle 0|. \tag{96}$$

for a function $f(z,\overline{z})$ such that $f(z,\overline{w})$ can be expanded as Taylor series with respect to z^i and $\overline{w}^j$, respectively.

A proof for this lemma is given in [33].

We expand a function $\exp\Phi(z,\overline{z})/\hbar$ as a power series,

$$e^{\Phi(z,\overline{z})/\hbar} = \sum_{\vec{m},\vec{n}} H_{\vec{m},\vec{n}} (z)^{\vec{m}} (\overline{z})^{\vec{n}}, \tag{97}$$

where $(z)^{\vec{n}} = (z^1)^{n_1} \cdots (z^N)^{n_N}$ and $(\overline{z})^{\vec{n}} = (\overline{z^1})^{n_1} \cdots (\overline{z^N})^{n_N}$. Since $\exp\Phi/\hbar$ is real and satisfies (82), the expansion coefficients $H_{\vec{m},\vec{n}}$ obey

$$\overline{H}_{\vec{m},\vec{n}} = H_{\vec{n},\vec{m}}, \tag{98}$$

$$H_{\vec{0},\vec{n}} = H_{\vec{n},\vec{0}} = \delta_{\vec{n},\vec{0}}. \tag{99}$$

Using this expansion, the following relations are obtained.

Proposition 4.5. *The right $*$-multiplication of $(\underline{a})^{\vec{n}}_* = (\partial\Phi/\hbar)^{\vec{n}}_*$ on $|\vec{0}\rangle\langle\vec{0}|$ is related to the right $*$-multiplication of $(a)^{\vec{n}}_* = (\overline{z})^{\vec{n}}_*$ on $|\vec{0}\rangle\langle\vec{0}|$ as follows,*

$$\begin{aligned} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{n}}_* &= |\vec{0}\rangle\langle\vec{0}| * \left(\frac{1}{\hbar}\partial\Phi\right)^{\vec{n}}_* \\ &= \vec{n}! \sum_{\vec{m}} H_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (\overline{z})^{\vec{m}}_* = \vec{n}! \sum_{\vec{m}} H_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (a)^{\vec{m}}_*. \end{aligned} \tag{100}$$

Similarly, the following relation holds,

$$\begin{aligned} (\underline{a}^\dagger)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| &= \left(\frac{1}{\hbar}\overline{\partial}\Phi\right)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| \\ &= \vec{n}! \sum_{\vec{m}} H_{\vec{m},\vec{n}} (z)^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}| = \vec{n}! \sum_{\vec{m}} H_{\vec{m},\vec{n}} (a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\langle\vec{0}|. \end{aligned} \tag{101}$$

The proof of this proposition is given in [36]

From this proposition, if there exists the inverse matrix $H^{-1}_{\vec{m},\vec{n}}$, then the following relations also holds,

Corollary 4.6.

$$|\vec{0}\rangle\langle\vec{0}| * (a)^{\vec{n}}_* = \sum_{\vec{m}} \frac{1}{\vec{m}!} H^{-1}_{\vec{n},\vec{m}} |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{m}}_*, \tag{102}$$

$$(a^\dagger)^{\vec{n}}_* * |\vec{0}\rangle\langle\vec{0}| = \sum_{\vec{m}} \frac{1}{\vec{m}!} H^{-1}_{\vec{m},\vec{n}} (\underline{a}^\dagger)^{\vec{m}} * |\vec{0}\rangle\langle\vec{0}|, \tag{103}$$

where $H^{-1}_{\vec{n},\vec{m}}$ is the inverse matrix of the matrix $H_{\vec{n},\vec{m}}$, $\sum_{\vec{k}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k},\vec{n}} = \delta_{\vec{m},\vec{n}}$.

We introduce bases of the Fock representation as follows,

$$|\vec{m}\rangle\underline{\langle\vec{n}|} := \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{n}}_* = \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (z)^{\vec{m}}_* * e^{-\Phi/\hbar} * \left(\frac{1}{\hbar}\partial\Phi\right)^{\vec{n}}_*. \tag{104}$$

By using (100), the bases are also written as

$$|\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H_{\vec{n},\vec{k}}(z)^{\vec{m}}_{*} * e^{-\Phi/\hbar} * (\overline{z})^{\vec{k}}_{*}$$

$$= \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H_{\vec{n},\vec{k}}(z)^{\vec{m}}(\overline{z})^{\vec{k}} e^{-\Phi/\hbar}. \tag{105}$$

The completeness of the bases are formally shown as

$$\sum_{\vec{n}} |\vec{n}\rangle\underline{\langle\vec{n}|} = \sum_{\vec{m},\vec{n}} H_{\vec{n},\vec{m}}(z)^{\vec{n}}(\overline{z})^{\vec{m}} e^{-\Phi/\hbar}$$

$$= e^{\Phi/\hbar} e^{-\Phi/\hbar}$$

$$= 1. \tag{106}$$

The bases are orthogonal to each other under the $*$-products,

$$|\vec{m}\rangle\underline{\langle\vec{n}|} * |\vec{k}\rangle\underline{\langle\vec{l}|} = \frac{1}{\sqrt{\vec{m}!\vec{n}!\vec{k}!\vec{l}!}}(a^{\dagger})^{\vec{m}}_{*} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{n}}_{*} * (a^{\dagger})^{\vec{k}}_{*} * |\vec{0}\rangle\langle\vec{0}| * (\underline{a})^{\vec{l}}_{*}$$

$$= \delta_{\vec{n},\vec{k}} |\vec{m}\rangle\underline{\langle\vec{l}|}. \tag{107}$$

The behavior of the bases under the complex conjugation is different from usual, because the bra vectors are not Hermitian conjugate with the ket vectors in general, as noticed above,

$$\overline{|\vec{m}\rangle\underline{\langle\vec{n}|}} = \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k}} H_{\vec{k},\vec{n}}(z)^{\vec{k}}(\overline{z})^{\vec{m}} e^{-\Phi/\hbar}$$

$$= \sqrt{\frac{\vec{n}!}{\vec{m}!}}\sum_{\vec{k},\vec{l}} \sqrt{\frac{\vec{k}!}{\vec{l}!}} H_{\vec{k},\vec{n}} H^{-1}_{\vec{m},\vec{l}}\, |\vec{k}\rangle\underline{\langle\vec{l}|}. \tag{108}$$

The creation and annihilation operators $a^{\dagger}_{i}, \underline{a}_{i}$ act on the bases as follows,

$$a^{\dagger}_{i} * |\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{m_i + 1}\,|\vec{m} + \vec{e}_i\rangle\underline{\langle\vec{n}|}, \tag{109}$$

$$\underline{a}_{i} * |\vec{m}\rangle\underline{\langle\vec{n}|} = \sqrt{m_i}\,|\vec{m} - \vec{e}_i\rangle\underline{\langle\vec{n}|}, \tag{110}$$

$$|\vec{m}\rangle\underline{\langle\vec{n}|} * a^{\dagger}_{i} = \sqrt{n_i}\,|\vec{m}\rangle\underline{\langle\vec{n} - \vec{e}_i|}, \tag{111}$$

$$|\vec{m}\rangle\underline{\langle\vec{n}|} * \underline{a}_{i} = \sqrt{n_i + 1}\,|\vec{m}\rangle\underline{\langle\vec{n} + \vec{e}_i|}, \tag{112}$$

where $\vec{e}_i$ is a unit vector, $(\vec{e}_i)_j = \delta_{ij}$. The action of a_i and $\underline{a}^{\dagger}_{i}$ is derived by the Hermitian conjugation of the above equations.

The creation and annihilation operators can be expanded with respect to the bases as

$$a^{\dagger}_{i} = \sum_{\vec{n}} \sqrt{n_i + 1}\,|\vec{n} + \vec{e}_i\rangle\underline{\langle\vec{n}|}, \tag{113}$$

$$\underline{a}_{i} = \sum_{\vec{n}} \sqrt{n_i + 1}\,|\vec{n}\rangle\underline{\langle\vec{n} + \vec{e}_i|}, \tag{114}$$

$$a_{i} = \sum_{\vec{m},\vec{n},\vec{k}} \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H^{-1}_{\vec{k}+\vec{e}_i,\vec{n}} |\vec{m}\rangle\underline{\langle\vec{n}|}, \tag{115}$$

$$\underline{a}^{\dagger}_{i} = \sqrt{\frac{\vec{m}!}{\vec{n}!}}(k_i + 1) H_{\vec{m},\vec{k}+\vec{e}_i} H^{-1}_{\vec{k},\vec{n}} |\vec{m}\rangle\underline{\langle\vec{n}|}. \tag{116}$$

5. EXAMPLES

In this section, some examples of the Fock representations are given.

Example 1 : Fock representation of noncommutative $\mathbb{C}P^N$
We give an explicit expression of the twisted Fock representation of noncommutative of $\mathbb{C}P^N$. In this case, the twisted Fock representation on an open set is essentially the same as the representation given in [33, 34, 24, 35]. (In a context of a Fuzzy $\mathbb{C}P^N$, which is a different approach to noncommutative $\mathbb{C}P^N$, the Fock representations are discussed in [1, 2, 11].)

Let denote ζ^a $(a = 0, 1, \ldots, N)$ homogeneous coordinates and $\bigcup U_a$ $(U_a = \{[\zeta^0 : \zeta^1 : \cdots : \zeta^N]\} | \zeta^a \neq 0)$ an open covering of $\mathbb{C}P^N$. We define inhomogeneous coordinates on U_a as

$$z_a^0 = \frac{\zeta^0}{\zeta^a}, \cdots, z_a^{a-1} = \frac{\zeta^{a-1}}{\zeta^a}, z_a^{a+1} = \frac{\zeta^{a+1}}{\zeta^a}, \cdots, z_a^N = \frac{\zeta^N}{\zeta^a}. \tag{117}$$

We choose a Kähler potential on U_a which satisfies the condition (82)

$$\Phi_a = \ln(1 + |z_a|^2), \tag{118}$$

where $|z_a|^2 = \sum_i |z_a^i|^2$. As we saw in Section 3 (64), a star product on U_a is given as follows

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1 \bar{k}_1} \cdots g_{j_n \bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f\right) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g, \tag{119}$$

where

$$c_n(\hbar) = \frac{\Gamma(1 - n + 1/\hbar)}{n!\Gamma(1 + 1/\hbar)}, \quad D^{\bar{i}} = g^{\bar{i}j}\partial_j, \quad D^i = g^{i\bar{j}}\partial_{\bar{j}}.$$

On U_a, creation and annihilation operators are given as

$$a_{a,i}^\dagger = z_a^i, \quad \underline{a_{a,i}} = \frac{1}{\hbar}\partial_i \Phi_a = \frac{\bar{z}_a^i}{\hbar(1 + |z_a|^2)}, \quad a_{a,i} = \bar{z}_a^i, \quad \underline{a_{a,i}}^\dagger = \frac{1}{\hbar}\partial_{\bar{i}} \Phi_a = \frac{z_a^i}{\hbar(1 + |z_a|^2)}, \tag{120}$$

and a vacuum is

$$|\vec{0}\rangle_{a\underline{a}}\langle\vec{0}| = e^{-\Phi_a/\hbar} = (1 + |z_a|^2)^{-1/\hbar}. \tag{121}$$

Bases of the Fock representation on U_a are constructed as

$$\begin{aligned} |\vec{m}\rangle_{a\underline{a}}\langle\vec{n}| &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}} (a_a^\dagger)_*^{\vec{m}} * |\vec{0}\rangle_{a\underline{a}}\langle\vec{0}| * (\underline{a_a})_*^{\vec{n}} \\ &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\hbar^{|n|}}} (z_a)_*^{\vec{m}} * e^{-\Phi_a/\hbar} * (\partial\Phi_a)_*^{\vec{n}}. \end{aligned} \tag{122}$$

By using (95), (96) and the following relation which is shown in [33],

$$\begin{aligned} (\partial\Phi_a)_*^{\vec{n}} &= \frac{\hbar^{|n|}\Gamma(1/\hbar + 1)}{\Gamma(1/\hbar - |n| + 1)} (\partial\Phi_a)^{\vec{n}} \\ &= \frac{\hbar^{|n|}\Gamma(1/\hbar + 1)}{\Gamma(1/\hbar - |n| + 1)} \left(\frac{\bar{z}_a}{1 + |z_a|^2}\right)^{\vec{n}}, \end{aligned} \tag{123}$$

the bases can be explicitly written as

$$|\vec{m}\rangle_{a\underline{a}}\langle\vec{n}| = \frac{\Gamma(1/\hbar + 1)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar - |n| + 1)} (z_a)^{\vec{m}} (\bar{z}_a)^{\vec{n}} e^{-\Phi/\hbar}. \tag{124}$$

By comparing this equation and (105), $H_{\vec{m},\vec{n}}$ is obtained as

$$H_{\vec{m},\vec{n}} = \delta_{\vec{m},\vec{n}} \frac{\Gamma(1/\hbar + 1)}{\vec{m}!\Gamma(1/\hbar - |m| + 1)}, \tag{125}$$

and it is easily seen that this formally satisfies $e^{\Phi_a/\hbar} = \sum H_{\vec{m},\vec{n}}(z_a)^{\vec{m}}(\overline{z})^{\vec{n}}$.

Let us consider transformations between the Fock representations on U_a and U_b ($a < b$). Discussions of such transformations for general Kähler manifolds are given in [36]. The transformations for the coordinates and the Kähler potential on $U_a \bigcap U_b$ are

$$z_a^i = \frac{z_b^i}{z_b^a}, \ (i = 0,1,\ldots,a-1,a+1,\ldots,b-1,b+1,\ldots,N), \qquad z_a^b = \frac{1}{z_b^a}, \tag{126}$$

$$\Phi_a = \Phi_b - \ln z_b^a - \ln \overline{z}_b^a. \tag{127}$$

Thus, $|\vec{m}\rangle_{aa}\underline{\langle\vec{n}|}$ is written on $U_a \bigcap U_b$ as

$$\begin{aligned} |\vec{m}\rangle_{aa}\underline{\langle\vec{n}|} = & \frac{\Gamma(1/\hbar+1)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar-|n|+1)} e^{-\Phi_b/\hbar} \\ & \times (z_b^0)^{m_0}\cdots(z_b^{a-1})^{m_{a-1}}(z_b^a)^{1/\hbar-|m|}(z_b^{a+1})^{m_{a+1}}\cdots(z_b^{b-1})^{m_{b-1}}(z_b^{b+1})^{m_{b+1}}\cdots(z_b^N)^{m_N} \\ & \times (\overline{z}_b^0)^{n_0}\cdots(\overline{z}_b^{a-1})^{n_{a-1}}(\overline{z}_b^a)^{1/\hbar-|n|}(\overline{z}_b^{a+1})^{n_{a+1}}\cdots(\overline{z}_b^{b-1})^{n_{b-1}}(\overline{z}_b^{b+1})^{n_{b+1}}\cdots(\overline{z}_b^N)^{n_N}, \end{aligned} \tag{128}$$

where

$$\vec{m} = (m_0,\ldots,m_{a-1},m_{a+1},\ldots,m_N), \tag{129}$$

$$\vec{n} = (n_0,\ldots,n_{a-1},n_{a+1},\ldots,n_N). \tag{130}$$

We should treat $(z_b^a)^{1/\hbar-|m|}$ and $(\overline{z}_b^a)^{1/\hbar-|n|}$ carefully, because if they are not monomials some trick is needed to express them as the twisted Fock representation. The trick is given in [36], but another way is described here.

To avoid such kind of problems concerning $(z_b^a)^{1/\hbar-|m|}$ and $(\overline{z}_b^a)^{1/\hbar-|n|}$, we can introduce a slightly different representation from the above twisted Fock representation of $\mathbb{C}P^N$. Let us consider the case that the noncommutative parameter is the following value,

$$1/\hbar = L \in \mathbb{Z}, \ L \geq 0, \tag{131}$$

Then, we define F_a^L on U_a as a subspace of a local twisted Fock algebra F_{U_a},

$$F_a^L = \{\sum_{\vec{m},\vec{n}} A_{\vec{m}\vec{n}}|\vec{m}\rangle_{aa}\underline{\langle\vec{n}|} \mid A_{\vec{m}\vec{n}} \in \mathbb{C}, \ |m| \leq L, \ |n| \leq L\}. \tag{132}$$

The bases on U_a are related to those on U_b as,

$$\sqrt{\frac{(L-|n|)!}{(L-|m|)!}}|\vec{m}\rangle_{aa}\underline{\langle\vec{n}|} = \sqrt{\frac{(L-|n'|)!}{(L-|m'|)!}}|\vec{m'}\rangle_{bb}\underline{\langle\vec{n'}|}, \tag{133}$$

where

$$\vec{m'} = (m_0,\cdots,m_{a-1},L-|m|,m_{a+1},\cdots,m_{b-1},m_{b+1},\cdots,m_N), \tag{134}$$

$$\vec{n'} = (n_0,\cdots,n_{a-1},L-|n|,n_{a+1},\cdots,n_{b-1},n_{b+1},\cdots,n_N). \tag{135}$$

Using the expression of (133), we can define $|\vec{m}\rangle_{aa}\underline{\langle\vec{n}|}$ on the whole of U_b. Therefore, the operators in F_a^L can be extended to the whole of $\mathbb{C}P^N$ by using the relation like (133).

Under the condition (131), the creation and annihilation operators on F_{U_a} are changed from the definition (70). Similarly to (113) and (114), let us define a creation operator ${a_{a,i}^L}^\dagger$ and

an annihilation operator $\underline{a}^L_{a,i}$ restricted on F^L_a by

$$a^{L\ \dagger}_{a,i} = \sum_{0\le|n|\le L-1} \sqrt{n_i+1}|\vec{n}+\vec{e}_i\rangle_{aa}\underline{\langle\vec{n}|} = z^i_a\left[1-\left(\frac{|z_a|^2}{1+|z_a|^2}\right)^L\right], \tag{136}$$

$$\underline{a}^L_{a,i} = \sum_{0\le|n|\le L-1} \sqrt{n_i+1}|\vec{n}\rangle_{aa}\underline{\langle\vec{n}+\vec{e}_i|} = L\frac{\bar{z}^i_a}{1+|z_a|^2}. \tag{137}$$

By the restriction on F^L_a, $a^{L\ \dagger}_{a,i}$ is shifted from z^i_a. These operators satisfy the following commutation relation,

$$\begin{aligned}[\underline{a}^L_{a,i},\ a^{L\ \dagger}_{a,j}] &= \delta_{ij}\left(\sum_{0\le|n|\le L}|\vec{n}\rangle_{aa}\underline{\langle\vec{n}|} - \sum_{|n|=L}(n_i+1)|\vec{n}\rangle_{aa}\underline{\langle\vec{n}|}\right)\\ &= \delta_{ij} - \delta_{ij}\left(\frac{|z_a|^2}{1+|z_a|^2}\right)^L\left(1+L\frac{|z^i_a|^2}{|z_a|^2}\right).\end{aligned} \tag{138}$$

Example 2 : Fock representation of noncommutative $\mathbb{C}H^N$
Here, we give an explicit expression of the Fock representation of noncommutative of $\mathbb{C}H^N$ [33, 34].

We choose a Kähler potential satisfies the condition (82)

$$\Phi = -\ln(1-|z|^2), \tag{139}$$

where $|z|^2 = \sum_i^N |z^i|^2$. A star product is given as the same representation (119), but

$$c_n(\hbar) = \frac{\Gamma(1/\hbar)}{n!\Gamma(n+1/\hbar)}, \quad D^{\bar{i}} = g^{\bar{i}j}\partial_j, \quad D^i = g^{i\bar{j}}\partial_{\bar{j}}. \tag{140}$$

The creation and annihilation operators are given as

$$a^\dagger_i = z^i, \quad \underline{a_i} = \frac{1}{\hbar}\partial_i\Phi = \frac{\bar{z}^i}{\hbar(1-|z|^2)}, \quad a_i = \bar{z}^i, \quad \underline{a_i}^\dagger = \frac{1}{\hbar}\partial_{\bar{i}}\Phi = \frac{z^i}{\hbar(1-|z|^2)}. \tag{141}$$

and a vacuum is

$$|\vec{0}\rangle\langle\vec{0}| = e^{-\Phi/\hbar} = (1-|z|^2)^{1/\hbar}. \tag{142}$$

Bases of the Fock representation on $\mathbb{C}H^N$ are constructed as

$$\begin{aligned}|\vec{m}\rangle\underline{\langle\vec{n}|} &= \frac{1}{\sqrt{\vec{m}!\vec{n}!}}(a^\dagger)^{\vec{m}}_* * |\vec{0}\rangle\underline{\langle\vec{0}|} * (\underline{a})^{\vec{n}}_*\\ &= \frac{1}{\sqrt{\vec{m}!\vec{n}!\hbar^{|n|}}}(z)^{\vec{m}}_* * e^{-\Phi/\hbar} * (\partial\Phi)^{\vec{n}}_*.\end{aligned} \tag{143}$$

By using (95), (96) and the following relation which is shown in [33],

$$(\partial\Phi)^{\vec{n}}_* = \frac{(-\hbar)^{|n|}\Gamma(1/\hbar+|n|)}{\Gamma(1/\hbar)}\left(\frac{\bar{z}}{1-|z|^2}\right)^{\vec{n}}, \tag{144}$$

the bases can be explicitly written as

$$|\vec{m}\rangle\underline{\langle\vec{n}|} = \frac{(-1)^{|n|}\Gamma(1/\hbar+|n|)}{\sqrt{\vec{m}!\vec{n}!}\Gamma(1/\hbar)}(z)^{\vec{m}}(\bar{z})^{\vec{n}}(1-|z|^2)^{1/\hbar}. \tag{145}$$

These are defined globally.

For $\mathbb{C}H^N$, we can represent a trace on the Fock space by the integration. The trace density on $\mathbb{C}H^N$ is given by the usual Riemannian volume form

$$\mu_g = \frac{1}{(1-|z|^2)^{N+1}}. \tag{146}$$

The integration of the vacuum state becomes

$$\begin{aligned} c_0 &= \int_{\mathbb{C}H^N} dz^{2D} \mu_g |0\rangle\langle 0| \\ &= \int_{\mathbb{C}H^N} dz^{2D} \frac{1}{(1-|z|^2)^{\frac{1}{\hbar}-(N+1)}} \\ &= \pi^N \frac{\Gamma(1/\hbar - N)}{\Gamma(1/\hbar)}. \end{aligned} \tag{147}$$

By normalizing the integrations by c_0, we can give the trace by the following integration,

$$\mathrm{Tr}_{\mathbb{C}H^N} |\vec{m}\rangle\underline{\langle\vec{n}|} = \frac{\Gamma(1/\hbar)}{\pi^N \Gamma(1/\hbar - N)} \int_{\mathbb{C}H^N} dz^{2D} \mu_g |\vec{m}\rangle\underline{\langle\vec{n}|} = \delta_{\vec{m}\vec{n}}. \tag{148}$$

6. Summary and Outlook

In the deformation quantization with separation of variables, which is a method to obtain noncommutative Kähler manifolds, basically, infinite dimensional system of PDE should be solved to get explicit expressions of star products in this method. As examples of Kähler manifolds on which the system of PDE can be solved, we considered $\mathbb{C}P^N$ and $\mathbb{C}H^N$. Indeed the star products on them were explicitly given by using the gamma functions. Next, the twisted Fock representations of general noncommutative Kähler manifolds were constructed. The noncommutative Kähler manifolds studied in this article were given by deformation quantization with separation of variables. Using this type of deformation quantization, the twisted Fock representation which constructed based on two sets of creation and annihilation operators was introduced with the concrete expressions of them on a local coordinate chart. The corresponding functions were expressed by the local complex coordinates, the Kähler potentials and partial derivatives of them with respect to the coordinates. The dictionary to translate bases of the twisted Fock representation into functions were given as table 1. They were defined on a local coordinate chart, but they can be extended by using the transition functions which are described in [36]. This extension is achieved by essentially the fact that the star products with separation of variables have a trivial transition function. We also gave examples of the twisted Fock representation of Kähler manifolds, $\mathbb{C}P^N$ and $\mathbb{C}H^N$.

Owing to the representation of deformation quantization of noncommutative Kähler manifolds, a new method is given to construct a field theory whose observable can be estimated as finite number. For example, on the noncommutative homogeneous Kähler manifolds given by deformation quantization with separation of variables, gauge theories are constructed, which is based on a description of formal power series [23]. Such formal power series is not good for physics, because physical quantities are expressed as formal power series. Now, gauge theories on the manifolds is possible to be constructed by using the twisted Fock representations. Indeed gauge theories on the noncommutative $\mathbb{C}P^N$ given in this way were already studied in [35]. We can extend this way to more general Kähler manifolds easily.

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ON SOME LIE GROUPS CONTAINING SPIN GROUP IN CLIFFORD ALGEBRA

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ABSTRACT. In this paper we consider some Lie groups in complexified Clifford algebras. Using relations between operations of conjugation in Clifford algebras and matrix operations we prove isomorphisms between these groups and classical matrix groups (symplectic, orthogonal, linear, unitary) in the cases of arbitrary dimension and arbitrary signature. Also we obtain isomorphisms of corresponding Lie algebras which are direct sums of subspaces of quaternion types. Spin group is a subgroup of all considered groups and it coincides with one of them in the cases $n \leq 5$. We present classical matrix Lie groups that contain spin group in the case of arbitrary dimension.

INTRODUCTION

In this paper we prove isomorphisms between five Lie groups in complexified Clifford algebra and classical matrix groups in the case of arbitrary dimension and arbitrary signature. Also we obtain isomorphisms of corresponding Lie algebras. We further develop results of the paper [20]. In [20] you can find statements only in the cases of fixed signatures when the corresponding real Clifford algebra has faithful and irreducible representations over $\mathbb{R}$ and $\mathbb{R}\oplus\mathbb{R}$. In the present paper we prove isomorphisms in all remaining cases, when real Clifford algebra has faithful and irreducible representations over $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{H}\oplus\mathbb{H}$.

Lie groups which are considered in this paper can be useful in different questions of field theory. Note that the following three groups $\mathrm{G}^{2i1}_{p,q}$, $\mathrm{G}^{23}_{p,q}$, $\mathrm{G}^{2}_{p,q}$ are subgroups of pseudo-unitary group (see [21], [27], [14]). In [13] new symmetry of Dirac equation [8], [9] with respect to the pseudo-unitary group was considered. Spin group $\mathrm{Spin}_{+}(p,q)$ is a subgroup of all five considered Lie groups. Moreover, group $\mathrm{Spin}_{+}(p,q)$ coincides with group $\mathrm{G}^{2}_{p,q}$ in the cases of dimensions $n \leq 5$. We discuss it in details in the Section 5. Salingaros vee group (it consists of $\pm$ basis elements of real Clifford algebra) [17], [18], [19], [11], [2], [3] is a subgroup of spin group and, so, also of five considered groups.

We mention a series of articles [1], [2], [3], where many interesting facts about so-called transposition anti-involution are discussed. We consider such conjugation in real and complexified Clifford algebras and call it Hermitian conjugation in [15]. In [15] we were interested, mostly, in some particular problems related to applications in field theory; in [1], [2], [3] you can find more detailed description of corresponding algebraic structures. Note that operation of Hermitian conjugation of Clifford algebra elements is well-known, especially, in particular cases. For example, P. Dirac [8], [9] uses it in the case of signature $(p,q) = (1,3)$ in the theory of Dirac equation for electron. We should note that information about Hermitian conjugation in the Section 2 of the present paper is related to results of the paper [2].

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In particular, one finds information about connection between Hermitian conjugation (or so-called transposition anti-involution) and matrix operations of corresponding matrix representations: in [2] for representations based on the fixed idempotent and the basis of corresponding left ideal, in the present paper (see formulas (4)-(7)) for fixed matrix representations, in [15] for complexified Clifford algebras and their representations based on the fixed idempotent and the basis of corresponding left ideal. In [2], [3] some interesting facts about group $G^{\varepsilon}_{p,q} = \{U \in C\ell_{p,q} \,|\, U^{\dagger}U = e\}$ in real Clifford algebras were considered. In [15] we consider analogue of this group $\{U \in \mathbb{C} \otimes C\ell_{p,q} \,|\, U^{\dagger}U = e\}$ (so-called unitary group) in complexified Clifford algebras. The group $G^{\varepsilon}_{p,q}$ is subgroup of this group. Note, that group $G^{\varepsilon}_{p,q}$ coincides in particular cases with some of groups which are considered in the present paper: with the group $\mathrm{G}^{23}_{p,q}$ in the case of signature $(n,0)$, with the group $\mathrm{G}^{12}_{p,q}$ in the case of signature $(0,n)$. Note, that some of groups which are considered in the present paper are related to automorphism groups of the scalar products on the spinor spaces (see [16], [12], [3]), but we do not use this fact in the present paper. In [12] one finds isomorphisms between groups $\mathrm{G}^{12}_{p,q}$, $\mathrm{G}^{23}_{p,q}$ and classical matrix Lie groups. In the present paper we also obtain these isomorphisms and also isomorphisms for groups $\mathrm{G}^{2i1}_{p,q}$, $\mathrm{G}^{2i3}_{p,q}$, $\mathrm{G}^{2}_{p,q}$ using other techniques based on relations between operations of conjugations in Clifford algebras and corresponding matrix operations. In particular, we generalized the notion of additional signature [25] to the case of real Clifford algebras. We also study the corresponding Lie algebras with the use of techniques of quaternion types [22], [23], [24] in Clifford algebras.

Let us consider the real Clifford algebra $C\ell_{p,q}$ and complexified Clifford algebra $\mathbb{C} \otimes C\ell_{p,q}$, $p+q=n$, $n \geq 1$ [6]. The construction of real and complexified Clifford algebras is discussed in details in [12], [5], [10].

Let us remind the basic notation. Let e be the identity element and let e^a, $a = 1, \ldots, n$ be generators[1] of the Clifford algebra $C\ell_{p,q}$, $e^a e^b + e^b e^a = 2\eta^{ab} e$, where $\eta = ||\eta^{ab}||$ is the diagonal matrix with $+1$ appearing p times on the diagonal and -1 appearing q times on the diagonal. Elements $e^{a_1 \ldots a_k} = e^{a_1} \cdots e^{a_k}$, $a_1 < \cdots < a_k$, $k = 1, \ldots, n$, together with the identity element e, form a basis of Clifford algebra. Any Clifford algebra element $U \in C\ell_{p,q}$ can be written in the form[2]

$$U = ue + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \cdots + u_{1 \ldots n} e^{1 \ldots n}, \tag{1}$$

where $u, u_a, u_{a_1 a_2}, \ldots, u_{1 \ldots n}$ are real numbers. For arbitrary element $U \in \mathbb{C} \otimes C\ell_{p,q}$ of complexified Clifford algebra we use the same notation (1), where $u, u_a, u_{a_1 a_2}, \ldots, u_{1 \ldots n}$ are complex numbers.

We denote by $C\ell^k_{p,q}$ the vector spaces that span over the basis elements $e^{a_1 \ldots a_k}$. Elements of $C\ell^k_{p,q}$ are said to be elements of grade k. We have $C\ell_{p,q} = \bigoplus_{k=0}^{n} C\ell^k_{p,q}$. Clifford algebra is a Z_2-graded algebra and it is represented as the direct sum of even and odd subspaces:

$$C\ell_{p,q} = C\ell^{(0)}_{p,q} \oplus C\ell^{(1)}_{p,q}, \quad C\ell^{(i)}_{p,q} C\ell^{(j)}_{p,q} \subseteq C\ell^{(i+j) \bmod 2}_{p,q}, \quad \text{where} \quad C\ell^{(i)}_{p,q} = \bigoplus_{k \equiv i \bmod 2} C\ell^k_{p,q}, \quad i,j = 0,1.$$

Let us consider the Clifford algebra $C\ell_{p,q}$ as the vector space and represent it in the form of the direct sum of four subspaces of *quaternion types* 0, 1, 2 and 3 (see [22], [23], [24]):

$$C\ell_{p,q} = \overline{\mathbf{0}} \oplus \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus \overline{\mathbf{3}}, \qquad \text{where} \quad \overline{\mathbf{s}} = \bigoplus_{k \equiv s \bmod 4} C\ell^k_{p,q}, \quad s = 0,1,2,3.$$

We represent complexified Clifford algebra $\mathbb{C} \otimes C\ell_{p,q}$ in the form of the direct sum of eight subspaces: $\mathbb{C} \otimes C\ell_{p,q} = \overline{\mathbf{0}} \oplus \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus \overline{\mathbf{3}} \oplus i\overline{\mathbf{0}} \oplus i\overline{\mathbf{1}} \oplus i\overline{\mathbf{2}} \oplus i\overline{\mathbf{3}}$.

[1] Note that e^a is not exponent. We use notation with upper indices [4].

[2] We use Einstein's summation convention: there is a sum over index a.

In [20] we discussed a recurrent method of construction of matrix representation of real Clifford algebra in the cases of signatures $p-q \equiv 0,1,2 \mod 8$. In the Section 1 of the present paper we generalize this method to the case of arbitrary signature.

In the Section 2 of the present paper we give some information about Hermitian conjugation in real and complexified Clifford algebras (see also [15] and [2] for the case of real Clifford algebras). In [25] we introduced the notion of additional signature (k,l) of complexified Clifford algebra. In the Section 3 of the present paper we generalize this notion to the case of real Clifford algebras.

In the Section 4 we prove isomorphisms between five Lie groups in Clifford algebra and classical matrix Lie groups (some of these isomorphisms are known, see above). We study corresponding Lie algebras. In the Section 5 of the present paper we discuss connection between groups $\mathrm{Spin}_+(p,q)$ and $\mathrm{G}^2_{p,q}$. In the Section 6 we summarize results of [20] and the present paper.

1. Recurrent method of construction of matrix representations of real Clifford algebras in the case of arbitrary signature

In [20] we discussed a recurrent method of construction of matrix representation only in the cases of signatures $p-q \equiv 0,1,2 \mod 8$ (the case of faithful and irreducible representations over $\mathbb{R}$ and $\mathbb{R}\oplus\mathbb{R}$). Now we generalize this method to the case of arbitrary signature. Clifford algebra has faithful and irreducible representations over $\mathbb{R}$, $\mathbb{R}\oplus\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}\oplus\mathbb{H}$ in different cases. However, items 1-4 are the same as in [20].

We have the following well-known isomorphisms $C\ell_{p,q} \simeq \mathrm{L}_{p,q}$, where we denote by $\mathrm{L}_{p,q}$ the following matrix algebras

$$\mathrm{L}_{p,q} = \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}},\mathbb{R}), & \text{if } p-q \equiv 0;2 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{R}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{R}), & \text{if } p-q \equiv 1 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \text{if } p-q \equiv 3;7 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-2}{2}},\mathbb{H}), & \text{if } p-q \equiv 4;6 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H}), & \text{if } p-q \equiv 5 \mod 8. \end{cases}$$

We want to construct faithful and irreducible matrix representation $\beta : C\ell_{p,q} \to \mathrm{L}_{p,q}$ of all real Clifford algebras with some additional properties related to symmetry and skew-symmetry of corresponding matrices (we discuss it in the next section). In some particular cases we construct β in the following way:

- In the case $C\ell_{0,0}$: $e \to 1$.
- In the case $C\ell_{0,1}$: $e \to 1$, $e^1 \to i$.
- In the case $C\ell_{1,0}$: $e \to \mathbf{1} = \mathrm{diag}(1,1)$, $e^1 \to \mathrm{diag}(1,-1)$.
- In the case $C\ell_{0,2}$: $e \to 1$, $e^1 \to i$, $e^2 \to j$.
- In the case $C\ell_{0,3}$: $e \to \mathbf{1} = \mathrm{diag}(1,1)$, $e^1 \to \mathrm{diag}(i,-i)$, $e^2 \to \mathrm{diag}(j,-j)$, $e^3 \to \mathrm{diag}(k,-k)$.

For basis element $e^{a_1\dots a_k}$ we use the matrix that equals the product of matrices corresponding to generators $e^{a_1},\dots,e^{a_k}$. For identity element e, we always use identity matrix $\mathbf{1}$ of corresponding size.

Suppose that we have a faithful and irreducible matrix representation $\beta : C\ell_{p,q} \to \mathrm{L}_{p,q}$

$$e^a \to \beta^a, \quad a = 1,\dots,n, \tag{2}$$

where $\beta^a = \beta(e^a)$, $a = 1,2,\dots,n$. Now we want to construct the matrix representations of other Clifford algebras $C\ell_{p+1,q+1}$, $C\ell_{q+1,p-1}$, $C\ell_{p-4,q+4}$ with generators e^a ($a = 1,2,\dots,n+2$

in the first case and $a = 1,2,\ldots,n$ in the last two cases) with the use of matrices β^a. Using the following items 1-4 and Cartan periodicity of real Clifford algebras we obtain matrix representations of all real Clifford algebras. We call it recurrent method of construction of matrix representations[3].

1. Let us consider $C\ell_{p+1,q+1}$. If $p-q \not\equiv 1 \mod 4$, then for p generators with squares $+1$ and q generators with squares -1 we have

$$e^a \to \begin{pmatrix} \beta^a & 0 \\ 0 & -\beta^a \end{pmatrix}, \qquad a = 1,2,\ldots,p,p+2,p+3,\ldots,p+q+1.$$

And for two remaining generators we have

$$e^{p+1} \to \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad e^{p+q+2} \to \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

2. If $p-q \equiv 1 \mod 4$, then matrices (2) are block-diagonal and we have the following matrix representation of $C\ell_{p+1,q+1}$. For $p+q$ generators e^a, $a = 1,2,\ldots,p,p+2,p+3,\ldots,p+q+1$ we have the same as in the previous item and for remaining two generators we have

$$e^{p+1} \to \begin{pmatrix} \beta^1\ldots\beta^n\Omega & 0 \\ 0 & -\beta^1\ldots\beta^n\Omega \end{pmatrix}, \quad e^{p+q+2} \to \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix},$$

where

$$\Omega = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

3. We construct matrix representation of $C\ell_{q+1,p-1}$ using

$$e^1 \to \beta^1, \qquad e^i \to \beta^i\beta^1, \quad i = 2,\ldots,n.$$

4. We construct matrix representation of $C\ell_{p-4,q+4}$ using

$$e^i \to \beta^i\beta^1\beta^2\beta^3\beta^4, \quad i = 1,2,3,4, \qquad e^j \to \beta^j, \quad j = 5,\ldots,n.$$

In [20] we gave some examples in the cases $p-q \equiv 0,1,2 \mod 8$. Now let us give some examples in the cases of other signatures:

$C\ell_{1,2}$**:**

$$e^1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e^2 \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e^3 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$C\ell_{1,3}$**:**

$$e^1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e^2 \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e^3 \to \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, e^4 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$C\ell_{4,0}$**:**

$$e^1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e^2 \to \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, e^3 \to \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, e^4 \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$C\ell_{0,4}$**:**

$$e^1 \to \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, e^2 \to \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix}, e^3 \to \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, e^4 \to \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

[3]Another method of construction of matrix representations is based on the idempotent and basis of corresponding left ideal (see, for example, [1], [2], [3], [15]).

2. Relation between operations of conjugation in Clifford algebra and matrix operations

Consider the following well-known involutions in real $C\ell_{p,q}$ and complexified Clifford algebra $\mathbb{C}\otimes C\ell_{p,q}$:

$$\hat{U} = U|_{e^a\to -e^a}, \quad \tilde{U} = U|_{e^{a_1\dots a_r}\to e^{a_r}\dots e^{a_1}},$$

where U has the form (1). The operation $U\to\hat{U}$ is called *grade involution* and $U\to\tilde{U}$ is called *reversion*. Also we have operation of *complex conjugation*

$$\bar{U} = \bar{u}e + \bar{u}_a e^a + \sum_{a_1<a_2}\bar{u}_{a_1a_2}e^{a_1a_2} + \sum_{a_1<a_2<a_3}\bar{u}_{a_1a_2a_3}e^{a_1a_2a_3} + \dots,$$

where we take complex conjugation of complex numbers $u_{a_1\dots a_k}$. Superposition of reversion and complex conjugation is *pseudo-Hermitian conjugation of Clifford algebra elements*[4]

$$U^{\ddagger} = \tilde{\bar{U}}.$$

In the real Clifford algebra $C\ell_{p,q}\subset\mathbb{C}\otimes C\ell_{p,q}$ we have $U^{\ddagger}=\tilde{U}$, because $\bar{U}=U$.

Let us consider in complexified $\mathbb{C}\otimes C\ell_{p,q}$ and real $C\ell_{p,q}\subset\mathbb{C}\otimes C\ell_{p,q}$ Clifford algebras the linear operation (involution) $\dagger:\mathbb{C}\otimes C\ell_{p,q}\to\mathbb{C}\otimes C\ell_{p,q}$ such that $(\lambda e^{a_1\dots a_k})^{\dagger}=\bar{\lambda}(e^{a_1\dots a_k})^{-1}$, $\lambda\in\mathbb{C}$. We call this operation *Hermitian conjugation of Clifford algebra elements*[5]. We use notation $e_a=\eta_{ab}e^b=(e^a)^{-1}=(e^a)^{\dagger}$ and $e_{a_1\dots a_k}=(e^{a_1\dots a_k})^{-1}=(e^{a_1\dots a_k})^{\dagger}$. This operation is well-known and many authors use it, for example, in different questions of field theory in the case of signature $(p,q)=(1,3)$. For more details, see [15], and for the case of real Clifford algebras see [1], [2], [3] (so-called transposition anti-involution in real Clifford algebras).

Note that we have the following relation between operation of Hermitian conjugation of Clifford algebra elements $\dagger$ and other operations in complexified Clifford algebra $\mathbb{C}\otimes C\ell_{p,q}$ (see [15])

$$\begin{aligned}
U^{\dagger} &= e_{1\dots p}U^{\ddagger}e^{1\dots p}, && \text{if } p \text{ is odd},\\
U^{\dagger} &= e_{1\dots p}\hat{U}^{\ddagger}e^{1\dots p}, && \text{if } p \text{ is even},\\
U^{\dagger} &= e_{p+1\dots n}U^{\ddagger}e^{p+1\dots n}, && \text{if } q \text{ is even},\\
U^{\dagger} &= e_{p+1\dots n}\hat{U}^{\ddagger}e^{p+1\dots n}, && \text{if } q \text{ is odd}.
\end{aligned} \tag{3}$$

We have the following well-known isomorphisms:

$$\begin{aligned}
\mathbb{C}\otimes C\ell_{p,q} &\simeq \mathrm{Mat}(2^{\frac{n}{2}},\mathbb{C}), && \text{if } n \text{ is even},\\
\mathbb{C}\otimes C\ell_{p,q} &\simeq \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), && \text{if } n \text{ is odd}.
\end{aligned}$$

Hermitian conjugation of Clifford algebra elements corresponds to Hermitian conjugation of matrix $\beta(U^{\dagger})=(\beta(U))^{\dagger}$ for faithful and irreducible matrix representations over $\mathbb{C}$ and $\mathbb{C}\oplus\mathbb{C}$ of complexified Clifford algebra, based on the fixed idempotent and basis of corresponding left ideal, see [15].

[4]Pseudo-Hermitian conjugation of Clifford algebra elements is related to pseudo-unitary matrix groups as Hermitian conjugation is related to unitary groups, see [27], [15].

[5]Note that it is not Hermitian conjugation of matrix but it is related to this operation.

It is not difficult to prove that for matrix representation β of real Clifford algebra $C\ell_{p,q}$ from the previous section we have

$$\begin{aligned} \beta(U^\dagger) &= (\beta(U))^T, \qquad p-q \equiv 0,1,2 \mod 8, \\ \beta(U^\dagger) &= (\beta(U)^\dagger, \qquad p-q \equiv 3,7 \mod 8, \\ \beta(U^\dagger) &= (\beta(U))^*, \qquad p-q \equiv 4,5,6 \mod 8, \end{aligned} \tag{4}$$

where T is the operation of matrix transposition, $*$ is the operation of conjugate transpose of matrices of quaternions, $U^\dagger$ is the Hermitian conjugate of a Clifford algebra element U, $\beta(U)^\dagger$ is the Hermitian conjugate of the corresponding matrix. Note, that in [2] analogous statement is proved for matrix representations based on the fixed idempotent and the basis of corresponding left ideal. In this paper we do not use idempotent to construct matrix representation. We use the recurrent method of construction of matrix representation (see previous section). So, we must verify statement for fixed matrices in some cases of small n (see the beginning of the previous section) and then, using the method of mathematical induction, we must verify that, using items 1-4, we obtain matrix representations of other Clifford algebras with the same property (4). We omit the detailed proof.

In the cases $p-q \equiv 0,1,2 \mod 8$ for the real Clifford algebra $C\ell_{p,q} \subset \mathbb{C} \otimes C\ell_{p,q}$ we have

$$\begin{aligned} U^T &= e_{1\ldots p}\tilde{U}e^{1\ldots p}, \qquad \text{if } p \text{ is odd}, \\ U^T &= e_{1\ldots p}\tilde{\tilde{U}}e^{1\ldots p}, \qquad \text{if } p \text{ is even}, \\ U^T &= e_{p+1\ldots n}\tilde{U}e^{p+1\ldots n}, \qquad \text{if } q \text{ is even}, \\ U^T &= e_{p+1\ldots n}\tilde{\tilde{U}}e^{p+1\ldots n}, \qquad \text{if } q \text{ is odd}, \end{aligned} \tag{5}$$

where $U^T = \beta^{-1}(\beta^T(U)) = U^\dagger$.

In the cases $p-q \equiv 3,7 \mod 8$ for the real Clifford algebra $C\ell_{p,q}$ we have

$$\begin{aligned} U^\dagger &= e_{1\ldots p}\tilde{U}e^{1\ldots p}, \qquad \text{if } p \text{ is odd}, \\ U^\dagger &= e_{1\ldots p}\tilde{\tilde{U}}e^{1\ldots p}, \qquad \text{if } p \text{ is even}, \\ U^\dagger &= e_{p+1\ldots n}\tilde{U}e^{p+1\ldots n}, \qquad \text{if } q \text{ is even}, \\ U^\dagger &= e_{p+1\ldots n}\tilde{\tilde{U}}e^{p+1\ldots n}, \qquad \text{if } q \text{ is odd}. \end{aligned} \tag{6}$$

In the cases $p-q \equiv 4,5,6 \mod 8$ for the real Clifford algebra we have

$$\begin{aligned} U^* &= e_{1\ldots p}\tilde{U}e^{1\ldots p}, \qquad \text{if } p \text{ is odd}, \\ U^* &= e_{1\ldots p}\tilde{\tilde{U}}e^{1\ldots p}, \qquad \text{if } p \text{ is even}, \\ U^* &= e_{p+1\ldots n}\tilde{U}e^{p+1\ldots n}, \qquad \text{if } q \text{ is even}, \\ U^* &= e_{p+1\ldots n}\tilde{\tilde{U}}e^{p+1\ldots n}, \qquad \text{if } q \text{ is odd}, \end{aligned} \tag{7}$$

where $U^* = \beta^{-1}(\beta^*(U)) = U^\dagger$. We use formulas (5) - (7) in the next sections.

3. Additional signature of real Clifford algebra

In [25] we introduced the notion of additional signature (k,l) of complexified Clifford algebra. Now we want to generalize this notion to the case of real Clifford algebras.

Suppose we have faithful and irreducible matrix representation β over $\mathbb{C}$ or $\mathbb{C} \oplus \mathbb{C}$ of complexified Clifford algebra. We can always use such matrix representation that all matrices $\beta^a = \beta(e^a)$ are symmetric or skew-symmetric. Let k be the number of symmetric matrices among $\{\beta^a\}$ for matrix representation β, and l be the number of skew-symmetric matrices

among $\{\beta^a\}$. Let $e^{b_1},\dots,e^{b_k}$ denote the generators for which the matrices are symmetric. Analogously, we have $e^{c_1},\dots,e^{c_l}$ for skew-symmetric matrices.

We use the notion of additional signature of Clifford algebra when we study relation between matrix representation and operations of conjugation. In complexified Clifford algebra we have (see [25])

$$
\begin{aligned}
U^T &= e_{b_1\dots b_k}\tilde{U}e^{b_1\dots b_k}, \qquad k \text{ is odd},\\
U^T &= e_{b_1\dots b_k}\tilde{\tilde{U}}e^{b_1\dots b_k}, \qquad k \text{ is even},\\
U^T &= e_{c_1\dots c_l}\tilde{U}e^{c_1\dots c_l}, \qquad l \text{ is even},\\
U^T &= e_{c_1\dots c_l}\tilde{\tilde{U}}e^{c_1\dots c_l}, \qquad l \text{ is odd}.
\end{aligned}
\tag{8}
$$

Numbers k and l depend on the matrix representation β. But they can take only certain values despite dependence on the matrix representation.

In [25] we proved that in complexified Clifford algebra we have only the following possible values of additional signature:

$n \mod 8$	$(k \mod 4, l \mod 4)$
0	$(0,0)$, $(1,3)$
1	$(1,0)$
2	$(1,1)$, $(2,0)$
3	$(2,1)$

$n \mod 8$	$(k \mod 4, l \mod 4)$
4	$(3,1)$, $(2,2)$
5	$(3,2)$
6	$(3,3)$, $(0,2)$
7	$(0,3)$

Now we want to use the notion of additional signature in real Clifford algebra. Let us consider faithful and irreducible matrix representation over $\mathbb{R}$, $\mathbb{R}\oplus\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{H}\oplus\mathbb{H}$ of real Clifford algebra. We can always use such matrix representation that all matrices $\beta^a=\beta(e^a)$ are symmetric or skew-symmetric (for example, we can use matrix representation $\beta : C\ell_{p,q}\to \mathrm{L}_{p,q}$ from the Section 1). Let k be the number of symmetric matrices among $\{\beta^a\}$ for matrix representation β, and l be the number of skew-symmetric matrices among $\{\beta^a\}$. Let $e^{b_1},\dots,e^{b_k}$ denote the generators for which the matrices are symmetric. Analogously, we have $e^{c_1},\dots,e^{c_l}$ for skew-symmetric matrices.

Note, that in the cases $p-q\equiv 0,1,2 \mod 8$ for the real Clifford algebra $C\ell_{p,q}$ the formulas (8) are valid and they coincide with formulas (5) (we have $p=k$ and $q=l$). In the cases $p-q\equiv 3,7 \mod 8$ for the real Clifford algebra we can use formulas (8). The proof is similar to the proof for the complexified Clifford algebra. Actually, it is sufficient to prove these formulas for basis elements because of linearity of operation of conjugation. For example, if k is odd, then

$$
\begin{aligned}
(e^{a_1\dots a_m})^T &= (e^{b_k})^{-1}\dots(e^{b_1})^{-1}e^{a_1\tilde{\dots}a_m}e^{b_1}\dots e^{b_k} =\\
&= (e^{b_k})^{-1}\dots(e^{b_1})^{-1}e^{a_m}\dots e^{a_1}e^{b_1}\dots e^{b_k} =\\
&= (-1)^{mk-j}e^{a_m}\dots e^{a_1} = (-1)^{m-j}e^{a_m}\dots e^{a_1}
\end{aligned}
$$

where j is the number of elements $e^{a_1},\dots,e^{a_m}$ that have symmetric matrix representation. We can analogously prove remaining three formulas from (8).

But formulas (8) are not valid in real Clifford algebra of signatures $p-q\equiv 4,5,6 \mod 8$ because of the properties of operation of transposition for quaternionic matrices: $(AB)^T\neq B^TA^T$ for quaternionic matrices A and B.

So, it makes sense to use the notion of additional signature of real Clifford algebra only in the cases $p-q\equiv 3,7 \mod 8$. We don't want to obtain all possible values of additional signature, but we want to obtain some possible values. If we consider matrix representation

β from the Section 1, then in the case $(p,q)=(0,1)$ we have $(k,l)=(1,0)$. Note, that we use only transformations of types 1, 3 and 4 from Section 1 to obtain matrix representations of other Clifford algebras of signatures $p-q\equiv 3,7 \mod 8$. If we use transformation of type 1, then $(p,q)\to(p+1,q+1)$ and $(k,l)\to(k+1,l+1)$. When we use transformation of type 3, we have $(p,q)\to(q+1,p-1)$ and $(k,l)\to(l+1,k-1)$ because β^1 is a symmetric matrix. When we use transformation of type 4, we have $(p,q)\to(p-4,q+4)$ and it is not difficult to obtain that k and l do not change the parity. So, for matrix representation β we always have: if p is even and q is odd, then k is odd and l is even (note, that the same is true in complexified Clifford algebra, see table above). For example, for $(p,q)=(1,2)$ we have $(k,l)=(2,1)$, for $(p,q)=(3,0)$ we have $(k,l)=(2,1)$, for $(p,q)=(2,3)$ we have $(k,l)=(3,2)$ and so on. We will use this fact in the proof of Theorem 6.2 in the following section.

4. THEOREMS

Let us consider the following subsets of complexified Clifford algebra [20]

$$\begin{aligned}
\mathrm{G}^{2i1}_{p,q} &= \{U\in C\ell^{(0)}_{p,q}\oplus iC\ell^{(1)}_{p,q} : U^{\ddagger}U=e\},\\
\mathrm{G}^{2i3}_{p,q} &= \{U\in C\ell^{(0)}_{p,q}\oplus iC\ell^{(1)}_{p,q} : \hat{U}^{\ddagger}U=e\},\\
\mathrm{G}^{23}_{p,q} &= \{U\in C\ell_{p,q} \,|\, \tilde{U}U=e\},\\
\mathrm{G}^{12}_{p,q} &= \{U\in C\ell_{p,q} \,|\, \tilde{\hat{U}}U=e\},\\
\mathrm{G}^{2}_{p,q} &= \{U\in C\ell^{(0)}_{p,q} \,|\, \tilde{U}U=e\}.
\end{aligned} \tag{9}$$

They can be considered as Lie groups. Their Lie algebras are

$$\overline{\mathbf{2}}\oplus i\overline{\mathbf{1}},\quad \overline{\mathbf{2}}\oplus i\overline{\mathbf{3}},\quad \overline{\mathbf{2}}\oplus\overline{\mathbf{3}},\quad \overline{\mathbf{2}}\oplus\overline{\mathbf{1}},\quad \overline{\mathbf{2}} \tag{10}$$

respectively.

Note, that we have the following properties (see [22], [23], [24])

$$\begin{aligned}
&[\overline{\mathbf{k}},\overline{\mathbf{k}}]\subseteq\overline{\mathbf{2}}, \qquad k=0,1,2,3;\\
&[\overline{\mathbf{k}},\overline{\mathbf{2}}]\subseteq\overline{\mathbf{k}}, \qquad k=0,1,2,3;\\
&[\overline{\mathbf{0}},\overline{\mathbf{1}}]\subseteq\overline{\mathbf{3}},\quad [\overline{\mathbf{0}},\overline{\mathbf{3}}]\subseteq\overline{\mathbf{1}},\quad [\overline{\mathbf{1}},\overline{\mathbf{3}}]\subseteq\overline{\mathbf{0}},
\end{aligned} \tag{11}$$

where $[U,V]=UV-VU$ is the commutator of arbitrary Clifford algebra elements U and V.

It is not difficult to calculate dimensions of these Lie algebras, because we know that $\dim C\ell^{k}_{p,q}=C^k_n=\frac{n!}{k!(n-k)!}$. So, for example,

$$\dim\overline{\mathbf{2}}=\sum_{k\equiv 2 \mod 4} C^k_n = 2^{n-2}-2^{\frac{n-2}{2}}\cos\frac{\pi n}{4}.$$

Let us represent Lie groups, corresponding Lie algebras, and their dimensions in the following table.

Lie group	Lie algebra	dimension
$\mathrm{G}^{2i1}_{p,q}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{1}}$	$2^{n-1}-2^{\frac{n-1}{2}}\cos\frac{\pi(n+1)}{4}$
$\mathrm{G}^{2i3}_{p,q}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{3}}$	$2^{n-1}-2^{\frac{n-1}{2}}\sin\frac{\pi(n+1)}{4}$
$\mathrm{G}^{23}_{p,q}$	$\overline{\mathbf{2}}\oplus\overline{\mathbf{3}}$	$2^{n-1}-2^{\frac{n-1}{2}}\sin\frac{\pi(n+1)}{4}$
$\mathrm{G}^{12}_{p,q}$	$\overline{\mathbf{2}}\oplus\overline{\mathbf{1}}$	$2^{n-1}-2^{\frac{n-1}{2}}\cos\frac{\pi(n+1)}{4}$
$\mathrm{G}^{2}_{p,q}$	$\overline{\mathbf{2}}$	$2^{n-2}-2^{\frac{n-2}{2}}\cos\frac{\pi n}{4}$

Theorem 4.1. [20] *We have the following Lie group isomorphisms*

$$\mathrm{G}^{2i1}_{p,q} \simeq \mathrm{G}^{12}_{q,p}, \qquad \mathrm{G}^{2i3}_{p,q} \simeq \mathrm{G}^{23}_{q,p}, \qquad \mathrm{G}^{2}_{p,q} \simeq \mathrm{G}^{12}_{p,q-1} \simeq \mathrm{G}^{12}_{q,p-1}, \qquad \mathrm{G}^{2}_{p,q} \simeq \mathrm{G}^{2}_{q,p}.$$

Proof. We must use transformation $e^a \to e^a e^n$ or $e^a \to ie^a$, $a = 1,2,\ldots,n$ in different cases. ■

According to Theorem 4.1 it is sufficient to consider only groups $\mathrm{G}^{12}_{p,q}$ and $\mathrm{G}^{23}_{p,q}$. In [20] we proved isomorphisms between these groups and classical matrix Lie groups in the cases of signatures $p-q \equiv 0,1,2 \mod 8$. Now let us consider cases $p-q \equiv 3,7 \mod 8$ and $p-q \equiv 4,5,6 \mod 8$.

Theorem 4.2. *We have the following Lie group isomorphisms. In the cases of signatures* $p-q \equiv 3,7 \mod 8$ *(n is odd)*

$$\mathrm{G}^{23}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n-1}{2}}), & (p,q) = (n,0); \\ \mathrm{U}(2^{\frac{n-3}{2}}, 2^{\frac{n-3}{2}}), & n \equiv 3,7 \mod 8,\ q \neq 0; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}, \mathbb{C}), & n \equiv 5 \mod 8; \\ \mathrm{O}(2^{\frac{n-1}{2}}, \mathbb{C}), & n \equiv 1 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 3,7 \mod 8$ *(n is odd)*

$$\mathrm{G}^{12}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n-1}{2}}), & (p,q) = (0,n); \\ \mathrm{U}(2^{\frac{n-3}{2}}, 2^{\frac{n-3}{2}}), & n \equiv 1,5 \mod 8,\ p \neq 0; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}, \mathbb{C}), & n \equiv 3 \mod 8; \\ \mathrm{O}(2^{\frac{n-1}{2}}, \mathbb{C}), & n \equiv 7 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 1,5 \mod 8$ *(n is odd)*

$$\mathrm{G}^{2i1}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n-1}{2}}), & (p,q) = (n,0); \\ \mathrm{U}(2^{\frac{n-3}{2}}, 2^{\frac{n-3}{2}}), & n \equiv 1,5 \mod 8,\ q \neq 0; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}, \mathbb{C}), & n \equiv 3 \mod 8; \\ \mathrm{O}(2^{\frac{n-1}{2}}, \mathbb{C}), & n \equiv 7 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 1,5 \mod 8$ *(n is odd)*

$$\mathrm{G}^{2i3}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n-1}{2}}), & (p,q) = (0,n); \\ \mathrm{U}(2^{\frac{n-3}{2}}, 2^{\frac{n-3}{2}}), & n \equiv 3,7 \mod 8,\ p \neq 0; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}, \mathbb{C}), & n \equiv 5 \mod 8; \\ \mathrm{O}(2^{\frac{n-1}{2}}, \mathbb{C}), & n \equiv 1 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 2,6 \mod 8$ *(n is even)*

$$\mathrm{G}^{2}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n-2}{2}}), & (p,q) = (0,n),(0,n); \\ \mathrm{U}(2^{\frac{n-4}{2}}, 2^{\frac{n-4}{2}}), & n \equiv 2,6 \mod 8,\ p,q \neq 0; \\ \mathrm{Sp}(2^{\frac{n-4}{2}}, \mathbb{C}), & n \equiv 4 \mod 8; \\ \mathrm{O}(2^{\frac{n-2}{2}}, \mathbb{C}), & n \equiv 0 \mod 8. \end{cases}$$

We use the standard notation of classical matrix groups [7]

$$\begin{aligned} \mathrm{U}(n) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) : A^\dagger A = \mathbf{1}\}, \\ \mathrm{U}(p,q) &= \{A \in \mathrm{Mat}(p+q,\mathbb{C}) : A^\dagger \eta A = \eta\}, \\ \mathrm{Sp}(n,\mathbb{C}) &= \{A \in \mathrm{Mat}(2n,\mathbb{C}) : A^T \Omega A = \Omega\}, \\ \mathrm{O}(n,\mathbb{C}) &= \{A \in \mathrm{Mat}(n,\mathbb{C}) : A^T A = \mathbf{1}\}, \end{aligned}$$

where Ω is the block matrix $\Omega = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.

Proof. We consider Lie group $\mathrm{G}^{23}_{p,q}$ in Clifford algebra $C\ell_{p,q} \simeq \mathrm{Mat}(\frac{n-1}{2}, \mathbb{C})$, $p-q \equiv 3,7 \mod 8$. We use complex matrix representation $\beta : U \to \beta(U)$, $U \in C\ell_{p,q}$ in the cases $p-q \equiv 3,7 \mod 8$ from Section 1 such that $\beta(U^\dagger) = (\beta(U))^\dagger$.

In the case of signature $(n,0)$ we have $\tilde{U} = U^\dagger$ (see (6)). From $\tilde{U}U = e$ we obtain $U^\dagger U = \mathbf{1}$ and isomorphism with group $\mathrm{U}(2^{\frac{n-1}{2}})$.

Consider the case $q \neq 0$. If p is odd and q is even, then $\tilde{U} = e_{1\dots p} U^\dagger e^{1\dots p}$. In the case $p \equiv 1 \mod 4$ (in this case we have $q \equiv 2 \mod 4$ and $n \equiv 3 \mod 4$) let us consider matrix $M = \beta^{1\dots p}$. We have $M^2 = (-1)^{\frac{p(p-1)}{2}}\mathbf{1} = \mathbf{1}$, $M^\dagger = M^{-1}$, $\mathrm{tr}M = 0$. Thus, the spectrum of matrix M consists of the same number of 1 and -1. So there exists such unitary matrix $T^\dagger = T^{-1}$ that $J = T^{-1}MT$, where $J = \mathrm{diag}(1,\dots,1,-1,\dots,-1)$ is the diagonal matrix with the same number of 1 and -1 on the diagonal. Now we consider transformation $T^{-1}\beta^a T = \gamma^a$ and obtain another matrix representation γ of Clifford algebra with $\gamma^{1\dots p} = J$. From $\tilde{U}U = e$ we obtain $U^\dagger JU = J$ and isomorphism with the group $\mathrm{U}(2^{\frac{n-3}{2}}, 2^{\frac{n-3}{2}})$. The case $p \equiv 3 \mod 4$ (in this case we have $q \equiv 0 \mod 4$, $n \equiv 3 \mod 4$) is similar (we must consider element $M = i\beta^{1\dots p}$).

Now let p be even and q be odd. Then $n \equiv 1 \mod 4$. We know that k is odd and l is even in these cases (see previous section). So we can use formula $U^{\mathrm{T}} = e_{b_1\dots b_k}\tilde{U}e^{b_1\dots b_k}$. Let us consider matrix $M = \beta^{b_1\dots b_k}$. We have $M^T = M^{-1}$, $\mathrm{tr}M = 0$, $M^2 = \mathbf{1}$ (or $M^2 = -\mathbf{1}$). Thus there exists such orthogonal matrix $T^T = T^{-1}$ that $J = T^{-1}MT$ (or $\Omega = T^{-1}MT$, because for Ω we also have $\Omega^T = \Omega$, $\Omega^2 = -\mathbf{1}$, $\mathrm{tr}\Omega = 0$). We consider transformation $T^{-1}\beta^a T = \gamma^a$ and obtain another matrix representation γ of Clifford algebra with $\gamma^{b_1\dots b_k} = J$ (or $\gamma^{b_1\dots b_k} = \Omega$). So we obtain condition $U^T JU = J$ (or $U^T \Omega U = \Omega$). Thus we obtain in the case $n \equiv 1 \mod 8$ isomorphism with group $\mathrm{O}(2^{\frac{n-3}{2}}, 2^{\frac{n-3}{2}}, \mathbb{C}) \simeq \mathrm{O}(2^{\frac{n-1}{2}}, \mathbb{C})$ and in the case $n \equiv 5 \mod 8$ isomorphism with group $\mathrm{Sp}(2^{\frac{n-3}{2}}, \mathbb{C})$. We take into account that $\dim \mathrm{G}^{23}_{p,q} = 2^{n-1} - 2^{\frac{n-1}{2}} \sin\frac{\pi(n+1)}{4}$, $\dim \mathrm{O}(2^{\frac{n-1}{2}}, \mathbb{C}) = 2^{n-1} - 2^{\frac{n-1}{2}}$, $\dim \mathrm{Sp}(2^{\frac{n-3}{2}}, \mathbb{C}) = 2^{n-1} + 2^{\frac{n-1}{2}}$.

We can obtain isomorphisms for the group $\mathrm{G}^{12}_{p,q}$ analogously. The statement for the groups $\mathrm{G}^{2i1}_{p,q}$, $\mathrm{G}^{2i3}_{p,q}$ and $\mathrm{G}^{2}_{p,q}$ follows from Theorem 4.1. ■

Theorem 4.3. *We have the following Lie group isomorphisms. In the cases of signatures* $p-q \equiv 4,5,6 \mod 8$

$$\mathrm{G}^{23}_{p,q} \simeq \begin{cases} \mathrm{Sp}(2^{\frac{n-2}{2}}), & (p,q) = (n,0),\ n \text{ is even;} \\ \mathrm{Sp}(2^{\frac{n-4}{2}}, 2^{\frac{n-4}{2}}), & n \equiv 4,6 \mod 8,\ q \neq 0; \\ \mathrm{O}(2^{\frac{n-2}{2}}, \mathbb{H}), & n \equiv 0,2 \mod 8; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-3}{2}}), & (p,q) = (n,0),\ n \text{ is odd;} \\ \mathrm{Sp}(2^{\frac{n-5}{2}}, 2^{\frac{n-5}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-5}{2}}, 2^{\frac{n-5}{2}}), & n \equiv 5 \mod 8,\ q \neq 0; \\ \mathrm{O}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{O}(2^{\frac{n-3}{2}}, \mathbb{H}), & n \equiv 1 \mod 8; \\ \mathrm{GL}(2^{\frac{n-3}{2}}, \mathbb{H}), & n \equiv 3,7 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 4,5,6 \mod 8$

$$\mathrm{G}^{12}_{p,q} \simeq \begin{cases} \mathrm{Sp}(2^{\frac{n-2}{2}}), & (p,q) = (0,n),\ n \text{ is even;} \\ \mathrm{Sp}(2^{\frac{n-4}{2}}, 2^{\frac{n-4}{2}}), & n \equiv 2,4 \mod 8,\ p \neq 0; \\ \mathrm{O}(2^{\frac{n-2}{2}}, \mathbb{H}), & n \equiv 0,6 \mod 8; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-3}{2}}), & (p,q) = (0,n),\ n \text{ is odd;} \\ \mathrm{Sp}(2^{\frac{n-5}{2}}, 2^{\frac{n-5}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-5}{2}}, 2^{\frac{n-5}{2}}), & n \equiv 3 \mod 8,\ p \neq 0; \\ \mathrm{O}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{O}(2^{\frac{n-3}{2}}, \mathbb{H}), & n \equiv 7 \mod 8; \\ \mathrm{GL}(2^{\frac{n-3}{2}}, \mathbb{H}), & n \equiv 1,5 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 2,3,4 \mod 8$

$$\mathrm{G}^{2i1}_{p,q} \simeq \begin{cases} \mathrm{Sp}(2^{\frac{n-2}{2}}), & (p,q)=(n,0),\ n \text{ is even;} \\ \mathrm{Sp}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), & n \equiv 2,4 \mod 8,\ q \neq 0; \\ \mathrm{O}(2^{\frac{n-2}{2}},\mathbb{H}), & n \equiv 0,6 \mod 8; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-3}{2}}), & (p,q)=(n,0),\ n \text{ is odd;} \\ \mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), & n \equiv 3 \mod 8,\ q \neq 0; \\ \mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H}) \oplus \mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H}), & n \equiv 7 \mod 8; \\ \mathrm{GL}(2^{\frac{n-3}{2}},\mathbb{H}), & n \equiv 1,5 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 2,3,4 \mod 8$

$$\mathrm{G}^{2i3}_{p,q} \simeq \begin{cases} \mathrm{Sp}(2^{\frac{n-2}{2}}), & (p,q)=(0,n),\ n \text{ is even;} \\ \mathrm{Sp}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), & n \equiv 4,6 \mod 8,\ p \neq 0; \\ \mathrm{O}(2^{\frac{n-2}{2}},\mathbb{H}), & n \equiv 0,2 \mod 8; \\ \mathrm{Sp}(2^{\frac{n-3}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-3}{2}}), & (p,q)=(0,n),\ n \text{ is odd;} \\ \mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), & n \equiv 5 \mod 8,\ p \neq 0; \\ \mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H}) \oplus \mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H}), & n \equiv 1 \mod 8; \\ \mathrm{GL}(2^{\frac{n-3}{2}},\mathbb{H}), & n \equiv 3,7 \mod 8. \end{cases}$$

In the cases of signatures $p-q \equiv 3,4,5 \mod 8$

$$\mathrm{G}^{2}_{p,q} \simeq \begin{cases} \mathrm{Sp}(2^{\frac{n-3}{2}}), & (p,q)=(n,0),(0,n),\ n \text{ is odd;} \\ \mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), & n \equiv 3,5 \mod 8,\ p,q \neq 0; \\ \mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H}), & n \equiv 1,7 \mod 8; \\ \mathrm{Sp}(2^{\frac{n-4}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-4}{2}}), & (p,q)=(n,0),(0,n),\ n \text{ is even;} \\ \mathrm{Sp}(2^{\frac{n-6}{2}},2^{\frac{n-6}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-6}{2}},2^{\frac{n-6}{2}}), & n \equiv 4 \mod 8,\ p,q \neq 0; \\ \mathrm{O}(2^{\frac{n-4}{2}},\mathbb{H}) \oplus \mathrm{O}(2^{\frac{n-4}{2}},\mathbb{H}), & n \equiv 0 \mod 8; \\ \mathrm{GL}(2^{\frac{n-4}{2}},\mathbb{H}), & n \equiv 2,6 \mod 8. \end{cases}$$

We use the standard notation of classical matrix groups [7]

$$\begin{aligned} \mathrm{GL}(n,\mathbb{H}) &= \{A \in \mathrm{Mat}(n,\mathbb{H}) : \exists A^{-1}\}, \\ \mathrm{Sp}(n) &= \{A \in \mathrm{GL}(n,\mathbb{H}) : A^*A = \mathbf{1}\}, \\ \mathrm{Sp}(p,q) &= \{A \in \mathrm{GL}(p+q,\mathbb{H}) : A^*\eta A = \eta\}, \\ \mathrm{O}(n,\mathbb{H}) &= O^*(2n) = \{A \in \mathrm{GL}(n,\mathbb{H}) : A^* i\mathbf{1} A = i\mathbf{1}\}, \end{aligned}$$

where $*$ is the operation of conjugate transpose of matrix of quaternions.

Proof. Let us consider Lie group $\mathrm{G}^{23}_{p,q}$. In the cases $p-q \equiv 4,6 \mod 8$ we have $C\ell_{p,q} \simeq \mathrm{Mat}(2^{\frac{n-2}{2}},\mathbb{H})$. In the cases $p-q \equiv 5 \mod 8$ we have $C\ell_{p,q} \simeq \mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H})$ and we use block-diagonal representation β of Clifford algebra (see Section 1).

In the case of signature $(n,0)$ we have $U^\dagger = \tilde{U}$ and obtain $U^\dagger U = e$. We have $(\beta(U))^* = \beta(U^\dagger)$, where $*$ is the operation of conjugate transpose of matrices of quaternions. So we obtain isomorphism with the group $\mathrm{Sp}(2^{\frac{n-2}{2}})$ in the case of even n and isomorphism with the group $\mathrm{Sp}(2^{\frac{n-3}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-3}{2}})$ in the case of odd n.

Now let us consider the case $q \neq 0$. Let n be even. If p and q are even, then we have $U^\dagger = e_{p+1\ldots n}\tilde{U}e^{p+1\ldots n}$ and obtain $U^\dagger e^{p+1\ldots n}U = e^{p+1\ldots n}$. If p and q are odd, then we have $U^\dagger = e_{1\ldots p}\tilde{U}e^{1\ldots p}$ and obtain $U^\dagger e^{1\ldots p}U = e^{1\ldots p}$. Let us consider matrix $M = \beta^{p+1\ldots n}$ (or $\beta^{1\ldots p}$ in

the corresponding case). We have $M^* = M^{-1}$ and $M^2 = \pm\mathbf{1}$, so $M^* = \pm M$. It is known (see [28]) that any square quaternionic matrix of size m has the Jordan canonical form with (standard) m right eigenvalues (which are complex numbers with nonnegative imaginary parts) on the diagonal. If matrix is normal ($A^*A = AA^*$), then there exists a unitary matrix $T^* = T^{-1}$ such that T^*AT is a diagonal matrix with standard right eigenvalues on the diagonal. If this matrix is Hermitian, then these right eigenvalues are real. We obtain that there exists such element $T^* = T^{-1}$ that $\eta = T^{-1}MT$ or $i\mathbf{1} = T^{-1}MT$. We use transformation $T^{-1}\beta^a T = \gamma^a$ and obtain another matrix representation γ of Clifford algebra with $\gamma^{p+1\ldots n} = \eta$ or $i\mathbf{1}$ (or analogously for element $\gamma^{1\ldots p}$). We obtain $U^*\eta U = \eta$ or $U^*i\mathbf{1}U = i\mathbf{1}$ and isomorphisms with group $\mathrm{Sp}(2^{\frac{n-4}{2}}, 2^{\frac{n-4}{2}})$ or $\mathrm{O}(2^{\frac{n-2}{2}}, \mathbb{H})$. We take into account that $\dim \mathrm{G}^{23}_{p,q} = 2^{n-1} - 2^{\frac{n-1}{2}} \sin\frac{\pi(n+1)}{4}$, $\dim \mathrm{Sp}(2^{\frac{n-4}{2}}, 2^{\frac{n-4}{2}}) = 2^{n-1} + 2^{\frac{n-2}{2}}$ and $\dim \mathrm{O}(2^{\frac{n-2}{2}}, \mathbb{H}) = 2^{n-1} - 2^{\frac{n-2}{2}}$.

If $n \equiv 1,5 \mod 8$, then p is odd and q is even. So we can use formula $U^\dagger = e_{p+1\ldots n}\tilde{U}e^{p+1\ldots n}$ and obtain $U^\dagger e^{p+1\ldots n}U = e^{p+1\ldots n}$. Our considerations are like considerations in the case of even n but our matrix representation is block-diagonal. We obtain isomorphisms with groups $\mathrm{O}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{O}(2^{\frac{n-3}{2}}, \mathbb{H})$ or $\mathrm{Sp}(2^{\frac{n-5}{2}}, 2^{\frac{n-5}{2}}) \oplus \mathrm{Sp}(2^{\frac{n-5}{2}}, 2^{\frac{n-5}{2}})$ in different cases.

If $n \equiv 3,7 \mod 8$, then p is even and q is odd. So, we have $U^* = e_{1\ldots p}\tilde{\hat{U}}e^{1\ldots p}$ and $\tilde{U} = (e^{1\ldots p})^{-1}\hat{U}^*e^{1\ldots p}$. We have block-diagonal matrix representation β (see Section 1). The even part of an arbitrary Clifford algebra element has the form $\mathrm{diag}(A, A)$ and the odd part of an element has the form $\mathrm{diag}(B, -B)$. Then we obtain from $\tilde{U}U = e$:

$$(\mathrm{diag}(A-B, A+B))^* \mathrm{diag}(G, G)\mathrm{diag}(A+B, A-B) = \mathrm{diag}(G, G),$$

where $\mathrm{diag}(G, G)$ is matrix representation of element $e^{1\ldots p}$. So, we obtain $(A-B)^T G(A+B) = G$ and isomorphism with group $\mathrm{GL}(2^{\frac{n-3}{2}}, \mathbb{H})$ because for any invertible matrix A there exists such matrix B that $(A-B)^T G(A+B) = G$. Note, that in particular case $p = 0$ we have $U^* = \tilde{\hat{U}}$ and we obtain $\hat{U}^*U = e$, $(\mathrm{diag}(A-B, A+B))^*\mathrm{diag}(A+B, A-B) = \mathbf{1}$, $(A-B)^T(A+B) = \mathbf{1}$, and isomorphism with linear group again.

We can obtain isomorphisms for the group $\mathrm{G}^{12}_{p,q}$ analogously. The statement for the groups $\mathrm{G}^{2i1}_{p,q}$, $\mathrm{G}^{2i3}_{p,q}$ and $\mathrm{G}^{2}_{p,q}$ follows from Theorem 4.1. ■

5. Relation between group $\mathrm{G}^2_{p,q}$ and spin group

Note that spin group [12], [26]

$$\mathrm{Spin}_+(p,q) = \{U \in C\ell^{(0)}_{p,q} | \forall x \in C\ell^1_{p,q}, UxU^{-1} \in C\ell^1_{p,q}, \tilde{U}U = e\}$$

is a subgroup of all five considered Lie groups (9). Moreover, group $\mathrm{Spin}_+(p,q)$ coincides with group

$$\mathrm{G}^2_{p,q} = \{U \in C\ell^{(0)}_{p,q} \,|\, \tilde{U}U = e\}$$

in the cases of dimensions $n \leq 5$. Lie algebra $C\ell^2_{p,q}$ of Lie group $\mathrm{Spin}_+(p,q)$ is a subalgebra of algebras (10). Moreover, Lie algebra $C\ell^2_{p,q}$ coincides with Lie algebra $\overline{\mathbf{2}}$ in the cases of dimensions $n \leq 5$, because the notion of grade 2 and the notion of quaternion type 2 coincide in these cases.

Let us represent all isomorphisms of the group $\mathrm{G}^2_{p,q}$ in the following endless table. We use notation ${}^2\mathrm{Sp}(1) = \mathrm{Sp}(1) \oplus \mathrm{Sp}(1)$ and similar notation.

$p\diagdown q$	0	1	2	3	4	5	6	7
0	$\mathrm{O}(1)$	$\mathrm{O}(1)$	$\mathrm{U}(1)$	$\mathrm{Sp}(1)$	${}^2\mathrm{Sp}(1)$	$\mathrm{Sp}(2)$	$\mathrm{U}(4)$	$\mathrm{O}(8)$
1	$\mathrm{O}(1)$	$\mathrm{GL}(1,\mathbb{R})$	$\mathrm{Sp}(1,\mathbb{R})$	$\mathrm{Sp}(1,\mathbb{C})$	$\mathrm{Sp}(1,1)$	$\mathrm{GL}(2,\mathbb{H})$	$\mathrm{O}(4,\mathbb{H})$	$\mathrm{O}(8,\mathbb{C})$
2	$\mathrm{U}(1)$	$\mathrm{Sp}(1,\mathbb{R})$	${}^2\mathrm{Sp}(1,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{U}(2,2)$	$\mathrm{O}(4,\mathbb{H})$	${}^2\mathrm{O}(4,\mathbb{H})$	$\mathrm{O}(8,\mathbb{H})$
3	$\mathrm{Sp}(1)$	$\mathrm{Sp}(1,\mathbb{C})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{GL}(4,\mathbb{R})$	$\mathrm{O}(4,4)$	$\mathrm{O}(8,\mathbb{C})$	$\mathrm{O}(8,\mathbb{H})$	$\mathrm{GL}(8,\mathbb{H})$
4	${}^2\mathrm{Sp}(1)$	$\mathrm{Sp}(1,1)$	$\mathrm{U}(2,2)$	$\mathrm{O}(4,4)$	${}^2\mathrm{O}(4,4)$	$\mathrm{O}(8,8)$	$\mathrm{U}(8,8)$	$\mathrm{Sp}(8,8)$
5	$\mathrm{Sp}(2)$	$\mathrm{GL}(2,\mathbb{H})$	$\mathrm{O}(4,\mathbb{H})$	$\mathrm{O}(8,\mathbb{C})$	$\mathrm{O}(8,8)$	$\mathrm{GL}(16,\mathbb{R})$	$\mathrm{Sp}(16,\mathbb{R})$	$\mathrm{Sp}(16,\mathbb{C})$
6	$\mathrm{U}(4)$	$\mathrm{O}(4,\mathbb{H})$	${}^2\mathrm{O}(4,\mathbb{H})$	$\mathrm{O}(8,\mathbb{H})$	$\mathrm{U}(8,8)$	$\mathrm{Sp}(16,\mathbb{R})$	${}^2\mathrm{Sp}(16,\mathbb{R})$	$\mathrm{Sp}(32,\mathbb{R})$
7	$\mathrm{O}(8)$	$\mathrm{O}(8,\mathbb{C})$	$\mathrm{O}(8,\mathbb{H})$	$\mathrm{GL}(8,\mathbb{H})$	$\mathrm{Sp}(8,8)$	$\mathrm{Sp}(16,\mathbb{C})$	$\mathrm{Sp}(32,\mathbb{R})$	$\mathrm{GL}(64,\mathbb{R})$

Note that isomorphisms of the group $\mathrm{Spin}_+(p,q)$ are well-known (see, for example, [12], table on the page 224):

$p\diagdown q$	0	1	2	3	4	5	6
0	$\mathrm{O}(1)$	$\mathrm{O}(1)$	$\mathrm{U}(1)$	$\mathrm{SU}(2)$	${}^2\mathrm{SU}(2)$	$\mathrm{Sp}(2)$	$\mathrm{SU}(4)$
1	$\mathrm{O}(1)$	$\mathrm{GL}(1,\mathbb{R})$	$\mathrm{Sp}(1,\mathbb{R})$	$\mathrm{Sp}(1,\mathbb{C})$	$\mathrm{Sp}(1,1)$	$\mathrm{SL}(2,\mathbb{H})$	
2	$\mathrm{U}(1)$	$\mathrm{Sp}(1,\mathbb{R})$	${}^2\mathrm{Sp}(1,\mathbb{R})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{SU}(2,2)$		
3	$\mathrm{SU}(2)$	$\mathrm{Sp}(1,\mathbb{C})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{SL}(4,\mathbb{R})$			
4	${}^2\mathrm{SU}(2)$	$\mathrm{Sp}(1,1)$	$\mathrm{SU}(2,2)$				
5	$\mathrm{Sp}(2)$	$\mathrm{SL}(2,\mathbb{H})$					
6	$\mathrm{SU}(4)$						

So, in the cases $n \leq 5$ the tables coincide (note, that $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$). In the cases $n = 6$ condition $U^{-1}C\ell^1_{p,q}U \in C\ell^1_{p,q}$ in the definition of the group $\mathrm{Spin}_+(p,q)$ transforms into condition $\det\gamma(U) = 1$ for matrix representation γ and we obtain special groups.

Note that group $\mathrm{Spin}_+(p,q)$ in the cases $n \geq 7$ is not directly related to classical matrix groups (see [12], p.224). But we present classical matrix groups that contain this group in the cases of arbitrary dimension $n \geq 7$ and signature (p,q).

6. Conclusion

Using results of the present paper and [20], we can represent isomorphisms between considered five Lie groups and classical matrix groups in the following tables. There is $n \mod 8$ in the lines and there is $p-q \mod 8$ in the columns.

$$\mathrm{G}^{12}_{p,q} = \{U \in C\ell_{p,q} \,|\, \tilde{\tilde{U}}U = e\} :$$

$n\diagdown p-q$	$0,2$	$4,6$
$0,6$	$\mathrm{O}(2^{\frac{n-2}{2}},2^{\frac{n-2}{2}}), p \neq 0;$ $\mathrm{O}(2^{\frac{n}{2}}), p = 0.$	$\mathrm{O}(2^{\frac{n-2}{2}},\mathbb{H})$
$2,4$	$\mathrm{Sp}(2^{\frac{n-2}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), p \neq 0;$ $\mathrm{Sp}(2^{\frac{n-2}{2}}), p = 0.$

$n \diagdown p-q$	1	3,7	5
7	${}^2\mathrm{O}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), p\neq 0;$ ${}^2\mathrm{O}(2^{\frac{n-1}{2}}), p=0.$	$\mathrm{O}(2^{\frac{n-1}{2}},\mathbb{C})$	${}^2\mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H})$
3	${}^2\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{C})$	${}^2\mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), p\neq 0;$ ${}^2\mathrm{Sp}(2^{\frac{n-3}{2}}), p=0.$
1,5	$\mathrm{GL}(2^{\frac{n-1}{2}},\mathbb{R})$	$\mathrm{U}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), p\neq 0;$ $\mathrm{U}(\frac{n-1}{2}), p=0.$	$\mathrm{GL}(2^{\frac{n-3}{2}},\mathbb{H})$

$$\mathrm{G}^{23}_{p,q} = \{U \in C\ell_{p,q} \,|\, \tilde{U}U = e\} :$$

$n \diagdown p-q$	0,2	4,6
0,2	$\mathrm{O}(2^{\frac{n-2}{2}},2^{\frac{n-2}{2}}), q\neq 0;$ $\mathrm{O}(2^{\frac{n}{2}}), q=0.$	$\mathrm{O}(2^{\frac{n-2}{2}},\mathbb{H})$
4,6	$\mathrm{Sp}(2^{\frac{n-2}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), q\neq 0;$ $\mathrm{Sp}(2^{\frac{n-2}{2}}), q=0.$

$n \diagdown p-q$	1	3,7	5
1	${}^2\mathrm{O}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), q\neq 0;$ ${}^2\mathrm{O}(2^{\frac{n-1}{2}}), q=0.$	$\mathrm{O}(2^{\frac{n-1}{2}},\mathbb{C})$	${}^2\mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H})$
5	${}^2\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{C})$	${}^2\mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), q\neq 0;$ ${}^2\mathrm{Sp}(2^{\frac{n-3}{2}}), q=0.$
3,7	$\mathrm{GL}(2^{\frac{n-1}{2}},\mathbb{R})$	$\mathrm{U}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), q\neq 0;$ $\mathrm{U}(\frac{n-1}{2}), q=0.$	$\mathrm{GL}(2^{\frac{n-3}{2}},\mathbb{H})$

$$\mathrm{G}^{2i1}_{p,q} = \{U \in C\ell^{(0)}_{p,q} \oplus iC\ell^{(1)}_{p,q} : U^{\ddagger}U = e\} :$$

$n \diagdown p-q$	0,6	2,4
0,6	$\mathrm{O}(2^{\frac{n-2}{2}},2^{\frac{n-2}{2}}), q\neq 0;$ $\mathrm{O}(2^{\frac{n}{2}}), q=0.$	$\mathrm{O}(2^{\frac{n-2}{2}},\mathbb{H})$
2,4	$\mathrm{Sp}(2^{\frac{n-2}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), q\neq 0;$ $\mathrm{Sp}(2^{\frac{n-2}{2}}), q=0.$

$n \diagdown p-q$	7	1,5	3
7	${}^2\mathrm{O}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), q\neq 0;$ ${}^2\mathrm{O}(2^{\frac{n-1}{2}}), q=0.$	$\mathrm{O}(2^{\frac{n-1}{2}},\mathbb{C})$	${}^2\mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H})$
3	${}^2\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{C})$	${}^2\mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), q\neq 0;$ ${}^2\mathrm{Sp}(2^{\frac{n-3}{2}}), q=0.$
1,5	$\mathrm{GL}(2^{\frac{n-1}{2}},\mathbb{R})$	$\mathrm{U}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), q\neq 0;$ $\mathrm{U}(\frac{n-1}{2}), q=0.$	$\mathrm{GL}(2^{\frac{n-3}{2}},\mathbb{H})$

$$\mathrm{G}^{2i3}_{p,q} = \{U \in C\ell^{(0)}_{p,q} \oplus iC\ell^{(1)}_{p,q} : \hat{U}^{\ddagger}U = e\} :$$

$n \diagdown p-q$	$0,6$	$2,4$
$0,2$	$\mathrm{O}(2^{\frac{n-2}{2}},2^{\frac{n-2}{2}}), p\neq 0;$ $\mathrm{O}(2^{\frac{n}{2}}), p=0.$	$\mathrm{O}(2^{\frac{n-2}{2}},\mathbb{H})$
$4,6$	$\mathrm{Sp}(2^{\frac{n-2}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), p\neq 0;$ $\mathrm{Sp}(2^{\frac{n-2}{2}}), p=0.$

$n \diagdown p-q$	7	$1,5$	3
1	${}^2\mathrm{O}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), p\neq 0;$ ${}^2\mathrm{O}(2^{\frac{n-1}{2}}), p=0.$	$\mathrm{O}(2^{\frac{n-1}{2}},\mathbb{C})$	${}^2\mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H})$
5	${}^2\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{C})$	${}^2\mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), p\neq 0;$ ${}^2\mathrm{Sp}(2^{\frac{n-3}{2}}), p=0.$
$3,7$	$\mathrm{GL}(2^{\frac{n-1}{2}},\mathbb{R})$	$\mathrm{U}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), p\neq 0;$ $\mathrm{U}(\frac{n-1}{2}), p=0.$	$\mathrm{GL}(2^{\frac{n-3}{2}},\mathbb{H})$

$$\mathrm{G}^2_{p,q}=\{U\in C\ell^{(0)}_{p,q}\,|\,\tilde{U}U=e\}:$$

$n \diagdown p-q$	$1,7$	$3,5$
$1,7$	$\mathrm{O}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), p,q\neq 0;$ $\mathrm{O}(2^{\frac{n-1}{2}}),(n,0),(0,n).$	$\mathrm{O}(2^{\frac{n-3}{2}},\mathbb{H})$
$3,5$	$\mathrm{Sp}(2^{\frac{n-3}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-5}{2}},2^{\frac{n-5}{2}}), p,q\neq 0;$ $\mathrm{Sp}(2^{\frac{n-3}{2}}),(n,0),(0,n).$

$n \diagdown p-q$	0	$2,6$	4
0	${}^2\mathrm{O}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), p,q\neq 0;$ ${}^2\mathrm{O}(2^{\frac{n-2}{2}}),(n,0),(0,n).$	$\mathrm{O}(2^{\frac{n-2}{2}},\mathbb{C})$	${}^2\mathrm{O}(2^{\frac{n-4}{2}},\mathbb{H})$
4	${}^2\mathrm{Sp}(2^{\frac{n-4}{2}},\mathbb{R})$	$\mathrm{Sp}(2^{\frac{n-4}{2}},\mathbb{C})$	${}^2\mathrm{Sp}(2^{\frac{n-6}{2}},2^{\frac{n-6}{2}}), p,q\neq 0;$ ${}^2\mathrm{Sp}(2^{\frac{n-4}{2}}),(n,0),(0,n).$
$2,6$	$\mathrm{GL}(2^{\frac{n-2}{2}},\mathbb{R})$	$\mathrm{U}(2^{\frac{n-4}{2}},2^{\frac{n-4}{2}}), p,q\neq 0;$ $\mathrm{U}(\frac{n-2}{2}),(n,0),(0,n).$	$\mathrm{GL}(2^{\frac{n-4}{2}},\mathbb{H})$

Note that if we know isomorphisms between these Lie groups, then we know isomorphisms between corresponding Lie algebras. So, we also obtain isomorphisms between Lie algebras $\overline{\mathbf{2}}\oplus i\overline{\mathbf{1}}$, $\overline{\mathbf{2}}\oplus i\overline{\mathbf{3}}$, $\overline{\mathbf{2}}\oplus\overline{\mathbf{3}}$, $\overline{\mathbf{2}}\oplus\overline{\mathbf{1}}$, $\overline{\mathbf{2}}$ and corresponding classical matrix Lie algebras: linear $\mathrm{gl}(k,\mathbb{R})$, $\mathrm{gl}(k,\mathbb{H})$, unitary $\mathrm{u}(k)$, $\mathrm{u}(r,s)$, orthogonal $\mathrm{so}(k,\mathbb{R})$, $\mathrm{so}(r,s)$, $\mathrm{so}(k,\mathbb{C})$, $\mathrm{so}(k,\mathbb{H})$, symplectic $\mathrm{sp}(k)$, $\mathrm{sp}(r,s)$, $\mathrm{sp}(k,\mathbb{R})$, $\mathrm{sp}(k,\mathbb{C})$ or direct sums of such Lie algebras of corresponding dimensions in different cases.

As we have already mentioned in the Introduction, group $G^{\varepsilon}_{p,q}$ [2] coincides with the group $\mathrm{G}^{23}_{p,q}$ in the case of signature $(n,0)$ and with the group $\mathrm{G}^{12}_{p,q}$ in the case of signature $(0,n)$. In [12] (p. 236, Tables 1 and 2) one finds isomorphisms between classical matrix Lie groups and the groups $\mathrm{G}^{12}_{p,q}$ and $\mathrm{G}^{23}_{p,q}$ which are automorphism groups of the scalar products on the spinor spaces. In the present paper we use another technique. We obtain the same isomorphisms for groups $\mathrm{G}^{12}_{p,q}$ and $\mathrm{G}^{23}_{p,q}$ and also isomorphisms for groups $\mathrm{G}^{2i1}_{p,q}$, $\mathrm{G}^{2i3}_{p,q}$, $\mathrm{G}^{2}_{p,q}$ using the notion of additional signature and relations between operations of conjugation in Clifford algebra and the corresponding matrix operations. We also study the corresponding Lie algebras which are related to subspaces of quaternion types.

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WHAT THE KÄHLER CALCULUS CAN DO THAT OTHER CALCULI CANNOT

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Abstract. The progress of mathematics makes it sometimes fashionable to describe physical theory in more modern forms, not always deeper or more understandable. Of particular interest in this regard is relativistic quantum mechanics. Modern versions of it, like through the use of geometric calculus (mainly the work of Hestenes), may be more appealing than the original version of Dirac theory, but the physical contents remains virtually unchanged.

Enter the Kähler calculus (KC). Underlied by Clifford algebra of differential forms — like tangent Clifford algebra underlies the geometric calculus— it brings about a fresh new view of quantum mechanics. This view arises, almost without effort, from the equation which is in KC what the Dirac equation is in traditional quantum mechanics. One does not need to first hypothesize foundations of quantum mechanics, which makes the Dirac version unintelligible even when one is adept at computing with it. Many foundations come in the wash from the mathematics and very little additional input. Not only Kähler theory reproduces the main Dirac-related results more elegantly, but does so far more profoundly and shows the way to further developments. In this paper, we shall deal with differences between the Dirac and Kähler versions and, to a lesser extent, between Kähler and Hestenes.

We limit ourselves to scalar-valued differential forms. That is all that one needs to supersede the Dirac and geometric calculus versions of relativistic quantum mechanics. Hence, we shall give just a very brief inkling of KC with post-scalar-valued differential forms. Non-scalar-valuedness is needed for a unification of quantum and classical physics since the curvature and Einstein tensors are inadequate representations of what by their very nature are bivector-valued differential 2-forms and vector-valued differential 3-forms respectively.

We shall be specific about gems, both mathematical and (mainly) physical, contained in this calculus. We shall also explain the mathematical philosophy of Kähler on a variety of issues (vector fields, differential forms, Lie differentiation, unification of derivatives, product of tangent algebras with algebra of integrands). Not surprisingly, his philosophy is the same as É. Cartan's. We shall also illustrate how some of the results that one achieves with KC supersedes the less sophisticated results which one finds deep into very specialized books on group theory, harmonic function theory, complex variable theory, cohomology theory, relativistic quantum mechanics and even particle theory.

1. Introduction

This paper is being written in response to the unawareness of what the Kähler calculus (KC) has to offer, specially for the advancement of the physics paradigms. In section 2, we speak of this offer in connection with quantum physics, and, in section 3 in connection with physics unification, though only in relation to great overlooked ideas of the 20^{th} century that appear to come together under the KC.

It is a calculus where deep results, both in physics (Section 4) and mathematics (Section 5) follow from just a few definitions. It has not received the attention it deserves. It may be due to the language barrier (papers in German), or to a style that is no longer in use, or, in quantum physics, to misinterpretations, as shown in section 6. In section 7, we shall compare the Kähler calculus with the geometric calculus. In section 8, we explain the mathematical philosophy of Kähler, as it can help understand his work. In section 9, we enumerate a few specialized mathematics and physics books some of whose results are matched or superseded by relatively short proofs that do not require previous knowledge of the subject matter of those books.

2. Brief highlight of Kähler's Quantum Mechanics, meant for quantum physicists

This section is a very brief description of Kähler's Quantum Mechanics, which is a virtual concomitant of the KC. A more detailed description is to be found in Sec. 4.

Assume that you knew KC but not physics and that you asked yourself to solve equations of the form

$$\partial u = au, \tag{1}$$

where we do not need to know details other than ∂ is some Dirac-type derivative operator, that a and u are differential forms (respectively input and output) and that their juxtaposition means their Clifford product. Solutions u may be members of the Clifford algebra which are not necessarily members of ideals in this algebra. Assume further that, to start with, you choose a to be of the type $m + eA$ where A is a differential 1-form and where m and e are constants. Seek solutions in ideals $u\epsilon^{\pm}$ defined by idempotents $\epsilon^{\pm}$ related to time translation symmetry, $\epsilon^{\pm} = \frac{1}{2}(1 \mp idt)$. You can absorb dt in $\epsilon^{\pm}$, and, in particular the dt's in ∂u and au in any of the two equations

$$\partial(u\epsilon^{\pm}) = (m + eA)(u\epsilon^{\pm}). \tag{2}$$

The imaginary factor i has been ignored to avoid distractions, but should be there. clean this equation to, mainly, absorb dt and to leave just $\partial(u\epsilon^{\pm})/\partial dt$ on the left. With little effort, you get the Pauli–Dirac equation as a first approximation, and the Foldy-Wouthuysen transformation in the immediately following second approximation.

Assume on the other hand that, independently of those results, you specialize (2.2) to $A = -(edt/r)$ and solve the equation. You get the fine structure of the hydrogen atom. In neither of these two exercises there is a need for Pauli or Dirac matrices. Nowhere appears the need for negative energy solutions, since what makes the positron a positron is its pertaining to the left ideal defined by ϵ^{+}, not negative energies. Spinors and Hilbert spaces are emerging concepts, as they come in the wash of solving equation (1). Operator theory is not necessary for development of quantum mechanics, as these operators come embedded in the computations, each in its own idiosyncratic way. The concept of probability amplitude also would be an emerging concept, rather than one belonging to the foundations of quantum physics. The reason is that, in the basic Eq. (1), there are not even particles. This has to do with the conservation law, which has to do with two densities at the same time. This has to be seen as pertaining to a magnitude with two opposite signs. clearly this must be charged, as confirmed by results as those of which we have spoken above. This deserves more detailed explanation because it touches very explicitly the Copenhagen interpretation of Quantum Mechanics

A conservation law of a scalar-valued magnitude is usually given the form

$$\frac{\partial \rho}{\partial t} + div\ \mathbf{j} = 0. \tag{3}$$

In quantum mechanics, ρ and $\mathbf{j}$ are built from the wave function (non-relativistic) or the spinor solution of a Dirac equation. In the KC, we can always write

$$u = {}^{+}u\ \epsilon^{+} + {}^{-}u\ \epsilon^{-}. \tag{4}$$

The conservation law obtained from "basic KC theory" (see chapter 6 when posted in this web site) takes the form

$$\frac{\partial \rho_1}{\partial t} + div\ \mathbf{j}_1 + \frac{\partial \rho_2}{\partial t} + div\ \mathbf{j}_2 = 0, \tag{5}$$

where becomes ρ_1 and ρ_2 are at each and every point non-negative and non-positive. Of course, if a system is such that ${}^{+}u$ is zero, you only have "field of negative charge". The probability amplitude is what the negative charge amplitude ${}^{-}u$ looks like.

Probability amplitude thus is a derived concept. It cannot be taken as a basic tenet for an interpretation of quantum mechanics, specially since it was adopted in the conceptual fog that accompanied the birth of quantum mechanics.

3. A new paradigm that the Kähler calculus could make possible

There is synergy between the KC and great overlooked ideas in mathematics and physics in the 20th century, as well as new experimental evidence coming from microelectronics and which contradicts the limited and retrospectively inadequate experimental evidence on which the present theoretical paradigm was built. Imagine that one could put together those ideas in a way consistent with a more refined experimental evidence. That may be possible through the Kähler calculus. You be the judge.

During my first experience as a graduate student, the business of negative energy solutions and an infinite sea of such states made me loose interest in working at the cutting age of physics. I even left graduate school. Years later, I did return to a graduate program. Getting a Ph.D. is much easier when you have a passion for some topic in which there is not much competition, you have time and you have already published in refereed journals. That happened to me. I was a free spirit not constrained by what the National Science Foundation funded. It is a long story. But the success that I have had in disparate non-main stream field of physics was made possible by job opportunities that no longer exist, and by the support in different ways that Drs. Douglas and Marsha Torr provided me over decades. Thanks to that, I saw new opportunities for superseding physics in more than one area. Eventually, all that came together to form a picture of which I shall try here to give a glimpse. Unfortunately, the world of physics nowadays is a far more hostile environment for free spirits, which I have been. It is thus important to make public new profound scenarios like the one I uncovered, largely because of the right choice of mathematical tools.

Physicists have made great strides with the type of quantum physics with which I had problems. I do not regret this progress. After all, it does not hurt to have the enormous experimental evidence that development of the Dirac theory has brought. But one has to reinterpret it from the perspective of the far move relevant and reliable experimental evidence brought about by the microelectronics revolution. Like the Kähler calculus, it contradicts main tenets of the Bohr-Dirac-Feynman philosophy of quantum physics.

Let me now proceed with some of the great ideas announced in the title of this section, because they are precursors of a new vision of physics, whose developments have barely started.

3.1. **Julian Schwinger and source theory.**

Schwinger's source theory may be seen as a proxy for what the Kähler calculus will become when used to address the same issues. But it has received far less attention than it deserves. Source theory is difficult to define. Its major attractiveness is that the results of quantum electrodynamics are reproduced

without the irrelevance of divergent quantities and renormalizations". It emphasizes spacetime, but it is not operator field theory. Of course operators will nevertheless play some role; they also do in the KC, but not as fundamentally as in quantum field theory.

Like S-matrix theory, it also has phenomenological emphasis, which we do not view with enthusiasm. But the phenomenology might look less so when approached with the more formal perspective that the Kähler calculus provides. For instance, what we call the dominant energy-approximation, is not even conceived as an approximation in Dirac's equation where mass is an essential ingredient; it is not so in Kähler's theory, where it comes in because of phenomenology, even if very basic one. The day one shall know enough to actually compute the mass of the electron, we shall be justified in considering masses as pertaining to a higher category than phenomenology.

Schwinger points out that

> "... in general, particles must be created in order to study them, since most of them are unstable. In a general sense this is also true of high-energy stable particles, which must be created in that situation by some device, i.e. an accelerator. One can regard all such creation acts as collisions, in which the necessary properties are transferred from other particles to the one of interest... The other particles in the collision appear only to supply these attributes. They are, in an abstract sense, the source of the particle in question... We try to represent this abstraction of realistic processes numerically..."

And further down, he writes:

> "Unstable particles eventually decay and the decay process is a detection device. More generally, any detection device can be regarded as removing or annihilating the particle. Thus the source concept can again be used as an abstraction ... with the source acting negatively, as a sink."

What is the abstraction? Speaking of the creation of a particle with specified properties in a collision, Schwinger has this to say:

> " ...the source concept is the abstraction of all possible dynamical mechanisms whereby the particular particle can be produced."

A shallow immersion in source theory is all that one needs to realize that it is a calculus of integrals. In the KC, these are evaluations (read integrations) of differential forms. It has the flavor of the KC, not of operator field theory. In the second page where equations are given in his first book on source theory, he states:

> "To specify a weak source , we consider its effectiveness sin crating a particle with momentum $\mathbf{p}$, in the small range $(d\mathbf{p})$.An invariant measure of momentum space is
>
> $$d\omega_p = \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0,}, \qquad\qquad p_0 = +\sqrt{\mathbf{p}^2 + m^2.}$$
>
> We now define the source K in terms of the creation and annihilation probability amplitudes
>
> $$\langle 1_p \mid 0_- \rangle^K = \sqrt{d\omega_p} iK(p),$$
>
> $$\langle 0_+ \mid 1_p \rangle^K = \sqrt{d\omega_p} iK(-p),$$
>
> which conveys the idea that the source liberates or absorbs momentum $\mathbf{p}$ in the respective processes."

Of course, this is not Kähler notation but certainly illustrates the role that integrands play in source theory, as in the KC. Also significant is his principle of unity of the source, because it embodies a postulate with the same flavor as the Kähler equation in his general form, i.e. before we introduce a mass term (or any other specific term) in the input a. We shall come back to this. Let us say what Schwinger says in this regards. He considers what he calls a complete situation, namely one where particles are created by sources K_2, propagate in space and time and are detected by K_1. After a small computation yielding

$$\begin{aligned}\langle 0_+ \mid 0_- \rangle^K &\cong 1 + O(K^1)^2 + O(K^1)^2 \\ &\quad + \int d\omega_p \int (dx)(dx') iK_1(x) e^{ip(x-x')} iK_2(x'),\end{aligned}$$

he states:

> "We regard $K_1(x)$ and $K_2(x)$ as manifestations of the same physical mechanism, that is, they are the values of one general source in different spacetime regions. Therefore the only possible combination that can occur is the total source
>
> $$K = K_1 + K_2.$$
>
> This is a fundamental postulate, the principle of the unity of the source, which embodies the idea of the uniformity of nature."

This disquisition by Schwinger has the flavor of what the general Kähler equation is. Mass can only enter in applications. The Kähler equation that one should dream of has to be one where mass does not enter, since it must be valid anywhere. But practical calculations should not take place with that dream equation, certainly not if it is not even known For practical calculations, we choose an input, like when we use the dominant energy approximation. We shall hear a lot about this concept in the KC because this is what relativistic quantum mechanics is about (See chapter 4 in the web site of the Alterman event). Schwinger toys with the same idea, but at the level of interactions, rather than at a level of detail evolution of a system.

3.2. **Carver Mead and the concepts of electrons and photons.** Let us next deal with the conceptual revolution subjacent in the explosion of microelectronics knowledge of the last half a century. Carver Mead is emeritus professor at Caltech, main brain behind this microelectronics revolution. He is the 1999 Lemelson-MIT Prize for Invention and Innovation. But that is only a very small description of his many credits. Please google his name. He has in common with Schwinger that they are against the paradigm's description of the quantum world in terms of point particles and operators.

For Mead, an electron has the property of adapting to its environment, be it a hydrogen atom or a wire. He claims that experiments are regularly performed with neutrons that are one foot across. In his laboratory, he can make electrons that are ten feet long. He makes statements like "The electron ... is the thing that is wiggling, and the wave is the electron". His use of the term wave is not the standard one of classical electrodynamics (He would use the term non-coherent rather than classical). This characterization of electrons as waves is crucial in order not to misunderstand him in what follows.

Asked what should we think of a photon, Mead had this to say: "*John Cramer at the University of Washington was one of the first to describe it as a transaction between two atoms*". Then he was asked: "So that transaction is itself a wave?" Response: "The field that describes that transaction is a wave, that is right".

This vision of microphysics is totally at odds with present Heisenberg-Dirac-Feynman type formulation of quantum physics, and of cutting edge theories based on auxiliary bundles not directly related to the tangent bundle. It is consistent with the view that we have spoused at the end of the previous subsection. There is not such a think as

point particles. These are regions of space where the "background field" presents some concentrations of some sort, with some well defined algebraic (member of an ideal) or topologic-geometric invariant. To think of a photon as some kind of particle is stretching things too far. Things looked that way a century ago, when experimental technique was so primitive, as Mead would argue. And who could speak more authoritatively than he did about situations in an electronics laboratory? The photon is the field that is being transacted There will be some region —fuzzy or not, changing or not, it does not matter— where the transaction is taking place, not just a point of impact?

Those auxiliary bundles are directly tied to the point particle approach of modern physics since you cannot do tangent bundle physics, i.e. regular differential geometry, with them. They are a back road to the ideal of having theoretical physics be differential geometry. After all, is that not what Yang-Mills theory seeks to do? The problem is: what is behind those auxiliary bundles to which Yang-Mills theory resorts? Is there a need for them, or it is simply a matter of using them because one does not know better?. One certainly cannot do better if one does not first imagine what things could look like.

KC for quantum physics is in tune with a hypothetical geometrization of classical physics directly related to the tangent bundle. So it provides the connection between differential geometry and quantum physics. Why is that so?

From the equations of structure of a differential manifold, only the torsion is available for the electromagnetic field. In Riemann-plus-torsion geometry, one cannot match a two-index quantity (here electromagnetic field), with one which has three indices (here torsion). But the matching is possible in Finslerian structures. One can match the two differential form indices of the torsion with those of the electromagnetic field. Hence the 4-potential must be viewed as a differential 1-form, not as a 4-vector. This advocates Kähler's quantum mechanics, not Dirac's.

We now start to show the path towards the connection between classical and quantum physics.

3.3. **É. Cartan and the concept of differential form in electrodynamics.** The point just made about the mathematical language for physics is so important that we reinforce the argument for the KC calculus in quantum physics with an overlooked study that Cartan did of this topic in his second paper on the theory of affine connections. After representing the electromagnetic quantities in terms of differential forms, he argued that Maxwell's equations should not be viewed as relations at each point of the components of the quantities that enter those equations, i.e. the differential forms. They should be viewed as relating integrals, which is equivalent to viewing them as relations of integrands, not of antisymmetric multilinear functions of vectors. Does this not have the flavor of sources (extended objects rather than points) and of the Kähler calculus?

With the quotation that follows, Cartan then begins a discussion electromagnetic energy-momentum, where vector-valuedness becomes of the essence:

> "But Maxwell's equations (8) do not provide all the laws of electromagnetism.
>
> One knows that in Lorentz theory there is an electromagnetic energy-momentum that is represented by a sliding *vector*..."

At that point Cartan starts to use differential forms that are not scalar-valued.

Since classical physics requires the use also of non-scalar-valued differential forms (gravitation theory for sure) and classical physics is what it is because quantum physics is what it is, a comprehensive KC and concomitant physics will have to be one for non-scalar-valued differential forms, the essential objects in differential geometry. Kähler did not go with his calculus beyond simply defining a Dirac type equation for tensor-valued differential forms. It is not our intention to go into any valuedness other than scalar-valuedness

in this summer school, except in the very last day. And there are not other applications at this point except first results on unification of the non-gravitational interactions and the algebraic representation of leptons and quarks through ideals defined by primitive idempotents, beyond what Kähler did in this regard. But the right decision as to what type of valuedness is required comes from the interplay of Kähler's quantum physics with the classical geometrization of the electromagnetic interaction. That is where Einstein comes in the picture with his attempt at unification with teleparallelism (TP). Einstein's is not the only way. In fact, this author got into TP not as a postulate but through the study of the Lorentz force and Finsler geometry. We shall emphasize the Einstein route because his idea was fabulous and because, in the process, Cartan gets again in the picture.

3.4. **Einstein and Cartan on teleparallelism.** Einstein postulated teleparallelism, i.e. equality of vectors at a distance, because there was no equality of vectors at a distance in his general relativity theory. There was not equality in 1915, year of the birth of general relativity, because the concept of comparison of vectors at a distance was born with the Levi-Civita connection in 1917. but Levi-Civita's is a path-dependent comparison and does not, therefore, qualify as equality. The practical implementation of such equality is through the postulate of annulment of the affine curvature. The Riemannian curvature of a manifold with a metric remains in place, but only in a metric role, as before 1915, not in a metric role, as after 1917. It is obvious from the Cartan-Einstein correspondence that Cartan did no understand that. Einstein believed that the right choice of geometry would bring about the geometrization of electrodynamics, unification with gravity and possibly an alternative to the quantum physics in the Bohr mold.

Einstein's failed in his attempt at physical unification with the **postulate of TP**. And he did so because he did have only a very vague idea of how to connect TP with his **thesis of** what he called **logical homogeneity** of differential geometry and theoretical physics, and with his **view of particles as special regions of the field.** His postulate, thesis and view were in the right track, as later developments in differential geometry have shown. He did not listen at all to Cartan when, on December 3rd 1929, the latter rightly told him that certain identity used as an equation in physics ——which happens to be the first Bianchi identities when there is TP— had to be present in his system of equations. Neither did he pay much attention when in letter of February 2, 1930, Cartan told him the relation of the Ricci tensor to the torsion and its derivatives; this is information contained in the second equation of structure. Cartan was giving Einstein advice consistent with the thesis of logical homogeneity in almost pure form, which takes the form: make the equations of structure and Bianchi identities part of you field equations. But Cartan was not aware of the relation between electromagnetic field and torsion. The mathematics did not yet exist for understanding this relation. It was created precisely by Cartan a few years later with his theory "of metric-Finsler connections". We have used the prefix "metric" because Finslerian connections exist on Riemannian metrics; the Finslerian character of a geometry resides in the type of fibration of its bundle. Cartan put one foot in the door for an understanding of Finsler bundles, but it was an ignored differential topologist by the name of Yeaton H. Clifton —highly praised by S.S. Chern— who pushed the door wide open. If one is used to the thinking of É. Cartan i.e. doing geometry in the bundle rather than its sections, the Finslerian generalization of standard differential geometry is rather straight forward.

3.5. **Yeaton H. Clifton and the geometrization of electrodynamics.** Enter Clifton, a rather eccentric mathematician who did not care to publish. To remove this difficulty, Douglas G. Torr and myself published in his name. We did so with the title of one of

three papers of which he was the major contributor by far: "Finslerian Structure: the Cartan-Clifton method of the moving frame".

An explanation seems in order here. A reviewer of the second of those papers objected to our having used the name of Clifton in referring to the method, as if Cartan had been the creator of the method. But this is incorrect. The names Serret, Frenet, Demoulin, Ribaucour, Cotton and Darboux come to my mind as having been associated with moving frames. Cartan brought the method to new heights with his use in formulating modern differential geometry, aside from the fact that he was by far the best known of the mathematician associated with it. But this use was not viewed as rigorous and a horrible herd of hackers phagocytized it and replaced it with the present Babel tower of formulations of differential geometry, whose authors have missed the essential ideas of Cartan, and of the Erlangen program, and of the theory of integrability of exterior systems, etc. Clifton made the theory of affine connections rigorous, without loosing any of the flavor in Cartan's original work. But not only that. He also rigorously defined Finslerian connections, regardless of whether a metric is defined on it, or whether it is Riemannian or properly Finslerian. This is why his name should be attached to the so called Cartan's theory of moving frames.

Clifton did not quite realize that the equations of the autoparallels of Finsler connections contain the Lorentz force, certainly in addition to the gravitational force. Let us explain. A Finslerian torsion is of the form

$$\mathcal{R} = (R^{\mu}_{\nu\lambda}\omega^{\mu} \wedge \omega^{\lambda} + S^{\mu}_{\nu l}\omega^{\nu} \wedge \omega^{l}_{\nu})\boldsymbol{e}_{\mu}, \tag{6}$$

Greek and Latin indices being $(0, 1, 2, 3)$ and $(1, 2, 3)$ respectively. Let us ignore the S terms. The torsion then looks as if it pertains to the standard (non-Finslerian) connections with torsions. But they is not quite correct since neither ω^0 nor $\boldsymbol{e}_0$ mean in the Finsler bundles what they mean in the standard bundles. And here comes the amazing feature. Regardless of what the connection with torsion is, the $R^{i}_{\nu\lambda}$ do not contribute to the equation of the autoparallels; only the $R^{0}_{\nu\lambda}$ do. Furthermore, this equation takes the form of the equation of motion with Lorentz force with Ω^0 playing the role of F. Of course, the factor q/m does not appear, which is the reason why some claim that it is not possible to geometrize the equation of motion of electrodynamics. Where there is a solution, naysayers see a problem. A factor such as as q/m is compatible in principle with geometry because a particle is a configuration of a field (Here is Einstein again) with a well defined torsion, which does not contribute to its own acceleration. Of course, the art of geometrization has not yet reached the degree of sophistication necessary to deal with this issue. And, what a coincidence, as Kähler showed charge and energy are both conserved quantities under time translation symmetry.

3.6. **Of Kaluza-Klein space and Cartan-Clifton on Finsler bundles.** Here is an inkling of things to come. Enter again E. Cartan. In 1922, he did a straightforward computation that shows that the theory of connections is a theory of just moving frames, not frames and particles in an equal footing. One can get particles in the equations of structure by making propertime, τ, a fifth dimension, not just a concept applicable to curves. That means that the four-velocity, $\mathbf{u}$, is now outside spacetime, since it is the fifth element of a basis of vector tangent to the time-space-propertime manifold. $d\tau$ thus is a horizontal differential form dual to $\mathbf{u}$. And it further happens that the arena of quantum physics is not the subspace (t, x^i) but (x^i, τ). This is so because, for example, a hydrogen atom is identical to itself in spite of its state of inertial motion (that is, in flat spacetime, with constant velocity). This is similar to what happens in Finsler geometry where we have

$$d\mathbf{u} = d\boldsymbol{e}_0 = \omega^{i}_{0}\boldsymbol{e}_i. \tag{7}$$

There is here not only the information that $d\mathbf{u}$ equals $d\boldsymbol{e}_0$, but also that the $d\boldsymbol{e}_0$ and thus $d\mathbf{u}$ are horizontal invariants. For those who are not very knowledgeable in differential geometry suffice to say that, for the standard spacetime bundle, the horizontal invariants are the ω^μ's. For the Finslerian spacetime bundle, they are the ω^μ_i and the ω^i_0 (Seven "translation" differential forms, like seven is the dimension of the time-space-propertime manifold). This opens the door for the geometrization of quantum physics and unification with classical physics.

A project for a whole new paradigm lies in front of us. The main ideas have been supplied by a constellation of geniuses: Kähler, Schwinger, Mead, Einstein, E. Cartan and Clifton. Is there motivation for such a project? An increasing number of physicists think that the present paradigm is virtually exhausted. Others, who spend time in laboratories rather than in offices of ivory towers would side with Carver Mead, who, in year 2000, opened the first chapter of his book "Collective Electrodynamics" with the following statements: "It is my firm belief that the last seven decades of the 20^{th} century will go down in history as the dark ages of theoretical physics" (now eight and a half and counting).

4. Gems of the Kähler calculus for mathematical analysis

Kähler calculus contains important contributions to analysis and to the understanding of the foundations of certain areas of mathematics. The rich ensemble of results that he obtained, both in quantum physics and mathematics. The components of his tensor-valued differential forms have three series of indices , two of them of subscripts. This speaks of the subtleties of his treatment, which, for instance, distinguishes between differential forms and antisymmetric multilinear functions of vectors fields. His differential forms are integrands, i.e. functions of $r-$surfaces. In other words, we we have to distinguish between antisymmetric multilinear functions of vector fields and integrands, i.e. functions of r-surfaces. Acting on the first ones, Kähler's (as well as Cartan's) operator d yields covariant derivatives. It yields exterior derivatives acting upon the second ones. Because of Cartan's lesser use of tensor-calculus-like use of components, this distinction was less explicit in his work.

Lie differentiation of differential forms is another case in point. His treatment shows that it is a matter of knowing your partial derivatives well. it, and it helps one to better understand that it is dangerous (I am not saying it should not be done!) to deal with sums of terms involving differentials and where differentiations take place leaving constants different sets of coordinates. The brief incursion that follows in the next paragraph should suffice to illustrate this.

Consider $\frac{\partial}{\partial\phi}dx^l$, where the x^l is a Cartesian coordinate and ϕ is the azymuthal coordinate (say as in the spherical and cylindrical systems as well as in an infinity of other systems) Let us denote the coordinates of any such system as y^i, and let ϕ be y^n. We have

$$\frac{\partial}{\partial\phi}dx^i = \frac{\partial}{\partial y^n}\frac{\partial x^i}{\partial y^l}dy^l = \frac{\partial}{\partial y^l}\frac{\partial x^i}{\partial y^n}dy^l = d\frac{\partial x^i}{\partial\phi} = d\alpha^i, \tag{8}$$

where we have defined α^i as $\partial x^i/\partial\phi$. Let u be $u_l dx^l$ and compute $\partial u/\partial\phi$ where the u_l are functions of the x's. We have

$$\frac{\partial}{\partial\phi}u = \frac{\partial}{\partial\phi}(u_i dx^i) = \frac{\partial u_i}{\partial\phi}dx^i + u_i\frac{\partial dx^i}{\partial\phi}. \tag{9}$$

But

$$\frac{\partial u_i}{\partial\phi}dx^i = \frac{\partial u_i}{\partial x^l}\frac{\partial x^l}{\partial\phi}dx^i = \alpha^l\frac{\partial u}{\partial x^l}. \tag{10}$$

Hence

$$\frac{\partial u}{\partial \phi} = \alpha^l \frac{\partial u}{\partial x^l} + d\alpha^i u_i. \tag{11}$$

If we had followed the same process with a differential form of grade greater than one, we would have obtained

$$\frac{\partial u}{\partial \phi} = \alpha^l \frac{\partial u}{\partial x^l} + d\alpha^i \wedge e_i u. \tag{12}$$

The right hand side is known as the Lie derivative of u by the Lie operator $\alpha^i \partial/\partial x^i$, equivalently, by $\partial/\partial\phi$. But why even bother about defining a concept of Lie derivative? Kähler showed how to find ϕ for any given combination $\alpha_i(x)\partial/\partial x^i$.

Of course, the above is well known by experts on the theory of differential equations. The problem is that they only know these things which any undergraduate student in physics and mathematics should learn. As we shall see in a later section, this treatment of Lie differentiation has implications even for the foundations of quantum mechanics, as it establishes a difference between the Dirac and Kaehler formulations of phase factors for rotational symmetry. This difference raises the issue of what may be behind the $\hbar$. More on this later on (but just a little bit).

He went on to further develop the last equation, whose last two terms are not covariant. He converted the ringt hand side of the equation into the sum of two covariant terms. Then, if the metric does not depend on ϕ, the new terms thus obtained are for the field u what the orbital and spin angular momenta are for particles.

Another piece of promising mathematical analysis is constituted by his obtaining the strict harmonic differentials in 3-D Euclidean space punctured at the point that is used as the origin of coordinates. All that was done by Kähler. We now proceed to report on a couple of interested mathematical results obtained by this author.

Classical analysis theory teaches us that integration in the xy plane on a closed curve around a singularity is independent of the curve. One can thus choose to compute on a circle centered at the origin. The integration then takes place with respect to ϕ because differential 1-forms, α, reduce to

$$a = jd\phi$$

on such curves. Since $(dxdy)^2 = -1$, we have

$$d\phi = \frac{xdy - ydx}{\rho^2} = \frac{x - ydxdy}{\rho^2} dy = z^{-1}dy,$$

where $z = 1/(x + ydxdy)$. Since $d\phi \cdot d\phi = 1/\rho^2$, we further have

$$j = \rho^2(\alpha \cdot d\phi).$$

Given $\alpha(= fdx + gdy)$, we can write it as wdx, which defines w. We then compute

$$j = \rho^2(wdx) \cdot (z^{-1}dy)$$

and obtain $(wz)^{(2)}$ where the superscript (2) means the differential 2-form component (i.e. coefficient) of wz. Use this result in the intended integral and make the radius go to zero. The theorem of residues results.

But there may be much important implications if we were interested in the calculus of complex variables. We have just seen in the plane that a strict harmonic even differential form in the real plane plays the role of a holomorphic function in the complex plane. Then, in 3-D, the mathematics starts to look really interesting. We not only have the imaginary unit $dxdy$, but also $dydz$ and $dzdx$. And there is the "entanglement" of imaginary units since

$$(dxdy)(dydz) = dxdz.$$

In addition, we have a basis of strict harmonic differentials in 3-D and what amounts to a Laurent expansion in them. Hence there is a very geometric brand of calculus of several complex variables in 3-D. Higher dimension would be still more interesting because, in addition, we would have products of imaginary units that are differential forms of higher even grade. For the moment, $D > 3$ is of limited interest in this new vision because, apparently, strict harmonic differentials in higher dimension have not yet been worked out.

Still more relevant is the use of the traditional Helmholtz theorem in its "natural environment", i.e. not of vector fields but of differential 1-forms. It is an ackward theorem for those fields because gradients, curls and divergences and divergences are involved. Gradients and curls of vector fields are replaced with just exterior derivatives in a corresponding theorem for differential $1-$forms. This in turn can be eadily extended to arbitrary grade in Euclidean spaces of dimension $n > r$. Furthermore, Riemannian $r-$manifolds can always be viewed as r-surfaces in Euclidean spaces of dimension $N > n$ for sufficiently high N. Since one cannot then use in the proof of a Helmholtz theorem that a certain integral vanishes (the one that vanishes at infinity when integrating over the whole Euclidean space), an extra term (actually terms) contributes to the final result. One thus obtains a theorem of a Helmholz type but with a harmonic terms contribution, like in Hodge's theorem. But it is far more sophisticated than Hodge's, for one not only is proving the decomposition (closed, co-closed and harmonic), but actually specifying what they are in terms of integrands, as in Helmholtz theorem. So, we strike at the heart of cohomology theory without using co-homology theory.

What we have just reported illustrates that, with the KC calculus, one finds results that match or improve on theorems in last chapters of highly specialized books. One does so virtually without attempting to do so. It is only a matter of trying to compute with the KC what you would compute with other calculi in your everyday work with mathematics. New doors open themselves in front of you. And one achieves everything that we have reported through the use of only scalar-valued differential forms. Imagine what may happen if we were to use Clifford-valued differential forms. Can any other calculus compete with Kähler's?