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Multi–component Cahn–Hilliard Systems with Singular Potentials: Theoretical Results

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Abstract

We consider a system of nonlinear diffusion equations modelling (isothermal) phase segregation of an ideal mixture of $N \geq 2$ components occupying a bounded region $\Omega \subset \mathbb{R}^d$, $d \leq 3$. Our system is subject to a constant mobility matrix of coefficients, a free energy functional given in terms of *singular entropy* generated potentials and *localized* capillarity effects. We prove well-posedness and regularity results which generalize the ones obtained by Elliott and Luckhaus (IMA Preprint Ser 887, 1991). In particular, if $d \leq 2$, we derive the uniform strict separation of solutions from the singular points of the (entropy) nonlinearity. Then, even if $d = 3$, we prove the existence of a global (regular) attractor as well as we establish the convergence of solutions to single equilibria. If $d = 3$, this convergence requires the validity of the asymptotic strict separation property. This work constitutes the first part of an extended three-part study involving the phase behavior of multi-component systems, with a second part addressing the presence of nonlocal capillarity effects, and a final part concerning the numerical study of such systems along with some relevant application.

Keywords Multi-component Cahn–Hilliard equation · Singular potential · Strict separation property · Regularity · Global attractors · Convergence to equilibrium

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1 Introduction

The Cahn–Hilliard equation has been originally proposed to model phase separation phenomena in binary alloys (see [7, 8], see also [19, 38] and references therein). Since then, it has been observed that phase separation characterizes many important processes like, for instance, behavior of polymer mixtures, solid tumor growth, inpainting (see, for instance, [37] and its references). More recently, phase separation has become a paradigm in cell biology (see, e.g., [13, 14, 39] and references therein). Correspondingly, the theoretical and numerical literature on Cahn–Hilliard type equations has been growing a lot in this last decade. However, most of the theoretical contributions are devoted to binary mixtures (see [37] and references therein). Nonetheless, multi-component systems are ubiquitous in nature and possess important scientific and industrial applications. Thinking just of cell biology, we refer the reader to [40, 46] and their references. In spite of that, the thermodynamic and kinetic properties, and solution behavior of N-component mixtures with $N > 3$ have remained relatively unexplored, even though considerable progress has been made for binary and ternary systems in the last few decades (see [5, 44] and references therein). Rigorous analysis of the coupling of the two-phase Cahn–Hilliard theory with incompressible fluid flows on the basis of Korteweg stress tensor dynamics have been investigated by many authors (see, e.g., [2, 3, 20–23, 26–28, 30–33] and the references therein, just to give some examples of the most important developments), resulting in a fairly reasonable theoretical picture about the corresponding binary fluid behavior in most cases. Coupling of hydrodynamic models for fluid behavior with a N -component family of Cahn–Hilliard models has also been considered recently[1] (see, e.g., [5, 15, 35]). As such, existence of weak solutions to a class of N-component Cahn–Hilliard systems, subject to a mechanism of cross-diffusion between different chemical species and singular bulk potentials, has recently been studied in [17].

Furthermore, well-posedness for a hierarchy of N multi-species Cahn–Hilliard systems which are consistent with the standard Cahn–Hilliard equation for binary components was provided in [5], in the case when the capillarity effects are reflected through the presence of penalizing gradients, while the entropy of the system is related to a properly-constructed regular (i.e., polynomial like) bulk potential. We also recall that the case of regular bulk potential for a N-component Cahn–Hilliard system has been studied, for instance, in [9] and in [12]. In the former, the existence of a smooth global attractor was obtained, while in the latter the authors studied well-posedness and the existence of global and exponential attractors in the case of dynamic boundary conditions.

In the context of non-equilibrium thermodynamics, here we consider the model derived in [16] (cf. also [5, 18]). Let Ω be a bounded domain in $\mathbb{R}^d$, $d \leq 3$, with a

[1] Although, these theories are constrained to some degree due to the inability of meeting all conditions of physical and mathematical consistency [5].

smooth boundary $\partial\Omega$. We then focus on the following nonlinear diffusion system:

$$\begin{cases} \partial_t \mathbf{u} = \operatorname{div}(\boldsymbol{\alpha}\nabla\mu), & \text{in } \Omega\times(0,T), \\ \mu_i = -\gamma_i\Delta u_i + \Psi_{,u_i}(\mathbf{u}), \quad \forall i=1,\dots,N, & \text{in } \Omega\times(0,T), \\ (\boldsymbol{\alpha}\nabla\boldsymbol{\mu})\cdot\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega\times(0,T), \\ \nabla u_i\cdot\mathbf{n} = 0, \quad \forall i=1,\dots,N, & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\gamma_i > 0$, $i \in \{1, ..., N\}$ and $\mathbf{u} = (u_1, ..., u_N) \in [0, 1]^N$, $N \geq 2$, is the vector of mass fractions. The (constant) mobility matrix $\boldsymbol{\alpha}$ is a symmetric, positive semidefinite $N\times N$ matrix such that its kernel is given by $span\{\boldsymbol{\zeta}\}$ (where $\zeta_i = 1$, for $i = 1,\dots,N$). As in [16], we are also interested in a particular free energy related to the Boltzmann–Gibbs mixing entropy potential

$$\Psi(\mathbf{u}) := \theta\sum_{i=1}^{N} u_i\ln(u_i) - \frac{1}{2}\mathbf{u}\cdot\mathbf{Au} = \sum_{i=1}^{N}\psi(u_i) - \frac{1}{2}\mathbf{u}\cdot\mathbf{Au} = \Psi^1(\mathbf{u}) - \frac{1}{2}\mathbf{u}\cdot\mathbf{Au}, \tag{1.2}$$

where $\theta > 0$ is the absolute temperature for the mixture. Here, $\mathbf{A}$ is a constant symmetric $N\times N$ matrix with the largest eigenvalue $\lambda_{\mathbf{A}} > 0$. In what follows, our goal is in fact to extend the framework of [16] to include many other (physically relevant) entropy functions $\Psi^1 : [0, 1] \to \mathbb{R}_+$ (cf. [25]). An advantage of our approach is that both the physical and mathematical consistency of the model (1.1), as motivated by the work [5], is met for the general class of (singular) potentials considered subsequently. The total energy of the system is defined as follows

$$\mathcal{E}(\mathbf{u}) := \frac{1}{2}\sum_{i=1}^{N}\int_\Omega \gamma_i|\nabla u_i|^2dx + \int_\Omega \Psi(\mathbf{u})\,dx. \tag{1.3}$$

In particular, for the multi-component system (1.1), in addition to satisfying the hierarchy conditions of [5], the following consistency conditions are also strongly desirable:

(1) total mass conservation and energy dissipation hold for every energy solution. (i.e., the energy (1.3) is non-increasing for all time $t \geq 0$, as well as, $-c_* \leq \mathcal{E}(\mathbf{u}(t)) \leq C_*$, for any $t \geq t_0(\mathbf{u}_0)$, for some computable $c_*, C_* > 0$ *independent* of time and the initial conditions);
(2) physical separation of each chemical component u_i, $i \in \{1, ..., N\}$ from the endpoint values $\{0, 1\}$ of the bulk potential Ψ^1;
(3) in the absence of any external sources, for a given fixed initial condition $\mathbf{u}_0$, one has the convergence of the associated (unique) energy solution to a single equilibrium.

The main novelty of our work lies in that we establish properties (1) and (3) for the N-component system (1.1) and also property (2) in dimension two. More precisely, concerning (1), we extend and refine the well posedness result proven in [16]. In particular, in dimension two, we establish the so-called strict separation property (see [25] and references therein), that is, property (2). In order to demonstrate property

(3) (see [1, 11] for the binary case), we need to obtain suitable regularization results which also allow us to show the existence of a global attractor (see, e.g., [36, 43] and references therein). Finally, in subsequent contributions we will address the presence of nonlocal capillarity effects in the energy functional, as well as provide the numerical analysis of such systems along with some relevant applications to the applied sciences.

The structure of the paper is as follows. In Sect. 2, we state the required mathematical framework, specifying, in particular, the nature of the entropy functions we can handle. In Sect. 3.1 we recall the corresponding notion of weak (energy) solutions which can be constructed by variational techniques. Furthermore, we give a main summary of the main results, involving the long term behavior of energy solutions (see Sect. 3.3), along with the existence of a dissipative (asymptotically compact) semigroup in Sect. 3.2. The remaining Sects. 4, 5, 6 and 7 contain detailed proofs of the aforementioned results. The final section is an appendix which contains a number of technical results, assisting in the proofs of the main results.

2 The Mathematical Framework

The Sobolev spaces are denoted as usual by $W^{k,p}(\Omega)$, where $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, with norm $\|\cdot\|_{W^{k,p}(\Omega)}$. The Hilbert space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k(\Omega)}$. Moreover, given a (real) vector space X, we denote by $\mathbf{X}$ the space of N-component vectors each one belonging to X. In this case $|\mathbf{v}|$ is the Euclidean norm of $\mathbf{v} \in \mathbf{X}$. We then denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $\|\cdot\|$ the induced norm. We indicate by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the canonical inner product and its induced norm in the (real) Hilbert space H, respectively. We also define the spatial average of a measurable function $f : \Omega \to \mathbb{R}$ as

$$\overline{f} := |\Omega|_d^{-1} \int_\Omega f(x)dx,$$

where $|\Omega|_d$ stands for the d-dimensional Lebesgue measure of Ω. We then recall the following Sobolev–Gagliardo–Nirenberg's inequality for two-dimensional bounded domains:

$$\|f\|_{L^p(\Omega)} \leq C\sqrt{p}\|f\|_{H^1(\Omega)}, \quad 2 \leq p < \infty,$$

where $C > 0$ is a constant independent of p. Further, we introduce the affine hyperplane

$$\Sigma := \left\{ \mathbf{c}' \in \mathbb{R}^N : \sum_{i=1}^N c_i' = 1 \right\}, \tag{2.1}$$

and since only the nonnegative values for the u_i are physically relevant, we also define the Gibbs simplex

$$\mathbf{G} := \left\{ \mathbf{c}' \in \mathbb{R}^N : \sum_{i=1}^N c_i' = 1, \quad c_i' \geq 0, \quad i = 1, \ldots, N \right\}, \tag{2.2}$$

and the tangent space to the affine hyperplane Σ

$$T\Sigma := \left\{ \mathbf{d}' \in \mathbb{R}^N : \sum_{i=1}^{N} d_i' = 0 \right\}. \tag{2.3}$$

We introduce the following useful notation:

$$\begin{aligned}
\mathbf{H}_0 &:= \left\{ \mathbf{f} \in \mathbf{L}^2(\Omega) : \int_\Omega \mathbf{f}\, dx = 0 \text{ and } \mathbf{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \right\}, \\
\widetilde{\mathbf{H}}_0 &:= \left\{ \mathbf{f} \in \mathbf{L}^2(\Omega) : \mathbf{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \right\}, \\
\mathbf{V}_0 &:= \left\{ \mathbf{f} \in \mathbf{H}^1(\Omega) : \int_\Omega \mathbf{f}\, dx = 0 \text{ and } \mathbf{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \right\}, \\
\widetilde{\mathbf{V}}_0 &:= \left\{ \mathbf{f} \in \mathbf{H}^1(\Omega) : \ \mathbf{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \right\}.
\end{aligned}$$

Notice that the two spaces above are still Hilbert spaces with the same inner products defined in $\mathbf{L}^2(\Omega)$ and in $\mathbf{H}^1(\Omega)$, respectively. Furthermore, we have the Hilbert triplets $\mathbf{V}_0 \hookrightarrow\hookrightarrow \mathbf{H}_0 \hookrightarrow \mathbf{V}_0'$ and $\widetilde{\mathbf{V}}_0 \hookrightarrow\hookrightarrow \widetilde{\mathbf{H}}_0 \hookrightarrow \widetilde{\mathbf{V}}_0'$. These spaces are the right functional setting for the homogeneous Neumann Laplace problem where the forcing term belongs to $T\Sigma$, see Appendix 1. Indeed, the condition $\mathbf{f}(x) \in T\Sigma$ entails that the components of $\mathbf{f}$ are linearly dependent, and this forces the (weak) Laplace operator to be defined on $\mathbf{V}_0$. The spaces with the tilde symbol are useful when one does not need that the integral mean of the function is zero. We then introduce the Euclidean projection $\mathbf{P}$ of $\mathbb{R}^N$ onto $T\Sigma$, by setting, for $l = 1, \dots, N$,

$$(\mathbf{P}\mathbf{v})_l = \left(\mathbf{v} - \left(\frac{1}{N} \sum_{i=1}^{N} v_i \right) \boldsymbol{\zeta} \right)_l = \frac{1}{N} \sum_{m=1}^{N} (v_l - v_m).$$

Notice that the projector $\mathbf{P}$ is also an orthogonal $\mathbf{L}^2(\Omega)$-projector on $\widetilde{\mathbf{H}}_0$, being symmetric with respect to the $\mathbf{L}^2(\Omega)$-scalar product and idempotent.

Given a (closed) subspace V of a (real) Hilbert space H we denote by $V^{\perp_H}$ its orthogonal complement with respect to the H-topology. We then denote by the simple symbol $V^\perp$ the annihilator of V, i.e.

$$V^\perp := \{x \in H' : \ \langle x, y \rangle_{H',H} = 0 \quad \forall\, y \in V\}.$$

We define $\mathcal{B}(X, Y)$ ($\mathcal{B}(X)$ when $X = Y$) the set of linear bounded operators from the (real) Banach space X to the (real) Banach space Y, and $\mathcal{K}(X, Y)$ ($\mathcal{K}(X)$ when $X = Y$) for the compact operators from X to Y. Moreover, given an operator T from X to Y, we define by $T' : Y' \to X'$ the adjoint of T, whereas, in case of Hilbert spaces, we denote the Hilbert adjoint of T by $T^* : Y \to X$. We now observe that the assumptions on $\boldsymbol{\alpha}$ imply that $\boldsymbol{\alpha}$ is positive definite over $T\Sigma$. This will constitute the main assumption on the mobility matrix in this contribution.

(**M0**) In particular, we assume that there exists $l_0 > 0$ such that

$$\boldsymbol{\alpha}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \geq l_0 \boldsymbol{\eta} \cdot \boldsymbol{\eta}, \quad \forall \boldsymbol{\eta} \in T\Sigma. \tag{2.4}$$

Next, we define the set

$$\mathcal{K} := \left\{ \boldsymbol{\eta} \in \mathbf{H}^1(\Omega) : \sum_{i=1}^{N} \eta_i = 1, \quad \eta_i \geq 0, \quad \forall\, i = 1, \dots, N \right\}. \tag{2.5}$$

For the sake of simplicity we will adopt the compact notation $\mathbf{v} \geq k$, with $\mathbf{v} \in \mathbb{R}^N$ and $k \in \mathbb{R}$ to indicate the relation $v_i \geq k$ for any $i = 1, \dots, N$, as well a we will write $\mathbf{z} = \mathbf{v} + k$ to indicate that $z_i = v_i + k$, $i = 1, \dots, N$.

Concerning the entropy function, we recall that, according to (1.2), ψ is the Boltzmann-Gibbs entropy potential

$$(\boldsymbol{\phi}(\mathbf{u}))_i = \phi(u_i) := \psi'(u_i) = \theta(\ln u_i + 1), \quad i = 1, \dots, N.$$

More generally, $\psi' = \phi$ can be associated with a generalized (statistical) class of entropy potentials that are physically relevant (see [25, Section 2] and references therein). In particular, our analysis also includes the Tsallis' entropy formulation containing the usual Boltzmann-Gibbs (i.e. logarithmic) form (1.2) as a particular case. Namely, for given $q \in \mathbb{R}_+$, we define the q-logarithm of a real number $r > 0$, as

$$\ln_{\{q\}} r := \begin{cases} \ln r, & \text{if } q = 1, \\ \frac{r^{1-q}-1}{1-q}, & \text{if } q > 0,\ q \neq 1. \end{cases}$$

The Tsallis' entropy $S = S_q$ is then given by

$$S(u_i) := \int_\Omega \psi(u_i(x))dx, \ \ \psi(r) := -r \ln_{\{q\}}(1/r).$$

A class of relevant statistically generated entropy functional.[2] for the multi-component problem then becomes

$$\int_\Omega \Psi(\mathbf{u}(x))dx := \theta \sum_{i=1}^{N} S(u_i) - \frac{1}{2}\int_\Omega \mathbf{u} \cdot \mathbf{A}\mathbf{u}\, dx. \tag{2.6}$$

To simplify the exposition, we shall set $\theta = 1$ and $\gamma_i \equiv \gamma$ in what follows.[3] Also, we will denote by $\Psi_{,\mathbf{u}}(\mathbf{r})$ the gradient of $\Psi(\mathbf{r})$ in $\mathbb{R}^N$. We refer the reader to [25, Section 6.3] for some other important classes of (singular at 0) mixing potentials. Among examples of (entropy) densities we have functions of the following form:

2 See once again [25, Section 6.2].

3 All the results in this article clearly hold for any $\theta > 0$.

- $\psi(x) = \left(x^{1+\kappa} - x^{1-\kappa}\right)/2\kappa$, for some $\kappa \in (0,1)$;
- $\psi(x) = x^{\beta}\ln(x)$, for some $\beta \in (0,1)$;
- $\psi(x) = -x\ln^{1/\nu}\left(\frac{1}{x}\right)$, for some $\nu > 0$.

In order to include a large admissible class of entropy functionals in (2.6), we assume that

$$\psi \in C([0,1]) \cap C^2((0,1])$$

is such that

(E0) $\psi''(s) \geq \zeta > 0$, for all $s \in (0,1]$;
(E1) $\lim_{s\to 0^+}\psi'(s) = -\infty$;
(E2) there exist $C > 0$ and $\beta \in [1,2)$ such that

$$\phi'(s) = \psi''(s) \leq C\mathrm{e}^{C|\psi'(s)|^{\beta}}, \text{ for all } s \in (0,1].$$

We also extend $\psi(s) = +\infty$, for any $s \in (-\infty, 0)$, and extend ψ for all $s \in [1,\infty)$ so that ψ is still a C^2 function on $(0,+\infty)$ and property (**E0**) holds for any $s > 0$. To this aim we define

$$\psi(s) := As^3 + Bs^2 + Ds, \quad \text{for all } s \geq 1, \tag{2.7}$$

with

$$\begin{cases} A = \psi(1) - \psi'(1) + \frac{1}{2}\psi''(1), \\ B = -3\psi(1) + 3\psi'(1) - \psi''(1), \\ D = 3\psi(1) - 2\psi'(1) + \frac{1}{2}\psi''(1). \end{cases}$$

Note that $\psi''(s) \geq \psi''(1) \geq \zeta$ for any $s \geq 1$, as desired.

Following the general scheme developed in [24, Section 3.1], on account of (**E0**)-(**E1**), we can define an approximation of the potential ψ by means of a family $\{\psi_\varepsilon\}_{\varepsilon>0}$ of everywhere defined functions. Indeed, let

$$\psi_\varepsilon(s) = \frac{\varepsilon}{2}|\mathbb{T}_\varepsilon s|^2 + \psi(J_\varepsilon(s)), \quad s \in \mathbb{R},\ \varepsilon > 0, \tag{2.8}$$

where $J_\varepsilon = (I + \varepsilon\mathbb{T})^{-1} : \mathbb{R} \to (0,+\infty)$ is the resolvent operator and $\mathbb{T}_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon)$ is the Yosida approximation of $\mathbb{T}(s) := \psi'(s)$, for all $s \in \mathfrak{D}(\mathbb{T}) = (0,+\infty)$ (see also [24, Lemmas 3.7,3.8]). According to the general theory of maximal monotone operators (see, for instance, [4, 6, 42]), the following properties hold (see properties (i)-(iii) [24, Sec.3.1]):

(i) ψ_ε is convex and $\psi_\varepsilon(s) \nearrow \psi(s)$, for all $s \in \mathbb{R}$, as ε goes to 0;
(ii) $\psi'_\varepsilon(s) = \mathbb{T}_\varepsilon(s)$ and $\psi'_\varepsilon =: \phi_\varepsilon$ is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\varepsilon}$;
(iii) $|\psi'_\varepsilon(s)| \nearrow |\psi'(s)|$ for all $s \in (0,+\infty)$ and $|\psi'_\varepsilon(s)| \nearrow \infty$, for all $s \in (-\infty, 0]$, as $\varepsilon \searrow 0$.

Moreover

(iv) for any $\varepsilon \in (0, 1]$, there holds

$$\psi_\varepsilon''(s) \geq \frac{\zeta}{1+\zeta}, \quad \forall\ s \in \mathbb{R}.$$

This follows as in [24, Lemma 3.10]. Indeed, $\mathbb{T}_\varepsilon$ is clearly differentiable in $\mathbb{R}$ and from the differentiation formula for inverse functions (used for J_ε) we infer, for any $s \in \mathbb{R}$,

$$\psi_\varepsilon''(s) = \frac{1}{\varepsilon}\left(1 - \frac{1}{1+\varepsilon\psi''(J_\varepsilon(s))}\right) \geq \frac{\zeta}{1+\varepsilon\zeta} \geq \frac{\zeta}{1+\zeta},$$

having exploited assumption (**E0**) which we assumed by construction to be valid for any $s > 0$.

(v) For any compact subset $M \subset (0, 1]$, ψ_ε' converges uniformly to ψ' on M. This comes directly from property (iii) and the fact that ψ_ε', ψ' are continuous in M for any $\varepsilon > 0$.

(vi) For any $\varepsilon_0 > 0$ there exists $\tilde{K} = \tilde{K}(\varepsilon_0) > 0$ such that

$$\sum_{i=1}^{N} \psi_\varepsilon(r_i) \geq \frac{1}{4\varepsilon_0}|\mathbf{r}|^2 - \tilde{K}, \quad \forall\, \mathbf{r} \in \mathbb{R}^N, \quad \forall\, 0 < \varepsilon < \varepsilon_0.$$

This directly follows from a straightforward adaptation of [24, Lemma 3.11]. Indeed by de l'Hôpital's Theorem we have

$$\lim_{s\to+\infty} \frac{\psi_\varepsilon(s)}{s^2} = \lim_{s\to+\infty} \frac{\psi_\varepsilon'(s)}{2s} = \lim_{s\to+\infty} \frac{1}{2\varepsilon}\frac{s - J_\varepsilon(s)}{s} = \frac{1}{2\varepsilon},$$

since, recalling (2.7), J_ε is of order $\frac{1}{2}$ with respect to s for s sufficiently large, and thus $\lim_{s\to+\infty} \dfrac{J_\varepsilon(s)}{s} = 0$. Moreover, by assumption (**E1**) it is clear that J_ε is bounded on $(-\infty, 0)$, and thus again

$$\lim_{s\to-\infty} \frac{\psi_\varepsilon(s)}{s^2} = \lim_{s\to-\infty} \frac{1}{2\varepsilon}\frac{s - J_\varepsilon(s)}{s} = \frac{1}{2\varepsilon}.$$

Therefore, for any $\varepsilon_0 > 0$, there exists $M_{\varepsilon_0} > 0$ depending on ε_0 such that, by property (i), $\psi_\varepsilon(s) \geq \psi_{\varepsilon_0}(s) \geq \frac{1}{4\varepsilon_0}s^2$, for any $|s| \geq M_{\varepsilon_0}$ and any $0 < \varepsilon < \varepsilon_0$. Moreover, for the same ε_0 it also holds

$$\psi_\varepsilon(s) \geq \psi_{\varepsilon_0}(s) \geq \frac{1}{4\varepsilon_0}s^2 - \frac{M_{\varepsilon_0}^2}{4\varepsilon_0} - \max_{r\in[-M_{\varepsilon_0}, M_{\varepsilon_0}]} |\psi_{\varepsilon_0}(r)|,$$

for all $|s| \leq M_{\varepsilon_0}$ and for all $0 < \varepsilon < \varepsilon_0$. Thus, in the end, we get

$$\psi_\varepsilon(s) \geq \frac{1}{4\varepsilon_0}s^2 - K_{\varepsilon_0}, \quad \forall\, s \in \mathbb{R}, \quad \forall\, 0 < \varepsilon < \varepsilon_0,$$

with $K_{\varepsilon_0} := -\frac{M_{\varepsilon_0}^2}{4\varepsilon_0} - \max_{r\in[-M_{\varepsilon_0}, M_{\varepsilon_0}]} |\psi_{\varepsilon_0}(r)|$, from which property (vi) follows.

Remark 2.1 Introducing

$$\Psi_\varepsilon(\mathbf{r}) := \sum_{i=1}^{N} \psi_\varepsilon(r_i) - \frac{1}{2}\mathbf{r}^T\mathbf{A}\mathbf{r} = \Psi_\varepsilon^1(\mathbf{r}) - \frac{1}{2}\mathbf{r}^T\mathbf{A}\mathbf{r},$$

we have that, for any $\varepsilon_0 > 0$ sufficiently small, there exist $K = K(\varepsilon_0) > 0$ and $C = C(\varepsilon_0) > 0$, such that

$$\Psi_\varepsilon(\mathbf{r}) \geq C(\varepsilon_0)|\mathbf{r}|^2 - K, \quad \forall\, \mathbf{r} \in \mathbb{R}^N, \quad \forall\, \varepsilon \in (0, \varepsilon_0).$$

In particular, this comes from property (vi) and the fact that $-\frac{1}{2}\mathbf{r}^T\mathbf{A}\mathbf{r} \geq -\frac{\lambda_{\mathbf{A}}}{2}|\mathbf{r}|^2$. Notice that ε_0 has to be small enough so that $C(\varepsilon_0) := \frac{1}{4\varepsilon_0} - \frac{\lambda_{\mathbf{A}}}{2} \geq \frac{1}{8\varepsilon_0} > 0$.

3 Main Results

This section is divided into several subsections according to the nature of the results.

3.1 Well-Posedness and Regularity

First we introduce the following vector of generalized chemical potential differences:

$$\mathbf{w} := \mathbf{P}\mu,$$

and observe that, being $\boldsymbol{\alpha}\boldsymbol{\zeta} = \mathbf{0}$,

$$\operatorname{div}(\boldsymbol{\alpha}\nabla\mu) = \operatorname{div}\left(\sum_{l=1}^{N} \boldsymbol{\alpha}_{kl}\nabla\mu_l\right) = \sum_{l=1}^{N} \boldsymbol{\alpha}_{kl}\Delta\frac{1}{N}\sum_{m=1}^{N}(\mu_l - \mu_m) = \boldsymbol{\alpha}\Delta\mathbf{w}.$$

Therefore, system (1.1) can be rewritten as follows

$$\begin{cases} \partial_t\mathbf{u} = \boldsymbol{\alpha}\Delta\mathbf{w}, & \text{in } \Omega \times (0, T), \\ w_i = -\gamma\Delta u_i + \left(\mathbf{P}\left(\Psi_{,\mathbf{u}}(\mathbf{u})\right)\right)_i, & \text{in } \Omega \times (0, T), \forall\, i = 1, \ldots, N, \\ (\boldsymbol{\alpha}\nabla\mathbf{w}) \cdot \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0, T), \\ \nabla u_i \cdot \mathbf{n} = 0, & \text{on } \partial\Omega \times (0, T),\ \forall\, i = 1, \ldots, N, \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\mathbf{u}_0 \in \mathcal{K}$. Indeed, observe that $\sum_{i=1}^{N} u_i \equiv 1$ (just test the first equation of (3.1) with $\boldsymbol{\zeta}$), so that $\Delta\mathbf{u} \in T\Sigma$ and thus $\mathbf{P}\Delta\mathbf{u} = \Delta\mathbf{u}$.

In our well-posedness theorem we extend the result of [16, Theorem 1]. In particular, we show the instantaneous strict separation property in dimension two. This means that, for any $\tau > 0$, there exists $0 < \delta = \delta(\tau, T) < \frac{1}{N}$ such that $\delta < \mathbf{u}$ everywhere in $\overline{\Omega} \times [\tau, T]$. Notice that this condition, together with the constraint $\sum_{i=1}^{N} u_i = 1$, implies the existence of $\delta_1 := (N-1)\delta > 0$ such that $\mathbf{u} < 1 - \delta_1$ everywhere in $\overline{\Omega} \times [\tau, T]$, i.e., the solution is strictly separated from the pure phases 0 and 1. More precisely, we have

Theorem 3.1 *Assume (**M0**) and (**E0**)-(**E1**), and let $\mathbf{u}_0 \in \mathcal{K}$ and $T > 0$ be given. Suppose that*

$$\delta_0 < \overline{\mathbf{u}}_0, \tag{3.2}$$

for some $0 < \delta_0 < \frac{1}{N}$. Then there exists a unique pair $(\mathbf{u}, \mathbf{w})$, called weak or energy solution to (3.1), with the following properties

$$\begin{aligned}
\mathbf{u} &\in C([0,T]; \boldsymbol{H}^1(\Omega)) \cap L^\infty(0,T; \mathbf{H}^1(\Omega)) \cap L^4(0,T, \mathbf{H}^2(\Omega)),\\
\sqrt{t}\mathbf{u} &\in L^\infty(0,T; \mathbf{W}^{2,r}(\Omega)),\\
\partial_t \mathbf{u} &\in L^2(0,T; \mathbf{H}^1(\Omega)'), \quad \sqrt{t}\partial_t \mathbf{u} \in L^2(0,T; \mathbf{H}^1(\Omega)),\\
\mathbf{w} &\in L^2(0,T; \mathbf{H}^1(\Omega)), \quad \sqrt{t}\mathbf{w} \in L^\infty(0,T; \mathbf{H}^1(\Omega)),\\
\phi(u_i) &\in L^2(0,T; L^2(\Omega)),\\
\sqrt{t}\phi(u_i) &\in L^\infty(0,T; L^r(\Omega)), \quad i = 1, \ldots, N,
\end{aligned}$$

where $r \in [2, \infty)$ if $d = 2$ and $r \in [2, 6]$ if $d = 3$, which satisfies

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \tag{3.3}$$

$$\mathbf{u}(\cdot, t) \in \mathcal{K}, \quad \textit{for a.a. } t \in (0,T), \quad \overline{\mathbf{u}}(t) \equiv \overline{\mathbf{u}}_0, \tag{3.4}$$

$$0 < \mathbf{u}(x,t) < 1 \quad \textit{for a.a. } (x,t) \in \Omega \times (0,T), \tag{3.5}$$

and, for all $\xi \in C([0,T])$ and $\boldsymbol{\eta} \in \mathbf{H}^1(\Omega)$,

$$\int_0^T \xi(t) \left\{ \frac{d}{dt} < \mathbf{u}, \boldsymbol{\eta} > + (\boldsymbol{\alpha} \nabla \mathbf{w}, \nabla \boldsymbol{\eta}) \right\} dt = 0, \tag{3.6}$$

$$\int_0^T \xi(t) \left\{ (\mathbf{w} + \mathbf{P}(-\mathbf{A}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u})), \boldsymbol{\eta}) - \gamma(\nabla \mathbf{u}, \nabla \boldsymbol{\eta}) \right\} dt = 0. \tag{3.7}$$

Moreover, the following energy identity holds

$$\frac{d}{dt}\mathcal{E}(t) + (\boldsymbol{\alpha} \nabla \mathbf{w}(t), \nabla \mathbf{w}(t)) = 0, \quad \textit{for almost any } t \in [0,T]. \tag{3.8}$$

*Finally, if $d = 2$ and assumption (**E2**) holds, then, for any $\tau > 0$, there exists $0 < \delta = \delta(\tau, T) < \frac{1}{N}$ such that*

$$\delta < \mathbf{u}(x,t), \quad \forall (x,t) \in \overline{\Omega} \times [\tau, T].$$

Remark 3.2 Notice that (3.2) implies that there exists $\rho > 0$ such that $\rho < \overline{u}_{0,i} < 1-\rho$ for any $i = 1, \dots, N$. Indeed, we have for any $i = 1, \dots, N$,

$$\delta_0 < \min_{j=1,\dots,N} \overline{u}_{0,j} \le \overline{u}_{0,i} = 1-\sum_{j\neq i} \overline{u}_{0,j} \le 1-(N-1) \min_{j=1,\dots,N} \overline{u}_{0,j} < 1-(N-1)\delta_0,$$

and thus we can choose, e.g., $\rho = \delta_0$, being $N \ge 2$.

Remark 3.3 Notice that, arguing as in [16, Proposition 2.1], we easily obtain that $\sum_{i=1}^N u_i = 1$ and $\sum_{i=1}^N w_i = 0$. Moreover, by choosing $\boldsymbol{\eta} \equiv \boldsymbol{\eta}_i$, with $\boldsymbol{\eta}_i$ the i-th unit vector of the standard basis, we get that the total mass of each component u_i is preserved, i.e.,

$$\overline{\mathbf{u}}(t) \equiv \overline{\mathbf{u}}_0, \quad \forall\, t \ge 0.$$

Remark 3.4 As it will be clear from the proof (see Sect. 4 below), the quantity $\delta > 0$ only depends on the parameters of the problem, on ψ and on the $\mathbf{H}^1(\Omega)$-norm of the initial datum. Therefore, if we consider a set of initial data contained in a ball of $\mathbf{H}^1(\Omega)$ of radius $R > 0$, δ will depend on the initial datum only through R.

Remark 3.5 Notice that a continuous dependence estimate from which uniqueness directly follows has already been shown in [16, Section 3]. By this uniqueness result one can easily see that the weak solution is actually globally defined on $(0, \infty)$ and the properties in the above theorem hold for any $T > 0$.

The energy identity (3.8) allows us to extend the weak solution to all times $t \ge 0$. Also, on account of the dissipative nature of the system, we have the following uniform control of the energy $\mathcal{E}$.

Theorem 3.6 *Let the assumptions of Theorem 3.1 hold. Then the energy satisfies the following inequality*

$$\mathcal{E}(t) \le C_1 e^{-\omega t}\mathcal{E}(0) + C_2, \quad \forall\, t \ge 0, \tag{3.9}$$

where $C_1, C_2 > 0$ are positive constants and $\omega > 0$ is a universal constant.

The weak solution given by Theorem 3.1 instantaneously regularizes. Indeed, we have

Theorem 3.7 *Let the assumptions of Theorem 3.1 hold. Then the energy solution $(\mathbf{u}, \mathbf{w})$, defined for all $t \ge 0$, is such that, for any $\tau > 0$,*

$$\begin{aligned}
\mathbf{u} &\in C([0,\infty); \mathbf{H}^1(\Omega)) \cap L^\infty(\tau,\infty; \mathbf{W}^{2,p}(\Omega)),\\
\partial_t \mathbf{u} &\in L^2(t, t+1; \mathbf{H}^1(\Omega)), \quad \forall\, t \ge \tau,\\
\mathbf{w} &\in L^\infty(\tau,\infty; \mathbf{H}^1(\Omega)), \quad \forall\, t \ge \tau,\\
\phi(u_i) &\in L^\infty(\tau,\infty; L^p(\Omega)), \quad i = 1, \dots, N,
\end{aligned} \tag{3.10}$$

*where $p \in [2, \infty)$ if $d = 2$ and $p \in [2, 6]$ if $d = 3$. Moreover, if $d = 2$ and assumption **(E2)** holds, then there exists $0 < \delta = \delta(\tau) < \frac{1}{N}$ such that*

$$\delta < \mathbf{u}(x,t), \quad \forall\, (x,t) \in \overline{\Omega} \times [\tau, \infty).$$

3.2 Existence of the Regular Global Attractor

We now define a complete metric space which will be the phase space of the dissipative dynamical system associated with (3.1). For a given $\mathbf{M} \in \Sigma$, such that $M_i \in (0, 1)$, for any $i = 1, \dots, N$, we set

$$\mathcal{V}_{\mathbf{M}} := \left\{ \mathbf{u} \in \mathbf{H}^1(\Omega) : \ 0 \le \mathbf{u}(x) \le 1, \quad \text{for a.a. } x \in \Omega, \quad \overline{\mathbf{u}} = \mathbf{M}, \quad \sum_{i=1}^{N} u_i = 1 \right\},$$

endowed with the $\mathbf{H}^1$-topology. In particular we consider the one induced by the equivalent norm $\|\mathbf{u}\|_{\mathcal{V}_{\mathbf{M}}} = \|\nabla \mathbf{u}\| + |\overline{\mathbf{u}}|$. This is a complete metric space. Thus we can define a dynamical system $(\mathcal{V}_{\mathbf{M}}, S(t))$ where

$$S(t) : \mathcal{V}_{\mathbf{M}} \to \mathcal{V}_{\mathbf{M}}, \quad S(t)\mathbf{u}_0 = \mathbf{u}(t), \quad \forall t \ge 0.$$

Observe that $S(t)$ satisfies the following properties:

- $S(0) = Id_{\mathcal{V}_{\mathbf{M}}}$;
- $S(t + \tau) = S(t)S(\tau)$, for every $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$;
- $t \mapsto S(t)\mathbf{u}_0 \in C([0, \infty); \mathcal{V}_{\mathbf{M}})$, for every $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$;
- $\mathbf{u}_0 \mapsto S(t)\mathbf{u}_0 \in C(\mathcal{V}_{\mathbf{M}}; \mathcal{V}_{\mathbf{M}})$, for any $t \in [0, \infty)$.

In particular, the last property can be proved as follows. From [16, Section 3] (see also Remark 3.5) we deduce that, for any sequence $\{\mathbf{u}_{0,n}\}_{n\in\mathbb{N}} \subset \mathcal{V}_{\mathbf{M}}$ such that $\mathbf{u}_{0,n} \to \mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$ as $n \to \infty$,

$$\|S(t)\mathbf{u}_{0,n} - S(t)\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)'} = \|S(t)\mathbf{u}_{0,n} - S(t)\mathbf{u}_0\|_{\mathbf{V}_0'} \to 0,$$

as $n \to \infty$, for any $t \ge 0$ fixed. From this result, together with the $\mathbf{H}^2$-regularity (for any $t > 0$) and the interpolation estimate

$$\|\cdot\|_{\mathbf{H}^1(\Omega)} \le C \|\cdot\|_{\mathbf{H}^2(\Omega)}^{\frac{2}{3}} \|\cdot\|_{\mathbf{H}^1(\Omega)'}^{\frac{1}{3}}$$

we deduce that $\mathbf{u}_0 \mapsto S(t)\mathbf{u}_0 \in C(\mathcal{V}_{\mathbf{M}}; \mathcal{V}_{\mathbf{M}})$, for any $t \in (0, \infty)$. The case $t = 0$ is trivial.

Furthermore, we recall that the *global attractor* is the unique compact set $\mathcal{A} \subset \mathcal{V}_{\mathbf{M}}$ such that

- $\mathcal{A}$ is fully invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for every $t \ge 0$;
- $\mathcal{A}$ is attracting for the semigroup, i.e.,

$$\lim_{t\to\infty} [\text{dist}_{\mathcal{V}_{\mathbf{M}}}(S(t)B, \mathcal{A})] = 0$$

  for every bounded set $B \subset \mathcal{V}_{\mathbf{M}}$.

The dissipative inequality (3.9) and the instantaneous regularization of the energy solution allow us to prove

Theorem 3.8 *The dynamical system* $(\mathcal{V}_{\mathbf{M}}, S(t))$ *admits a (unique) connected global attractor* $\mathcal{A} \subset \mathcal{V}_{\mathbf{M}}$ *which is bounded in* $\mathbf{W}^{2,r}(\Omega)$*, for any* $r \geq 2$ *if* $d = 2$*,* $r = 6$ *if* $d = 3$*.*

Remark 3.9 The proof of this result is based on showing that the dynamical system $(\mathcal{V}_{\mathbf{M}}, S(t))$ admits a compact absorbing set $\mathcal{B}_0$ (see Sect. 6 below).

Remark 3.10 Notice that in dimension two, thanks to the validity of the strict separation property given by the extra assumption (**E2**) (which is not needed elsewhere), we can in principle prove the existence of an exponential attractor (and thus deduce that the global attractor is of finite fractal dimension). To obtain this result, one should demonstrate the existence of strong solutions when the initial data are sufficiently regular. This can be done through an approximating scheme similar to [31]. Having the existence (and uniqueness) of the strong solution, which is strictly separated from the initial time $t = 0$, one can prove the Lipschitz continuity of the semigroup $S(t)$ with respect to the initial datum, provided that the initial data belong to a sufficiently regular positively invariant bounded absorbing set, say $\mathbb{B}$. Then, one can also prove a smoothing property of $S(t)$ on $\mathbb{B}$ and, following [36, Section 3], get the existence of an exponential global attractor first on $\mathbb{B}$ and then on $\mathcal{V}_{\mathbf{M}}$, being $\mathbb{B}$ a suitable bounded absorbing set. We refer to [34, Section 3.3], in which this procedure is applied to the multi-component Allen-Cahn equation.

3.3 Convergence to Equilibrium

In this section we exploit and adapt the arguments of [1, Section 6]. We detail the main steps, postponing the proofs of the main results to Sects. 7 and 1.

We consider again the dynamical system $(\mathcal{V}_{\mathbf{M}}, S(t))$ and the ω-limit set $\omega(\mathbf{u}_0)$ of a given $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$

$$\omega(\mathbf{u}_0) = \{\mathbf{z} \in \mathbf{H}^{2r}(\Omega) \cap \mathcal{V}_{\mathbf{M}} : \exists\, t_n \nearrow \infty \text{ s.t. } \mathbf{u}(t_n) \to \mathbf{z} \text{ in } \mathbf{H}^{2r}(\Omega)\},$$

where $r \in [\frac{1}{2}, 1)$. In particular, for later purposes, we fix $r \in (\frac{d}{4}, 1)$. Thanks to Theorem 3.7, we have that $\mathbf{u} \in L^\infty(\tau, \infty; \mathbf{H}^2(\Omega))$ for any $\tau > 0$. Hence the sets $\bigcup_{t \geq \tau} S(t)\mathbf{u}_0$ are relatively compact in $\mathbf{H}^{2r}(\Omega)$. Since for a fixed $t_0 > 0$ we have

$$\omega(\mathbf{u}_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathbf{u}_0}^{\mathbf{H}^{2r}(\Omega)} = \bigcap_{s \geq t_0} \overline{\bigcup_{t \geq s} S(t)\mathbf{u}_0}^{\mathbf{H}^{2r}(\Omega)},$$

by standard results related to the intersection of non-empty, compact (in $\mathbf{H}^{2r}(\Omega)$), connected and nested sets, we infer that $\omega(\mathbf{u}_0)$ is non-empty, compact and connected in $\mathbf{H}^{2r}(\Omega)$. Moreover, it is easy to show that

$$\lim_{t \to \infty} \operatorname{dist}_{\mathbf{H}^{2r}(\Omega)}(S(t)\mathbf{u}_0, \omega(\mathbf{u}_0)) = 0. \tag{3.11}$$

Let us set

$$Z = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathcal{E}(\boldsymbol{u}) < +\infty\},$$

where $\mathcal{E}$ is defined in (1.3), and introduce the notion of stationary point. Given

$$\boldsymbol{f}_1 \in \mathbf{G} := \{\mathbf{v} \in \mathbf{L}^\infty(\Omega) : \ \mathbf{v}(x) \in T\Sigma \text{ for almost any } x \in \Omega\},$$

we say that $\mathbf{z} \in \mathbf{H}^2(\Omega) \cap Z$ is a stationary point if it solves the boundary value problem

$$\begin{cases} -\Delta \mathbf{z} + \mathbf{P}\Psi^1_{,\mathbf{z}}(\mathbf{z}) = \boldsymbol{f}_1 + \mathbf{PAz}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}}\mathbf{z} = \mathbf{0}, & \text{a.e. on } \partial\Omega, \\ \sum_{i=1}^N \mathbf{z}_i = 1, & \text{in } \Omega. \end{cases} \tag{3.12}$$

Let then $\mathcal{W}$ be the set of all the stationary points. As we shall see below, Theorem 8.1 guarantees that $\mathbf{z}$ is *strictly separated* from the pure phases, i.e., there exists $0 < \delta = \delta(\boldsymbol{f}) < \frac{1}{N}$ such that

$$\delta < \mathbf{z}(x) \tag{3.13}$$

for any $x \in \overline{\Omega}$. Thus all the stationary points in $\mathcal{W}$ are strictly separated, but possibly *not* uniformly. However, it can be proven that $\omega(\mathbf{u}_0) \subset \mathcal{W}$ and that $\omega(\mathbf{u}_0)$ is actually *uniformly* strictly separated from the pure phases. Indeed, we have (see Sect. 7.1 below for the proof).

Lemma 3.11 *For any $\mathbf{u}_0 \in \mathcal{V}_{\boldsymbol{M}}$ it holds $\omega(\mathbf{u}_0) \subset \mathcal{W}$, namely, each element $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$ is a solution to the steady-state equation (3.12), with $\mathbf{f}_1 = \overline{\mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty)}$. Moreover, it holds $\overline{\mathbf{u}}_\infty = \overline{\mathbf{u}}_0$, so that $\omega(\mathbf{u}_0) \subset \mathcal{V}_{\boldsymbol{M}}$, and there exists $\delta > 0$ so that*

$$\delta < \mathbf{u}_\infty, \quad \forall\, \mathbf{u}_\infty \in \omega(\mathbf{u}_0)$$

for any $x \in \overline{\Omega}$, i.e., the ω-limit set of $\mathbf{u}_0$ is uniformly strictly separated from the pure phases.

Remark 3.12 As already noticed, thanks to the costraints on $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$, the strict separation property also implies that

$$\mathbf{u}_\infty(x) < 1 - (N-1)\delta$$

for any $x \in \overline{\Omega}$.

Thanks to the choice of $r \in (\frac{d}{4}, 1)$, $\omega(\mathbf{u}_0)$ is compact in $\mathbf{L}^\infty(\Omega)$ and thus it is totally bounded in $\mathbf{L}^\infty(\Omega)$. This means that we can choose $\varepsilon > 0$ such that, e.g., $\varepsilon < \frac{\delta}{2}$, where δ is given by Lemma 3.11, and there exists a finite number, say M_0, of $\mathbf{L}^\infty$-open balls $B_{\varepsilon,n}$ of radius ε such that

$$\omega(\mathbf{u}_0) \subset \bigcup_{n=1}^{M_0} B_{\varepsilon,n} =: U_\varepsilon \subset \mathbf{L}^\infty(\Omega),$$

and $\omega(\mathbf{u}_0) \cap B_{\varepsilon,n} \neq \emptyset$ for any $n = 1, \ldots, M_0$. Note that U_ε is open in $\mathbf{L}^\infty(\Omega)$. Therefore, thanks to Lemma 3.11, we infer that, for any $\mathbf{v} \in U_\varepsilon$, for any $j = 1, \ldots, N$

and for some $\mathbf{u} \in \omega(\mathbf{u}_0)$ (depending on $\mathbf{v}$),

$$\begin{aligned} v_j(x) &\leq u_j(x) + |v_j(x) - u_j(x)| \\ &\leq 1 - (N-1)\delta + \|\mathbf{v} - \mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} \\ &\leq 1 - ((N-1)\delta - \varepsilon) \end{aligned}$$

and also

$$v_j(x) \geq u_j(x) - |v_j(x) - u_j(x)| \geq \delta - \|\mathbf{v} - \mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} \geq \delta - \varepsilon.$$

Thus, for almost any $x \in \Omega$, we have that

$$0 < \delta - \varepsilon \leq \mathbf{v}(x) \leq 1 - ((N-1)\delta - \varepsilon) < 1, \quad \forall \mathbf{v} \in U_\varepsilon.$$

Furthermore, by (3.11) and the embedding $\mathbf{H}^{2r}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$ we deduce that there exists $t^* \geq 0$ such that $\mathbf{u}(t) = S(t)\mathbf{u}_0 \subset U_\varepsilon$ for any $t \geq t^*$. This means that, also for the three-dimensional case, a strict separation property holds asymptotically, namely,

Theorem 3.13 *Let the assumptions of Theorems 3.1 and 3.7 hold. Then, for any* $\mathbf{M} \in (0,1)$, $\mathbf{M} \in \Sigma$, *and for any initial datum* $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$, *there exists* $\delta > 0$ *and* $t^* = t^*(\mathbf{u}_0)$ *such that the corresponding (unique) solution* $\mathbf{u}$ *satisfies:*

$$\delta < \mathbf{u}(x,t), \quad \textit{for any } (x,t) \in \overline{\Omega} \times [t^*, +\infty). \tag{3.14}$$

Remark 3.14 Note that, differently from the two-dimensional case with assumption (**E2**), the strict separation is not uniform with respect to the initial datum, Moreover, in the case $d = 3$, we only have $\mathbf{u} \in C([t^*, +\infty); \mathbf{H}^{3/2}(\Omega))$ which is *not* embedded into $C(\overline{\Omega} \times [t^*, +\infty))$. Nevertheless the separation holds everywhere in $\overline{\Omega} \times [t^*, +\infty)$, since surely $\mathbf{u}(t) \in C(\overline{\Omega})$ for any $t \geq t^*$.

In order to show that $\omega(\mathbf{u}_0)$ is actually a singleton we need to require a further assumption on ψ, that is,

(**E3**) ψ is (real) analytic in $(0,1)$.

Due to (3.14), the singularities of ψ and its derivatives no longer play any role in our analysis as we are only interested in the behavior of the solution $\mathbf{u}(t)$, as $t \to \infty$. Thus we can alter the function ψ outside the interval $I_\varepsilon = [\delta - \varepsilon, 1 - ((N-1)\delta - \varepsilon)]$ in such a way that the extension $\widetilde{\psi}$ is of class $C^3(\mathbb{R}^N)$ and additionally $|\widetilde{\psi}^{(j)}(s)|$, $j = 1, 2, 3$, are uniformly bounded on $\mathbb{R}$. Correspondingly we define $\widetilde{\Psi}(\mathbf{s}) := \sum_{i=1}^N \widetilde{\psi}(s_i) - \frac{1}{2}\mathbf{A}\mathbf{s} \cdot \mathbf{s}$. Observe that $\widetilde{\psi}_{|I_\varepsilon} = \psi$ and ψ is analytic in I_ε by assumption (**E3**). We then introduce the "reduced" energy $\widetilde{\mathcal{E}} : \mathbf{V}_0 \to \mathbb{R}$ by setting

$$\widetilde{\mathcal{E}}(\mathbf{z}) := \frac{1}{2}\int_\Omega |\nabla \mathbf{z}|^2 dx + \int_\Omega \widetilde{\Psi}(\mathbf{z} + \mathbf{M}) dx.$$

Observe that $\widetilde{\mathcal{E}}(\mathbf{u}_0 - \mathbf{M}) = \mathcal{E}(\mathbf{u}_0)$ for all $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}} \cap U_\varepsilon$, thanks to (3.14) and to the definition of the extension $\widetilde{\psi}$. We then recall the following fundamental lemma whose proof is based on [10] (see Sect. 7 below).

Lemma 3.15 (Łojasiewicz–Simon Inequality) *Let* $\mathbf{u}$ *be the global solution of* (3.1) *in the sense of Theorems* 3.1 *and* 3.7 *with* $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$, *and suppose* $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$. *Then there exist* $\theta \in (0, \frac{1}{2}]$, $C, \sigma > 0$ *such that*

$$\left|\widetilde{\mathcal{E}}(\mathbf{v}) - \widetilde{\mathcal{E}}(\mathbf{u}_\infty - \mathbf{M})\right|^{1-\theta} \leq C\|\widetilde{\mathcal{E}}'(\mathbf{v})\|_{\mathbf{V}_0'},$$

whenever $\|\mathbf{v} - \mathbf{u}_\infty + \mathbf{M}\|_{\mathbf{V}_0} \leq \sigma$.

Exploiting this tool, we are then able to prove that $\omega(\mathbf{u}_0)$ is a singleton. More precisely, we have (for the proof see Sect. 7.3 below).

Theorem 3.16 *Let the assumptions of Theorems* 3.1 *and* 3.7 *hold. Then, for any* $\mathbf{u}_0 \in \mathcal{V}_{\boldsymbol{M}}$, *it holds* $\omega(\mathbf{u}_0) = \{\mathbf{u}_\infty\}$, *where* $\mathbf{u}_\infty \in \mathcal{W} \cap \mathcal{V}_{\boldsymbol{M}}$, *i.e.,* $\mathbf{u}_\infty$ *is a solution to*

$$\begin{cases} -\Delta \mathbf{u}_\infty + \mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{u}_\infty) = \boldsymbol{f}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}} \mathbf{u}_\infty = \mathbf{0}, & \text{a.e. on } \partial\Omega, \\ \sum_{i=1}^N \mathbf{u}_{\infty,i} = 1, & \text{in } \Omega. \end{cases}$$

with $\boldsymbol{f} = \mathbf{PAu}_\infty + \overline{\mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty)}$. *Moreover,* $\overline{\mathbf{u}}_\infty = \mathbf{M}$, *there exists* $\delta > 0$ *so that*

$$\delta < \mathbf{u}_\infty(x), \quad \forall x \in \overline{\Omega},$$

and the (unique) weak solution $\mathbf{u}(t)$ *is such that*

$$\mathbf{u}(t) = S(t)\mathbf{u}_0 \underset{t\to\infty}{\longrightarrow} \mathbf{u}_\infty \quad \text{in } \mathbf{H}^{2r}(\Omega), \tag{3.15}$$

for any $r \in (0, 1)$.

4 Proofs of Subsection 3.1

Here we collect the proofs of Theorems 3.1, 3.6 and 3.7.

4.1 Proof of Theorem 3.1

$C_t\mathbf{H}^1_x$ regularity and the energy identity. First we want to show that $\mathbf{u} \in C([0, T]; \mathbf{H}^1(\Omega))$. This is not a trivial consequence of the other regularities. Indeed we can only get $\mathbf{u} \in C([0, T]; \boldsymbol{H}^{\frac{1}{2}}(\Omega))$ from $\mathbf{u} \in L^4(0, T; \mathbf{H}^2(\Omega)) \cap H^1(0, T, \mathbf{H}^1(\Omega)')$. The idea is to apply [41, Lemma 4.1]. Let us set $\mathbf{M} := \overline{\mathbf{u}}_0$ and introduce the functional

$$\mathcal{J}(\mathbf{v}) := \frac{1}{2}\|\nabla \mathbf{v}\|^2 + \int_\Omega \Psi^1(\mathbf{v} + \mathbf{M})dx,$$

whose effective domain in $\mathbf{H}_0$ is

$$\mathfrak{D}(\mathcal{J}) = \left\{\mathbf{v} \in \mathbf{V}_0 : \Psi^1(\mathbf{v} + \mathbf{M}) \in \mathbf{L}^1(\Omega)\right\}.$$

Clearly $\mathcal{J}$ is proper and convex, but also lower semicontinuous (w.r.t. $\mathbf{L}^2(\Omega)$ topology) in $\mathbf{H}_0$, thanks to the property that all functions in $\mathbf{H}_0$ have zero integral mean (see, e.g., [1, Proof of Lemma 4.1]). Moreover, we can immediately see that, for almost any fixed $t \geq 0$, the solution $\mathbf{u}(t)$ is such that $\mathbf{z}(t) := \mathbf{u}(t) - \mathbf{M} \in \mathfrak{D}(\mathcal{J})$ and

$$\mathbf{q}(t) := \mathbf{w}(t) + \mathbf{PAu}(t) - \overline{\mathbf{w}}(t) - \overline{\mathbf{PAu}}(t) \in \partial\mathcal{J}(\mathbf{z}(t)).$$

Indeed, by convexity of $\mathcal{J}$ and integration by parts, recalling that $\mathbf{w}(t) + \mathbf{PAu}(t) = -\Delta\mathbf{u}(t) + \mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{u}(t))$, we have that, for any $\mathbf{v} \in \mathfrak{D}(\mathcal{J})$,

$$\begin{aligned}
(\mathbf{q}(t), \mathbf{v} - \mathbf{z}(t)) &= (\mathbf{w}(t) + \mathbf{PAu}(t) - \overline{\mathbf{w}}(t) - \overline{\mathbf{PAu}}(t), \mathbf{v} - \mathbf{z}(t)) \\
&= (-\Delta\mathbf{u}(t) + \mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{u}(t)), \mathbf{v} - \mathbf{z}(t)) \\
&= (-\Delta\mathbf{z}(t) + \mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{z}(t) + \mathbf{M}), \mathbf{v} - \mathbf{z}(t)) \\
&= (-\Delta\mathbf{z}(t) + \Psi^1_{,\mathbf{u}}(\mathbf{z}(t) + \mathbf{M}), \mathbf{v} - \mathbf{z}(t)) \\
&= \lim_{\lambda\to 0} \frac{\mathcal{J}(\mathbf{z}(t) + \lambda(\mathbf{v} - \mathbf{z}(t))) - \mathcal{J}(\mathbf{z}(t))}{\lambda} \\
&\leq \mathcal{J}(\mathbf{v}) - \mathcal{J}(\mathbf{z}(t)),
\end{aligned}$$

where we crucially exploited the identities $\overline{\mathbf{v} - \mathbf{z}(t)} = 0$ and $\mathbf{v} - \mathbf{z}(t) = \mathbf{P}(\mathbf{v} - \mathbf{z}(t))$. Notice that it is essential that $\overline{\mathbf{q}}(t) = 0$, since by definition of subdifferential and recalling the identification $\mathbf{H}_0' \equiv \mathbf{H}_0$, we need $\mathbf{q}(t) \in \mathbf{H}_0$. Summing up, thanks to the regularity (3.10) and using the Hilbert triplet $\mathbf{V}_0 \hookrightarrow\hookrightarrow \mathbf{H}_0 \equiv \mathbf{H}_0' \hookrightarrow \mathbf{V}_0'$, we have:

- $\mathcal{J} : \mathbf{H}_0 \to (-\infty, +\infty]$ is a proper, convex, lower semicontinous functional;
- $\mathbf{z} = \mathbf{u} - \mathbf{M} \in H^1(0, T; \mathbf{V}_0') \cap L^2(0, T; \mathbf{V}_0)$;
- $\mathbf{q}(t) \in \partial\mathcal{J}(\mathbf{z}(t))$ for almost any $t \in (0, T)$ and $\mathbf{q} \in L^2(0, T; \mathbf{V}_0)$;
- by Poincaré's inequality and since Ψ^1 is bounded below, there exist $k_1, k_2 > 0$ such that
$$\mathcal{J}(\mathbf{r}) \geq k_1\|\mathbf{r}\|^2 - k_2, \quad \forall\, \mathbf{r} \in \mathbf{H}_0.$$

Therefore, we can apply [41, Lemma 4.1], with $H = \mathbf{H}_0$ and $V = \mathbf{V}_0$ and conclude that the function $\mathcal{J} : t \mapsto \mathcal{J}(\mathbf{z}(t)) \in AC([0, T])$ and

$$\int_s^t \langle \partial_t \mathbf{z}(r), \mathbf{q}(r)\rangle_{\mathbf{V}_0', \mathbf{V}_0} dr = \mathcal{J}(\mathbf{z}(t)) - \mathcal{J}(\mathbf{z}(s)).$$

Let us introduce now the functional

$$\tilde{\mathcal{J}}(\mathbf{v}) := \mathcal{J}(\mathbf{v} - \mathbf{M}) = \frac{1}{2}\|\nabla\mathbf{v}\|^2 + \int_\Omega \Psi^1(\mathbf{v})dx$$

and observe that $\mathcal{J}(\mathbf{z}(t)) = \tilde{\mathcal{J}}(\mathbf{u}(t))$. Moreover, being $\partial_t\mathbf{u}(t) \in \mathbf{H}^1(\Omega)'$ for almost any $t \in (0, T)$, we can consider its restriction in $\mathbf{V}_0'$ so that

$$\langle \partial_t\mathbf{u}(t), \mathbf{v}\rangle_{\mathbf{V}_0', \mathbf{V}_0} = \langle \partial_t\mathbf{u}(t), \mathbf{v}\rangle_{\mathbf{H}^1(\Omega)', \mathbf{H}^1(\Omega)}.$$

Since $\partial_t \mathbf{z}(t) \equiv \partial_t \mathbf{u}(t)$, we immediately deduce that

$$\langle \partial_t \mathbf{z}(t), \mathbf{v} \rangle_{\mathbf{V}_0', \mathbf{V}_0} = \langle \partial_t \mathbf{u}(t), \mathbf{v} \rangle_{\mathbf{V}_0', \mathbf{V}_0} = \langle \partial_t \mathbf{u}(t), \mathbf{v} \rangle_{\mathbf{H}^1(\Omega)', \mathbf{H}^1(\Omega)}.$$

Therefore, we obtain that $\tilde{\mathcal{J}}(\mathbf{u}(t)) \in AC([0, T])$ and

$$\int_s^t \langle \partial_t \mathbf{u}(r), \mathbf{w}(r) + \mathbf{PAu}(r) \rangle_{\mathbf{H}^1(\Omega)', \mathbf{H}^1(\Omega)} dr = \tilde{\mathcal{J}}(\mathbf{u}(t)) - \tilde{\mathcal{J}}(\mathbf{u}(s)), \tag{4.1}$$

recalling the definition of $\mathbf{q}$ and the property that $\langle \partial_t \mathbf{u}(t), \mathbf{c} \rangle_{\mathbf{H}^1(\Omega)', \mathbf{H}^1(\Omega)} \equiv 0$, for any $\mathbf{c} \in \mathbb{R}^N$. On account of the fact that Ψ^1 is bounded and $\mathbf{u} \in C([0, T]; \mathbf{H}^{\frac{1}{2}}(\Omega))$, the Lebesgue Dominated Convergence theorem implies that $t \mapsto \int_\Omega \Psi^1(\mathbf{u}(t)))$ is continuous. This guarantees, together with $\tilde{\mathcal{J}}(\mathbf{u}(t)) \in AC([0, T])$, that $t \mapsto \|\nabla \mathbf{u}\|^2$ is continuous, entailing

$$\mathbf{u} \in C([0, T]; \mathbf{H}^1(\Omega)).$$

Moreover, thanks to (4.1), we have

$$\frac{d}{dt} \tilde{\mathcal{J}}(\mathbf{u}(t)) = \langle \partial_t \mathbf{u}(t), \mathbf{w}(t)(r) + \mathbf{Pu}(t) \rangle_{\mathbf{H}^1(\Omega)', \mathbf{H}^1(\Omega)},$$

from which it is easy to show that the energy identity (3.8) holds, by testing (3.6) with $\mathbf{w}(t)$.

Existence of a solution. We consider the approximation (2.8). In particular, we define, for each $\varepsilon > 0$ sufficiently small,

$$\boldsymbol{\phi}_\varepsilon(\mathbf{y}) = \Psi^1_{\varepsilon, \mathbf{y}}(\mathbf{y}) = \{\psi_\varepsilon'(y_i)\}_{i=1,\dots,N}, \quad \forall \mathbf{y} \in \mathbb{R}^N.$$

We then fix $0 < \varepsilon < \varepsilon_0$ and look for a couple $(\mathbf{u}_\varepsilon, \mathbf{w}_\varepsilon)$, such that, for each $T > 0$,

$$\begin{gathered} \mathbf{u}_\varepsilon \in L^\infty(0, T; \mathbf{H}^1(\Omega)), \\ \partial_t \mathbf{u}_\varepsilon \in L^2(0, T; (\mathbf{H}^1(\Omega))'), \\ \mathbf{w}_\varepsilon \in L^2(0, T; \mathbf{H}^1(\Omega)), \end{gathered}$$

and, for all $\boldsymbol{\eta} \in \mathbf{H}^1(\Omega)$, satisfies, almost everywhere in $(0, T)$,

$$< \partial_t \mathbf{u}_\varepsilon, \boldsymbol{\eta} > + (\boldsymbol{\alpha} \nabla \mathbf{w}_\varepsilon, \nabla \boldsymbol{\eta}) = 0, \tag{4.2}$$
$$(\mathbf{w}_\varepsilon, \boldsymbol{\eta}) = \gamma(\nabla \mathbf{u}_\varepsilon, \nabla \boldsymbol{\eta}) + (\mathbf{P}(-\mathbf{Au}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\eta}). \tag{4.3}$$

We recall some results obtained in [16, Propositions 2.1-—2.3]:

- Conservation of mass:

$$\overline{\mathbf{u}}_\varepsilon = \overline{\mathbf{u}}_0.$$

- Conservation of total mass

$$\sum_{i=1}^{N} u_{\varepsilon,i} = 1, \qquad \forall\, x \in \Omega, \quad t > 0. \tag{4.4}$$

- Conservation of total chemical potential differences

$$\sum_{i=1}^{N} \mathbf{w}_{\varepsilon,i} = 0, \qquad \forall\, x \in \Omega, \quad t > 0. \tag{4.5}$$

- There exists $C > 0$ depending only on the initial data and independent of ε, such that, for any $t > 0$,

$$\int_\Omega \Psi_\varepsilon(\mathbf{u}_\varepsilon(t))dx + \|\nabla \mathbf{u}_\varepsilon(t)\|^2 + \int_0^t \|\nabla \mathbf{w}_\varepsilon(\tau)\|^2 d\tau \le C, \tag{4.6}$$

  and, by the conservation of mass and Poincaré's inequality, it holds

$$\|\mathbf{u}_\varepsilon(t)\|_{\mathbf{H}^1(\Omega)} \le C, \quad \text{for a.a. } t \in (0, T). \tag{4.7}$$

  Clearly, by (4.6), it is straightforward to infer

$$\|\partial_t \mathbf{u}_\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega)')} \le C.$$

- There exists a constant $C > 0$ depending on the initial data and T, but not on ε, such that

$$t\|\nabla \mathbf{w}_\varepsilon(t)\|^2 + \int_0^T s\|\nabla \partial_t \mathbf{u}_\varepsilon(s)\|^2 ds \le C, \tag{4.8}$$

  for almost any $t \in (0, T)$.

Notice that actually the proof in [16] is carried out for a different approximation ψ_ε of ψ, but the same proof by means of a Galerkin scheme can be adapted to the case of the approximation (2.8), thanks to properties (i)–(vi). In particular, (vi) is essential to guarantee that the approximated energy (i.e., (1.3) with Ψ_ε in place of Ψ) is bounded below by a constant (see also Remark 2.1). At this point, differently from what was done in [16], we follow some ideas coming from [29], in order to recover the control over $\overline{\mathbf{w}}_\varepsilon(t)$, which then gives the control of $\mathbf{w}_\varepsilon(t)$ in $\mathbf{H}^1(\Omega)$. In particular, as in the proof of [29, Lemma 3.3], we define

$$\mathbf{w}_{\varepsilon,0} := \mathbf{w}_\varepsilon - \boldsymbol{\lambda}_\varepsilon,$$

with, on account of the boundary conditions,

$$\boldsymbol{\lambda}_\varepsilon := \overline{\mathbf{w}}_\varepsilon = \overline{\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon))}.$$

Taking advantage of (4.3), we have, for all $\boldsymbol{\eta} \in \mathbf{H}^1(\Omega)$ and for almost all $t \in (0, T)$,

$$(\mathbf{w}_{\varepsilon,0} + \boldsymbol{\lambda}_\varepsilon, \boldsymbol{\eta}) = \gamma(\nabla \mathbf{u}_\varepsilon, \nabla \boldsymbol{\eta}) + (\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\eta}). \tag{4.9}$$

Then we exploit the convexity of Ψ^1_ε: for any $\mathbf{k} \in \mathbf{G}$, $\mathbf{G}$ being the Gibbs simplex, because $\mathbf{k} - \mathbf{u}_\varepsilon \in T\Sigma$ almost everywhere, we find

$$\begin{aligned} C &\geq \int_\Omega \Psi^1_\varepsilon(\mathbf{k}) \geq \int_\Omega \Psi^1_\varepsilon(\mathbf{u}_\varepsilon) + \int_\Omega \Psi^1_{\varepsilon,\mathbf{u}}(\mathbf{u}_\varepsilon) \cdot (\mathbf{k} - \mathbf{u}_\varepsilon) \\ &= \int_\Omega \Psi^1_\varepsilon(\mathbf{u}_\varepsilon) + \int_\Omega \mathbf{P}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon) \cdot (\mathbf{k} - \mathbf{u}_\varepsilon), \end{aligned} \tag{4.10}$$

where we used (see property (i) of ψ_ε)

$$\int_\Omega \Psi^1_\varepsilon(\mathbf{k}) \leq \int_\Omega \Psi^1(\mathbf{k}) \leq |\Omega| \max_{\mathbf{s}\in[0,1]} |\Psi^1(\mathbf{s})| = C.$$

Here and in the sequel $C > 0$ stands for a generic constant independent of ε. Any further dependency will be explicitly pointed out if needed.

Note that $\Psi^1_{\varepsilon,\mathbf{u}}(\mathbf{u}_\varepsilon) = \{\psi'_\varepsilon(u_{\varepsilon,i})\}_{i=1,\ldots,N}$. Moreover, we can choose $\boldsymbol{\eta} = \mathbf{k} - \mathbf{u}_\varepsilon$ in (4.9) to deduce, on account of (4.10), that

$$\begin{aligned} C &\geq \int_\Omega \Psi^1_\varepsilon(\mathbf{k}) \\ &\geq \int_\Omega \Psi^1_\varepsilon(\mathbf{u}_\varepsilon) + (\mathbf{P}(\mathbf{A}\mathbf{u}_\varepsilon), \mathbf{k} - \mathbf{u}_\varepsilon) \\ &\quad + \gamma\|\nabla \mathbf{u}_\varepsilon\|^2 + (\mathbf{w}_{\varepsilon,0}, \mathbf{k} - \mathbf{u}_\varepsilon) + (\boldsymbol{\lambda}_\varepsilon, \mathbf{k} - \mathbf{u}_\varepsilon), \end{aligned}$$

for almost all $t \in (0, T)$. Observe that

$$\int_\Omega \sum_{i=1}^N k_i^2 \leq \int_\Omega \left(\sum_{i=1}^N k_i\right)^2 = |\Omega|_d.$$

Then by Cauchy–Schwarz's, Young's and Poincaré's inequalities (all applied to $\mathbf{w}_{\varepsilon,0}$), recalling property (vi) of ψ_ε, we obtain

$$\begin{aligned} &(\boldsymbol{\lambda}_\varepsilon, \mathbf{k} - \mathbf{u}_\varepsilon) + \gamma\|\nabla \mathbf{u}_\varepsilon\|^2 - \tilde{K} \\ &\quad \leq (\boldsymbol{\lambda}_\varepsilon, \mathbf{k} - \mathbf{u}_\varepsilon) + \gamma\|\nabla \mathbf{u}_\varepsilon\|^2 + \int_\Omega \Psi^1_\varepsilon(\mathbf{u}_\varepsilon) \\ &\quad \leq -(\mathbf{P}(\mathbf{A}\mathbf{u}_\varepsilon), \mathbf{k} - \mathbf{u}_\varepsilon) - (\mathbf{w}_{\varepsilon,0}, \mathbf{k} - \mathbf{u}_\varepsilon) \\ &\quad \leq C\left(1 + \|\mathbf{u}_\varepsilon\| + \|\mathbf{u}_\varepsilon\|^2 + \|\nabla \mathbf{w}_\varepsilon\|(1 + \|\mathbf{u}_\varepsilon\|)\right) \leq C(1 + \|\nabla \mathbf{w}_\varepsilon\|), \end{aligned} \tag{4.11}$$

where in the last estimate we have exploited (4.7). By the conservation of mass and Remark 3.2, we also deduce that, for all $i = 1, \ldots, N$ and all $t \in [0, T]$,

$$0 < \delta_0 < \overline{\mathbf{u}}_{\varepsilon,i} < 1 - (N-1)\delta_0 < 1 - \delta_0.$$

Therefore we choose, for fixed $k, l = 1, \ldots, N$,

$$\mathbf{k} = \overline{\mathbf{u}}_\varepsilon + \delta_0 \mathrm{sign}(\boldsymbol{\lambda}_{\varepsilon,k} - \boldsymbol{\lambda}_{\varepsilon,l})(\boldsymbol{\eta}_k - \boldsymbol{\eta}_l) \in \mathbf{G}$$

in (4.11), where $\boldsymbol{\eta}_j$ is the j-th element of the standard orthonormal Euclidean basis of $\mathbb{R}^N$, i.e. $\boldsymbol{\eta}_j = (0, \ldots, 0, \underbrace{1}_{j\text{-th position}}, \ldots, 0)$. Thus, from (4.11) we get that

$$|(\boldsymbol{\lambda}_{\varepsilon,k} - \boldsymbol{\lambda}_{\varepsilon,l})(t)| \leq \frac{C}{\delta_0 |\Omega|_d}(1 + \|\nabla \mathbf{w}_\varepsilon\|) \tag{4.12}$$

Integrating then $|(\lambda_{\varepsilon,k} - \lambda_{\varepsilon,l})(t)|^2$ from 0 to T and using the identity

$$\boldsymbol{\lambda}_\varepsilon = \frac{1}{N}\left(\sum_{l=1}^{N}(\boldsymbol{\lambda}_{\varepsilon,k} - \boldsymbol{\lambda}_{\varepsilon,l})\right)_{k=1,\ldots,N},$$

we find

$$\int_0^T |\boldsymbol{\lambda}_\varepsilon(t)|^2 dt \leq C.$$

This, together with (4.6) and Poincaré's inequality, gives

$$\|\mathbf{w}_\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C. \tag{4.13}$$

Coming back to (4.12), we also deduce that

$$|\boldsymbol{\lambda}_\varepsilon(t)|^2 \leq C(1 + \|\nabla \mathbf{w}_\varepsilon(t)\|^2),$$

for almost any $t \in (0, T)$. Therefore, by (4.8), we infer, again by Poincar é's inequality, that

$$\|\sqrt{t}\mathbf{w}_\varepsilon\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \leq C. \tag{4.14}$$

We are now left with some estimates concerning the potential $\boldsymbol{\phi}_\varepsilon$. We follow some ideas in [29, Lemma 5.1]. Being ϕ'_ε bounded for $0 < \varepsilon < \varepsilon_0$ fixed, we have that

$$\nabla \phi_\varepsilon(u_{\varepsilon,i}) = \phi'_\varepsilon(u_{\varepsilon,i}) \nabla \mathbf{u}_{\varepsilon,i} \in \mathbf{L}^2(\Omega),$$

for almost any $t \in (0, T)$. Thus we can test (4.3) with $\boldsymbol{\eta} = \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon(t))$ to get

$$\sum_{i=1}^{N}(w_{\varepsilon,i}, \phi_\varepsilon(u_{\varepsilon,i})) = \sum_{i=1}^{N}\left(\gamma(\nabla u_{\varepsilon,i}, \phi_\varepsilon'(u_{\varepsilon,i})\nabla u_{\varepsilon,i})\right) + (\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)). \tag{4.15}$$

Observe that

$$(\mathbf{P}(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)) = \sum_{k=1}^{N}\int_\Omega \left(\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N}\sum_{l=1}^{N}\phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon(u_{\varepsilon,k})dx,$$

and

$$\begin{aligned}
&\sum_{k=1}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N}\sum_{l=1}^{N}\phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon(u_{\varepsilon,k}) \\
&= \frac{1}{N}\sum_{k,l=1}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon(u_{\varepsilon,k}) \\
&= \frac{1}{N}\sum_{k<l}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon(u_{\varepsilon,k})\sum_{k>l}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon(u_{\varepsilon,k}) \\
&= \frac{1}{N}\sum_{k<l}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\right)(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})) \\
&= \frac{1}{N}\sum_{k<l}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\right)^2.
\end{aligned}$$

Thanks to (4.4), we have

$$u_{\varepsilon,m} := \min_{i=1,\ldots,N} u_{\varepsilon,i} \leq \frac{1}{N} \leq \max_{i=1,\ldots,N} u_{\varepsilon,i} =: u_{\varepsilon,M}, \tag{4.16}$$

so that, being ϕ_ε monotone, we infer

$$\begin{aligned}
\frac{1}{N}\sum_{k<l}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon(u_{\varepsilon,l})\right)^2 &\geq \frac{1}{N}\left(\phi_\varepsilon(u_{\varepsilon,m})-\phi_\varepsilon(u_{\varepsilon,M})\right)^2\\
&\geq \frac{1}{N}\max_{i=1,\dots,N}\left(\phi_\varepsilon(u_{\varepsilon,i})-\phi_\varepsilon\left(\frac{1}{N}\right)\right)^2\\
&\geq \frac{1}{N}\max_{i=1,\dots,N}\left(\frac{1}{2}\phi_\varepsilon(u_{\varepsilon,i})^2-\phi_\varepsilon\left(\frac{1}{N}\right)^2\right)\\
&\geq \frac{1}{2N}\max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2-C,
\end{aligned}$$

owing to the basic inequality $(a-b)^2 \geq \frac{1}{2}a^2-b^2$. Notice that C is independent of ε provided that we choose ε sufficiently small. Indeed, since we have the pointwise convergence $\phi_\varepsilon(\frac{1}{N}) \to \phi(\frac{1}{N})$ as $\varepsilon \to 0$, then there exists $C>0$, independent of ε, such that $|\phi_\varepsilon(\frac{1}{N})| \leq C$ for any $\varepsilon \in (0,\varepsilon_0)$. Then, by (4.7) and the embedding $\mathbf{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$, we get

$$\sum_{i=1}^{N}(w_{\varepsilon,i},\phi_\varepsilon(u_{\varepsilon,i})) \leq \sum_{i=1}^{N}\|w_{\varepsilon,i}\|\|\phi_\varepsilon(u_{\varepsilon,i})\| \leq C\|\mathbf{w}_\varepsilon\|^2+\frac{1}{8N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2dx,$$

and

$$\begin{aligned}
|(\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon,\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon))| &\leq C\|\mathbf{u}_\varepsilon\|^2+\frac{1}{8N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2dx\\
&\leq C+\frac{1}{8N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2dx.
\end{aligned}$$

Therefore, on account of the above inequalities and recalling that $\phi_\varepsilon' \geq 0$, we deduce from (4.15) that

$$\frac{1}{4N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2dx \leq C\left(1+\|\mathbf{w}_\varepsilon\|^2\right), \tag{4.17}$$

which yields (see (4.13))

$$\|\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \leq C(T). \tag{4.18}$$

From this result, together with (4.7) and (4.13), by elliptic regularity, we infer from (4.3) that

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \leq C(T). \tag{4.19}$$

Concerning $\mathbf{u}_\varepsilon$ we can actually obtain more. Indeed, due to (4.19), we see that (4.3) also holds in a strong sense, that is,

$$\mathbf{w}_\varepsilon = -\gamma\Delta\mathbf{u}_\varepsilon + \mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \quad \text{a.e. in } \Omega\times(0,T).$$

We can then multiply this equation by $-\Delta\mathbf{u}_\varepsilon \in T\Sigma$. Then, recalling the properties of the projector $\mathbf{P}$, after an integration by parts, we get

$$\begin{aligned}\gamma\|\Delta\mathbf{u}_\varepsilon\|^2 &= -(\mathbf{w}_\varepsilon, \Delta\mathbf{u}_\varepsilon) + (-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \Delta\mathbf{u}_\varepsilon)\\ &= (\nabla\mathbf{w}_\varepsilon, \nabla\mathbf{u}_\varepsilon) + (-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \Delta\mathbf{u}_\varepsilon).\end{aligned}$$

Observe now that, integrating by parts, we have

$$-(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \Delta\mathbf{u}_\varepsilon) = (\nabla\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \nabla\mathbf{u}_\varepsilon) = \sum_{i=1}^N \left(\nabla u_{\varepsilon,i}, \phi_\varepsilon'(u_{\varepsilon,i})\nabla u_{\varepsilon,i}\right) \geq 0.$$

Thus, by standard inequalities and integration by parts, we obtain

$$\begin{aligned}\gamma\|\Delta\mathbf{u}_\varepsilon\|^2 &\leq (\nabla\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \nabla\mathbf{u}_\varepsilon) + \gamma\|\Delta\mathbf{u}_\varepsilon\|^2\\ &\leq \|\nabla\mathbf{w}_\varepsilon\|\|\nabla\mathbf{u}_\varepsilon\| + \|\nabla\mathbf{A}\mathbf{u}_\varepsilon\|\|\nabla\mathbf{u}_\varepsilon\|\\ &\leq C(1+\|\nabla\mathbf{w}_\varepsilon\|).\end{aligned}$$

Therefore, given that $\|\nabla\mathbf{w}_\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq C$ and using the conservation of mass, we find

$$\|\mathbf{u}_\varepsilon\|_{L^4(0,T;\mathbf{H}^2(\Omega))} \leq C.$$

Let us now set

$$\phi_\varepsilon^r(s) = \phi_\varepsilon(s)|\phi_\varepsilon(s)|^{r-2}, \quad \forall s\in\mathbb{R},$$

for a given $r \geq 2$. Notice that, being ϕ_ε' bounded, and ϕ_ε sublinear, by (4.19) and the embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, $u_{\varepsilon,i}\in L^\infty(\Omega)$, $i=1,\dots,N$, we have

$$\nabla\phi_\varepsilon^r(u_{\varepsilon,i}) = (r-1)\phi_\varepsilon'(u_{\varepsilon,i})|\phi_\varepsilon(u_{\varepsilon,i})|^{r-2}\nabla u_{\varepsilon,i} \in \mathbf{L}^2(\Omega)$$

for almost any $t\in(0,T)$. We then test (4.3) with $\boldsymbol{\eta} = \{\phi_\varepsilon^r(u_{\varepsilon,i}(t))\}_{i=1,\dots,N}$. This gives

$$\begin{aligned}\sum_{i=1}^N (w_{\varepsilon,i}, \phi_\varepsilon^r(u_{\varepsilon,i})) &= \sum_{i=1}^N \left(\gamma(\nabla u_{\varepsilon,i}, (r-1)\phi_\varepsilon'(u_{\varepsilon,i})|\phi_\varepsilon(u_{\varepsilon,i})|^{r-2}\nabla u_{\varepsilon,i})\right)\\ &\quad + (\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \{\phi_\varepsilon^r(u_{\varepsilon,i})\}_{i=1,\dots,N}).\end{aligned} \tag{4.20}$$

Observe that

$$(\mathbf{P}(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \{\phi_\varepsilon^r(u_{\varepsilon,i})\}_{i=1,\dots,N}) = \sum_{k=1}^N \int_\Omega \left(\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N}\sum_{l=1}^N \phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon^r(u_{\varepsilon,k})dx,$$

and

$$
\begin{aligned}
&\sum_{k=1}^{N}\left(\phi_\varepsilon(u_{\varepsilon,k})-\frac{1}{N}\sum_{l=1}^{N}\phi_\varepsilon(u_{\varepsilon,l})\right)\phi_\varepsilon^r(u_{\varepsilon,k})\\
&\quad=\frac{1}{N}\sum_{k,l=1}^{N}\big(\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon(u_{\varepsilon,l})\big)\,\phi_\varepsilon^r(u_{\varepsilon,k})\\
&\quad=\frac{1}{N}\sum_{k<l}^{N}\big(\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon(u_{\varepsilon,l})\big)\,\phi_\varepsilon^r(u_{\varepsilon,k})\sum_{k>l}^{N}\big(\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon(u_{\varepsilon,l})\big)\,\phi_\varepsilon^r(u_{\varepsilon,k})\\
&\quad=\frac{1}{N}\sum_{k<l}^{N}\big(\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon(u_{\varepsilon,l})\big)\,(\phi_\varepsilon^r(u_{\varepsilon,k})-\phi_\varepsilon^r(u_{\varepsilon,l}))\geq 0,
\end{aligned}
\tag{4.21}
$$

since ϕ_ε and ϕ_ε^r are monotone non-decreasing. This result together with (4.16) gives

$$
\begin{aligned}
&\frac{1}{N}\sum_{k<l}^{N}\big(\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon(u_{\varepsilon,l})\big)\,(\phi_\varepsilon^r(u_{\varepsilon,k})-\phi_\varepsilon^r(u_{\varepsilon,l}))\\
&\quad\geq\frac{1}{N}\big(\phi_\varepsilon(u_{\varepsilon,m})-\phi_\varepsilon(u_{\varepsilon,M})\big)\,(\phi_\varepsilon^r(u_{\varepsilon,m})-\phi_\varepsilon^r(u_{\varepsilon,M}))\\
&\quad=\frac{1}{N}\left|\phi_\varepsilon(u_{\varepsilon,m})-\phi_\varepsilon(u_{\varepsilon,M})\right|\left|\phi_\varepsilon^r(u_{\varepsilon,m})-\phi_\varepsilon^r(u_{\varepsilon,M})\right|\\
&\quad\geq\frac{1}{N}\max_{k=1,\dots,N}\left|\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon\left(\frac{1}{N}\right)\right|\left|\phi_\varepsilon^r(u_{\varepsilon,k})-\phi_\varepsilon^r\left(\frac{1}{N}\right)\right|\\
&\quad\geq\frac{1}{N}\max_{k=1,\dots,N}\left||\phi_\varepsilon(u_{\varepsilon,k})|^r-\phi_\varepsilon^r\left(\frac{1}{N}\right)\phi_\varepsilon(u_{\varepsilon,k})-\phi_\varepsilon\left(\frac{1}{N}\right)\phi_\varepsilon^r(u_{\varepsilon,k})+\left|\phi_\varepsilon\left(\frac{1}{N}\right)\right|^r\right|\\
&\quad\geq\frac{1}{N}\max_{k=1,\dots,N}\Bigg(|\phi_\varepsilon(u_{\varepsilon,k})|^r-\left|\phi_\varepsilon\left(\frac{1}{N}\right)\right|^{r-1}\left|\phi_\varepsilon(u_{\varepsilon,k})\right|\\
&\quad-\left|\phi_\varepsilon\left(\frac{1}{N}\right)\right|\left|\phi_\varepsilon(u_{\varepsilon,k})\right|^{r-1}+\left|\phi_\varepsilon\left(\frac{1}{N}\right)\right|^r\Bigg)\geq\frac{1}{2N}\max_{k=1,\dots,N}|\phi_\varepsilon(u_{\varepsilon,k})|^r-C,
\end{aligned}
$$

where, in the last step, Young's inequality has been used several times. We note once more that C is independent of ε if we choose ε sufficiently small. We then have, by Hölder's and Young's inequalities,

$$
\begin{aligned}
&\sum_{i=1}^{N}(w_{\varepsilon,i},\phi_\varepsilon^r(u_{\varepsilon,i}))\\
&\quad\leq\sum_{i=1}^{N}\|w_{\varepsilon,i}\|_{L^r(\Omega)}\|\phi_\varepsilon(u_{\varepsilon,i})\|_{L^r(\Omega)}^{r-1}\\
&\quad\leq C\|\mathbf{w}_\varepsilon\|_{L^r(\Omega)}^r+\frac{1}{8N}\int_\Omega\max_{k=1,\dots,N}|\phi_\varepsilon(u_{\varepsilon,k})|^r\,dx,
\end{aligned}
$$

and also, by (4.7) and Sobolev embeddings,

$$
\begin{aligned}
&|(\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon, \{\phi_\varepsilon^r(u_{\varepsilon,i})\}_{i=1,\dots,N})| \\
&\quad\le C\|\mathbf{u}_\varepsilon\|^r_{\boldsymbol{L}^r(\Omega)} + \frac{1}{8N}\int_\Omega \max_{i=1,\dots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx \\
&\quad\le C + \frac{1}{8N}\int_\Omega \max_{i=1,\dots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx.
\end{aligned}
$$

Combining the above inequalities and recalling that $\phi_\varepsilon'(u_{\varepsilon,i}) \ge 0$, we obtain from (4.20) that

$$
\frac{1}{4N}\int_\Omega \max_{i=1,\dots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx \le C\left(1 + \|\mathbf{w}_\varepsilon\|^r_{\boldsymbol{L}^r(\Omega)}\right). \tag{4.22}
$$

We now treat the cases $d = 2$ and $d = 3$ separately. In the case $d = 3$, by the Sobolev embedding $\mathbf{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^q(\Omega)$, $q \in [2, 6]$, we have

$$
\frac{1}{4N}\int_\Omega \max_{i=1,\dots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx \le C\left(1 + \|\mathbf{w}_\varepsilon\|^r_{\boldsymbol{L}^r(\Omega)}\right) \le C(r)\left(1 + \|\mathbf{w}_\varepsilon\|^r_{\mathbf{H}^1(\Omega)}\right),
$$

and multiplying the above inequality by $t^{\frac{r}{2}}$, since $t \in (0, T)$, we get

$$
\frac{t^{\frac{r}{2}}}{4N}\int_\Omega \max_{i=1,\dots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx \le C(T, r)\left(1 + \|\sqrt{t}\mathbf{w}_\varepsilon\|^r_{\mathbf{H}^1(\Omega)}\right).
$$

Therefore, recalling (4.14), we infer

$$
\|\sqrt{t}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\|_{L^\infty(0,T;\boldsymbol{L}^r(\Omega))} \le C(T, r),
$$

for any $r \in [2, 6]$. In the case $d = 2$, applying the two-dimensional Gagliardo–Nirenberg's inequality, we obtain from (4.22) that

$$
\frac{1}{4N}\int_\Omega \max_{i=1,\dots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx \le C\left(1 + r^{\frac{r}{2}}\|\mathbf{w}_\varepsilon\|^r_{\boldsymbol{H}^1(\Omega)}\right), \quad \forall r \ge 2, \tag{4.23}
$$

where C does not depend on r. Multiplying the above inequality by $t^{\frac{r}{2}}$, we exploit (4.14) to infer

$$
\|\sqrt{t}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\|_{L^\infty(0,T;\boldsymbol{L}^r(\Omega))} \le C, \quad \forall r \ge 2, \tag{4.24}
$$

where C does not depend on r if $d = 2$ and for $r \in [2, 6]$ if $d = 3$.

Summing up, we have obtained all the estimates which allow us to pass to the limit as $\varepsilon \to 0$. Being this step standard (see, e.g., [29]), we only present a sketch of the argument. By compactness we immediately deduce that, up to subsequences,

$$
\begin{aligned}
&\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly* in } L^\infty(0, T; \mathbf{H}^1(\Omega)), \\
&\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in } L^4(0, T; \mathbf{H}^2(\Omega)), \\
&\partial_t\mathbf{u}_\varepsilon \rightharpoonup \partial_t\mathbf{u} \quad \text{weakly in } L^2(0, T; \mathbf{H}^1(\Omega)'),
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{t}\partial_t \mathbf{u}_\varepsilon \rightharpoonup \sqrt{t}\partial_t \mathbf{u} \quad \text{weakly in } L^2(0,T;\mathbf{H}^1(\Omega)),\\
&\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{strongly in } L^2(0,T;\boldsymbol{L}^2(\Omega)),\\
&\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{a.e. in } \Omega\times(0,T),\\
&\mathbf{w}_\varepsilon \rightharpoonup \mathbf{w} \quad \text{weakly in } L^2(0,T;\mathbf{H}^1(\Omega)),\\
&\sqrt{t}\mathbf{w}_\varepsilon \rightharpoonup \sqrt{t}\mathbf{w} \quad \text{weakly* in } L^\infty(0,T;\mathbf{H}^1(\Omega)).
\end{aligned} \tag{4.25}
$$

Arguing then as in [29, Section 6], we also infer, exploiting (4.18), that, for any $k=1,\dots,N$,

$$
\phi_\varepsilon(u_{\varepsilon,k}) \to \phi(u_k) \quad \text{a.e. in } \Omega\times(0,T), \tag{4.26}
$$

$$
\phi_\varepsilon(u_{\varepsilon,k}) \rightharpoonup \phi(u_k) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{4.27}
$$

Thus the pair $(\mathbf{u},\mathbf{w})$ satisfies (3.3)–(3.7).

Observe now that, thanks to (4.24), up to subsequences, we have, for any $r\in[2,\infty)$ if $d=2$, $r\in[2,6]$ if $d=3$,

$$
\sqrt{t}\phi_\varepsilon(u_{\varepsilon,k}) \rightharpoonup \xi \quad \text{weakly* in } L^\infty(0,T;L^r(\Omega)),
$$

by (4.27) we can identify $\xi=\sqrt{t}\phi(u_k)$ and deduce, by weak lower semicontinuity, that

$$
\|\sqrt{t}\phi(u_k)\|_{L^\infty(0,T;L^r(\Omega))} \le C, \tag{4.28}
$$

for any $k=1,\dots,N$. In conclusion, by elliptic regularity and Sobolev embeddings, we can deduce from (3.7), thanks to (4.14) and (4.28), that

$$
\|\sqrt{t}\mathbf{u}\|_{L^\infty(0,T;\mathbf{W}^{2,r}(\Omega))} \le C, \tag{4.29}
$$

for any $r\in[2,\infty)$ if $d=2$, for $r\in[2,6]$ if $d=3$.

We are left to prove the (strict) separation property for the case $d=2$. Thanks to (4.23), we can pass to the limit by Fatou's Lemma, to obtain that

$$
\sqrt{t}\left(\int_\Omega \max_{i=1,\dots,N} |\phi(u_i)|^r dx\right)^{\frac{1}{r}} \le C\sqrt{r},
$$

for almost any $t\in(0,T)$. Therefore, for any $i=1,\dots,N$, any $\tau>0$ and any $r\in[2,\infty)$, we have

$$
\operatorname*{ess\,sup}_{\tau\le t\le T} \|\phi(u_i)\|_{L^r(\Omega)} \le C\sqrt{r}, \tag{4.30}
$$

with C independent of r but dependent on τ. From this we can directly exploit assumption (**E2**) (corresponding to assumption [25, (E2)]) and repeat the proof in [25, Theorem 3.1] (indeed, (4.30) corresponds to [25, (3.4)] and we can argue exactly in the same way from [25, (3.4)] on). This leads, for any $i=1,\dots,N$ and any fixed

$r \geq 2$, to the following bound

$$\operatorname*{ess\,sup}_{\tau \leq t \leq T} \|\phi'(u_i)\|_{L^r(\Omega)} \leq C(\tau, r).$$

Let $\tau > 0$ be given. Applying the chain rule to $\phi(u_k)$ (which is possible, for instance, by using again a truncation argument), we get

$$\nabla \phi(u_k) = \phi'(u_k) \nabla u_k, \quad \text{a.e. in } [\tau, T].$$

Then, for almost any $t \in [\tau, T]$, we have that

$$\|\nabla \phi(u_k)\|_{L^p(\Omega)} \leq \|\phi'(u_k)\|_{L^{2p}(\Omega)} \|\nabla u_k\|_{\mathbf{L}^{2p}(\Omega)} \leq C(T, \tau, p),$$

exploiting (4.29), (4.30) and the Sobolev embedding $W^{2,r}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ for every $p \in [2, \infty)$ and $r \geq 2$. Therefore, for any $k = 1, \ldots, N$, it holds

$$\|\phi(u_k)\|_{W^{1,p}(\Omega)} \leq C(T, \tau, p),$$

for every $p \in [2, \infty)$. Fix now, e.g., $p = 3$. This implies that, for any $k = 1, \ldots, N$,

$$\|\phi\,(u_k)\|_{L^\infty(\Omega)} \leq C(T, \tau), \quad \text{a.e. in } [\tau, T],$$

owing to the embedding $W^{1,3}(\Omega) \hookrightarrow L^\infty(\Omega)$. Note that, being

$$u_k \in L^\infty(\tau, T; W^{2,r}(\Omega)) \cap H^1(\tau, T; H^1(\Omega)),$$

for any $r \geq 2$, we have that $u_k \in C([\tau, T]; H^{3/2}(\Omega))$, implying $u_k \in C([\tau, T]; C(\overline{\Omega}))$. Therefore we can find $\delta_k = \delta_k(T, \tau) > 0$ such that, for $k = 1, \ldots, N$,

$$u_k > \delta_k, \quad \text{in } \overline{\Omega} \times [\tau, T],$$

and recalling the constraint $\sum_{k=1}^N u_k = 1$, this condition implies that, for any $k = 1, \ldots, N$,

$$1 - \sum_{i \neq k} u_i = u_k > \delta_k, \quad \text{in } \overline{\Omega} \times [\tau, T],$$

entailing that, necessarily, $\sum_{k=1}^N \delta_k < 1$. Moreover, we also deduce that, for any $k = 1, \ldots, N$,

$$u_k = 1 - \sum_{i \neq k} u_i < 1 - \sum_{i \neq k} \delta_i, \quad \text{in } \overline{\Omega} \times [\tau, T].$$

We can then find a common $\delta := \min_{k=1,\ldots,N} \delta_k = \delta(\tau, T) \in (0, \frac{1}{N})$ such that

$$\delta < \mathbf{u} < (1 - (N-1)\delta), \quad \text{in } \overline{\Omega} \times [\tau, T].$$

This concludes the proof.

4.2 Proof of Theorem 3.6

Let us take $\boldsymbol{\eta} = \mathbf{u}(t) - \overline{\mathbf{u}}(t)$ in equation (3.7). This gives

$$(\mathbf{P}\boldsymbol{\phi}(\mathbf{u}), \mathbf{u} - \overline{\mathbf{u}}) + \|\nabla \mathbf{u}\|^2 = (\mathbf{w}, \mathbf{u} - \overline{\mathbf{u}}) + (\mathbf{A}\mathbf{u}, \mathbf{u} - \overline{\mathbf{u}}). \tag{4.31}$$

Moreover, by convexity of Ψ^1, since $\mathbf{u} - \overline{\mathbf{u}} \in T\Sigma$, we have

$$(\mathbf{P}\boldsymbol{\phi}(\mathbf{u}), \mathbf{u} - \overline{\mathbf{u}}) = (\boldsymbol{\phi}(\mathbf{u}), \mathbf{u} - \overline{\mathbf{u}}) \geq \int_\Omega \Psi^1(\mathbf{u})dx - \int_\Omega \Psi^1(\overline{\mathbf{u}})dx,$$

but, being $\overline{\mathbf{u}} \equiv \overline{\mathbf{u}}_0$,

$$|\Psi^1(\overline{\mathbf{u}})| \leq C,$$

where $C > 0$ depends on $\overline{\mathbf{u}}_0$, applying standard inequalities, we find from (4.31)

$$\int_\Omega \Psi^1(\mathbf{u})dx + \|\nabla \mathbf{u}\|^2 \leq C + C\|\nabla \mathbf{u}\|\|\nabla \mathbf{w}\| + (\mathbf{A}\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{A}\mathbf{u})$$

and using (2.4) we get

$$\begin{aligned}
&\int_\Omega \Psi^1(\mathbf{u})dx - \frac{1}{2}(\mathbf{A}\mathbf{u}, \mathbf{u}) + \frac{1}{4}\|\nabla \mathbf{u}\|^2 \\
&\quad \leq C(\boldsymbol{\alpha}\nabla \mathbf{w}, \nabla \mathbf{w}) + \frac{1}{2}(\mathbf{A}\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{A}\mathbf{u}) \\
&\quad \leq C(1 + (\boldsymbol{\alpha}\nabla \mathbf{w}, \nabla \mathbf{w})) + C\|\mathbf{u}\|^2 \\
&\quad \leq C(1 + (\boldsymbol{\alpha}\nabla \mathbf{w}, \nabla \mathbf{w})) + \frac{1}{2}\int_\Omega \Psi(\mathbf{u})dx,
\end{aligned}$$

where in the last step we applied (vi) (recall that these estimates must be obtained in an approximated scheme, so for ε sufficiently small). Therefore, we get

$$\frac{1}{4}\|\nabla \mathbf{u}\|^2 + \frac{1}{2}\int_\Omega \Psi(\mathbf{u})dx \leq C(1 + (\boldsymbol{\alpha}\nabla \mathbf{w}, \nabla \mathbf{w})). \tag{4.32}$$

Combining (3.8) with (4.32) (multiplied by $\epsilon > 0$ sufficiently small), we end up with

$$\frac{d}{dt}\mathcal{E}(t) + \frac{\epsilon}{2}\mathcal{E}(t) \leq \frac{d}{dt}\mathcal{E}(t) + \frac{\epsilon}{2}\mathcal{E}(t) + (1 - \epsilon C)(\boldsymbol{\alpha}\nabla \mathbf{w}, \nabla \mathbf{w}) \leq C,$$

and the result follows from Gronwall's lemma.

5 Proof of Theorem 3.7

Again the rigorous proof has to be carried out using the same approximation scheme as in the proof of Theorem 3.1, i.e., by approximating the potential with ψ_ε and considering a Galerkin setting (see also [16]). For the sake of brevity, here we simply show the formal estimates. First, we observe that (3.8) entails

$$\|\mathbf{u}\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))} + \|\nabla \mathbf{w}\|_{L^2(t,t+1;\mathbf{L}^2(\Omega))} \le C, \quad \forall t \ge 0. \tag{5.1}$$

Then, arguing as in [16, (2.11)], we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{w}\|^2 + \|\nabla \partial_t \mathbf{u}\|^2 \le (\mathbf{A}\partial_t \mathbf{u}, \partial_t \mathbf{u}),$$

but, testing (3.6) with $\boldsymbol{\eta} = \mathbf{A}\partial_t \mathbf{u}(t)$ and applying Poincaré's inequality (recall that $\overline{\partial_t \mathbf{u}} \equiv 0$), we get

$$(\mathbf{A}\partial_t \mathbf{u}, \partial_t \mathbf{u}) = (\boldsymbol{\alpha}\nabla \mathbf{w}, \mathbf{A}\partial_t \mathbf{u}) \le C\|\nabla \partial_t \mathbf{u}\|\|\nabla \mathbf{w}\| \le \frac{1}{2}\|\nabla \partial_t \mathbf{u}\|^2 + C\|\nabla \mathbf{w}\|^2,$$

so that in the end we come up with

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{w}\|^2 + \frac{1}{2}\|\nabla \partial_t \mathbf{u}\|^2 \le C\|\nabla \mathbf{w}\|^2.$$

Due to (5.1), we can apply the uniform Gronwall's lemma (see, e.g., [43], by choosing, e.g., $r = \frac{\tau}{2}$) to deduce, for any given $\tau > 0$,

$$\|\nabla \mathbf{w}\|_{L^\infty(\tau,\infty;\mathbf{L}^2(\Omega))} + \|\nabla \partial_t \mathbf{u}\|_{L^2(t,t+1;\mathbf{L}^2(\Omega))} \le C \quad \forall t \ge \tau.$$

From now on we can repeat *verbatim* the arguments in the proof of Theorem 3.1, to get the regularity (3.10). The proof is finished.

6 Proof of Theorem 3.8

By Remark 3.9, we only need to show the existence of a compact absorbing set. From Theorem 3.6, we deduce that, for any $\mathbf{u}_0 \in \mathcal{V}_\mathbf{M}$, being Ψ bounded, there exist constants $C_3, C_4 > 0$ such that

$$\|S(t)\mathbf{u}_0\|^2_{\mathcal{V}_\mathbf{M}} \le C_3 e^{-\omega t}\|\mathbf{u}_0\|^2_{\mathcal{V}_\mathbf{M}} + C_4 \quad \forall t \ge 0.$$

This means that the set

$$\widetilde{\mathcal{B}}_0 := \left\{ \mathbf{u} \in \mathcal{V}_\mathbf{M} : \ \|\mathbf{u}\|_{\mathcal{V}_\mathbf{M}} \le \sqrt{\frac{C_3}{2} + C_4} := R_0 \right\}$$

is an absorbing set, i.e., for any bounded set $B \subset \mathcal{V}_\mathbf{M}$ there exists $t_e > 0$ depending on B such that $S(t)B \subset \widetilde{\mathcal{B}}_0 \ \ \forall t \ge t_e$. Checking the proof of Theorem 3.7, it is not difficult to realize that all the constants appearing in the regularization estimates only depend on the $\mathcal{V}_{\mathbf{M}}$-norm of $\mathbf{u}_0$, being Ψ bounded. This means in particular that there exists a bounded set

$$\mathcal{B}_0 := \left\{\mathbf{u} \in \widetilde{\mathcal{B}}_0 : \ \|\mathbf{u}\|_{\mathbf{W}^{2,r}(\Omega)} \le C_0\right\},$$

for some $C_0 > 0$ and any $r \ge 2$ if $d = 2$, $r = 6$ if $d = 3$, and a time t_{R_0}, depending only on R_0, such that $S(t)\widetilde{\mathcal{B}}_0 \subset \mathcal{B}_0$ for any $t \ge t_{R_0}$. This clearly implies that $\mathcal{B}_0$ is a compact absorbing set and ends the proof.

7 Proofs of Subsection 3.3

This section is devoted to show the convergence of any weak solution to a single stationary state. We first prove two fundamental lemmas stated in Subsect. 3.3. Then we demonstrate Theorem 3.16.

7.1 Proof of Lemma 3.11

Let us consider $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$. By definition of ω-limit set there exists a sequence $t_n \to \infty$ such that $\mathbf{u}(t_n) = S(t)\mathbf{u}_0 \to \mathbf{u}_\infty$ in $\mathbf{H}^{2r}(\Omega)$ as $n \to \infty$. We then define the sequence of trajectories $\mathbf{u}_n(t) := \mathbf{u}(t + t_n)$ and $\mathbf{w}_n(t) := \mathbf{w}(t + t_n)$. Observe that $\mathbf{u}_n$ solves (3.1) with initial datum $\mathbf{u}(t_n) \in \mathcal{V}_{\mathbf{M}}$. By Theorem 3.7 applied to $\mathbf{u}$, we immediately infer that $\mathbf{u}_n$ is uniformly (in n) bounded in $L^\infty(0, \infty; \mathbf{H}^2(\Omega))$, $\mathbf{w}_n$ is uniformly bounded in $L^\infty(0, \infty; \mathbf{H}^1(\Omega)) \cap H^1(0, \infty; \mathbf{H}^1(\Omega))$ and $\boldsymbol{\phi}(\mathbf{u}_n)$ is uniformly bounded in $L^\infty(0, \infty; \mathbf{L}^2(\Omega))$. From these bounds, by passing to the limit, up to subsequences, in the equations solved by $\mathbf{u}_n$, we infer the existence of $\mathbf{u}^*$ which is a strong solution to (3.1) (i.e., a weak solution with the regularity given in Theorem 3.7 with $\tau = 0$). In particular, concerning the initial datum, $\mathbf{u}^*(0) = \lim_{n\to\infty} \mathbf{u}_n(0) = \lim_{n\to\infty} \mathbf{u}(t_n) = \mathbf{u}_\infty$, where the limit is intended in the sense of $\mathbf{H}^{2r}(\Omega)$. We thus have $\lim_{n\to\infty} \mathcal{E}(\mathbf{u}_n(t)) = \mathcal{E}(\mathbf{u}^*(t))$ for all $t \ge 0$. Thanks to the energy identity, we see that the energy $\mathcal{E}(\mathbf{u}(t))$ is nonincreasing in time, thus there exists E_∞ such that $\lim_{t\to\infty} \mathcal{E}(\mathbf{u}(t)) = E_\infty$. Therefore, the convergence also holds for the energy along the subsequence $\{t + t_n\}_n$ so that

$$\mathcal{E}(\mathbf{u}^*(t)) = \lim_{n\to\infty} \mathcal{E}(\mathbf{u}_n(t)) = E_\infty, \quad \forall t \ge 0, \tag{7.1}$$

entailing that $\mathcal{E}(\mathbf{u}^*(t)))$ is constant in time. Passing then to the limit as $n \to \infty$, we infer

$$E_\infty + \int_s^t (\boldsymbol{\alpha}\nabla\mathbf{w}^*(\tau), \nabla\mathbf{w}^*(\tau))d\tau \le E_\infty, \quad \forall t \ge s,$$

where $\mathbf{w}^*$ is the chemical potential corresponding to $\mathbf{u}^*$, implying that $\nabla\mathbf{w}^* = 0$ almost everywhere in $\Omega \times (0, \infty)$, and thus, by comparison in $(3.1)_1$, $\partial_t\mathbf{u}^* = 0$ almost everywhere in $\Omega \times (0, \infty)$. As a consequence, we infer that $\mathbf{u}^*$ is constant in time, namely $\mathbf{u}^*(t) \equiv \mathbf{u}^*(0) = \mathbf{u}_\infty$ for all $t \ge 0$, and $\mathbf{w}^*$ is constant is space and time. This means that $\mathbf{u}_\infty$ satisfies (3.12) for $\mathbf{f}_1 = \overline{\mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty)} \in \mathbf{G}$ and thus satisfies (8.4) with

$\mathbf{f} = \mathbf{f}_1 + \mathbf{PAu}_\infty$. Then $\mathbf{u}_\infty \in \mathcal{W}$. This shows that $\omega(\mathbf{u}_0) \subset \mathcal{W}$. Concerning the mean value, it is easy to see that $\overline{\mathbf{u}^*}(t) = \lim_{n\to\infty} \overline{\mathbf{u}}_n(t) \equiv \overline{\mathbf{u}}_0$ for any $t \geq 0$, thus $\overline{\mathbf{u}}_\infty = \overline{\mathbf{u}}_0$. Moreover, it is useful to notice that

$$\mathcal{E}(\mathbf{u}_\infty) = E_\infty = \lim_{s\to\infty} \mathcal{E}(\mathbf{u}(s)) = \inf_{s\geq 0} \mathcal{E}(\mathbf{u}(s)) \leq \mathcal{E}(\mathbf{u}(t)), \quad \forall t \geq 0.$$

By uniqueness of the solution to the steady-state equation (8.4) with $\mathbf{f} = \mathbf{f}_1 + \mathbf{PAu}_\infty$ (see Theorem 8.1 below), we preliminarily know that for *any* $\mathbf{u}_\infty \in \omega(\mathbf{u}_0) \subset \mathcal{W}$ there exists $\delta_0(\mathbf{u}_\infty) > 0$ depending on $\mathbf{u}_\infty$, such that $\delta_0 < \mathbf{u}_\infty$ for any $x \in \overline{\Omega}$. Since $\omega(\mathbf{u}_0)$ is compact in $\mathbf{H}^{2r}(\Omega)$ and we have fixed $r \in (\frac{d}{4}, 1)$, by the continuous embedding $\mathbf{H}^{2r}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$ we can deduce that $\omega(\mathbf{u}_0)$ is compact in $\mathbf{C}(\overline{\Omega})$. Assume then by contradiction that we cannot find a quantity $\delta > 0$ such that the separation property holds *uniformly* on $\omega(\mathbf{u}_0)$. This means that for any $m \in \mathbb{N}$ there exist $\mathbf{u}_m \in \omega(\mathbf{u}_0)$, $x_m \in \overline{\Omega}$ and $j_m = 1, \dots, N$ such that

$$0 < u_{m,j_m}(x_m) \leq \frac{1}{m}.$$

Since j_m ranges in a finite set, we can find a (nonrelabeled) subsequence and a fixed index $\overline{J}$ such that

$$0 < u_{m,\overline{J}}(x_m) \leq \frac{1}{m}, \quad \forall m \in \mathbb{N}. \tag{7.2}$$

Being $\{u_m\}_m \subset \omega(\mathbf{u}_0)$ a bounded sequence in $\mathbf{L}^\infty(\Omega)$ (indeed it is contained in the $\mathbf{L}^\infty$-ball of radius $\sqrt{N}$, since $0 < u_{m,i} < 1,\ i = 1, \dots, N$) by compactness there exists a (nonrelabeled) subsequence such that $\mathbf{u}_m \to \mathbf{u}_\infty \in \omega(\mathbf{u}_0)$ in $\mathbf{C}(\overline{\Omega})$ as $m \to \infty$. But this implies also that $u_{m,\overline{J}} \to u_{\infty,\overline{J}}$ *uniformly* as $m \to \infty$. On the other hand, by the Bolzano-Weierstrass Theorem, we can extract a further (nonrelabeled) subsequence such that

$$x_m \to x_\infty \in \overline{\Omega}, \quad \text{as } m \to \infty.$$

We can now pass to the limit, since it holds

$$\begin{aligned} |u_{m,\overline{J}}(x_m) - u_{\infty,\overline{J}}(x_\infty)| &\leq |u_{m,\overline{J}}(x_m) - u_{\infty,\overline{J}}(x_m)| + |u_{\infty,\overline{J}}(x_m) - u_{\infty,\overline{J}}(x_\infty)| \\ &\leq \max_{x\in\overline{\Omega}} |u_{m,\overline{J}}(x) - u_{\infty,\overline{J}}(x)| + |u_{\infty,\overline{J}}(x_m) - u_{\infty,\overline{J}}(x_\infty)| \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

Indeed, the first term converges to zero by the uniform convergence described above, whereas the second one converges since $u_{\infty,\overline{J}} \in C(\overline{\Omega})$. Passing then to the limit in (7.2), we infer

$$u_{\infty,\overline{J}}(x_\infty) = 0.$$

Since $\mathbf{u}_\infty \in \omega(\mathbf{u}_0) \subset \mathcal{W}$, there exists $\delta_\infty > 0$ such that $\min_{x\in\overline{\Omega}} u_{\infty,\overline{J}}(x) \geq \delta_\infty > 0$, a contradiction. We thus conclude that all the elements of $\omega(\mathbf{u}_0)$ are *uniformly* strictly separated. The proof is finished.

7.2 Proof of Lemma 3.15

The first Fréchet derivative of $\widetilde{\mathcal{E}}$ reads as follows (recall that $\widetilde{\Psi}$ is smooth):

$$\langle \widetilde{\mathcal{E}}'(\mathbf{u}), \mathbf{h}\rangle_{\mathbf{V}_0', \mathbf{V}_0} = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{h}\, dx + \int_\Omega \widetilde{\Psi}_{,\mathbf{u}}(\mathbf{u} + \mathbf{M}) \cdot \mathbf{h}\, dx$$

where $\widetilde{\Psi}_{,\mathbf{u}}(\mathbf{u}) := (\widetilde{\psi}'(u_i) - (\mathbf{A}\mathbf{u})_i)_{i=1,\dots,N}$. Notice that $\mathbf{u}_\infty - \mathbf{M} \in \omega(\mathbf{u}_0) - \mathbf{M}$ is a critical point for $\widetilde{\mathcal{E}}$. Indeed, for $\mathbf{u}_0 \in \mathcal{V}_\mathbf{M}$, thanks to the fact that, for any $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$, by Lemma 3.11, there exists a set $\tilde{U} \subset I_\varepsilon$ (I_ε is defined in Sect. 3.3) such that $\mathbf{u}_\infty(x) \in \tilde{U}$ for any $x \in \overline{\Omega}$ and due to the definition of $\widetilde{\psi}$, we have $\widetilde{\mathcal{E}}|_{\omega(\mathbf{u}_0)-\mathbf{M}} = \mathcal{E}|_{\omega(\mathbf{u}_0)}$. Therefore,

$$\begin{aligned}
\langle \widetilde{\mathcal{E}}'(\mathbf{u}_\infty - \mathbf{M}), \mathbf{h}\rangle_{\mathbf{V}_0', \mathbf{V}_0} &= \int_\Omega \nabla \mathbf{u}_\infty \cdot \nabla \mathbf{h}\, dx + \int_\Omega \widetilde{\Psi}_{,\mathbf{u}}(\mathbf{u}_\infty) \cdot \mathbf{h}\, dx \\
&= \int_\Omega \nabla \mathbf{u}_\infty \cdot \nabla \mathbf{h}\, dx + \int_\Omega \Psi_{,\mathbf{u}}(\mathbf{u}_\infty) \cdot \mathbf{h}\, dx \\
&= \int_\Omega \big(-\Delta \mathbf{u}_\infty + \mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty)\big) \cdot \mathbf{h}\, dx \\
&= \int_\Omega \big(-\Delta \mathbf{u}_\infty + P_0(\mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty))\big) \cdot \mathbf{h}\, dx = 0, \quad \forall \mathbf{h} \in \mathbf{V}_0,
\end{aligned}$$

where P_0 is the L^2-projector onto the subspace with zero spatial average. Recall that $\mathbf{u}_\infty$ satisfies (8.4) with $\mathbf{f} = \mathbf{P}\mathbf{A}\mathbf{u}_\infty + \overline{\mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty)}$ (see Lemma 3.11).

Concerning the second Fréchet derivative, it is easy to show that

$$\begin{aligned}
&\langle \widetilde{\mathcal{E}}''(\mathbf{u})\mathbf{h}_1, \mathbf{h}_2\rangle_{\mathbf{V}_0', \mathbf{V}_0} \\
&\quad = \int_\Omega \nabla \mathbf{h}_1 \cdot \nabla \mathbf{h}_2\, dx + \int_\Omega \left(\sum_{i=1}^N \widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{h}_{1,i}\mathbf{h}_{2,i} - \mathbf{A}\mathbf{h}_1 \cdot \mathbf{h}_2\right) dx, \qquad (7.3)
\end{aligned}$$

for all $\mathbf{u}, \mathbf{h}_1, \mathbf{h}_2 \in \mathbf{V}_0$.

Let us set $\mathcal{L} := \widetilde{\mathcal{E}}'' \in \mathcal{B}(\mathbf{V}_0, \mathbf{V}_0')$ (omitting the dependence on $\mathbf{u}$, which will be pointed out if necessary) and consider the operator $\mathbb{A}$ as defined in Appendix 1. First we observe that, for all $\mathbf{z} \in Ker(\mathcal{L}) \subset \mathbf{V}_0$, setting $\mathbf{v} := (\widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{z}_i - (\mathbf{A}\mathbf{z})_i)_{i=1,\dots,N} \in \mathbf{L}^2(\Omega)$, we have

$$\langle \mathbb{A}\mathbf{z}, \mathbf{h}\rangle_{\mathbf{V}_0', \mathbf{V}_0} = -(\mathbf{v}, \mathbf{h}) = -(P_0\mathbf{P}\mathbf{v}, \mathbf{h}), \quad \forall \mathbf{h} \in \mathbf{V}_0.$$

This means, recalling the identification $\mathbf{V}_0 \hookrightarrow\hookrightarrow \mathbf{H}_0 \equiv \mathbf{H}_0' \hookrightarrow \mathbf{V}_0'$,

$$\mathbb{A}\mathbf{z} = -P_0\mathbf{P}\mathbf{v} \in \mathbf{H}_0,$$

implying that $\mathbf{z} \in \mathfrak{D}(\widetilde{\mathbb{A}})$ (see Appendix 1 for the definition of $\widetilde{\mathbb{A}}$). This entails $Ker(\mathcal{L}) \subset \mathfrak{D}(\widetilde{\mathbb{A}})$. We now introduce the operator $\mathbb{Q} \in \mathcal{B}(\mathbf{V}_0')$ such that, for any $\mathbf{z} \in \mathbf{V}_0'$,

$$\langle \mathbb{Q}\mathbf{z}, \mathbf{w}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = \left\langle \mathbf{z}, P_0\mathbf{P}\left(\widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{w}_i - (\mathbf{A}\mathbf{w})_i\right)_{i=1,\dots,N}\right\rangle_{\mathbf{V}_0',\mathbf{V}_0}, \quad \forall\, \mathbf{w} \in \mathbf{V}_0,$$

which is well defined being $\widetilde{\psi}$ of class $C^3(\mathbb{R})$ and $\mathbf{u} \in \mathbf{V}_0$. Note that, for any $\mathbf{z} \in \mathfrak{D}(\mathcal{L}) = \mathfrak{D}(\mathbb{A}) = \mathbf{V}_0$, we have

$$\mathcal{L}\mathbf{z} = \mathbb{A}\mathbf{z} + \mathbb{Q}\mathbf{z},$$

since, for $\mathbf{z} \in \mathbf{H}_0$, being $\mathbf{A}$ symmetric,

$$\begin{aligned}\langle \mathbb{Q}\mathbf{z}, \mathbf{w}\rangle_{\mathbf{V}_0',\mathbf{V}_0} &= \left(\mathbf{z}, P_0\mathbf{P}\left(\widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{w}_i - (\mathbf{A}\mathbf{w})_i\right)_{i=1,\dots,N}\right)\\ &= \left(P_0\mathbf{P}\left(\widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{z}_i - (\mathbf{A}\mathbf{z})_i\right)_{i=1,\dots,N}, \mathbf{w}\right), \quad \forall\, \mathbf{w} \in \mathbf{V}_0, \quad (7.4)\end{aligned}$$

so that

$$\mathbb{Q}\mathbf{z} = P_0\mathbf{P}\left(\widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{z}_i - (\mathbf{A}\mathbf{z})_i\right)_{i=1,\dots,N} \in \mathbf{H}_0$$

as in the definition of $\mathcal{L}$. Note that, thanks to the regularity of $\mathbf{u} \in \mathbf{V}_0$ and $\widetilde{\psi}''$, if $\mathbf{z} \in \mathbf{V}_0$ we also have $\mathbb{Q}\mathbf{z} \in \mathbf{V}_0$. As observed in the Appendix, both $\widetilde{\mathbb{A}}^{-1} : \mathbf{V}_0' \to \mathfrak{D}(\mathbb{A}) = \mathbf{V}_0 \hookrightarrow \mathbf{V}_0'$ ($\mathbb{A}$ considered as unbounded on $\mathbf{V}_0'$) and $\widetilde{\mathbb{A}}^{-1} : \mathbf{H}_0 \to \mathfrak{D}(\widetilde{\mathbb{A}})$ are compact in $\mathbf{V}_0'$ and $\mathbf{H}_0$, respectively. In particular, we have that both $\mathbb{A}^{-1}\mathbb{Q}$ and $\mathbb{Q}\mathbb{A}^{-1}$ are compact operators on $\mathbf{V}_0'$. Thus we can apply Theorem 8.2 to $A = \mathcal{L} : \mathfrak{D}(\mathbb{A}) \hookrightarrow \mathbf{V}_0' \to \mathbf{V}_0'$, with $T = \mathbb{A}^{-1} \in \mathcal{B}(\mathbf{V}_0')$ to deduce that $\mathcal{L}$ is a Fredholm operator, implying that $Ker(\mathcal{L}) \subset \mathbf{V}_0'$ is finite dimensional and $Range(\mathcal{L})$ is closed in $\mathbf{V}_0'$. Moreover, it is easy to see that the operator $\mathbb{A}$ is selfadjoint with respect to the Hilbert adjoint. Indeed, setting $L_{\mathbf{V}_0} = \mathbb{A}^{-1} : \mathbf{V}_0' \to \mathbf{V}_0$ the Riesz isomorphism from $\mathbf{V}_0'$ to $\mathbf{V}_0$, we have that the operator $\mathbb{A}$ is symmetric, since, for any $\mathbf{w}, \mathbf{z} \in \mathfrak{D}(\mathbb{A}) = \mathbf{V}_0$,

$$\begin{aligned}(\mathbb{A}\mathbf{w}, \mathbf{z})_{\mathbf{V}_0'} &= (L_{\mathbf{V}_0}\mathbb{A}\mathbf{w}, L_{\mathbf{V}_0}\mathbf{z})_{\mathbf{V}_0}\\ &= (\mathbf{w}, L_{\mathbf{V}_0}\mathbf{z})_{\mathbf{V}_0} = \langle \mathbf{z}, \mathbf{w}\rangle_{\mathbf{V}_0',\mathbf{V}_0}\\ &= (\mathbf{w}, \mathbf{z})_{\mathbf{H}_0} = \langle \mathbf{w}, \mathbf{z}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = (\mathbf{w}, \mathbb{A}\mathbf{z})_{\mathbf{V}_0'}.\end{aligned}$$

Being $Range(\mathbb{A}) = \mathbf{V}_0'$, a well known result implies that $\mathbb{A}$ is selfadjoint. Therefore, we can write the Hilbert adjoint of $\mathcal{L}$ as $\mathcal{L}^* = \mathbb{A} + \mathbb{Q}^*$. Now, by the Closed Range Theorem, recalling that $\mathcal{L}' = \mathbb{A}^{-1}\mathcal{L}^*\mathbb{A}$ ($\mathcal{L}'$ being the adjoint of $\mathcal{L}$, since $\mathbf{V}_0 \equiv \mathbf{V}_0''$ via the canonical map), observe that

$$\begin{aligned}Range(\mathcal{L}) &= \{\mathbf{y}^* \in \mathbf{V}_0' : \langle \mathbf{y}^*, \mathbf{x}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = 0, \quad \forall\, \mathbf{x} \in Ker(\mathcal{L}')\}\\ &= \{\mathbf{y}^* \in \mathbf{V}_0' : \langle \mathbf{y}^*, \mathbb{A}^{-1}\mathbf{z}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = 0, \quad \forall\, \mathbf{z} \in Ker(\mathcal{L}^*)\}\\ &= \{\mathbf{y}^* \in \mathbf{V}_0' : \langle \mathbf{y}^*, \mathbf{q}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = 0, \quad \forall\, \mathbf{q} \in Ker(\mathcal{L})\} = (Ker(\mathcal{L}))^{\perp}, \quad (7.5)\end{aligned}$$

where $\perp$ is intended to be the annihilator of the set. The last isomorphism is due to the fact that, on account of $\mathbb{Q}(\mathbb{A}^{-1}\mathbf{w}) \in \mathbf{V}_0$ for $\mathbf{w} \in \mathbf{V}_0'$, it holds

$$\mathbb{Q}^*\mathbf{w} = \mathbb{A}(\mathbb{Q}(\mathbb{A}^{-1}\mathbf{w})).$$

Thus

$$\mathcal{L}^*\mathbf{z} = \mathbb{A}\mathbf{z} + \mathbb{Q}^*\mathbf{z} = 0 \Longleftrightarrow \mathbf{z} + \mathbb{Q}\mathbb{A}^{-1}\mathbf{z} = 0 \Longleftrightarrow \mathbb{A}\mathbf{w} + \mathbb{Q}\mathbf{w} = 0, \quad \mathbf{w} = \mathbb{A}^{-1}\mathbf{z},$$

that is,

$$Ker(\mathcal{L}^*) = \mathbb{A}Ker(\mathcal{L}).$$

Let us denote by $\widetilde{\mathbb{Q}} \in \mathcal{B}(\mathbf{H}_0)$ the restriction to $\mathbf{H}_0$ of $\mathbb{Q}$, so that from (7.4) we get

$$\widetilde{\mathbb{Q}}\mathbf{z} = \mathbb{Q}\mathbf{z} = P_0\mathbf{P}\left(\widetilde{\psi}''(u_i + \mathbf{M}_i)\mathbf{z}_i - (\mathbf{A}\mathbf{z})_i\right)_{i=1,\dots,N} \in \mathbf{H}_0$$

for any $\mathbf{z} \in \mathbf{H}_0$. Again note that, for any $z \in \mathfrak{D}(\widetilde{\mathbb{A}}) \hookrightarrow \mathbf{H}_0$, it holds

$$\mathcal{L}_{|\mathfrak{D}(\widetilde{\mathbb{A}})}\mathbf{z} = \widetilde{\mathbb{A}}\mathbf{z} + \widetilde{\mathbb{Q}}\mathbf{z},$$

and both $\widetilde{\mathbb{A}}^{-1}\widetilde{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}\widetilde{\mathbb{A}}^{-1}$ are compact operators on $\mathbf{H}_0$ being compositions of a compact and a bounded operator on $\mathbf{H}_0$. Therefore we can apply Theorem 8.2 with $T = \widetilde{\mathbb{A}}^{-1} \in \mathcal{B}(\mathbf{H}_0)$ and deduce that also $\mathcal{L}_{|\mathfrak{D}(\widetilde{\mathbb{A}})}$ is a Fredholm operator. Namely, since clearly in this case $\mathcal{L}_{|\mathfrak{D}(\widetilde{\mathbb{A}})}$ is selfadjoint (without distinction with the Hilbert adjoint, since $\mathbf{H}_0 \equiv \mathbf{H}_0'$), we immediately deduce that

$$Range(\mathcal{L}_{|\mathfrak{D}(\widetilde{\mathbb{A}})}) = Ker(\mathcal{L}_{|\mathfrak{D}(\widetilde{\mathbb{A}})})^{\perp_{\mathbf{H}_0}} = Ker(\mathcal{L})^{\perp_{\mathbf{H}_0}}, \tag{7.6}$$

where the last identity is due to the fact that $Ker(\mathcal{L}) \subset \mathfrak{D}(\widetilde{\mathbb{A}})$. We now have all the ingredients to apply [10, Corollary 3.11]. To this aim, let us fix $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$ and set $\mathbf{u}_\infty^0 = \mathbf{u}_\infty - \mathbf{M} \in \mathfrak{D}(\widetilde{\mathbb{A}})$. Adopting the notation of [10, Corollary 3.11], we set $V_0 := Ker(\mathcal{L}(\mathbf{u}_\infty^0))$ and define the projection $P \in \mathcal{B}(\mathbf{V}_0)$ as to be the orthogonal $\mathbf{V}_0$-projection on V_0. Set also $V_1 = Ker(P) = V_0^{\perp_{\mathbf{V}_0}}$, so that we have the direct (orthogonal in $\mathbf{V}_0$) sum $\mathbf{V}_0 = V_0 \oplus V_1$. In this way, [10, Hypothesis3.2] is verified. Let us verify [10, Hypothesis 3.4]. Firstly we set $W := \mathbf{V}_0'$, $U = \mathbf{V}_0$ and notice that the adjoint of P, $P' : \mathbf{V}_0' \to \mathbf{V}_0'$, is such that

$$Range(P') = V_1^\perp = V_0', \quad Ker(P') = V_0^\perp = V_1', \quad \mathbf{V}_0' = V_0' \oplus V_1'.$$

We then have

(I) $W = \mathbf{V}_0' \hookrightarrow \mathbf{V}_0'$ by construction;
(II) $P'W = P'\mathbf{V}_0' = V_0' \subset \mathbf{V}_0' = W$;
(III) $\widetilde{\mathcal{E}}' \in C^1(U, \mathbf{V}_0')$ since $\widetilde{\Psi} \in C^3(\mathbb{R})$;
(IV) by (7.5), $Range(\mathcal{L}(\mathbf{u}_\infty^0)) = V_0^\perp = V_1' \cap \mathbf{V}_0' = V_1' \cap W$.

We are left to verify the assumptions of [10, Corollary 3.11]. Set $X := \mathfrak{D}(\widetilde{\mathbb{A}})$ and $Y := \mathbf{H}_0$.

(1) Clearly, since $V_0 = Ker(\mathcal{L}(\mathbf{u}_\infty^0)) \subset \mathfrak{D}(\widetilde{\mathbb{A}})$, $PX \subset \mathfrak{D}(\widetilde{\mathbb{A}}) = X$. Moreover, $P'Y = P'\mathbf{H}_0 \subset \mathbf{H}_0$. Indeed, $\mathbf{h} \in \mathbf{H}_0 \hookrightarrow \mathbf{V}_0'$ can be written as the sum of

$$\mathbf{h}_1 = P_{\mathbf{H}_0}\mathbf{h} \in Ker(\mathcal{L}(\mathbf{u}_\infty^0)),$$

where $P_{\mathbf{H}_0} \in \mathcal{B}(\mathbf{H}_0)$ is the $\mathbf{H}_0$-orthogonal projection onto $V_0 = Ker(\mathcal{L}(\mathbf{u}^0_\infty))$, and

$$\mathbf{h}_2 = (I - P_{\mathbf{H}_0})\mathbf{h} \in Range(\mathcal{L}(\mathbf{u}^0_\infty)_{|\mathfrak{D}(\widetilde{\mathbb{A}})})$$

by (7.6), so that $(\mathbf{h}_1, \mathbf{h}_2)_{\mathbf{H}_0} = 0$ (notice that (7.6) is essential to reach this conclusion, since P is defined as a projection with respect to the inner product in $\mathbf{V}_0$ and not in $\mathbf{H}_0$). Then we have

$$\begin{aligned}\langle P'\mathbf{h}, \mathbf{v}\rangle_{\mathbf{V}_0',\mathbf{V}_0} &= \langle \mathbf{h}, P\mathbf{v}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = (\mathbf{h}, P\mathbf{v})_{\mathbf{H}_0}\\ &= (\mathbf{h}_1 + \mathbf{h}_2, P\mathbf{v})_{\mathbf{H}_0} = (\mathbf{h}_1, P\mathbf{v})_{\mathbf{H}_0} = \langle P'\mathbf{h}_1, \mathbf{v}\rangle_{\mathbf{V}_0',\mathbf{V}_0}\\ &= (\mathbf{h}_1, \mathbf{v})_{\mathbf{H}_0} = \langle \mathbf{h}_1, \mathbf{v}\rangle_{\mathbf{V}_0',\mathbf{V}_0},\end{aligned}$$

since $\mathbf{h}_2$ is orthogonal to $Ker(\mathcal{L})$ with respect to the inner product in $\mathbf{H}_0$ and due to the fact that $\mathbf{h}_1 \in Ker(\mathcal{L}) \subset V_1^\perp = Range(P')$ and thus $P'\mathbf{h}_1 = \mathbf{h}_1$. This is possible for $V_1 \subset Ker(\mathcal{L})^{\perp_{\mathbf{H}_0}}$. Therefore, $P'\mathbf{h} = \mathbf{h}_1 \in \mathbf{H}_0$, i.e., $P'Y \subset Y$. Summing up, X and Y are invariant under the action of P and P', respectively.

(2) Note that for any $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$, by Lemma 3.11, there exists a set $\tilde{U} \subset I_\varepsilon$ (I_ε defined in Sect. 3.3) such that $\mathbf{u}_\infty(x) \in \tilde{U}$ for any $x \in \overline{\Omega}$ and ψ is analytic in $\tilde{U}$ (recall that $\widetilde{\psi}_{|J_\varepsilon} = \psi$). It thus holds that for each $\mathbf{u}^0_\infty \in \omega(\mathbf{u}_0) - \overline{\mathbf{u}}_0$ the function $\widetilde{\mathcal{E}}'$ is real analytic in a neighborhood of $\mathbf{u}^0_\infty$ in X (see, e.g., [10, Proof of Corollary 4.6]) with values in $Y = \mathbf{H}_0$. Notice that this is possible since $\widetilde{\mathcal{E}}(\mathbf{u}^0_\infty) = \mathcal{E}(\mathbf{u}_\infty)$. The essential properties exploited are the fact that $\mathfrak{D}(\widetilde{\mathbb{A}}) \subset \mathbf{L}^\infty(\Omega)$ and $Y \subset \mathbf{L}^\infty(\Omega)$. Indeed, notice that the first summand in $\widetilde{\mathcal{E}}'$, after an integration by parts, is the linear operator $\mathbf{u} \in U \cap X \mapsto -\Delta\mathbf{u} \in \mathbf{H}_0$, which is analytic.

(3) As already seen, $Ker(\mathcal{L}(\mathbf{u}^0_\infty)) \subset \mathfrak{D}(\widetilde{\mathbb{A}}) = X$ is finite-dimensional.

(4) Recall that $Ker P' = V_1' = V_0^\perp$, and assume $\mathbf{w} \in V_0^\perp \cap \mathbf{H}_0$. Then, for any $\mathbf{z} \in V_0 = Ker((\mathcal{L}(\mathbf{u}^0_\infty))$, it holds, being $\mathbf{w} \in \mathbf{H}_0$,

$$0 = \langle \mathbf{w}, \mathbf{z}\rangle_{\mathbf{V}_0',\mathbf{V}_0} = (\mathbf{w}, \mathbf{z}),$$

i.e., $\mathbf{w} \in V_0^{\perp_{\mathbf{H}_0}}$. On the other hand, by the same argument, $V_0^{\perp_{\mathbf{H}_0}} \subset V_0^\perp \cap \mathbf{H}_0$. Thus we deduce $V_0^{\perp_{\mathbf{H}_0}} = V_0^\perp \cap \mathbf{H}_0$. Then, from (7.6), we immediately infer

$$Range(\mathcal{L}(\mathbf{u}^0_\infty)_{|\mathfrak{D}(\widetilde{\mathbb{A}})}) = V_0^{\perp_{\mathbf{H}_0}} = V_0^\perp \cap \mathbf{H}_0 = Ker P' \cap Y,$$

as desired.

Therefore, all the assumptions of [10, Corollary 3.11] are satisfied and the proof is finished.

7.3 Proof of Theorem 3.16

First of all, as already noticed, for $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$, thanks to the fact that for any $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$, by Lemma 3.11, there exists a set $\tilde{U} \subset I_\varepsilon$ such that $\mathbf{u}_\infty(x) \in \tilde{U}$ for any $x \in \overline{\Omega}$ and due to the definition of $\widetilde{\psi}$, we have that $\widetilde{\mathcal{E}}|_{\omega(\mathbf{u}_0)-\mathbf{M}} = \mathcal{E}|_{\omega(\mathbf{u}_0)}$. Arguing as in (7.1), $\mathcal{E}(\mathbf{u}_\infty) = E_\infty = \lim_{s\to\infty}\mathcal{E}(S(s)\mathbf{u}_0)$ for any $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$, so that $\mathcal{E}|_{\omega(\mathbf{u}_0)}$, and thus $\widetilde{\mathcal{E}}|_{\omega(\mathbf{u}_0)-\mathbf{M}}$ is constant, equal to E_∞. By Lemma 3.15, the Lojasiewicz-Simon inequality is valid for any $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$. This means, recalling what we just noticed about $\widetilde{\mathcal{E}}|_{\omega(\mathbf{u}_0)-\mathbf{M}}$, that there exist constants $\theta \in (0, \frac{1}{2}]$, $C > 0$, $\sigma > 0$ such that

$$\left|\widetilde{\mathcal{E}}(\mathbf{v}) - E_\infty\right|^{1-\theta} \le C\|\widetilde{\mathcal{E}}'(\mathbf{v})\|_{\mathbf{V}_0'},$$

for any $\mathbf{v} \in \mathbf{V}_0$ such that $\|\mathbf{v} + \mathbf{M} - \mathbf{u}_\infty\|_{\mathbf{V}_0} \le \sigma$. Clearly, this can be restated as

$$\left|\widetilde{\mathcal{E}}(\boldsymbol{\xi} - \mathbf{M}) - E_\infty\right|^{1-\theta} \le C\|\widetilde{\mathcal{E}}'(\mathbf{w} - \mathbf{M})\|_{\mathbf{V}_0'}, \tag{7.7}$$

for any $\boldsymbol{\xi} \in \mathbf{V}_0 + \mathbf{M}$ such that $\|\boldsymbol{\xi} - \mathbf{u}_\infty\|_{\mathbf{V}_0} \le \sigma$. Since $\mathbf{H}^{2r}(\Omega) \hookrightarrow\hookrightarrow \mathbf{H}^1(\Omega)$, $\omega(\mathbf{c}_0)$ is compact in $\mathbf{H}^1(\Omega)$, thus we can find a finite number M_1 of $\mathbf{H}^1(\Omega)$-open balls B_m, $m = 1, \dots, M_1$ of radius σ, centered at $\mathbf{u}_m \in \omega(\mathbf{u}_0)$, such that

$$\omega(\mathbf{u}_0) \subset \widetilde{U} := \bigcup_{m=1}^{M_1} B_m.$$

Note that θ and C in (7.7) depend on the choice of $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$, but being $\mathbf{u}_m$ in finite number, we can easily deduce that (7.7) holds *uniformly* for any $\boldsymbol{\xi} \in \mathbf{V}_0 + \mathbf{M} \cap \widetilde{U}$. From (3.11) and the embedding $\mathbf{H}^{2r}(\Omega) \hookrightarrow\hookrightarrow \mathbf{H}^1(\Omega)$, we deduce that there exists $\tilde{t} > 0$ such that $\mathbf{u}(t) = S(t)\mathbf{u}_0 \in \widetilde{U}$ for any $t \ge \tilde{t}$. Recalling the definition of U_ε given in Sect. 3.3, we have

$$\mathbf{u}(t) = S(t)\mathbf{u}_0 \in U_\varepsilon, \quad \forall t \ge t^*,$$

therefore we can choose $\bar{t} := \max\{\tilde{t}, t^*\}$ and $\mathbf{U} = \widetilde{U} \cap U_\varepsilon$ such that $\mathbf{u}(t) \in \mathbf{U}$ for any $t \ge \bar{t}$, implying (note that $\mathbf{u}(t) - \mathbf{u}_\infty \in \mathbf{V}_0$ for any $t \ge 0$):

$$\|\mathbf{u}(t) - \mathbf{u}_\infty(t)\|_{\mathbf{V}_0} \le \|\mathbf{u}(t) - \mathbf{u}_\infty(t)\|_{\mathbf{H}^1(\Omega)} \le \sigma, \quad \forall t \ge \bar{t}.$$

Since then $\mathbf{u}(t) \in \mathbf{V}_0 + \mathbf{M} \cap \widetilde{U}$, it holds

$$|\mathcal{E}(\mathbf{u}(t)) - E_\infty|^{1-\theta} \le C\|\widetilde{\mathcal{E}}'(\mathbf{u}(t) - \mathbf{M})\|_{\mathbf{V}_0'}, \quad \forall t \ge \bar{t},$$

recalling that, since $\mathbf{u}(t) \in U_\varepsilon$ for any $t \ge \bar{t}$, $\widetilde{\mathcal{E}}(\mathbf{u}(t) - \mathbf{M}) = \mathcal{E}(\mathbf{u}(t))$, thanks to (3.14) and the definition of $\widetilde{\psi}$. Observe now that, for any $t \ge \bar{t}$,

$$\begin{aligned}
&\langle \widetilde{\mathcal{E}}'(\mathbf{u}(t) - \mathbf{M}), \mathbf{h}\rangle_{\mathbf{V}_0', \mathbf{V}_0} \\
&\quad = \int_\Omega \nabla\mathbf{u}(t) \cdot \nabla\mathbf{h}\, dx + \int_\Omega \widetilde{\Psi}_{,\mathbf{u}}(\mathbf{u}(t)) \cdot \mathbf{h}\, dx \\
&\quad = \int_\Omega \nabla\mathbf{u}(t) \cdot \nabla\mathbf{h}\, dx + \int_\Omega \Psi_{,\mathbf{u}}(\mathbf{u}(t)) \cdot \mathbf{h}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \big(-\Delta \mathbf{u}(t) + \mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}(t))\big) \cdot \mathbf{h}\, dx \\
&= \int_\Omega (-\Delta \mathbf{u}(t) + P_0(\mathbf{P}\Psi_{\mathbf{u}}(\mathbf{u}(t)))) \cdot \mathbf{h}\, dx = (\mathbf{w}(t) - \overline{\mathbf{w}(t)}, \mathbf{h}) \\
&\le \|\nabla \mathbf{w}\| \|\mathbf{h}\| \le C\sqrt{(\boldsymbol{\alpha}\nabla \mathbf{w}(t), \nabla \mathbf{w}(t))} \|\mathbf{h}\|_{\mathbf{V}_0}, \quad \forall \mathbf{h} \in \mathbf{V}_0,
\end{aligned}
$$

where $\mathbf{w}(t)$ is defined in (3.7). Note that we exploited Poincaré's inequality and (2.4). This means that

$$
\|\widetilde{\mathcal{E}}'(\mathbf{u}(t) - \mathbf{M})\|_{\mathbf{V}_0'} \le C\sqrt{(\boldsymbol{\alpha}\nabla \mathbf{w}(t), \nabla \mathbf{w}(t))}, \quad \forall t \ge \bar{t}. \tag{7.8}
$$

Recalling the energy identity, setting $H(t) := |\mathcal{E}(\mathbf{u}(t)) - E_\infty|^\theta$, by (7.8), we have that

$$
\begin{aligned}
-\frac{d}{dt} H(t) &= -\theta \frac{d\mathcal{E}(\mathbf{u}(t))}{dt} |\mathcal{E}(\mathbf{u}(t)) - E_\infty|^{\theta - 1} \\
&\ge \theta \frac{(\boldsymbol{\alpha}\nabla \mathbf{w}(t), \nabla \mathbf{w}(t))}{C\|\widetilde{\mathcal{E}}'(\mathbf{u}(t) - \mathbf{M})\|_{\mathbf{V}_0'}} \\
&\ge C\sqrt{(\boldsymbol{\alpha}\nabla \mathbf{w}(t), \nabla \mathbf{w}(t))}, \quad \forall t \ge \bar{t}.
\end{aligned}
$$

Being H a non nonincreasing nonnegative function such that $H(t) \to 0$ as $t \to \infty$, we can integrate from $\bar{t}$ to $+\infty$ and deduce that (see (2.4)),

$$
\int_{\bar{t}}^{\infty} \|\nabla \mathbf{w}(t)\| dt \le C \int_{\bar{t}}^{\infty} \sqrt{(\boldsymbol{\alpha}\nabla \mathbf{w}(t), \nabla \mathbf{w}(t))} dt \le C H(\bar{t}) < +\infty,
$$

i.e., $\nabla \mathbf{w} \in L^1((\bar{t}, +\infty); \mathbf{L}^2(\Omega))$, entailing by comparison $\partial_t \mathbf{u} \in L^1((\bar{t}, +\infty); \mathbf{H}^1(\Omega)')$. Hence, there exists $\mathbf{u}_\infty \in \omega(\mathbf{u}_0)$ such that

$$
\mathbf{u}(t) = \mathbf{u}(\bar{t}) + \int_{\bar{t}}^{t} \partial_t \mathbf{u}(\tau) d\tau \longrightarrow \mathbf{u}_\infty \quad \text{in } \mathbf{H}^1(\Omega)', \quad \text{as } t \to \infty,
$$

and, by uniqueness of the limit, we conclude that $\omega(\mathbf{u}_0) = \{\mathbf{u}_\infty\}$. We also have $\lim_{t\to\infty} \mathbf{u}(t) = \mathbf{u}_\infty$ in $\mathbf{H}^{2r}(\Omega)$ for a fixed $r \in (\frac{d}{4}, 1)$ (the one used in the definition of the ω-limit set). On the other hand, thanks to the embedding $\mathbf{H}^{2r}(\Omega) \hookrightarrow \mathbf{H}^1(\Omega)'$, which is valid for all $r \in (0, 1)$, we deduce that the convergence to the equilibrium actually holds for any $r \in (0, 1)$. Recalling Lemma 3.11, the proof is finished.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

Appendix

The Homogeneous Neumann Laplace Operator on $T\Sigma$

First we need to consider the following elliptic problem: given $\mathbf{f} \in \mathbf{V}_0'$ find $\mathbf{u} \in \mathbf{V}_0$ such that

$$(\nabla \mathbf{v}, \nabla \mathbf{w}) = \langle \mathbf{f}, \mathbf{w} \rangle_{\mathbf{V}_0', \mathbf{V}_0} \quad \forall\, \mathbf{w} \in \mathbf{V}_0. \tag{8.1}$$

Using Hilbert triplet $\mathbf{V}_0 \hookrightarrow\hookrightarrow \mathbf{H}_0 \equiv \mathbf{H}_0' \hookrightarrow \mathbf{V}_0'$, thanks to Poincaré's inequality, we see that the bilinear form $\mathbf{a} : \mathbf{V}_0 \times \mathbf{V}_0$, $\mathbf{a}(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v})$ is coercive and continuous, so that we can apply Lax-Milgram Lemma and deduce that the bounded operator $\mathbb{A} : \mathbf{V}_0 \to \mathbf{V}_0'$, such that $\langle \mathbb{A}\mathbf{v}, \mathbf{w} \rangle_{\mathbf{V}_0', \mathbf{V}_0} = (\nabla \mathbf{v}, \nabla \mathbf{w})$, is invertible with bounded inverse $\mathbb{A}^{-1} : \mathbf{V}_0' \to \mathbf{V}_0$. Since by Poincaré's inequality, $(\nabla \cdot, \nabla \cdot)$ is an inner product on $\mathbf{V}_0$, we see that $\mathbb{A}$ is the Riesz isomorphism from $\mathbf{V}_0$ to $\mathbf{V}_0'$. Notice that we can also define (without relabeling) the unbounded operator $\mathbb{A} : \mathbf{V}_0' \to \mathbf{V}_0'$, so that $\mathfrak{D}(\mathbb{A}) = \mathbf{V}_0$ which acts exactly as $\mathbb{A}$ previously defined. In this case, being $\mathfrak{D}(\mathbb{A})$ compactly embedded in $\mathbf{V}_0'$, the inverse operator $\mathbb{A}^{-1} : \mathbf{V}_0' \to \mathfrak{D}(\mathbb{A})$ is compact in $\mathbf{V}_0'$.

Problem (8.1) is the well-posed weak formulation of the following boundary value problem

$$\begin{cases} -\Delta \mathbf{v} = \mathbf{f}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}} \mathbf{v} = \mathbf{0}, & \text{a.e. on } \partial\Omega, \end{cases} \tag{8.2}$$

where $\mathbf{f} \in \mathbf{H}_0$. We can then define the selfadjoint unbounded operator $\widetilde{\mathbb{A}} : \mathbf{H}_0 \to \mathbf{H}_0$, such that

$$\mathfrak{D}(\widetilde{\mathbb{A}}) := \{\mathbf{u} \in \mathbf{V}_0 : \mathbb{A}\mathbf{u} \in \mathbf{H}_0\}, \qquad \widetilde{\mathbb{A}}\mathbf{u} = \mathbb{A}\mathbf{u} = -\Delta \mathbf{u}, \quad \forall\, \mathbf{u} \in \mathfrak{D}(\widetilde{\mathbb{A}}).$$

Notice that, by elliptic regularity, we have

$$\mathfrak{D}(\widetilde{\mathbb{A}}) := \{\mathbf{u} \in \mathbf{V}_0 \cap \mathbf{H}^2(\Omega) : \partial_{\mathbf{n}} \mathbf{u} = 0, \quad \text{a.e. on } \partial\Omega\}.$$

In conclusion, note that we can easily exploit the properties of the Hilbert triplet to infer, being $\mathbf{V}_0 \hookrightarrow\hookrightarrow \mathbf{H}_0$, that $\widetilde{\mathbb{A}}^{-1} \in \mathcal{B}(\mathbf{H}_0)$ is compact. We now study the following problem: for $\lambda > 0$ fixed,

$$\begin{cases} -\Delta \mathbf{v} + \lambda \mathbf{v} = \mathbf{f}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}} \mathbf{v} = \mathbf{0}, & \text{a.e. on } \partial\Omega, \end{cases} \tag{8.3}$$

with $\mathbf{f} \in \widetilde{\mathbf{H}}_0$. We can repeat exactly the same argument as before, but in this case we introduce the operator $\mathbb{A}_\lambda : \widetilde{\mathbf{V}}_0 \to \widetilde{\mathbf{V}}_0'$, such that

$$\langle \mathbb{A}_\lambda \mathbf{v}, \mathbf{w} \rangle_{\widetilde{\mathbf{V}}_0', \widetilde{\mathbf{V}}_0} = (\nabla \mathbf{v}, \nabla \mathbf{w}) + \lambda (\mathbf{v}, \mathbf{w}),$$

is invertible with bounded inverse $\mathbb{A}_\lambda^{-1} : \widetilde{\mathbf{V}}_0' \to \widetilde{\mathbf{V}}_0$. The corresponding unbounded operator $\widetilde{\mathbb{A}}_\lambda : \widetilde{\mathbf{H}}_0 \to \widetilde{\mathbf{H}}_0$ is such that

$$\mathfrak{D}(\widetilde{\mathbb{A}}_\lambda) := \{\mathbf{u} \in \widetilde{\mathbf{V}}_0 : \mathbb{A}\mathbf{u} \in \widetilde{\mathbf{H}}_0\},$$
$$\widetilde{\mathbb{A}}_\lambda \mathbf{u} = \mathbb{A}_\lambda \mathbf{u} = -\Delta \mathbf{u} + \lambda \mathbf{u} \quad \forall \mathbf{u} \in \mathfrak{D}(\widetilde{\mathbb{A}}_\lambda).$$

Again, elliptic regularity yields

$$\mathfrak{D}(\widetilde{\mathbb{A}}_\lambda) := \{\mathbf{u} \in \widetilde{\mathbf{V}}_0 \cap \mathbf{H}^2(\Omega) : \partial_{\mathbf{n}} \mathbf{u} = 0, \quad \text{a.e. on } \partial\Omega\},$$

so that the unbounded operator $\widetilde{\mathbb{A}}_\lambda$ is invertible with (bounded) compact inverse.

A Neumann Problem with Singular Nonlinearity

Consider the following stationary problem: given

$$\boldsymbol{f} \in \{\mathbf{v} \in \mathbf{L}^\infty(\Omega) : \ \mathbf{v}(x) \in T\Sigma, \quad \text{for a.a. } x \in \Omega\},$$

find $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap Z$ such that

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{u}) = \boldsymbol{f}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}} \mathbf{u} = \mathbf{0}, & \text{a.e. on } \partial\Omega, \\ \sum_{i=1}^N \mathbf{u}_i = 1, & \text{in } \Omega. \end{cases} \tag{8.4}$$

Then the following result holds.

Theorem 8.1 *Problem* (8.4) *has a unique solution* $\mathbf{u}$ *which is also strictly separated from the pure phases, i.e., there exists* $0 < \delta = \delta(\boldsymbol{f}) < \frac{1}{N}$ *such that*

$$\delta < \mathbf{u}(x) \tag{8.5}$$

for any $x \in \overline{\Omega}$.

Proof Uniqueness is straightforward. Consider two solutions $\mathbf{u}_1, \mathbf{u}_2$ corresponding to the same value of $\boldsymbol{f}$, it is enough to test (8.4)$_1$ (written for $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$) with $\mathbf{u} \in \widetilde{\mathbf{V}}_0$ to get, by convexity of Ψ^1 (notice that $\psi'' \geq C > 0$) and being $\mathbf{P}$ selfadjoint,

$$\|\nabla \mathbf{u}\|^2 + C\|\mathbf{u}\|^2 \leq \|\nabla \mathbf{u}\|^2 + (\Psi^1_{,\mathbf{u}}(\mathbf{u}_1) - \Psi^1_{,\mathbf{u}}(\mathbf{u}_2), \mathbf{u}) = 0,$$

so that $\mathbf{u} = 0$.

Concerning existence, we consider the following approximated problem, where Ψ^1_ε is the same approximation exploited in the proof of Theorem 3.1: for any $\varepsilon > 0$, find $\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap Z$ solution to the problem

$$\begin{cases} -\Delta \mathbf{u}_\varepsilon + \mathbf{P}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon) = \boldsymbol{f}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}} \mathbf{u}_\varepsilon = \mathbf{0}, & \text{a.e. on } \partial\Omega, \\ \sum_{i=1}^N u_{\varepsilon,i} = 1, & \text{in } \Omega. \end{cases} \tag{8.6}$$

The rest of the proof is divided into three steps.

Existence of the approximating solution $\mathbf{u}_\varepsilon$. It follows from an application of the Leray-Schauder fixed point Theorem. First we consider the following problem: for any given $\mathbf{v} \in \widetilde{\mathbf{V}}_0$, find $\mathbf{u}_\varepsilon \in \mathfrak{D}(\widetilde{\mathbb{A}}_1) \subset \widetilde{\mathbf{V}}_0$ such that

$$\begin{cases} -\Delta \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \boldsymbol{f} - \mathbf{P}\boldsymbol{\phi}_\varepsilon\left(\mathbf{v} + \frac{1}{N}\right) + \mathbf{v}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}} \mathbf{u}_\varepsilon = \mathbf{0}, & \text{a.e. on } \partial\Omega. \end{cases}$$

The problem above is clearly equivalent to finding $\mathbf{u}_\varepsilon \in \mathfrak{D}(\widetilde{\mathbb{A}}_1) \subset \widetilde{\mathbf{V}}_0$ such that

$$\widetilde{\mathbb{A}}_1 \mathbf{u}_\varepsilon = \boldsymbol{f} - \mathbf{P}\boldsymbol{\phi}_\varepsilon\left(\mathbf{v} + \frac{1}{N}\right) + \mathbf{v} \in \widetilde{\mathbf{H}}_0,$$

where $\widetilde{\mathbb{A}}_1$ is $\widetilde{\mathbb{A}}_\lambda$ with $\lambda = 1$. Being the operator $\mathbb{A}_1$ invertible, we obtain that the map

$$T : \widetilde{\mathbf{V}}_0 \to \widetilde{\mathbf{V}}_0, \quad \mathbf{v} \mapsto T(\mathbf{v}) = \mathbf{u}_\varepsilon$$

is well defined. Moreover, we also have that the (linear) operator T is compact, since by elliptic regularity $\mathbf{u}_\varepsilon \in \mathbf{H}^2(\Omega) \cap \widetilde{\mathbf{V}}_0 \hookrightarrow\hookrightarrow \widetilde{\mathbf{V}}_0$. Continuity of the operator T is immediate: assume that $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathbf{V}}_0$ is such that $\mathbf{v}_n \to \mathbf{v} \in \widetilde{\mathbf{V}}_0$ as $n \to \infty$ and define $\mathbf{u}_n = T(\mathbf{v}_n)$ and $\mathbf{u} = T(\mathbf{v})$. Then we get

$$\begin{aligned} &\|\nabla(\mathbf{u}_n - \mathbf{u})\|^2 + \|\mathbf{u}_n - \mathbf{u}\|^2 \\ &\quad = \left(\boldsymbol{\phi}_\varepsilon\left(\mathbf{v} + \frac{1}{N}\right) - \boldsymbol{\phi}_\varepsilon\left(\mathbf{v}_n + \frac{1}{N}\right), \mathbf{u}_n - \mathbf{u}\right) + (\mathbf{u}_n - \mathbf{u}, \mathbf{v}_n - \mathbf{v}) \\ &\quad \leq \frac{1}{2}\|\mathbf{u}_n - \mathbf{u}\|^2 + C(\varepsilon)\|\mathbf{v}_n - \mathbf{v}\|^2, \end{aligned}$$

where we have used the fact that ψ'_ε is Lipschitz continuous with constant ε^{-1} (see Sect. 2). From this we clearly deduce that $T(\mathbf{v}_n) \to T(\mathbf{v})$. Consider now the family of operators $\{sT\}_{s\in(0,1)}$. We show that if there exist fixed points $\mathbf{u} \in \widetilde{\mathbf{V}}_0$ for the family, i.e., such that $sT(\mathbf{u}) = \mathbf{u}$ (for some $s \in (0,1)$), these points are uniformly bounded. First notice that $\frac{1}{s}\mathbf{u} = T(\mathbf{u}) \in \mathfrak{D}(\mathbb{A}_1)$ satisfies

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{u} + s\mathbf{P}\boldsymbol{\phi}_\varepsilon\left(\mathbf{u} + \frac{1}{N}\right) = s\boldsymbol{f} + s\mathbf{u}, & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}}\mathbf{u} = \mathbf{0}, & \text{a.e. on } \partial\Omega. \end{cases}$$

On account of the (strict) convexity of ψ_ε, we have

$$\begin{aligned} &\left(\mathbf{P}\boldsymbol{\phi}_\varepsilon\left(\mathbf{u} + \tfrac{1}{N}\right), \mathbf{u}\right) = \left(\boldsymbol{\phi}_\varepsilon\left(\mathbf{u} + \tfrac{1}{N}\right), \mathbf{u}\right) \\ &\quad = \sum_{i=1}^N \left(\psi_\varepsilon'\left(u_i + \tfrac{1}{N}\right) - \psi_\varepsilon'(\tfrac{1}{N}), u_i\right) + \sum_{i=1}^N \left(\psi_\varepsilon'\left(\tfrac{1}{N}\right), u_i\right) \\ &\quad \geq C\|\mathbf{u}\|^2 + \sum_{i=1}^N \left(\psi_\varepsilon'\left(\tfrac{1}{N}\right), u_i\right), \end{aligned}$$

for some $C > 0$. Then, by standard estimates, we deduce

$$\begin{aligned} &\|\nabla\mathbf{u}\|^2 + (C+1)\|\mathbf{u}\|^2 \leq \|\nabla\mathbf{u}\|^2 + \|\mathbf{u}\|^2 + s\left(\mathbf{P}\boldsymbol{\phi}_\varepsilon\left(\mathbf{u} + \tfrac{1}{N}\right), \mathbf{u}\right) - s\sum_{i=1}^N \left(\psi_\varepsilon'\left(\tfrac{1}{N}\right), u_i\right) \\ &\quad = s(\mathbf{f} + \mathbf{u}, \mathbf{u}) - s\sum_{i=1}^N \left(\psi_\varepsilon'\left(\tfrac{1}{N}\right), u_i\right) \leq (\|\mathbf{f}\| + C)\|\mathbf{u}\| + \|\mathbf{u}\|^2, \end{aligned}$$

where we exploited the fact that $|\psi_\varepsilon'(\frac{1}{N})| < +\infty$, being ψ_ε' continuous. This clearly implies

$$\|\mathbf{u}\|_{\widetilde{\mathbf{V}}_0} \leq C,$$

as desired. Therefore, the Leray–Schauder fixed point Theorem entails the existence of $\tilde{\mathbf{u}}_\varepsilon \in \widetilde{\mathbf{V}}_0$ such that $\tilde{\mathbf{u}}_\varepsilon = T(\tilde{\mathbf{u}}_\varepsilon) \in \mathfrak{D}(\mathbb{A}_1)$, i.e.,

$$\begin{cases} -\Delta\tilde{\mathbf{u}}_\varepsilon = \boldsymbol{f} - \mathbf{P}\boldsymbol{\phi}_\varepsilon\left(\tilde{\mathbf{u}}_\varepsilon + \frac{1}{N}\right), & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}}\tilde{\mathbf{u}}_\varepsilon = \mathbf{0}, & \text{a.e. on } \partial\Omega. \end{cases}$$

At this point is enough to set $\mathbf{u}_\varepsilon = \tilde{\mathbf{u}}_\varepsilon + \frac{1}{N}$ to obtain the existence of a solution to (8.6) with the desired properties, since $-\Delta\tilde{\mathbf{u}}_\varepsilon = -\Delta\mathbf{u}_\varepsilon$ and $\partial_{\mathbf{n}}\mathbf{u}_\varepsilon = \partial_{\mathbf{n}}\tilde{\mathbf{u}}_\varepsilon$. Uniqueness of the solution $\mathbf{u}_\varepsilon$ can be proven exactly as in the case of problem (8.4) above.

Uniform-in-ε estimates. Let us test (8.6) with $\mathbf{u}_\varepsilon$, integrate over Ω and then integrate by parts. This gives

$$\|\nabla\mathbf{u}_\varepsilon\|^2 + (\mathbf{P}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \mathbf{u}_\varepsilon) = (\mathbf{f}, \mathbf{u}_\varepsilon) \leq \|\mathbf{f}\|\|\mathbf{u}_\varepsilon\|. \tag{8.7}$$

Observe that

$$(\mathbf{P}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \mathbf{u}_\varepsilon) = \sum_{k=1}^N \int_\Omega \left(\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N} \sum_{l=1}^N \phi_\varepsilon(u_{\varepsilon,l}) \right) u_{\varepsilon,k}\, dx,$$

and

$$\begin{aligned}
&\sum_{k=1}^N \left(\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N} \sum_{l=1}^N \phi_\varepsilon(u_{\varepsilon,l}) \right) u_{\varepsilon,k} \\
&\quad = \frac{1}{N} \sum_{k,l=1}^N \big(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\big) u_{\varepsilon,k} \\
&\quad = \frac{1}{N} \sum_{k<l}^N \big(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\big) u_{\varepsilon,k} \sum_{k>l}^N \big(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\big) u_{\varepsilon,k} \\
&\quad = \frac{1}{N} \sum_{k<l}^N \big(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})\big) (u_{\varepsilon,k} - u_{\varepsilon,l}) \\
&\quad = \frac{1}{N} \sum_{k<l}^N \int_0^1 \psi_\varepsilon''(s u_{\varepsilon,k} + (1-s) u_{\varepsilon,l})(u_{\varepsilon,k} - u_{\varepsilon,l})^2 ds \\
&\quad \geq \frac{C}{N} \sum_{k<l}^N |u_{\varepsilon,k} - u_{\varepsilon,l}|^2,
\end{aligned}$$

where $C > 0$ is independent of ε. Note now that, for any $k = 1, \ldots, N$,

$$\left(\mathbf{u}_\varepsilon - \frac{1}{N} \right)_k = (\mathbf{P}\mathbf{u}_\varepsilon)_k = \frac{1}{N} \sum_{l=1}^N (u_{\varepsilon,k} - u_{\varepsilon,l}),$$

entailing that there exists another $C > 0$ independent of ε such that

$$\left| \mathbf{u}_\varepsilon - \frac{1}{N} \right|^2 \leq C \sum_{k<l}^N |u_{\varepsilon,k} - u_{\varepsilon,l}|^2.$$

Thus, from the previous result, we get

$$(\mathbf{P}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon), \mathbf{u}_\varepsilon) \geq C \int_\Omega \left| \mathbf{u}_\varepsilon - \tfrac{1}{N} \right|^2 dx,$$

and this allows us to conclude from (8.7) that

$$\|\nabla \mathbf{u}_\varepsilon\|^2 + C^\sharp \left\| \mathbf{u}_\varepsilon - \frac{1}{N} \right\|^2 = (\mathbf{f}, \mathbf{u}_\varepsilon) \leq C(1 + \|\mathbf{f}\|^2) + \frac{C^\sharp}{2} \left\| \mathbf{u}_\varepsilon - \frac{1}{N} \right\|^2,$$

for some $C^\sharp > 0$ independent of ε. Being $\frac{1}{N}$ a constant, this clearly implies that

$$\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^1(\Omega)} \leq C, \tag{8.8}$$

for some $C > 0$ independent of ε. Thanks to this bound we can repeat word by word the argument used to get (4.17) (with $\mathbf{f}$ in place of $\mathbf{w}_\varepsilon$ and the matrix $\mathbf{A} = \mathbf{0}$). This gives

$$\frac{1}{4N}\int_\Omega \max_{i=1,\ldots,N} \phi_\varepsilon(u_{\varepsilon,i})^2 dx \leq C\left(1 + \|\mathbf{f}\|^2\right) \leq C, \tag{8.9}$$

uniformly with respect to ε. Then, by comparison in (8.6), together with elliptic regularity, we find

$$\|\mathbf{u}_\varepsilon\|_{\mathbf{H}^2(\Omega)} \leq C, \tag{8.10}$$

uniformly with respect to ε. We can also redo the argument leading to (4.22), to get, for any $r \geq 2$,

$$\frac{1}{4N}\int_\Omega \max_{i=1,\ldots,N} |\phi_\varepsilon(u_{\varepsilon,i})|^r dx \leq C\|\mathbf{f}\|^r_{L^r(\Omega)} \leq C|\Omega|_d\|\mathbf{f}\|^r_{\mathbf{L}^\infty(\Omega)}, \tag{8.11}$$

where $C > 0$ does not depend neither on ε nor on r. From this we infer

$$\|\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\|_{\mathbf{L}^r(\Omega)} \leq (4C|\Omega|_d)^{\frac{1}{r}}\|\mathbf{f}\|_{\mathbf{L}^\infty(\Omega)}.$$

Letting ε go to 0. By standard compactness arguments we can then pass to the limit as $\varepsilon \to 0$, along a suitable subsequence, and deduce the existence of a (unique) solution $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap Z$ to (8.4). In particular, we have that

$$\|\boldsymbol{\phi}(\mathbf{u})\|_{\mathbf{L}^r(\Omega)} \leq (4C|\Omega|_d)^{\frac{1}{r}}\|\mathbf{f}\|_{\mathbf{L}^\infty(\Omega)},$$

for any $r \geq 2$. Letting then $r \to \infty$, we deduce

$$\|\boldsymbol{\phi}(\mathbf{u})\|_{\mathbf{L}^\infty(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{L}^\infty(\Omega)}. \tag{8.12}$$

The embedding $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{C}(\overline{\Omega})$ and (8.12) imply (8.5), since $|\psi'(s)| \to \infty$ when $s \to 0$ and $\sum_{i=1}^N u_i = 1$. □

Unbounded Fredholm Operators

Here we report a characterization of unbounded Fredholm operators (see [45]). First, we say that a densely defined closed operator A on a Banach space X, $A : \mathfrak{D}(A) \hookrightarrow X \to X$ is said to be an (unbounded) Fredholm operator if it satisfies the conditions:

- $Range(A)$ is closed in X;
- $\dim Ker(A) < +\infty$;

- codim$Range(A) < +\infty$.

The following characterization holds (see [45, Theorem VII, (ii)])

Theorem 8.2 *Let A be a closed densely defined operator on X. A is Fredholm if and only if it is invertible modulo a compact operator in $\mathcal{K}(X)$, i.e., if there exists an operator $T \in \mathcal{B}(X)$ and two compact operators $K_1, K_2 \in \mathcal{K}(X)$ such that*

$$AT = I_X + K_1 \text{ on } X, \quad \text{and} \quad TA = I_X + K_2 \text{ on } \mathfrak{D}(A),$$

where I_X is the identity on X.

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