Stochastic Pinning Controllability of Noisy Complex Networks

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Abstract—**Our ability to coordinate the behavior in networks of complex dynamical systems in often challenged by the presence of noise affecting the individual dynamics and the communication links. In the literature, conservative global conditions guaranteeing the almost sure convergence toward the desired trajectory of a virtual node, the pinner, have been derived. In this paper, we identify the minimal conditions on the individual dynamics, interconnection topology, and noise intensities, so that the network exponentially converges onto the pinner's trajectory. Specifically, we broaden the Master Stability Function approach to deal with networks of coupled stochastic differential equations, and provide necessary and sufficient conditions for local exponential pinning controllability of networks of stochastic systems. Interestingly, our analyses show that noise can be either beneficial or detrimental for pinning controllability, depending on how it diffuses in each node. Our analytical findings are illustrated with representative numerical examples.**

Index Terms—**Stochastic complex networks, pinning control, synchronization, stochastic differential equations, communication noise.**

I. INTRODUCTION

In the last decades, the emergence of spontaneous coordinated behaviors in networks of coupled dynamical systems has attracted the interest of diverse scientific communities. A substantial research effort contributed to uncover how ensembles of interconnected systems may spontaneously coordinate to achieve consensus on a point of the space space [1]–[6] or to converge towards a common synchronous trajectory [7]–[13]. Synchronization and consensus have been widely studied for their relevance in several domains of application, including social networks analyses [14], [15], animal group dynamics [16], [17], power grids [18], [19], and metabolic networks [20], [21]. Formally, analytical conditions have been derived guaranteeing convergence toward a common solution of the individual dynamics, which, in general, cannot be arbitrarily imposed. Then, a major control problem still remained in all the engineering applications where the network nodes are prescribed to synchronize onto a predefined desired trajectory. This is the case, for instance, of formation control problems, where a subset of the network nodes, commonly denoted as *leaders*, is aware of the target trajectory of the formation and sets the reference for the remaining nodes, the *followers* [22]. The reference trajectory may also not be assigned by a node of the network, but rather come from an external reference signal. For instance, in power grids we would ideally drive the generators' network to nominal synchronous operation at the reference frequency [18].

A possible solution to this control problem could be to inject a feedback control signal to each node in the network. However, this is not always feasible in applications, as the desired solution may not be accessible to all the network nodes. To cope with this constraint, in the literature the so-called *pinning control* [23]–[25] has been proposed, which only requires to directly control a small fraction of the network nodes. Specifically, conditions for the local convergence toward the synchronization manifold have been provided in [26], [27] by linearizing the dynamics around the pinner's trajectory. Later works have provided conditions on the number of pinned nodes required to achieve global convergence of the whole network [28]–[31] or of a maximal node subset [32], [33]. Furthermore, decentralized adaptive strategies have been developed to tune both the coupling and the control gains in a distributed fashion [34], [35].

Most of the existing results on pinning control of complex networks have been obtained in a deterministic setting, in which the individual node dynamics are driven by ordinary differential equations, and the interaction topology is either static or varies according to known dynamics. However, this is seldom true in applications. For instance, in formation control problems, ground robots might be subject to perturbations due to internal factors such as fluctuations of the battery level, possible component failures, chassis vibrations, or environmental factors such as rough terrain, wind gusts, and sensor measurement errors [36], [37], while, in biology, deterministic models are not capable of capturing the cell-to-cell fluctuations in genetic switching [38]. Furthermore, the communication among the network nodes in engineered networks might be subject to quantization or measurement uncertainty [39], [40]. Recent work have tried to extend the pinning control framework to deal with the presence of noise on the individual dynamics and on the communication [41]–[44]. In particular, the authors of [44] provided sufficient conditions for global pinning controllability of complex networks in a stochastic setting. Specifically, they suggested that noise can have a beneficial impact on the convergence toward a desired stochastic trajectory, while it negatively affects controllability when it disturbs the communications.

Statement of Contribution: our manuscript contributes a novel method to investigate pinning controllability in complex networks affected by noise. Namely, we broaden the Master Stability Function (MSF) approach to cope with the presence of mismodeling noise, that is, noise on the individual dynamics that accounts for modeling uncertainties, and of communication noise on the coupling protocol. With respect to the

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existing approaches, our method allows to

- Provide the minimal requirements for pinning controllability. Indeed, the global results available in the literature are unavoidably conservative, thus requiring restrictive assumptions on the individual dynamics, the coupling structure of the network, and the control gain of the pinner. Our approach, instead, allows to establish necessary and sufficient conditions for the local exponential pinning controllability of noisy complex networks.
- Clarify the impact of noise on pinning controllability. Existing work [44], based on the derived sufficient conditions, posited that, while the mismodeling noise is beneficial for control, the noise acting on the communication hinders pinning controllability. Our necessary and sufficient conditions allow to better specify the effect of noise, which strongly depends on the way it diffuses in each node. Indeed, we observe that i) mismodeling noise can also be detrimental for synchronization, and this happens when it diffuses unevenly on the node state variables, and ii) pinning controllability can benefit from the presence of noise in the communication when it acts on all the node states.

We emphasize that the sharpness of our condition goes hand in hand with their applicability in practical applications on large network systems. Indeed, as for the deterministic MSF [26], the computation of our Stochastic Master Stability Function (SMSF) only requires evaluating the sample Lyapunov exponents of a low-dimensional system, whose size coincides with that of the node.

Outline of the paper: in Section II, we review the stability properties of stochastic differential equations and introduce pinning control in the context of deterministic networks. In Section III, we introduce our stochastic complex network model and then derive necessary and sufficient conditions for exponential pinning controllability in a stochastic sense. Section IV illustrates through a paradigmatic example of nonlinear stochastic network the impact of mismodeling and communication noise on pinning controllability. Finally, conclusions are drawn in Section V.

II. Mathematical Preliminaries

A. Stochastic differential equations

Let us consider the following stochastic Itô equation:

$$\mathrm{d}z(t) = \varphi(z(t), t)\mathrm{d}t + \gamma(z(t), t)\mathrm{d}b(t), \tag{1}$$

where $z \in \mathbb{R}^m$, $\varphi : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^m$ and $\gamma : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^{m\times q}$ are nonlinear vector fields commonly denoted as drift and diffusion functions, respectively, and $b(t)$ is a q-dimensional Wiener process [45]. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual properties

1) $\emptyset \in \mathcal{F}$, where $\emptyset$ denotes the empty set;
2) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$, where $A^C := \Omega - A$;
3) $A_1 \in \mathcal{F},\ A_2 \in \mathcal{F} \Rightarrow A_1 \bigcup A_2 \in \mathcal{F}$.

Given a vector v, we denote by $\|v\|$ its Euclidean norm. The following Lemma provides sufficient conditions on φ and γ ensuring the existence and uniqueness of the solution $z(t)$ of equation (1) on every finite sub-interval $[t_0, T]$ of $[t_0, +\infty]$:

Lemma 1 (Existence and uniqueness of a global solution)**.** *[45, Ch. 2, Thm. 3.6] If*

1) *(Lipschitz condition) for every real number $T > t_0$ and integer $\delta \geq 1$, there exists a positive constant $K_{T,\delta}$ such that*
$$\max(\|\varphi(x,t) - \varphi(y,t)\|^2, \|\gamma(x,t) - \gamma(y,t)\|^2) \leq K_{T,\delta}\|x-y\|^2$$
for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^n$ fulfilling $\max(\|x\|, \|y\|) \leq \delta$, and
2) *(Growth condition) for every $T > t_0$ there exists a positive constant K_T such that*
$$z^T\varphi(x,t) + \frac{1}{2}\|\gamma(z,t)\|^2 \leq K_T(1 + \|z\|^2)$$
for all $z \in \mathbb{R}^n$ and $t \in [t_0, T]$,

then there exists a unique global solution $z(t)$ to equation (1), with $z(t)$ being a real-valued measurable $\{\mathcal{F}_t\}$-adapted process with finite variance.

Definition 1 (Equilibrium of a stochastic equation)**.** $z(t) = \bar{z}$ is an equilibrium of the stochastic Itô equation (1) if both

$$\varphi(\bar{z}, t) = 0 \text{ and } \gamma(\bar{z}, t) = 0 \quad \forall t.$$

If $\bar{z} = 0$, the equilibrium is also called the *trivial solution* of the stochastic system (1).

Definition 2 (Sample Lyapunov exponent and almost sure stability)**.** Let us consider a stochastic Itô process of the form (1) having the trivial solution $\bar{z} = 0$. The sample Lyapunov exponent [45, Eq. (5.4)] associated to $\bar{z}$ is

$$\mathrm{Lyap}(t, z(t_0)) := \frac{1}{t}\log\left(\|z(t; t_0, z(t_0))\|\right). \tag{2}$$

The trivial solution is *locally almost sure exponentially stable* [45, Def. 3.1] if there exists $\varepsilon > 0$ such that

$$\limsup_{t\to+\infty} \mathrm{Lyap}(t, z(t_0)) < 0 \text{ almost surely,} \tag{3}$$

for all $z(t_0) : \|z(t_0)\| < \varepsilon$, while it is *globally almost sure exponentially stable* if (3) holds for all $z(t_0) \in \mathbb{R}^n$.

Note that the $\limsup$ and the strict inequality in (3) grant that the norm of $z(t)$ (almost surely) vanishes at least exponentially, and therefore the computation of the left-hand side of inequality (3) can be used to estimate the rate of convergence toward the trivial solution of (1).

B. Pinning control of deterministic complex networks

Here, we consider a controlled complex network of $N > 1$ identical dynamical systems, diffusively coupled through an undirected and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are the set of the network nodes and edges, respectively. The network dynamics are described by

$$\mathrm{d}x_i(t) = \left(f(x_i(t), t) + \sigma\sum_{j=1}^{N} a_{ij}(h(x_j(t), t) - h(x_i(t), t)) + u_i(t)\right)\mathrm{d}t, \tag{4}$$

where $x_i \in \mathbb{R}^n$ is the state of node i, $f(x_i,t): \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ describes the individual dynamics of each node, $h(x_i,t): \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the coupling function, a_{ij} is the ij-th element of the adjacency matrix $A \in \mathbb{R}^{N\times N}$ associated to $\mathcal{G}$, and σ is the coupling strength. The scope of the external input u_i is to synchronize the network nodes onto a desired reference trajectory, denoted $x_r(t)$ from now on, which is a solution of the uncoupled dynamics

$$\mathrm{d}x_r(t) = f(x_r(t),t)\mathrm{d}t.$$

The pinning control strategy consists in a proportional feedback control action that only acts on the proper subset $\mathcal{P} \subset \mathcal{V}$ of *pinned nodes* that can be directly controlled, that is,

$$u_i(t) = -p_i k\big(h(x_i(t),t) - h(x_r(t),t)\big),$$

where $k>0$ is the control gain, and $p_i = 1$ if $i \in \mathcal{P}$, while it is zero otherwise.

III. Stochastic pinning controllability

A. Stochastic complex network model

Here, we consider that network (4) is affected by two types of noise, one acting on the individual node dynamics and the other on the communication links. In the absence of coupling and control inputs, the individual dynamics of the i-th node are described by

$$\mathrm{d}x_i(t) = f(x_i(t),t)\mathrm{d}t + \sigma_m g(x_i(t),t)\mathrm{d}b_m(t), \tag{5}$$

where the stochastic term $\sigma_m g(x_i(t),t)\mathrm{d}b_m(t)$ represents modeling uncertainties on the individual dynamics. Specifically, b_m is a scalar Wiener process representing the *mismodeling noise*, $g: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the diffusion function determining how the Wiener increment $\mathrm{d}b_m$ propagates on the state of the i-th node, and σ_m modulates its variance.

When coupled and controlled, in writing the dynamics of the i-th node we also need to account for the possible uncertainties in the communication protocol. Therefore, in the presence of both mismodeling and communication noise, the overall network dynamics are described by the following nonlinear stochastic differential equations of Itô type:

$$\begin{aligned}\mathrm{d}x_i(t) = &\Bigg(f(x_i(t),t) + \sigma\sum_{j=1}^{N} a_{ij}\Big(h(x_j(t),t) - h(x_i(t),t)\Big)\\ &- p_i k\Big(h(x_i(t),t) - h(x_r(t),t)\Big)\Bigg)\mathrm{d}t + \sigma_m g(x_i(t),t)\mathrm{d}b_m(t)\\ &+ \sigma_c\Bigg(\sum_{j=1}^{N} a_{ij}\Big(h(x_j(t),t) - h(x_i(t),t)\Big) - p_i k\Big(h(x_i(t),t)\\ &- h(x_r(t),t)\Big)\Bigg)\mathrm{d}b_c(t),\end{aligned} \tag{6}$$

for all $i \in \mathcal{V}$, where the last addend models possible uncertainties on the communication, with b_c being a one-dimensional Wiener process, independent of b_m, representing the *communication noise*, and σ_c modulating its intensity. To enforce the existence of a unique global solution, throughout the manuscript we assume that functions f, g, and h fulfill the assumptions of Lemma 1.

As in the deterministic case, pinning control assumes that the reference trajectory $x_r(t)$ is a solution of the uncoupled dynamics (5). To provide a definition of pinning controllability for the controlled network (6), we need to introduce the control error $e(t) = [e_1^T(t),\dots,e_N^T(t)]^T$, with $e_i(t) := x_i(t) - x_r(t)$. The error dynamics can be written as

$$\begin{aligned}\mathrm{d}e_i = &\Bigg(f(x_r+e_i) - f(x_r) + \sigma\sum_{j=1}^{N} a_{ij}\Big(h(x_r+e_j)\\ &- h(x_r+e_i)\Big) - p_i k\Big(h(x_r+e_i) - h(x_r)\Big)\Bigg)\mathrm{d}t\\ &+ \sigma_m\Big(g(x_r+e_i) - g(x_r)\Big)\mathrm{d}b_m\\ &+ \sigma_c\Bigg(\sigma\sum_{j=1}^{N} a_{ij}\Big(h(x_r+e_j) - h(x_r+e_i)\Big)\\ &- p_i k\Big(h(x_r+e_i) - h(x_r)\Big)\Bigg)\mathrm{d}b_c,\end{aligned} \tag{7}$$

for $i = 1,\dots,N$, where we omitted the explicit dependence on time t. Noting that $e(t) = 0$ is the trivial solution of (7), we are ready to give the following definition:

Definition 3. System (6) is *exponentially pinning controllable in a stochastic sense* onto a solution $x_r(t)$ of (5) if there exist a value of the control gain k, a set of pinned nodes $\mathcal{P}$, and a positive scalar ε such that

$$\limsup_{t\to+\infty}\frac{1}{t}\mathrm{Lyap}(t,e(0)) < 0 \text{ almost surely} \tag{8}$$

for all $e(0): \|e(0)\| < \varepsilon$.

B. Pinning controllability analysis

A direct application of the definition of exponential pinning controllability would require the computation of the sample Lyapunov exponent of the set of stochastic equations (7). However, this could be computationally prohibitive for large networks, as the order of the error dynamics would be nN, thus scaling with the network size. Moreover, even if such a computation were performed, this would only assess whether our control goal can be achieved with the current selection of the control gain k and of the set of pinned nodes $\mathcal{P}$. A negative result would not provide any insight on how to tune the control parameters to enforce pinning controllability.

In what follows, we derive a necessary and sufficient condition for exponential pinning controllability that overcomes the above limitations. Specifically, we aim at broadening the Master Stability Function approach firstly applied to pinning control of deterministic networks in [26] to deal with the presence of mismodeling and communication noise. Toward this goal, we can linearize the error dynamics (7) at the origin, and write them in compact matrix as

$$\begin{aligned}\mathrm{d}e(t) = &\big(\mathbb{I}_N \otimes f_x(x_r(t),t) - \mathcal{M}\otimes h_x(x_r(t),t)\big)e(t)\mathrm{d}t\\ &+ \sigma_m\big(\mathbb{I}_N \otimes g_x(x_r(t),t)\big)e(t)\mathrm{d}b_m(t)\\ &- \sigma_c\big(\mathcal{M}\otimes h_x(x_r(t),t)\big)e(t)\mathrm{d}b_c(t),\end{aligned} \tag{9}$$

where $\otimes$ is the Kronecker product, $\mathbb{I}_N$ is the $N\times N$ identity matrix, f_x, h_x, and g_x are the Jacobian of f, h, and g, respectively, and the matrix $\mathcal{M}$ is the augmented Laplacian matrix defined as

$$\mathcal{M} = \begin{bmatrix} \sigma d_1 + p_1 k & -\sigma a_{12} & \dots & -\sigma a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma a_{N1} & -\sigma a_{N2} & \cdots & \sigma d_N + p_N k \end{bmatrix}$$

where d_i is the degree of node i.

Let us now consider the transformation T that diagonalizes $\mathcal{M}$, (*i.e.,* $T\mathcal{M}T^{-1} = \Lambda$, with Λ being the diagonal matrix containing the eigenvalues of $\mathcal{M}$) and define a transformed variable $\xi(t) := (T \otimes \mathbb{I}_n)e$. This transformation does not affect the terms $\mathbb{I}_N \otimes f_x(x_s(t), t)$ and $\mathbb{I}_N \otimes g_x(x_s(t), t)$ in equation (9). Hence, the dynamics of $\xi(t)$ can be obtained from (9) by applying the Itô's formula as follows:

$$\begin{aligned} \mathrm{d}\xi(t) = &\big(\mathbb{I}_N \otimes f_x(x_r(t), t) - \Lambda \otimes h_x(x_r(t), t)\big)\xi(t)\mathrm{d}t \\ &+ \sigma_m \big(\mathbb{I}_N \otimes g_x(x_r(t), t)\big)\xi(t)\mathrm{d}b_m(t) \\ &- \sigma_c \big(\Lambda \otimes h_x(x_r(t), t)\big)\xi(t)\mathrm{d}b_c(t). \end{aligned} \tag{10}$$

The dynamics matrix of system (10) is in block-diagonal form. Namely, it is composed by N diagonal blocks of size $n \times n$, with each block being associated to an eigenvalue of $\mathcal{M}$. Being $\mathcal{G}$ undirected and connected, if at least one node is pinned, these eigenvalues are all real and positive [28], and can then be sorted in ascending order as

$$0 < \lambda_1(\mathcal{M}) \leq \lambda_2(\mathcal{M}) \leq \ldots \leq \lambda_N(\mathcal{M}).$$

We can now rewrite (10) as the following set of n-dimensional stochastic differential equations:

$$\begin{aligned} \mathrm{d}\xi_i(t) = &\big(f_x(x_r(t), t) - \lambda_i(\mathcal{M})h_x(x_r(t), t)\big)\zeta(t)\mathrm{d}t \\ &+ \sigma_m \big(g_x(x_r(t), t)\big)\xi_i(t)\mathrm{d}b_m(t) \\ &- \lambda_i(\mathcal{M})\sigma_c h_x(x_r(t), t)\xi_i(t)\mathrm{d}b_c(t), \end{aligned} \tag{11}$$

for $i = 1, \ldots, N$. By replacing λ_i with a nonnegative scalar η, we define the *stochastic master stability equation* as the following parametric equation:

$$\begin{aligned} \mathrm{d}\zeta(t) = &\big(f_x(x_r(t), t) - \eta h_x(x_r(t), t)\big)\zeta(t)\mathrm{d}t \\ &+ \sigma_m \big(g_x(x_r(t), t)\big)\zeta(t)\mathrm{d}b_m(t) \\ &- \eta\sigma_c h_x(x_r(t), t)\zeta(t)\mathrm{d}b_c(t). \end{aligned} \tag{12}$$

The *stochastic master stability function* $\mathrm{SMSF}(\eta)$ is the function that associates to each value of η the sample Lyapunov exponent associated to the trivial solution $\zeta(t) = 0$ of (12). We are now ready to provide our main stability result.

Theorem 1. *Network* (6) *is exponentially pinning controllable in a stochastic sense onto the solution* $x_r(t)$ *of* (5) *if and only if there exist a value of the control gain k, a set of pinned nodes* $\mathcal{P}$*, and a positive scalar* ε *such that* $\mathrm{SMSF}(\lambda_i(\mathcal{M})) < 0$ *for all* $i = 1, \ldots, N$.

Proof. Notice that, by direct comparison of equations (11) and (12), $\mathrm{SMSF}(\lambda_i(\mathcal{M})) < 0$ for all $i = 1, \ldots, N$ is equivalent to the local almost sure exponential stability of $\xi(t)$, see Definition 2. Since $e(t) = (T^{-1} \otimes \mathbb{I}_n)\xi(t)$, and from Definition 3, the thesis follows. □

The above necessary and sufficient condition highlights the contribution of each term of the stochastic network dynamics to pinning controllability. In particular, we observe that the SMSF depends on the individual dynamics through the Jacobian f_x and g_x of the drift and diffusion functions, respectively, and on the coupling function through its Jacobian h_x. The network topology, together with the selection of the set of pinned nodes $\mathcal{P}$ and the control gain k, impacts on the pinning controllability through the eigenvalues of the augmented Laplacian $\mathcal{M}$.

Remark 1. An intrinsic advantage of the proposed stochastic master stability function approach is computational. Indeed, independent of the network size, checking pinning controllability requires to compute the sample Lyapunov exponents of the n-dimensional system (12) for selected values of η, rather then studying the stability of the nN-dimensional error system (7).

Remark 2. The diffusion function g appears in the stochastic master stability equation (12) only through its Jacobian g_x. This means that a mismodeling noise whose intensity is independent on the value of the state does not affect the exponential pinning controllability of network (6).

Remark 3. In the absence of noise, that is, when $\sigma_m = \sigma_c = 0$, the SMSF coincides with the Master Stability Function (MSF) for deterministic networks. The value of the MSF when $\eta = 0$ is the maximum Lyapunov exponent of the uncoupled and uncontrolled system. This implies that, in a deterministic setting, if the MSF is negative at the origin, then the pinner and the uncontrolled network are both locally converging towards the same equilibrium point, and this is the only case in which an uncontrolled network would converge toward x_r. On the contrary, in the presence of noise this is not necessarily true, and a negative value of the SMSF at $\eta = 0$ does not imply that the pinner is converging towards an equilibrium point, as the individual dynamics might not admit any equilibrium. This may determine the so-called *noise-induced synchronization* of an uncontrolled network, a phenomenon that is discussed in Section IV-A.

The case of affine stochastic networks

When the drift and output functions f and g are both affine, we can provide two relevant corollaries of Theorem 1. For clarity, we start with the simplest case of networks of scalar linear systems, and then we generalize the results to n-dimensional affine dynamics.

Linearly coupled scalar linear systems: in this case, the individual dynamics are given by

$$\mathrm{d}x_i(t) = a x_i(t)\mathrm{d}t + \sigma_m x_i(t)\mathrm{d}b_m(t), \tag{13}$$

where $a \in \mathbb{R}$, while the stochastic complex network (6) can be rewritten as

$$\begin{aligned} \mathrm{d}x_i(t) = &\Big(a x_i(t) + \sigma \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) \\ &- p_i k(x_i(t) - x_r(t))\Big)\mathrm{d}t + \sigma_m x_i(t)\mathrm{d}b_m(t) \\ &+ \sigma_c\Big(\sigma \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) \\ &- p_i k(x_i(t) - x_r(t))\Big)\mathrm{d}b_c(t). \end{aligned} \tag{14}$$

Corollary 1. *Network* (14) *is exponentially pinning controllable in a stochastic sense onto the solution* $x_r(t)$ *of* (13) *if and only if there exist a value of the control gain* k*, and a set of pinned nodes* $\mathcal{P}$ *such that*

$$2a-\sigma_m^2<0 \vee \lambda_1(M)>\left(\sqrt{1+(2a-\sigma_m^2)\sigma_c^2}-1\right)/\sigma_c^2 \quad (15)$$

Proof. The SMSF for network (14) is the sample Lyapunov exponent of the following linear stochastic differential equation:

$$\mathrm{d}\zeta(t)=(a-\eta)\zeta(t)\mathrm{d}t+\sigma_m\zeta(t)\mathrm{d}b_m(t)-\eta\sigma_c\zeta(t)\mathrm{d}b_c(t). \quad (16)$$

Next, we show that the unique solution of (16) is

$$\zeta(t)=\zeta(0)\exp(z(t)), \quad (17)$$

where

$$\begin{aligned}z(t)=&\left(a-\eta-\frac{\sigma_m^2}{2}-\frac{\eta^2\sigma_c^2}{2}\right)t+\sigma_m\left(b_m(t)-b_m(0)\right)\\&-\eta\sigma_c(b_c(t)-b_c(0)).\end{aligned} \quad (18)$$

Indeed, as $\exp(z(t))=1$, for $t=0$ we find the identity $\zeta(0)=\zeta(0)$. Now, notice that from (18) we get

$$\mathrm{d}z(t)=\left(a-\eta-\frac{\sigma_m^2}{2}-\frac{\eta^2\sigma_c^2}{2}\right)\mathrm{d}t+\sigma_m\mathrm{d}b_m(t)-\eta\sigma_c\mathrm{d}b_c(t). \quad (19)$$

Then, applying the multi-dimensional Itô's formula [45, Theorem 6.4] to (17), and considering (19), we obtain

$$\begin{aligned}\mathrm{d}\zeta(t)=&\left(\frac{\partial\zeta(t)}{\partial z}\left(a-\eta-\frac{\sigma_m^2}{2}-\frac{\eta^2\sigma_c^2}{2}\right)\right.\\&\left.+\frac{1}{2}\frac{\partial^2\zeta(t)}{\partial z^2}\left(\sigma_m^2+\eta^2\sigma_c^2\right)\right)\mathrm{d}t\\&+\frac{\partial\zeta(t)}{\partial z}(\sigma_m\mathrm{d}b_m(t)-\eta\sigma_c\mathrm{d}b_c(t)).\end{aligned} \quad (20)$$

Noting that $\partial\zeta(t)/\partial z=\partial^2\zeta(t)/\partial z^2=\zeta(t)$, we obtain that (20) is equivalent to (16), and therefore (17) is a solution of (16). The solution is unique as the stochastic equation (16) fulfills the assumptions of Lemma 1. Therefore, by combining (17) and (18), we find that the stochastic master stability function of (14) is

$$\mathrm{SMSF}(\eta)=-\frac{\sigma_c^2}{2}\eta^2-\eta+a-\frac{\sigma_m^2}{2}. \quad (21)$$

From Theorem 1, the thesis follows. □

The deterministic MSF for this system can be viewed as a specific instance of (21) when $\sigma_m=\sigma_c=0$, and it is therefore a line with slope -1 that starts at a for $\eta=0$, *i.e.,* a Type II MSF. The introduction of the mismodeling noise has the effect of shifting downwards the MSF, while the communication noise has the effect of turning the straight line into a downward parabola, as illustrated in Figure 1. This means that, for the linear network (14), noise can only be beneficial for pinning controllability.

We notice that the mismodeling noise acting on all the nodes of the linear network (14) has a stabilizing effect. Indeed, as long as $\sigma_m^2/2>a$, the pinner trajectory converges to zero with probability one even when the drift component of the system is unstable ($a>0$). This property can be well explained by interpreting the noisy system as a Markov chain: as long as the noise intensity is sufficiently large, the unique absorbing state of the chain is the trivial solution, that will eventually be reached. In this case, the network is exponentially pinning controllable regardless of the coupling configuration and even if it is not pinned. This happens as a stochastic stability function that is negative at the origin ($\mathrm{SMSF}(0)<0$) implies that all the nodes eventually reach zero even if disconnected from the pinner.

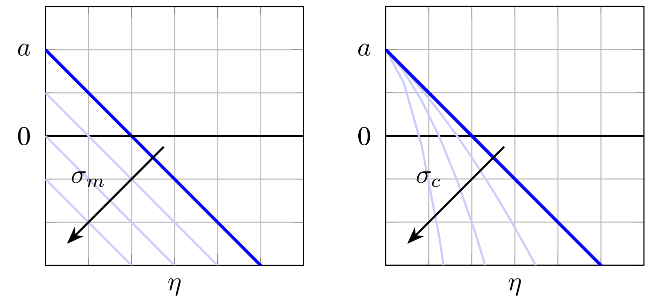

Fig. 1. (Left panel) SMSF of the linear network (14) for $\sigma_c=0$ and σ_m equal to $\sqrt{a}$, $\sqrt{2a}$, and $\sqrt{3a}$, respectively (grey lines, left panel), and for $\sigma_m=0$ and σ_c equal to $\sqrt{a}/2$, $\sqrt{a}$, and $2\sqrt{a}$, respectively (grey lines, right panel). In both panels, the blue line identifies the deterministic MSF and the black arrow points towards increasing values of the noise.

The effect of the noise on the communication is less trivial. Indeed, it does not affect the pinner and, therefore, if $a>0$ the network cannot synchronize towards the pinner's trajectory when uncontrolled. Nonetheless, by increasing the communication noise it becomes easier to pinning control the network, as smaller values of $\lambda_1(\mathcal{M})$ are required to fulfill condition (15), see the right panel in Figure 1.

Linearly coupled n*-dimensional affine systems:* in this case, the individual dynamics are given by

$$\mathrm{d}x_i(t)=(Fx_i(t)+b)\,\mathrm{d}t+\sigma_m\,(x_i(t)+c)\,\mathrm{d}b_m(t), \quad (22)$$

where $F\in\mathbb{R}^{n\times n}$, and $b,c\in\mathbb{R}^n$, and the stochastic complex network (6) can be rewritten as

$$\begin{aligned}\mathrm{d}x_i(t)=&\Big(Fx_i(t)+b+\sigma\sum_{i=1}^N a_{ij}(x_j(t)-x_i(t))\\&-p_ik(x_i(t)-x_r(t))\Big)\mathrm{d}t+\sigma_m(x_i(t)+c)\mathrm{d}b_m(t)\\&+\sigma_c\Big(\sigma\sum_{i=1}^N a_{ij}(x_j(t)-x_i(t))\\&-p_ik(x_i(t)-x_r(t))+d\Big)\mathrm{d}b_c(t).\end{aligned} \quad (23)$$

Corollary 2. *Network* (23) *is exponentially pinning controllable in a stochastic sense onto the solution* $x_r(t)$ *of* (22) *if and only if there exist a value of the control gain* k*, and a set of pinned nodes* $\mathcal{P}$ *such that*

$$\lambda_1(\mathcal{M})+\frac{1}{2}\sigma_c^2\lambda_1^2(\mathcal{M})>\max\left(\Re\left(\mathrm{eig}\left(F\right)\right)\right)-\frac{1}{2}\sigma_m^2,$$

where $\Re$ *is the real-part operator and* $\mathrm{eig}(F)$ *is the spectrum of* F*.*

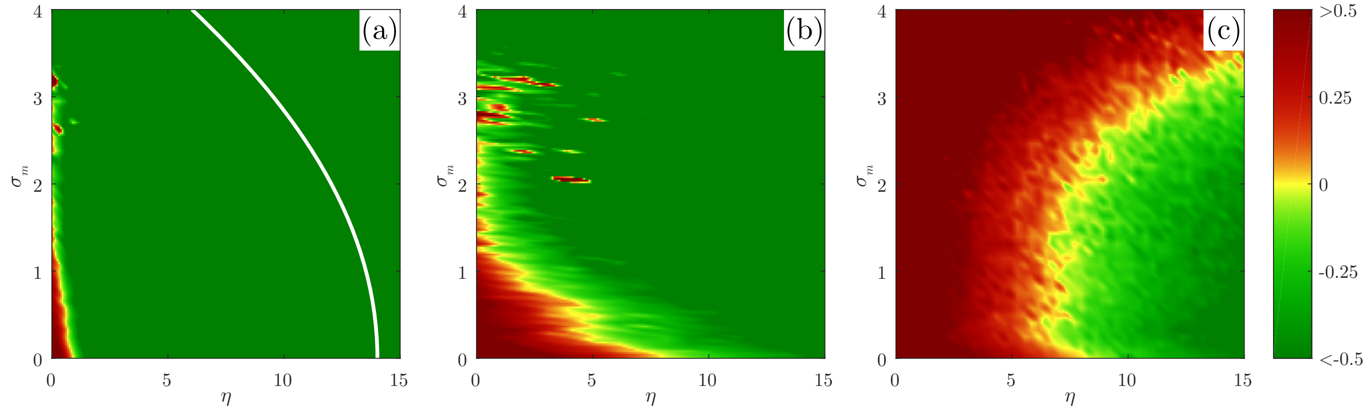

Fig. 2. Stochastic Master Stability Function of the network of stochastic Lorenz systems (6)-(24) for $\sigma_c = 0$ and $\sigma_m \in [0, 4]$ when $h(x_i, t) = x_i, g(x_i, t) = x_i$ (a), $h(x_i, t) = [x_{i1}\ 0\ 0]^T, g(x_i, t) = x_i$ (b), and $h(x_i, t) = [x_{i1}\ 0\ 0]^T, g(x_i, t) = [0\ x_{i3}\ 0]^T$ (c). In panel (a), the points on the left of the white line do not fulfil the conditions for global pinning controllability derived in [44].

Proof. Following the same line of arguments as in the proof of Corollary 1, we can compute the SMSF as

$$\mathrm{SMSF}(\eta) = \max\left(\Re\left(\mathrm{eig}\left(F\right)\right)\right) - \eta - \frac{1}{2}\sigma_m^2 - \frac{1}{2}\sigma_c^2\eta^2.$$

From Theorem 1, the thesis follows. □

IV. NUMERICAL EXAMPLES

Here, we illustrate how our theoretical findings can be used to assess the impact of noise on pinning controllability in the case of nonlinear drift functions. To this aim, we consider as individual dynamics a paradigmatic example of nonlinear noisy system, that is, a stochastic version of the well-known Lorenz system, which was used as a testbed in the recent literature on stochastic complex networks [40], [44]. Namely, in the general equation (6), we consider the following drift function [46]:

$$f(x_i(t), t) = \begin{bmatrix} s(x_{i2}(t) - x_{i1}(t)) \\ sx_{i1}(t) - x_{i2}(t) - x_{i1}(t)x_{i3}(t) \\ x_{i1}(t)x_{i2}(t) - \beta(x_{i3}(t) + \rho + s) \end{bmatrix}, \quad (24)$$

where $\beta = 8/3$, $\rho = 28$, and $s = 10$ are selected so that, in the absence of noise, coupling and control, the dynamics of each node admits a chaotic attractor. In all our numerical analyses, the simulations of the stochastic differential equations are performed with the standard Euler-Maruyama weak integrator [47] with a time step of 10^{-4}. The transversal sample Lyapunov exponent is computed through an optimized version of the heuristic discrete QR method described in [48], which is easy to implement and numerically efficient.

In what follows, we first evaluate separately the impact of mismodeling and communication noise on pinning controllability, respectively, and then illustrate how the control gains can be tuned in the presence of both noise types. When discussing our numerical results, we will refer to the *stability set* $\mathcal{S}$ of the SMSF as the set of values $\eta > 0$ such that the SMSF is negative, that is,

$$\mathcal{S} := \{\eta \geq 0 : \mathrm{SMSF}(\eta) < 0\}$$

A. Effect of the mismodeling noise ($\sigma_m \neq 0$, $\sigma_c = 0$)

Here, we discuss how the impact of mismodeling noise may change dramatically with different selections of the diffusion function g. In our first numerical experiment, we consider the case in which both the coupling and diffusion functions are the identity, that is, $h(x_i, t) = x_i$ and $g(x_i, t) = x_i$. In this scenario, we observe that the mismodeling noise generally improves pinning controllability, with the stability set tending to become larger and larger as σ_m increases, see the green region in Figure 2(a). Interestingly, when the mismodeling noise is sufficiently high, we notice that the SMSF becomes negative even for $\eta = 0$. This means that all the nodes converge towards the trajectory of the pinner even when uncoupled. In a deterministic setting, this would only be possible when all the nodes are converging towards a common (asymptotically stable) equilibrium point. In the presence of mismodeling noise, the nodes can converge toward a nontrivial (e.g. chaotic) pinner trajectory.

To illustrate this paradoxical phenomenon, called *noise-induced synchronization* [49]–[55], we considered the simplest network of $N = 2$ uncoupled nodes that are also disconnected from the pinner, that is, $\mathcal{P} = \emptyset$. As illustrated in Figure 3, the trajectory of the pinner converges towards a noisy counterpart of the Lorenz chaotic attractor, and the pinning error norm asymptotically goes to zero, even though the network is uncontrolled. Figure 2(a) also highlights the unavoidable conservativeness of the sufficient global conditions reported in [44]. As an example, when $\sigma_m = 1.72$, our results show that exponential pinning controllability is feasible also for a disconnected network, while to fulfill the assumptions of Theorem 4.3 in [44], we would need a coupling configuration such that $\lambda_1(\mathcal{M}) > 12.43$.

Next, we focus on a case in which the function h is not quadratic-Lipschitz ($h(x_i, t) = [x_{i1}\ 0\ 0]^T$), and therefore the sufficient conditions for pinning controllability reported in [44] cannot be applied. In Figure 2(b) we observe that, as long as the mismodeling noise equally affects all the state variables, it generally facilitates the achievement of the control goal. On the contrary, in panel (c), we notice that, when it only acts on a subset of the state variables ($g(x_i, t) = [0\ x_{i3}\ 0]^T$), it

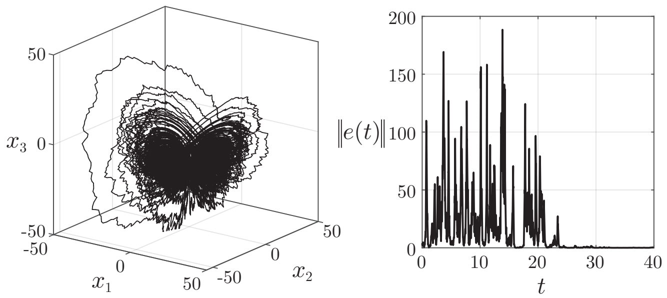

Fig. 3. Simulation of $N = 2$ uncoupled and uncontrolled stochastic Lorenz systems when $\sigma_m = 1.72$. Phase portrait of the pinner dynamics (left panel) and time evolution of the pinning error norm (right panel).

becomes detrimental for pinning controllability. More specifically, although low noise seems to be moderately beneficial for control, as the noise intensity further increases, the stability set $\mathcal{S}$ becomes smaller and smaller, and exponential pinning controllability becomes unfeasible for $\sigma_m > 3.71$.

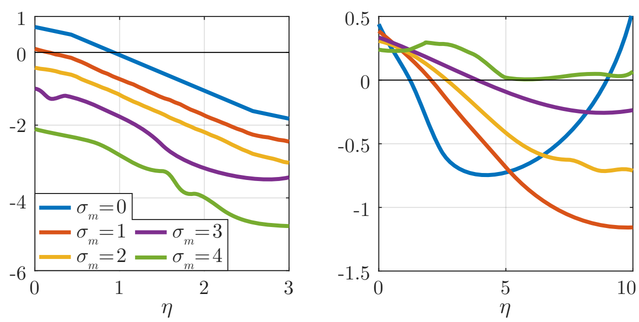

Fig. 4. Stochastic Master Stability Function associated to the network of stochastic Lorenz system (6)-(24) for $\sigma_c = 0$ and selected values of the noise intensity σ_m when $h(t, x_i) = g(t, x_i) = x_i$ (left panel) and $h(t, x_i) = g(t, x_i) = [0\ 0\ x_{i3}]^T$ (right panel).

Overall, we observe that the effect of the mismodeling noise, when it diffuses through the identity ($g(x_i, t) = x_i$), is to shift downward the SMSF, thus enhancing pinning controllability. This is analytically proved for stochastic networks of linear systems in Section III, and numerically demonstrated in the left panel of Figure 4 for a network of stochastic Lorenz systems. On the other hand, when the noise diffuses unevenly across the state variables, we noticed that it can be either beneficial or detrimental. The right panel of Figure 4 shows the SMSF for the the stochastic complex network (6)-(24) when $h(x_i, t) = g(x_i, t) = [0\ 0\ x_{i3}]^T$. In this case, the SMSF is not shifted downward, but it is rather flattened. For low values of σ_m this renders the stability set unbounded by paying the price of an increase of the minimum η such that the SMSF is negative. As the noise further increases, the SMSF becomes flatter and flatter, until $\mathrm{SMSF}(\eta) > 0$ for all η ($\mathcal{S} = \emptyset$), thus making convergence toward $x_r(t)$ impossible.

B. Effect of the communication noise ($\sigma_m = 0$, $\sigma_c \neq 0$)

We now discuss the impact of the communication noise σ_c on the stochastic master stability function in the absence of mismodeling noise. Notice that the communication noise acts on both the links between the controlled network nodes and those connecting the pinner with the network, and therefore existing results on pinning controllability of stochastic networks cannot be directly applied [44]. In our numerical investigation, we evaluate the SMSF of network (6)-(24) when the coupling function is the identity, and therefore the communication noise affects all the state variables, and when the nodes only communicate through the first state variable. As illustrated in the left panel of Figure 5, when $h(x_i, t) = x_i$, in the absence of noise the SMSF linearly decreases with η. As the communication noise increases, pinning controllability improves, with the SMSF decreasing faster and faster with η.

A qualitatively different behavior is observed when $h(x_i, t) = [x_{i1}\ 0\ 0]^T$. In the absence of noise, this choice of the coupling function yields a monotone decreasing MSF (see Figure 5, right panel), which, according to the classification in [56], is called of type II. As the communication noise increases, the monotonicity is lost, and thus the SMSF becomes of type III. More specifically, by increasing σ_c, the stability set $\mathcal{S}$ becomes a finite interval, whose width gets smaller and smaller. On the other hand, the minimum η for which the SMSF is negative reduces. This means that, with this coupling function, noise can be either detrimental or beneficial for pinning controllability depending on the network topology $\mathcal{G}$, the set of pinned nodes $\mathcal{P}$, and the coupling and control gains σ and k, respectively.

While the negative effect of the noise in this coupling

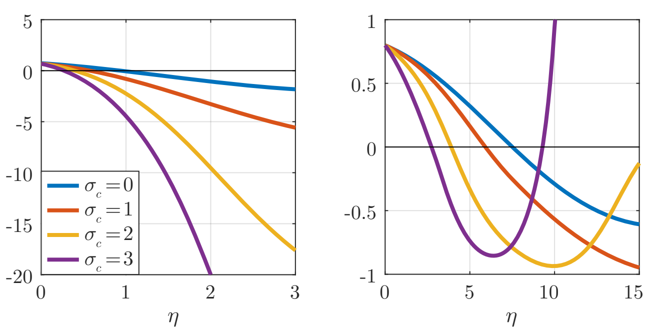

Fig. 5. Stochastic Master Stability Function associated to the network of stochastic Lorenz system (24) for $\sigma_m = 0$ and selected values of the noise intensity σ_c when $h(t, x_i) = x_i$ (left panel) and $h(t, x_i) = [x_{i1}\ 0\ 0]^T$ (right panel).

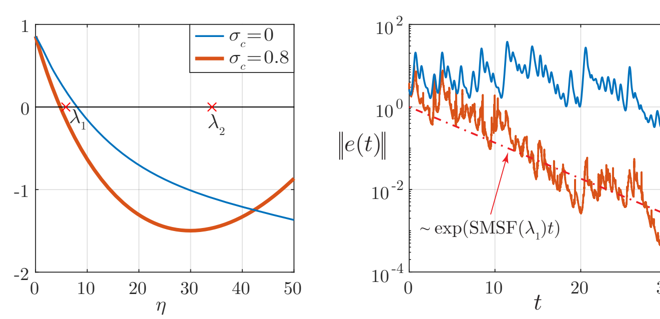

Fig. 6. Network (6)-(24) when $N = 2$ and $\mathcal{P} = \{1\}$. SMSF (left panel) and time evolution of the pinning error norm (right panel) in the absence of noise (blue lines) and when $\sigma_m = 0$ and $\sigma_c = 0.8$ (orange lines).

configuration is apparent as it turns a type II MSF into a type III SMSF, we introduce a simple example to illustrate that it may also be beneficial for pinning controllability. Namely, we consider $N = 2$ coupled nodes with $\mathcal{P} = \{1\}$, and assume the coupling and control gains to be $\sigma = 15$ and $k = 25$, respectively. As a result, the two eigenvalues of the augmented Laplacian are $\lambda_1(\mathcal{M}) = 5.9$ and $\lambda_2(\mathcal{M}) = 34.1$. Figure 6 illustrates how, in the absence of noise, the assumption of Theorem 1 are not met and the error does not converge towards zero, while the introduction of a moderate communication noise ($\sigma_c = 0.8$) yields exponential pinning controllability.

C. Control design examples

To illustrate how our necessary and sufficient condition for exponential pinning controllability can be used for tuning the control gain k, we focus on a network of $N = 3$ Lorenz systems with $\mathcal{P} = \{1\}$, coupled as depicted in the left panel of Figure 7. We select the coupling and diffusion functions as $h(x_i, t) = g(x_i, t) = [0\ 0\ x_{i3}]^T$, and assume the coupling gain to be $\sigma = 10$. The variances σ_m and σ_c of the mismodeling and communication noises are both set to 0.5.

From Theorem 1, local exponential pinning controllability is guaranteed if $\mathrm{SMSF}(\lambda_i(\mathcal{M})) < 0$ for all i. Figure 7 shows that $\mathrm{SMSF}(\eta) < 0$ for all $\eta > 1.78$. Therefore, to enforce pinning controllability, we need to select k so that $\lambda_1(\mathcal{M}) > 1.78$, with $\mathcal{M}$ in this case being

$$\begin{bmatrix} 10-k & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix}$$

Notice that, by selecting $k > 8.30$, we obtain $\lambda_1(\mathcal{M}) > 1.78$ as prescribed by Theorem 1. Hence, we set $k = 20$, thus getting $\lambda_1(\mathcal{M}) = 2.68$ and $\mathrm{SMSF}(\lambda_1(\mathcal{M})) = -0.21$. Figure 8 illustrates that, consistently with the theoretical predictions, the choice $k = 20$ yields an exponential convergence of the pinning error norm to zero, with a rate that is approximately equal to $\mathrm{SMSF}(\lambda_1(\mathcal{M}))$.

Note that the same method to tune the control gains can be employed for networks of arbitrary size, while maintaining the computational burden negligible. To illustrate this point, we considered the randomly generated network of $N = 1000$ nodes reported in the left panel of Figure 9, where the pinned nodes (30% of the nodes randomly selected) are depicted in black. Assuming that the coupling strength is $\sigma = 0.1$, we

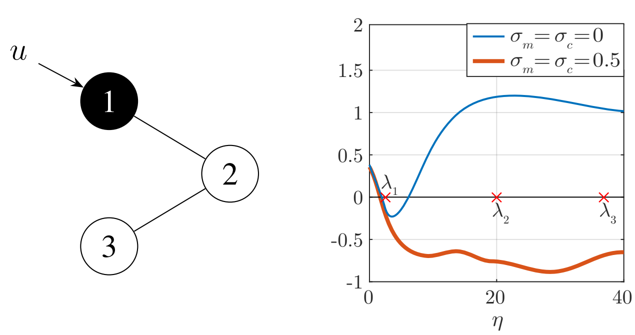

Fig. 7. First control design example. Topology (left panel) and SMSF of the network (6)-(24) when $h(x_i, t) = g(x_i, t) = [0\ 0\ x_{i3}]^T$ (right panel).

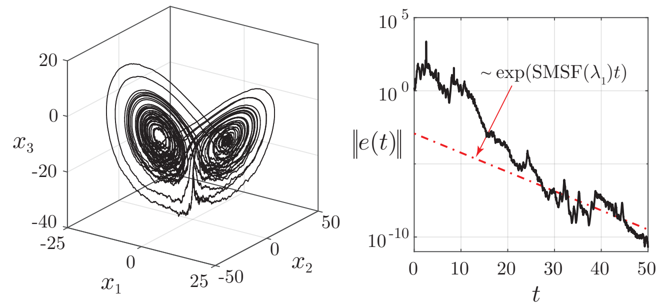

Fig. 8. First control design example. Pinner trajectory (left panel) and time evolution of the pinning error norm (right panel).

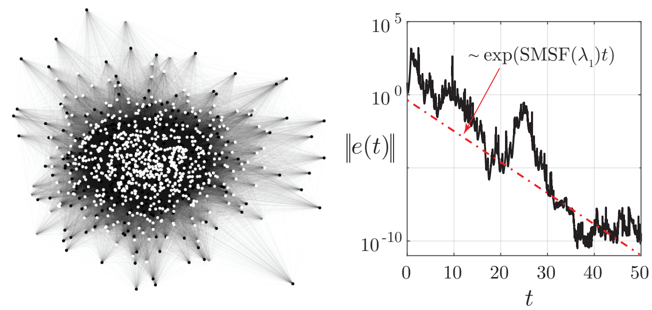

Fig. 9. Second control design example. Network topology generated with the Watts-Strogatz algorithm [57] with complete rewiring and average degree 100 (left panel) and time evolution of the pinning error norm (right panel).

can enforce pinning controllability by selecting $k \in (7.8, 24]$, which yields $\lambda_1(\mathcal{M}) > 1.78$ and $\lambda_N(\mathcal{M}) < 40$. Indeed, this implies that $\mathrm{SMSF}(\lambda_i(\mathcal{M})) < 0$ for all i, see the orange line in Figure 7. Note that computing the eigenvalues of matrix $\mathcal{M}$ only required 0.1s with our laptop (i7-4810MQ CPU @ 2.8 GHz), this being the time required to assess pinning controllability for a given selection of k. Performing a single simulation with $k = 20$ (reported in the right panel of Fig. 9) took 2543 s. Thus, a brute force trial-and-error approach to tune the control gain k based on simulations would not be practically viable.

V. CONCLUSIONS

In this paper, we derived necessary and sufficient conditions for exponential pinning controllability of stochastic complex networks. Our theoretical findings address the fundamental need in control of stochastic networks of assessing the impact of noise on our ability of driving the network toward a desired trajectory. Specifically, we expanded the master stability function approach to deal with the presence of mismodeling noise, accounting for the presence of unmodeled individual dynamics, and of the communication noise on the coupling protocol. A key contribution of our manuscript has been to provide the minimal conditions guaranteeing exponential pinning controllability in a stochastic sense. Indeed, the existing stability results provided sufficient conditions, and therefore nothing could be concluded when those conditions were not met. Furthermore, we showed that the existing pinning controllability conditions are very conservative, and could only be applied when the coupling function fulfils the quadratic-Lipschitz condition [44].

A previous study observed instances in which mismodeling noise is beneficial for pinning control, while communication noise was found to be detrimental [44]. As anticipated, the sharpness of our stability conditions allowed for better clarifying the impact of noise on pinning controllability. In particular, we found that the way noise affects the network dynamics is highly dependent on the coupling and diffusion functions. More specifically, mismodeling noise is beneficial only when it diffuses evenly on all the state variables, otherwise it may also make pinning controllability unfeasible independent of the network topology. When noise uniformly diffuses across the node state variables, a phenomenon called noise-induced synchronization may emerge: the network converges towards a non-trivial solution (e.g. a chaotic trajectory) of the pinner's dynamics even in the absence of coupling and control. This phenomenon has been already observed in biological systems, in which a suitable noise intensity facilitates synchronization of human brain waves [58].

As for the communication noise, we showed that it is not necessarily deleterious for synchronization. On the contrary, we analytically proved that for networks of affine stochastic systems it improves pinning controllability, and numerical evidence suggests that this is also true for nonlinear stochastic systems if the coupling function is the identity. When the network nodes are unevenly coupled on their state variables (e.g. they are coupled only through a subset of the state vector), then the impact of communication noise is less trivial. Indeed, the range of values for which the stochastic master stability function is negative, which we called the stability set, becomes smaller, but it shifts closer to the origin. This means that communication noise can have either a positive or negative effect on pinning controllability depending on the specific interconnection topology of the network we aim at controlling. Ongoing work [59] is devoted to explore how the method presented in this manuscript can be employed to explain the spontaneous emergence of a synchronous behavior in the absence of a control input.

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