Z. Angew. Math. Phys. (2023) 74:180
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https://doi.org/10.1007/s00033-023-02061-2

Zeitschrift für angewandte Mathematik und Physik ZAMP

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Asymmetric equilibrium configurations of a body immersed in a 2d laminar flow

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Abstract. We study the equilibrium configurations of a possibly asymmetric fluid–structure interaction problem. The fluid is confined in a bounded planar channel and is governed by the stationary Navier–Stokes equations with laminar inflow and outflow. A body is immersed in the channel and is subject to both the lift force from the fluid and to some external elastic force. Asymmetry, which is motivated by natural models, and the possibly non-vanishing velocity of the fluid on the boundary of the channel require the introduction of suitable assumptions to prevent collisions of the body with the boundary. With these assumptions at hand, we prove that for sufficiently small inflow/outflow there exists a unique equilibrium configuration. Only if the inflow, the outflow and the body are all symmetric, the configuration is also symmetric. A model application is also discussed.

Mathematics Subject Classification. 35Q35, 76D05, 74F10.

1. Introduction

Let $L > H > 0$ and consider the rectangle $R = (-L, L)\times(-H, H)$. Let $B \subset R$ be a closed smooth domain having barycenter at the origin $(x_1, x_2) = (0, 0)$ such that $\mathrm{diam}(B) \ll L, H$. We study the behavior of a stationary laminar (horizontal) fluid flow going through R and filling the domain $\Omega_h = R \setminus B_h$, where $B_h = B + he_2$ for some h (a vertical translation of B), see Fig. 1. Note that $B_0 = B$.

The fluid is governed by the stationary 2D Navier–Stokes equations

$$-\mu\Delta u + u\cdot\nabla u + \nabla p = 0, \quad \nabla\cdot u = 0 \quad \text{in} \quad \Omega_h, \tag{1.1}$$

complemented with inhomogeneous Dirichlet boundary conditions on $\partial\Omega_h = \partial B_h \cup \partial R$, see (2.4). Here, $\mu > 0$ is the kinematic viscosity, u is the velocity vector field, p is the scalar pressure.

The body B is subject to two vertical forces. The first force (the lift) is due to the fluid flow and tends to move B away from its original position B_0; it is expressed through a boundary integral over ∂B, see (3.1). The second force is mechanical (elastic) and acts as a restoring force tending to maintain B in B_0. When there is no inflow/outflow, the body is only subject to the restoring force and remains in B_0 which is the unique equilibrium position. But, as soon as there is a fluid flow, these two forces start competing and one may wonder if the body remains in B_0 or, at least, if the equilibrium position remains unique.

We show that if the inflow/outflow is sufficiently small, then the equilibrium position of B remains unique and coincides with B_h for some h close to zero. We point out that, contrary to [3,8,10], *we make no symmetry assumptions neither on B nor on the laminar inflow/outflow*. Therefore, not only the overall configuration will be asymmetric but also some of the techniques developed in these papers do not work and B_h may be different from B_0. The motivation for studying asymmetric configurations comes from nature. Only very few bodies are perfectly symmetric, and most fluid flows, although laminar in the horizontal direction, are asymmetric in the vertical direction: think of an horizontal wind depending on the altitude or the water flow in a river depending on the distance from the banks. Figure 2 shows two front waves in sandstorms that have no vertical symmetry although the wind is (almost) horizontally laminar.

Published online: 21 August 2023

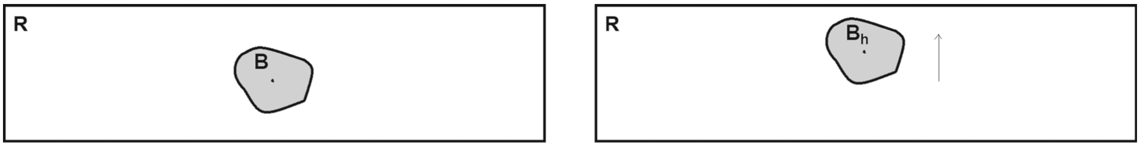

FIG. 1. Rectangle R and the body B with its vertical displacements B_h

FIG. 2. Front wave of two wind storms

In Sect. 2, we give a detailed description of our model and we prove that, for small Reynolds numbers, the Navier–Stokes equations are uniquely solvable in any Ω_h, see Theorem 2.2. The related a priori bounds depend on h, and this is one crucial difference compared to the (symmetric) Poiseuille inflow/outflow considered in [3]. It is well known [5] that to solve inhomogeneous Dirichlet problems for the Navier–Stokes equations, one needs to find a solenoidal extension of the boundary data and to transform the original problem in an homogeneous Dirichlet problem with an additional source term. For the existence issue, one can use the classical Hopf extension, but there are infinitely many other possible choices for the solenoidal extension. One of them, introduced in [12], was used in [3] to write the lift force as a volume integral by means of the solution of an auxiliary Stokes problem. For asymmetric flows, the same solenoidal extension does not allow to estimate all the boundary terms and, in order to obtain refined bounds for the solution to the Navier–Stokes equations in Ω_h, we build a new explicit solenoidal extension that also plays a fundamental role in the analysis of the subsequent fluid–structure interaction (FSI) problem.

The main physical interest in FSI problems is to determine the ω-limit of the associated evolution equations because this allows to forecast the long-time behavior of the structure. Since the evolution Navier–Stokes equations are dissipative, one is led to investigate if the global attractor exists, see [7,15]: the main difficulty is that the corresponding phase space is time dependent and semigroup theory does not apply. The global attractor contains stationary solutions of the evolution FSI problem that we call equilibrium configurations, which are investigated in the present work.

In Sect. 3, we introduce the lift force and the restoring force and we set up the steady-state FSI problem. Our main result, namely Theorem 3.1, states that, for small Reynolds numbers, the equilibrium position is unique and may differ from B_0. By exploiting the strength of the restoring force, uniqueness for the FSI problem is obtained without assuming uniqueness for (1.1). To prove this result, we need some bounds on the lift force in proximity of collisions of B_h with ∂R: these bounds are collected in Theorem 3.2 and proved in Sect. 4 by using the very same solenoidal extension introduced in Sect. 2. The remaining part of the proof of Theorem 3.1 is divided in two steps. In Subsection 5.1, we prove some properties of the global force exerted on the body B. These properties are then used in Subsection 5.2

to complete the proof by means of an implicit function argument, combined with some delicate bounds involving derivatives of moving boundary integrals. We emphasize that for our FSI problem we cannot use the explicit expression of the lift derivative as in [17] because the displacements B_h within R do not follow the normal of ∂B_h, in particular if ∂B_h contains some vertical segments. Instead, based on the general approach introduced in [2] (see also the previous work [14]), we compute with high precision the lift variation with respect to the vertical displacement parameter h of B_h by acting directly on the strong form of the FSI problem.

Section 6 contains the symmetric version of Theorem 3.1, see Theorem 6.1 which states that, under symmetry assumptions on the inflow/outflow and on B, for small Reynolds numbers the equilibrium position is unique and coincides with B_0. This extends former results in [3,8,10] to a wider class of symmetric frameworks.

As an application of our results, in Sect. 7 we consider a model where B_h represents the cross-section of the deck of a suspension bridge [6], while Ω_h is filled by the air and represents either a virtual box around the deck or a wind tunnel around a scaled model of the bridge. Since the deck may have a nonsmooth boundary, we also explain how to extend our results to the case where B is merely Lipschitz.

2. Fluid boundary value problem

Let R and B be as in Sect. 1 (Fig. 1) with

$$B \text{ of class } W^{2,\infty}. \tag{2.1}$$

On the one hand, (2.1) ensures the regularity $(u,p) \in H^2(\Omega_h) \times H^1(\Omega_h)$ for the solutions to (1.1), see [14, Theorem 2.1] and Theorem 2.2. On the other hand, in engineering applications B is usually a polygon with rounded corners, see Sect. 7, which belongs to $W^{2,\infty}$ but not to C^2. Let

$$\delta_b := -\min_{(x_1,x_2)\in\partial B} x_2 > 0, \quad \delta_t := \max_{(x_1,x_2)\in\partial B} x_2 > 0, \quad \tau := \max_{(x_1,x_2)\in\partial B} |x_1|. \tag{2.2}$$

Since we consider vertical displacements B_h within R, we have $h \in (-H+\delta_b, H-\delta_t)$ and $B_h \subset [-\tau,\tau] \times [h-\delta_b, h+\delta_t]$ for any such h. Then, $\partial\Omega_h = \partial B_h \cup \partial R$. The bottom and top parts of ∂R are, respectively,

$$\Gamma_b = [-L, L] \times \{-H\} \quad \text{and} \quad \Gamma_t = [-L, L] \times \{H\},$$

while its lateral left and right parts are, respectively,

$$\Gamma_l = \{-L\} \times [-H, H] \quad \text{and} \quad \Gamma_r = \{L\} \times [-H, H].$$

Let $V_{\text{in}}, V_{\text{out}} \in W^{2,\infty}(-H,H) \subset C^0[-H,H]$ satisfy

$$\begin{gathered} V_{\text{in}}(-H) = V_{\text{out}}(-H) = 0, \quad V_{\text{in}}(H) = V_{\text{out}}(H) = U \geq 0, \\ \int_{-H}^{H} V_{\text{in}}(x_2)\mathrm{d}x_2 = \int_{-H}^{H} V_{\text{out}}(x_2)\mathrm{d}x_2. \end{gathered} \tag{2.3}$$

For some $\lambda \geq 0$, we consider the boundary value problem

$$\begin{gathered} -\mu\Delta u + u\cdot\nabla u + \nabla p = 0, \qquad \nabla\cdot u = 0 \quad \text{in} \quad \Omega_h, \\ u_{|\partial B_h} = u_{|\Gamma_b} = 0, \quad u_{|\Gamma_t} = \lambda U e_1, \quad u_{|\Gamma_l} = \lambda V_{\text{in}}(x_2) e_1, \quad u_{|\Gamma_r} = \lambda V_{\text{out}}(x_2) e_1. \end{gathered} \tag{2.4}$$

Note that $u_{|\partial R} \in C^0(\partial R)$ and (2.3)-(2.4) are compatible with the Divergence Theorem. The role of $\lambda \geq 0$ in the boundary conditions is to measure with a unique parameter the strength of both the inflow and the outflow. Hence, $\lambda \asymp \mathrm{Re}$ where Re is the Reynolds number.

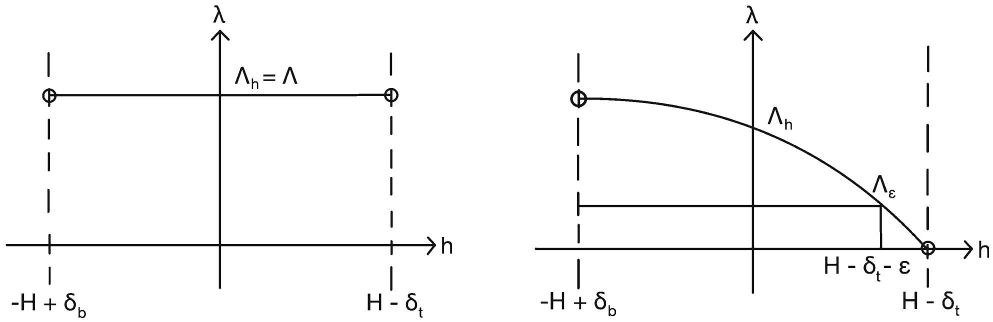

FIG. 3. Qualitative behavior of $\Lambda = \Lambda(h)$ for $U = 0$ (left) and $U = 1$ (right)

Definition 2.1. We say that $(u, p) \in H^2(\Omega_h) \times H^1(\Omega_h)$ is a strong solution to (2.4) if the differential equations are satisfied a.e. in Ω_h and the boundary conditions are satisfied as restrictions (recall that $H^2(\Omega_h) \subset C^0(\overline{\Omega_h})$).

We now state an apparently classical existence and uniqueness result which, however, has some novelties. First, since the domain Ω_h is only Lipschitzian, the regularity of the solution is obtained through a geometric reflection. More importantly, the explicit upper bound for the blow-up of the H^1-norm of the unique solution to (2.4) in proximity of collision: when B approaches Γ_t, the norm remains bounded while when B approaches Γ_b we estimate its blow-up. This refined bound requires the construction of a suitable solenoidal extension of the boundary data. Note that, up to normalization, we can reduce to the cases where

$$U \in \{0, 1\}. \tag{2.5}$$

In order to state the result, we define the distances of the body B_h to Γ_b and Γ_t, respectively, by

$$\varepsilon_b(h) := H - \delta_b + h, \qquad \varepsilon_t(h) := H - \delta_t - h. \tag{2.6}$$

Hence, $0 < \varepsilon_b(h), \varepsilon_t(h) \leq 2H - \delta_b - \delta_t$ for any $h \in (-H + \delta_b, H - \delta_t)$. Throughout the paper, any (positive) constant depending only on μ, B_0, L, H will be denoted by C and, when it depends also on h, by C_h. We may now state

Theorem 2.2. *Let $h \in (-H + \delta_b, H - \delta_t)$ and assume (2.3) with (2.5). Then, (2.4) admits a strong solution (u, p) for any $\lambda \geq 0$ and there exists $\Lambda = \Lambda(h) > 0$ such that the solution is unique if $\lambda \in [0, \Lambda(h))$; if $U = 0$, $\Lambda(h)$ can be chosen independent of h, i.e., $\Lambda(h) \equiv \Lambda > 0$. Moreover, there exist $C > 0$ and $C_h > 0$ such that the unique solution (when $\lambda < \Lambda(h)$) satisfies*

$$\|u\|_{H^1(\Omega_h)} \leq C(1 + U(\varepsilon_t(h))^{-3/2})\lambda, \tag{2.7}$$

$$\|u\|_{H^2(\Omega_h)} + \|p\|_{H^1(\Omega_h)} \leq C_h \lambda. \tag{2.8}$$

A priori bounds such as (2.7) and (2.8) are available for any $\lambda \geq 0$ and any strong solution of (2.4), but with different powers of λ.

Before giving the proof, let us explain qualitatively the main differences between the cases $U = 0$ and $U = 1$. For $U = 0$, the a priori bound (2.7) is *independent* of h, so that the graph of $\Lambda(h)$ looks like Fig. 3 (left). For $U = 1$, (2.7) depends on h and $\Lambda(h)$ itself may depend on h, see Fig. 3 (right) and (2.20).

Proof. Existence of weak solutions. For later use, we first define weak solution for the forced Navier–Stokes equations

$$-\mu\Delta u + u \cdot \nabla u + \nabla p = f, \qquad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_h, \tag{2.9}$$

which reduces to (2.4) when $f = 0$. We say that $u \in H^1(\Omega_h)$ is a weak solution to (2.9) with $f \in L^2(\Omega_h)$ if u is a solenoidal vector field satisfying the boundary conditions in the trace sense and

$$\mu \int_{\Omega_h} \nabla u : \nabla \varphi + \int_{\Omega_h} u \cdot \nabla u \cdot \varphi = \int_{\Omega_h} f \cdot \varphi \tag{2.10}$$

for all $\varphi \in W(\Omega_h) := \{\varphi \in H_0^1(\Omega_h) : \nabla \cdot \varphi = 0 \text{ a.e. in } \Omega_h\}$. For any weak solution u, there exists a unique associated $p \in L_0^2(\Omega_h)$ (i.e., with zero mean value), satisfying

$$\mu \int_{\Omega_h} \nabla u : \nabla \psi + \int_{\Omega_h} u \cdot \nabla u \cdot \psi - \int_{\Omega_h} p \nabla \cdot \psi = \int_{\Omega_h} f \cdot \psi \tag{2.11}$$

for all $\psi \in H_0^1(\Omega_h)$ (Lemma IX.1.2, [5]). In (2.24), we introduce an ad-hoc solenoidal extension matching our geometric framework which is not optimal for our current purpose. This is why we use here the well-known Hopf's extension s that reduces the effect of the nonlinearity and allows to prove existence for any $\lambda \geq 0$. Hence, we recast (2.4) as (2.9) with homogeneous boundary conditions, namely

$$-\mu \Delta v + v \cdot \nabla v + \nabla p = f, \quad \nabla \cdot v = 0 \quad \text{in} \quad \Omega_h, \qquad v_{|\partial \Omega_h} = 0, \tag{2.12}$$

where $f = \mu \Delta s - s \cdot \nabla v - v \cdot \nabla s - s \cdot \nabla s$. Then, there exists $v \in W(\Omega_h)$ satisfying (2.10) for any $\lambda \geq 0$ (Theorem IX.4.1, [5]). This is equivalent to say that the vector field $u = v + s \in H^1(\Omega_h)$ and the associated pressure $p \in L^2(\Omega_h)$ satisfy (2.10)-(2.11) with $f = 0$. Moreover, $\nabla \cdot u = 0$, $u_{|\partial \Omega_h} = s_{|\partial \Omega_h}$ and

$$\begin{aligned} \|u\|_{H^1(\Omega_h)} &\leq C(\|\nabla v\|_{L^2(\Omega_h)} + \|s\|_{H^1(\Omega_h)}) \\ &\leq C\left(\left(1 + \tfrac{1}{\mu}\right)\|s\|_{H^1(\Omega_h)} + \tfrac{1}{\mu}\|s\|^2_{H^1(\Omega_h)}\right) \leq C_h(\lambda + \lambda^2), \end{aligned} \tag{2.13}$$

$$\|p\|_{L^2(\Omega_h)} \leq C(\mu \|u\|_{H^1(\Omega_h)} + \|u\|^2_{H^1(\Omega_h)}) \leq C_h(\lambda + \lambda^4). \tag{2.14}$$

In these bounds and the ones below, we only emphasize the smallest and largest powers of λ, as for any polynomial. These bounds are not part of the statement, but they will be used later in the present proof.

Regularity. We claim that any weak solution (u, p) to (2.4) satisfies $(u, p) \in H^2(\Omega_h) \times H^1(\Omega_h)$. This would be straightforward if $\Omega_h \in W^{2,\infty}$, see [14], but R is only Lipschitzian. Here, we take advantage of the particular shape of R and use a reflection argument as in [9]. We construct a new domain $\Omega_h^t = R^t \setminus B_h^t$, obtained by reflecting Ω_h across Γ_t, where $R^t = (-L, L) \times [H, 3H)$ and B_h^t is the reflection of B_h with respect to Γ_t. Define $(u^t, p^t) : \Omega_h^t \to \mathbb{R}^2 \times \mathbb{R}$ by

$$\begin{aligned} &u_1^t(x_1, H + x_2) = u_1(x_1, H - x_2), \quad u_2^t(x_1, H + x_2) = -u_2(x_1, H - x_2), \\ &p^t(x_1, H + x_2) = p(x_1, H - x_2) \qquad \text{for all } (x_1, x_2) \in (-L, L) \times [0, 2H), \end{aligned}$$

which satisfies

$$-\mu \Delta u^t + u^t \cdot \nabla u^t + \nabla p^t = 0, \quad \nabla \cdot u^t = 0 \quad \text{in} \quad \Omega_h^t. \tag{2.15}$$

Therefore, the couple

$$(\overline{u}, \overline{p}) = \begin{cases} (u, p) & \text{in} \quad \Omega_h, \\ (u^t, p^t) & \text{in} \quad \Omega_h^t, \end{cases}$$

satisfies the Navier–Stokes equations

$$-\mu \Delta \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = 0, \quad \nabla \cdot \overline{u} = 0 \quad \text{in} \quad \big\{(-L, L) \times (-H, 3H)\big\} \setminus \{B_h \cup B_h^t\}.$$

Similarly, let $\Omega_h^b = R^b \setminus B_h^b$ with $R^b = (-L, L) \times (-3H, -H]$ and B_h^b is the reflection of B_h with respect to Γ_b. Define $(u^b, p^b) : \Omega_h^b \to \mathbb{R}^2 \times \mathbb{R}$ by

$$\begin{aligned} &u_1^b(x_1, -H - x_2) = u_1(x_1, -H + x_2), \quad u_2^b(x_1, -H - x_2) = -u_2(x_1, -H + x_2), \\ &p^b(x_1, -H - x_2) = p(x_1, -H + x_2) \qquad \text{for all } (x_1, x_2) \in (-L, L) \times [0, 2H), \end{aligned}$$

which satisfies the corresponding of (2.15) in Ω_h^b. Thanks to these two vertical reflections, we obtain a solution in $\Omega_h^s = \big\{(-L,L)\times(-3\,H,3\,H)\big\}\setminus\{B_h\cup B_h^t\cup B_h^b\}$.

With the same principle, we then perform two horizontal reflections of Ω_h^s with respect to $x_1=\pm L$. At the end of this procedure, let

$$\widetilde{\Omega}_h = \big\{(-3L,3L)\times(-3H,3H)\big\}\setminus\{B_h \text{ and its eight reflections}\}$$

and $(\widetilde{u},\widetilde{p}):\widetilde{\Omega}_h\to\mathbb{R}^2\times\mathbb{R}$ be the extension of (u,p), so that

$$-\mu\Delta\widetilde{u}+\widetilde{u}\cdot\nabla\widetilde{u}+\nabla\widetilde{p}=0,\quad \nabla\cdot\widetilde{u}=0 \quad\text{in}\quad \widetilde{\Omega}_h,\qquad \widetilde{u}_{|_{\partial B_h}}=0 \tag{2.16}$$

and $\tilde{u}$ satisfies further boundary conditions that we do not need to make explicit. After introducing a suitable solenoidal extension, we can proceed as in the first part of the proof and obtain the existence of a solution $(\widetilde{u},\widetilde{p})\in H^1(\widetilde{\Omega}_h)\times L^2(\widetilde{\Omega}_h)$ satisfying the bounds (2.13)-(2.14). Hence, $\widetilde{u}\cdot\nabla\widetilde{u}\in L^{3/2}(\widetilde{\Omega}_h)$ and

$$\|\widetilde{u}\cdot\nabla\widetilde{u}\|_{L^{3/2}(\widetilde{\Omega}_h)}\le\|\widetilde{u}\|_{L^6(\widetilde{\Omega}_h)}\|\nabla\widetilde{u}\|_{L^2(\widetilde{\Omega}_h)}\le C\|\widetilde{u}\|^2_{H^1(\widetilde{\Omega}_h)}\le C_h(\lambda^2+\lambda^4) \tag{2.17}$$

with $C_h=C(\widetilde{\Omega}_h)$. By applying [14] and [5, Theorems IV.4.1 and IV.5.1] to the Stokes problem (2.16), we infer that $(\widetilde{u},\widetilde{p})\in W^{2,3/2}(\Omega')\times W^{1,3/2}(\Omega')$ for any $\Omega'\subset\widetilde{\Omega}_h$ and

$$\begin{aligned}&\|\widetilde{u}\|_{W^{2,3/2}(\Omega')}+\|\widetilde{p}\|_{W^{1,3/2}(\Omega')}\\&\le C_h(\|\widetilde{u}\cdot\nabla\widetilde{u}\|_{L^{3/2}(\widetilde{\Omega}_h)}+\|\widetilde{u}\|_{W^{1,3/2}(\widetilde{\Omega}_h)}+\|\widetilde{p}\|_{L^{3/2}(\widetilde{\Omega}_h)})\le C_h(\lambda+\lambda^4)\end{aligned} \tag{2.18}$$

with $C_h=C(\Omega',\widetilde{\Omega}_h)$. We recall that $(\widetilde{u},\widetilde{p})=(u,p)$ in Ω_h. Then, using Sobolev embedding $W^{2,3/2}\hookrightarrow W^{1,6}$ in $\mathbb{R}^2$ and a bootstrap argument we obtain that $(u,p)\in H^2(\Omega_h)\times H^1(\Omega_h)$. Moreover, from (2.17)-(2.18) we get

$$\begin{aligned}\|u\|_{H^2(\Omega_h)}+\|p\|_{H^1(\Omega_h)}&\le C_h(\|\widetilde{u}\cdot\nabla\widetilde{u}\|_{L^2(\Omega')}+\|\widetilde{u}\|_{H^1(\Omega')}+\|\widetilde{p}\|_{L^2(\Omega')})\\&\le C_h(\|\widetilde{u}\|_{L^3(\Omega')}\|\nabla\widetilde{u}\|_{L^6(\Omega')}+\|\widetilde{u}\|_{H^1(\Omega')}+\|\widetilde{p}\|_{L^2(\Omega')})\\&\le C_h(\|\widetilde{u}\|_{H^1(\Omega')}\|\widetilde{u}\|_{W^{2,3/2}(\Omega')}+\|\widetilde{u}\|_{H^1(\Omega')}+\|\widetilde{p}\|_{L^2(\Omega')})\\&\le C_h(\lambda+\lambda^4)\end{aligned}$$

with $C_h=C(\Omega_h,\widetilde{\Omega}_h)$. This also proves (2.8) whenever $\lambda<\Lambda(h)$.

Uniqueness. Let u_1 and u_2 be two weak solutions to (2.4), let $w=u_1-u_2$, then

$$\mu\int_{\Omega_h}\nabla w:\nabla\varphi+\int_{\Omega_h}w\cdot\nabla w\cdot\varphi=-\int_{\Omega_h}(w\cdot\nabla u_2+u_2\cdot\nabla w)\cdot\varphi$$

for all $\varphi\in W(\Omega_h)$. Then, take $\varphi=w$ so that the latter yields

$$\begin{aligned}\mu\|\nabla w\|^2_{L^2(\Omega_h)}&=-\int_{\Omega_h}w\cdot\nabla u_2\cdot w\le\|\nabla u_2\|_{L^2(\Omega_h)}\|w\|^2_{L^4(\Omega_h)}\\&\le C_h(1+\tfrac{1}{\mu})(\lambda+\lambda^2)\|\nabla w\|^2_{L^2(\Omega_h)},\end{aligned} \tag{2.19}$$

where we used Hölder, Ladyzhenskaya and Poincaré inequalities and (2.13). Hence, there exists $\Lambda=\Lambda(h)>0$ (uniformly upper-bounded with respect to h) such that

$$\lambda\in[0,\Lambda(h))\iff C_h(1+\tfrac{1}{\mu})(\lambda+\lambda^2)<\mu \tag{2.20}$$

and this condition implies $\|\nabla w\|_{L^2(\Omega_h)}=0$ and, in turn, $w=0$ since $w_{|_{\partial\Omega_h}}=0$.

Refined bounds. For $\lambda\in[0,\Lambda(h))$, in all the above bounds we can drop the largest power of λ and they all become linear upper bounds. We treat separately the cases $U=1$ and $U=0$ and we make explicit the dependence of the constant C_h in (2.13) on h.

When $U=1$, we claim that the unique strong solution u to (2.4) satisfies

$$\|u\|_{H^1(\Omega_h)}\le C\big(1+(\varepsilon_t(h))^{-3/2}\big)\lambda \tag{2.21}$$

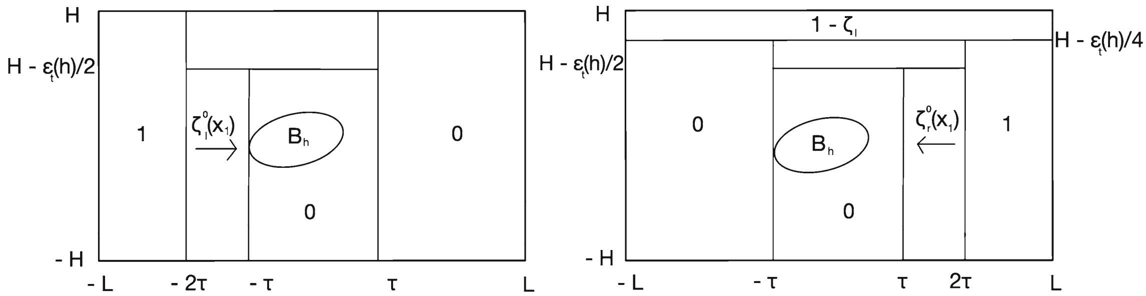

FIG. 4. Cutoff functions ζ_l (left) and ζ_r (right) on $\overline{R}$ when $U = 1$

with $C > 0$ independent of h. To this end, we introduce a different (and explicit) solenoidal extension. Consider the cutoff functions $\zeta_l, \zeta_r \in C^\infty(\mathbb{R}^2)$, with $0 \le \zeta_l, \zeta_r \le 1$, defined piece-wise in the rectangles of Fig. 4 by

$$\zeta_l(x_1, x_2) = \begin{cases} 0 & \text{in} \quad [-\tau, \tau] \times \left[-H, H - \frac{\varepsilon_t(h)}{2}\right], \\ 0 & \text{in} \quad [\tau, L] \times [-H, H], \\ 1 & \text{in} \quad [-L, -2\tau] \times [-H, H], \\ \zeta_l^0(x_1) & \text{in} \quad [-2\tau, -\tau] \times \left[-H, H - \frac{\varepsilon_t(h)}{2}\right], \\ C^\infty\text{-completion} & \text{in} \quad [-2\tau, -\tau] \times \left[H - \frac{\varepsilon_t(h)}{2}, H\right], \end{cases} \tag{2.22}$$

where ζ_l^0 is a function only of x_1, and

$$\zeta_r(x_1, x_2) = \begin{cases} 0 & \text{in} \quad [-\tau, \tau] \times \left[-H, H - \frac{\varepsilon_t(h)}{2}\right], \\ 0 & \text{in} \quad [-L, -\tau] \times \left[-H, H - \frac{\varepsilon_t(h)}{4}\right], \\ 1 & \text{in} \quad [2\tau, L] \times \left[-H, H - \frac{\varepsilon_t(h)}{4}\right], \\ \zeta_r^0(x_1) & \text{in} \quad [\tau, 2\tau] \times \left[-H, H - \frac{\varepsilon_t(h)}{2}\right], \\ 1 - \zeta_l(x_1, x_2) & \text{in} \quad [-L, L] \times \left[H - \frac{\varepsilon_t(h)}{4}, H\right], \\ C^\infty\text{-completion} & \text{in} \quad [-\tau, 2\tau] \times \left[H - \frac{\varepsilon_t(h)}{2}, H - \frac{\varepsilon_t(h)}{4}\right], \end{cases} \tag{2.23}$$

where ζ_r^0 is a function only of x_1.

Then, letting $\nabla^\perp = (-\partial_2, \partial_1)$, consider the vector field $s : R \to \mathbb{R}^2$ defined by

$$s(x_1, x_2) := -\lambda \nabla^\perp \left(\zeta_l(x_1, x_2) \int_{-H}^{x_2} V_{\text{in}}(z) \mathrm{d}z + \zeta_r(x_1, x_2) \int_{-H}^{x_2} V_{\text{out}}(z) \mathrm{d}z \right), \tag{2.24}$$

which is solenoidal and satisfies the boundary conditions in (2.4). Rewriting s as

$$s(x_1, x_2) = \lambda \left(-\nabla^\perp \zeta_l \int_{-H}^{x_2} V_{\text{in}} - \nabla^\perp \zeta_r \int_{-H}^{x_2} V_{\text{out}} + (\zeta_l V_{\text{in}} + \zeta_r V_{\text{out}}) e_1 \right),$$

its partial derivatives read

$$\partial_1 s = \lambda \left(-\nabla^\perp \partial_1 \zeta_l \int_{-H}^{x_2} V_{\rm in} - \nabla^\perp \partial_1 \zeta_r \int_{-H}^{x_2} V_{\rm out} + (\partial_1 \zeta_l V_{\rm in} + \partial_1 \zeta_r V_{\rm out}) e_1 \right),$$

$$\partial_2 s = \lambda \Big(-\nabla^\perp \partial_2 \zeta_l \int_{-H}^{x_2} V_{\rm in} - \nabla^\perp \partial_2 \zeta_r \int_{-H}^{x_2} V_{\rm out} - \nabla^\perp \zeta_l V_{\rm in} - \nabla^\perp \zeta_r V_{\rm out}$$

$$+ (\partial_2 \zeta_l V_{\rm in} + \partial_2 \zeta_r V_{\rm out} + \zeta_l \tfrac{d}{dx_2} V_{\rm in} + \zeta_r \tfrac{d}{dx_2} V_{\rm out}) e_1 \Big).$$

Using that $V_{\rm in}, V_{\rm out} \in W^{2,\infty}(-H,H)$ and that ζ_l, ζ_r are smooth, it follows that

$$\begin{gathered} \|s\|_{L^\infty(\Omega_h)}, \|s\|_{L^2(\Omega_h)}, \|s\|_{L^4(\Omega_h)}, \|\nabla s\|_{L^2(\Omega_h)}, \|\Delta s\|_{L^2(\Omega_h)} \le C_h \lambda, \\ \|s \cdot \nabla s\|_{L^2(\Omega_h)} \le C_h \lambda^2 \le C_h \lambda. \end{gathered} \tag{2.25}$$

We need to quantify the dependence of $C_h > 0$ on $\varepsilon_b(h)$ and $\varepsilon_t(h)$. On the one hand, we notice that, by construction, both ζ_l and ζ_r depend on x_2 only in

$$\Omega_{\varepsilon_t(h)} := [-2\tau, 2\tau] \times \left[H - \tfrac{\varepsilon_t(h)}{2}, H \right]. \tag{2.26}$$

In this domain, the x_1-derivatives of ζ_l and ζ_r are uniformly bounded with respect to h while the x_2-derivatives blow-up as $\varepsilon_t(h)$ goes to zero, for instance, we have

$$|\partial_2 \zeta_l|, |\partial_2 \zeta_r| \le C(\varepsilon_t(h))^{-1}, \qquad |\partial_2^2 \zeta_r|, |\partial_2^2 \zeta_l| \le C(\varepsilon_t(h))^{-2}.$$

Therefore, in $\Omega_{\varepsilon_t(h)}$

$$\begin{gathered} |s| \le C(1 + (\varepsilon_t(h))^{-1})\lambda, \quad |\partial_1 s| \le C(1 + (\varepsilon_t(h))^{-1})\lambda, \\ |\partial_2 s| \le C((\varepsilon_t(h))^{-1} + (\varepsilon_t(h))^{-2})\lambda. \end{gathered}$$

On the other hand, the cutoff functions depend only on x_1 in $\Omega_h \setminus \Omega_{\varepsilon_t(h)}$ and their x_1 and x_2-derivatives are uniformly bounded with respect to h. Therefore, in $\Omega_h \setminus \Omega_{\varepsilon_t(h)}$

$$|s|, |\partial_1 s|, |\partial_2 s| \le C\lambda.$$

Gathering all together, we refine the bounds in (2.25) as

$$\begin{aligned} &\|s\|_{L^\infty(\Omega_h)} \le C(1 + (\varepsilon_t(h))^{-1})\lambda, \\ &\|s\|_{L^2(\Omega_h)} \le C\lambda + C \left(\int_{\Omega_{\varepsilon_t(h)}} (\varepsilon_t(h))^{-2} \right)^{1/2} \lambda \le C(1 + (\varepsilon_t(h))^{-1/2})\lambda, \\ &\|s\|_{L^4(\Omega_h)} \le C(1 + (\varepsilon_t(h))^{-3/4})\lambda, \qquad \|\nabla s\|_{L^2(\Omega_h)} \le C(1 + (\varepsilon_t(h))^{-3/2})\lambda, \\ &\|\Delta s\|_{L^2(\Omega_h)}, \ \|s \cdot \nabla s\|_{L^2(\Omega_h)} \le C(1 + (\varepsilon_t(h))^{-5/2})\lambda, \end{aligned} \tag{2.27}$$

with all the constants $C > 0$ independent of h. Then, testing (2.12) with $v = u - s$ we obtain

$$\mu \|\nabla v\|^2_{L^2(\Omega_h)} = -\int_{\Omega_h} v \cdot \nabla s \cdot v - \int_{\Omega_h} s \cdot \nabla s \cdot v - \mu \int_{\Omega_h} \nabla s : \nabla v \tag{2.28}$$

We want to estimate, when possible, only s and not ∇s since the bounds for s are less singular in terms of $\varepsilon_t(h)$. Hence, since $\nabla \cdot v = \nabla \cdot s = 0$ and using integration by parts, we rewrite (2.28) as

$$\mu \|\nabla v\|^2_{L^2(\Omega_h)} = \int_{\Omega_h} v \cdot \nabla v \cdot s + \int_{\Omega_h} s \cdot \nabla v \cdot s - \mu \int_{\Omega_h} \nabla s : \nabla v. \tag{2.29}$$

We split the first integral in the right-hand side over $\Omega_{\varepsilon_t(h)}$ and $\Omega_h \setminus \Omega_{\varepsilon_t(h)}$. On the one hand, since $v_{|\Gamma_t} = 0$, Poincaré inequality

$$\|v\|_{L^2(\Omega_{\varepsilon_t(h)})} \le \frac{\varepsilon_t(h)}{2}\|\nabla v\|_{L^2(\Omega_{\varepsilon_t(h)})},$$

and Hölder inequality yield

$$\begin{aligned}\int\limits_{\Omega_{\varepsilon_t(h)}} (v\cdot\nabla v)\cdot s &\le \|v\|_{L^2(\Omega_{\varepsilon_t(h)})}\|\nabla v\|_{L^2(\Omega_{\varepsilon_t(h)})}\|s\|_{L^\infty(\Omega_{\varepsilon_t(h)})}\\ &\le C\varepsilon_t(h)\|\nabla v\|^2_{L^2(\Omega_{\varepsilon_t(h)})}(1+(\varepsilon_t(h))^{-1})\lambda \le C\lambda\|\nabla v\|^2_{L^2(\Omega_{\varepsilon_t(h)})},\end{aligned}$$

where we used that $\|s\|_{L^\infty(\Omega_{\varepsilon_t(h)})} \le C(1+(\varepsilon_t(h))^{-1})\lambda$ and $\varepsilon_t(h) \le 2H - \delta_b - \delta_t$. On the other hand, since $v_{|\Gamma_l,\Gamma_r} = 0$, Poincaré and Hölder inequalities yield

$$\begin{aligned}\int\limits_{\Omega_h\setminus\Omega_{\varepsilon_t(h)}} (v\cdot\nabla v)\cdot s &\le \|v\|_{L^2(\Omega_h\setminus\Omega_{\varepsilon_t(h)})}\|\nabla v\|_{L^2(\Omega_h\setminus\Omega_{\varepsilon_t(h)})}\|s\|_{L^\infty(\Omega_h\setminus\Omega_{\varepsilon_t(h)})}\\ &\le C\lambda\|\nabla v\|^2_{L^2(\Omega_h\setminus\Omega_{\varepsilon_t(h)})},\end{aligned}$$

where we used that $\|s\|_{L^\infty(\Omega_h\setminus\Omega_{\varepsilon_t(h)})} \le C\lambda$. Therefore, from (2.27) and (2.29) we infer

$$\begin{aligned}\mu\|\nabla v\|^2_{L^2(\Omega_h)} &\le C\lambda\|\nabla v\|^2_{L^2(\Omega_h)} + \|s\|^2_{L^4(\Omega_h)}\|\nabla v\|_{L^2(\Omega_h)} + \mu\|\nabla s\|_{L^2(\Omega_h)}\|\nabla v\|_{L^2(\Omega_h)}\\ &\le C\lambda\|\nabla v\|^2_{L^2(\Omega_h)} + C(1+(\varepsilon_t(h))^{-3/2})(\lambda+\lambda^2)\|\nabla v\|_{L^2(\Omega_h)}.\end{aligned}$$

Then, for $\lambda \in [0,\Lambda(h))$ with $\Lambda(h)$ as in (2.20) we have

$$\|\nabla v\|_{L^2(\Omega_h)} \le C(1+(\varepsilon_t(h))^{-3/2})\lambda \tag{2.30}$$

and

$$\|u\|_{H^1(\Omega_h)} \le \|\nabla v\|_{L^2(\Omega_h)} + \|s\|_{H^1(\Omega_h)} \le C(1+(\varepsilon_t(h))^{-3/2})\lambda,$$

which proves (2.21).

When $U = 0$, we claim that the unique strong solution u to (2.4) satisfies

$$\|u\|_{H^1(\Omega_h)} \le C\lambda \tag{2.31}$$

with $C > 0$ independent of h, which will imply that $\Lambda(h) \equiv \Lambda$ can be also taken independent of h. In this case, we shall define the cut-off functions and the solenoidal extension differently depending if $h \le 0$ or $h > 0$. If $h \le 0$, we define ζ_l, ζ_r as in (2.22)-(2.23) (see Fig. 5) replacing $\varepsilon_t(h)$ with the distance of B_0 to Γ_t, namely $\varepsilon_t(0) = H - \delta_t$. The solenoidal extension s is then defined as in (2.24). By construction both ζ_l and ζ_r depend on x_2 only in $\Omega_{\varepsilon_t(0)}$, defined as in (2.26) with $\varepsilon_t(h)$ replaced by $\varepsilon_t(0)$. In this domain, both x_1 and x_2-derivatives of ζ_l and ζ_r are uniformly bounded with respect to h, for instance, we have

$$|\partial_2\zeta_l|, |\partial_2\zeta_r| \le C(\varepsilon_t(0))^{-1} \le C, \qquad |\partial_2^2\zeta_r|, |\partial_2^2\zeta_l| \le C(\varepsilon_t(0))^{-2} \le C.$$

Since in $\Omega_h \setminus \Omega_{\varepsilon_t(0)}$ the cutoff functions depend only on x_1, we infer that s, $\partial_1 s$ and $\partial_2 s$ are uniformly bounded with respect to h in all Ω_h and

$$\|s\|_{L^\infty(\Omega_h)}, \|s\|_{L^2(\Omega_h)}, \|s\|_{L^4(\Omega_h)}, \|\nabla s\|_{L^2(\Omega_h)} \le C\lambda. \tag{2.32}$$

Repeating the same computations as in the case $U = 1$ and using (2.32), we obtain (2.31) for $h \le 0$.

If $h > 0$, we make a vertical reflection $x_2 \mapsto -x_2$ and we consider the new cutoff functions defined piece-wise in the rectangles of Fig. 5, where $\varepsilon_b(0) = H - \delta_b$.

Then, we consider the vector field $s : R \to \mathbb{R}^2$ defined by

$$s(x_1,x_2) := \lambda\nabla^\perp\left(\zeta_l(x_1,x_2)\int\limits_{x_2}^{H} V_{\text{in}}(z)\mathrm{d}z + \zeta_r(x_1,x_2)\int\limits_{x_2}^{H} V_{\text{out}}(z)\mathrm{d}z\right),$$

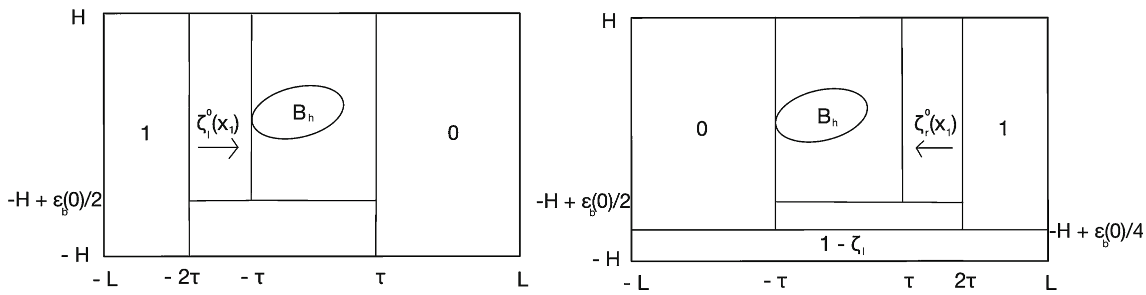

FIG. 5. Cutoff functions ζ_l (left) and ζ_r (right) on $\overline{R}$ when $U = 0$ for $h > 0$

which is solenoidal and satisfies the boundary conditions in (2.4). By the same argument used when $h \leq 0$, s, $\partial_1 s$ and $\partial_2 s$ are uniformly bounded with respect to h in Ω_h. Therefore, using again (2.32), we obtain (2.31) for $h < 0$. □

Remark 2.3. We stated (2.7) and (2.8) only in case of uniqueness because, in what follows, λ will be taken small and higher powers of λ can be upper estimated with the first power.

The reflection method used to obtain the regularity result has its own interest. The rectangular shape of the domain is crucial and the technique fails for other polygons. However, in the case of convex polygons, in particular also for a rectangle, one can obtain the more C^∞-regularity result by using Theorem 2 in [13], see also [11, Section 7.3.3] and [4].

3. Equilibrium configurations of a FSI problem

By Theorem 2.2, for any $(\lambda, h) \in [0, +\infty) \times (-H + \delta_b, H - \delta_t)$ there exists at least a strong solution $(u, p) = (u(\lambda, h), p(\lambda, h))$ to (2.4). The fluid described by (u, p) in Ω_h exerts on B_h a force perpendicular to the direction of the inflow, called *lift* (see [16]). Since the inflow in (2.4) is horizontal, the lift is vertical and given by

$$\mathcal{L}(\lambda, h) = -e_2 \cdot \int_{\partial B_h} \mathbb{T}(u, p) n, \tag{3.1}$$

where $\mathbb{T}$ is the fluid stress tensor, namely

$$\mathbb{T}(u, p) := \mu(\nabla u + \nabla u^T) - p\mathbb{I}\,,$$

and n is the unit outward normal vector to $\partial\Omega_h$, which, on ∂B_h, points toward the interior of B_h. In fact, $\mathcal{L}(\lambda, h)$ is a multi-valued function when uniqueness for (2.4) fails. However, we keep this simple notation instead of writing $\mathcal{L}(\lambda, h, u(\lambda, h), p(\lambda, h))$, in which also the dependence on the particular solution (u, p) is emphasized. The regularity of the solution (see Theorem 2.2) and the smoothness of ∂B_h yield $\mathbb{T}(u, p)_{|\partial B_h} \in H^{1/2}(\partial B_h) \subset L^1(\partial B_h)$; hence, the integral in (3.1) is finite. In fact, the lift can also be defined for merely weak solutions, see (7.5) in Sect. 7. Note that (3.1) holds for any $\lambda \geq 0$ and any solution to (2.4), but our main result on the FSI problem focuses on small inflows, see Theorem 3.1.

Aiming to model, in particular, a wind flow hitting a suspension bridge, the body B may also be subject to a (possibly nonsmooth) vertical restoring force f tending to maintain B in the equilibrium position B_0 (for $h = 0$); see Sect. 7. We assume that f depends only on the position h, that $f \in C^0(-H + \delta_b, H - \delta_t)$ with $f(0) = 0$ and

$$\exists \gamma > 0 \quad \text{s.t.} \quad \frac{f(h_1) - f(h_2)}{h_1 - h_2} \geq \gamma \quad \forall h_1, h_2 \in (-H + \delta_b, H - \delta_t),\ h_1 \neq h_2. \tag{3.2}$$

Moreover, we assume that there exists $K > 0$ such that

$$\limsup_{h \to -H+\delta_b} f(h)(H - \delta_b + h)^{3/2} \le -K,$$
$$\liminf_{h \to H-\delta_t} \frac{f(h)}{\max\{(H - \delta_t - h)^{-3/2}, U(H - \delta_t - h)^{-3}\}} \ge K. \tag{3.3}$$

The assumption (3.3) is somehow technical and prevents collisions of B with the horizontal boundary $\Gamma_b \cup \Gamma_t$, at least for small inflow/outflow. It can probably be relaxed but, so far, only few (numerical) investigations on the effect of proximity to collisions of hydrodynamic forces (such as the lift), acting on non-spherical bodies, have been tackled, see [20] and references therein. The presence of U in (3.3) highlights the different behavior of f when B is close to Γ_t for $U = 0$ or $U = 1$. In the first case, f has the same strength close to Γ_b and Γ_t. Conversely, for $U = 1$, the asymmetry of the boundary conditions requires a different strength of f, which is stronger when B is close to Γ_t than when B is close to Γ_b. Overall, (3.2)-(3.3) model the fact that B is not allowed to go too far away from the equilibrium position B_0.

Since we are interested in the equilibrium configurations of the FSI problem, we consider the boundary-value problem (2.4) coupled with a compatibility condition stating that the restoring force balances the lift force, namely

$$-\mu\Delta u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_h$$
$$u_{|\partial B_h} = u_{|\Gamma_b} = 0, \quad u_{|\Gamma_t} = \lambda U e_1, \quad u_{|\Gamma_l} = \lambda V_{\text{in}}(x_2) e_1, \quad u_{|\Gamma_r} = \lambda V_{\text{out}}(x_2) e_1,$$
$$f(h) = -e_2 \cdot \int_{\partial B_h} \mathbb{T}(u, p) n. \tag{3.4}$$

Our main result concerns the existence and uniqueness of the solution to (3.4) for small values of λ that we expect to be stable.

Theorem 3.1. *Let $f \in C^0(-H+\delta_b, H-\delta_t)$ satisfy (3.2)-(3.3) with $f(0) = 0$ and V_{in}, $V_{\text{out}} \in W^{2,\infty}(-H, H)$ satisfy (2.3) with (2.5). There exist $\Lambda_1 > 0$ and a unique $\mathfrak{h} \in C^0[0, \Lambda_1)$ such that for $\lambda \in [0, \Lambda_1)$ the FSI problem (3.4) admits a unique solution $(u(\lambda, h), p(\lambda, h), h) \in H^2(\Omega_h) \times H^1(\Omega_h) \times (-H+\delta_b, H-\delta_t)$ given by*

$$(u(\lambda, \mathfrak{h}(\lambda)), p(\lambda, \mathfrak{h}(\lambda)), \mathfrak{h}(\lambda)).$$

We emphasize that Theorem 3.1 ensures uniqueness of the equilibrium configuration for the FSI problem (3.4) in the *uniform* interval $[0, \Lambda_1)$ even in absence of uniqueness for (2.4) that, instead, is only ensured in the possibly *non-uniform* interval $[0, \Lambda(h))$. The proof of Theorem 3.1 is given in Sect. 5. It is fairly delicate because if $U = 0$ (as for symmetric inflow/outflow), then from (2.21) we infer that the H^1-norm is uniformly bounded with respect to h. However, if $U = 1$, the same norm obviously blows up when B_h approaches Γ_t, which affects the bounds for the lift in (3.1). As already mentioned, very little is known when a body approaches a collision, see again [20] and references therein. Therefore, the next statement has its own independent interest; it provides some upper bounds and shows that, probably, the lift behaves differently for homogeneous and inhomogeneous boundary data.

Theorem 3.2. *Assume (2.5) and let $\lambda \in [0, \Lambda_0]$ for some $\Lambda_0 > 0$. Let (u, p) be a strong solution to (2.4) (see Theorem 2.2) and let $\mathcal{L}(\lambda, h)$ be as in (3.1). There exists $C > 0$ (independent of λ, h, u, p) such that, for any $(\lambda, h) \in [0, \Lambda_0] \times (-H + \delta_b, H - \delta_t)$,*

$$|\mathcal{L}(\lambda, h)| \le C\Big((\varepsilon_b(h))^{-3/2} + \max\{(\varepsilon_t(h))^{-3/2}, U(\varepsilon_t(h))^{-3}\}\Big)\lambda \tag{3.5}$$

with $\varepsilon_b(h)$ and $\varepsilon_t(h)$ defined in (2.6). In fact, $\mathcal{L}(\lambda, h)$ is defined in all $[0, +\infty) \times (-H+\delta_b, H-\delta_t)$, possibly as a multi-valued function, but (3.5) would hold with different powers of λ.

The proof of Theorem 3.2 is given in the next section.

4. Proof of theorem 3.2

We rewrite the lift (3.1), which is a boundary integral, as a volume integral. This can be done by considering $w \in H^1(\Omega_h)$ that satisfies

$$\nabla \cdot w = 0 \quad \text{in} \quad \Omega_h, \qquad w_{|\partial B_h} = e_2, \qquad w_{|\partial R} = 0. \tag{4.1}$$

The divergence theorem ensures that (4.1) admits infinitely many solutions. Testing (2.4) with one such solution w (recall that $\nabla \cdot \mathbb{T} = \mu \Delta u - \nabla p$) yields

$$\int_{\Omega_h} u \cdot \nabla u \cdot w = \int_{\Omega_h} \nabla \cdot \mathbb{T}(u,p) \cdot w = -\mu \int_{\Omega_h} \nabla u : \nabla w + \int_{\partial \Omega_h} \mathbb{T}(u,p) n \cdot w$$

and, using the boundary conditions on w,

$$-e_2 \cdot \int_{\partial B_h} \mathbb{T}(u,p) n = -\int_{\Omega_h} u \cdot \nabla u \cdot w - \mu \int_{\Omega_h} \nabla u : \nabla w. \tag{4.2}$$

Among the infinitely many solutions of (4.1), we select one obtained by using a solenoidal extension similar to the ones introduced in Sect. 2. We consider a cutoff function $\chi \in C^\infty(\overline{R})$ with $0 \le \chi \le 1$ such that

$$\chi(x_1,x_2) = \begin{cases} 1 & \text{in} \quad [-\tau,\tau] \times [h-\delta_b, h+\delta_t], \\ 0 & \text{in} \quad \Omega_h \setminus \left([-2\tau, 2\tau] \times \left[h - \delta_b - \frac{\varepsilon_b(h)}{2}, h + \delta_t + \frac{\varepsilon_t(h)}{2} \right] \right), \\ \chi^0(x_1) & \text{in} \quad ([-2\tau,-\tau] \cup [\tau, 2\tau]) \times [h-\delta_b, h+\delta_t], \\ C^\infty\text{-completion} & \text{elsewhere}. \end{cases}$$

We put $w = \nabla^\perp(x_1 \chi)$. Clearly $w \in H^1(\Omega_h)$ satisfies (4.1) and supp $w \subseteq \Omega_w = \Omega_{w,b} \cup \Omega_{w,c} \cup \Omega_{w,t}$ with

$$\Omega_{w,b} := [-2\tau, 2\tau] \times \left[h - \delta_b - \frac{\varepsilon_b(h)}{2}, h - \delta_b \right], \qquad \Omega_{w,c} := [-2\tau, 2\tau] \times [h - \delta_b, h + \delta_t],$$
$$\Omega_{w,t} := [-2\tau, 2\tau] \times \left[h + \delta_t, h + \delta_t + \frac{\varepsilon_t(h)}{2} \right].$$

Moreover, from the definition of χ it follows that w and its x_1 and x_2-derivatives are uniformly bounded with respect to h in $\Omega_{w,c}$, while in $\Omega_{w,b}$

$$\begin{gathered} |w| \le C(1 + (\varepsilon_b(h))^{-1}), \quad |\partial_1 w| \le (1 + (\varepsilon_b(h))^{-1}), \\ |\partial_2 w| \le ((\varepsilon_b(h))^{-1} + (\varepsilon_b(h))^{-2}) \end{gathered} \tag{4.3}$$

and in $\Omega_{w,t}$

$$\begin{gathered} |w| \le C(1 + (\varepsilon_t(h))^{-1}), \quad |\partial_1 w| \le (1 + (\varepsilon_t(h))^{-1}), \\ |\partial_2 w| \le ((\varepsilon_t(h))^{-1} + (\varepsilon_t(h))^{-2}). \end{gathered} \tag{4.4}$$

B_h close to Γ_b. We consider the case when h is close to $-H + \delta_b$; hence, $\varepsilon_b(h)$ is close to zero. This implies that $\varepsilon_t(h) \ge 1$ and the bounds in (4.4) become uniform. Choosing in (4.2) the previously constructed w, we observe that the integrals in the right-hand side are defined only on Ω_w. Let us split these integrals over the regions $\Omega_{w,b}$, which is shrinking as $\varepsilon_b(h)$ goes to zero, and $\Omega_w \setminus \Omega_{w,b}$. On the one hand, Hölder inequality and (2.7) yield

$$\begin{aligned} &\left| \int_{\Omega_w \setminus \Omega_{w,b}} u \cdot \nabla u \cdot w + \mu \int_{\Omega_w \setminus \Omega_{w,b}} \nabla u : \nabla w \right| \\ &\le C \|u\|^2_{H^1(\Omega_h} \|w\|_{L^\infty(\Omega_w \setminus \Omega_{w,b})} + \mu \|\nabla u\|_{L^2(\Omega_h)} \|\nabla w\|_{L^2(\Omega_w \setminus \Omega_{w,b})} \\ &\le C(\|u\|^2_{H^1(\Omega_h)} + \|u\|_{H^1(\Omega_h)}) \le C\lambda \end{aligned} \tag{4.5}$$

for $\lambda \in [0, \Lambda_0]$, using that w and its derivatives are uniformly bounded with respect to h in $\Omega_w \backslash \Omega_{w,b}$. On the other hand, since $w \equiv 0$ in $\Omega^0_{w,b} := [-2\tau, 2\tau] \times [-H, h - \delta_b - \frac{\varepsilon_b(h)}{2}]$ and $u_{|\Gamma_b} = 0$, Poincaré inequality for u in $\Omega_{w,b} \cup \Omega^0_{w,b}$, the Hölder inequality and (2.7) yield

$$
\begin{aligned}
&\left| \int_{\Omega_{w,b}} u \cdot \nabla u \cdot w \right| = \left| \int_{\Omega_{w,b} \cup \Omega^0_{w,b}} u \cdot \nabla u \cdot w \right| \\
&\leq \varepsilon_b(h) \|\nabla u\|^2_{L^2(\Omega_{w,b} \cup \Omega^0_{w,b})} \|w\|_{L^\infty(\Omega_{w,b})} \leq C\|u\|^2_{H^1(\Omega_h)} \leq C\lambda
\end{aligned}
\tag{4.6}
$$

and

$$
\left| \int_{\Omega_{w,b}} \nabla u : \nabla w \right| \leq \|u\|_{H^1(\Omega_h)} \|\nabla w\|_{L^2(\Omega_{w,b})} \leq C(\varepsilon_b(h))^{-3/2} \lambda, \tag{4.7}
$$

for $\lambda \in [0, \Lambda_0]$, using that $\|w\|_{L^\infty(\Omega_{w,b})} \leq C(\varepsilon_b(h))^{-1}$ and $\|\nabla w\|_{L^2(\Omega_{w,b})} \leq C(\varepsilon_b(h))^{-3/2}$ for $\varepsilon_b(h)$ close to zero, due to (4.3).

Putting together (4.5)-(4.7), then there exists $\eta_b > 0$ sufficiently small such that, for any $(\lambda, h) \in [0, \Lambda_0] \times (-H + \delta_b, -H + \delta_b + \eta_b)$,

$$
|\mathcal{L}(\lambda, h)| \leq C(\varepsilon_b(h))^{-3/2} \lambda. \tag{4.8}
$$

We remark that the same blow-up rate in (4.8) could be obtained without taking advantage of Poincaré inequality in (4.6) but using directly $u \in H^1 \subset L^4$. This idea, however, will be crucial to obtain a better blow-up rate for the lift in the case when the body is close to Γ_t, that we now analyze.

B_h *close to* Γ_t. We consider the case when h is close to $H - \delta_t$; hence, $\varepsilon_t(h)$ is close to zero. Analogously to what done in the previous case, we split the integrals over the regions $\Omega_{w,t}$, which is shrinking as $\varepsilon_t(h)$ goes to zero, and $\Omega_w \setminus \Omega_{w,t}$. On the one hand, Hölder inequality yields

$$
\begin{aligned}
&\left| \int_{\Omega_w \backslash \Omega_{w,t}} u \cdot \nabla u \cdot w + \mu \int_{\Omega_w \backslash \Omega_{w,t}} \nabla u : \nabla w \right| \\
&\leq C\|u\|^2_{H^1(\Omega_h)} \|w\|_{L^\infty(\Omega_w \backslash \Omega_{w,t})} + \mu \|\nabla u\|_{L^2(\Omega_h)} \|\nabla w\|_{L^2(\Omega_w \backslash \Omega_{w,t})} \\
&\leq C(\|u\|^2_{H^1(\Omega_h)} + \|u\|_{H^1(\Omega_h)})
\end{aligned}
$$

using that w and its derivatives are uniformly bounded with respect to h in $\Omega_w \backslash \Omega_{w,t}$. On the other hand, since $w \equiv 0$ in $\Omega^0_{w,t} := [-2\tau, 2\tau] \times [h + \delta_t + \frac{\varepsilon_t(h)}{2}, H]$ and $u = v + s$ with $v_{|\Gamma_t} = 0$, Poincaré inequality for v in $\Omega_{w,t} \cup \Omega^0_{w,t}$ and Hölder inequality yield

$$
\begin{aligned}
&\left| \int_{\Omega_{w,t}} u \cdot \nabla u \cdot w \right| = \left| \int_{\Omega_{w,t} \cup \Omega^0_{w,t}} v \cdot \nabla u \cdot w + \int_{\Omega_{w,t} \cup \Omega^0_{w,t}} s \cdot \nabla u \cdot w \right| \\
&\leq \varepsilon_t(h) \|\nabla v\|_{L^2(\Omega_h)} \|\nabla u\|_{L^2(\Omega_h)} \|w\|_{L^\infty(\Omega_{w,t})} + \|s\|_{L^2(\Omega_h)} \|\nabla u\|_{L^2(\Omega_h)} \|w\|_{L^\infty(\Omega_{w,t})} \\
&\leq C\|\nabla v\|_{L^2(\Omega_h)} \|u\|_{H^1(\Omega_h)} + C\|s\|_{L^2(\Omega_h)} \|u\|_{H^1(\Omega_h)} (\varepsilon_t(h))^{-1}
\end{aligned}
$$

and

$$
\left| \int_{\Omega_{w,t}} \nabla u : \nabla w \right| \leq \|u\|_{H^1(\Omega_h)} \|\nabla w\|_{L^2(\Omega_{w,t})} \leq \|u\|_{H^1(\Omega_h)} (\varepsilon_t(h))^{-3/2},
$$

using that $\|w\|_{L^\infty(\Omega_{w,t})} \le C(\varepsilon_t(h))^{-1}$ and $\|\nabla w\|_{L^2(\Omega_{w,t})} \le C(\varepsilon_t(h))^{-3/2}$ for $\varepsilon_t(h)$ close to zero, due to (4.4). Now we shall distinguish the cases $U = 1$ and $U = 0$. When $U = 1$, using (2.7), (2.27) and (2.30) we obtain, for $\lambda \in [0, \Lambda_0]$,

$$\left| \int\limits_{\Omega_w \setminus \Omega_{w,t}} u \cdot \nabla u \cdot w + \mu \int\limits_{\Omega_w \setminus \Omega_{w,t}} \nabla u : \nabla w \right| \le C(\varepsilon_t(h))^{-3}\lambda \tag{4.9}$$

and

$$\left| \int\limits_{\Omega_{w,t}} u \cdot \nabla u \cdot w \right| \le C(\varepsilon_t(h))^{-3}\lambda, \quad \left| \int\limits_{\Omega_{w,t}} \nabla u : \nabla w \right| \le C(\varepsilon_t(h))^{-3}\lambda. \tag{4.10}$$

When $U = 0$, using (2.7) and (2.32), we obtain, for $\lambda \in [0, \Lambda_0]$,

$$\left| \int\limits_{\Omega_w \setminus \Omega_{w,t}} u \cdot \nabla u \cdot w + \mu \int\limits_{\Omega_w \setminus \Omega_{w,t}} \nabla u : \nabla w \right| \le C\lambda \tag{4.11}$$

and

$$\left| \int\limits_{\Omega_{w,t}} u \cdot \nabla u \cdot w \right| \le C(\varepsilon_t(h))^{-1}\lambda, \quad \left| \int\limits_{\Omega_{w,t}} \nabla u : \nabla w \right| \le C(\varepsilon_t(h))^{-3/2}\lambda. \tag{4.12}$$

Putting together (4.9)-(4.12), then there exists $\eta_t > 0$ sufficiently small such that, for $(\lambda, h) \in [0, \Lambda_0] \times (H - \delta_t - \eta_t, H - \delta_t)$,

$$|\mathcal{L}(\lambda, h)| \le C \max\{(\varepsilon_t(h))^{-3/2}, U(\varepsilon_t(h))^{-3}\}\lambda. \tag{4.13}$$

For $h \in [-H + \delta_b + \eta_b, H - \delta_t - \eta_t]$, $\varepsilon_b(h)$ and $\varepsilon_t(h)$ are uniformly bounded from below with respect to h. Therefore, by combining (4.8) and (4.13), there exists $C > 0$ independent of h such that, for any $(\lambda, h) \in [0, \Lambda_0] \times (-H + \delta_b, H - \delta_t)$,

$$|\mathcal{L}(\lambda, h)| \le C((\varepsilon_b(h))^{-3/2} + \max\{(\varepsilon_t(h))^{-3/2}, U(\varepsilon_t(h))^{-3}\})\lambda.$$

5. Proof of theorem 3.1

5.1. Continuity and monotonicity of the global force

In Sect. 3, we have defined the lift $\mathcal{L}(\lambda, h)$ as a possibly multi-valued function of $(\lambda, h) \in [0, +\infty) \times (-H + \delta_b, H - \delta_t)$. Let f be the restoring force satisfying (3.2)-(3.3). Then, the global force acting on B_h is the function $\phi : [0, +\infty) \times (-H + \delta_b, H - \delta_t) \to \mathbb{R}$ defined by

$$\phi(\lambda, h) = f(h) - \mathcal{L}(\lambda, h). \tag{5.1}$$

We first focus on the λ-dependence by maintaining h fixed, and we prove the Lipschitz-continuity of the map $\lambda \mapsto \phi(\lambda, h)$.

Proposition 5.1. *Let $\overline{h} = H - \max\{\delta_b, \delta_t\}$. There exist $\overline{\lambda} > 0$ and $h^* \in (0, \overline{h})$ such that $\lambda \mapsto \phi(\lambda, h)$ is Lipschitz continuous in $[0, \overline{\lambda})$ for all $h \in [-h^*, h^*]$.*

Proof. To begin, let us take $\overline{\lambda}$ and h^* sufficiently small so that Theorem 2.2 guarantees the uniqueness for (2.4) whenever $\lambda < \overline{\lambda}$ and $|h| \le h^*$ (see Fig. 3). Hence, $\mathcal{L}(\lambda, h)$ is a one-valued function on $[0, \overline{\lambda}) \times [-h^*, h^*]$. Since f does not depend on λ, we only need to show that $\lambda \mapsto \mathcal{L}(\lambda, h)$ is Lipschitz continuous in a neighborhood of $\lambda = 0$, possibly smaller than $[0, \overline{\lambda})$.

For $\lambda_1,\lambda_2\in[0,\overline{\lambda})$ consider, respectively, the solutions $(u(\lambda_1),p(\lambda_1))$ and $(u(\lambda_2),p(\lambda_2))$ to (2.4). Let

$$v:=u(\lambda_1)-u(\lambda_2),\qquad q:=p(\lambda_1)-p(\lambda_2), \tag{5.2}$$

so that (v,q) satisfies

$$\begin{gathered}-\mu\Delta v+v\cdot\nabla v+\nabla q=-v\cdot\nabla u(\lambda_2)-u(\lambda_2)\cdot\nabla v,\qquad \nabla\cdot v=0\quad\text{in}\quad\Omega_h,\\ v_{|_{\Gamma_t}}=(\lambda_1-\lambda_2)Ue_1,\ v_{|_{\Gamma_l}}=(\lambda_1-\lambda_2)V_{\text{in}}(x_2)e_1,\ v_{|_{\Gamma_r}}=(\lambda_1-\lambda_2)V_{\text{out}}(x_2)e_1,\\ v_{|_{\partial B_h}}=v_{|_{\Gamma_b}}=0.\end{gathered} \tag{5.3}$$

Let $v_\lambda:=v-s_\lambda$, where $s_\lambda\in W^{1,\infty}(\Omega_h)\cap H^2(\Omega_h)$ is a solenoidal extension of v that can be constructed as s in (2.24) and, hence, it satisfies the estimates (2.25), namely

$$\begin{gathered}\|\nabla s_\lambda\|_{L^2(\Omega_h)}\le C_h|\lambda_1-\lambda_2|,\ \|\Delta s_\lambda\|_{L^2(\Omega_h)}\le C_h|\lambda_1-\lambda_2|,\\ \|s_\lambda\|_{L^\infty(\Omega_h)}\le C_h|\lambda_1-\lambda_2|,\ \|s_\lambda\cdot\nabla s_\lambda\|_{L^2(\Omega_h)}\le C_h|\lambda_1-\lambda_2|^2.\end{gathered} \tag{5.4}$$

We then rewrite (5.3) as

$$-\mu\Delta v_\lambda+v_\lambda\cdot\nabla v_\lambda+\nabla q=g,\quad \nabla\cdot v_\lambda=0\quad\text{in}\quad\Omega_h,\qquad v_{\lambda|_{\partial\Omega_h}}=0, \tag{5.5}$$

where

$$g:=\mu\Delta s_\lambda-v\cdot\nabla(u(\lambda_2)+s_\lambda)-u(\lambda_2)\cdot\nabla v+s_\lambda\cdot\nabla s_\lambda-s_\lambda\cdot\nabla v.$$

From Theorem 2.2, we know that $v,u(\lambda_2)\in H^2(\Omega_h)\hookrightarrow L^\infty(\Omega_h)$, so that $g\in L^2(\Omega_h)$. Moreover,

$$\begin{aligned}\|g\|_{L^2(\Omega_h)}&\le\mu\|\Delta s_\lambda\|_{L^2(\Omega_h)}+\big(\|\nabla u(\lambda_2)\|_{L^2(\Omega_h)}+\|\nabla s_\lambda\|_{L^2(\Omega_h)}\big)\|v\|_{L^\infty(\Omega_h)}\\ &\quad+\|u(\lambda_2)\|_{L^\infty(\Omega_h)}\|\nabla v\|_{L^2(\Omega_h)}+\|s_\lambda\cdot\nabla s_\lambda\|_{L^2(\Omega_h)}+\|s_\lambda\|_{L^\infty(\Omega_h)}\|\nabla v\|_{L^2(\Omega_h)}\\ &\le C_h|\lambda_1-\lambda_2|+C_h(\lambda_2+|\lambda_1-\lambda_2|)\|v\|_{H^2(\Omega_h)}\\ &\quad+C_h\lambda_2\|v\|_{H^2(\Omega_h)}+C_h|\lambda_1-\lambda_2|^2+C_h|\lambda_1-\lambda_2|\cdot\|v\|_{H^2(\Omega_h)},\end{aligned}$$

where we used Hölder inequality (first step), the estimates (2.7)-(2.8)-(5.4) and the embeddings $H^2\hookrightarrow H^1,L^\infty$ (second step). Thus, by extending the solution as in the proof of Theorem 2.2, recalling [14] and applying [5, Theorem IV.5.1] to (5.5), we obtain

$$\|v_\lambda\|_{H^2(\Omega_h)}+\|q\|_{H^1(\Omega_h)}\le C_h|\lambda_1-\lambda_2|+C_h\big(\lambda_2+|\lambda_1-\lambda_2|\big)\|v\|_{H^2(\Omega_h)}. \tag{5.6}$$

Hence, there exists a possibly smaller $\overline{\lambda}>0$ such that if $\lambda_1,\lambda_2\in[0,\overline{\lambda})$, the second term in the right-hand side of (5.6) can be absorbed in the left-hand side and

$$\|v_\lambda\|_{H^2(\Omega_h)}+\|q\|_{H^1(\Omega_h)}\le C_h|\lambda_1-\lambda_2|, \tag{5.7}$$

for some $C_h>0$ also depending on $\overline{\lambda}$. Since the lift (3.1) is linear with respect to u and p, we have

$$\mathcal{L}(\lambda_1,h)-\mathcal{L}(\lambda_2,h)=-e_2\cdot\int_{\partial B_h}\mathbb{T}(v,q)n$$

with v and q defined in (5.2). Therefore, using the Trace Theorem and (5.7), we infer that, for any $\lambda_1,\lambda_2\in[0,\overline{\lambda})$ and a fixed $h\in[-h^*,h^*]$, we have

$$\begin{aligned}|\mathcal{L}(\lambda_1,h)-\mathcal{L}(\lambda_2,h)|&\le C_h\left(\|\nabla v\|_{L^1(\partial B_h)}+\|q\|_{L^1(\partial B_h)}\right)\\ &\le C_h\left(\|v\|_{H^2(\Omega_h)}+\|q\|_{H^1(\Omega_h)}\right)\le C_h|\lambda_1-\lambda_2|.\end{aligned}$$

This shows that $\lambda\mapsto\mathcal{L}(\lambda,h)$ is Lipschitz continuous in $[0,\overline{\lambda})$ for all $h\in[-h^*,h^*]$. □

We now focus on the h-dependence of ϕ by maintaining λ fixed. Although we prove a slightly stronger result, we state:

Proposition 5.2. *Let $\overline{h}=H-\max\{\delta_b,\delta_t\}$. There exist $h_0\in(0,h^*]$ and $\lambda_0\in(0,\overline{\lambda}]$ (see Proposition 5.1) such that $h\mapsto\phi(\lambda,h)$ is continuous and strictly increasing in $[-h_0,h_0]$ for all $\lambda\in[0,\lambda_0)$.*

Proof. Recall that $R = (-L, L) \times (-H, H)$. Let $0 < r_1 < r_2$ and $D_{r_i}(0)$ be the open disk centered at $(0, 0)$ with radius r_i. Choose $h_0 \in (0, h^*)$ in such a way that $B_h \subset D_{r_1}(0) \subset D_{r_2}(0) \subset R$ whenever $|h| \leq h_0$; in later steps, we may need to choose a possibly smaller h_0 that, however, we continue calling h_0. Let $\sigma \in W^{2,\infty}(R, \mathbb{R}^2)$ be defined by

$$\sigma(x_1, x_2) = F(|x|)e_2, \tag{5.8}$$

with $F \equiv 1$ in $[0, r_1]$, $F \equiv 0$ in $[r_2, +\infty)$ and $F \in W^{2,\infty}(r_1, r_2)$ is the polynomial of third degree such that $F(r_1) = 1$ and $F(r_2) = F'(r_1) = F'(r_2) = 0$. For $h \in [-h_0, h_0]$, with h_0 small, we view the fluid domain Ω_h as a variation of Ω_0 via the diffeomorphism $\mathrm{Id} + h\sigma$, that is,

$$\Omega_h = (\mathrm{Id} + h\sigma)(\Omega_0).$$

In particular, $\partial B_h = \partial B_0 + he_2$ with unit outer normal vector $n(h) = n(0) \circ (\mathrm{Id} + he_2)$. Let $J(h)$ denote the Jacobian matrix of the diffeomorphism $\mathrm{Id} + h\sigma$, that is,

$$J(h) = I + h\frac{F'(|x|)}{|x|}\begin{pmatrix} 0 & 0 \\ x_1 & x_2 \end{pmatrix}$$

with I the 2×2 identity matrix. Fixing $\lambda \in [0, \overline{\lambda})$, the lift in (3.1) can be written as

$$\mathcal{L}(\lambda, h) = -e_2 \cdot \int\limits_{\partial B_0 + he_2} \mathbb{T}(u(h), p(h))n(h)$$

with $\mathbb{T}(u(h), p(h)) = \mathbb{T}(u(\lambda, h), p(\lambda, h))$. Letting

$$U(h) = u(h) \circ (\mathrm{Id} + h\sigma), \quad P(h) = p(h) \circ (\mathrm{Id} + h\sigma)$$

with σ as in (5.8), we transform the moving boundary integral into a fixed boundary integral, namely

$$\mathcal{L}(\lambda, h) = -e_2 \cdot \int\limits_{\partial B_0} \mathbb{T}(U(h), P(h))(n(0) \circ (\mathrm{Id} + he_2)).$$

Note that $(U(0), P(0)) = (u(0), p(0))$. We now claim that

$$h \mapsto (U(h), P(h)) \in H^2(\Omega_0) \times H^1(\Omega_0) \text{ belongs to } C^1(-h_0, h_0). \tag{5.9}$$

To this end, let $M(h) = (J^{-1}(h))^T$ and we rewrite (2.4) as

$$\begin{aligned}
&- \mu\nabla \cdot (|\det J(h)|M^T(h)M(h)\nabla U(h)) \\
&+ U(h) \cdot |\det J(h)|M(h)\nabla U(h) + \nabla \cdot (|\det J(h)|M(h)P(h)) = 0 \qquad && \text{in} \quad \Omega_0, \\
&|\det J(h)|M(h)\nabla \cdot U(h) = 0 && \text{in} \quad \Omega_0,
\end{aligned}$$

complemented with the same boundary conditions. This can also be expressed as

$$\mathcal{H}(h, U(h), P(h)) = 0 \tag{5.10}$$

where $\mathcal{H} : (-h_0, h_0) \times H^2(\Omega_0) \times H^1(\Omega_0) \to L^2(\Omega_0) \times H^1(\Omega_0)$ is defined by $\mathcal{H}(h, \xi, \varpi) = (\mathcal{H}_1(h, \xi, \varpi), \mathcal{H}_2(h, \xi, \varpi))$ with

$$\begin{aligned}
\mathcal{H}_1(h, \xi, \varpi) = &- \mu\nabla \cdot (|\det J(h)|M^T(h)M(h)\nabla\xi) \\
&+ \xi \cdot |\det J(h)|M(h)\nabla\xi + \nabla \cdot (|\det J(h)|M(h)\varpi), \\
\mathcal{H}_2(h, \xi, \varpi) = &\, |\det J(h)|M(h)\nabla \cdot \xi.
\end{aligned} \tag{5.11}$$

Due to the expression (5.8), we are able to compute $|\det J(h)|M(h)$ and $|\det J(h)|M^T(h)M(h)$ explicitly at second order for $h \to 0$. In fact,

$$|\det J(h)| = 1 + h\frac{F'(|x|)}{|x|}x_2,$$
$$M(h) = I + \frac{h}{|\det J(h)|}\frac{F'(|x|)}{|x|}\begin{pmatrix} 0 & -x_1 \\ 0 & -x_2 \end{pmatrix} = I + h\frac{F'(|x|)}{|x|}\begin{pmatrix} 0 & -x_1 \\ 0 & -x_2 \end{pmatrix} + O(h^2)$$

yield

$$\begin{aligned} |\det J(h)|M(h) &= I + h\frac{F'(|x|)}{|x|}\begin{pmatrix} x_2 & -x_1 \\ 0 & 0 \end{pmatrix} =: I + hR_0, \\ |\det J(h)|M^T(h)M(h) &= I + h\frac{F'(|x|)}{|x|}\begin{pmatrix} x_2 & -x_1 \\ -x_1 & -x_2 \end{pmatrix} + h^2(F'(|x|))^2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + O(h^3) \\ &=: I + hR_1 + h^2R_2 + O(h^3), \end{aligned} \tag{5.12}$$

where $O(h^3)$ contains terms having at least third order with respect to h as $h \to 0$. Note that the expression of $|\det J(h)|M(h)$ in (5.12) is exact and obtained without any Taylor expansion for $h \to 0$. We have that $\mathcal{H}$ is C^1 in a neighborhood of $(0, U(0), P(0))$ since the mappings $h \mapsto \det J(h)$ and $h \mapsto M(h)$ are $C^1(-h_0, h_0)$ with values in $C^1(R, \mathbb{R}^4)$.

For $h \in (-h_0, h_0)$, we consider the linearized operator $\Upsilon = D_{(\xi,\varpi)}\mathcal{H}(h, U(h), P(h))$ defined through the Jacobian matrix of $\mathcal{H}$. For any

$$(\chi, \Pi) \in \mathcal{X} \times \mathcal{Y} := (H^2(\Omega_0) \cap H^1_0(\Omega_0)) \times (H^1(\Omega_0) \cap L^2_0(\Omega_0)),$$

we have $\Upsilon(\chi, \Pi) = (\Upsilon_1(\chi, \Pi), \Upsilon_2(\chi, \Pi))$ with

$$\begin{aligned} \Upsilon_1(\chi, \Pi) =& -\mu\nabla \cdot (|\det J(h)|M^T(h)M(h)\nabla\chi) + \chi \cdot |\det J(h)|M(h)\nabla U(h) \\ &+ U(h) \cdot |\det J(h)|M(h)\nabla\chi + \nabla \cdot (|\det J(h)|M(h)\Pi), \\ \Upsilon_2(\chi, \Pi) =& |\det J(h)|M(h)\nabla \cdot \chi. \end{aligned}$$

The linear operator Υ is bounded from $\mathcal{X} \times \mathcal{Y}$ into $L^2(\Omega_0) \times \mathcal{Y}$. To show that Υ is an isomorphism, given $(\varphi_1, \varphi_2) \in L^2(\Omega_0) \times \mathcal{Y}$, we have to prove that there exists a unique solution $(\chi, \Pi) \in \mathcal{X} \times \mathcal{Y}$ to

$$\begin{aligned} & -\mu\Delta\chi + \chi \cdot \nabla U(h) + U(h) \cdot \nabla\chi + \nabla\Pi \\ & + h\left(-\mu\nabla \cdot R_1\nabla\chi + \chi \cdot R_0\nabla U(h) + U(h) \cdot R_0\nabla\chi + \nabla \cdot (R_0\Pi)\right) \\ & - h^2\mu\nabla \cdot R_2\nabla\chi + O(h^3) = \varphi_1 && \text{in } \Omega_0, \\ & \nabla \cdot \chi + hR_0\nabla \cdot \chi = \varphi_2 && \text{in } \Omega_0. \end{aligned}$$

This linear elliptic problem admits a unique solution provided that

$$|h| < h_0 \qquad \text{and} \qquad \|U(h)\|_{H^2(\Omega_0)} < r$$

for $h_0, r > 0$ small enough. For $|h| < h_0$ and σ as in (5.8), we have

$$\begin{aligned} c\|u(h)\|_{H^2(\Omega_h)} \leq& \|U(h)\|_{H^2(\Omega_0)} \leq C\|u(h)\|_{H^2(\Omega_h)}, \\ c\|p(h)\|_{H^1(\Omega_h)} \leq& \|P(h)\|_{H^1(\Omega_0)} \leq C\|p(h)\|_{H^1(\Omega_h)}, \end{aligned} \tag{5.13}$$

with constants $0 < c \leq C$ independent of h. Then, by taking $\lambda \in [0, \overline{\lambda})$, the bound (2.8), where the constant C_h is uniformly bounded for $|h| < h_0$, yields the needed smallness condition for $U(h)$, so that Υ is an isomorphism. Therefore, by applying the Implicit Function Theorem to (5.10), we conclude (5.9).

Moreover, the derivatives $U'(h)$ and $P'(h)$, whose existence follows from (5.9), satisfy

$$\Upsilon(U'(h), P'(h)) = -\partial_h\mathcal{H}(h, U(h), P(h)). \tag{5.14}$$

From (5.12), we know that for any h (resp. $h \to 0$)

$$\frac{d}{dh}(|\det J(h)|M(h)) = R_0, \quad \frac{d}{dh}(|\det J(h)|M^T(h)M(h)) = R_1 + 2hR_2 + O(h^2).$$

Then, recalling the definition (5.11), (5.14) and the fact that Υ is an isomorphism imply that $(U'(h), P'(h))$ is uniquely determined by the linear elliptic problem

$$\begin{aligned} & -\mu\Delta U'(h) + U'(h)\cdot\nabla U(h) + U(h)\cdot\nabla U'(h) + \nabla P'(h) \\ & = S_0(U(h),P(h)) + hS_1(U'(h),P'(h),U(h)) + O(h^2) \qquad \text{in } \Omega_0, \\ & \nabla\cdot U'(h) = -R_0\nabla\cdot U(h) - hR_0\nabla\cdot U'(h) \qquad \text{in } \Omega_0, \\ & (U'(h),P'(h)) \in \mathcal{X}\times\mathcal{Y}, \end{aligned} \tag{5.15}$$

with

$$\begin{aligned} S_0(U(h),P(h)) =& \mu\nabla\cdot R_1\nabla U(h) - U(h)\cdot R_0U(h) - \nabla\cdot(R_0P(h)), \\ S_1(U'(h),P'(h),U(h)) =& \mu\nabla\cdot(R_1\nabla U'(h) + 2R_2\nabla U(h)) - U'(h)\cdot R_0\nabla U(h) \\ & - U(h)\cdot R_0\nabla U'(h) - \nabla\cdot(R_0P'(h)). \end{aligned}$$

For $h \in (-h_0, h_0)$, with h_0 small, we have

$$\begin{aligned} & \|U'(h)\|_{H^2(\Omega_0)} + \|P'(h)\|_{H^1(\Omega_0)} \\ & \le C(\|U'(h)\cdot\nabla U(h) + U(h)\cdot\nabla U'(h) + S_0(U(h),P(h))\|_{L^2(\Omega_0)} + \|R_0\nabla\cdot U(h)\|_{H^1(\Omega_0)}). \end{aligned}$$

Since $(U(h), P(h)) \in H^2(\Omega_0)\times H^1(\Omega_0)$ due to (5.13) and Theorem 2.2, we bound the right-hand side of the above expression as

$$\begin{aligned} & \|U'(h)\cdot\nabla U(h) + U(h)\cdot\nabla U'(h)\|_{L^2(\Omega_0)} \le C\|\nabla U'(h)\|_{L^2(\Omega_0)}\|U(h)\|_{H^2(\Omega_0)}, \\ & \|S_0(U(h),P(h))\|_{L^2(\Omega_0)} + \|R_0\nabla\cdot U(h)\|_{H^1(\Omega_0)} \le C(\|U(h)\|_{H^2(\Omega_0)} + \|P(h)\|_{H^1(\Omega_0)}), \end{aligned}$$

where in the second inequality we used that $\sigma \in W^{2,\infty}(R,\mathbb{R}^2)$, see (5.8). Testing the first equation in (5.15) with $U'(h)$, using (5.13) and (2.13)-(2.14) yield

$$\begin{aligned} \|\nabla U'(h)\|_{L^2(\Omega_0)} & \le C(\|U(h)\|_{H^1(\Omega_0)} + \|U(h)\|^2_{H^1(\Omega_0)} + \|P(h)\|_{L^2(\Omega_0)}) \\ & \le C(\|U(h)\|_{H^1(\Omega_0)} + \|U(h)\|^2_{H^1(\Omega_0)}). \end{aligned}$$

Summarizing, we obtain

$$\begin{aligned} & \|U'(h)\|_{H^2(\Omega_0)} + \|P'(h)\|_{H^1(\Omega_0)} \\ & \le C\left(\|U(h)\|_{H^2(\Omega_0)}(1 + \|U(h)\|_{H^1(\Omega_0)} + \|U(h)\|^2_{H^1(\Omega_0)}) + \|P(h)\|_{H^1(\Omega_0)}\right) \\ & \le C(\lambda + \lambda^3) \le C\lambda \end{aligned} \tag{5.16}$$

for any $\lambda \in [0, \overline{\lambda})$, where in the second inequality we used (5.13) and (2.7)-(2.8).

Finally, we estimate the variation in the lift for small values of h, say $|h| < h_0$. By taking $h_1, h_2 \in (-h_0, h_0)$, from the trace theorem we have

$$\begin{aligned}|\mathcal{L}(\lambda, h_1) - \mathcal{L}(\lambda, h_2)| &= \left| \int_{\partial B_0} \mathbb{T}(U(h_1), P(h_1))(n(0) \circ (\mathrm{Id} + h_1 e_2)) - \mathbb{T}(U(h_2), P(h_2))(n(0) \circ (\mathrm{Id} + h_2 e_2)) \right| \\ &\le \int_{\partial B_0} |\mathbb{T}(U(h_1), P(h_1)) - \mathbb{T}(U(h_2), P(h_2))| \\ &\quad + \int_{\partial B_0} |\mathbb{T}(U(h_2), P(h_2))| \cdot |n(0) \circ (\mathrm{Id} + h_1 e_2) - n(0) \circ (\mathrm{Id} + h_2 e_2)| \\ &\le C(\|U(h_1) - U(h_2)\|_{H^2(\Omega_0)} + \|P(h_1) - P(h_2)\|_{H^1(\Omega_0)}) \\ &\quad + C(\|U(h_2)\|_{H^2(\Omega_0)} + \|P(h_2)\|_{H^1(\Omega_0)})|h_1 - h_2|.\end{aligned}$$

Then, (5.16) and the mean value theorem yield

$$\begin{aligned}&|\mathcal{L}(\lambda, h_1) - \mathcal{L}(\lambda, h_2)| \\ &\le C\lambda|h_1 - h_2| + C(\|u(h_2)\|_{H^2(\Omega_{h_2})} + \|p(h_2)\|_{H^1(\Omega_{h_2})})|h_1 - h_2| \le C\lambda|h_1 - h_2|\end{aligned}$$

using (5.13) and (2.8) in Ω_{h_2}. Then, the monotonicity property (3.2) ensures that, if $-h_0 < h_2 < h_1 < h_0$,

$$\phi(\lambda, h_1) - \phi(\lambda, h_2) = f(h_1) - f(h_2) - \mathcal{L}(\lambda, h_1) + \mathcal{L}(\lambda, h_2) \ge (\gamma - C\lambda)(h_1 - h_2).$$

There exists $\lambda_0 \in (0, \overline{\lambda}]$ such that $\gamma - C\lambda_0 \ge \gamma/2$. Therefore, $h \mapsto \phi(\lambda, h)$ is continuous and strictly increasing in $[-h_0, h_0]$ (with a possible smaller h_0) for all $\lambda \in [0, \lambda_0)$. □

5.2. Conclusion of the proof

Let $(u(\lambda, h), p(\lambda, h))$ be a solution to (2.4), and let $\phi(\lambda, h)$ be the corresponding global force in (5.1). Then, the triple (u, p, h) is a solution to (3.4) if and only if

$$(u(\lambda, h), p(\lambda, h)) \,\text{solves (2.4) and}\, \phi(\lambda, h) = 0.$$

Therefore, Theorem 3.1 follows once we prove:

Proposition 5.3. *Let ϕ be as in (5.1) and (λ_0, h_0) be as in Proposition 5.2. Then, there exist $\Lambda_1 \in (0, \lambda_0]$ and a unique $\mathfrak{h} \in C^0[0, \Lambda_1)$ such that, for all $\lambda \in [0, \Lambda_1)$, $\phi(\lambda, h) = 0$ if and only if $h = \mathfrak{h}(\lambda)$. Moreover, $\|\mathfrak{h}\|_{L^\infty(0,\Lambda_1)} \le h_0$.*

Proof. We prove the result in two steps, namely by analyzing the behavior of ϕ in two different subregions of $[0, \lambda_0) \times (-H + \delta_b, H - \delta_t)$.

We start by considering the case when $|h|$ is close to 0. Let again $\overline{h} = H - \max\{\delta_b, \delta_t\}$. We claim that there exists $\widetilde{\lambda} \in (0, \lambda_0]$ and a unique $\mathfrak{h} \in C^0[0, \widetilde{\lambda})$ such that

$$\forall (\lambda, h) \in [0, \widetilde{\lambda}) \times [-h_0, h_0] \qquad \phi(\lambda, h) = 0 \iff h = \mathfrak{h}(\lambda). \tag{5.17}$$

To this end, we notice that Theorem 2.2 implies that, when $\lambda = 0$, the unique solution to (2.4) is $(u, p) = (0, 0)$, regardless of the value of $h \in (-H + \delta_b, H - \delta_t)$. Hence, $\phi(0, 0) = 0$. Moreover, by Proposition 5.2 we know that $h \mapsto \phi(0, h)$ is continuous and strictly increasing in $[-h_0, h_0]$. These two facts imply that

$$\phi(0, -h_0) < 0 < \phi(0, h_0). \tag{5.18}$$

In turn, by Proposition 5.1 we know that $\lambda \mapsto \phi(\lambda, h)$ is continuous in $[0, \overline{\lambda})$ for all $h \in [-h_0, h_0]$. By (5.18) and by compactness, we then infer that there exists $\widetilde{\lambda} \in (0, \lambda_0]$ such that

$$\phi(\lambda, -h_0) < 0 < \phi(\lambda, h_0) \qquad \forall \lambda \in [0, \widetilde{\lambda}) \tag{5.19}$$

and, by invoking again Proposition 5.2, that $h \mapsto \phi(\lambda, h)$ is continuous and strictly increasing in $[-h_0, h_0]$ for all $\lambda \in [0, \widetilde{\lambda})$. Together with (5.19), this implies that for all $\lambda \in [0, \widetilde{\lambda})$ there exists a unique $\mathfrak{h}(\lambda) \in [-h_0, h_0]$ such that $\phi(\lambda, \mathfrak{h}(\lambda)) = 0$. This defines the function $\lambda \mapsto \mathfrak{h}(\lambda)$ in the interval $[0, \widetilde{\lambda})$. Its continuity follows by the (separated) continuities proved in Propositions 5.1 and 5.2. The proof of (5.17) is so complete.

We now claim that there exists $\Lambda_1 \in (0, \widetilde{\lambda}]$ such that

$$\phi(\lambda, h) \neq 0 \qquad \forall (\lambda, h) \in [0, \Lambda_1) \times \Big[(-H + \delta_b, H - \delta_t) \setminus [-h_0, h_0]\Big]. \tag{5.20}$$

Recall that in this set $\phi(\lambda, h)$ may be multi-valued, see Theorem 3.2. In order to prove (5.20), from (3.2)-(3.3) we know that there exists $K_0 \in (0, K]$ such that

$$\begin{aligned} f(h) &\leq -K_0 (\varepsilon_b(h))^{-3/2} && \text{for} \quad h \in (-H + \delta_b, -h_0), \\ f(h) &\geq K_0 \max\{(\varepsilon_t(h))^{-3/2}, U(\varepsilon_t(h))^{-3}\} && \text{for} \quad h \in (h_0, H - \delta_t), \end{aligned} \tag{5.21}$$

while from Theorem 3.2 there exists (a different) $C > 0$ such that

$$\begin{aligned} \mathcal{L}(\lambda, h) &\geq -C (\varepsilon_b(h))^{-3/2} \lambda && \text{for} \quad h \in (-H + \delta_b, -h_0), \\ \mathcal{L}(\lambda, h) &\leq C \max\{(\varepsilon_t(h))^{-3/2}, U(\varepsilon_t(h))^{-3}\} \lambda && \text{for} \quad h \in (h_0, H - \delta_t). \end{aligned} \tag{5.22}$$

Gathering (5.21)-(5.22) together yields

$$\begin{aligned} \phi(\lambda, h) &\leq (-K_0 + C\lambda)(\varepsilon_b(h))^{-3/2} && \text{for} \quad h \in (-H + \delta_b, -h_0), \\ \phi(\lambda, h) &\geq (K_0 - C\lambda) \max\{(\varepsilon_t(h))^{-3/2}, U(\varepsilon_t(h))^{-3}\} && \text{for} \quad h \in (h_0, H - \delta_t). \end{aligned}$$

Then, there exists $\Lambda_1 \in (0, \widetilde{\lambda}]$ such that (5.20) holds and the statement of the proposition follows from (5.17) and (5.20). □

Remark 5.4. In fact, the proof of (5.20) shows that if $\lambda > 0$ is small, then

$$h_0 < h < H - \delta_t \implies \phi(\lambda, h) > 0 \quad \text{and} \quad -H + \delta_b < h < -h_0 \implies \phi(\lambda, h) < 0.$$

From a physical point of view, this means that, for small Reynolds numbers, the global force $\phi = \phi(\lambda, h)$ in (5.1) pushes downwards the body if B_h is close to the upper boundary Γ_t, whereas it pushes the body upwards if B_h is close to the lower boundary Γ_b.

6. Symmetric configuration

We consider here a symmetric framework for (3.4), that is, when

$$(x_1, x_2) \in \partial B \iff (x_1, -x_2) \in \partial B$$

and the boundary data are symmetric with respect to the line $x_2 = 0$. Therefore, the FSI problem (3.4) is modified on Γ_b and reads

$$\begin{gathered} -\mu \Delta u + u \cdot \nabla u + \nabla p = 0, \qquad \nabla \cdot u = 0 \quad \text{in} \quad \Omega_h \\ u_{|\partial B_h} = 0, \quad u_{|\Gamma_b} = u_{|\Gamma_t} = \lambda U e_1, \quad u_{|\Gamma_l} = \lambda V_{\text{in}}(x_2) e_1, \quad u_{|\Gamma_r} = \lambda V_{\text{out}}(x_2) e_1, \\ f(h) = -e_2 \cdot \int_{\partial B_h} \mathbb{T}(u, p) n, \end{gathered} \tag{6.1}$$

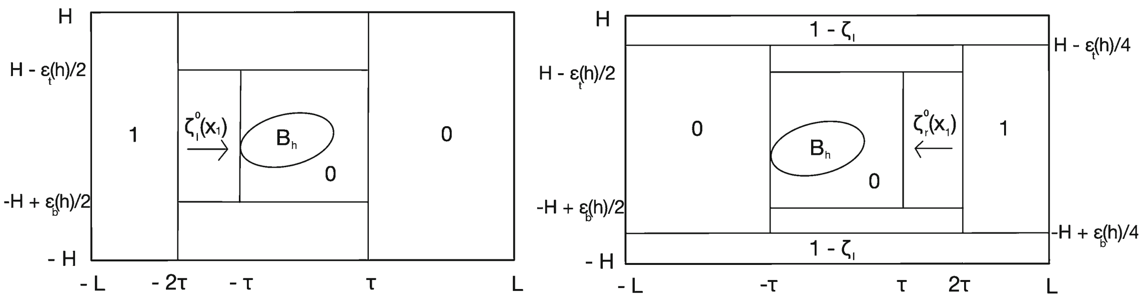

FIG. 6. Cutoff functions ζ_l (left) and ζ_r (right) on $\overline{R}$ when $U = 1$ for the symmetric configuration

with $\lambda \geq 0$, $U \in \{0,1\}$ (up to normalization). Here, $V_{\text{in}}, V_{\text{out}} \in W^{2,\infty}(-H,H)$ are now even functions satisfying

$$V_{\text{in}}(\pm H) = V_{\text{out}}(\pm H) = U, \qquad \int_{-H}^{H} V_{\text{in}}(x_2)\mathrm{d}x_2 = \int_{-H}^{H} V_{\text{out}}(x_2)\mathrm{d}x_2. \tag{6.2}$$

In this symmetric framework, $\delta_b = \delta_t = \delta$ and $h \in (-H+\delta, H-\delta)$. Then, we prove that the unique curve $\mathfrak{h}(\lambda)$ found in Theorem 3.1 reduces to $\mathfrak{h}(\lambda) \equiv 0$, namely that the unique equilibrium position is symmetric. Again, we expect this position to be stable, at least for small λ.

Theorem 6.1. *Let V_{in}, $V_{\text{out}} \in W^{2,\infty}(-H,H)$ be even functions satisfying (6.2) and $f \in C^0(-H+\delta, H-\delta)$ satisfying $f(0) = 0$ and (3.2)-(3.3) with $\delta_b = \delta_t = \delta$. There exists $\Lambda_1 > 0$ such that for $\lambda \in [0, \Lambda_1)$ the FSI problem (6.1) admits a unique strong solution $(u(\lambda,h), p(\lambda,h), h) \in H^2(\Omega_h) \times H^1(\Omega_h) \times (-H+\delta, H-\delta)$ given by*

$$\big(u^0(\lambda,0), p^0(\lambda,0), 0\big),$$

where $(u^0(\lambda,0), p^0(\lambda,0))$ is the unique solution to the first two lines in (6.1) for $h = 0$ and has the following symmetries:

$$u_1^0(x_1,-x_2) = u_1^0(x_1,x_2), \quad u_2^0(x_1,-x_2) = -u_2^0(x_1,x_2), \quad p^0(x_1,-x_2) = p^0(x_1,x_2).$$

Proof. The first step is to obtain the counterpart of Theorem 2.2. The case $U = 0$ is already included in the original statement. When $U = 1$, we construct the cutoff functions ζ_l and ζ_r in a slightly different way with Fig. 4 replaced by Fig. 6. We define the solenoidal extension as in (2.24), which satisfies the boundary conditions in (6.1).

With this construction, the refined bound (2.7) is replaced by

$$\|u\|_{H^1(\Omega_h)} \leq C((\varepsilon_b(h))^{-3/2} + (\varepsilon_t(h))^{-3/2})\lambda.$$

Hence, in both cases $U \in \{0,1\}$, by arguing as in the proof of Theorem 2.2, we infer that there exists $\Lambda = \Lambda(h) > 0$ such that for $\lambda \in [0, \Lambda(h))$ the solution (u,p) to

$$\begin{gathered} -\mu\Delta u + u\cdot\nabla u + \nabla p = 0, \qquad \nabla\cdot u = 0 \quad \text{in} \quad \Omega_h \\ u_{|\partial B_h} = 0, \quad u_{|\Gamma_b} = u_{|\Gamma_t} = \lambda U e_1, \quad u_{|\Gamma_l} = \lambda V_{\text{in}}(x_2)e_1, \quad u_{|\Gamma_r} = \lambda V_{\text{out}}(x_2)e_1, \end{gathered} \tag{6.3}$$

is unique for any $h \in (-H+\delta, H-\delta)$. This proves the counterpart of Theorem 2.2.

In particular, for $h = 0$ there exists a unique solution (u^0, p^0) to (6.3) in Ω_0. Since Ω_0 is symmetric with respect to the line $x_2 = 0$, the couple $(u^*, p^*) : \Omega_0 \to \mathbb{R}^2 \times \mathbb{R}$ defined by

$$u_1^*(x_1,x_2) = u_1^0(x_1,-x_2), \quad u_2^*(x_1,x_2) = -u_2^0(x_1,-x_2), \quad p^*(x_1,x_2) = p^0(x_1,-x_2),$$

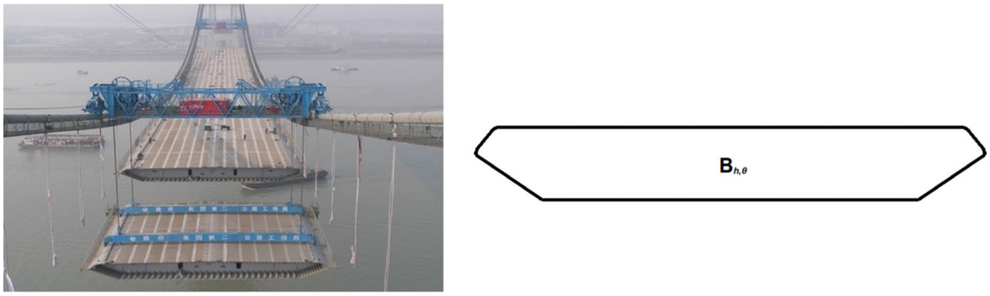

FIG. 7. Left: erection of a suspension bridge. Right: sketch of a cross section

also satisfies (6.3) for $h = 0$ (see also [10]). Therefore, by uniqueness $(u^0, p^0) = (u^*, p^*)$ is also symmetric and, thanks to all these symmetries, we obtain

$$\mathcal{L}(\lambda, 0) = -e_2 \cdot \int_{\partial B_0} \mathbb{T}(u^0(\lambda, 0), p^0(\lambda, 0))n = 0,$$

which implies

$$\phi(\lambda, 0) = f(0) = 0 \qquad \text{for} \quad \lambda \in [0, \Lambda(h)). \tag{6.4}$$

From Theorem 3.1, we know that there exist $\Lambda_1 > 0$ and a unique curve $\mathfrak{h} \in C^0[0, \Lambda_1)$ such that for $\lambda \in [0, \Lambda_1)$ the unique solution to (6.1) is given by

$$(u(\lambda, \mathfrak{h}(\lambda)), p(\lambda, \mathfrak{h}(\lambda)), \mathfrak{h}(\lambda)).$$

Thanks to (6.4), $\mathfrak{h}(\lambda) \equiv 0$ and this solution coincides with $(u^0(\lambda, 0), p^0(\lambda, 0), 0)$. □

7. An application: equilibrium positions of the deck of a bridge

A suspension bridge is usually erected starting from the anchorages and the towers. Then, the sustaining cables are installed between the two couples of towers and the hangers are hooked to the cables. Once all these components are in position, they furnish a stable working base from which the deck can be raised from floating barges. We refer to [18, Section 15.23] for full details. The deck segments are put in position one aside the other (see Fig. 7, left) and have the shape of rectangles while their cross-section resembles to smoothened irregular hexagons (see Fig. 7, right) that satisfy (2.1).

This cross section B plays the role of the obstacle in (2.4) while Ω_h is the region filled by the air. This region can be either be a virtual box around the deck of the bridge or a wind tunnel around a scaled model of the bridge. In both cases, we may refer to inflow and outflow also as windward and leeward, respectively: $\lambda V_{\text{in}} e_1$ represents the laminar horizontal windward while $\lambda V_{\text{out}} e_1$ is the leeward. Typically, the higher is the altitude the stronger is the wind. Therefore, in this application we consider specific laminar shear flows, which are the Couette flows. Thus, the inflow and outflow now read

$$V_{\text{in}}(x_2) = V_{\text{out}}(x_2) = \frac{U}{2H}(x_2 + H) \qquad \text{for} \qquad x_2 \in [-H, H], \tag{7.1}$$

and satisfy (2.3). The windward creates both vertical and torsional displacements of the deck. However, the cross section of the suspension bridge is also subject to some elastic restoring forces tending to maintain the deck in its original position B_0. These forces are of three different kinds. There is an upwards restoring force due to the elastic action of both the hangers and the sustaining cables of the bridge. The hangers behave as nonlinear springs which may slacken [1, 9-VI] so that they have no downwards action and they be nonsmooth. There is the weight of the deck which acts constantly downwards: this is why there is no odd requirement on the restoring force considered in the model. There is also a nonlinear resistance to both elastic bending and stretching of the whole deck for which B merely represents a cross-section. Moreover, since the boundary of the channel R is virtual and our physical model breaks down in case of collision of B with ∂R, we require that there exists an "unbounded force" preventing collisions.

Overall, the position of B depends on both the displacement parameter h and the angle of rotation θ with respect to the horizontal axis. With the addition of this second degree of freedom, we have $B = B_{h,\theta}$ and $\Omega = \Omega_{h,\theta}$. A "plastic" regime leading to the collapse of the bridge is reached when $\theta = \pm\frac{\pi}{4}$ (see [1]) since the sustaining cables of the bridge attain their maximum elastic tension. The strong point of the analysis carried out in this paper is that it applies independently of the part of ∂B closest to ∂R. Therefore, for any $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, we can apply our general theory considering the family of bodies $B_{h,\theta}$ simply by adapting it to the rotating scenario. The only difference now is that, when the body is free to rotate, the collision with Γ_b and Γ_t occurs at $h = -H + \delta_b(\theta)$ and $h = H - \delta_t(\theta)$, where $\delta_b(\theta)$ and $\delta_t(\theta)$ are positive functions of θ. For $\theta = 0$, $\delta_b(0)$ and $\delta_t(0)$ are as in (2.2) while, for $\theta \neq 0$,

$$\delta_b(\theta) := -\min_{(x_1,x_2)\in\partial B_{0,\theta}} x_2 > 0, \qquad \delta_t(\theta) := \max_{(x_1,x_2)\in\partial B_{0,\theta}} x_2 > 0,$$

both being independent of h. Due to the possible complicated shape of B, these functions are not easy to be determined explicitly. For this reason, we define the set of non-contact values of (h, θ) by

$$A = \{(h,\theta) \in (-H,H) \times (-\tfrac{\pi}{4}, \tfrac{\pi}{4}) \ : \ B_{h,\theta} \subset R\}. \tag{7.2}$$

Clearly, $(0, 0) \in A$ and $(h, \theta) \in \partial A$ if and only if $B_{h,\theta} \cap \partial R \neq \emptyset$. We assume that, for some $K > 0$, $f \in C^0(A)$ satisfies

$$\begin{aligned} &\limsup_{d(B_{h,\theta},\Gamma_b)\to 0} \ f(h,\theta)(d(B_{h,\theta},\Gamma_b))^{3/2} \le -K, \\ &\liminf_{d(B_{h,\theta},\Gamma_t)\to 0} \ \frac{f(h,\theta)}{\max\{(d(B_{h,\theta},\Gamma_t))^{-3/2}, U(d(B_{h,\theta},\Gamma_t))^{-3}\}} \ge K, \end{aligned} \tag{7.3}$$

where $d(\cdot,\cdot)$ is the distance function. Assumption (7.3) generalizes (3.3) taking into account the rotational degree of freedom. Moreover, we assume that

$$\begin{aligned} &\exists \gamma > 0 \quad \text{s.t.} \quad \frac{f(h_1,\theta) - f(h_2,\theta)}{h_1 - h_2} \ge \gamma \qquad \forall (h_1,\theta),(h_2,\theta) \in A, \\ &f(0,0) = 0, \qquad f(h,\theta)\theta > 0 \quad \forall (h,\theta) \in A \quad \text{with} \quad \theta \neq 0. \end{aligned} \tag{7.4}$$

In fact, the second line in (7.4) is not mathematically needed, but, from a physical point of view, it states that the restoring force does not act at equilibrium and tends to maintain B in an horizontal position. A straightforward consequence of Theorem 3.1, in the case of the interaction between the wind and the deck of a suspension bridge, is the following:

Corollary 7.1. *Let V_{in}, V_{out} be as in (7.1) and $f \in C^0(A)$ satisfy (7.3)-(7.4). There exist $\Lambda_1 > 0$ and a unique $\mathfrak{h} \in C^0[0, \Lambda_1)$ such that, for $\lambda \in [0, \Lambda_1)$ and $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, the FSI problem (3.4) admits a unique solution $(u_\theta(\lambda, h), p_\theta(\lambda, h), h) \in H^2(\Omega_{h,\theta}) \times H^1(\Omega_{h,\theta}) \times (-H, H)$, with $(h, \theta) \in A$, given by*

$$(u_\theta(\lambda, \mathfrak{h}(\lambda)), p_\theta(\lambda, \mathfrak{h}(\lambda)), \mathfrak{h}(\lambda)).$$

Here, (3.4) is understood with h replaced by the couple (h, θ).

The deck of a suspension bridge, in particular its cross section, may have a nonsmooth boundary. If B is not $W^{2,\infty}$, but it is only Lipschitzian, Theorem 2.2 ceases to hold and we only know that (u, p) is a weak solution to (2.4) so that (3.1) does not hold in a "strong" sense. Indeed, since $u \in H^1(\Omega_h)$, see (2.7), we may rewrite the first equation in (2.4) as $-\mu\Delta u + \nabla p = f$ with $f \in L^q(\Omega_h)$ for all $q < 2$. Hence, $f \in H^{-\epsilon}(\Omega_h)$ for any $\epsilon > 0$. By applying [19, Theorem 7], we then deduce that $u \in H^{1+s}(\Omega_h)$ and $p \in H^s(\Omega_h)$ for all $s < 1/2$, but, still, this does not allow to consider the trace of $\mathbb{T}(u, p)$ as an integrable function over ∂B_h. However, following [10] we may define the lift L through a generalized formula. Indeed, from $u \in H^1(\Omega_h)$ we know that $\mathbb{T}(u,p) \in L^2(\Omega_h)$ and, since Ω_h is a bounded domain, $\mathbb{T}(u,p) \in L^{3/2}(\Omega_h)$. Moreover, from the first equation in (2.4) we obtain $\nabla\cdot\mathbb{T}(u,p) \in L^{3/2}(\Omega_h)$. Therefore, $\mathbb{T}(u,p) \in E_{3/2}(\Omega_h) := \{f \in L^{3/2}(\Omega_h) \mid \nabla\cdot f \in L^{3/2}(\Omega_h)\}$. By Theorem III.2.2 in [5], we know that $\mathbb{T}(u,p)n_{|\partial\Omega_h} \in W^{-2/3,3/2}(\partial\Omega_h)$. Hence, if ∂B_h is Lipschitzian and (u,p) is a weak solution to (2.4), then the lift exerted by the fluid over B_h is

$$\mathcal{L}(\lambda,h) = -e_2 \cdot \langle \mathbb{T}(u,p)n, 1\rangle_{\partial B_h}, \tag{7.5}$$

where $\langle\cdot,\cdot\rangle_{\partial B_h}$ denotes the duality pairing between $W^{-2/3,3/2}(\partial B_h)$ and $W^{2/3,3}(\partial B_h)$.

Acknowledgements

The authors warmly thank the anonymous referee for the careful proofreading and several useful remarks. The authors were partially supported by the PRIN project *Direct and inverse problems for partial differential equations: theoretical aspects and applications* and by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Funding Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest There are no conflicts of interest.

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(Received: March 20, 2023; revised: June 26, 2023; accepted: July 3, 2023)