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Stefania Ugolini *Editors*

Quantum and Stochastic Mathematical Physics

Sergio Albeverio, Adventures of a Mathematician, Verona, Italy, March 25–29, 2019

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To Sergio, our teacher and friend

Introduction

Sergio (Angelo, Ernesto) Albeverio was born on January 17, 1939 in Lugano, Ticino, Switzerland. His parents were Luigi and Olivetta (Brighenti) Albeverio.

In 1958 he started his studies in mathematics and physics at Eidgenössische Technische Hochschule (ETH) in Zürich, ending in 1962 with a Diploma (Master) in theoretical physics, in the area of statistical mechanics (generalized Ising models) under the supervision of Markus Fierz and David Ruelle.

During his Ph.D. in the "Seminar für Theoretische Physik" he worked on problems related to the axiomatic approach to quantum field theory. In 1966 he got the Dr. rer. nat. (Ph.D.) under the supervision of Res Jost and Markus Fierz with a work on a model of quantum mechanical scattering theory with singular interactions, while working as their Assistant.

At that time, ETH was still a small university with a limited number of students. About 6000 students were enrolled there and no more than 20 people were affiliated to the Theoretical Physics Seminar in Hochstrasse 60 in Zürich. Arthur Jaffe wrote a beautiful article about Hochstrasse 60.[1] He was touched by the unique charm of this place, which was also an exceptional working environment. Moreover, a paper published by IHES explained that Hochstrasse 60 was one out of three or four environments in the world where theoretical physics had an exceptional quality and it was strongly influenced by mathematical physics. Outstanding scientists worked there and had a deep impact on the student Sergio Albeverio. The first professor in theoretical physics at the ETH was Wolfgang Pauli. His successors were two of his former assistants, Markus Fierz (former coworker of Pauli, well known for his work in quantum field theory and general relativity) and Res Jost (well known for his work on scattering theory, the inverse problem, and axiomatic quantum field theory). At that time, David Ruelle (well known for his work on dynamical systems, statistical mechanics and scattering in axiomatic quantum field theory) also worked at the Seminar as a Privatdozent.

It was at ETH that Sergio first met Philippe Blanchard, who was a student in one of the courses he taught, and later Ludwig Streit. The *Seminar für Theoretische*

[1] See https://www.arthurjaffe.com/Assets/documents/ETH_First_Visit.htm.

Physik was a very international environment, not without its own multicultural Swiss charm. Sergio came from Ticino, the Italian-speaking part of Switzerland, Philippe Blanchard came from France, and Ludwig Streit from Austria. The three were to become long time collaborators and good friends; Blanchard is also the godfather of Albeverio's daughter Mielikki.

> *[P. Blanchard] "I want to mention the importance of Res Jost, an exceptional human, crucial for our scientific career and development. He was not only an excellent scholar but also a wonderful person."*

Philippe Blanchard describes the seminar as "Un endroi heureux et un espace extraordinaire". Hochstrasse 60 really was a place where one was happy and where exceptional things happened. The institute was located in a very narrow street that went downhill.

> *[P. Blanchard] "At one point I parked my car there in winter ... I put the handbrake on and went to the institute to discuss things with Res Jost. At some point, a couple of policemen came in and asked Jost if he knew a certain Philippe Blanchard ... [and said] that my car had gone downhill. Fortunately, nobody got killed or hurt. They wanted to confiscate my license and put me on trial. Jost ...told them that it did not make any sense to confiscate my license as ... my accident only happened because I did not park correctly and that consequently the only reasonable thing to do was to disallow me from parking. The policemen ... finally decided not to prosecute me."*

> *[L. Streit] "Fierz was Sergio's advisor for his thesis. He was also a "homme de lettres", fascinating, with a very distinct sense of humor, and always open. One day, while walking up from the main physics building to the lecture halls, Fierz asked me: "Do you know by any chance how one measures cosmological distances? Come to my office and I shall give you a private lecture about it". Actually, he could have given me a private lecture on a baroque author as well."*

> *[P. Blanchard] "Before his lectures, Fierz would often tell me that he hadn't had time to prepare the lecture and that although he wasn't sure what his lecture was going to be about, he had an idea. Then, when the lecture started, he would talk about what seemed to be hundreds-year-old physical problems. The lectures contained at least as much information as if he had prepared his course. Zurich really had incredibly talented people at that time."*

With Philippe Blanchard and Ludwig Streit, Sergio Albeverio shared his interest in Quantum Mechanics and the stochastic approach to Quantum Field Theory. They solved fundamental problems together, creating new scientific connections and schools in many countries, e.g. France, Japan and Portugal and, last but not least, Germany. Working as professors at Bielefeld University (from 1977 to 1979), they all started to realize their dream of developing a collaborating community between the institutes of Physics and Mathematics. With the appointment of Sergio as full Professor in Bochum in 1979 and later in Bonn in 1997, this project continued leading to the co-creation of the Institute BiBoS.

As Dr. rer. nat., Sergio spent the academic year 1967–1968 at Imperial College (IC) London, where he was a lecturer (giving a course on multi particle systems) at the invitation of Paul Matthews, Abdus Salam and Ray F. Streater. Ray motivated Sergio to study the works of James Glimm and Arthur Jaffe on the constructive theory of fields and to give lectures on this subject.

He started to do this with great enthusiasm, but due to the poor health of his parents he decided to return to Lugano, to be near to them, taking up a mathematics and physics teacher job at the local high school. Tragically, both parents passed away within few months, terrible blows for Sergio. His well-known capability of communicating complex problems to experienced as well as non-experienced scientists might depend not only on his generosity in sharing his thoughts and new ideas (which was so often mentioned when interviewing his colleagues) but also on this unexpected part of his life.

[T. Lindstrøm] "He's willing to share and he can explain things to you in a way that he sort of reasons with you. He can orientate his explanation to your own needs."

[M. Röckner] "When he started to give lectures, I was immediately fascinated by the way he explained the subject and I had to work very hard, because I wasn't able to understand many of the things he said. One thing I knew was that he had a deep knowledge of the topics. Moreover, the way he presented them was somehow motivating to go deeper into the subject."

In the fall of '69 Sergio left Lugano spending first a couple of months at the ETH Zürich, before crossing the ocean to work at Princeton University. He was accompanied by the artist Solvejg Manzoni, whom he married shortly after. Sergio spent the years 1970 and 1971 in the Departments of Mathematics and Physics as associate researcher with Arthur S. Wightman and Edward Nelson. There he met Barry Simon and Elliot Lieb, who definitely influenced him and who strengthened his interest in linking mathematics and physics, a task to which he dedicated his whole scientific life. Sergio was particularly influenced by Edward Nelson and his stochastic approach to quantum mechanics (stochastic mechanics theory) and quantum field theory. In the US he met other established and promising scientists, including Raphael Høegh-Krohn, who soon became one of his main friends and collaborators.

[F. Gesztesy] "Sergio came with a guest. Now you can almost guess who he was, he was Raphael Høegh-Krohn. We all know he is one of the longtime collaborators of Sergio's. Raphael was a bear of a man—imagine the biggest Russian bear—and he instantly broke the ice by putting his arm around my shoulders, which was quite impactful because in those days I was really shy and, I mean, that really changed everything."

The development of the stochastic approach to quantum physics they dealt with in Princeton motivated Sergio and Raphael to extend the theory of Dirichlet forms to infinite dimensions. This idea began to form when Sergio visited Raphael in Oslo in 1972 and later between 1974 and 1977. They understood the importance of Dirichlet forms, linking techniques coming from potential analysis to those of stochastics.

The intense collaboration of Sergio and Raphael in Oslo produced outstanding results in several areas of infinite dimensional analysis and mathematical physics, leading to important applications in quantum field theory. In those years they developed new theories that nowadays are associated to their names, such as the mathematical construction of Feynman path integrals, the study of point interactions, the development of non-standard analysis. The combined efforts of the two researchers, deeply linked by their common interests and ideals, although very different in character, culminated in their famous paper of 1977, in which they laid the foundations of Dirichlet forms in infinite dimensions. The novel techniques in this work found manifold applications, e.g. in quantum field theory and in the construction of Malliavin calculus. This intensive collaboration lasted until the sudden premature death of Raphael Høegh-Krohn in 1988.[2]

> *[S. Albeverio] "It was a particular constellation, with Raphael I had so much in common. In mathematics and physics we had a similar background, even if we came from different schools (Oslo and Aarhus Universities and Courant Institute for Raphael). We had namely the same taste and orientation, we did not have to discuss much to agree. I learned a lot from Raphael both in science and culture. I was also humanly enriched by the close contact with him. He was passionate, slept very little. Together we did the kind of very hard work you can only do when you are enthusiastic and in complete agreement."*

The environment at Oslo University was outstandingly rich and the discussions Sergio had with Jens Erik Fenstad, professor in logic—with whom Sergio shared an interest in philosophy too—and his Ph.D. student, Tom Lindstrøm, also led to new research and a joint book on non-standard analysis. Moreover, Sergio collaborated with a Norwegian student of Raphael's, Helge Holden, and a former student of Ludwig Streit's, Fritz Gesztesy, writing a joint book on solvable models in quantum mechanics. Even after leaving Oslo, Sergio shared with Raphael many students and coworkers.

During their period in Oslo, Sergio and Raphael also had an active exchange with the scientific community of the Soviet Union, in particular with Felix Berezin, Roland Dobrushin, Israel Gelfand, Robert Minlos, Yakov Sinai, Anatoly Vershik and Viktor Maslov. This was enhanced by many invitations to undertake research visits and participate in conferences and schools in the Soviet Union. The connection lasted many years and is still active in the new millennium with joint works with Minlos—on Ising models, Gibbs measures and quantum lattice systems—and with many other mathematicians from former Soviet Union. After moving to Germany in 1977, Sergio supported and collaborated with many researchers of the former Soviet Union in the framework of BiBoS, SFB-projects, von Humboldt grants as well as Volkswagen and DFG collaboration grants. Part of them later became professors in the UK, Germany and Scandinavia, and enriched mathematics and science in their later developments.

> *[A. Khrennikov] "[…] After this Sergio invited me to come for an Alexander von Humboldt fellowship and this opportunity was my way to science because otherwise I would not be a scientist anymore, since in Russia there was no possibility. Here I would like to underline that I was not alone. Sergio has really helped many, many scientists from Russia, Ukraine and all other republics of former Soviet Union in that really terrible time."*

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[2] For a description of the outstanding results of Raphael. Høegh-Krohn see the two volumes:

Ideas and Methods in Mathematical Analysis, Stochastics, and Applications: In Memory of Raphael Hoegh-Krohn (1938-1988), Volume 1.

Ideas and Methods in Quantum and Statistical Physics: In Memory of Raphael Hoegh-Krohn (1938–1988), Volume 2.

By S. Albeverio, H. Holden, J.E. Fenstad, T. Lindstrøm (eds). Cambridge University Press, 1992. These volumes contain also the article by Sergio " On the scientific work of Raphael Høegh-Krohn".

It is therefore no surprise that in 2002 the Norwegian Academy of Science awarded Sergio the title of Doctor honoris causa in Mathematics on the occasion of Niels H. Abel's 200th birthday, and that in 2019 the University of Stockholm honoured him in the same way.

In the Soviet Union Sergio also met a big Italian community working in statistical mechanics, coming especially from Rome. This and further strong connections with the Italian scientific community later lead to nominations to professorships for chiara fama at the University of Rome and University of Trento (that he was not able to accept) and long stays at those places as well as e.g. at Scuola Normale Superiore (Pisa) and SISSA (Trieste). He currently belongs to the Ph.D. board of the Department of Mathematics of the University of Milan (Università degli Studi). In 2021 he was elected as Foreign Member of the Accademia Nazionale dei Licei (Rome) and member of the Academia Europaea (London).

In Princeton Sergio had already started a long-life friendship with the Italian physicists and mathematicians Gian Fausto Dell'Antonio and Francesco Guerra.

> *[F. Guerra] "I met Sergio when I arrived in Princeton in September 1970. I was immediately impressed by his great humanity. [...] He spontaneously helped me and my family in the settlement. In 1971 E. Nelson held a high-level special course on Euclidean quantum field theory. Sergio's help was crucial. In particular, his personal notes of the course were a true masterpiece. [...] I learned Nelson's Euclidean theory from Sergio's explanations and his fundamental notes."*

Gianfausto Dell'Antonio invited him to spend the year 1973 in Naples at the Institute of Theoretical Physics, University of Naples there he worked on constructive field theory and stochastic mechanics, under the lasting influence of his Princeton period.

> *[G. Dell'Antonio] "We became friends in Naples. Then there were countless encounters. Sergio is able to present his very complicated works in a simple way. 65 years of friendship."*
>
> *[R. Figari] "Sergio's presence in Naples was fundamental for my scientific career. Everything came from him: scientific problems and collaborators."*

The frequent visits in Italy soon spread from southern to northern Italy. Starting with the new millennium, for several years Sergio had, in parallel to his Professorship in Bochum and later in Bonn, an assignment at the University of Trento.

> *[G. Da Prato] "I have met Sergio many times and I have had many scientific discussions with him. I learned from Sergio the theory of Dirichlet forms in infinite dimensions and the stochastic quantization, thanks to the various discussions we had and also through a course he taught when I invited him in Pisa".*
>
> *[L. Tubaro] "When we understood who he was, we invited him to our conferences, which he attended several times. In the early 2000s I invited him in Trento. He gave a course there. Some of my Ph.D. students actually studied with him".*
>
> *[A. Teta] "Although we have never written a work together, we have always been friends and scientifically very close."*
>
> *[S. Albeverio] " In Oslo and in Italy I have always felt like being at home".*

From 1972 to 1977 Sergio was also a guest in France, where together with Raphael Høegh-Krohn he visited several research institutes, including the University and

CNRS in Aix-Marseille (Luminy) invited by Daniel Kastler and Raymond Stora. They started a long standing collaboration with mathematicians and physicists in Marseille and Philippe Blanchard on topics from quantum physics to representation theory and stochastic modeling. In particular they started a program of extending the interplay of Laplacian perturbed by potentials with Gaussian measures and Poisson measures to the consideration of pseudo differential operation and infinitely divisible laws. Later, being Professor in Bielefeld, Bochum and Bonn, Sergio engaged several generations of collaborators, extending his research to the theory of S(P)DEs with non-Gaussian additive noise. Further, he found the desired applications to quantum field theory as well as to many other areas, including neurobiology and finance.

In 1977 he got a position as Associate Professor in the Department of Mathematics, University of Bielefeld (in the section Analysis/Potential Theory). Having as colleagues Philipp Blanchard and Ludwig Streit at the Physics Department, the research connection between the Mathematics and Physics Departments was strengthened by involving scientists from all over the world. The theory of Dirichlet forms in infinite dimension, started in Oslo together with Raphael Høegh-Krohn, was shared with doctoral students, like e.g. Michael Röckner, who was later appointed as Full Professor in Bielefeld and continued and enlarged the school on this and many other topics.

[M. Röckner] "I wouldn't be the scientist and probably also not the person I am today without Sergio because he had so much influence on my scientific life and on the development of my personality and character. He was just a pure inspiration from the first day."

In 1979 Sergio was appointed as a Full Professor at the Faculty of Mathematics of Ruhr-University Bochum. This soon made the city Bochum an international center for every level students and visitors in Mathematics and Physics coming from all over the world. Indeed, Sergio contributed to widen the department's connections with the USA, U.K., Italy, France, Poland, Portugal, Scandinavia, Soviet Union, Spain, Bulgaria, Chekia, China (Beijing, Wuhan), Japan (Kyoto, Osaka, Tokyo, Kyushu, Hiroshima, Nagoya, Nara, Sendai), Mexico (CINVESTAV), Russia (St. Petersburg, Moscow), Saudi Arabia (Dharhan), Tunisia (Tunis), Ukraine, Uzbekistan, thanks to many projects funded by institutions like the Deutsche Forschungsgemeinschaft, European Community, Alexander V. Humboldt Society, JSPS, VolkswagenStiftung

[A. Khrennikov] "I came to Bochum and there I met such a friendly and unusual scientific school [...] I think Sergio's school was really unique."

[Y. Kondratiev] "Sergio belongs to the older generation of great mathematicians like Skorokhod and so on. The role of Sergio in my scientific life is deep and special. I would like to stress that Sergio was also a prominent example of a deep scientist with wide areas of interests. I am always happy to discuss with Sergio not only particular mathematical problems, but also several aspects in psychology, philosophy, history, and physics. It is my honor to call Sergio my teacher and friend."

[S. Paycha] "Sergio's characterizing feature is that he sprinkles all over science. He's a delicate person and I think that's also why he can work with so many people because he's very respectful and very delicate and I think he has left his footprint in so many areas, but a very delicate footprint and very deep for this delicacy. If I may add, you see you are many women. I'm a woman. There are not many women in math. I think he supported women. […]

He is very careful to be inclusive. He takes everybody with him. Women tend to step back and he takes them back in."

This activity was then continued in 1997 at the University of Bonn where he moved as Full Professor together with all the members of his Stochastics and Mathematical Physics Group. In Bonn he also created with Volker Jentsch the Interdisziplinäres Zentrum für Komplexe Systeme (IZKS) of the University of Bonn, working on complex systems, stochastic traffic models, models of urban development and other applications of mathematics and of the theory of extreme events.

Furthermore, he was also a founding member of the Excellence Cluster in Mathematics, which then gave rise to the Hausdorff Center for Mathematics (HCM), where he is still leading many scientific projects. As Professor Emeritus he is a focus of present research at HCM.

[M. Röckner] "Another point that I want to stress is, of course, what is called Sergio's family. This is a huge family. All of us belong to this family, but many other people belong to this family. When he organized conferences (and he organized so many conferences—unbelievable) he always linked his family to other groups and families all the time. [...] I would like to mention that he is still influencing young people; as you know in Bonn he is working with very young people again."

The present volume contains an absolutely non-exhaustive collection of papers aiming to describe some of the main research areas where Sergio gave significant contributions.

We start with "Wick powers in stochastic PDEs: an introduction" by Giuseppe Da Prato and Luciano Tubaro. This article gives a clear mathematical description of Wick powers, an important tool in constructive field theory, certainly one of the first scientific interests of Sergio, which was initiated by a lecture of Edward Nelson at Princeton University. At the same time Nelson's influence led Sergio to study the emergent theory of stochastic mechanics. To this regard the article "The Albeverio-Høegh-Krohn paradox in Nelson stochastic mechanics" by Francesco Guerra presents one of Sergio's contributions in this area.

During the fruitful collaboration with Raphael Høegh-Krohn outstanding results in different areas were obtained. The development of the theory of Dirichlet forms in infinite dimensions is one of the most famous. The article "Energy Forms and Quantum Dynamics" by Ludwig Streit presents the origins of the physical applications, while "The emergence of non-commutative potential theory" by Fabio Cipriani describes the consecutive developments related to potential theory.

Dirichlet forms also opened up for studying quantum mechanical systems with point interactions.

A generalization of Sergio's point interactions and its applications is presented in "Contact interactions and Gamma convergence" by Gianfausto Dell'Antonio. The main methodology and results concerning the three-particle interaction are illustrated in the article by Rodolfo Figari and Alessandro Teta "On the Hamiltonian for three bosons with point interactions".

The theme of constructive quantum fields was taken up again by Sergio Albeverio and Raphael Høegh-Krohn in their studies in a mathematical rigorous way. In particular, the mathematical theory of Feynman path integral is attributed to their pioneering book published in 1976. The article "Mathematical theory of Feynman path integrals" by Sonia Mazzucchi gives an overview of Sergio's contributions, while "Gauge theories in low dimensions: reminiscences of work with Sergio Albeverio" by Ambar Sengupta discusses the application of functional integration techniques to Gauge Theories.

The search of new and powerful mathematical techniques allowing to tackle the unsolved problems in quantum field theory led Sergio Albeverio and Raphael Hoegh-Krohn to the study of non-standard analysis in collaboration with Jens Erik Fenstad and Tom Lindstrøm. A survey of Sergio Albeverio's work in this area is presented in "The Allure of Infinitesimals: Sergio Albeverio and Nonstandard Analysis" by Tom Lindstrøm.

"Sergio's work in statistical mechanics: from quantum particles to geometric stochastic analysis" by Alexei Daletskii, Yuri Kondratiev and Tanja Pasurek discusses some of Sergio's contributions to quantum statistical mechanics and related areas.

A review of Sergio Albeverio's research in hydrodynamics in another setting is presented in "Hydrodynamic Models" by Benedetta Ferrario and Franco Flandoli, where bidimensional Euler and Navier-Stokes equations with space-time white noise are investigated from the point of view of statistical mechanics. Recent results in this field are also presented in "On strong solution to the 2d stochastic Ericksen-Leslie system: a Ginzburg-Landau approximation approach" by Zdzislaw Brzezniak, Gabriel Deugoué and Paul André Razafimandimby.

Some results on the theory of stochastic PDEs related to pseudo–differential operators developed by Sergio are contained in the paper "Stability properties of mild solutions of SPDEs related to Pseudo Differential Equations" by Vidyadhar Mandrekar and Barbara Rüdiger.

Sergio's analysis of stochastic processes extends to other types of state spaces, like p-adics and, more generally, non-Archimedian ones. The paper "Random Processes on non-Archimedian Spaces" by Witold Karwowski describes Sergio's work and its relation with other approaches. According to his method, Lévy processes on p-adic numbers and other non-Archimedean spaces are constructed from semigroups in the sense of a study of Chapman-Kolmogorov equations.

Considered the huge number of Sergio's publications and the limits in time and space, some other areas of Sergio's activities have to be necessarily left out. They are partly listed at the end of Sergio's interview, to which we refer.

A second Springer Volume, titled *Geometry & Invariance in Stochastic Dynamics* (2021) and written by both young and long-time collaborators of Sergio, collects the proceedings of the Verona Conference 2019, also related to Sergio's 80th birthday.

The editors of the present volume sincerely thank the authors of all contributions. Their generous effort was fundamental in creating this special volume dedicated to Sergio Albeverio's incredible adventure in Science. Many thanks also to the colleagues who agreed to be interviewed. Their touching testimonies enriched the historical and human memory of Sergio's scientific career.

We also thank Marina Reizakis from Springer-Verlag for her enthusiastic incentive to undertake this project as well as for her mediation work among the Verona conference participants, which eventually led to the shared choice of the title of the present volume.

Finally, we thank Sergio Albeverio for his discreet and delicate support to this book.

Contents

Wick Powers in Stochastic PDEs: An Introduction

Giuseppe Da Prato and Luciano Tubaro

For the eighty years of Sergio

Abstract Using Wick polynomials we define a two-points function $:x_N^n:, \ n, \ N \in \mathbb{N}$ and prove the existence of the limit $\lim_{N\to\infty} :x_N^n: =: x^n :$. We also give a simple proof of the Nelson estimate.

Keywords Wick powers · Stochastic quantization · Nelson estimate · White noise

Mathematics Subject Classification 37L55 · 60J65 · 60H40

1 Introduction

Consider the following stochastic differential equation in the Hilbert space $H = L^2(0, 2\pi)$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$),

$$dX = \left[\tfrac{1}{2}\,(X_{\xi\xi} - X) - X^3\right] dt + dW(t), \quad X(0) = x \in L^2(0, 2\pi), \tag{1.1}$$

where $\xi \in [0, 2\pi]$, X is 2π periodic, $W(t)$ is a cylindrical Wiener process and $X_{\xi\xi}$ denotes the second derivative of X with respect to ξ.

Denote by $(e_k)_{k\in\mathbb{Z}}$ the complete orthonormal system of $L^2(0, 2\pi)$,

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$$e_k(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{\imath k\xi}, \quad \xi \in [0, 2\pi],\ k \in \mathbb{Z}$$

and define

$$W(t) = \sum_{k\in\mathbb{Z}} \beta_k(t) e_k,$$

where $(\beta_k(t))_{k\in\mathbb{Z}}$ is a family of standard Brownian motions mutually independent in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Let us write Eq. (1.1) in the following mild form

$$X(t) = e^{tA}x - \int_0^t e^{(t-s)A} X^3(s)\, ds + W_A(t), \tag{1.2}$$

where[1]

$$Ax = \tfrac{1}{2}\,(x_{\xi\xi} - x), \quad x \in \{y \in H^2(0, 2\pi) :\ y(0) = y(2\pi),\ y_\xi(0) = y_\xi(2\pi)\}$$

and

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) = \sum_{k\in\mathbb{Z}} e_k \int_0^t e^{-\frac{1}{2}\,(t-s)(1+|k|^2)} d\beta_k(s). \tag{1.3}$$

It is easy to see that the *stochastic convolution* $W_A(t)$ is a Gaussian random variable in $L^2(0, 2\pi)$ with mean 0 and covariance operator

$$C(t) = C(1 - e^{tA}), \qquad t \geq 0$$

where

$$C = -\tfrac{1}{2}\, A^{-1}.$$

Notice that

$$Ce_k = \frac{1}{1+|k|^2}\, e_k, \qquad k \in \mathbb{Z},$$

so that $C(t)$ is a trace class operator. Moreover, one can see that the Gibbs probability measure (on $L^2(0, 2\pi)$)

$$\nu(dx) = Z^{-1} \exp\left\{-\frac{1}{2}\int_0^{2\pi} x^4(\xi)\, d\xi\right\}\, \mu(dx), \tag{1.4}$$

[1] $H^2(0, 2\pi)$ is the usual Sobolev space.

where

$$Z = \int_{L^2(0,2\pi)} \exp\left\{ -\frac{1}{2} \int_0^{2\pi} y^4(\xi)\, d\xi \right\} \mu(dy)$$

and μ is the Gaussian measure with mean 0 and covariance operator C, is the invariant measure of the Markov semigroup associated to the process $X(t)$.

It is not difficult to solve Eq. (1.2) by a fixed point argument, see e.g. [8].

Try now to generalize this result to the two dimensional case by considering the equation

$$dX = \left[\frac{1}{2} (\Delta_\xi X - X) - X^3 \right] dt + dW(t), \quad X(0) = x \tag{1.5}$$

in the Hilbert space $H = L^2((0, 2\pi)^2)$. Proceeding as before we consider the complete orthonormal system $(e_k)_{k\in\mathbb{Z}^2}$ in $L^2((0, 2\pi)^2)$,

$$e_k(\xi) = \tfrac{1}{2\pi}\, e^{\imath\langle k,\xi\rangle}, \quad k = (k_1, k_2) \in \mathbb{Z}^2,\ \xi \in [0, 2\pi]^2$$

and define

$$W(t) = \sum_{k\in\mathbb{Z}^2} \beta_k(t) e_k,$$

where $(\beta_k(t))_{k\in\mathbb{Z}^2}$ is a family of standard Brownian motions mutually independent in $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Again, we write Eq. (1.5) in mild form

$$X(t) = e^{tA}x - \int_0^t e^{(t-s)A} X^3(s)\, ds + W_A(t), \tag{1.6}$$

where

$$Ax = \tfrac{1}{2}\, (\Delta_\xi x - x), \quad x \in \{y \in H^2((0, 2\pi)^2) :\ y, y_{\xi_1}, y_{\xi_2} \text{ periodic in } \xi_1, \xi_2\}$$

and

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) = \sum_{k\in\mathbb{Z}^2} e_k \int_0^t e^{-\frac{1}{2}\,(t-s)(1+|k|^2)} d\beta_k(s). \tag{1.7}$$

But in this case the operator

$$C = -\tfrac{1}{2}\, A^{-1}$$

is not of trace class. In other words the stochastic convolution $W_A(t)$ is not a well defined random variable with values in $L^2((0, 2\pi)^2)$. One can easily see that it is well defined and Gaussian in every Sobolev space $H^{-\varepsilon}((0, 2\pi)^2)$ with $\varepsilon > 0$; thus, it is natural to try to solve Eq. (1.6) in this space. However, a problem will arise since the nonlinear term x^3 is not well defined in $H^{-\varepsilon}((0, 2\pi)^2)$ which is a space of distributions.

For this reason the function x^3 is replaced by the following one

$$:x^3:= \lim_{N\to\infty} \left([x_N]^3 - 3\rho_N^2 x_N\right),$$

where

$$x_N = \sum_{|k|\le N} \langle e_k, x\rangle e_k$$

and

$$\rho_N = \frac{1}{2\pi}\left[\sum_{|k|\le N} \frac{1}{1+|k|^2}\right]^{1/2}.$$

One can show that the limit above does exists in $L^2(\mathcal{H}, \mu)$ where $\mathcal{H}$ is a suitable extension of the space H and μ is a Gaussian measure of covariance C, see Sect. 3 below for details. In this way we have changed the original problem with the following one

$$dX = \left[\tfrac{1}{2}\,(\Delta_\xi X - X) - :X^3:\right] dt + dW(t), \quad X(0) = x. \tag{1.8}$$

This is the so called *renormalization* procedure. This choice is physically justified in quantum field theory and somebody believes that it is natural even in other situations as: reaction diffusion and Ginzburg–Landau equations, see e.g. [3].

In the pioneering papers devoted to the stochastic quantisation a modified equation was studied

$$\begin{cases} dX = -\dfrac{1}{2}\,\left(C^{-\varepsilon}X + \lambda C^{1-\varepsilon}X^n\right) dt + C^{\frac{1-\varepsilon}{2}} dW(t) \\[2ex] X(0) = x \in H, \end{cases} \tag{1.9}$$

where $\epsilon > 0$, see Jona Lasinio and Mitter see [14, 15]: for further developments the reader can look at [4, 7, 9, 16]. For a different approach, based on Dirichlet forms and Malliavin calculus, see [2, 17] respectively.

Equation (1.8) in dimensions 2 was solved in suitable Besov spaces in [5]. In dimensions 3 the problem is more challenging, it requires the theory of rough paths and its developments, see [12, 13]. For the existence of the equivalent of Gibbs measure (1.4) see [1].

In the past few years, some attention has been payed to the so called *stochastic quantisation*, see Parisi and Wu [18], in order to compute integrals of the form

$$\int_H f(x)\,\nu(dx)$$

where ν is the invariant measure of (1.8) defined as (1.4), using the ergodic theorem

$$\int_H f(x)\,\nu(dx) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))\,dt.$$

The renormalization has a long story, also in connection with the constructive field theory in the euclidean framework, see Glimm and Jaffe [10], Simon [19] and references therein.

In this paper we shall describe the renormalization of the power and the Nelson estimate, following essentially the ideas in Simon [19]. We shall proceed similarly as in [6], where we presented a reformulation of the theory in the space $H^{-\alpha}((0,2\pi)^2)$, $\alpha > 0$, but here we prefer to enlarge the space $L^2((0,2\pi)^2)$ introducing the product space

$$\mathcal{H} = \prod_{k\in\mathbb{Z}^2} \mathbb{R}_k, \quad \mathbb{R}_k = \mathbb{R},$$

identifying H with $\ell^2(\mathbb{Z}^2) \subset (\mathbb{R}^2)^\infty$ and setting

$$\mu = \bigotimes_{k\in\mathbb{Z}^2} N_{(1+|k|^2)^{-1}},$$

where $N_{(1+|k|^2)^{-1}}$ represents the one-dimensional Gaussian measure with mean 0 and variance $(1+|k|^2)^{-1}$. This is essentially equivalent to work in the space of distributions, but it avoids for instance the use of the Minlos theorem.

Let us describe the content of this paper. In Sect. 2 we shall define for every integer n the Wick product $:\phi^n:$ with respect to the Gaussian measure μ. As shown here, this definition corresponds, roughly speaking, to subtract to ϕ^n some divergent term.

In Sect. 3 we give a new (at our knowledge) simpler proof of the Nelson estimate. It allows to define the Gibbs measure

$$\nu(d\phi) = \frac{\exp\{-\frac{1}{2}\langle 1, :\phi^4:\rangle\}}{\int_{\mathcal{H}} \exp\{-\frac{1}{2}\langle 1, :\psi^4:\rangle\}\,\mu(d\psi)}\,\mu(d\phi). \tag{1.10}$$

2 Wick Polynomials

Let $H = L^2(\mathcal{O})$ be a complex Hilbert space, where $\mathcal{O} := [0,2\pi]\times[0,2\pi]$. The scalar product and the norm in H are respectively denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$.

We consider the self-adjoint operator Q in H defined by

$$Qe_h = \lambda_h e_h, \quad \lambda_h = (1 + |h|^2)^{-1}, \quad h \in \mathbb{Z}^2,$$

where $(e_h)_{h\in\mathbb{Z}^2}$ is the orthonormal basis in H,

$$e_h(\xi) = \tfrac{1}{2\pi}\, e^{i\langle h,\xi\rangle}, \quad \xi \in \mathcal{O},\ h = (h_1, h_2).$$

For all $N \in \mathbb{N}$ we set[2]

$$Q_N x = \sum_{|h|\le N} \lambda_h \langle x, e_h\rangle e_h.$$

We embed H in $\mathcal{H} := \mathbb{R}^{\mathbb{Z}^2}$,

$$H \to \mathbb{R}^{\mathbb{Z}^2}, \quad x = \sum \in H \mapsto j(x) = (x_h)_{h\in\mathbb{Z}^2},$$

with $x_h = \langle x, e_h\rangle$ and consider the measures μ, μ_N in $\mathcal{H}$ defined as

$$\mu = \bigotimes_{h\in\mathbb{Z}^2} N_{\lambda_h}, \quad \mu_N = \bigotimes_{|h|\le N} N_{\lambda_h}.$$

Note that $\operatorname{Tr} Q = \infty$, whereas $\operatorname{Tr} Q^{1+\epsilon} < \infty$, $\forall\, \epsilon > 0$. The following asymptotic behaviour of ρ_N is basic

$$\rho_N^2 = O(\log N). \tag{2.1}$$

It follows from

$$\frac{1}{(2\pi)^2} \sum_{|h|\le N} \frac{1}{1+|h|^2} \sim \int_0^N \frac{r}{1+r^2}\, dr = \frac{1}{2} \log(1 + N^2).$$

2.1 Approximations

For all $x \in H$, $N \in \mathbb{N}$ we set

$$x_N(\xi) = \sum_{|h|\le N} \langle x, e_h\rangle e_h(\xi), \quad \xi \in \mathcal{O}.$$

---

[2] $\{|h| \le N\} = \{h \in \mathbb{Z}^2 :\ |h| \le N\}$.

Obviously $x_N \in L^\infty(\mathcal{O})$. It is useful to express $x_N(\xi)$, $\xi \in \mathcal{O}$, in terms of the *white noise* function of those elements of H of norm 1, because then several computations with the Wiener's chaos becomes easy.[3] To this purpose, write

$$x_N(\xi) = \left\langle x, \sum_{|h|\le N} \overline{e_h(\xi)}\, e_h \right\rangle = \left\langle Q^{-1/2}x, \sum_{|h|\le N} \lambda_h^{1/2}\, \overline{e_h(\xi)}\, e_h \right\rangle = W_{\sum_{|h|\le N} \lambda_h^{1/2} \overline{e_h(\xi)}\, e_h}.$$

We note that, by the Parseval identity we have

$$\left| \sum_{|h|\le N} \lambda_h^{1/2}\, \overline{e_h(\xi)}\, e_h \right|^2 = 4\pi^2 \sum_{|h|\le N} \lambda_h = 4\pi^2 \ \mathrm{Tr}\ Q_N = \rho_N^2.$$

Then we write

$$x_N(\xi) = \rho_N W_{\eta_N(\xi)}, \quad \xi \in \mathcal{O},$$

where

$$\eta_N(\xi) = \frac{1}{\rho_N} \sum_{|h|\le N} \lambda_h^{1/2}\, \overline{e_h(\xi)}\, e_h, \quad \rho_N^2 = \sum_{|h|\le N} \lambda_h,$$

so that

$$|\eta_N(\xi)| = 1, \quad N \in \mathbb{N}. \tag{2.2}$$

Note moreover that

$$\begin{aligned}\langle \eta_N(\xi_1), \eta_N(\xi_2)\rangle &= \frac{1}{\rho_N^2} \sum_{|h|\le N} \lambda_h\ e_h(\xi_1)\overline{e_h(\xi_2)} \\ &= \frac{1}{\rho_N^2}\ \gamma_N(\xi_1 - \xi_2), \quad \xi_1, \xi_2 \in \mathcal{O}, \quad N \in \mathbb{N},\end{aligned} \tag{2.3}$$

where

$$\gamma_N = \sum_{|h|\le N} \lambda_h\ e_h, \quad N \in \mathbb{N}, \quad \xi \in \mathcal{O}. \tag{2.4}$$

We set also

$$\gamma = \sum_{h\in\mathbb{Z}^2} \lambda_h\ e_h. \tag{2.5}$$

Clearly $\gamma \in L^2(\mathcal{O})$ and it is easy to see that it coincides with the kernel of the integral operator Q,

[3] For all $f, g \in H$ such that $|f| = |g| = 1$ we have $\int_H H_n(f)\, H_n(g)\, d\mu = [\langle f, g\rangle]^n$.

$$Qx(\xi) = \int_{\mathcal{O}} \gamma(\xi - \xi_1)\, x(\xi_1)\, d\xi_1, \quad x \in H,\ \xi \in \mathcal{O}. \tag{2.6}$$

Notice that γ is not bounded but it belongs to $L^p(\mathcal{O})$ for all $p \geq 1$. We have in fact

Proposition 2.1 *For all $p \geq 2$ we have*

$$|\gamma|_{L^p(\mathcal{O})} \leq (2\pi)^{\frac{p-2}{2}} \left[\sum_{h\in\mathbb{Z}^2} \left(\frac{1}{1+|h|^2} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}} =: a_p. \tag{2.7}$$

Proof Let us consider the mapping

$$\Gamma : \{\lambda_j\}_{j\in\mathbb{Z}^2} \to \sum_{h\in\mathbb{Z}^2} \lambda_h e_h.$$

Then

$$\Gamma : \ell^1(\mathbb{Z}^2) \ \to \ L^\infty(\mathcal{O}), \quad \text{with norm } (2\pi)^{-1},$$

$$\Gamma : \ell^2(\mathbb{Z}^2) \ \to \ L^2(\mathcal{O}), \quad \text{with norm } 1.$$

By the Riesz–Thorin theorem if $p > 2$ and $q = \frac{p}{p-1}$ we have

$$\Gamma : \ell^q(\mathbb{Z}^2) \ \to \ L^p(\mathcal{O}), \quad \text{with norm less or equal to } (2\pi)^{\frac{p-2}{2}},$$

and the conclusion follows. □

Let us estimate for further use $|\gamma - \gamma_N|_{L^p}$.

Lemma 2.2 *If $p \geq 2$, we have*

$$|\gamma - \gamma_N|_{L^p(\mathcal{O})} \leq b_p N^{-\frac{2}{p}}, \tag{2.8}$$

where $b_p = (p-1)(2\pi)^{\frac{p-2}{2}}$.

Proof We have in fact

$$\begin{aligned} |\gamma - \gamma_N|_{L^p(\mathcal{O})} &\leq (2\pi)^{\frac{p-2}{2}} \sum_{|h|\geq N} \left(\frac{1}{1+|h|^2} \right)^{\frac{p}{p-1}} \\ &\leq (2\pi)^{\frac{p-2}{2}} \int_N^{+\infty} \frac{2r}{(1+r^2)^{\frac{p}{p-1}}} dr \\ &= (p-1)(2\pi)^{\frac{p-2}{2}} (1+N^2)^{-\frac{1}{p-1}} \\ &\leq (p-1)(2\pi)^{\frac{p-2}{2}} N^{-\frac{2}{p}}. \end{aligned}$$

□

Let us finally recall a basic hypercontractivity estimate, see [11] (also [6]).

Theorem 2.3 *Let $n, m \in \mathbb{N}$, and $u \in L^2_n(\mathcal{H}, \mu)$ (the Wiener chaos of order n). Then we have*

$$\left(\int_{\mathcal{H}} |u(x)|^{2m} \mu(dx)\right)^{\frac{1}{2m}} \leq (2m-1)^{\frac{n}{2}} \left(\int_{\mathcal{H}} |u(x)|^2 \mu(dx)\right)^{\frac{1}{2}}. \tag{2.9}$$

2.2 Wick Polynomials

For any $n, N \in \mathbb{N}$ and any $x \in H$ define

$$:x_N^n:(\xi) = \sqrt{n!}\ \rho_N^n\ H_n(W_{\eta_N(\xi)}), \quad \xi \in \mathcal{O}, \tag{2.10}$$

where H_n is the Hermite polynomial of order n.[4]

In particular we have

$$:x_N^1:(\xi) = x_N(\xi),$$

$$:x_N^2:(\xi) = [x_N(\xi)]^2 - \rho_N^2,$$

$$:x_N^3:(\xi) = [x_N(\xi)]^3 - 3\rho_N^2 x_N(\xi),$$

$$:x_N^4:(\xi) = [x_N(\xi)]^4 - 3\rho_N^2 [x_N(\xi)]^2 + 6\rho_N^4.$$

2.3 Existence of $\lim_{N\to\infty} :x_N^n:$

Let us start with a few useful identities. By the embedding of H in $\mathcal{H}$ defined in the beginning of Sect. 2, in the sequel we use both notations such that $\int_H \cdot \mu(dx)$ and $\int_{\mathcal{H}} \cdot \mu(dx)$.

Lemma 2.4 *We have*

$$\int_H |\ :x_N^n:\ |^2\ \mu(dx) = 4\pi^2\ n!\ \rho_N^{2n}. \tag{2.11}$$

Proof Taking into account (2.10) we have

[4] Let $F(t, \xi) = e^{-\frac{t^2}{2}+t\xi}$, $\xi \in \mathbb{R}$, and write $F(t, \xi) = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(\xi)$, $t, \xi \in \mathbb{R}$. For any $n \in \mathbb{N}$, H_n is called the Hermite polynomial of degree n.

$$\int_H |\,{:}x_N^n{:}\,|^2\,\mu(dx) = n!\,\rho_N^{2n}\int_{\mathcal{O}} d\xi \int_H H_n^2(W_{\eta_N(\xi)})\,d\mu = 4\pi^2\,n!\,\rho_N^{2n}.$$

□

This Lemma shows that we cannot expect that there exists the limit

$$\lim_{N\to\infty}\,{:}x_N^n{:}\quad \text{in } L^2(\mathcal{H},\mu;H).$$

We are going to show that for any $z\in H$, there exists instead the limit (Theorem 2.6)

$$\lim_{N\to\infty}\langle{:}x_N^n{:},z\rangle_{\mathcal{H}}\quad \text{in } L^2(\mathcal{H},\mu),$$

and the limit

$$\lim_{N\to\infty}\,{:}x_N^n{:}\quad \text{in } L^2(\mathcal{H},\mu;D((-A)^{-\epsilon}),\ \forall\,\epsilon>0,$$

see Theorem 2.8.

Lemma 2.5 *For any $z\in H$ we have*

$$\int_H |\langle{:}x_N^n{:},z\rangle|^2\mu(dx) = n!\langle\gamma_N^n * z,z\rangle. \tag{2.12}$$

Proof Write

$$\begin{aligned}\int_H |\langle z,{:}x_N^n{:}\rangle|^2\mu(dx) &= \int_H \left|\int_{\mathcal{O}}\langle{:}x_N^n(\xi){:},z(\xi)\rangle d\xi\right|^2\mu(dx)\\ &= n!\rho_N^{2n}\int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)}d\xi d\xi_1\\ &\quad\int_H H_n(W_{\eta_N(\xi)}(x))H_n(W_{\eta_N(\xi_1)}(x))\mu(dx)\\ &= n!\rho_N^{2n}\int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)}\,[\langle\eta_N(\xi),\eta_N(\xi_1)\rangle]^n\,d\xi d\xi_1\\ &= n!\int_{\mathcal{O}\times\mathcal{O}}\gamma_N^n(\xi-\xi_1)z(\xi)\overline{z(\xi_1)}d\xi d\xi_1\\ &= n!\langle\gamma_N^n * z,z\rangle.\end{aligned}$$

□

Theorem 2.6 *Let $M>N$ and $z\in H$. Then we have*

$$\int_H |\langle z, :x_N^n: - :x_M^n:\rangle|^2 \mu(dx) = n!\langle(\gamma_M^n - \gamma_N^n) * z, z\rangle. \tag{2.13}$$

Moreover there exists $c_n > 0$ such that

$$\int_H |\langle z, :x_N^n: - :x_M^n:\rangle|^2 \mu(dx) \leq \frac{c_n}{N}\, |z|^2. \tag{2.14}$$

Therefore there exists $:x_N^n:\in \mathcal{H}$ such that

$$\lim_{N\to\infty} \langle z, :x_N^n:\rangle_{\mathcal{H}} := \langle :x^n:, z\rangle_{\mathcal{H}}, \quad \text{in } L^2(\mathcal{H}, \mu). \tag{2.15}$$

Proof Let $N > M$, and set

$$L_{N,M} = \int_H |\langle z, :x_N^n:\rangle - \langle z, :x_M^n:\rangle|^2 \mu(dx).$$

Then we have

$$\begin{aligned}
L_{N,M} = n!\rho_M^n &\int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)} d\xi d\xi_1 \\
&\times \int_H \left[\rho_M^n H_n(W_{\eta_M(\xi)}(x)) - \rho_N^n H_n(W_{\eta_N(\xi)} x)\right] \\
&\times \left[\rho_M^n H_n(W_{\eta_M(\xi_1)}(\phi)) - \rho_N^n H_n(W_{\eta_N(\xi_1)}(x))\right] \mu(dx) \\
&\times n! \int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)} \Big\{ \rho_M^{2n} \left[\langle \eta_M(\xi), \eta_M(\xi_1)\rangle\right]^n - \rho_M^n \rho_N^n \left[\langle \eta_M(\xi), \eta_N(\xi_1)\rangle\right]^n \\
&- \rho_M^n \rho_N^n \left[\langle \eta_M(\xi), \eta_N(\xi)\rangle\right]^n - \rho_N^{2n} \left[\langle \eta_N(\xi), \eta_N(\xi_1)\rangle\right]^n \Big\} d\xi d\xi_1 \\
= n! &\int_{\mathcal{O}\times\mathcal{O}} \left[\gamma_M^n(\xi - \xi_1) - 2\gamma_N^n(\xi - \xi_1) + \gamma_N^n(\xi - \xi_1)\right] z(\xi)\overline{z(\xi_1)} d\xi d\xi_1 \\
= n! &\langle(\gamma_N^n - \gamma_M^n) * z, z\rangle.
\end{aligned}$$

Therefore (2.13) is proved.

It remains to prove (2.14). We have in fact

$$|\gamma_M^n - \gamma_N^n|_{L^1(\mathcal{O})} \le \sum_{j=0}^{n-1} \int_{\mathcal{O}} |\gamma_M - \gamma_N|\, |\gamma_M^j|\, |\gamma_N^{n-1-j}|\, d\xi.$$

Using the Hölder estimate, and taking into account (2.7) and (2.8), we obtain

$$|\gamma_M^n - \gamma_N^n|_{L^1(\mathcal{O})} \le \sum_{j=0}^{n-1} |\gamma_M - \gamma_N|_{L^2(\mathcal{O})} |\gamma_M|^j_{L^{4j}(\mathcal{O})} |\gamma_M|^{n-1-j}_{L^{4(n-1-j)}(\mathcal{O})}$$

$$\le \frac{2b_2}{N} \sum_{j=0}^{n-1} a_{4j}^j\, a_{4(n-1-j)}^{n-1-j},$$

which implies (2.14) and consequently (2.15). □

Remark 2.7 $:x_n:$ does not belong to $L^2(\mathcal{H}, \mu; H)$. In fact by (2.12) we have

$$\int_{\mathcal{H}} |\, :x_n:\, |^2 \mu(dx) = \sum_{k\in\mathbb{Z}^2} \int_{\mathcal{H}} |\langle e_k, :x_n:\rangle|^2 \mu(dx) = n!\ \ \mathrm{Tr}\, [Q^{\otimes n}] = +\infty.$$

However we are able to define $Q^\varepsilon :x^n:$ as an element of $L^2(\mathcal{H}, \mu; H)$ for any $\varepsilon > 0$, as the next Theorem shows.

The following result can be proved as Theorem 2.6.

Theorem 2.8 *Let $M > N$. Then we have*

$$\int_H |Q^\varepsilon :x_N^n: - Q^\varepsilon :x_M^n:\, |^2 \mu(dx) = n! \left(\sum_{k\in\mathbb{Z}^2} \frac{1}{(1+|k|^2)^{1+2\varepsilon}} \right)^n. \tag{2.16}$$

Thus there exists the limit

$$\lim_{N\to\infty} Q^\varepsilon :x_N^n: = Q^\varepsilon :x^n:, \quad in\ L^2(\mathcal{H}, \mu; H). \tag{2.17}$$

3 The Nelson Estimate

We fix now an *even* integer $n \in \mathbb{N}$ and set

$$U(x) = \langle :x^n:, 1\rangle_{\mathcal{H}}, \quad U_N(x) = \langle :x_N^n:, 1\rangle_{\mathcal{H}}, \quad x \in \mathcal{H}.$$

By Lemma 2.2 there exists $a > 0$ such that

$$\|U - U_N\|_{L^2(\mathcal{H},\mu)} \le \frac{a}{\sqrt{N}}, \quad \forall\, N \in \mathbb{N}. \tag{3.1}$$

Since $U, U_N \in L^2_n(\mathcal{H}, \mu)$, by Theorem 2.3 it follows that

$$\|U - U_N\|_{L^p(\mathcal{H},\mu)} \le \frac{ap^n}{\sqrt{N}} \tag{3.2}$$

Moreover let $c_n > 0$ be such that $H_n(\theta) \le -c_n$. Then there exists $b > 0$ such that

$$U_N(x) \ge -b(\log N)^n, \quad x \in \mathcal{H}. \tag{3.3}$$

Proposition 3.1 *For any $p \ge 1$ we have $e^{-U} \in L^p(\mathcal{H}, \mu)$.*

Proof It is enough to prove the proposition for $p = 1$. We first note that,

$$\int_{\mathcal{H}} e^{-U} d\mu = \int_0^{+\infty} \mu(e^{-U} > t)dt = \int_0^{+\infty} \mu(U < -\log t)dt. \tag{3.4}$$

Set

$$F(t) = \mu(U < -\log t), \quad t \ge 0,$$

and notice that if $u(x) < -\log t$ we have

$$U(x) \le -\log t < -\log t + 1 \le -b(\log N(t))^n \le U_{N(t)}(x), \tag{3.5}$$

provided $N(t)$ is chosen such as

$$-b(\log N(t))^n \ge -\log t + 1,$$

that is

$$N(t) = \exp\left\{\left(\tfrac{\log t - 1}{b}\right)^{1/n}\right\}. \tag{3.6}$$

Now, by (3.5) it follows by the Markov inequality that for any $p \ge 2$,

$$F(t) = \mu(U \le -\log t) \le \mu\left(|U - U_{N(t)}| \ge 1\right) \le \|U - U_{N(t)}\|^p_{L^p(\mathcal{H},\mu)}.$$

By (3.2) and (3.6)

$$F(t) \le a^p p^{np} N(t)^{-p/2} \le a^p p^{np} \exp\left\{-\frac{p}{2}\left(\frac{\log t - 1}{b}\right)^{\frac{1}{n}}\right\}.$$

Finally, we choose $p = p(t)$ such that for some $M, \lambda > 0$,

$$F(t) = \mu(U < -\log t) \le Mt^{-(\lambda+1)}, \quad t > 0, \tag{3.7}$$

and so, by (3.4), we see that $\int_{\mathcal{H}} e^{-U} d\mu < +\infty$. □

Proposition 3.2 *We have*

$$\lim_{N\to\infty} \int_{\mathcal{H}} e^{U_N} \mu(dx) = \int_{\mathcal{H}} e^{U} \mu(dx). \tag{3.8}$$

Proof Let $N_0 \in \mathbb{N}$ be fixed and set

$$V(x) = \min\left\{U, U_{N_0}\right\}, \quad V_N(x) = \min\left\{U_N, U_{N_0}\right\}.$$

Then we have

$$\|V - V_N\|_{L^2(\mathcal{H},\mu)} \le \|U - U_N\|_{L^2(\mathcal{H},\mu)},$$

and

$$V_N(x) \ge -b(\log N)^n.$$

Now, arguing as in the proof of Proposition 3.1 (see (3.7)), we find

$$\int_{\mathcal{H}} e^{-V_{N_0}} d\mu \le \int_{\mathcal{H}} e^{-V} d\mu \le 1 + \frac{M}{\lambda},$$

and the conclusion follows. □

Final Remark. In a recent paper by Sergio Albeverio et al. with the title: Non-local Markovian symmetric forms on infinite dimensional spaces, ArXiv:2006.13571v2 [math.PR], a general formulation on the non-local type stochastic quantisation is considered. There, the above considerations are performed by embedding the state spaces into $\mathbb{R}^n$, which is analogous to the present article.

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The Albeverio–Høegh-Krohn Paradox in Nelson Stochastic Mechanics

Francesco Guerra

Dedicated to Sergio Albeverio on the occasion of his eightieth birthday.

Abstract We discuss an important paradox introduced by Albeverio and Høegh-Krohn in the frame of the stochastic formulation of quantum mechanics proposed by Nelson. The solution of the paradox is based on subtle stability properties of the stochastic processes associated to quantum states. Then we recall the handling made by Fermi of very short range interactions, in atomic physics and nuclear physics, in order to explain the spectroscopic shifts of the very high levels of alkaline atoms immersed in an extraneous gas, and the effects of slow neutrons in inducing artificial radioactivity on massive nuclei.

Keywords Stochastic mechanics · Quantum ground state · Excited states · Stability properties · Short range potentials

1 Introduction

In the first part of this paper, I will discuss about Nelson stochastic mechanics and the Albeverio–Høegh-Krohn paradox.

Since localized potentials play a basic role in the explanation of the Albeverio–Høegh-Krohn paradox, in the second short part, I will include some considerations on historical aspects related to the topics of zero range quantum interactions, which have been also an important subject in the research by Albeverio.

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Of course, the treatment here will be very short and schematic, with emphasis on some historical and pedagogical aspects.

The paradox I am alluding to is formulated in a short sentence in the paper "A remark on the connection between stochastic mechanics and the heat equation", by Albeverio and Høegh-Krohn [1]. Here we see the front page of the original 1973 preprint, issued at the Institute of Theoretical Physics of the University of Naples, as shown by the *signature* of the IBM typewriter held there by comparison with other preprints at approximately the same time.

A remark on the connection between stochastic mechanics and the heat equation

by

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ABSTRACT

We prove that the solution of Nelson's stochastic mechanics equation associated with any stationary solution $\underline{\psi}$ of the Schrödinger equation is the homogeneous Markoff process of the heat equation with Dirichlet boundary condition on the hyper-surface $\underline{\psi} = 0$.

I discussed these topics with Raphael and Sergio in 1973, while Sergio, after a stay in Princeton and Oslo, was visiting in Naples, under invitation by Dell'Antonio, with the support of the INFN (*Italian* National Institute for Nuclear Physics), and Raphael was participating to a small meeting on quantum field theory there. Some of the participants to the Naples meeting, as Robert Schrader, Lon Rosen, and myself, in few days reached the conference on Constructive quantum field theory, organized in Erice by Velo and Wightman [2].

I have a vivid remembrance of our enthusiasm during this period, when quantum field theory was being confirmed as THE theory of elementary particles, in the form leading to the Standard Model, and probabilistic methods were proving to be very useful in QUANTUM field theory, in the shape of the so called Euclidean (Quantum) Field Theory, with applications not only at the mathematical constructive level [3, 4], but also as a basis for the modern renormalization group, and the investigation of critical exponents.

The short sentence in the Albeverio–Høegh-Krohn paper reads "that in the stationary case for higher eigenvalues the stochastic mechanics equation has several solutions, namely those obtained by starting the process in one or some of the domains given by the hypersurface $\phi(x) = 0$". This hypersurface is made by the points where the quantum mechanical density is zero. Therefore, the paradox points to the fact that in some cases there are more solutions in the stochastic mechanics scheme than those provided by quantum mechanics. This of course would be a disaster for stochastic mechanics. The paper [1] is also very important from an historical point of view because it can be seen as the starting point for the extensive research by the Authors about Dirichlet forms (see for example [5]).

2 Quantum Mechanics and Stochastic Mechanics

There are many ways to see the connection between stochastic mechanics and quantum mechanics. The simplest way is to show how to associate a stochastic process to each quantum state. For the sake of simplicity, we consider the elementary case of the harmonic oscillator. The quantum state space is given by square integrable wave functions on the real line

$$\mathcal{H} \ni \psi : \mathbb{R} \ni x \to \psi(x), \int |\psi(x)|^2 \, dx < \infty. \tag{1}$$

The Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(q), \tag{2}$$

the potential is here given by

$$V(x) = \frac{1}{2} m\omega^2 x^2, \tag{3}$$

p, q are the momentum and the position represented by the operators

$$(p\psi)(x) = -i\hbar \frac{\partial \psi}{\partial x}, \tag{4}$$

$$(q\psi)(x) = x\psi(x), \tag{5}$$

m is the mass and ω a constant. To any normalized wave function evolving according to Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = (H\psi)(x,t), \tag{6}$$

we associate a stochastic process $t \to q(t)$ as follows.

First of all let us make the transition to the so called Madelung equations [6]. By defining

$$\psi(x,t) = \rho(x,t)^{\frac{1}{2}} e^{\frac{i}{\hbar}S(x,t)}, \tag{7}$$

the Schrödinger equation for the complex wave function $\psi(x,t)$ is equivalent to the following couple of equations for the real functions $\rho(x,t)$ and $S(x,t)$

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla\cdot(\rho v) = 0, \tag{8}$$

$$\frac{\partial}{\partial t}S(x,t) + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{9}$$

where $v(x,t) = \nabla S(x,t)/m$.

The first of the Madelung equations is nothing but a continuity equation, assuring the particle conservation. The second is a quantum generalization of the Hamilton–Jacobi equation of classical mechanics, where to the classical potential $V(x)$ a new potential has been added, called the quantum De Broglie potential for well founded historical reasons.

For each quantum state, the associated stochastic process $q(t)$, according to Nelson [7–10], is constructed through the following procedure.

The density of the process is given exactly by $\rho(x,t) = |\psi(x,t)|^2$, so that at each time we have for the averages

$$\mathbb{E}(F(q(t),t)) = \int F(x,t)\rho(x,t)\,dx. \tag{10}$$

The kinematics of the process is described by the Ito forward stochastic differential equation

$$dq(t) = v_+(q(t),t)\,dt + dw(t), \tag{11}$$

which replaces the classical definition of the velocity

$$dq(t) = v(t)\,dt. \tag{12}$$

The Brownian motion $dw(t)$ is normalized so that

$$\mathbb{E}(dw(t)|q(t)=x) = 0, \tag{13}$$

$$\mathbb{E}(dw(t)dw(t)|q(t)=x) = \frac{\hbar}{m}\,dt. \tag{14}$$

There is a system of forward v_+, backward v_-, current v, osmotic u velocity fields defined by

$$v_+(x,t) = \lim_{\Delta t \to 0} (\Delta t)^{-1} \mathbb{E}(q(t+\Delta t) - q(t) | q(t) = x), \tag{15}$$

$$v_-(x,t) = \lim_{\Delta t \to 0} (\Delta t)^{-1} \mathbb{E}(q(t) - q(t-\Delta t) | q(t) = x), \tag{16}$$

$$v(x,t) = \frac{1}{2}(v_+(x,t) + v_-(x,t)) = \frac{1}{m}\nabla S(x,t), \tag{17}$$

$$u(x,t) = \frac{1}{2}(v_+(x,t) - v_-(x,t)) = \frac{\hbar}{2m}\nabla \log \rho, \tag{18}$$

where Δt is a vanishing positive time increment, $\mathbb{E}(.|q(t) = x)$ is a conditional expectation, and the expression for the osmotic velocity is a consequence of Bayes Theorem. Of course, here ∇ is simply given by the derivative $\partial/\partial x$, since the system is one-dimensional.

It turns out that for the acceleration field $a(x,t)$ defined by

$$a(x,t) = \frac{1}{2}((D_+D_- + D_-D_+)q(t))(x,t), \tag{19}$$

the following dynamical equation is verified

$$ma(x,t) = -\nabla V(x), \tag{20}$$

where $V(x)$ is the potential, identical to Newton second principle of dynamics. We have introduced the forward and backward derivation operators

$$D_+F(x,t) = \lim_{\Delta t \to 0} \mathbb{E}(F(q(t+\Delta t), t+\Delta t) - F(q(t),t)|q(t) = x)/\Delta t, \tag{21}$$

$$D_-F(x,t) = \lim_{\Delta t \to 0} \mathbb{E}(F(q(t),t) - F(q(t-\Delta t), t-\Delta t)|q(t) = x)/\Delta t. \tag{22}$$

It turns out that

$$D_+ = \frac{\partial}{\partial t} + v_+(x,t)\cdot\nabla + \frac{\hbar}{2m}\Delta, \tag{23}$$

$$D_- = \frac{\partial}{\partial t} + v_-(x,t)\cdot\nabla - \frac{\hbar}{2m}\Delta, \tag{24}$$

where Δ is the Laplacian. Moreover, the dynamical equation for the acceleration $a(x,t)$ is completely equivalent to the second Madelung equation.

This ends the procedure of association of a Nelson stochastic process to a quantum state. But of course, Nelson motivation for the introduction of stochastic mechanics was much more ambitious. In fact one can easily show that stochastic mechanics is a completely independent theory. The two starting points are the assumption of the Ito stochastic differential equation, and the stochastic second principle of dynamics. Then, through the mild assumption that the current velocity field is a gradient, it can be immediately verified that the wave function can be defined so that the Schrödinger equation follows, from the dynamical assumptions. In a sense, the dynamical scheme introduced by Nelson is able to predict the existence and the form of the quantum De Broglie potential, as a simple consequence of the involved stochastic processes.

One can even introduce a stochastic variational principle (Guerra–Morato–Nelson) of Lagrangian type, so that the gradient assumption for the current velocity field is no longer necessary, see [9, 11]. The right equation of motion follows.

3 The Paradox

Now let us introduce the paradox, already foreshadowed in the first Nelson paper, which seems to seriously undermine the one way connection between Schrödinger states and Nelson processes. For the harmonic oscillator, consider the ground state wave function $\psi_0(x)$. It is given as an easily calculated Gaussian form [10]. In this case the association is perfect. All features of the ground state process $q_0(t)$ can be explicitly given, in particular its stationary Gaussian density, and the transition probabilities. Take now the first excited state, which is of the form $\psi_1(x) = x\psi_0(x)$ with the proper normalization. Here in the notation we take away irrelevant time oscillating terms. Also in this case the association is perfect. The Nelson process does never cross the origin, where there is a node for the wave function. Call $q_1(t)$ the process associated to the first excited state. All its features are perfectly well known, including the transition probabilities, as shown for example in [12, 13].

Albeverio–Høegh-Krohn statement of the paradox, as recalled at the end of the Introduction, in this case is equivalent to the following. For a value of the parameter ϵ, $0 \le \epsilon \le 2$, introduce the stochastic process $q_\epsilon(t)$, where the drifts are exactly the same as for $q_1(t)$, while the density is given by $\rho_\epsilon(x) = (2-\epsilon)\rho_1(x)$ for $x \le 0$, and $\rho_\epsilon(x) = \epsilon\rho_1(x)$ for $x \ge 0$. Notice that the change from $\rho_1(x)$ to $\rho_\epsilon(x)$ is a simple rescaling for $x < 0$ or $x > 0$. As a consequence, the osmotic velocity, which depends only on $\nabla\rho/\rho$ is coherently given by the same value for both processes. Therefore the statement according to which "in the stationary case for higher eigenvalues the stochastic mechanics equation has several solutions" is perfectly true. However this does not mean that we have several stochastic processes associated to the same quantum state. As a matter of fact, in Nelson scheme we must have $\rho(x,t) = |\psi(x,t)|^2$, and here this is true only in the case $\epsilon = 1$, where the q_ϵ process is exactly the q_1 process. We leave as an exercise to determine the wave function (and the potential) to which the generic q_ϵ process is associated.

A mild extension of the Albeverio–Høegh-Krohn reasoning leads to the following statement of a paradox pointing in the opposite direction: the possibility that different wave functions correspond to the same stochastic process.

Let us now consider the modified wave function $\psi_2(x) = |x|\psi_0(x)$. It seems that the stochastic process $q_2(t)$ associated to ψ_2 is the same as q_1. In fact, the two processes have the same density, and the same drifts for $x > 0$ and $x < 0$. But the two wave functions ψ_1 and ψ_2 are completely different and have a different evolution according to Schrödinger equation.

Therefore, it seems that the same stochastic process is associated to two completely different wave functions The paradox is very precise, clean and simple to understand, and in a sense astonishing. It seems that Nelson must relinquish his claims. Only the nodeless ground state has a precisely defined associated stochastic process.

As any well thought paradox, the investigation on the way out leads to a deeper understanding of the underlying structure, in this case stochastic mechanics.

4 Quantum and Stochastic Stability

Clearly, the problem with the nodes has to do with the singular nature of the drifts at $x = 0$, in our case. The difficulty is very simple. The dynamical equations should be valid also at $x = 0$, but we can not operate there, because of the singularities in the drifts.

However, we can invoke the physical continuity of quantum mechanics in the Hilbert space metric. We sketch the complete argument.

The problem is at $x = 0$. Let us regularize $\psi_2(x) = |x|\psi_0(x)$ in the form $\psi_2'(x, \epsilon) = \sqrt{x^2 + \epsilon^2}\psi_0(x)$, with the help of a small parameter ϵ. There is no singularity in the drifts at $x = 0$, while $S(x) = 0$ but for irrelevant time oscillating term. A simple calculation shows that the De Broglie potential now acquires a term

$$\frac{\hbar^2}{2m}\frac{\Delta\sqrt{x^2+\epsilon^2}}{\sqrt{x^2+\epsilon^2}} = \frac{\hbar^2}{2m}\frac{\epsilon^2}{(x^2+\epsilon^2)^2}. \tag{25}$$

By insisting that $q_2(t)$ stays stationary in the limit $\epsilon \to 0$, we have to take into account the contribution to the effective potential coming from the piece of the De Broglie potential that we have calculated. It turns out that this contribution is zero for $x \neq 0$, while it becomes infinite for $\epsilon \to 0$ at $x = 0$. Therefore, an infinite barrier appears at $x = 0$ and the paradox disappears. In fact, with the infinite barrier, the two wave functions ψ_1 and ψ_2 are completely equivalent.

In conclusion, we see that the node problem is simply solved by invoking quantum and stochastic stability.

As a matter of fact, Carlen has shown how to deal with processes with singular drifts in stochastic mechanics in the fundamental paper [14], an output of his Princeton Thesis, by using suitable limiting procedures.

On the other hand, in [15], we have introduced a large class of stochastic processes with singular drifts by starting from processes associated to regular drift, introducing a suitable metric, and taking the closure.

The procedure exploited here, for avoiding the Albeverio–Høegh-Krohn paradox, is a simple empirical procedure whose complete justification relies on the rigorous methods of [14, 15].

Let us now make an additional short observation.

Stochastic mechanics is a creation of Nelson, but there are precursors, notably Fényes [16] and Weizel [17].

Fenyes was involved in the study of the statistical models for atoms and molecules, of the type introduced by Thomas [18] and Fermi [19]. So it seems that statistical atomic models provided a stimulating atmosphere also for the development of stochastic mechanics.

5 Slow Electrons and Slow Neutrons in Rome 1934

Our final remarks are devoted to some considerations on the introduction of very localized interaction in atomic physics and nuclear physics, in Rome 1934, by Fermi, which are worth to be known.

We will deal with slow electrons in Rome in the Spring of 1934, and slow neutrons in the Fall of the same year.

At the beginning of 1934, Amaldi and Segrè in Rome, are involved in very important spectroscopic research on a phenomenon they have discovered: the behavior of the high level absorption series of alkaline vapours in the high pressure atmosphere of extraneous gases (essentially Nitrogen and Hydrogen) [20]. The point is that higher order Bohr orbits (or quantum states) contain a high number of molecules of the extraneous gas, which strongly perturbs the energy levels.

Fermi gives a theoretical framework for the interpretation of the phenomenon, and finds a very general connection between the displacement of the high energy levels under the action of the extraneous gas and the limiting cross section of electrons against molecules of the extraneous perturbating gas at extremely low energy [21]. Fermi knew well the behavior of slow electrons (when the De Broglie wave length is much higher than the potential range). He knew in particular that in some cases the cross section increases enormously with the decrease of the velocity of the electrons, as for example with Mercury, due to complex quantum resonance effects.

We can consider a page, 162, of his paper on Nuovo Cimento [21] dedicated to slow electrons. We can not resist to the temptation to compare it with a page, 208, of a different article of some months later, dedicated to the completely different phenomenon of slow neutrons [22].

U non subisca in esso considerevoli variazioni. Otteniamo, tenendo conto anche di (6)

$$\Delta\psi + \frac{8\pi^2 m}{h^2}(W-U)\overline{\psi} - \frac{8\pi^2 m}{h^2}\overline{\Sigma V_i \psi} = 0. \tag{8}$$

Per calcolare l'ultimo valor medio si osservi che, entro una delle buche di potenziale, $W - U$ è trascurabile di fronte al potenziale V della buca. Inoltre la ψ in una regione circostante alla buca di potenziale, a distanza da questa grande in confronto al raggio ρ della buca, ma piccola in confronto alla lunghezza d'onda di DE BROGLIE, ha valore praticamente costante ed eguale in prima approssimazione a $\overline{\psi}$; si riconosce dunque che la ψ nell'interno della buca e nelle sue vicinanze dipende praticamente solo dalla distanza r dal centro della buca. Ponendo perciò in questa regione

$$\psi = \frac{u(r)}{r} \tag{9}$$

la funzione u (r) soddisferà in prima approssimazione all'equazione differenziale:

$$u''(r) = \frac{8\pi^2 m}{h^2} V(r) u(r). \tag{10}$$

All'esterno della buca, dove $V(r)$ si annulla, la u è dunque una funzione lineare di r. E siccome il valore di ψ lontano dalla buca

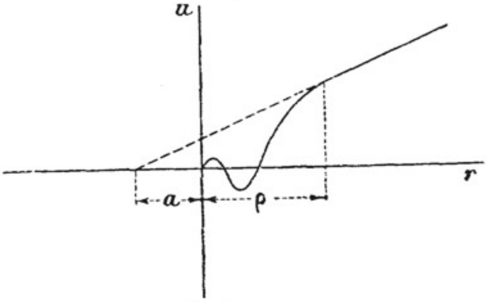

Fig. 1

deve tendere approssimativamente al valore $\overline{\psi}$, si potrà porre, all'esterno della buca:

$$u(r) = (a + r)\overline{\psi} \tag{11}$$

dove a è una lunghezza, il cui significato è chiarito nella fig. 1. In essa sono riportati in ascisse i valori di r e in ordinate quelli di u.

riteniamo utile sviluppare alcune conseguenze generali della teoria da tener presenti nella discussione di questo problema.

Ammettiamo, come si assume generalmente, che le forze agenti tra un neutrone e un nucleo si estendano circa fino alla distanza del raggio nucleare. Se così stanno le cose, la lunghezza d'onda di DE BROGLIE è, per neutroni veloci, dell'ordine del raggio d'azione, e quindi per neutroni lenti molto maggiore. La nota teoria dell'urto, nella quale il nucleo è trattato come una buca di potenziale,

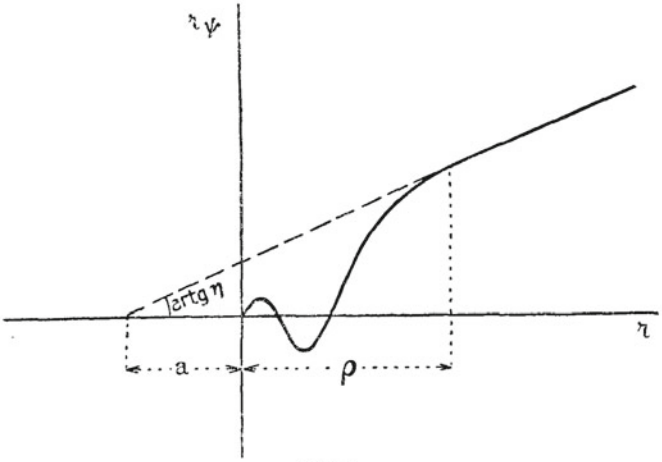

Fig. 3

prende in questo caso una forma particolarmente semplice. Sia ψ l'autofunzione *s* corrispondente alla energia 0. Nella fig. 3 il prodotto $r\psi$ è rappresentato come funzione del raggio vettore r, e ρ rappresenta il raggio d'azione. La curva ha una forma irregolare per $r < \rho$, mentre che per $r > \rho$ diviene una linea retta. Supponiamo di normalizzare ψ in modo che sia $\psi(0) = 1$. Sia allora l'equazione della linea retta che rappresenta $r\psi$ per grandi valori di r:

$$r\psi = \eta(a + r)$$

ove il significato geometrico di η e di a è chiaro dalla figura. Queste due grandezze potrebbero calcolarsi facilmente conoscendo la forma della buca di potenziale che rappresenta il nucleo. Si trova allora che la sezione d'urto elastico, al limite per basse velocità, è

$$\sigma_{el} = 4\pi a^2 \qquad (1)$$

Fermi exploits the same quantum mechanical treatment, and in particular the same figure, for completely different phenomena, where completely different particles and forces are involved (electrons with electromagnetic forces, neutrons with nuclear forces). However, in each case the slowness produces an enormous increment of the cross section.

There is no doubt that the familiarity with slow electrons produces a very favourable conceptual atmosphere for the consideration of the effects of slow neutrons.

Moreover, in these spectroscopic research, in particular with the extraordinary case where Mercury is involved as perturbating gas, Pontecorvo is strongly involved [23]. No doubt that Pontecorvo will play a very important role also in the discovery of the slow neutron effect, in the Fall of 1934.

Even if the Fermi local potential is confined in a very small region, nevertheless it can have dramatic effects. Let only recall that for Cadmium, the absorption cross section for slow neutrons can be hundredths of time larger that the geometric cross section, due to quantum nuclear resonance phenomena.

This lesson coming from very far in time, but with Fermi as lecturer, must be considered very carefully by the numerous people involved in the important topic of studying quantum mechanical very short range interactions.

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Energy Forms and Quantum Dynamics

Ludwig Streit

Abstract We review Dirichlet form methods for the formulation of quantum dynamics where in many cases perturbations of the Laplacian are much more singular than e.g. the Kato class.

Keywords Dirichlet forms · Quantum physics · Polymer models

1 Introduction

In the 1960s Sergio and I both embarked on lifelong expeditions into the world of mathematical physics, first getting acquainted at the famous "Hochstrasse" in Zurich. Arthur Jaffe, a couple of years later, gives a vivid and illustrated account of this very special place, saying "... The experiences that unfolded over those ten weeks in Zürich shaped much of my scientific life ..." [1]. He was not the only one ...

At our time there, the quest for a mathematically sound fundamental theory of matter, combining quantization and relativity, was in the doldrums, in spite of the fact that "elementary particle physics", invoking heuristic arguments and unconcerned with inconsistencies, was eminently successful, producing, e.g. in the realm of quantum electrodynamics, numerical results of impressive precision.

Mathematical physics on the other hand looked in vain for relativistic quantum (field) theories that could do more than describe the trivial dynamics of non-interacting bosons or fermions. As Jost wrote in his book on quantum field theory [2]: "We had very compelling reasons for not mentioning any models except free fields. No interesting models are known ...".

The crux of the matter was encoded in Haag's theorem [3] which states—in simple terms—that the mathematical framework of the above mentioned "free fields" is inadequate to describe realistic interactive dynamics. Unknown representations of

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the dynamical variables, fields and other observables, were needed to construct the dynamics.

The latter, in quantum physics, is to be given as a unitary group, its self-adjoint generator H, the "Hamiltonian", should model the energy of the system. Heuristically this would be attempted by a "perturbative" ansatz

$$H = H_0 + V$$

where the positive self-adjoint operator H_0 would describe the dynamics of free particles, and V their interaction. If V were in a suitable sense small w.r. to H_0, H too would then be self-adjoint and methods of perturbation theory would admit reliable approximate calculations. Unfortunately this is far from being the case in quantum field theory, not even in various interesting models in the much simpler non-relativistic quantum mechanics.

How to replace this perturbative construction of dynamics? There were, both in the same year of 1960, two papers that were suggesting a different approach [4, 5]. Araki's paper had the charm of mathematical rigor, while the paper of Coester and Haag rather appealed to the physicists' intuition. Their common feature was to invoke the lowest energy eigenstate ("ground state" or "vacuum"), substituting it, instead of the interaction energy V, to define the dynamics in canonical quantum field theories.

I was intrigued with these papers at the time, if only for lack of alternatives, and was musing about how to reduce this ansatz to ordinary non-relativistic quantum mechanics, again eliminating the potential V as a dynamical input by the choice of a suitable ground state, such as e.g. of the harmonic oscillator. And what if I used some excited state of the harmonic oscillator instead, what kind of dynamics would I get then? At the time that earned me the comment "good question" from my advisor, and things stopped there for me—until Sergio and Raphael came around with their seminal paper [6] of 1976. It was on quantum field theory in terms of a ground state measure and I was fortunate that they generously shared their insights on what this would produce for the simpler case of non-relativistic quantum mechanics.

2 Dirichlet Forms for Quantum Mechanics

The framework for quantum non-relativistic dynamics in terms of the ground state was laid out in [7]. Schroedinger theory in its typical form will have a Hilbert space $L^2(\mathbb{R}^n, dx)$ and in suitable units the energy operator H is a perturbation of the Laplacian Δ_x by a "potential" $V(x)$

$$H = -\Delta_x + V(x) \tag{1}$$

chosen such that the operator H is self-adjoint and semi-bounded.

Fixing constants such that $H \geq 0$, we single out a zero-energy solution of the Schroedinger equation:

$$(-\Delta_x + V(x))\,\psi_0(x) = 0. \tag{2}$$

Under very general conditions on V this function will be without zeroes, and we proceed to construct the so-called ground state representation of Schroedinger theory by the isomorphism

$$U : \varphi \in L^2\left(\mathbb{R}^n, dx\right) \leftrightarrow f \in L^2\left(\mathbb{R}^n, \psi_0^2(x)dx\right)$$

by setting

$$f(x) = \varphi(x)/\psi_0(x).$$

A straightforward calculation shows that

$$\int \varphi(x)\,(-\Delta_x + V(x))\,\varphi(x)dx = \int \nabla f(x) \cdot \nabla f(x)\psi_0^2(x)dx$$

i.e.

$$(\varphi, H\varphi)_{L^2(\mathbb{R}^n,dx)} = (\nabla f, \nabla f)_{L^2\left(\mathbb{R}^n,\psi_0^2(x)dx\right)}\,.$$

Under the unitary map U, the operator $H = -\Delta_x + V$ takes the form

$$H' := \nabla^*\nabla,$$

no more mention of a potential here, the interaction is encoded in the measure

$$d\mu(x) := \psi_0^2(x)dx.$$

In fact from ψ_0 we can recuperate the potential by Eq. (2):

$$V(x) = \frac{\Delta\psi_0}{\psi_0}. \tag{3}$$

So far so good for "well-behaved" potentials V; the interesting question is now the inverse one. Starting from the expression

$$\left(f, H'f\right) := \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla f(x)d\mu(x),$$

for what measures μ will this produce a unique self-adjoint positive operator H', and by the inverse unitary map to $L^2(\mathbb{R}^n, dx)$, a Schroedinger energy operator H? The answer to that, in [7] and subsequent studies, produced as we shall see a vast extension of admissible quantum dynamics, way beyond traditional perturbation theory.

Specifically, for a positive Radon measure μ on $\mathbb{R}^n$, we can define

$$\varepsilon(f, g) = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x) d\mu(x)$$

for $f, g \in C_0^1(\mathbb{R}^n)$, continuously differentiable functions with compact support. When will this definition give rise to unique self-adjoint operators H' and H, via

$$\varepsilon(f, g) =: \left(f, H' g\right)?$$

This is the case whenever the form ε is has a closure $\bar{\varepsilon}$ on $L^2(d\mu)$, equivalently when the gradient ∇ is a closable operator from $L^2(d\mu)$ to $L^2(d\mu) \bigotimes \mathbb{R}^n$. It then has a densely defined adjoint ∇^*, and there is a unique self-adjoint operator H', with

$$H' = \nabla^* \nabla.$$

To connect with the standard Schroedinger ansatz as in (1) consider absolutely continuous measures

$$d\mu(x) = \rho(x) dx$$

with

$$\psi_0(x) := \sqrt{\rho(x).}$$

We can ask for conditions on ρ or ψ_0 such that the measure μ is admissible in the above sense.

A simple sufficient criterion for admissibility is $\ln \rho \in D(\nabla)$ because then, with $\beta(x) = \nabla \ln \rho$

$$\nabla^* := -\nabla - \beta$$

is a densely defined adjoint of the gradient, hence the form ε is closable. For $f \in C_0^2(\mathbb{R}^n)$

$$H' f = (-\Delta - \beta \cdot \nabla) f.$$

The condition $\beta \in L^2(d\mu) \bigotimes \mathbb{R}^n$ amounts to $\nabla \psi_0 \in L^2(dx) \bigotimes \mathbb{R}^n$, but this admissibility condition can be weakened considerably, a restricted local integrability condition is sufficient as in the following theorem [7].

2.1 Very Singular Interactions

Theorem 1 *Let $d\mu = \psi_0^2 dx$ with $\psi_0 \in L^2_{loc}(\mathbb{R}^n, dx)$ and non-zero almost everywhere, with partial derivatives $\partial_i \psi_0 \in L^2_{loc}(\mathbb{R}^n - N, dx)$ for some closed null set $N \subset \mathbb{R}^n$.*

Then there are unique self-adjoint operators, H' *with* $D(H') \subset D(\bar{\varepsilon}) \subset L^2(d\mu)$ *with*

$$\bar{\varepsilon}(f, g) = \left(f, H'g\right),$$

and H *on* $L^2(dx)$, *with*

$$\bar{\varepsilon}(f_1, f_2) = (\varphi_1, H\varphi_2) \text{ with } \varphi_i = f_i\psi_0$$

whenever $f_1 \subset D(\bar{\varepsilon})$ *and* $f_2 \in D(H')$.

Example 1 For $n = 3$ and $g \geq 0$ the function

$$\psi_0(x) = \sqrt{\frac{g}{2\pi}}\frac{e^{-g|x|}}{|x|} \tag{4}$$

satisfies the conditions of the above theorem, with $N = \{0\}$. $\psi_0(x) \in L^2(dx)$, making it an eigenstate

$$H\psi_0 = 0.$$

On the other hand one finds (in the sense of distributions)

$$\Delta\psi_0(x) = g^2\psi_0(x) - \sqrt{4\pi g}\delta(x),$$

so that for functions φ vanishing at zero

$$H\varphi = \left(-\Delta + g^2\right)\varphi$$

acts like the "free Hamiltonian" $-\Delta$, shifted by a trivial constant g^2. The bound state ψ_0 is thus the result of a "zero-range interaction". These kinds of interactions—aka "pseudopotentials"—are often invoked more or less heuristically as limits of short range interactions with bound state energy $E_0 = -g^2$, but the definition of a potential even in the sense of distributions will fail as one sees from (3). On the other hand, using the ground state representation as above, one can even extend the formalism to one of n particles with zero-range interactions, setting

$$\psi_0\left(x_1, \ldots, x_n\right) = const. \sum_{i<k}\frac{e^{-g|x_i - x_k|}}{|x_i - x_k|}. \tag{5}$$

For much more on such singular perturbations see e.g. [8], and its extensive list of references.

2.2 Barriers

Example 2 Under suitable conditions on the potential V ground state wave functions do not have zeroes. So what if one chose e.g. not the ground state e_0 of the harmonic oscillator Hamiltonian, but the following one

$$\psi_0(x) = e_1(x) = const.\, x \exp\left(-\frac{x^2}{2}\right)?$$

As it turns out, the construction outlined above will have all the odd numbered Hermite functions e_{2n+1} as eigenvectors (all of them with $e_{2n+1}(0) = 0$), as well as their symmetrizations $e_{2n+1}(|x|)$, with the same energy. Hence, equivalently, one can choose the one sided functions $e^{(+)}_{2n+1}(x)$ and $e^{(-)}_{2n+1}(x)$ as a complete set of eigenvectors, restricting e_{2n+1} to the positive and negative half-line, respectively. So the effect of the zero is to decompose the problem; with

$$H = H^{(+)} \bigoplus H^{(-)}$$

acting on

$$L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \bigoplus L^2(\mathbb{R}_-).$$

Sufficient for such a separation is that $\psi_0(x) = O(|x|)$ at least on one side of zero; $\psi_0(x) = O(|x|^\alpha)$ with $\alpha < 1$ would not have this effect. A generalization of separating barriers for quantum dynamics (and diffusions) in $\mathbb{R}^n$ can be found in [9]; and a remarkable application to the Titius–Bode law of planetary spacings in [10].

2.3 Approximation by Regular Perturbations

For many computations in quantum mechanics such as e.g. of scattering matrices it is helpful to approximate singular interactions by more regular ones in a controlled fashion.

Technically one might want to approximate Hamiltonians H arising from energy forms

$$(f, H'g) = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x) \psi^2(x)\, dx$$

by sequences of the usual, perturbative form, $H_n = \Delta + V_n$ which would arise from smoother ground states ψ_n. Strong resolvent convergence in particular is of interest because of its relevance for spectra and wave operators.

Theorem 2 *[11] Consider an increasing sequence of admissible $\psi_n \in L^2_{loc}(\mathbb{R}^n)$, $n = 1, 2, \ldots, \infty$, such that $0 < \psi_1 \le \psi_2 \le \ldots \le \psi_n \nearrow \psi_\infty$ for almost all x in $\mathbb{R}^n$.*

Then the corresponding operators H_n converge to H_∞ in strong resolvent sense.

There is an analogous theorem also for decreasing sequences [11].

Coming back to the zero range "pseudopotentials" with ground states as in equations (4) and more generally (5), it would be very helpful to be able to approximate these dynamics reliably using standard potentials. This can be done on the basis of the above theorem as follows.

Example 3 Approximating the ground state (4) by the sequence

$$\psi_n(x) = \ const.\, \frac{e^{-g|x|}}{|x| + n^{-1}},$$

the conditions of the above theorem are fulfilled and for $n = 1, 2, \ldots$ we obtain a sequence of approximating Hamiltonians

$$H_n = -\Delta + V_n$$

with well-defined potentials

$$V_n = \frac{\Delta\psi_n}{\psi_n}.$$

3 Back to Quantum Field Theory

The stumbling block for the construction of interacting relativistic quantum fields was—and to some extent still is—the need to find appropriate substitutes for the Gaussian "vacuum measures", ground states that are at the heart of the (bosonic) free field models.

Finding such measures would be a task for the developing discipline(s) of stochastic and infinite dimensional analysis, and in 1975 Klauder directed my attention to Hida and the budding research area of White Noise Analysis. Its fundamental concept is a probability measure μ on the space $S'(R^n)$ of Schwartz distributions with characteristic function

$$C(f) = \int\limits_{S^*(R^n)} d\mu\,(\omega) \exp\left(i\,\langle\omega, f\rangle\right) = \exp\left(-\frac{1}{2}\int f^2(t)dt\right) \qquad f\ \in S(R^n).$$

In finite dimensional analysis spaces S and S^* of test and generalized functions form a Gelfand triple

$$S(R^n) \subset L^2(R^n, dt) \subset S^*(R^n).$$

Likewise a space denoted by (S) of smooth "Hida test functions" can be constructed within the Hilbert space $(L^2) \equiv L^2(S^*, d\mu)$, and by duality a Gelfand triple with corresponding space $(S)^*$ of generalized functions:

$$(S) \subset \left(L^2\right) \subset (S)^* .$$

This construction can e.g. be found in [12]. Elements Φ of $(S)^*$ are characterized by the fact that their action $F(z, f) \equiv \left\langle \Phi, e_{zf} \right\rangle$ on exponential functions $e_{zf} \equiv e^{iz\langle \omega, f \rangle} \in (S)$ is an entire function of 2nd order growth in z, and continuous in f. For details on this see e.g. [13]. This criterion is particularly useful since objects like $F(z, f)$ are often computed more or less heuristically in various fields of physics, often under the name of "source functionals"; Feynman integrals are a prominent example which acquire rigorous mathematical meaning once the above conditions are satisfied.

As for measures on the space $S^*(R^n)$, there is an important theorem by Kondratiev, stating that, as in finite dimensional analysis, positive $\Phi \in (S)^*$ are indeed measures [14]. For any such Φ there is a unique measure ν_Φ on the Borel sets of $(S)^*$ such that

$$\left\langle \Phi, e_f \right\rangle = \int\limits_{S^*(R^n)} \exp\left(i \left\langle \omega, f \right\rangle\right) d\nu_\Phi\left(\omega\right) .$$

Coming back to quantum field theory, starting in the late sixties, there finally was some progress on this front—"constructive quantum field theory" began to produce ground state measures for Euclidean Bose fields—would they be in the Hida–Kondratiev class? For the free field this could be verified by direct computation using the characterization theorem. For non-trivial models such as the Hoegh-Krohn model [15], $P(\varphi)_2$, and others [16], there exist moment bounds [17, 18] for the Euclidean fields φ, such as

$$\mathbb{E}\left(\varphi^n(f)\right) \leq a^n \left(n!\right)^{1/k} \|f\|^n \tag{6}$$

for some fixed positive m and $k \geq 2$. Hence by the above characterization theorem and Kondratiev's, the expectations of these fields are expectations with respect to a white noise measure. These measures furthermore admit local Dirichlet forms [19] and the ensuing stochastic partial differential equations ("stochastic quantization") [20].

4 Dirichlet Forms for the Edwards Model

In the memorable year of 1968 Jost assembled a stellar workshop in Varenna, focusing on the budding successes and techniques of constructive quantum field theory. It was there that Varadhan presented his construction of the Edwards model in two spatial dimensions [21].

Informally, in the Edwards model, the Wiener measure $d\mu_0$ has an additional weight factor

$$d\mu_g(x) = \frac{1}{Z}\exp\left(-g\int_0^T ds\int_0^T dt\delta\left(x(s)-x(t)\right)\right)d\mu_0(x) \tag{7}$$

suppressing self-intersections $x(s) = x(t)$ of the paths, with

$$Z = \mathbb{E}\left(\exp\left(-g\int_0^T ds\int_0^T dt\delta\left(x(s)-x(t)\right)\right)\right).$$

The self-intersection local time

$$L = \int_0^T ds\int_0^T dt\delta\left(x(s)-x(t)\right)$$

becomes increasingly singular for higher-dimensional Brownian motion [22]. In particular a proper definition has to pass through a regularization of the Dirac distribution, renormalizations, and limit taking. A common regularization is by the replacement

$$\delta_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{d/2}}e^{-\frac{|x|^2}{2\varepsilon}}, \quad \varepsilon > 0,$$

$$L_\varepsilon(T) := \int_0^T dt\int_0^T ds\,\delta_\varepsilon(x(t)-x(s)), \tag{8}$$

others would use $|t-s| > \varepsilon$ to control the singularity at $t = s$.

Varadhan showed for the centered local time

$$L_\varepsilon^c = L_\varepsilon - \mathbb{E}(L_\varepsilon)$$

the existence of $L_0^c = \lim_{\varepsilon\searrow 0} L_\varepsilon^c$ and that

$$Z = \mathbb{E}\left(\exp(-gL_0^c)\right) < \infty,$$

so that for the centered local time the Edwards model for $d = 2$ becomes well-defined.

Much more recently we extended this result to fractional Brownian motion $x(t) = B^H(t)$ where B^H is a centered Gaussian process with covariance

$$\mathbb{E}(B_t^H B_s^H) = \frac{\delta_{ij}}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),\ i, j = 1,\ldots,d,\ s, t \geq 0, 0 < H < 1.$$

The Edwards model can be constructed [23] for dimensions d and Hurst parameter H as long as $Hd \leq 1$.

So what about a local Dirichlet form on the $L^2(d\mu)$ space with the Edwards measure μ? There again Sergio et al. [24] showed the way. For the Varadhan model, i.e. for $d = 2$ and the authors show the existence of a local Dirichlet form and of a diffusion process whose invariant measure is the two-dimensional Edwards measure [25].

Following once again the lead of Sergio, we managed to extend this result [25, 26] to fractional Brownian motion for $Hd \leq 1$ and more recently to fractional Brownian loops and trees.

5 Conclusion

Looking back now it is patent that I have been learning from Sergio throughout my professional life—*ad multos annos, dear master and friend.*

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The Emergence of Noncommutative Potential Theory

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Abstract We review origins and developments of Noncommutative Potential Theory as underpinned by the notion of energy form. Recent and new applications are shown to approximation properties of von Neumann algebras.

Keywords Noncommutative Dirichlet form · Noncommutative Potential Theory

Mathematics Subject Classification 46L89 · 31C25 · 46L10 · 60J45 · 81R15

1 Introduction

Our intent here is to trace some of the main steps of Noncommutative Potential Theory, starting from the seminal works by Albeverio and Hoegh-Krohn [2, 3]. The point of view adopted in treating Potential Theory it is essentially the one of Dirichlet forms, i.e. the point of view of Energy. The justification for this is that, not only the motivating situations to develop a potential theory on operator algebras came from Mathematical Physics but also that the concept of Energy seems to have a unifying character with respect to the different aspects of the subject.

The present exposition is thought to be addressed to researcher not necessarily familiar with the tools of operator algebras and, in this respect, we privileged the illustration of examples and applications instead to provide the details of the proofs.

In this presentation several aspects of the theory has been necessarily sacrificed and for them we refer to other presentations [27, 28]. In particular, the construction of Fredholm modules and Dirac operators from Dirichlet forms and the realization of Dirichlet spaces as instances of Connes' Noncommutative Geometry [42] can

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be found in [27, 31, 34, 46, 94, 97] while the study of energy states, potentials and multipliers of noncommutative Dirichlet spaces has been initiated in [35]. The details of the theory on KMS symmetric Markovian semigroups on C*-algebras can be found in [28].

The recent developments of the theory of noncommutative Dirichlet forms show a not rare situation in Mathematics in which a theory born to solve specific problems, as time goes by, applies to, apparently far away, others. In this respect we review in Sect. 7 the recent close relationships among spectral characteristics of noncommutative Dirichlet forms and approximation properties of von Neumann algebras such as Haagerup Property (H), amenability and Property (T). In particular a new characterization of the Murray–von Neumann Property (Γ) is proved in terms of absence of a Poincaré inequality for elementary Dirichlet forms.

2 Commutative Potential Theory

2.1 Classical Potential Theory

Classical Potential Theory concerns properties of the Dirichlet integral

$$\mathcal{D} : L^2(\mathbb{R}^d, m) \to [0, +\infty] \qquad \mathcal{D}[u] := \int_{\mathbb{R}^d} |\nabla u|^2 \, dm$$

as a lower semicontinuous quadratic form on the Hilbert space $L^2(\mathbb{R}^d, m)$, which is finite on the Sobolev space $H^1(\mathbb{R}^d)$. The associated positive, self-adjoint operator is the Laplace operator

$$\Delta = -\sum_{k=1}^{d} \partial_k^2 \qquad \mathcal{D}[u] = \|\sqrt{\Delta} u\|_2^2$$

which generates the heat semigroup $e^{-t\Delta}$ whose Gaussian kernel

$$e^{-t\Delta}(x, y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}$$

is the fundamental solution of the heat equation $\partial_t u + \Delta u = 0$. The Brownian motion (Ω, P_x, B_t) is the stochastic processes associated to the semigroup by the relation

$$(e^{-t\Delta} u)(x) = \mathbb{E}_x(u \circ B_t)$$

which is also directly connected to the Dirichlet integral by the identity

$$\mathcal{D}[u] = \lim_{t\to 0^+} \frac{\mathbb{E}_m(|u\circ B_t - u\circ B_0|^2)}{2t}.$$

The polar sets, i.e. those sets which are avoided by the Brownian motion, can be characterized as those which have vanishing electrostatic capacity, defined in terms of the Dirichlet integral itself as

$$\mathrm{Cap}(A) := \inf\{\mathcal{D}[u] + \|u\|_2^2 : u \in H^1(\mathbb{R}^n), \quad 1_A \leq u\}$$

for any open set $A \subseteq \mathbb{R}^n$ and then as

$$\mathrm{Cap}(B) := \inf\{\mathrm{Cap}(A) : B \subseteq A, \quad A \text{ open}\}$$

for any other measurable set $B \subseteq \mathbb{R}^n$. The heat semigroup is Markovian on $L^2(\mathbb{R}^n, m)$ in the sense that it is strongly continuous, contractive, positivity preserving and satisfies $e^{-t\Delta}u \leq 1$ whenever u is a real function such that $u \leq 1$. By these properties it can be extended to a contractive and positivity preserving semigroup on any $L^p(\mathbb{R}^n, m)$ for $p \in [1, +\infty]$ which is strongly continuous for $p \in [1, +\infty)$ and weakly*-continuous for $p = +\infty$. The Markovianity of the heat semigroup is equivalent to the following property, also called Markovianity, of the Dirichlet integral

$$\mathcal{D}[u \wedge 1] \leq \mathcal{D}[u] \qquad u = \bar{u} \in L^2(\mathbb{R}^n, m)$$

which can be easily checked using differential calculus and the definition of the Dirichlet integral. All others above properties can be proved by the explicit knowledge of the Green kernel of heat semigroup which, for $d \geq 3$ at least, equals

$$\Delta^{-1}u(x) = \int_{\mathbb{R}^d} G(x, y)u(y)\, m(dy) \qquad G(x, y) = c_d \cdot |x - y|^{2-d}.$$

2.2 Beurling–Deny Potential Theory [10, 22, 53, 58, 75, 99, 100]

A turning point in the development of potential theory was represented by two seminal papers by Beurling and Deny [9, 10]. They developed a *kernel free potential theory* based on the notion of *energy* on general locally compact measured spaces (X, m). The whole theory relies on the notion of *regular Dirichlet form* which is required to be a lower semicontinuous quadratic functional on $L^2(X, m)$ satisfying

- *Markovianity*:

$$\mathcal{E} : L^2(X, m) \to [0, +\infty] \qquad \mathcal{E}[u \wedge 1] \leq \mathcal{E}[u]$$

- *regularity*: $\mathcal{F} \cap C_0(X)$ is a form core uniformly dense in $C_0(X)$

and where the form domain $\mathcal{F} := \{u \in L^2(X,m) : \mathcal{E}[u] < +\infty\}$ is assumed to be L^2-dense. The lower semicontinuity of $\mathcal{E}$ on $L^2(X,m)$, being equivalent to the closedness of the densely defined quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(X,m)$, implies the existence of a nonnegative, self-adjoint operator $(L, D(L))$ which generates a Markovian semigroup e^{-tL} on $L^2(X,m)$.

2.2.1 Beurling–Deny Decomposition

One of the first fundamental results in the Beurling–Deny analysis concerns the structure of a general regular Dirichlet form: these can be uniquely realized as a sum of three Markovian forms (each of which not necessarily closed)

$$\mathcal{E} = \mathcal{E}^d + \mathcal{E}^j + \mathcal{E}^k$$

where the *jumping part* has the form

$$\mathcal{E}^j[u] = \int_{X\times X\setminus\Delta_X} |u(x) - u(y)|^2 \, J(dx, dy)$$

for a positive measure J supported off the diagonal Δ_X of $X \times X$, the *killing part* appears as

$$\mathcal{E}^k[u] = \int_X |u(x)|^2 \, k(dx)$$

for some positive measure k on X and the *diffusion part is strongly local* in the sense that

$$\mathcal{E}^d[u+v] = \mathcal{E}^d[u] + \mathcal{E}^d[v]$$

whenever u is constant in a neighborhood of the support of v.

Two turning point in the development of Potential Theory took place on the probabilistic side when Fukushima associated a Hunt stochastic process (Ω, P_x, X_t) to a regular Dirichlet form in such a way that

$$\mathcal{E}[u] = \lim_{t\to 0^+} \frac{\mathbb{E}(|u \circ X_t - u \circ X_0|^2)}{2t} \qquad u \in \mathcal{F}$$

and when Silverstein introduced the notion of *extended Dirichlet space*, especially for the connections with the boundary theory and the random time change of symmetric Hunt processes.

The process is a stochastic dynamical system which represents the semigroup through

$$(e^{-tL}u)(x) = \mathbb{E}_x(u \circ X_t).$$

A basic tool in the development of the Beurling–Deny theory of a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is the capacity one associates to it exactly in the same way we have seen above in the case of Dirichlet integrals. A key point to construct the associated stochastic processes is the fact that the regularity property of the Dirichlet form allows to prove that the capacity associated to it is in fact a *Choquet capacity* which implies that Borel sets are capacitable.

From the point of view of the process, the three different summands of the Beurling–Deny decomposition have a nice and useful probabilistic interpretation: the measure J counts the jumps of the process, the measure k specifies the rate at which the process is killed inside X and a Dirichlet form is strongly local if and only if the associated process is a diffusion, i.e. it has continuous sample paths. In Sect. 5.2 we will show an independent, algebraic way to prove the above decomposition of Dirichlet forms.

3 Operator Algebras

3.1 C*-Algebras as Noncommutative Topology [6, 54, 89, 105]

A C*-algebra is A is a Banach $*$-algebra in which norm and involution conspire as follows

$$\|a^*a\| = \|a\|^2 \qquad a \in A.$$

This notion generalizes topology in an algebraic form in the sense that, by a theorem of Gelfand, a *commutative* C*-algebra A is isomorphic to the algebra $C_0(X)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space X, called the *spectrum* of A. In $C_0(X)$ the product of functions is defined pointwise, the involution is given by pointwise complex conjugation and the norm is the uniform one.

The simplest example of a *noncommutative* C*-algebra is the full matrix algebra $M_n(\mathbb{C})$ where the product is the usual rows-by-columns, the involution of a matrix A is defined as its matrix adjoint A^* and the norm $\|A\|$ is given by the operator norm (the largest singular value of A, i.e. the square root of the largest eigenvalue of A^*A).

Finite dimensional C*-algebras are isomorphic to finite direct sums of full matrix algebras. The simplest examples of noncommutative, *infinite dimensional* C*-algebras are those of the algebra $B(h)$ of all bounded operators and its subalgebra of all compact operators $\mathcal{K}(h)$ on an infinite dimensional Hilbert space h, the norm being the operator one.

A morphism $\alpha : A \to B$ between C*-algebras A, B is a norm continuous $*$-algebras morphism. A first example of the deep interplay that the algebraic and the analytic structures on a C*-algebra give rise, is the fact that *-algebra morphisms are automatically norm continuous. Morphisms between commutative C*-algebras $C_0(X)$ and $C_0(Y)$ correspond to homeomorphisms $\phi : Y \to X$ by $\alpha(f) = f \circ \phi$.

A morphism of type $\pi : A \to B(h)$ is called a representation of A on the Hilbert space h. It is called faithful if it is an injective map and in this case A can be identified with the C*-subalgebra $\pi(A) \subseteq B(h)$. Any C*-algebra admits a faithful representation.

A C*-algebras is, in particular, an ordered vector space where the closed cone is given by

$$A_+ := \{a^*a \in A : a \in A\}.$$

When A is represented as a subalgebra of some $B(h)$, the positive elements of A are positive, self-adjoint operators on h. In $C_0(X)$, the positive elements are just the nonnegative functions.

3.2 von Neumann Algebras as Noncommutative Measure Theory [55, 84, 89]

A von Neumann algebra M is a C*-algebra which admits a predual M_* as a Banach space in the sense that $(M_*)^* = M$.

Any commutative, σ-finite[1] von Neumann algebra is isomorphic to the algebra of (classes of) essentially bounded measurable functions $L^\infty(X, m)$ on a measured standard space (X, m) with $L^1(X, m)$ as predual space. This commutative situation forces to regard the theory of von Neumann algebras as a noncommutative generalization of Lebesgue measure theory. Even if this is a fruitful point of view, other natural constructions suggest to look at the theory as a generalization of Euclidean Geometry and as a generalization of Harmonic Analysis.

The simplest example of a noncommutative von Neumann algebra is that of the space $B(h)$ of all bounded operators acting on a Hilbert space h having dimension greater than one. The predual of $B(h)$ is given by the Banach space $L^1(h)$ of trace-class operators on h and the duality is given by

$$\langle A, B\rangle := \mathrm{Tr}(AB) \qquad A \in B(h), \quad B \in L^1(h).$$

All C*-algebras are isomorphic to norm-closed subalgebras of some $B(h)$ and all von Neumann algebras are isomorphic to subalgebras of some $B(h)$, closed in its weak*-topology. A first fundamental results of von Neumann asserts that for any subset $S \subseteq B(h)$, its *commutant*

$$S' := \{a \in B(h) : ab = ab, \ \text{ for all } \ b \in B(h)\}$$

[1] A von Neumann algebra is σ-finite if all collections of mutually disjoint orthogonal projections have at most a countable cardinality. von Neumann algebras acting on separable Hilbert spaces are σ-finite (the converse being in general not true).

is a von Neumann algebra. A second fundamental result of von Neumann asserts that an involutive subalgebra $M \subseteq B(h)$ is a von Neumann algebras iff it is weakly*-closed and iff it coincides with its double commutant $M = M'' := (M')'$. A key aspect is that for an involutive subalgebra $M \subseteq B(h)$ its weak*-closure coincides with its double commutant $(M')'$.

The center of a von Neumann algebra is defined as

$$\mathrm{Center}(M) := \{x \in M : xy = yx, \ \ y \in M\}.$$

and M is called a *factor* if its center reduces to the one dimensional algebra $\mathbb{C} \cdot 1_M$ of scalar multiples of the unit of M.

3.3 Weights, Traces, States and the GNS Representation [54, 89]

A *positive functional* on a C^*-algebra A is a linear map $\tau : A \to \mathbb{C}$ such that

$$\tau(a) \geq 0 \qquad a \in A_+.$$

These are automatically bounded and are called *states* when having norm one. In case A has a unit, a positive functional is a state as soon as $\tau(1_A) = 1$ as it follows from $0 \leq a \leq \|a\|_A \cdot 1_A$. Positive functionals are noncommutative analog of finite, positive Borel measures on locally compact spaces: in fact, by the Riesz Representation Theorem, a positive functional on a commutative C*-algebra $C_0(X)$ corresponds, via Lebesgue integration, to a finite, positive Borel measure m on X

$$\tau(a) = \int_X a\, dm \qquad a \in C_0(X),$$

which is a probability if and only if τ is a state. To accommodate the analog of possibly unbounded positive Borel measures, one has to consider *weights* on A defined as functions $\tau : A_+ \to [0, +\infty]$ which are *homogeneous* and *additive* in the sense

$$\tau(\lambda a) = \lambda \tau(a), \qquad \tau(a + b) = \tau(a) + \tau(b) \qquad a, b \in A_+, \quad \lambda \geq 0.$$

If a weight is everywhere finite, then it can be extended to a positive linear functional on A. A weight is called a *trace* if it is invariant under inner automorphisms in the sense

$$\tau(uau^*) = \tau(a), \qquad a \in A_+$$

for all unitaries $u \in \widetilde{A} = A \oplus \mathbb{C}$ (recall that A is a two-sided ideal in $\widetilde{A}$). This is equivalent to require that τ is *central* in sense that

$$\tau(a^*a) = \tau(aa^*) \qquad a \in A.$$

If τ is finite this reduces to

$$\tau(ab) = \tau(ba) \qquad a, b \in A.$$

A weight is *faithful* if it vanishes $\tau(a) = 0$ on $a \in A_+$ only when $a = 0$. In the commutative case, faithful weights correspond to fully supported positive Borel measures.

A weight is *densely defined* if the ideal $A^\tau := \{a \in A_+ : \tau(a^*a) < +\infty\}$ is dense in A. If a trace is lower-semicontinuous, than it is *semifinite* in the sense that

$$\tau(a) = \sup\{\tau(b) \in A_+ : b \le a\} \qquad b \in A_+.$$

On a von Neumann algebra, a weight is *normal* if

$$\tau(\sup_{i\in I} a_i) = \sup_{i\in I} \tau(a_i)$$

for any net $\{a_i : i \in I\} \subset A_+$ admitting a least upper bound in A_+. The predual Banach space M_* of a von Neumann algebra M can be shown to the space of all normal continuous functionals on M. A von Neumann algebra is said to be *finite* (resp. *semi-finite*) if, for every non-zero $a \in A_+$, there exists a finite (resp. semi-finite) normal trace τ such that $\tau(a) > 0$ and it is said *properly infinite* (resp. *purely infinite*) if the only finite (resp. semi-finite) normal trace on A is zero. On a semi-finite von Neumann algebra, there exists a semi-finite faithful normal trace.

In this exposition we will be essentially concerned with semi-finite von Neumann algebras and in particular with those which are *σ-finite* in the sense that they admit a faithful, normal state. If h is a separable Hilbert space, then any von Neumann algebra $A \subseteq B(h)$ is σ-finite. In fact, for any Hilbert base $\{e_k \in h : k \in \mathbb{N}\}$, a faithful, normal state is provided by

$$\tau(x) := \sum_{k\in\mathbb{N}} (e_k|xe_k)_h \qquad x \in A.$$

In a way similar to the one by which a probability measure m on X give rise to the Hilbert space $L^2(X, m)$ and to the representation of continuous functions in $C_0(X)$ as multiplication operators on it, a densely defined weight τ on a C*-algebra A give rise to a Hilbert space $L^2(A, \tau)$ on which the elements $a \in A$ act as bounded operators. This is called the Gelfand–Neimark–Segal or GNS-representation of A associated to τ.

In fact, the sesquilinear form $x, y \mapsto \tau(x^*y)$ on the vector space A, satisfies the *Cauchy–Schwarz inequality*

$$|\tau(x^*y)|^2 \le \tau(x^*x)\tau(y^*y) \qquad x, y \in A^\tau$$

and a Hilbert space $L^2(A, \phi)$ can be constructed from the inner product space A^τ by separation and completion. Since A^τ is an ideal of A, the left regular action $b \mapsto ab$ of A onto itself give rise to an action of A onto A^τ and then to a representation of A on the GNS Hilbert space. If τ is faithful, the identity map of A give rise to an injective, bounded map $A \to L^2(A, \tau)$ and if A is unital, the vector $\xi_\phi \in L^2(A, \tau)$ image of the identity $1_A \in A$, allows to represent the state ϕ by $\tau(x) = (\xi_\tau | x\xi_\tau)_2$. This vector, uniquely determined by this property, is *cyclic* in the sense that $\overline{A\xi_\tau} = L^2(A, \tau)$ and *separating* in the sense that if $a \in A$ and $a\xi_\tau = 0$ then $a = 0$.

The von Neumann algebra $L^\infty(A, \tau) := (\pi_{\mathrm{GNS}}(A))'' \subseteq B(L^2(A, \tau))$ obtained by w*-completion, is called the *von Neumann algebra generated by* τ *on* A. The GNS-representation can then be extended to a normal representation of $L^\infty(A, \tau)$. As notations are aimed to suggest, this is a generalization of the usual construction of Lebesgue–Riesz measure theory.

In the case of the trace functional Tr on $\mathcal{K}(h)$, the associated GNS space is $L^2(\mathcal{K}(h), \mathrm{Tr}) = L^2(h)$ the space of Hilbert–Schmidt operators on which compact operators in $\mathcal{K}(h)$ act by left composition.

The Hilbert space of the GNS representation of a faithful trace is naturally endowed with a closed convex cone $L^2_+(A, \tau)$, which provides an order structure on $L^2(A, \tau)$. It is defined as the closure of A^τ. In the commutative case $A = C_0(X)$, this is just the cone of square integrable, positive functions. $L^2(X, m)$. The construction of a suitable closed, convex cone from a faithful state on a C*-algebra or from a faithful normal state on a von Neumann algebra will be done later on.

We conclude this section mentioning that a noncommutative integration theory for traces on C*-algebras has been developed in [85, 98] giving rise to an interpolation scale of spaces $L^p(A, \tau)$ between the von Neumann algebra $L^\infty(A, \tau)$ and its predual $L^1(A, \tau)$. The elements of this spaces can realized as closed operators on $L^2(A, \tau)$.

3.4 Morphisms of Operator Algebras [89]

The most obvious notion of morphism to form a category of C*-algebras is certainly that of continuous morphisms of involutive algebras. However, this category risks to have a poor amount of morphisms. For example, if $\alpha : A \to B$ is a morphism and B is commutative then $\alpha(ab - ba) = \alpha(a)\alpha(b) - \alpha(b)\alpha(a) = 0$ so that if the algebra generated by commutators $[a, b] := ab - ba$ is dense in A then $\alpha = 0$. This is the case for example of $A = \mathcal{K}(h)$ or more generally for the so called *stable* C*-algebras.

We illustrate now a much more well behaved notion of morphism between C* and von Neumann algebras, i.e. *completely positive map* (an even more general and fundamental notion of morphism is that of *Connes correspondence*, which we will meet later on in this lectures). This notion is of probabilistic nature in the sense that, among commutative von Neumann algebras, completely positive maps are just the transformations associated to *positive kernels*. Notice, *en passant*, that another basic tool in operator algebra theory which is of clearly probabilistic nature is the notion of *conditional expectation*. See discussion in [42, Chap. 5, Appendix B].

Beside to any C*-algebra A we may consider its *matrix ampliations* $A \otimes \mathbb{M}_n(\mathbb{C})$, $n \geq 1$. A linear map $T : A \to B$ is said to be *completely positive*, or CP map, if its ampliations

$$T \otimes I_n : A \otimes \mathbb{M}_n(\mathbb{C}) \to B \otimes \mathbb{M}_n(\mathbb{C})$$

are positive for any $n \geq 1$. $*$-algebra morphisms (such as representations) are completely positive maps. If A or B is commutative, all positive maps (in particular, states) are automatically completely positive. Complete positivity is however a much more demanding property than just positivity. While the general structure of positive maps is rather elusive, even in a finite dimensional setting, the structure of CP maps is completely described by the Stinespring Theorem [103]. We may consider, without loss of generality, the case of a CP map $T : A \to B(h)$. The result ensures the existence of a representation $\pi : A \to B(k)$ on a Hilbert space k and that of a bounded operator $V : h \to k$ such that

$$Ta = V^*\pi(a)V \qquad a \in A.$$

In case A is unital and $T1_A = 1_A$, then $V^*V = I_h$ so that V is an isometry which can be considered as an immersion of h into k. V^* is then the projection of k onto h and the CP map T results as the *compression of the restriction of a representation*. The Stinespring construction can be considered as a generalization of the GNS representation. One starts endowing the vector space $A \otimes_{\mathrm{alg}} h$ by the sesquilinear form

$$(a \otimes \xi | b \otimes \eta) := (\xi | T(a^*b)\eta)_h \qquad a, b \in A, \quad \xi, \eta \in h$$

and checks that the CP property just ensures that this form is positive definite. Cutting out its kernel and completing the normed space obtained, one gets the Hilbert space k. The representation of A on k is an ampliation of the left regular representation of A as it is induced by the map $a(b \otimes \xi) \mapsto ab \otimes \xi$.

A positivity preserving map $\phi : M \to N$ between von Neumann algebras, is *normal* if $\phi(\sup_\alpha x_\alpha) = \sup_\alpha \phi(x_\alpha)$ for all bounded monotone increasing nets of self adjoint elements $\{x_\alpha\} \subset M$. The property is equivalent to the continuity with respect to weak*-topology of the algebras.

3.5 Positivity Preserving and Markovian Semigroups on Operator Algebras [15]

A strongly continuous semigroup $\{T_t : t > 0\}$ of contractions on a unital C*-algebra A

$$T_t : A \to A \qquad T_t \circ T_s = T_{t+s}, \quad T_0 = I, \quad \lim_{t \to 0^+} \|a - T_t a\|_A = 0, \quad a \in A$$

is said to be *Markovian* if it is *positivity preserving and subunital*

$$0 \le a \le 1_A \quad \Rightarrow \quad 0 \le T_t a \le 1_A \qquad a \in A.$$

If A is endowed with a densely defined trace τ, the semigroup is said to be τ-symmetric if

$$\tau(a^*(T_t b)) = \tau((T_t a^*)b) \qquad a, b \in A \cap L^1(A, \tau).$$

In case A is a von Neumann algebra, one requires the trace to be normal and the semigroup to be point-weak*-continuous in the sense

$$\lim_{t \to 0^+} \eta(a - T_t a) = 0 \qquad a \in A, \;\; \eta \in A_*.$$

In case the C*-algebra A does not have a unit, one can understand positivity preserving and Markovianity embedding A into a larger unital C*-algebra $\widetilde{A}$ and there using the unit $1_{\widetilde{A}}$ instead of 1_A. For example one can choose $A \oplus \mathbb{C}$.

The generator $(L, D(L))$ of a Markovian semigroup on a C*-algebra (resp. a von Neumann algebra) A is a norm (resp. weak*) closed, densely defined operator on A defined as

$$D(L) := \left\{ a \in A : \exists \lim_{t \to 0^+} \frac{a - T_t a}{t} \in A \right\} \qquad La := \lim_{t \to 0^+} \frac{a - T_t a}{t} \quad a \in D(L)$$

where the limit is understood in the norm (resp. weak*)-topology. Norm continuous semigroups are exactly those which have bounded generators and these are classified in [23, 76]. *Completely positive, completely contractive or completely Markovian semigroups* are defined as those semigroups on A whose ampliations to the algebras $A \otimes \mathbb{M}_n(\mathbb{C})$ are positive, contractive or Markovian for all $n \ge 1$. Completely Markovian semigroups are also called *dynamical semigroups* especially in Mathematical Physics and Quantum Probability (see [44]).

Remark 3.1 Notice that, on von Neumann algebras, *strongly continuous semigroups are automatically norm continuous* as it follows by a direct application of [56, Theorem 1]. Since semigroups with bounded generators have rather limited applications, this is the reasons for which *on von Neumann algebra the natural continuity of a semigroups is the point-weak*-continuity*.

4 Noncommutative Potential Theory

In this section, we let (A, τ) be a C*-algebra endowed with a densely defined, lower semicontinuous faithful trace and consider the GNS representation π_{GNS} acting on the space $L^2(A, \tau)$. We will indicate by $L^\infty(A, \tau)$ the von Neumann algebra $(\pi_{\mathrm{GNS}}(A))'' \subseteq B(L^2(A, \tau))$ generated by A through the GNS representation.

Recall that the little Lipschitz algebra is defined as

$$\mathrm{Lip}_0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f(0) = 0, |f(t) - f(s)| \le |t - s|, t, s \in \mathbb{R}\}.$$

If $a = a^* \in A$ and $f \in \mathrm{Lip}_0(\mathbb{R})$, then $f(a) \in A$ acquires a meaning thank to the fact that C*-algebras are closed under continuous functional calculus. Since, by assumption, $A^\tau := A \cap L^2(A, \tau)$ is dense in A and a fortiori in $L^2(A, \tau)$, if $a = a^* \in L^2(A, \tau)$ then $f(a) \in L^2(A, \tau)$ may be defined as the limit in $L^2(A, \tau)$ of the sequence $f(a_n) \in L^2(A, \tau)$ associated to a sequence $a_n \in A \cap L^2(A, \tau)$ converging to a in $L^2(A, \tau)$.

4.1 Dirichlet Forms on C-Algebras with Trace d'après Albeverio–Hoegh-Krohn*

In this section we define Dirichlet forms and Markovian semigroups on the space $L^2(A, \tau)$ and discuss the connection between them and the Markovian semigroups on the von Neumann algebra $L^\infty(A, \tau)$, where (A, τ) is a C*-algebra A endowed with a densely defined, lower semicontinuous faithful trace, introduced in [2]. Even if we will not discuss them in this notes, we mention that Guido et al. in [65] provided the extension of this theory to the case of *non-symmetric Dirichlet forms*.

Definition 4.1 A *Dirichlet form* is a lower semicontinuous functional

$$\mathcal{E} : L^2(A, \tau) \to (-\infty, +\infty]$$

with domain $\mathcal{F} := \{a \in L^2(A, \tau) : \mathcal{E}[a] < +\infty\}$ satisfying the properties

(i) $\mathcal{F}$ is dense in $L^2(A, \tau)$
(ii) $\mathcal{E}[a^*] = \mathcal{E}[a]$ for all $a \in L^2(A, \tau)$ (reality)
(iii) $\mathcal{E}[f(a)] \le \mathcal{E}[a]$ for all $a = a^* \in L^2(A, \tau)$ and all $f \in \mathrm{Lip}_0(\mathbb{R})$ (Markovianity).
A Dirichlet form is said to be
(iv) *regular* if its domain $\mathcal{F}$ is dense in A
(v) *complete Dirichlet form* if the ampliation $\mathcal{E}^n$ on the algebra $(A \otimes M_n(\mathbb{C}), \tau \otimes \mathrm{tr}_n)$ defined

$$\mathcal{E}^n : L^2(A \otimes M_n(\mathbb{C}), \tau \otimes \mathrm{tr}_n) \to (-\infty, +\infty]$$
$$\mathcal{E}^n[[a_{i,j}]_{i,j=1}^n] := \sum_{i,j=1}^n \mathcal{E}[a_{i,j}]$$

is a Dirichlet forms for all $n \ge 1$.
A strongly continuous, self-adjoint semigroup $\{T_t : t > 0\}$ on $L^2(A, \tau)$ is said
(vi) *positivity preserving* if $T_t a \in L^2_+(A, \tau)$ for all $a \in L^2_+(A, \tau)$
(vii) *Markovian* if it is positivity preserving and for $a = a^* \in A \cap L^2(A, \tau)$

$$0 \le a \le 1_{\tilde{A}} \quad \Longrightarrow \quad 0 \le T_t a \le 1_{\tilde{A}} \qquad t > 0$$

(viii) *completely Markovian* if the extensions $T_t^n := T_t \otimes I_n$ to $L^2(A \otimes M_n(\mathbb{C}), \tau \otimes \mathrm{tr}_n)$ are Markovian semigroups for all $n \ge 1$.

Remark 4.2

(1) If in the Markovianity condition one considers as f the zero function in $\mathrm{Lip}_0(\mathbb{R})$, one verifies that Dirichlet forms are nonnegative.
(2) It may be checked that Markovianity is equivalent to the single contraction property

$$\mathcal{E}[a \wedge 1] \le \mathcal{E}[a] \qquad a = a^* \in L^2(A, \tau)$$

in which only the *unit contraction* $f(t) := t \wedge 1$ is involved. A geometric Hilbertian interpretation of this fact will be vital to extend the theory beyond the trace case.
(3) A nice characterization of elements of type $f(a)$ for a fixed $a = a^* \in L^2(A, \tau)$ and $f \in \mathrm{Lip}_0(\mathbb{R})$ has been shown in [2] as those hermitian $b = b^* \in L^2(A, \tau)$ such that

$$b^2 \le a^2, \qquad |b \otimes 1 - 1 \otimes b|^2 \le |a \otimes 1 - 1 \otimes a|^2.$$

(4) Since $L^2(A \otimes M_n(\mathbb{C}), \tau \otimes \mathrm{tr}_n) = L^2(A, \tau) \otimes L^2(M_n(\mathbb{C}), \mathrm{tr}_n)$, the ampliations are equivalently defined as

$$\mathcal{E}[a \otimes m] := \mathcal{E}[a] \cdot \|m\|^2_{\mathrm{HS}} \qquad a \otimes m \in L^2(A, \tau) \otimes L^2(M_n(\mathbb{C}), \mathrm{tr}_n).$$

The first fundamental result of the Albeverio–Hoegh-Krohn work [2] is the following correspondence which generalize that of Beurling–Deny in the commutative case.

Theorem 4.3 *There exists a one-to-one correspondence among*

(i) Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on $L^2(A, \tau)$
(ii) Markovian semigroups $\{T_t : t > 0\}$ on $L^2(A, \tau)$
(iii) τ-symmetric, Markovian semigroups $\{S_t : t > 0\}$ on the von Neumann algebra $L^\infty(A, \tau)$. Moreover, the semigroups are completely Markovian if and only if the quadratic form is a completely Dirichlet form.

The correspondence between semigroups and quadratic forms on $L^2(A, \tau)$ is given by the relation

$$\mathcal{E}[a] = \lim_{t \to 0} t^{-1}\big(a|(I - T_t)a\big)_{L^2(A,\tau)} \qquad a \in L^2(A, \tau)$$

where both sides are finite precisely when $a \in \mathcal{F}$. The correspondence between the semigroup on the Hilbert space $L^2(A, \tau)$ and the one the von Neumann algebra $L^\infty(A, \tau)$ is given by

$$S_t a = T_t a \qquad a \in A \cap L^2(A, \tau).$$

Remark 4.4 The above correspondence is exactly the original one proved in [2] even if the result still holds true if one start with a semi-finite von Neumann algebra (M, τ) and a densely defined, semifinite trace on it.

We prefer the first presentation since it prepares the ground (i) to naturally introduce and discuss the notion of *regularity* of a noncommutative Dirichlet forms, which, as in the Beurling–Deny theory, is the key notion to develop a rich potential theory [35] and (ii) to develop the intrinsic differential calculus of Dirichlet spaces (see Sect. 5 below and [32]).

The second fundamental result of the Albeverio–Hoegh-Krohn work is the following

Theorem 4.5 *Let the C*-algebra A be represented as acting on a Hilbert space h. Let K be a self-adjoint (non necessarily bounded) operator on h and $m_i \in L^2(h)$ be Hilbert–Schmidt operators for $i = 1, 2, \ldots$. Then the quadratic form*

$$\mathcal{E}[a] := \sum_{i=1}^{\infty} \mathrm{Tr}(|[a, m_i]|^2) + \mathrm{Tr}(K|a|^2) \qquad a \in L^2(A, \tau)$$

is a completely Dirichlet form provided it is densely defined.

This result is fundamental not only because it provides a tool to construct a large class of examples but also because it suggests, at least in one direction, a correspondence between completely Dirichlet forms and unbounded derivations $a \mapsto i[a, m]$ on the C*-algebra A (see Sect. 5 below).

The proofs in [2] of both theorems are based on a careful analysis of the normal contractions on A.

4.1.1 Dirichlet Energy Forms on Clifford C*-Algebras

Here we illustrate the first example of a noncommutative Dirichlet form. It has been created to represents the quadratic form of a physical Hamiltonian of an assembly of electrons and positrons. In particular, its definition and the study of its properties has been introduced by L. Gross [63, 64] in connection with the problem of existence and uniqueness of the ground state of physical Hamiltonians describing Fermions.

Let h be a complex Hilbert space and J a conjugation on it (i.e. an anti-linear, anti-unitary operator such that $J^2 = I$). Systems whose number of particles is not a priori bounded above are described by the Fock space

$$\mathfrak{F}(h) := \bigoplus_{n=0}^{\infty} h^{\otimes^n}.$$

Particles system obeying a Fermi–Dirac statistics are described by the Fermi–Fock subspace

$$\mathfrak{F}_-(h) := P_-(\mathfrak{F}(h)),$$

where the orthogonal projection P_- is defined by

$$P_-(f_1 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_{\pi} \varepsilon_\pi f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$$

where the sum is over all permutations $(1, \ldots, n) \mapsto (\pi(1), \ldots, \pi(n))$. For $f \in h$, the creation operator $a^*(f)$, defined as $a^*(f) := \sqrt{n+1} P_-(f \otimes g_1 \otimes \cdots \otimes g_n)$, is bounded with norm $\|a^*(f)\| = \|f\|_h$. Together with the annihilation operator defined by $a(f) := (a^*(f))^*$, it satisfies the canonical anti-commutation CAR relations

$$a(f)a(g) + a(g)a(f) = 0, \qquad a(f)a^*(g) + a^*(g)a(f) = (f|g)\mathrm{I}_h \qquad f, g \in h$$

which represents the Pauli's Exclusion Principle. The Clifford C*-algebra $Cl(h)$ is defined as the C*-algebra generated by the fields operators

$$b(f) := a^*(f) + a(Jf) \qquad f \in h$$

i.e. as the intersection of all C*-subalgebras of $B(\mathfrak{F}_-(h))$ containing the fields $\{b(f) : f \in h\}$. It is highly noncommutative since it is a simple C*-algebra in the sense that it has no nontrivial closed, bilateral ideals. The Fock vacuum vector $\Omega := 1 \oplus 0 \cdots \in \mathfrak{F}_-(h)$ defines a trace vector state on it by

$$\tau_0(A) := (\Omega | A\Omega) \qquad A \in Cl(h)$$

and the natural map $D : Cl(h) \to \mathfrak{F}_-(h)$ given by $A \mapsto A\Omega$, extends to a unitary map from $L^2(Cl(h), \tau_0)$ onto $\mathfrak{F}_-(h)$, called the Segal isomorphism. This natural isomorphism allows to transfer on the Fermi–Fock space the order structure one has on $L^2(Cl(h), \tau_0)$ and vice versa, to study on $L^2(Cl(h), \tau_0)$ operators originally created on $\mathfrak{F}_-(h)$. This procedure is especially useful in combination with second quantization, where a self-adjoint operator $(A, D(A))$ on h give rise to a self-adjoint operator $(d\Gamma(A), D(d\Gamma(A)))$ on $\mathfrak{F}_-(h)$ as follows. First define self-adjoint operators $(A_n, D(A_n))$ on $P_-(h^{\otimes^n})$ for $n \geq 0$ setting $A_0 = 0$ and

$$A_n(P_-(f_1 \otimes \cdots \otimes f_n)) := \sum_{k=1}^{n} P_-(f_1 \otimes \cdots \otimes Af_k \otimes \cdots f_n) \qquad f_1 \otimes \cdots \otimes f_n \in D(A_n) := D(A)^{\otimes^n}.$$

The direct sum of the A_n is essentially self-adjoint because it is symmetric and it has a dense set of analytic vectors formed by finite sums of anti-symmetrized products of analytic vectors of A. The self-adjoint closure $d\Gamma(A) := \overline{\oplus_{k=0}^{\infty} A_n}$ of this sum is called the *second quantization* of A and is denoted by $(d\Gamma(A), D(d\Gamma(A)))$. The main example of this procedure concerns the Number operator $d\Gamma(I)$.

Theorem 4.6 *(Clifford Dirichlet form) Let* $(A, D(A))$ *be a self-adjoint operator on* h*, commuting with* J *and satisfying* $A \geq mI_h$ *for some* $m > 0$*. Then*

(i) the quadratic form $(\mathcal{E}, \mathcal{F})$ *of the operator* $H := D^{-1}d\Gamma(A)D$

$$\mathcal{E}[\xi] := (\xi|H\xi)_{L^2(Cl(h),\tau_0)} = (D\xi|d\Gamma(A)D\xi)_{\mathfrak{F}_-(h)} \qquad \xi \in D(H)$$

is a completely Dirichlet form on $L^2(Cl(h), \tau_0)$*;*

(ii) the completely Markovian semigroup e^{-tH} *is hypercontractive in the sense that it is bounded from* $L^2(Cl(h), \tau_0)$ *to* $L^4(Cl(h), \tau_0)$ *as soon as* $mt \geq (\ln 3)/2$*;*

(iii) $\inf \sigma(H)$ *is an isolated eigenvalue of finite multiplicity.*

The main point is to prove the result for $A = I_h$ so that the Hamiltonian is $H = Dd\Gamma(I_h)D^{-1} = DND^{-1}$ is unitarily equivalent to the Number operator $N = d\Gamma(I_h)$. We shall see later a proof based on the structure of the Dirichlet form of the Number operator. The interest in noncommutative Dirichlet forms originated in QFT to extend to Fermions the non perturbative techniques of Nelson, Segal, Glimm, Gross, Jaffe, Simon and others, elaborated for Bosons systems.

We conclude this section noticing that the *Clifford von Neumann algebra* $L^\infty(Cl(h), \tau_0)$ generated by the GNS representation of the Clifford C*-algebra provided by the Fock vacuum state, is isomorphic to *the hyperfinite* II_1*-factor* (usually denoted by R) to which τ_0 extends to a normal tracial state. While the *hyperfinitness* (see Sect. 7.3 below) is a reflection of the fact that $L^\infty(Cl(h), \tau_0)$ is generated by the net of finite-dimensional subalgebras corresponding to finite dimensional subspaces of the Hilbert space h, its *uniqueness* is a fundamental result of [40].

As any von Neumann algebra, R is generated by its projections $p \in R$ (which are defined as the self-adjoint elements $p = p^*$ satisfying $p^2 = p$). However, while projections in a type I von Neumann algebra as $B(h)$ have traces which can assume integers values only (equal to the dimension of their ranges), the trace of a projection in R may assume any real value $\tau_0(p) \in [0, 1]$, interpreted as a *real dimension* of the range of p. This is the reason by which von Neumann regarded R as exhibiting a *Euclidean continuous geometry*.

5 Dirichlet Forms and Differential Calculus: Bimodules and Derivations

In this section we show that on C*-algebras endowed with a densely defined, lower semicontinuous, faithful trace (A, τ), *completely Dirichlet forms are representations of a differential calculus* (see [32, 91, 92]). In fact they can be constructed, on one side, and determine, on the other side, closable derivations on the C*-algebra A. This was suggested, at least in one direction, by the result of Albeverio–Hoegh-Krohn illustrated in Theorem 4.5 above.

To specify what a derivation on a C*-algebra A is, let us recall the notion of A-bimodule $\mathcal{H}$: this is an Hilbert space together with two continuous commuting actions (say left and right) of A

$$A \times \mathcal{H} \ni (a, \xi) \mapsto a\xi \in \mathcal{H}, \qquad \mathcal{H} \times A \ni (\xi, b) \mapsto \xi b \in \mathcal{H}.$$

The commutativity says that $(a\xi)b = a(\xi b)$ for all $a, b \in A$ and $\xi \in \mathcal{H}$. If the left and right actions coincides, $a\xi = \xi a$, then $\mathcal{H}$ is called a A-mono-module. Equivalently, a A-bimodule is a representation on the Hilbert space $\mathcal{H}$ of the maximal or projective tensor product C*-algebra $A \otimes_{\max} A^\circ$. Here A° denote the *opposite* C*-algebra coinciding with A as linear space with involution but in which the product is reversed in order: $x^\circ y^\circ := (yx)^\circ$.

A *symmetric* A-bimodule $(\mathcal{H}, \mathcal{J})$ is a A-bimodule $\mathcal{J}$ together with a conjugation $\mathcal{J}$ such that

$$\mathcal{J}(a\xi b) = b^*(\mathcal{J}\xi)a^* \qquad a, b \in A, \quad \xi \in \mathcal{H}.$$

Definition 5.1 (Derivation on C*-algebras) A derivation on a C*-algebra A is a linear map $\partial : D(\partial) \to \mathcal{H}$ defined on a subalgebra $D(\partial) \subseteq A$ with values into a A-bimodule $\mathcal{H}$ satisfying the Leibnitz rule

$$\partial(ab) = (\partial a)b + a(\partial b) \qquad a, b \in D(\partial) \subseteq A.$$

The derivation is called symmetric if $D(\partial)$ is involutive, $\mathcal{H}$ is symmetric and

$$\mathcal{J}(\partial a) = \partial a^* \qquad a \in D(\partial).$$

Here we review some examples of derivations.

Gradient operator. Let M be a Riemannian manifold and consider the Hilbert space $L^2_{\mathbb{C}}(TM) := L^2(TM) \otimes_{\mathbb{R}} \mathbb{C}$ obtained complexifying the Hilbert space of square integrable vector fields. This is a mono-module[2] over the commutative C*-algebra of continuous function $C_0(M)$ where the action is defined pointwise and it can be endowed with the involution $\mathcal{J}(\chi \otimes z) := \chi \otimes \bar{z}$. If $H^1(M)$ denotes the first Sobolev space, then a symmetric derivation is defined by the gradient operator

$$\nabla : C_0(M) \cap H^1(M) \to L^2_{\mathbb{C}}(TM).$$

Difference operator. Let X be a locally compact Hausdorff space and let j be a Radon measure on $X \times X$ supported off the diagonal. Left and right commuting actions of $C_0(X)$ on $L^2(X \times X, j)$ may be defined as

$$(af)(x, y) := a(x)f(x, y), \qquad (fb)(x, y) = f(x, y)b(y)$$
$$a, b \in C_0(X), \quad f \in L^2(X \times X, j)$$

[2] A mono-module is a bi-module in which the left and right actions coincide.

and one may check that

$$j : C_c(X) \to L^2(X \times X, j) \qquad (\partial_j a)(x, y) := a(x) - a(y)$$

is a symmetric derivation on $C_0(X)$ once the conjugation is defined as

$$(\mathcal{J} f)(x, y) := \overline{f(y, x)}.$$

Killing measure. Let X be a locally compact Hausdorff space and let k be a Radon measure on X. Consider $L^2(X, k)$ as a $C_0(X)$-bimodule where the left action is the usual pointwise one while the right action is the trivial one so that $\xi b := 0$ for all $\xi \in L^2(X, k)$ and $b \in C_0(X)$. If one considers as $\mathcal{J}$ just the pointwise complex conjugation of functions in $L^2(X, k)$, then one may easily check that the map

$$\partial_k : C_c(X) \to L^2(X, k) \qquad \partial a := a$$

is a symmetric derivation on $C_0(X)$.

Commutators I. Let (A, τ) be a C*-algebra endowed with a faithful, semifinite trace and recall that $A_\tau := \{a \in A : \tau(a^* a) < +\infty\}$ is a bilateral ideal in A. Then if one consider on the Hilbert space $L^2(A, \tau)$ the natural left and right actions of A and the conjugation $\mathcal{J} a := a^*$, one obtains a symmetric A-bimodule. Moreover any $b \in A$ give rise to a symmetric derivation

$$\partial_b : A_\tau \to L^2(A, \tau) \qquad \partial_b a := i[a, b] = i(ab - ba).$$

If a sequence $\{b_k \in A : k \geq 1\}$ is fixed and one consider the direct sum of symmetric A-bimodules $\oplus_{k=1}^\infty L^2(A, \tau)$, then the direct sum

$$\partial := \oplus_{k=1}^\infty \partial_{b_k} : D(\partial) \to \oplus_{k=1}^\infty L^2(A, \tau)$$

is a symmetric derivation defined on the involutive subalgebra $D(\partial)$ of those $a \in A_\tau$ such that the series $\oplus_{k_1}^\infty \|[a, b_k]\|^2_{L^2(A,\tau)}$ converges.

Commutators II. As a variation of the above construction, suppose that A is represented on the Hilbert space h. Then the space of Hilbert–Schmidt operators $L^2(h)$ is a A-bimodule for the left $a\xi$ and right ξb actions of $a, b \in A$ on $\xi \in L^2(h)$ given by composition as operators on h. A natural involution is defined by $\mathcal{J}\xi := \xi^*$ on $\xi \in L^2(h)$ and then a symmetric derivation is given by

$$\partial_\xi : A \to L^2(h) \qquad \partial_\xi a := i(a\xi - \xi a).$$

If a sequence $\{\xi_k \in L^2(h) : k \geq 1\}$ of Hilbert–Schmidt operators is fixed and one consider the direct sum of symmetric A-bimodules $\oplus_{k=1}^\infty L^2(h)$, then the direct sum

$$\partial := \oplus_{k=1}^\infty \partial_{\xi_k} : A \to \oplus_{k=1}^\infty L^2(h)$$

is a symmetric derivation defined on the involutive subalgebra $D(\partial)$ of those $a \in A_\tau$ such that the series $\oplus_{k=1}^{\infty} \|a\xi_k - \xi_k a\|^2_{L^2(h)}$ converges. This last derivation is clearly related to the one appearing in the Albeverio–Hoegh-Krohn Theorem above. We will come back on this construction later on.

Next result shows that closable derivations give rise to Dirichlet forms [32].

Theorem 5.2 *Let (A, τ) be a C^*-algebra endowed with a densely defined, lower semicontinuous, faithful trace and let $(\partial, D(\partial), \mathcal{H}, \mathcal{J})$ be a symmetric derivation, densely defined on a domain $D(\partial) \subset A \cap L^2(A, \tau)$, which is closable as an operator from $L^2(A, \tau)$ to $\mathcal{H}$. Then the closure of the quadratic form*

$$\mathcal{E}[a] := \|\partial a\|^2_{\mathcal{H}} \qquad a \in \mathcal{F} := D(\partial)$$

is a completely Dirichlet form.

The proof of the above result goes through the establishment of *noncommutative chain rule* [32] for closable derivation, by which one has

$$\partial f(a) = ((L_a \otimes R_a)(\tilde{f}))(\partial a) \qquad a = a^* \in A \cap L^2(A, \tau), \quad f \in \mathrm{Lip}_0(\mathbb{R})$$

Here L_a (resp. R_a) are the representation of $\mathrm{C}(sp(a))$, continuous, complex valued functions on the spectrum $sp(a)$ of a, uniquely defined by

$$L_a(f)\xi = \begin{cases} f(a)\xi \text{ if } f(0) = 0 \\ \xi \text{ if } f \equiv 1 \end{cases} \qquad f \in C(sp(a)) \quad \xi \in \mathcal{H}$$

and

$$R_a(f)\xi = \begin{cases} \xi f(a) \text{ if } f(0) = 0 \\ \xi \text{ if } f \equiv 1 \end{cases} \qquad f \in C(sp(a)) \quad \xi \in \mathcal{H}.$$

$L_a \otimes R_a$ is the tensor product representation of $C(sp(a)) \otimes C(sp(a)) = C(sp(a) \times sp(a))$. When $I \subseteq \mathbb{R}$ is a closed interval and $f \in \mathrm{C}^1(I)$, we will denote by $\tilde{f} \in \mathrm{C}(I \times I)$ the *differential quotient* on $I \times I$, sometimes called the *quantum derivative* of f, defined by

$$\tilde{f}(s,t) = \begin{cases} \frac{f(s)-f(t)}{s-t} & \text{if } \; s \neq t \\ f'(s) & \text{if } \; s = t. \end{cases} \tag{5.1}$$

Since commutators in A are bounded derivations in the above sense, the above result provides an independent proof of the Albeverio–Hoegh-Krohn Theorem 4.5 above.

A derivation for the Clifford–Dirichlet form. As a further application of the above result, let us show that the quadratic form $\mathcal{E}_N$ of the Number operator N of Fermions, when seen on the space $L^2(Cl(h), \tau_0)$ via the Segal isomorphism D, is a completely Dirichlet form on the Clifford algebra $(Cl(h), \tau_0)$. Recall first that N and $\mathcal{E}_N$ can be written as

$$N = \sum_{k=1}^{\infty} a^*(f_k)a(f_k) \qquad \mathcal{E}_N[\psi] := \sum_{k=1}^{\infty} \|a(f_k)\psi\|^2$$

for any orthonormal base of $\{f_k : k \geq 1\} \subset h$. Let us denote by L_b and R_b the left and right actions on $L^2(Cl(h), \tau_0)$ of an element of the Clifford algebra $b \in Cl(h)$. The symmetry $\beta := \Gamma(-I_h)$ of $\mathfrak{F}_-(h)$ induces an idempotent automorphism $\gamma \in B(\mathfrak{F}_-(h))$

$$\gamma(A) := \beta A \beta \qquad A \in B(\mathfrak{F}_-(h)).$$

Since $b(f)\beta = -b(f)$, we have $\gamma(b(f)) = -b(f)$ for all $f \in h$ so that γ leaves $Cl(h)$ globally invariant and then $\gamma \in \mathrm{Aut}(Cl(h))$. Since $Cl(h) \subset L^2(Cl(h), \tau_0)$ and $\beta\Omega = \Omega$ we have $D(\gamma(b)) = \beta b \beta \Omega = \beta b \Omega = \beta D(b)$ so that

$$\gamma(b) = (D^{-1}\beta D)(b) \qquad b \in Cl(h).$$

For $f \in h$ and $\xi \in Cl(h)$ we have $DL_{b(f)}D^{-1}(D\xi) = D(L_{b(f)}\xi) = L_{b(f)}(\xi\Omega) = L_{b(f)}(D\xi)$ so that, since $Cl(h)$ is dense in $L^2(Cl(h), \tau_0)$, we obtain

$$DL_{b(f)}D^{-1} = a^*(f) + a(Jf) = b(f).$$

Since $\beta(P_-(f_1 \otimes \cdots \otimes f_n)) = P_-\beta(f_1 \otimes \cdots \otimes f_n) = (-1)^n P_-(f_1 \otimes \cdots \otimes f_n)$ and $\beta\Omega = \Omega$, by the CAR relations we have that $a^*(f) - a(Jf)\beta$ commutes with all of $b(g)$ for $g \in h$ and then with all elements of the Clifford algebra. Then, for $b \in Cl(h)$ we have $(a^*(f) - a(Jf))\beta(D(b)) = (a^*(f) - a(Jf))\beta(b\Omega) = b(a^*(f) - a(Jf))\beta\Omega = b(a^*(f) - a(Jf))\Omega = ba^*(f)\Omega = b(a^*(f) + a(Jf))\Omega = bb(f)\Omega = (R_{b(f)}b)\Omega = D(R_{b(f)}b)$ and since $Cl(h)$ is dense in $L^2(Cl(h), \tau_0)$, we obtain

$$DR_{b(f)}D^{-1} = a^*(f) - a(Jf)\beta$$

which can be rewritten as

$$a^*(f) - a(Jf) = DR_{b(f)}D^{-1}\beta = DR_{b(f)}(D^{-1}\beta D)D^{-1}.$$

By summation we have

$$a(Jf) = \frac{1}{2}D(L_{b(f)} - R_{b(f)}(D^{-1}\beta D))D^{-1}$$

and changing f with Jf we get

$$a(f) = D\Big(\frac{1}{2}\left(L_{b(Jf)} - R_{b(Jf)}(D^{-1}\beta D)\right)\Big)D^{-1}.$$

Consider now on the Hilbert space $\mathcal{H}_\gamma := L^2(Cl(h), \tau_0)$ the $Cl(h)$-bimodule structure where the right action of $Cl(h)$ is the usual one while the left one is twisted by the automorphism γ

$$b_1 \cdot \xi \cdot b_2 := \gamma(b_1)\xi b_2 \qquad b_1, b_2 \in Cl(h), \quad \xi \in L^2(Cl(h), \tau_0)$$

(dots indicate actions in this new bimodule structure). The definition is justified by the fact that, as one can easily check, this new left action is continuous and commutes with the right one. Let us now check that for any fixed $f \in h$, the map

$$\partial_f : Cl(h) \to \mathcal{H}_\gamma \qquad \partial_f := \frac{i}{2}(L_{b(Jf)} - R_{b(Jf)}(D^{-1}\beta D)),$$

more explicitly given by a module commutator

$$\begin{aligned}
\partial_f(b) &= \frac{1}{2}(L_{b(Jf)} - R_{b(Jf)}(D^{-1}\beta D))(b) \\
&= \frac{1}{2}(L_{b(Jf)}(b) - R_{b(Jf)}(D^{-1}\beta D)(b)) \\
&= \frac{1}{2}(b(Jf)b - R_{b(Jf)}(\gamma(b)) \\
&= \frac{1}{2}(b(Jf)b - \gamma(b)b(Jf)) \\
&= \frac{1}{2}(b(Jf) \cdot b - b \cdot b(Jf))
\end{aligned}$$

is a derivation in the sense that the Leibniz rule holds true:

$$\begin{aligned}
\partial_f(ab) &= \frac{1}{2}\{b(Jf) \cdot (ab) - (ab) \cdot b(Jf)\} \\
&= \frac{1}{2}\{(b(Jf) \cdot a - a \cdot b(Jf)) \cdot b + a \cdot (b(Jf) \cdot b - b \cdot b(Jf))\} \\
&= (\partial_f a) \cdot b + a \cdot (\partial_f b) \qquad a, b \in Cl(h).
\end{aligned}$$

To define a symmetry $\mathcal{J}_\gamma$ such that $(\mathcal{H}_\gamma, \mathcal{J}_\gamma)$ is a symmetric bimodule, notice first that the automorphism γ of the Clifford algebra $Cl(h)$ can be extended to a symmetry acting on the Hilbert space $L^2(Cl(h), \tau_0)$. In fact as $\beta\Omega = \Omega$, the trace τ_0 is γ-invariant: $\tau_0(\gamma(b)) = (\Omega|\gamma(b)\Omega) = (\beta\Omega|b\beta\Omega) = (\Omega|b\Omega) = \tau_0(b)$ for all $b \in Cl(h)$. Consequently γ is isometric with respect to L^2-norm

$$\begin{aligned}
\|\gamma(b)\|_2^2 &= \tau_0((\gamma(b))^*\gamma(b)) = \tau_0(\gamma(b^*)\gamma(b)) \\
&= \tau_0(\gamma(b^*b)) = \tau_0(b^*b) = \|b\|_2^2 \qquad b \in Cl(h)
\end{aligned}$$

so that by density it extends to an isometry on the whole $L^2(Cl(h), \tau_0)$ such that $\gamma^2 = I$. Further, as for all $a, b \in Cl(h)$ we have

$$(\gamma(a)|b)_2 = \tau_0((\gamma(a))^* b) = \tau_0((\gamma(a^*))b) = \tau_0(\gamma(a^*\gamma(b)))$$
$$= \tau_0(a^*\gamma(a^*)) = (a|\gamma(b))_2,$$

it results that γ is a symmetry on $L^2(Cl(h), \tau_0)$ which commutes with the symmetry $\mathcal{J}$

$$\gamma^* = \gamma, \quad \gamma^2 = I, \quad \gamma \circ \mathcal{J} = \mathcal{J} \circ \gamma.$$

A new conjugation is then defined by $\mathcal{J}_\gamma := \mathcal{J} \circ \gamma = \gamma \circ \mathcal{J}$ on $\mathcal{H}_\gamma$ in such a way that $(\mathcal{H}_\gamma, \mathcal{J}_\gamma)$ is a symmetric bimodule over the Clifford algebra $Cl(h)$, as it results from the following identities for $a, b, c \in Cl(h)$

$$\mathcal{J}_\gamma(a \cdot b \cdot c) = \mathcal{J}(\gamma(\gamma(a)bc)) = \mathcal{J}(a\gamma(b)\gamma(c))$$
$$= (\gamma(c))^*(\gamma(b))^* a^* = (\gamma(c^*))(\mathcal{J}_\gamma(b))a^* = c^* \cdot \mathcal{J}_\gamma(b) \cdot a^*.$$

Since for $f \in h$ one has $\mathcal{J}_\gamma(b(Jf)) = \gamma(b(Jf)^*) = \gamma(b(f)) = -b(f)$, it follows that

$$\begin{aligned}\mathcal{J}_\gamma(\partial_f b) &= \frac{1}{2}\mathcal{J}_\gamma((b(Jf) \cdot b - b \cdot b(Jf))) \\ &= \frac{1}{2}(b^* \cdot \mathcal{J}_\gamma(b(Jf)) - \mathcal{J}_\gamma(b(Jf)) \cdot b^*) \\ &= \frac{1}{2}(b(f) \cdot b^* - b^* b(f)) \\ &= \partial_{Jf}(b^*).\end{aligned}$$

Consequently, if $f \in h$ is J-real (in the sense that $Jf = f$) then ∂_f is $\mathcal{J}_\gamma$-symmetric

$$\mathcal{J}_\gamma(\partial_f b) = \partial_J(b^*) \qquad b \in Cl(h).$$

Choosing a Hilbert base $\{f_k : k \geq 1\} \subset h$ made by J-real vectors $Jf_k = f_k$, we can represent the quadratic form of the Number operator on the space $L^2(Cl(h), \tau_0)$ as

$$\mathcal{E}_{Cl}[b] := \sum_{k=1}^{\infty} \|\partial_{f_k} b\|^2_{\mathcal{H}_\gamma} \qquad b \in \mathcal{F}_{Cl}$$

where the form domain is obviously $\mathcal{F}_{Cl} := D^{-1}(D(\sqrt{N}))$. Setting $\mathcal{H}_{Cl} := \bigoplus_{k=1}^{\infty} \mathcal{H}_\gamma$ as a direct sum of symmetric $Cl(h)$-bimodules, we have that $\partial_{Cl} := \bigoplus_{k=1}^{\infty} \partial_{f_k}$ is a symmetric derivation of the Clifford algebra into $\mathcal{H}_{Cl}$ such that

$$\mathcal{E}_{Cl}[b] = \|\partial_{Cl} b\|^2_{\mathcal{H}_{Cl}} \qquad b \in \mathcal{F}_{Cl}$$

and which is densely defined and closable as sum of bounded derivations. By Theorem 5.2 above, the associated semigroup is thus completely Markovian.

Dirichlet form on noncommutative tori. This is a fundamental example appearing in Noncommutative Geometry [42] in which the relevant *algebra of coordinates* A_θ of the space is a noncommutative deformation of the algebra of continuous functions on a classical torus. For any fixed $\theta \in [0, 1]$, A_θ, called *noncommutative* 2-*torus*, is defined as the universal C*-algebra generated by two unitaries U and V, satisfying the relation

$$VU = e^{2i\pi\theta}UV.$$

When $\theta = 0$ one recovers the algebra of continuous functions on the 2-torus. All elements of A_θ can be written as a series $\sum_{m,n\in\mathbb{Z}} c_{m,n}U^m V^n$ with complex coefficients. A tracial state is specified by

$$\tau : A_\theta \to \mathcal{C} \qquad \tau(U^m V^n) = \delta_{m,0}\delta_{n,0} \qquad m, n \in \mathbb{Z}$$

so that

$$L^2(A_\theta, \tau)) = \Big\{ \sum_{m,n\in\mathbb{Z}} c_{m,n}U^m V^n : \sum_{m,n\in\mathbb{Z}} |c_{m,n}|^2 < +\infty \Big\} \simeq l^2(\mathbb{Z}^2).$$

A densely defined, closed form is given by

$$\mathcal{E}\Big[\sum_{m,n\in\mathbb{Z}} c_{m,n}U^m V^n \Big] = \sum_{m,n\in\mathbb{Z}} (m^2 + n^2)|c_{m,n}|^2$$

on the domain $\mathcal{F} \subset l^2(\mathbb{Z}^2)$ where the above series converges (i.e. the first Sobolev space). To check that we are dealing with a Dirichlet form, one may observe that it is a "square of a derivation", taking values in the direct sum of two standard bimodules $L^2(A, \tau) \oplus L^2(A, \tau)$ and given by the direct sum

$$\partial(a) = \partial_1(a) \oplus \partial_2(a)$$

of the derivations ∂_1 and ∂_2 from A_θ into $L^2(A_\theta, \tau)$ defined by

$$\partial_1(U^m V^n) = imU^m V^n \ , \qquad \partial_1(U^m V^n) = inU^m V^n \qquad n, m \in \mathbb{Z}.$$

The associated Markovian semigroup $\{T_t : t \geq 0\}$, characterized by

$$T_t(U^m V^n) = e^{-t(m^2+n^2)}U^m V^n \qquad m, n \in \mathbb{Z},$$

is clearly conservative, in the sense that $T_t 1_{A_\theta} = 1_{A_\theta}$, because $\mathcal{E}[1_{A_\theta}] = 0$. Even if, at a Hilbert space level, the Dirichlet form and its associated Markovian semigroup are clearly isomorphic to the Dirichlet integral and the heat semigroup on the classical (commutative) torus $\mathbb{T}^2$, written in Fourier series terms, their potential theoretic properties arise from the order structure of A_θ which may differ completely from those of $C(\mathbb{T}^2)$. The properties of A_θ depend a lot upon the rationality/irrationality and diophantine approximation properties of the parameter $\theta \in [0, 1]$. For example consider the spectrum of the self-adjoint element $S := U + U^* + V + V^* \in A_\theta$. If $\theta = 0$ we have $A_\theta = C(\mathbb{T}^2)$ so that since $\mathbb{T}^2$ is connected and S a real continuous function, its spectrum is a closed interval of the real line. When θ is irrational, however, the spectrum of S is typically a Cantor set (so that, under "commutative spectacles" we would look at A_θ as a rather fragmented space).

5.1 The Derivation Determined by a Dirichlet Form

The following result, in combination with the previous one, establishes a one-to-one correspondence between Dirichlet forms and closable derivations on C*-algebras [32, 91, 92]. It says that derivations are differential square roots of Dirichlet forms. It can be considered as a generalization of the construction of the (differential first order) Dirac operator from the (differential second order) Hodge–de Rham Laplacian of a Riemannian manifold.

Theorem 5.3 *Let $(\mathcal{E}, \mathcal{F})$ be a completely Dirichlet form on a C*-algebra endowed with a densely defined, lower semicontinuous faithful trace (A, τ) Then there exists a densely defined, L^2-closable derivation $(\partial, D(\partial))$ with values in a symmetric A-bimodule $(\mathcal{H}, \mathcal{J})$ such that $D(\partial)$ is a form core and*

$$\mathcal{E}[a] = \|\partial a\|^2_{\mathcal{H}} \qquad a \in D(\partial).$$

The bimodule $(\mathcal{H}, \mathcal{J})$ is called the *tangent bimodule* associated to $(\mathcal{E}, \mathcal{F})$.

5.1.1 Conditionally Negative Definite Functions and Dirichlet Forms on Dual of Discrete Groups [32]

Let $\lambda : \Gamma \to B(\ell^2(\Gamma))$ be the left regular representation of a countable discrete group Γ

$$(\lambda_s f)(t) := f(s^{-1}t) \qquad s, t \in \Gamma$$

and consider the reduced C*-algebra $C^*_r(\Gamma)$ defined as the smallest C*-subalgebra of $B(L^2(\Gamma))$ containing the unitary operators λ_s for all $s \in \Gamma$. Extending λ to $c_c(\Gamma)$ as a convolution

$$(\lambda(f)g)(t) := (f * g)(t) = \sum_{s\in\Gamma} f(ts^{-1})g(s) \qquad f, g \in c_c(\Gamma), \qquad t \in \Gamma,$$

we can identify $c_c(\Gamma)$ as a dense involutive subalgebra of $C^*_r(\Gamma)$. The involution of an element $f \in c_c(\Gamma)$ is given by $f^*(s) = \overline{f(s^{-1})}$. A faithful tracial state is determined by

$$\tau(\lambda(f)) = f(e) \qquad f \in c_c(\Gamma)$$

where $e \in \Gamma$ is the unit of the group. Since

$$(\lambda(f)|\lambda(g))_{L^2(C_r^*(\Gamma),\tau)} = (f|g)_{l^2(\Gamma)} \qquad f, g \in c_c(\Gamma),$$

we can identify $L^2(C_r^*(\Gamma), \tau)$ with $l^2(\Gamma)$ at the Hilbert space level and represent the positive cone $L^2_+(C_r^*(\Gamma), \tau)$ as the one of all square integrable, positive definite functions on Γ. The von Neumann algebra $L(\Gamma)$ generated by the unitaries $\{\lambda_s : s \in \Gamma\}$ on $l^2(\Gamma)$ is called the *group von Neumann algebra* of Γ. It is a finite von Neumann algebra since the tracial state τ on $C_r^*(\Gamma)$ extends to a normal tracial state on it and it is a factor if and only if the conjugacy class

$$\{sts^{-1} \in \Gamma : s \in \Gamma\}$$

of any $t \in \Gamma$ is an infinite set. In this setting, any *conditionally negative definite function* $l : \Gamma \to \mathbb{C}$, i.e. a normalized, symmetric function

$$l(e) = 0, \qquad l(s^{-1}) = \overline{l(s)} \qquad s \in G$$

satisfying, for $s_1, \ldots, s_n \in \Gamma$, $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{k=1}^{n} \overline{c_j} l(s_j^{-1} s_k) c_k \leq 0 \qquad \text{whenever} \qquad \sum_{k=1}^{n} c_k = 0,$$

determines a completely Dirichlet form

$$\mathcal{E}_l[a] := \sum_{s \in \Gamma} l(s)|a(s)|^2 \qquad a \in l^2(\Gamma)$$

defined on the domain $\mathcal{F}_l$ where the above sum converges and whose associated completely Markovian semigroup is given by

$$(T_t a)(s) = e^{-tl(s)} a(s) \qquad a \in l^2(\Gamma).$$

To describe the associated derivation recall that any negative definite function can be represent by a 1-cocycle $c : \Gamma \to \mathcal{H}_\pi$ of a unitary representation $\pi : \Gamma \to \mathcal{H}_\pi$, i.e. a function satisfying

$$c(st) = c(s) + \pi(s)(c(t)) \qquad s, t \in \Gamma,$$

as follows

$$l(s) = \|c(s)\|^2_{\mathcal{H}_\pi} \qquad s \in \Gamma.$$

On the Hilbert space $\mathcal{H}_\pi \otimes l^2(\Gamma)$ a $C^*_r(\Gamma)$-bimodule structure is then specified by the left action $\pi \otimes \lambda$ and by the right action $id \otimes \rho$ where ρ is the right regular action of Γ and id is the trivial action on $\mathcal{H}_\pi$ assigning the identity operator to any element of Γ. Using the natural isomorphism $\mathcal{H}_\pi \otimes l^2(\Gamma) \simeq l^2(\Gamma, \mathcal{H}_\pi)$, the derivation representing $\mathcal{E}_l$ is identified by

$$(\partial_l a)(s) = a(s)c(s) \qquad a \in c_c(\Gamma), \quad s \in \Gamma.$$

When $\Gamma = \mathbb{Z}^n$, the C*-algebra $C^*_r(\Gamma)$ coincides with the algebra $C(\mathbb{T}^n)$ of continuous functions on the torus and if one considers the negative definite function $l(z_1, \ldots, z_n) := |z_1|^2 + \cdots + |z_n|^2$, one recovers as Dirichlet form just the standard Dirichlet integral on $\mathbb{T}^n$ (see Sect. 2.1).

5.2 Decomposition of Derivation, Beurling–Deny Formula Revisited [32, 33]

Since an A-bimodule is just a representation of the C*-algebra $A \otimes_{\max} A$, one disposes of all the tools that representation theory offers, such has decomposition theory, to analyze derivations and their associated Dirichlet forms. In the commutative situation, for example, one obtains, by an algebraic approach, a refinement of the Beurling–Deny decomposition of Dirichlet forms.

In this section we introduce notions of bounded, approximately bounded and completely unbounded derivations and we prove that any derivation canonically split into a sum of the latter.

Let $\partial : D(\partial) \to \mathcal{H}$ be a densely defined derivation on a C*-algebra A and denote by $\mathcal{L}_A(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$ which commute both with left and right actions of A.

An element $B \in \mathcal{L}_A(\mathcal{H})$ will be said to be ∂-bounded if the map $B \circ \partial$ extends to a bounded map from A into $\mathcal{H}$. Notice that if this is the case, $B \circ \partial$ is a derivation. A projection $p \in \mathcal{L}_A(\mathcal{H})$ will be said to be approximately ∂-bounded if it is the increasing limit of a net of ∂-bounded projections. As $\mathcal{H}$ is assumed to be separable, this means that one can write the A-bimodule $p(\mathcal{H})$ as an at most countable direct sum $\bigoplus_n \mathcal{H}_n$ of A-bimodules such that $p \circ \partial$ decomposes as a direct sum $\bigoplus_n \partial_n$ of bounded derivations $\partial_n := p_n \circ \partial$ where $p_n \in \mathcal{L}_A(\mathcal{H})$ is the ∂-bounded projection onto the A-submodule $\mathcal{H}_n$. A projection $p \in \mathcal{L}_A(\mathcal{H})$ will be said to be completely ∂-unbounded if 0 is the only ∂-bounded projection smaller than p. The derivation ∂ will be said to be bounded (resp. approximately bounded, resp. completely unbounded) if the identity operator $1_\mathcal{H}$ is a ∂-bounded (resp. approximately ∂-bounded, resp. completely ∂-unbounded) projection.

Then one can prove that there exists a greatest approximately ∂-bounded projection $P_a \in \mathcal{L}_A(\mathcal{H})$ and that every ∂-bounded $B \in \mathcal{L}_A(\mathcal{H})$ satisfies $B \circ P_a = B$.

Setting $\mathcal{H}_a := P_a(\mathcal{H})$ and $\mathcal{H}_c := (1 - P_a)(\mathcal{H})$ we have the decomposition of $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_c$ into its approximately bounded and completely unbounded sub-A-bimodules. Correspondingly, setting $\partial_a := P_a \circ \partial$ and $\partial_c := (1 - P_a) \circ \partial$ we have the decomposition of the derivation $\partial = \partial_a \oplus \partial_c$ into its approximately bounded and completely unbounded components. Finally, any Dirichlet form can be canonically splitted as a sum of its approximately bounded and completely unbounded parts

$$\mathcal{E}[a] = \|\partial a\|^2_{\mathcal{H}} = \|\partial_a a\|^2_{\mathcal{H}_j} + \|\partial_c a\|^2_{\mathcal{H}_c} \qquad a \in \mathcal{F}.$$

This is a (purely algebraic) generalization of the Beurling–Deny decomposition of Dirichlet forms on a commutative C*-algebra $A = C_0(X)$. In particular the completely unbounded part $\mathcal{E}_c$ can be identified with the diffusive part and the approximately bounded part $\mathcal{E}_a$ correspond to the sum $\mathcal{E}_j + \mathcal{E}_k$ of the jumping and killing parts. In the commutative situation $\mathcal{E}_c$ can also be characterized as the part of the Dirichlet form whose $C_0(X)$-bimodule $\mathcal{H}_c$ is the largest sub-$C_0(X)$-mono-module of the $C_0(X)$-bimodule $\mathcal{H}$ corresponding to $\mathcal{E}$, i.e. the largest submodule on which the left and right actions coincide. Moreover, since as any $C_0(X)$-mono-module, $\mathcal{H}_c$ is the direct integral $\int_X \mathcal{H}_x \, \mu(dx)$ of $C_0(X)$-mono-modules $\mathcal{H}_x$ whose actions are the simplest possible

$$a\xi = \xi a = a(x)\xi \qquad a \in C_0(X), \quad \xi \in \mathcal{H}_x, \quad x \in X,$$

in the corresponding splitting $\partial = \int_X \partial_x \mu(dx)$, the derivations ∂_x of $C_0(X)$ satisfy the Leibniz property

$$\partial_x(ab) = (\partial_x a)b(x) + a(x)(\partial_x b) \qquad a, b \in \mathcal{F} \cap C_0(X).$$

5.3 *Noncommutative Potential Theory and Curvature in Riemannian Geometry [33]*

Classical Potential Theory arose to understand properties of the potential energy functions in electromagnetism and in classical gravity. The properties of these functions were encoded in properties of the Laplace–Beltrami operators and in those of the Dirichlet integrals of Euclidean domains. Dealing with Nonlinear Elasticity or Riemannian Geometry [74], one is naturally lead to consider other Laplace type operators and associated quadratic energy forms acting on sections of vector bundles over Riemannian manifolds. In this section we describe briefly the strict relation between curvature and a distinguished noncommutative Dirichlet form.

Topological and geometric aspects of a Riemannian manifold (V, g) are related to the Hodge–de Rham operator $\Delta_{\mathrm{HdR}} = dd^* + d^*d$ acting on the space $L^2(\Lambda^*(V))$ of square integrable sections of the exterior bundle $\Lambda^*(V)$. It generalizes the Laplace–Beltrami operator acting on functions but its quadratic form cannot be directly considered as a Dirichlet form, essentially because exterior forms do not realize a C*-algebra. However, the geometric aspects of V are more deeply connected to the Dirac operator D and its square D^2, the so called Dirac Laplacian, acting on sections of the Clifford bundle $Cl(V)$ essentially because it is the construction of this space and operators that depends on the metric g.

Recall that the fibers of $Cl(V)$ are the Clifford algebras $Cl(T_xV)$ of the Hilbert space (T_xV, g_x). Since the exterior algebra $\Lambda^*_x V$ is nothing but the antisymmetric Fock space $\mathfrak{F}_-(T_xV)$, globalizing the Segal isomorphism, we met in a previous section, we have a canonical isomorphism of vector bundles between $Cl(V)$ and $\Lambda^*(V)$. The difference is that, by construction, the fibers of the bundle $Cl(V)$ form now C*-algebras. As a consequence, the space $C^*_0(V, g)$ of continuous sections vanishing at infinity of the Clifford bundle form, by pointwise product on V, a C*-algebra naturally associated to the Riemannian manifold (V, g). Moreover, denoting by $\Omega_x \in \mathfrak{F}_-(T_xV)$ the vacuum vector and by $\tau_x(\cdot) = (\Omega_x | \cdot \Omega_x)$ the associated trace on $Cl(T_xV)$, using the Riemannian measure m_g, we get on the Clifford *-algebra a densely defined, lower semicontinuous, faithful trace

$$\tau(\omega) := \int_V m_g(dx)\tau_x(\omega_x)$$

and the ordered Hilbert space $L^2(Cl(V, g), \tau)$. The Levi-Civita connection of (V, g) can be lifted to a metric connection on the Clifford bundle and represented, at the analytical level, by the covariant derivative ∇ acting between the smooth sections of the Hermitian bundles $Cl(V, g)$ and $Cl(V, g) \otimes T^*V$.

Theorem 5.4 *([49, 50, 101]). The closure of the Bochner integral*

$$\mathcal{E}_B[\omega] := \int_V |\nabla\omega(x)|^2 \, m_g(dx) \qquad \omega \in C^\infty_c(Cl()V, g)$$

is a completely Dirichlet form on $L^2(Cl(V, g), \tau)$.

The self-adjoint operator $\Delta_B := \nabla^*\nabla$ associated to $\mathcal{E}_B$, called the Bochner or connection Laplacian, thus generates a completely Markovian semigroup on $L^2(Cl(V, g), \tau)$ which is strongly Feller in the sense that it reduces to a strongly continuous Markovian semigroup on the Clifford algebra $C^*_0(V, g)$.

We may base the proof of the above result on Theorem 5.2 above. Notice first that on the Hilbert space $L^2(Cl(V, g) \otimes T^*V)$ we may consider the $C^*_0(V, g)$-bimodule structure given by

$$\sigma_1 \cdot (\sigma_2 \otimes \omega) \cdot \sigma_3 := (\sigma_1 \cdot \sigma_2 \cdot \sigma_3) \otimes \omega$$

$$J(\sigma_2 \otimes \omega) := \sigma_2^* \otimes \bar{\omega}$$

for $\sigma_1, \sigma_3 \in C_0^*(V, g)$ and $\sigma_2 \otimes \omega \in L^2(Cl(V, g) \otimes T^*V)$, where $\sigma_2 \to \sigma_2^*$ is the extension of the involution on the Clifford algebra and $\omega \to \bar{\omega}$ is the involution on the complexified cotangent bundle. Denoting by i_X the contraction operator with respect to a smooth vector field X, we may consider the covariant derivative along X given by $\nabla_X := i_X \circ \nabla$. Since, by definition, the Levi-Civita connection is a metric connection, we have the identity

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma \qquad X(\sigma|\sigma) = (\nabla_X\sigma|\sigma) + (\sigma|\nabla_X\sigma)$$

for any smooth section σ of the Clifford bundle and any smooth function f. Since the contraction i_X commutes with actions of the Clifford algebra we have

$$i_X(\nabla(\sigma \cdot \sigma)) = i_X((\nabla\sigma) \cdot \sigma + \sigma \cdot (\nabla\sigma))$$

for all smooth vector fields X so that

$$\nabla(\sigma \cdot \sigma) = (\nabla\sigma) \cdot \sigma + (\nabla\sigma) \cdot \sigma,$$

from which the Leibniz property follows by polarization. Notice that this result is independent upon the Riemannian curvature of the manifold. The situation changes drastically if we consider the Dirac Laplacian D^2 on $L^2(Cl(V, g), \tau)$ or better the Dirac quadratic form

$$\mathcal{E}_D[\sigma] := \int_V m_g(dx)\, |D\sigma(x)|^2.$$

Recall that the Dirac operator D is defined locally for smooth sections σ by

$$(D\sigma)(x) := \sum_{k=1}^{n} e_k(x) \cdot (\nabla_{e_k}\sigma)(x),$$

where the vector fields e_k form an orthonormal base in a neighborhood of $x \in V$. At an Hilbert space level, the Dirac operator on the Clifford bundle is isomorphic to the de Rham operator on the exterior bundle

$$D \simeq d + d^*$$

and the Dirac Laplacian is isomorphic to the Hodge–de Rham Laplacian

$$D^2 \simeq (d + d^*)^2 = dd^* + d^*d.$$

Differently from the Bochner Laplacian, the potential theoretic properties of D^2 depend, however, strongly on the sign of the curvature

Theorem 5.5 *([33]) The following conditions are equivalent*

(i) *the Dirac quadratic form $\mathcal{E}_D$ on $L^2(Cl(V,g),\tau)$ is a completely Dirichlet form*
(ii) *the curvature operator of V is nonnegative $\widehat{R} \geq 0$*
(iii) *Dirac heat semigroup e^{-tD^2} on the Clifford algebra $C_0^*(V,g)$ is completely Markovian.*

To describe the main steps of the proof let us recall that the curvature tensor R of the metric defines the curvature operator $\widehat{R}$ on the Hermitian bundle $\Lambda^2 V$ as follows

$$\widehat{R}_x : \Lambda_x^2 V \to \Lambda_x^2 V \qquad (\widehat{R}_x(v_1 \wedge v_2)|v_3 \wedge v_4)_{\Lambda_x^2 V} := R_x(v_1, v_2, v_3, v_4)$$
$$v_1, v_2, v_3, v_4 \in T_x V.$$

The curvatures identities imply that $\widehat{R}_x$ is symmetric and thus self-adjoint when extended on the Hilbert space obtained complexifying $\Lambda_x^2 V$. By the Bochner identity we have

$$\mathcal{E}_D = \mathcal{E}_B + Q_R$$

where Q_R is the quadratic form on $L^2(Cl(V,g),\tau)$ given by

$$Q_R[\sigma] = \int_V m(dx)\, Q_R(x)[\sigma_x] \qquad Q_R(x)[\sigma_x] = \sum_{\alpha=1}^{n(n-1)/2} \mu_\alpha(x) \|[\eta_\alpha(x), \sigma_x]\|^2$$

where the norms are those of the Hilbert spaces $L^2(Cl(T_x V, g_x), \tau_x)$, $\eta_\alpha(x) \in \Lambda_x^2 V$ are eigenvectors of $\widehat{R}_x$ and $\mu_\alpha(x) \in \mathbb{R}$ the corresponding eigenvalues. To prove that (i) implies (ii) one notices that, by the Albeverio–Hoegh-Krohn Theorem 4.5 or by the fact that commutators are bounded derivations, if the curvature operator is nonnegative, then all the eigenvalues are nonnegative and Q_R appears as a superposition of Dirichlet forms. Since, by the Davies–Rothaus Theorem 5.4 above, $\mathcal{E}_B$ is a Dirichlet form, we have that $\mathcal{E}_D$ is a Dirichlet form too. The proof that (i) implies (ii) the main idea is to use the decomposition theory of derivations to prove that Q_R is a Dirichlet forms because it coincides with the approximately bounded part of the Dirichlet form $\mathcal{E}_D$. Then, a careful analysis of the structure of the Dirichlet forms on the Clifford algebras of finite dimensional Euclidean spaces allows to conclude that all the eigenvalues or the curvature operator are nonnegative. Since D^2 is, by construction, a symmetric operator on $L^2(Cl(V,g),\tau)$, if we assume that the Dirac heat semigroup is completely Markovian on the Clifford algebra $C_0^*(V,g)$ then we get that it is a completely Markovian on $L^2(Cl(V,g),\tau)$ so that the quadratic form $\mathcal{E}_D$ is completely Dirichlet. To prove that $\widehat{R} \geq 0$ implies that the Dirac semigroup leaves globally invariant the Clifford algebra and that there is strongly continuous one uses (i) *ellipticity* of D^2 to deduce that e^{-tD^2} transforms compactly supported smooth sections of the Clifford bundle into bounded continuous sections, (ii) *Markovianity*, to reduce the problems to the algebra $C_0(V)$ of continuous functions and (iii) the fact that $\widehat{R} \geq 0$ implies that the Ricci curvature is nonnegative so that on $C_0(V)$ the Feller property holds true by a classical result.

5.4 Voiculescu Dirichlet Form in Free Probability

In this section we describe a Dirichlet form appearing in Free Probability Theory discovered by Voiculescu [107].

Let (M, τ) be a noncommutative probability space, i.e. a von Neumann algebra endowed with a faithful, normal trace state. Let us fix a unital *-subalgebra $1_M \in B \subset M$ and a finite set $X := \{X_1, \ldots, X_n\} \subset M$ of noncommutative random variables, i.e. self-adjoint elements of M, *algebraically free with respect to* B. Let us consider the *-subalgebra $B[X] \subset M$ generated by X and B (regarded as the algebra of *noncommutative polynomials in the variables* X with *coefficients* in the algebra B) and the von Neumann subalgebra $N \subset M$ generated by $B[X]$. Let $HS(L^2(N, \tau)) = L^2(N, \tau) \otimes L^2(N, \tau)$ be the Hilbert N-bimodule of Hilbert–Schmidt operators on $L^2(N, \tau)$ and $1_M \otimes 1_M \in HS(L^2(N, \tau))$ the rank one projection onto the multiples of the unit $1_M \in M \subset L^2(M, \tau)$.

Within this framework, Voiculescu introduced a natural differential calculus with associated Dirichlet form.

Theorem 5.6 *There exists a unique derivation* $\partial_{X_j} : B[X] \to HS(L^2(M, \tau))$ *for any fixed* $j = 1, \ldots, n$ *such that*

(i) $\partial_{X_j} X_k = \delta_{jk} 1_M \otimes 1_M \qquad k = 1, \ldots, n.$
(ii) $\partial_{X_j} b = 0$ *for all* $b \in B$.
(iii) The derivation $(\partial_{X_j}, B[X])$ *is densely defined in* $L^2(M, \tau)$*, symmetric and it is closable if* $1_M \otimes 1_M \in D(\partial^*_{X_j})$.
(iv) If $1_M \otimes 1_M \in \bigcap_{j=1}^n D(\partial^*_{X_j})$ *the quadratic form* $(\mathcal{E}, \mathcal{F})$ *defined as*

$$\mathcal{E}[a] := \sum_{j=1}^n \|\partial_{X_j} a\|^2_{HS(L^2(M,\tau))} \qquad a \in \mathcal{F} := B[X]$$

is closable and its closure is completely Dirichlet form.

Under the assumption $1_M \otimes 1_M \in \bigcap_{j=1}^n D(\partial^*_{X_j})$, the *Noncommutative Hilbert Transform of X with respect to B* is defined as

$$\mathcal{I}(X : B) := \Big(\sum_{j=1}^n \partial_{X_j}\Big) 1_M \otimes 1_M \in L^2(M, \tau)$$

and the *Relative Free Fisher information of X with respect to B* is defined as

$$\Phi^*(X : B) := \|\mathcal{I}(X : B)\|^2_{L^2(M,\tau)}.$$

It has been shown by Biane [11] that this Dirichlet form is the Hessian of the Relative Free Fischer information on the domain where the Relative Free Fisher information is finite. Moreover, if $B = \mathbb{C}$ and still under the assumption that $1_M \otimes$

$1_M \in \bigcap_{j=1}^n D(\partial_{X_j}^*)$, one has the following surprising spectral characterization of *semicircular random variables* X.

Theorem 5.7 *([11]) A Free Poincaré inequality holds true for some* $c > 0$

$$c \cdot \|a - \tau(a)\|_2^2 \leq \mathcal{E}[a] \qquad a \in L^2(M, \tau)$$

if and only if the random variable X *is centered, it has unital covariance and semi-circular distribution.*

In the case of semi-circular systems, the self-adjoint operator associated to the Dirichlet form $\mathcal{E}$ is unitarily equivalent to the Number operator on the Free Fock space, it generates the Free Ornstein–Uhlenbeck semi-group and a logarithmic Sobolev inequality holds true (see [12]).

6 Dirichlet Forms on Standard Forms of von Neumann Algebras

The theory of noncommutative Dirichlet forms illustrated so far has been introduced by Albeverio and Hoegh-Krohn and developed independently by Sauvageot [91, 92, 97] and by Lindsay and Davies [47, 48]. We have seen that it can be applied to several fields in which the relevant algebra of observables, to retain a physical language, is no more commutative but it requires, however, that the reference weight or state is a trace.

In this section we describe the extension of the theory to cases in which the reference functional is a normal state on a von Neumann algebra. In a forthcoming section we will describe how this theory may be used to study KMS-symmetric semigroups on C*-algebras as it is required for applications to Quantum Statistical Mechanics, Quantum Field Theory, Quantum Probability and Noncommutative Geometry. Notice that by a fundamental result of Dell'Antonio [52], the von Neumann algebras appearing in Quantum Field Theory are typically of type III so that no normal trace is available on them.

The exposition is based on the approach given in [24, 26] which work on general standard forms of σ-finite von Neumann algebras. An approach based on the Haagerup standard form [66, 67] of von Neumann algebras is given in [59, 60]. This last one has been generalized in [61] to consider the reference positive functional on a von Neumann algebra to be a weight. In this respect one ought to consult also [102, Appendix] for the correction of a result in [61].

The Potential Theory developed by Beurling and Deny and in particular the one to one correspondence between Dirichlet forms and symmetric Markovian semigroups on a measured space (X, m), relies on the geometric properties of the cone $L^2_+(X, m)$ of positive functions in the Hilbert space $L^2(X, m)$. This is a *closed, convex cone* which is *self-polar* in the sense that

$$a \in L^2_+(X, m) \quad \text{if and only if} \quad (a|b) \geq 0 \quad \text{for all} \quad b \in L^2_+(X, m).$$

The theory of noncommutative Dirichlet forms developed by Albeverio and Hoegh-Krohn on C*-algebras endowed with a faithful, semifinite trace (A, τ) is based on analogous properties of the cone $L^2_+(A, \tau)$ defined as the closure in the GNS Hilbert space $L^2(A, \tau)$ of the cone $\{a \in A_+ : \tau(a) < +\infty\}$. This cone determines, in particular, an anti-unitary involution J_τ on $L^2(A, \tau)$ which extends the isometric involution $a \mapsto a^*$ of A to the von Neumann algebra $L^\infty(A, \tau)$. The whole structure $(L^\infty(A, \tau), L^2(A, \tau), L^2_+(A, \tau), J_\tau)$ realizes *the standard form* of the von Neumann algebra $L^\infty(A, \tau)$ in the following sense.

Definition 6.1 (Standard form of a von Neumann algebra) ([5, 39, 66]).
A *standard form* $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ of a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, consists of a *self-polar cone* $\mathcal{P} \subset \mathcal{H}$ and an anti-unitary involution J, satisfying:

(i) $J\mathcal{M}J = \mathcal{M}'$;
(ii) $JxJ = x^*, \quad \forall x \in \mathcal{M} \cap \mathcal{M}'$ (the center of $\mathcal{M}$);
(iii) $J\eta = \eta, \quad \forall \eta \in \mathcal{P}$;
(iv) $xJxJ(\mathcal{P}) \subseteq \mathcal{P}, \quad \forall x \in \mathcal{M}$.

The J-*real part* of $\mathcal{H}$ is defined as $\mathcal{H}^J := \{\xi \in \mathcal{H} : J\xi = \xi\}$ and one has the decomposition $\mathcal{H} = \mathcal{H}^J \oplus i\mathcal{H}^J$. Moreover, one may define the *positive part* $\xi_+ \in \mathcal{P}$ of J-real vector $\xi \in \mathcal{H}^J$ as the Hilbert projection of ξ onto the positive cone $\mathcal{P}$, its *negative part* $\xi_- \in \mathcal{P}$ as the difference $\xi_- := \xi - \xi_+$ and its modulus by $|\xi| := \xi_+ + \xi_- \in \mathcal{P}$ so that $\xi = \xi_+ - \xi_-$ and $(\xi_+|\xi_-) = 0$.

Standard form of commutative von Neumann algebras. One may readily checks that $(L^\infty(X, m), L^2(X, m), L^2_+(X, m), J)$ is a standard form of the commutative von Neumann algebra $L^\infty(X, m)$ (once the anti-unitary involution is given by the complex conjugation: $Ja = \overline{a}$) and that the above notions related to the order structure assume the familiar meaning.

Hilbert–Schmidt standard form. A noncommutative example is provided by Hilbert–Schmidt standard form

$$(B(h), L^2(h), L^2_+(h), J)$$

of the algebra $B(h)$ of all bounded operators on a Hilbert space h. In the Hilbert space $L^2(h)$ of all Hilbert–Schmidt operators on h, the cone $L^2_+(h)$ of the positive ones is self-polar and the involution J associates to the Hilbert–Schmidt operator a its adjoint a^*.

Essential properties of the standard form of a von Neumann algebra are its *existence and uniqueness* (modulo unitaries preserving the positive cones). These properties authorize to denote the standard form of a von Neumann algebra M simply by

$$(M, L^2(M), L^2_+(M), J).$$

These main results are also of practical use because different standard forms may show different advantages (or inconveniences). In the commutative case uniqueness is a reflection of the fact that the von Neumann algebra $L^\infty(X, m)$ is determined by the class of zero m-measure sets only, so that the algebra can be represented standardly on the space $L^2(X, m')$ of any measure m' equivalent to m.

Standard form of semifinite von Neumann algebras. In case the von Neumann algebras $\mathcal{M}$ is semifinite, a standard form may be constructed by the GNS representation associated to a normal, semifinite trace τ on M as

$$(M, L^2(M, \tau), L^2_+(M, \tau), J_\tau).$$

To construct the standard form of a von Neumann algebra M starting from a normal state $\omega_0 \in M_{*+}$ one need to recall some aspects of the Tomita–Takesaki Modular Theory of von Neumann algebras [104, 105]. We may assume that M is represented in a Hilbert space $\mathcal{H}$ so that $M \subseteq B(\mathcal{H})$ and ω_0 is represented by a cyclic and separating vector $\xi_0 \in h$ as $\omega_0(x) = (\xi_0|x\xi_0)_\mathcal{H}$ for $x \in M$ (for example, $\mathcal{H}$ can be assumed to be the GNS space $L^2(M, \omega_0)$). The anti-linear map $S(x\xi_0) := x^*\xi_0$, densely defined on $M\xi_0 \subseteq \mathcal{H}$, is a closable operator on $\mathcal{H}$ and we may consider the polar decomposition of its closure $\bar{S}$

$$\bar{S} = J\Delta_0^{1/2}$$

where the square root of the self-adjoint *modular operator* $\Delta_0 = \bar{S}^*\bar{S}$ provides its positive part and the *modular conjugation* J is an anti-unitary operator on $\mathcal{H}$ providing the phase. Using these tools one proves that

$$\mathcal{P} := \overline{\{xJxJ \in \mathcal{H} : x \in M\}}$$

is a self-polar cone in $\mathcal{H}$ coinciding with $\overline{\Delta_0^{1/4}M_+\xi_0}$ and that $(M, \mathcal{H}, \mathcal{P}, J)$ is a standard form. When ω_0 is a trace state then S is isometric so that the modular operator Δ_0 reduces to the identity, $S = J$ and $\mathcal{P} = \overline{M_+\xi_0}$. The modular operator Δ_0 measures how much the state ω_0 differs from a trace state in that only in this case Δ_0 reduces to the identity.

The denomination *modular* used for the operator Δ_0 originates from the following example.

Modular operator and standard form of group von Neumann algebra. Let (G, m_H) be a locally compact group and consider the convolution algebra $C_c(G)$ acting by left convolution λ_G on $L^2(G, m_H)$ and define the *group von Neumann algebra* $L(G)$ as $\lambda_G(C_c(G))''$. The Haar measure m_H determines an additive, homogeneous, lower semicontinuous functional ω_H on the positive part $L(G)_+$, called the *Plancherel weight* (see [106, Chap. VII]). It is a trace if and only if G is unimodular and a trace state if and only if G is discrete. Since on $L(G)$ the involution is determined by $\lambda_G(a)^* := \lambda_G(a^*)$ where $a^*(s) := \overline{a(s^{-1})}$ for $s \in G$ and $a \in C_c(G)$, one may check that the modular operator Δ_H on $L^2(G, m_H)$ associated to the Plancherel weight ω_H is given by the multiplication operator by the *modular function* $G \ni s \mapsto dm_H(\cdot s^{-1})/dm_H$.

Modular operators and Gibbs states. On the von Neumann algebra $B(h)$ any normal state ω_0 can be represented by a self-adjoint, positive, compact operator $\rho \in B(h)$ having unit trace, called *density matrix*, as follows

$$\omega_0(x) = \mathrm{Tr}\,(x\rho) \qquad x \in B(h).$$

setting $H := -\ln\rho$ we have $\rho = e^{-H}$ so that ω_ρ appears as the Gibbs equilibrium state of the dynamical system whose time evolution is given by the automorphisms group

$$\alpha_t(x) = e^{-itH} x e^{+itH} \qquad x \in B(h)$$

generated by the Hamiltonian H. We now use the Hilbert–Schmidt standard form of $B(h)$ to compute the action of the modular operator. Since $\omega_0(x) = \mathrm{Tr}\,(x\rho) = \mathrm{Tr}\,(\rho^{1/2}x\rho^{1/2}) = (\rho^{1/2}|x\rho^{1/2})_{L^2(h)}$ we have that compact operator $\xi_0 = \rho^{1/2} \in L^2(h)$ in the Hilbert–Schmidt class is the cyclic and separating vector representing ω_0. To recover the action of the modular operator notice that, by definition, we have $J\Delta_0^{1/2}(x\xi_\rho) = x^*\xi_\rho$ for $x \in \mathcal{B}(h)$. Then $\Delta_0^{1/2}(x\rho^{1/2}) = J(x^*\rho^{1/2}) = \rho^{1/2}x$ for all $x \in \mathcal{B}(h)$ so that

$$\Delta_0^{1/2}\xi = \rho^{1/2}\xi\rho^{-1/2} \qquad \xi \in D(\Delta_0^{1/2}) := \{\eta \in L^2(h) : \rho^{1/2}\eta\rho^{-1/2} \in L^2(h)\}.$$

Notice that $(xJxJ)(\xi) = x\xi x^*$ for all $x \in B(h)$ and $\xi \in L^2(h)$ so that $\mathcal{P} = L^2_+(h)$.

Another crucial property of the standard form is that any normal state $\omega \in \mathcal{M}_{*+}$ can be represented as the vector state of a unique, unit vector $\xi_\omega \in \mathcal{P}$ in the standard positive cone, i.e. $\omega(x) = (\xi_\omega|x\xi_\omega)_{\mathcal{H}}$ for all $x \in \mathcal{M}$.

6.1 Tomita–Takesaki Theory and Connes' Radon–Nikodym Theorem [38, 104, 105]

Let (M,ω) be a von Neumann algebra with a faithful, normal state and denote by $\pi_\omega : M \to B(L^2(M,\omega))$ the associated GNS representation. The Tomita–Takesaki Theorem then ensure that

$$J_\omega\pi_\omega(M)J_\omega = \pi_\omega(M)',$$

$$\Delta_\omega^{-it}\pi_\omega(M)\Delta_\omega^{it} = \pi_\omega(M) \qquad t \in \mathbb{R}.$$

Moreover, setting

$$\sigma_t^\omega : M \to M \qquad \sigma_t^\omega(x) := \pi_\omega^{-1}(\Delta_\omega^{-it}\pi_\omega(x)\Delta_\omega^{it}) \qquad x \in M, \qquad t \in \mathbb{R}$$

one gets a w*-continuous group $\sigma^\omega \in \mathrm{Aut}(M)$ of automorphisms of the von Neumann algebra that *satisfies and it is uniquely determined* by the following *modular condition*

$$\omega(x\sigma^\omega_{-i}(y)) = \omega(yx)$$

for all $x, y \in M$ which are analytic with respect to σ^ω. A fundamental theorem due to Connes [[40], Theorem 1.2.1], which has to be considered as the noncommutative generalization of the Radon–Nikodym Theorem, states that the modular automorphism group of a von Neumann algebra is essentially unique: for any pair $\phi, \psi \in M_{*+}$ of faithful, normal states on M, there exists a canonical 1-cocycle $u : \mathbb{R} \to \mathcal{U}(M)$ for σ^ϕ_t,with values in the unitary group of M

$$u_{t_1+t_2} = u_{t_1}\sigma^\phi_{t_1}(u_{t_2}) \qquad t_1, t_2 \in \mathbb{R},$$

such that

$$\sigma^\psi_t(x) = u_t\sigma^\phi_t(x)u^*_t \qquad x \in M, \quad t \in \mathbb{R}.$$

6.1.1 Modular Operators on Type I Factors

In case of the von Neumann algebra $\mathcal{B}(h)$ and the normal state $\omega(x) := \mathrm{Tr}(\rho x)$ associated to a positive, trace-class operator $\rho \in \mathcal{B}(h)$ with unit trace, one checks that the modular group is given by

$$\sigma^\omega_t(x) = \rho^{it}x\rho^{-it}, \qquad x \in \mathcal{B}(h), \qquad t \in \mathbb{R}$$

and that the modular condition follows from the trace property of Tr

$$\omega(yx) = \mathrm{Tr}(\rho yx) = \mathrm{Tr}(\rho x\rho y\rho^{-1}) = \omega(x\sigma^\omega_{-i}(y)).$$

In the particular case of a matrix algebra $M_n(\mathbb{C})$, denoting by e_{jk} the matrix units, if the density matrix ρ is diagonal with eigenvalues $\lambda_1, \dots, \lambda_n > 0$, one has

$$\sigma^\omega_t(e_{jk}) = \Big(\frac{\lambda_j}{\lambda_k}\Big)^{it} e_{jk} \qquad j, k = 1, \dots, n, \quad t \in \mathbb{R}.$$

6.2 Symmetric Embeddings [5, 24]

In the commutative case, the standard form of a probability space (X, m), we have the natural embeddings $L^\infty(X, m) \subseteq L^2(X, m)$, $L^2(X, m) \subseteq L^1(X, m)$ and $L^\infty(X, m) \subseteq L^1(X, m)$.

These may be generalized to the standard form of any von Neumann algebra M, using the modular operators associated to any fixed faithful normal state $\omega \in M_{*+}$.

Definition 6.2 (Symmetric embeddings) The *symmetric embeddings* associated to the standard form $(M, \mathcal{H}, \mathcal{P}, J)$ and a cyclic and separating vector $\xi_\omega \in \mathcal{P}$ are defined as follows:

(i) $i_\omega : M \to \mathcal{H} \quad i_\omega(x) := \Delta_\omega^{1/4} x\xi_\omega, \qquad x \in M;$
(ii) $i_{\omega*} : \mathcal{H} \to M_* \quad \langle i_{\omega*}(\xi), y\rangle = (i_\omega(y^*)|\,\xi) = (\Delta_\omega^{1/4} y^*\xi_\omega|\,\xi), \quad \xi \in \mathcal{H}, \ \ y \in M;$
(iii) $j_\omega : M \to M_* \quad \langle j_\omega(x), y\rangle = (J_\omega y\xi_\omega|x\xi_\omega), \qquad x, y \in M.$

These maps are well defined because $\mathcal{M}\xi_\omega \subseteq D(\Delta_\omega^{1/2})$, by the very definition of the modular operator, and because $D(\Delta_\omega^{1/2}) \subseteq D(\Delta_\omega^{1/4})$ by the Spectral Theorem.

The essential feature of these embeddings is that they preserve the order structures of $\mathcal{M}, \mathcal{H}$ and M_* provided by the positive cones of these spaces. In particular i_ω establishes a one to one homeomorphism between the set $[0, 1_M] := \{x = x^* \in M : 0 \le x \le 1_M\} \subset M_+$ and its image $i_\omega([0, 1_M]) = \{\xi \in \mathcal{P} : 0 \le \xi \le \xi_\omega\} := [0, \xi_\omega] \subset \mathcal{P}$.

In the following we will indicate by $\xi \wedge \xi_0$ the Hilbert projection onto the closed, convex set $\{\xi \in \mathcal{H}^J : \xi \le \xi_\omega\}$ of a J-real vector $\xi \in \mathcal{H}^J$. In the commutative case and when ξ_ω is given by the constant function 1 and $a \in L^2_{\mathbb{R}}(X, m)$ is a real function, then $a \wedge 1$ reduces to the so called *unit contraction* of a given by $(a \wedge 1)(x) = \inf(a(x), 1)$ for $x \in X$. Using this geometric operation we may rephrase on the standard form of any von Neumann algebra M, the Markovianity of Dirichlet forms one considers in the commutative setting $L^\infty(X, m)$.

Definition 6.3 (Dirichlet forms [24]) Let $(M, \mathcal{H}, \mathcal{P}, J)$ be a standard form of a von Neumann algebra M and $\xi_\omega \in \mathcal{P}$ a cyclic and separating vector representing the normal state $\omega \in M_{*+}$.

A quadratic form $\mathcal{E} : \mathcal{H} \to (-\infty, +\infty]$ is said to be J-*real* if

$$\mathcal{E}[J\xi] = \mathcal{E}[\xi] \qquad \xi \in \mathcal{H}$$

and *Markovian* with respect to ξ_ω if it is J-real and

$$\mathcal{E}[\xi \wedge \xi_\omega] \le \mathcal{E}[\xi] \qquad \forall \xi \in \mathcal{H}^J. \tag{6.1}$$

In case $\mathcal{E}[\xi_\omega] = 0$, the Markovianity condition is equivalent to

$$\mathcal{E}(\xi_+|\xi_-) \le 0 \qquad \xi = J\xi \in \mathcal{H}.$$

A *densely defined, lower semicontinuous Markovian form is called a Dirichlet form* with respect to ξ_ω or ω.

The quadratic form $\mathcal{E}$ is called a *completely Dirichlet form* if any of its matrix extension $\mathcal{E}_n$ on $\mathcal{H} \otimes L^2(\mathbb{M}_n(\mathbb{C})$, given by

$$\mathcal{E}_n[[a_{i,j}]_{i,j=1}^n] = \sum_{i,j=1}^n \mathcal{E}[a_{i,j}] \qquad [a_{i,j}]_{i,j=1}^n \in \mathcal{H} \otimes L^2(\mathbb{M}_n(\mathbb{C}),$$

is a Dirichlet form w.r.t. the state $\omega_0 \otimes \tau_n$, where τ_n is the unique tracial state on $\mathbb{M}_n(\mathbb{C})$.

In particular, if $\mathcal{E}$ is Markovian and $\mathcal{E}[\xi]$ is finite then $\mathcal{E}[\xi \wedge \xi_\omega]$ is finite too. Also, in the commutative setting Dirichlet forms are automatically completely Dirichlet forms. In other words, under the Hilbertian projection $\xi \mapsto \xi \wedge \xi_\omega$, the value of the quadratic form does not increase. As noticed above, this definition reduces to the usual one in the commutative setting. We are going to see that in any standard form, Dirichlet forms represent an infinitesimal characterization of strongly continuous, symmetric Markovian semigroups.

Theorem 6.4 *(Characterization of Markovian semigroups by Dirichlet forms [24, 59, 60]) Let $(M, \mathcal{H}, \mathcal{P}\, J)$ be a standard form of a von Neumann algebra M and $\xi_\omega \in \mathcal{P}$ the cyclic vector representing a state $\omega \in M_{*+}$. Let $\{T_t : t \geq 0\}$ be a J-real, symmetric, strongly continuous, semigroup on the Hilbert space $\mathcal{H}$ and $\mathcal{E} : \mathcal{H} \to (-\infty, +\infty]$ the associated J-real, lower semibounded, closed quadratic form. Then, the following properties are equivalent*

- *(i)* $\{T_t : t \geq 0\}$ *is Markovian with respect to* ξ_ω*;*
- *(ii)* $\mathcal{E}$ *is a Dirichlet form with respect to* ξ_ω*.*

In particular, Dirichlet forms are automatically nonnegative and Markovian semigroups are automatically contractive.

Dirichlet forms not only determine and are determined by strongly continuous Markovian semigroups on the standard Hilbert space, but they are also in one-to-one correspondence with point-weak*-continuous, completely positive, subunital (abbreviated with Markovian) semigroups on the von Neumann algebra, satisfying a certain *modular symmetry property* which is a deformation of the modular condition.

Theorem 6.5 *(Markovian semigroups on standard forms of von Neumann algebras [24]) Let $(\mathcal{M}, \mathcal{H}, \mathcal{P}\, J)$ be a standard form of a von Neumann algebra $\mathcal{M}$, $\omega \in M_{*+}$ a faithful state and $\xi_\omega \in \mathcal{P}$ its representing cyclic vector. Then there exists a one-to-one correspondence between*

- *(i) Markovian (with respect to ξ_ω) semigroups $\{T_t : t > 0\}$ on $L^2(M)$ and*
- *(ii) Markovian semigroups $\{S_t : t > 0\}$ on M which are ω-modular symmetric in the sense that, for all x, y in a weak*-dense, σ^ω-invariant, *-subalgebra of M_{σ^ω}, one has*

$$\omega\Big(y S_t(x)\Big) = \omega\Big(\sigma^\omega_{\frac{i}{2}}(x) S_t(\sigma^\omega_{-\frac{i}{2}}(y))\Big). \tag{6.2}$$

The correspondence is provided by the symmetric embedding through the relation

$$i_\omega : M \to L^2(M) \qquad i_\omega \circ S_t = T_t \circ i_\omega .$$

Remark 6.6 A careful analysis of the family of closed cones $\overline{\{\Delta^\alpha_\omega x \xi_\omega : x \in M_+\}}$ in $L^2(M)$ for $\alpha \in (0, 1/2)$, indicates that if instead of the symmetric embedding, the so called *GNS embedding* $x \mapsto x\xi_\omega$ of M into $L^2(M)$ is used, the resulting semigroup automatically commutes with modular operator. In application to convergence to equilibrium in Quantum Statistical Mechanics this situation should be avoided and this is the reason why the self-polar cone corresponding to $\alpha = 1/4$ is used.

6.2.1 Elementary Dirichlet Forms

As a first example of a Dirichlet form with respect to a not necessarily trace state, we illustrate a construction that can be considered a generalization of the one of Albeverio–Hoegh-Krohn in Theorem 4.5 above. Elementary Dirichlet form will find application to approximation/rigidity properties of von Neumann algebras in Sect. 7.6.

Let $(\mathcal{M}, \mathcal{H}, \mathcal{P}\ J)$ be a standard form and $\xi_0 \in \mathcal{P}$ a cyclic vector. Consider, for fixed $a_k \in \mathcal{M}$, $\mu_k, \nu_k > 0$ and $k = 1, \ldots, n$, the operators

$$\partial_k : \mathcal{H} \to \mathcal{H} \qquad d_k := i(\mu_k a_k - \nu_k j(a_k^*))$$

and the bounded quadratic form

$$\mathcal{E}[\xi] := \sum_{k=1}^{n} \|\partial_k \xi\|^2 \qquad \xi \in \mathcal{H}.$$

Then $\mathcal{E}$ is J-real and $\mathcal{E}(\xi_+|\xi_-) \leq 0$ for all J-real $\xi \in \mathcal{H}$ if and only if

$$\sum_{k=1}^{n} [\mu_k^2 a_k^* a_k - \nu_k^2 a_k a_k^*] \in \mathcal{M} \cap \mathcal{M}'.$$

Moreover, if the above condition holds true, $\mathcal{E}$ is a Dirichlet form with $\mathcal{E}[\xi_0] = 0$ if and only if the numbers $(\mu_k/\nu_k)^2$, $k = 1, \ldots, n$, are eigenvalues of the modular operator Δ_{ξ_0}, corresponding the eigenvectors $a_k\xi_0 \in \mathcal{H}$. Conditions like the one above are considered in the framework of q-deformed CCR relations and related factor von Neumann algebras [13].

The construction above provides a, possibly unbounded, Dirichlet form even when $n = \infty$, provided $\mathcal{E}$ is densely defined.

6.2.2 Quantum Ornstein–Uhlenbeck and Quantum Brownian Motion Semigroups [29, 71]

We describe here the construction of a Dirichlet form, on the Neumann algebra $B(h)$, which generates a Markovian semigroups appearing in quantum optics.

On the Hilbert space $h := l^2(\mathbb{N})$ consider the standard form $(B(h), L^2(h), L^2_+(h), J)$. Let $\{e_n : n \geq 0\} \subset h$ be the canonical Hilbert basis, and denote by $| e_m\rangle\langle e_n|$, $n, m \geq 0$, the partial isometries, having $\mathbb{C}e_n$ as initial space and $\mathbb{C}e_m$ as final one.

Fix the parameters $\mu > \lambda > 0$, set $\nu := \lambda^2/\mu^2$ and let $\omega_\nu(x) := \mathrm{Tr}(\rho_\nu x)$ the normal state on $B(h)$ represented by the density matrix

$$\rho_\nu := (1-\nu)\sum_{n\geq 0} \nu^n |e_n\rangle\langle e_n|.$$

The state ω_ν is then represented by the cyclic vector $\xi_\nu := \rho_\nu^{1/2} = (1-\nu)^{1/2} \sum_{n\geq 0} \nu^{n/2}|e_n\rangle\langle e_n|$. The *creation* and *annihilation operators*, a^* and a on h, are defined by

$$a^* e_n := \sqrt{n+1}e_{n+1}, \qquad ae_n := \begin{cases} \sqrt{n}e_{n-1}, & \text{if } n > 0; \\ 0, & \text{if } n = 0. \end{cases}$$

They are adjoint of one another on their common domain $D(a) = D(a^*) = \big\{e \in h : \sum_{n\geq 0} \sqrt{n}|\langle e|\, e_n\rangle|^2 < \infty\big\}$ and satisfy the CCR

$$a\,a^* - a^* a = I.$$

The quadratic form given by

$$\mathcal{E}[\xi] := \|\mu a\xi - \lambda\xi a\|^2 + \|\mu a\xi^* - \lambda\xi^* a\|^2,$$

densely defined in $L^2(h)$ on the subspace of finite rank operators

$$D(\mathcal{E}) := \text{linear span}\{| e_m\rangle\langle e_n|, n, m \geq 0\},$$

is closable and Markovian so that its closure is a Dirichlet form with respect to ω_ν, generating the so called *quantum Ornstein–Uhlenbeck* Markovian semigroup. Moreover, since, as it is easy to check one has $\mathcal{E}[\xi_\nu] = 0$, it results that the cyclic vector is left invariant by the semigroup.

When $\lambda = \mu$, the role of the invariant state ω_ν has to be played by the normal, semifinite trace τ on $B(h)$. However, even in this case, using the Albeverio–Hoegh-Krohn criterion, it is possible to prove that the closure of the unbounded quadratic form

$$\mathcal{E}[\xi] := \|a\xi - \xi a\|^2 + \|a\xi^* - \xi^* a\|^2, \qquad \xi \in D(\mathcal{E})$$

is a Dirichlet form. The associated τ-symmetric Markovian semigroup on $B(h)$, may be dilated by a Quantum Stochastic Process, known as the *Quantum Brownian motion*. This represents a sort of *bridge* between pairs of classical stochastic processes of quite different type. In fact on a suitable, invariant, maximal abelian subalgebra (masa), this semigroup reduces to the semigroup of a classical Brownian motion while on another masa, it reduces to the semigroup of a classical birth and death process.

7 Application to Approximation/Rigidity Properties of von Neumann Algebras

In this section we describe three results showing that the spectral properties of Dirichlet forms are naturally and deeply connected with those fundamental properties of von Neumann algebras having to do with the ideas of approximation and rigidity.

7.1 Amenable Groups

In 1929, von Neumann discovered a far reaching explanation of the Banach–Tarski paradox in terms of a property, called *amenability*, of a group of Euclidean motions in $\mathbb{R}^n$ which holds true in dimension $n = 1, 2$ but it does not in higher dimensions.

Definition 7.1 A discrete group Γ is *amenable* if there exist a left-translation invariant probability measure on Γ.

This property is equivalent to the existence of a sequence of finitely supported, positive definite functions ϕ_n on Γ, pointwise converging to the constant function 1,

$$\lim_n \phi_n(t) = 1 \qquad \text{for all } t \in \Gamma,$$

and to the existence of a *proper*, conditionally negative definite function $\psi : \Gamma \to \mathbb{C}$ (see Sect. 5.1.1). Recall that a function $\phi : \Gamma \to \mathbb{C}$ is positive definite if the matrices $[\phi(s_j^{-1}s_k)]_{j,k=1}^n$ are positive definite for all $s_1, \ldots, s_k \in \Gamma$, i.e. if for all $c_1, \ldots c_n \in \mathbb{C}$ one has

$$\sum_{j,k=1}^n \bar{c}_j \phi(s_j^{-1}s_k)c_k \geq 0.$$

Since positive definite functions are coefficients of unitary representations and the constant function 1 is the coefficient of the trivial representation, amenability is also equivalent to the fact that the trivial representation is weakly contained in the left regular one, i.e. it is unitarily equivalent to a subrepresentation of a multiple of the regular representation.

The amenability of a group Γ can be translated in terms of a corresponding property of its associated group von Neumann algebra $L(\Gamma)$.

To introduce this property in complete generality, we recall the fundamental notions of bimodule and correspondence.

7.2 Bimodules and Connes Correspondences [4, 42, 43, 90]

A Banach *M-bimodule* E on a C*-algebra M is a Banach space E together with a pair of norm continuous, commuting actions of M.

If the left action of $x \in M$ on $\xi \in E$ is denoted by $x\xi$ and the right action of $y \in M$ on $\xi \in E$ is denoted by ξy, the required commutation reads $(x\xi)y = x(\xi y)$.

In case M is a von Neumann algebra and E is a *dual* bimodule, in the sense that it is the dual Banach space of a predual one, the left and right actions are required to be continuous with respect to the weak*-topology of E.

A *Connes correspondence* on M is a Hilbert space $\mathcal{H}$ which is an M-bimodule.

Denote by M° the *opposite algebra* of M: it coincides with M as a vector space but the product is taken in the reverse order $x^\circ y^\circ := (yx)^\circ$ for $x^\circ, y^\circ \in M^\circ$. By convention, elements $y \in M$, when regarded as elements of the opposite algebra are denoted by $y^\circ \in M^\circ$. Let $M \otimes_{\max} M^\circ$ the maximal tensor product of M and M° considered as C*-algebras.

A correspondence on M is nothing but a representation

$$\pi : M \otimes_{\max} M^\circ \to B(\mathcal{H}) \qquad \pi(x \otimes y^\circ)\xi = x\xi y$$

such that $M \ni x \mapsto \pi(x \otimes 1_M)$ and $M \ni x \mapsto \pi(1_M \otimes x^\circ)$ provide normal representations. Correspondences of von Neumann algebras may be conveniently thought of both as generalization of unitary representations of groups.

Among the correspondences of von Neumann algebras, the following are of central importance.

The *identity* or *standard* correspondence of a von Neumann algebra M is provided by its standard representation $(M, L^2(M), L^2_+(M), J)$. Here beside the left action of M on $L^2(M)$, denoted by $x\xi$ for $x \in M$ and $\xi \in L^2(M)$, we have the right action defined by $\xi x := Jx^* J\xi$.

The *coarse correspondence* is the M-bimodule given by $L^2(M) \otimes \overline{L^2(M)}$ with actions

$$x(\xi \otimes \overline{\eta})y = x\xi \otimes \overline{\eta y} \qquad x, y \in M, \quad \xi, \eta \in L^2(M).$$

This is also called the *Hilbert–Schmidt correspondence* by the identification of $L^2(M) \otimes \overline{L^2(M)}$ with the Hilbert space $HS(L^2(M))$ of Hilbert–Schmidt operator on $L^2(M)$. In this terms the actions are given by $xTy \in HS(L^2(M))$ for $x, y \in M$ and $T \in HS(L^2(M))$.

Correspondences of von Neumann algebras may also be fruitfully thought as generalization of completely positive maps. In fact, suppose that on M a faithful, normal state ω is fixed and consider a not necessarily self-adjoint, completely Markovian map $T : L^2(M, \omega) \to L^2(M, \omega)$, assuming, to simplify, that $T\xi_\omega = \xi_\omega$. Then the functional determined by

$$\Phi_T : M \otimes_{\max} M^\circ \to \mathbb{C} \qquad \Phi_T(x \otimes y^\circ) := (i_\omega(y^*)|Ti_\omega(x))$$

is a state on $M \otimes_{\max} M^\circ$ which, by the GNS construction, give rise to a representation of $M \otimes_{\max} M^\circ$, thus to a correspondence $\mathcal{H}_T$ on M. The unit, cyclic vector $\xi_T \in \mathcal{H}_T$ representing Φ_t thus satisfies

$$(i_\omega(y^*)|Ti_\omega(x)) = (\xi_T|x\xi_T y)_{\mathcal{H}_T} \qquad x, y \in M.$$

A fundamental operation that is defined on correspondences is their *relative tensor product*, by which any M-N-correspondence $\mathcal{H}_N$ and any N-P-correspondence $\mathcal{K}_P$ may tensorized, in this order, to produce an M-P-correspondence denoted by $\mathcal{H} \otimes \mathcal{K}_P$. The advantages to translate into the common language of correspondences problems of apparently different origin concerning von Neumann algebras, are the possibility to let them play into a shared ground on one side, and the possibility to use the tools of representation theory, for example to introduce notions like containment, weak containment and convergence.

7.3 Amenable von Neumann Algebras

Definition 7.2 ([23, 41, 42]) A C* or von Neumann algebra M is said to be *amenable* if for every dual Banach M-bimodule E, all derivations $\delta : M \to X$, that is maps satisfying the Leibniz property

$$\delta(ab) = (\delta a)b + a(\delta b) \qquad a, b \in M,$$

are inner, i.e. there exists $\xi \in E$ such that

$$\delta(x) = x\xi - \xi x \qquad x \in M.$$

This property was introduced by Johnson and Ringrose in their works on cohomology of operator algebras. As the result of an enormous amount of efforts, it has been shown that amenability is equivalent to approximation properties:

(i) a C*-algebra A is amenable if and only if it is *nuclear* in the sense that its identity map can be approximated in the point-norm topology,

$$\lim_n \|\psi_n \circ \phi_n(a) - a\| = 0 \qquad \text{for all } a \in A,$$

$$\psi_n : A \to \mathbb{M}_{k_n}(\mathbb{C}) \qquad \phi_n : \mathbb{M}_{k_n}(\mathbb{C}) \to A;$$

(ii) a von Neumann algebra M is *weakly nuclear* if and only if its identity map can be approximated in the point-ultraweak topology,

$$\lim_n \eta(\psi_n \circ \phi_n(a) - a) = 0 \qquad \text{for all } a \in A, \quad \eta \in M_*,$$

by the composition of suitable contractive, completely positive maps

$$\psi_n : A \to \mathbb{M}_{k_n}(\mathbb{C}) \qquad \phi_n : \mathbb{M}_{k_n}(\mathbb{C}) \to A;$$

(iii) a von Neumann algebra M is amenable if and only if it is *hyperfinite* in the sense that it is generated by an increasing sequence of finite-dimensional subalgebras.

Among the examples of amenable von Neumann algebras, one may recall (i) the group von Neumann algebra of a locally compact amenable group, (ii) the crossed product of an abelian von Neumann algebra by an amenable locally compact group, (iii) the commutant von Neumann algebra of any continuous unitary representation of a connected locally compact group and (iv) the von Neumann algebra generated by any representation of a nuclear C*-algebra.

7.4 Amenability and Subexponential Spectral Growth Rate of Dirichlet Forms [36]

To illustrate a first connection between approximation properties of von Neumann algebras and spectral properties of Dirichlet form, we first recall a definition.

Definition 7.3 (Spectral growth rate of Dirichlet forms [36]) Let (N, ω) be an infinite dimensional, σ-finite, von Neumann algebra with a fixed faithful, normal state on it.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(N, \omega)$ and let $(L, D(L))$ be the associated nonnegative, self-adjoint operator. Assume that its spectrum $\sigma(L) = \{\lambda_k \geq 0 : k \in \mathbb{N}\}$ is *discrete*, set

$$\Lambda_n := \{k \in \mathbb{N} : \lambda_k \in [0, n]\}, \qquad \beta_n := \sharp(\Lambda_n), \qquad n \in \mathbb{N}$$

and define the *spectral growth rate* of $(\mathcal{E}, \mathcal{F})$ as

$$\Omega(\mathcal{E}, \mathcal{F}) := \limsup_{n \in \mathbb{N}} \sqrt[n]{\beta_n}.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to have

exponential growth if $(\mathcal{E}, \mathcal{F})$ has discrete spectrum and $\Omega(\mathcal{E}, \mathcal{F}) > 1$
subexponential growth if $(\mathcal{E}, \mathcal{F})$ has discrete spectrum and $\Omega(\mathcal{E}, \mathcal{F}) = 1$
polynomial growth if $(\mathcal{E}, \mathcal{F})$ has discrete spectrum and, for some $c, d > 0$ and all $n \in \mathbb{N}$,

$$\beta_n \leq c \cdot n^d.$$

intermediate growth if it has subexponential growth but not polynomial growth.

It is easy to see that the subexponential growth property can be formulated in terms of *nuclearity* of the Markovian semigroup $\{e^{-tL} : t > 0\}$ on $L^2(N, \omega)$:

Lemma 7.4 *The Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *has discrete spectrum and subexponential spectral growth rate if and only if the Markovian semigroup* $\{e^{-tL} : t > 0\}$ *on* $L^2(N, \omega)$ *is nuclear, or trace-class, in the sense that:*

$$\text{Trace}\,(e^{-tL}) < +\infty \qquad t > 0.$$

Here is the announced connection between amenability and spectral properties.

Theorem 7.5 *([36]) Let* N *be a* σ*-finite von Neumann algebra. If there exists a normal, faithful state* $\omega \in M_{*+}$ *and a Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(N, \omega)$ *having subexponential spectral growth, then* N *is amenable.*

Let us sketch the main points of the proof assuming, to simplify, that $\mathcal{E}[\xi_\omega] = 0$. Let $N\overline{\otimes}N^\circ$ the *von Neumann spatial tensor product* of N and N°. It turns out that the coarse representation of $N \otimes_{\max} N^\circ$, defined by

$$\pi_{\text{co}} : N \otimes_{\max} N^\circ \to \mathcal{B}(L^2(N, \omega) \otimes L^2(N, \omega))$$
$$\pi_{\text{co}}(x \otimes y^o)(\xi \otimes \eta) := x\xi \otimes \eta y \qquad x, y \in N, \quad \xi, \eta \in L^2(N, \omega),$$

give rise to the spatial tensor product of the von Neumann algebras

$$(\pi_{\text{co}}(N \otimes_{\max} N^\circ))'' = N\overline{\otimes}N^\circ.$$

Moreover, the normal extension of the coarse representation π_{co} of $N \otimes_{\max} N^\circ$ to $N\overline{\otimes}N^\circ$ is the standard representation of $N\overline{\otimes}N^\circ$ so that

$$L^2(N, \omega) \otimes L^2(N, \omega) \simeq L^2(N\overline{\otimes}N^\circ, \omega \otimes \omega^\circ)$$

and the positive cone $L^2_+(N\overline{\otimes}N^\circ, \omega \otimes \omega^\circ)$ can be identified with the cone of all positive, Hilbert–Schmidt operators on $L^2(N, \omega)$. In particular, since, by assumption, e^{-tL} is a positive, Hilbert–Schmidt operator for all $t > 0$, we have

$$e^{-tL} \in L^2_+(N\overline{\otimes}N^\circ, \omega \otimes \omega^\circ) \qquad t > 0.$$

Since $\mathcal{E}$ is a complete Dirichlet form, its associated semigroup is completely positive and this implies that the linear functional Φ_t, determined by

$$\Phi_t : N \otimes_{\max} N^\circ \to \mathbb{C} \qquad \Phi_t(x \otimes y^\circ) := (i_\omega(y^*)|e^{-tL} i_\omega(x)),$$

is positive and actually a state since $\mathcal{E}[\xi_\omega] = 0$. By the continuity properties of the symmetric embeddings and the above identifications, we have

$$\Phi_t(z) = \left(e^{-tL}\middle| i_{\omega\otimes\omega^\circ}(z)\right)_{L^2(N\overline{\otimes}N^\circ, \omega\otimes\omega^\circ)} \qquad z \in N\overline{\otimes}N^\circ.$$

Since $i_{\omega\otimes\omega^\circ}$ is positive preserving and e^{-tL} is a positive element of the standard cone, we have that Φ_t is a normal state on $N\overline{\otimes}N^\circ$ and can thus be represented by a unique positive unit vector $\Omega_t \in L^2_+(N\overline{\otimes}N^\circ, \omega\otimes\omega^\circ)$ as

$$\Phi_t(z) = \Big(\Omega_t|\pi_{\rm co}(z)\Omega_t\Big)_{L^2(N\overline{\otimes}N^\circ,\omega\otimes\omega^\circ)} \qquad z \in N\overline{\otimes}N^\circ.$$

By the strong continuity of the Markovian semigroup e^{-tL} on $L^2(N, \omega)$, we then have

$$\lim_{t\downarrow 0}\Big(\Omega_t|\pi_{\rm co}(z)\Omega_t\Big)_{L^2(N,\omega)\otimes L^2(N,\omega)} = (\xi_\omega|\pi_{\rm id}(z)\xi_\omega) = 1 \qquad z \in N\otimes_{\max} N^\circ.$$

This proves that the trivial representation $\pi_{\rm id}$ of $N\otimes_{\max} N^\circ$, given by $\pi_{\rm id}(z) := I_{L^2(N,\omega)}$ for all $z \in N\otimes_{\max} N^\circ$, is weakly contained in the coarse representation $\pi_{\rm co}$ and thus N is amenable by a characterization of amenability due to Popa [90].

This approach by correspondences to relate spectral properties of Dirichlet forms to approximation properties of von Neumann algebras allows to treat also the *relative* case in which one deals with embeddings of subfactors $B \subset N$ on one side and with the a subexponential spectral growth rate of Dirichlet form r*elative to the subalgebra* B, on the other side. In these situations the essential spectrum of Dirichlet forms is not empty. (See [36]).

7.5 *Haagerup Approximation Property and Discrete Spectrum of Dirichlet Forms*

The free group of two generators $\mathbb{F}_2$ is non amenable but in 1979 Haagerup proved in [68] that its word-length function l is conditionally negative definite and proper. This allowed him to prove that the group von Neumann algebra $L(\mathbb{F}_2)$ and the group C*-algebra of $\mathbb{F}_2$ have the Grothendieck Metric Approximation Property. Moreover, the above properties of the length function of free groups also determine the following properties. This specific case opened the study of the following class of groups, larger than the class of amenable ones.

Definition 7.6 A countable, discrete group Γ is said to have the *Haagerup Approximation Property* if there exists a sequence of positive definite functions in $c_0(\Gamma)$, uniformly convergent on compact subsets, to the constant function 1 (see for example [17]). This property is equivalent to the existence of a *proper, conditionally negative definite function on* Γ.

Clearly all amenable groups have the Property (H). A series of contribution [18–21, 70], allowed to isolate the following property of von Neumann algebras that for group algebras $L(\Gamma)$ of discrete groups is equivalent to the Haagerup Approximation Property of Γ.

Definition 7.7 A von Neumann algebra with standard form $(M, L^2(M), L^2_+(M), J)$ is said to have the *Haagerup Approximation Property* (HAP) if there exists a sequence of completely positive contractions $T_k : L^2(M) \to L^2(M)$ strongly converging to the identity operator

$$\lim_k \|\xi - T_k\xi\|_{L^2(M)} = 0 \qquad \xi \in L^2(M).$$

Here is the announced connection between (HAP) and spectral properties.

Theorem 7.8 *([19]) Let N be a σ-finite von Neumann algebra. Then the following properties are equivalent*

(i) M has the Property (HAP)
(ii) there exists on $L^2(M)$ a completely Dirichlet form $(\mathcal{E}, \mathcal{F})$ with respect to some faithful, normal state $\omega \in M_{*+}$, having discrete spectrum.

Remark 7.9

(i) Property (H) may be formulated in a number of slightly different, equivalent ways also for not necessarily σ-finite von Neumann algebras too in such a way that the above spectral characterization remains anyway true.
(ii) The construction of Dirichlet forms out of negative definite functions on groups and the above characterization of the Haagerup Approximation Property, indicate that Dirichlet forms for arbitrary von Neumann algebras play a role parallel to the one played by the continuous, negative definite functions on groups (see discussion in [19]).

7.6 Property (Γ) and Poincaré Inequality for Elementary Dirichlet Forms

Another instance of the interactions among structural properties of a von Neumann algebra M and spectral properties of Dirichlet forms on $L^2(M)$ may be shown reformulating the Murray–von Neumann Property (Γ).

By an *elementary* completely Dirichlet form on a finite von Neumann algebra (M, τ), endowed with a normal, trace state, we mean one of type

$$\mathcal{E}_F[\xi] := \sum_{x\in F} \|x\xi - \xi x\|^2_{L^2(M,\tau)} \qquad \xi \in L^2(M, \tau)$$

for some finite, symmetric set $F = F^* \subset M$. The unit, cyclic vector $\xi_\tau \in L^2(M, \tau)$ representing the trace is central so that $\mathcal{E}_F[\xi_\tau] = 0$ and $\lambda_0 = 0$ is an eigenvalue for all elementary Dirichlet forms. Elementary Dirichlet forms are everywhere defined and thus bounded.

Definition 7.10 ([38]) A finite von Neumann algebra endowed with its normal, tracial state (M, τ), has the Property (Γ) if for any $\varepsilon > 0$ and any finite set $F \subset M$

there exists a unitary $u \in M$ with $\tau(u) = 0$ such that $\|(ux - xu)\xi_\tau\|_2 < \varepsilon$ for all $x \in F$.

This property was the first invariant introduced by Murray and von Neumann [84] to show the existence of non hyperfinite II_1 factors. For example, the group von Neumann algebra $L(S_\infty)$ of the countable discrete group S_∞ of finite permutations of a countable set and the Clifford von Neumann algebra of a separable Hilbert space are both isomorphic to the hyperfinite II_1 factor R, which fullfill the Property (Γ). This latter cannot be isomorphic to the group algebra $L(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$ with $n \geq 2$ generators which is a II_1 factor but does not have the Property (Γ) (and in fact it is not hyperfinite).

It is well known [90] that the absence of the Property (Γ) for a II_1 factor with separable predual, is a *rigidity property* equivalent to the existence of a spectral gap for suitable self-adjoint, finite, convex combinations of inner automorphisms, as unitary operators on $L^2(M, \tau)$.

We now show how the Property (Γ) can be also naturally interpreted in terms of a spectral property of elementary Dirichlet forms.

Theorem 7.11 *([37]) A finite von Neumann algebra endowed with its normal, tracial state* (M, τ)*, has the Property* (Γ) *if and only if for any elementary completely Dirichlet form*

$$\mathcal{E}_F[\xi] = \sum_{x \in F} \|x\xi - \xi x\|^2_{L^2(M,\tau)} \qquad \xi \in L^2(M, \tau),$$

associated to a finite set $F = F^* \subset M$*, the eigenvalue* $\lambda_0 := 0$ *is not isolated in the spectrum.*

Otherwise stated, (M, τ)*, does not have the Property* (Γ) *if and only if there exists an elementary Dirichlet form* $\mathcal{E}_F$ *such that the eigenvalue* $\lambda_0 = 0$ *is isolated (spectral gap) or, equivalently,* $\mathcal{E}_F$ *satisfies, for some* $c_F > 0$*, a Poincaré inequality*

$$c_F \cdot \|\xi - (\xi_\tau|\xi)\xi_\tau\|_2^2 \leq \mathcal{E}_F[\xi] \qquad \xi \in L^2(M, \tau).$$

Proof If J denotes the symmetry on $L^2(M, \tau)$ which extends the involution of M, then for $u, x \in M$, setting $\xi := x\xi_\tau \in M\xi_\tau$, we have $(ux - xu)\xi_\tau = u\xi - Ju^*x^*\xi_\tau = u\xi - Ju^*Jx\xi_\tau = u\xi - \xi u$. Since $\xi_\tau \in L^2(M, \tau)$ is cyclic, i.e. $M\xi_\tau$ is dense in $L^2(M, \tau)$, if (M, τ) does not have the Property Γ there exists $\varepsilon > 0$ and an elementary Dirichlet form $\mathcal{E}_F$ such that for all unitaries $u \in M$ we have

$$\varepsilon \cdot \|u\xi_\tau - \tau(u)\xi_\tau\|_2^2 \leq \mathcal{E}_F[u\xi_\tau].$$

For any $y = y^* \in M$ such that $\|y\|_M \leq 1/\sqrt{2}$, consider the unitaries $u_\pm := y \pm i\sqrt{1_M - y^2}$ so that $y = (u_+ + u_-)/2$. Since $\sqrt{1 - y^2} = \phi(y)$ with $\phi(s) := \sqrt{1 - s^2}$ and $|\phi'(s)| \leq 1$ for $|s| \leq 1/\sqrt{2}$, it follows by the Markovianity of the Dirichlet form that $\mathcal{E}_F[\sqrt{1 - y^2}\xi_\tau] \leq \mathcal{E}_F[y\xi_\tau]$. Since $\mathcal{E}_F$ is J-real we then have

$$\mathcal{E}_F[u_\pm\xi_\tau] = \mathcal{E}_F[(y \pm i\sqrt{1-y^2})\xi_\tau] = \mathcal{E}_F[y\xi_\tau] + \mathcal{E}_F[\sqrt{1-y^2}\xi_\tau] \le 2\mathcal{E}_F[y\xi_\tau]$$

and

$$\begin{aligned}&\varepsilon \cdot (\|u_+\xi_\tau - \tau(u_+)\xi_\tau\|_2^2 + \|u_-\xi_\tau - \tau(u_-)\xi_\tau\|_2^2)\\ &\le \mathcal{E}_F[u_+\xi_\tau] + \mathcal{E}_F[u_-\xi_\tau] \le 4\mathcal{E}_F[y\xi_\tau].\end{aligned}$$

Thus for all $y \in M$ we have

$$\begin{aligned}\|y\xi_\tau - \tau(y)\xi_\tau\|_2^2 &= \|u_+\xi_\tau - \tau(u_+)\xi_\tau + u_-\xi_\tau - \tau(u_-)\xi_\tau\|_2^2\\ &\le 2\big(\|u_+\xi_\tau - \tau(u_+)\xi_\tau\|_2^2 + \|u_-\xi_\tau - \tau(u_-)\xi_\tau\|_2^2\big)\\ &\le 8\varepsilon^{-1} \cdot \mathcal{E}_F[y\xi_\tau].\end{aligned}$$

and, by the density of $M\xi_\tau$ in $L^2(M,\tau)$, a Poincaré inequality holds true with $c_F = \varepsilon/8$. □

By classical results of Connes [40], obtained along his classification of injective factors, one can relate the existence of spectral gap for an elementary Dirichlet form to fundamental properties of II_1 factors (M,τ) with separable predual: the following properties are equivalent

(i) the subgroup $\mathrm{Inn}(M)$ of inner automorphisms is closed in $\mathrm{Aut}(M)$ (M is called a *full factor*)
(ii) the C*-algebra $C^*(M, M')$ generated by M and its commutant M', acting standardly on $L^2(M,\tau)$, contains the ideal of compact operators
(iii) there exists an elementary Dirichlet form $\mathcal{E}_F$ on $L^2(M,\tau)$ satisfying a Poincaré inequality.

7.7 Property (T)

Groups having the Kazdhan property (T) show, in many instances, a very rigid character. By their definition, all continuous, negative definite functions on them are bounded (see [17]) and they can be characterized by any of the following properties: (i) whenever a sequence of continuous, positive definite functions converges to 1 uniformly on compact subsets, then it converges uniformly, (ii) if a representation contains the trivial representation weakly, then it contains it strongly, (iii) every continuous, isometric action on an affine Hilbert space has a fixed point.

In von Neumann algebra theory, the Property (T) of a group Γ with infinite conjugacy classes, were first considered by Connes to show that the factor $L(\Gamma)$ has a countable fundamental group. The same author characterized countable, discrete groups Γ having the Property (T) through a specific property of $L(\Gamma)$. Later Connes and Jones [43] identified a property (T) for general von Neumann algebras in strong analogy with one of the above characterizations for the groups case. They key point

was the replacement of the notion of group representation by that of correspondence for general von Neumann algebras:
M has the property (T) if all correspondences sufficiently close to the standard one must contain it.

In Sect. 10 below, we will describe a recent result by Skalski and Viselter to a Dirichlet form characterization of the property (T) of von Neumann algebras of locally compact quantum groups.

8 KMS-Symmetric Semigroups on C*-Algebras

We have seen that the extension of the theory of Dirichlet forms introduced by Albeverio and Hoegh-Krohn and developed and applied by Sauvageot [93, 97] and by Davies [46], Davies and Rothaus [49, 50] and by Davies and Lindsay [47, 48], can be applied to several fields in which the relevant algebra of observables, to retain a physical language, is no more commutative. This theory concerns, however, C*-algebras or von Neumann algebras endowed with a well behaved trace functional. To have a theory suitable to be applied to other fields one has to face the problem to give a meaning to Markovianity of Dirichlet forms with respect non tracial states. For example,

(i) equilibria in Quantum Statistical Mechanics or Quantum Field Theory are described by states obeying the Kubo–Martin–Schwinger condition which are not trace at finite temperature
(ii) in Noncommutative Geometry the algebra generated by the "coordinate functions" of a noncommutative space may have a natural relevant state which is not a trace, as it is the case of the Haar state of several Compact Quantum Groups.

In this section we describe this extension of the theory of Dirichlet forms which deals with Markovianity with respect to KMS states on C*-algebras and with any normal, faithful states on von Neumann algebras. In the next sections we shall have occasion to describe applications were this generalized theory is due.

8.1 KMS-States on C-Algebras*

Let A be a C*-algebra and let $\{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous automorphism group on it, often interpreted as a dynamical system.

Definition 8.1 (**KMS-states**) ([69, 72]) Let $\alpha := \{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous group of automorphisms of a C*-algebra A and $\beta \in \mathbb{R}$. A state ω is said to be a (α, β)-KMS state if it is α-invariant and if the following *KMS-condition* holds true:

$$\omega(a\alpha_{i\beta}(b)) = \omega(ba)$$

for all a, b in a norm dense, α-invariant *-algebra of analytic element for α. If M is a von Neumann algebra and $\alpha := \{\alpha_t : t \in \mathbb{R}\}$ is a w^*-continuous group of automorphisms, a state ω is said to be a (α, β)-KMS state if ω is α-invariant, normal and the KMS-condition above holds true for all a, b in a $\sigma(M, M_*)$-dense, α-invariant *-subalgebra of A_α. KMS states corresponding to $\beta = 0$ are just the traces over M.

Notice that any faithful normal state ω on a von Neumann algebra M is a $(\sigma^\omega, -1)$-KMS state, i.e. a KMS state for the modular group σ^ω at inverse temperature $\beta = -1$. In this case, in fact, the KMS condition coincides with modular condition.

Definition 8.2 (**KMS-symmetric Markovian semigroups on C*-algebras**) ([25, 28]) Let $\alpha := \{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous group of automorphisms of a C*-algebra A and ω be a fixed (α, β)-KMS state, for some $\beta \in \mathbb{R}$.

A bounded map $R : A \to A$ is said to be (α, β)-*KMS symmetric with respect to* ω if

$$\omega\Big(bR(a)\Big) = \omega\Big(\alpha_{-\frac{i\beta}{2}}(a)R(\alpha_{+\frac{i\beta}{2}}(b))\Big) \tag{8.1}$$

for all a, b in a norm dense, α-invariant *-algebra of analytic elements for α.

A strongly continuous semigroup $\{R_t : t \geq 0\}$ on A is said to be (α, β)-*KMS symmetric with respect to* ω if R_t is (α, β)-KMS symmetric with respect to ω for all $t \geq 0$.

In the von Neumann algebra case, ω is assumed to be normal, maps and semigroups to be point-weak*-continuous and the subalgebra B to be weak*-dense.

Let $\alpha := \{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous group of automorphisms of a C*-algebra A and ω be a fixed (α, β)-KMS state, for some $\beta \in \mathbb{R}$. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ be the corresponding GNS-representation, $\widehat{\omega}$ the normal extension of ω to the von Neumann algebra $M := \pi_\omega(A)''$ and $\widehat{\alpha} := \{\widehat{\alpha}_t : t \in \mathbb{R}\}$ be the induced weak*-continuous group of automorphisms of M. Comparing the KMS condition for $\widehat{\omega}$ with respect to $\widehat{\alpha}$ to its modular condition, one readily observe that the modular group of $\widehat{\omega}$ is given by

$$\sigma_t^{\widehat{\omega}} = \widehat{\alpha}_{-\beta t} \qquad t \in \mathbb{R}.$$

The following is a key consequence of the (α, β)-KMS-symmetry of a map.

Lemma 8.3 *([25]) A map $R : A \to A$ which is (α, β)-KMS symmetric with respect to ω, leaves globally invariant the kernel* $\ker(\pi_\omega)$ *of the GNS-representation of ω.*

This result allows to study KMS symmetric maps and semigroups on the von Neumann algebra associated to the GNS representation of the KMS state.

Theorem 8.4 *(von Neumann algebra extension of KMS-symmetric semigroups) ([25]) Let $\{R_t : t \geq 0\}$ be a strongly continuous semigroup on A, (α, β)-KMS symmetric with respect to ω. Then there exists a unique point-weak*-continuous semigroup $\{S_t : t \geq 0\}$ on M determined by*

$$S_t(\pi_\omega(a)) = \pi_\omega(R_t(a)), \quad a \in A, \qquad t \geq 0. \tag{8.2}$$

This extension is $\widehat{\omega}$-modular symmetric in the sense that

$$\hat{\omega}\Big(S_t(x)\sigma^{\widehat{\omega}}_{-\frac{i}{2}}(y)\Big) = \widehat{\omega}\Big(\sigma^{\widehat{\omega}}_{+\frac{i}{2}}(x)S_t(y)\Big) \qquad t \geq 0, \tag{8.3}$$

for all x, y in a weak-dense, $\sigma^{\widehat{\omega}}$-invariant *-algebra of analytic elements $\sigma^{\widehat{\omega}}$. Moreover, if $\{R_t : t \geq 0\}$ is positive, completely positive, Markovian or completely Markovian, then $\{S_t : t \geq 0\}$ shares the same properties.*

As a consequence, a Dirichlet form on $L^2(A, \omega)$ is determined by the semigroup on A

Corollary 8.5 *Let $(L, D(L))$ be the generator of the semigroup $\{R_t : t \geq 0\}$ on A. Then the Dirichlet form on $L^2(A, \omega)$ associated to the strongly continuous extension of the $\widehat{\omega}$-modular symmetric semigroup $\{S_t : t \geq 0\}$ on M, satisfies the relation*

$$\mathcal{E}[i_\omega(\pi_\omega(a))] = (i_\omega(\pi_\omega(a))|i_\omega(\pi_\omega(La)))_{L^2(A,\tau)} \qquad a \in D(L).$$

By this result one may study properties of the semigroup R on the C*-algebra through the associated Dirichlet form $\mathcal{E}$ on A. Notice that by definition we have the coincidence of the spaces $L^2(A, \omega) = L^2(M, \widehat{\omega})$.

This result suggests also that one can approach the construction of Markovian semigroups (α, β)-symmetric with respect to a (α, β)-KMS state ω on a C*-algebra A, through the construction of Dirichlet forms on $L^2(A, \omega)$. The advantage being that working with quadratic forms instead that linear operators often allows to relax domain constrains to prove closability. To finalize this approach, however, once obtained from the Dirichlet form on $L^2(A, \tau)$ the Markovian semigroup on the von Neumann algebra $L^\infty(A, \tau)$, one has to face the problem to show that the C*-algebra A is left invariant and that on it the semigroup is not only w*-continuous but in fact strongly continuous. This last problem may be solved case by case as for the Ornstein–Uhlenbeck semigroup in [29] for example. We notice, however, that even in classical potential theory, on Riemmanian manifolds the construction of the heat semigroup on the algebra of continuous functions requires a certain amount of substantial potential analysis [45].

9 Application to Quantum Statistical Mechanics

After the proof, in the early nineties of the last century, by Stroock and Zegarlinski, of the equivalence between the Dobrushin-Shlosman mixing condition and the uniform logarithmic Sobolev inequalities for classical spin systems with continuous spin space, efforts were directed to obtain for quantum spin systems similar results, within the framework of the studies of the convergence to equilibrium. See for example [77–83].

In this section we describe just one of these constructions of Markovian semigroups by Dirichlet forms for KMS states of quantum spin systems, provided by Park and his school [7, 8, 86–88].

9.1 Heisenberg Quantum Spin Systems

Let us describe briefly, the quantum spin system and its dynamics. The observables at sites of the lattice $\mathbb{Z}^d$ are elements of the algebra $M_2(\mathbb{C})$ and the C*-algebra of observables confined in the finite region $X \subset \mathbb{Z}^d$ is

$$A_X := \bigotimes_{x\in X} M_2(\mathbb{C}).$$

If $\mathcal{L}$ denotes the net of all finite subsets of $\mathbb{Z}^d$, directed by inclusion, the system $\{A_X : X \in \mathcal{L}\}$ is in a natural way a net of C*-algebras so that the algebra of all *local observables* given by

$$A_0 := \bigcup_{X\in\mathcal{L}} A_X,$$

is naturally normed and its norm completion is a C*-algebra A (quasi-local observables).

Interactions among particles in finite regions is represented by a family of self-adjoint elements $\Phi := \{\Phi_X : X \in \mathcal{L}\} \subset A_X$. Using the Pauli's matrices $\sigma_j^x \in M_2(\mathbb{C})$, $j = 0, 1, 2, 3$, at the sites $x \in \mathbb{Z}^d$, in the *isotropic, translation invariant, Heisenberg model*, for example, in addition to an external potential represented by a one-body interaction of strength $h \in \mathbb{R}$

$$\Phi(\{x\}) := h\sigma_3^x,$$

particles interact only by a two-body potential so that $\Phi(X) = 0$ whenever $|X| \geq 3$ and

$$\Phi(\{x, y\}) := J(x-y)\sum_{i=1}^{3}\sigma_i^x\sigma_i^y \qquad x \neq y$$

for a parameter $\lambda > 0$ and a function $J : \mathbb{Z}^d \to \mathbb{R}$ describing the strength of the interaction between pairs of particles, under the assumption

$$\sum_{x\in\mathbb{Z}^d} e^{\lambda|x|}|J(x)| < +\infty.$$

For any fixed $Y \in \mathcal{L}$, the derivation

$$A_0 \ni a \mapsto i\big[\Phi_Y, a\big] \in A$$

extends to a bounded derivation on A and generates a uniformly continuous group of automorphisms of A, representing the time evolution of the observables, interacting with those particles confined in Y. To take into account simultaneously, the mutual influences among particles in different regions, one verifies that the superposition

$$D(\delta) := A_0 \qquad \delta(a) := \sum_{Y\in\mathcal{L}} i\big[\Phi_Y, a\big], \tag{9.1}$$

is a closable derivation on A whose closure is the generator of a strongly continuous group $\alpha^\Phi := \{\alpha_t^\Phi : t \in \mathbb{R}\}$ of automorphisms of A.

9.2 Markovian Approach to Equilibrium

The above interactions provide the existence of (α^Φ, β)-KMS-states ω at any inverse temperature $\beta > 0$. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ be the GNS representations of the state ω, M the von Neumann algebra $\pi_\omega(A)''$ and $(M, L^2(A,\omega), L^2_+(A,\omega), J_\omega)$ the corresponding standard form. In the following we will use the smearing function $f_0(t) := 1/\cosh(2\pi t)$.

Theorem 9.1 *Suppose that ω is a (α^Φ, β)-KMS-state at an inverse temperature satisfying*

$$\beta < \frac{\lambda}{\|\Phi\|_\lambda} \tag{9.2}$$

where

$$\|\Phi\|_\lambda := \sup_{x\in\mathbb{Z}^d} \sum_{x\in X\in\mathcal{L}} |X|4^{|X|}e^{\lambda D(X)}\|\Phi_X\|_{A_X} \tag{9.3}$$

is finite under the exponential decay assumption on the strength J. Then the quadratic forms associated to the self-adjoint elements $a_j^x := \pi_\omega(\sigma_j^x)$

$$\mathcal{E}_{x,j}[\xi] = \int_{\mathbb{R}} \|(\sigma_{t-i/4}(a_j^x) - j(\sigma_{t-i/4}(a_j^x)))\xi\|^2\, f_0(t)dt \tag{9.4}$$

are bounded completely Dirichlet forms and

$$\mathcal{E} : L^2(A,\omega) \to [0,+\infty] \qquad \mathcal{E}[\xi] := \sum_{x\in\mathbb{Z}^d}\sum_{j=0}^{3} \mathcal{E}_{x,j}[\xi] \tag{9.5}$$

is a completely Dirichlet form on $L^2(A,\omega)$.

Concerning the proof, a first observation is that the $\mathcal{E}_{x,j}$ are bounded Dirichlet form as uniformly convergent continuous superposition of elementary completely Dirichlet forms. The quadratic form $\mathcal{E}$ is Markovian and closed as pointwise monotone limit of bounded completely Dirichlet forms. The only point that is left to be shown is the fact that it is densely defined, i.e. $\mathcal{E}$ is finite on a dense domain in $L^2(A,\omega)$. This is a consequence of the fact that under the current hypotheses on the strength on the interaction, the dynamics has *finite speed propagation* in the sense that, denoting by $d(x, X)$ distance of the site $x \in \mathbb{Z}^d$ from the region $X \in \mathcal{L}$, we have

$$\|[\alpha_t^\Phi(a), b]\| \leq 2\|a\| \cdot \|b\| \cdot |X| \cdot e^{-\left(\lambda d(x,X)-2|t|\|\Phi\|_\lambda\right)} \qquad a \in A_{\{x\}}, b \in A_X, t \in \mathbb{R}. \tag{9.6}$$

Concerning the ergodic behaviour of the semigroups associated to the Dirichlet forms above, the following result shows how these properties are deeply connected to the other fundamental properties of the KMS-state.

Corollary 9.2 *([87, Theorem 2.1]) Within the assumption of Theorem 3.3, the following properties are equivalent:*

(i) *ω is an extremal (α^Φ, β)-KMS-state;*
(ii) *ω is a factor state in the sense that the von Neumann algebra $M := \pi_\omega(A)''$ is a factor;*
(iii) *the Markovian semigroup $\{T_t : t \geq 0\}$ is ergodic in the sense that the subspace of $L^2(A, \omega)$ where it acts as the identity operator is reduced to the scalar multiples of the cyclic vector $\xi_\omega \in L^2(A, \omega)$ representing the KMS state ω.*

Extremality, i.e. the impossibility to decompose a KMS state as convex, nontrivial superposition of other KMS states (see [16]), is the mathematical translation of the notion of *pure phase* in Statistical Mechanics.

Ergodicity of Markovian semigroups were considered by Gross [63] to prove the uniqueness of the ground state of physical Hamiltonians in Quantum Field Theory. Later, Albeverio and Hoegh-Krohn [3] established a Frobenious type theory for positivity preserving maps on von Neumann algebras with trace and in [26] a Perron type theory was provided for positivity preserving maps on the standard form of general von Neumann algebras.

10 Applications to Quantum Probability

As pointed out in Introduction, one of the major achievement of the theory of commutative potential theory is the correspondence between regular Dirichlet forms and symmetric Markov–Hunt processes on metrizable spaces. In noncommutative potential theory we do not dispose at moment of a complete theory but we have at least a clear connection between Dirichlet forms of translation invariant, symmetric, Markovain semigroups and Lévy's Quantum Stochastic Processes on Compact Quantum Groups.

10.1 *Compact Quantum Groups d'après S. L. Woronowicz [108]*

In the following $m_A : A \otimes_{\text{alg}} A \to A$ will denote the extension of the product operation of A.

Let us recall that a compact quantum group $\mathbb{G} := (A, \Delta)$ is a unital C*-algebra $A =: C(\mathbb{G})$ together with a

(i) *coproduct* $\Delta : A \to A \otimes_{\max} A$, a unital, *-homomorphism which is
(ii) *coassociative* $(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta$ and satisfies
(iii) *cancellation rules*: the closed linear span of $(1 \otimes A)\Delta(A)$ and $(A \otimes 1)\Delta(A)$ is $A \otimes A$.

An example of the above structure arise from a compact group G by dualization of its structure. In fact, setting $A := C(G)$ we have $A \otimes_{\max} A = C(G \times G)$ and a coproduct defined by

$$(\Delta f)(s, t) := f(st) \qquad f \in C(G), \quad s, t \in G.$$

A *unitary co-representation* of $\mathbb{G}$ is a unitary matrix $U = [u_{jk}] \in M_n(A)$ such that

$$\Delta u_{jk} = \sum_{i=1}^{n} u_{ji} \otimes u_{ik} \qquad j, k = 1, \ldots, n.$$

Denote by $\widehat{\mathbb{G}}$ the set of all equivalence classes of unitary co-representations of $\mathbb{G}$. If a family of inequivalent irreducible, unitary co-representations $\{U^s : s \in \mathbb{G}\}$ of $\mathbb{G}$ exhausts all of $\widehat{\mathbb{G}}$, then the algebra of *polynomials*, defined by the linear span of the coefficients of all unitary co-representations

$$\text{Pol}(\mathbb{G}) := \text{linear span}\{u_{jk} \in A : [u_{jk}] \in \widehat{\mathbb{G}}\}$$

is a *Hopf*-algebra*, dense in A, with *counit* ϵ and *antipode* S determined by

$$\epsilon(u_{jk}) := \delta_{jk}, \quad S(u_{jk}) := u_{kj}^* \qquad [u_{jk}] \in \widehat{\mathbb{G}}$$

and satisfying the rules

$$(\epsilon \otimes id)\Delta(a) = a, \qquad (id \otimes \epsilon)\Delta(a) = a,$$
$$m_A(S \otimes id)\Delta(a) = \epsilon(a)1_A = m_A(id \otimes S)\Delta(a).$$

The C*-algebra $C(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ is commutative if and only if it is of the form $C(G)$ for some compact group G. In this case counit and antipode are defined by

$$\epsilon(f) := f(e), \qquad S(f)(s) := f(s^{-1}) \qquad s \in G,$$

where $e \in G$ is the group unit.

Combining the tensor product with the coproduct, one may introduce new operations that in the case of compact group reduce to the well known classical ones.

The *convolution* $\xi * \xi' \in A^*$ of *functionals* $\xi, \xi' \in A^*$ is defined by

$$\xi * \xi' := (\xi \otimes \xi') \circ \Delta$$

and the *convolution* $\xi * a \in A^*$ of a *functional* $\xi \in A^*$ and an element $a \in A$ is defined by

$$\xi * a = (id \otimes \xi)(\Delta a) \qquad a * \xi := (\xi \otimes id)(\Delta a).$$

By a fundamental result of Woronowicz, on a compact quantum group $\mathbb{G}$ there exists a unique (Haar) state $h \in A^*_+$ which is both *left and right translation invariant* in the sense that

$$a * h = h * a = h(a)1_A \qquad a \in A = C(\mathbb{G}).$$

In the commutative case the Haar state reduces to the integral with respect to the Haar probability measure. However, in general, the Haar state is not even a trace but it is a $(\sigma, -1)$-KMS state with respect to a suitable automorphisms group $\sigma_t \in \mathrm{Aut}(A)$, $t \in \mathbb{R}$,

$$h(ab) = h(\sigma_{-i}(b)a) \qquad a, b \in \mathrm{Pol}(\mathbb{G}).$$

By a result of Woronowicz, the antipode $S : \mathrm{Pol}(\mathbb{G}) \to C(\mathbb{G})$ is a densely defined, closable operator on A and its closure $\bar{S}$ admits the polar decomposition

$$\bar{S} = R \circ \tau_{i/2}$$

where

(i) $\tau_{i/2}$ generates a *-automorphisms group $\tau := \{\tau_t : t \in \mathbb{R}\}$ of the C*-algebra A and
(ii) R is a linear, anti-multiplicative, norm preserving involution on A commuting with τ, called *unitary antipode*.

10.1.1 $SU_q(2)$ Compact Quantum Group

The compact quantum group $SU_q(2)$ with $q \in (0, 1]$, is defined as the universal C*-algebra generated by the coefficients of a matrix

$$U = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$$

subject to the relations ensuring unitarity: $UU^* = U^*U = I$. Then one may check that, in terms of the generators α, γ, all the other relevant objects are determined by

(i) comultiplication: $\Delta(\alpha) := \alpha \otimes \alpha + \gamma \otimes \gamma, \quad \Delta(\gamma) := \gamma \otimes \alpha + \alpha^* \otimes \gamma$
(ii) counit: $\epsilon(\alpha) := 1, \quad \epsilon(\gamma) := 0$
(iii) antipode: $S(\alpha) := \alpha^*, \quad S(\gamma) := -q\gamma, \quad S(u_{j,k}) := (-q)^{(j-k)} u_{-k,j}$ for $[u_{jk}] \in \widehat{\mathbb{G}}$
(iv) Haar state: $h(u_{jk}) := 0$ for $[u_{jk}] \in \widehat{\mathbb{G}}$
(v) automorphisms group: $\sigma_z(u_{jk}) := q^{2iz(j+k)} u_{jk}$ for $[u_{jk}] \in \widehat{\mathbb{G}}$ and $z \in \mathbb{C}$
(vi) unitary antipode: $R(u_{jk}) := q^{k-j} u^*_{jk}$ for $[u_{jk}] \in \widehat{\mathbb{G}}$.

When $q = 1$ one recovers the classical compact group $SU(2)$.

10.1.2 Countable Discrete Groups as CQGs

Let Γ be a countable discrete group and $\lambda : \Gamma \to B(l^2(\Gamma))$ its left regular representation

$$\lambda_s : l^2(\Gamma) \to l^2(\Gamma) \qquad \lambda_s(\delta_t) := \delta_{st} \qquad s, t \in \Gamma.$$

The reduced C*-algebra $C^*_r(\Gamma) \subset B(l^2(\Gamma))$ is the smallest C*-algebra containing all the unitary operators λ_s for $s \in \Gamma$. If instead of the regular representation one uses the direct sum of all cyclic unitary representation of Γ, the resulting algebra is called the universal C*-algebra. It is isomorphic to the regular one if and only if Γ is amenable.

A compact quantum group structure on $C^*_r(\Gamma)$ is obtained extending to a *-homomorphism Δ from $C^*_r(\Gamma)$ to $C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ the map defined by $\Delta(\lambda_s) := \lambda_s \otimes \lambda_s$ for $s \in \Gamma$. The linear span of the unitaries λ_s for $s \in \Gamma$ is a dense *-Hopf algebra on which counit and antipode are defined as $\epsilon(\lambda_s) = 1$ and $S(\lambda_s) := \lambda_{s^{-1}}$ for $s \in \Gamma$. The compact quantum group $C^*_r(\Gamma)$ is *cocommutative* in the sense that the comultiplication Δ is invariant under the flip of the left and right factors of $C^*_r(\Gamma) \otimes C^*_r(\Gamma)$. A theorem of Woronowicz ensures that any cocommutative compact quantum group $C(\mathbb{G})$ is essentially the C*-algebra of a countable discrete group in the sense that there exists a countable discrete group Γ and *-homomorphisms $C^*(\Gamma) \to C(\mathbb{G}) \to C^*_r(\Gamma)$. The CQG $C^*_r(\Gamma)$ is of Kac type and the Haar state coincides with the trace determined by $\tau(\delta_s) = 0$ for $s \neq e$ and $\tau(\delta_e) = 1$.

10.2 Lévy Processes on Compact Quantum Groups [30]

The Lévy processes on compact groups are among the most investigated stochastic processes in classical probability. We briefly describe in this section a class of quantum stochastic processes, in the sense of [1] (see also [62]), on compact quantum groups that generalize the classical Lévy processes.

Let (P, Φ) be a von Neumann algebra with a faithful, normal state, also called a *noncommutative probability space*.

(i) A *random variable* on $\mathbb{G}$ is a *-algebra homomorphism $j : \mathrm{Pol}(\mathbb{G}) \to P$

(ii) the *distribution* of the random variable is the state $\phi_j := \Phi \circ j$ on Pol($\mathbb{G}$)
(iii) the *convolution* $j_1 * j_2$ of the random variable $j_1, j_2 : \mathrm{Pol}(\mathbb{G}) \to P$ is the random variable

$$j_1 * j_2 := m_P \circ (j_1 \otimes j_2) \circ \Delta$$

where m_P is the product in P.

A Quantum Stochastic Process [1] is a family of random variables $\{j_{s,t} : 0 \le s \le t\}$ satisfying

(i) $j_{tt} = \epsilon 1_P$ for all $0 \le t$
(ii) *increment property*: $j_{rs} * j_{st} = j_{rt}$ for all $0 \le r \le s \le t$
(iii) *weak continuity*: $j_{tt} \to j_{ss}$ in distribution as $t \to s$ decreasing.

Definition 10.1 (Quantum Lévy Processes) A Lévy process on a CQG $\mathbb{G}$ is a quantum stochastic process on the Hopf-algebra Pol($\mathbb{G}$) such that it has

(i) *independent increments* in the sense that for disjoint intervals $(s_k, t_k]$, $k = 1, \ldots, n$

$$\Phi(j_{s_1 t_1}(a_1) \cdots j_{s_n t_1}(a_n)) = \Phi(j_{s_1 t_1}(a_1)) \cdots \Phi(j_{s_n t_1}(a_n))$$

(ii) *stationary increments* in the sense that the distribution $\phi_{st} = \Phi \circ j_{st}$ depends only on $t - s$.

Theorem 10.2 *Under a suitable probabilistic notion of equivalence of quantum stochastic processes, equivalence classes of Lévy processes* $\{j_{s,t} : 0 \le s \le t\}$ *on a compact quantum group* $\mathbb{G}$ *are in one-to-one correspondence with those Markovian semigroups* $\{S_t : 0 < t\}$ *on the* C^**-algebra* $C(\mathbb{G})$ *which are translation invariant in the sense that*

$$\Delta \circ S_t = (id \otimes S_t) \circ \Delta \qquad t > 0.$$

To illustrate the main steps of the correspondence, notice first that the distributions of the process $\phi_t := \Phi \circ j_{0t}$ form a continuous convolution semigroup on Pol($\mathbb{G}$)

$$\phi_0 = \epsilon, \qquad \phi_s * \phi_t = \phi_{s+t}, \qquad \lim_{t \to 0^+} \phi_t(a) = \epsilon(a) \qquad a \in \mathrm{Pol}(\mathbb{G})$$

and that the *generating functional* of the process is then defined as

$$G : D(G) \to \mathbb{C} \qquad G(a) := \frac{d}{dt}\phi_t(a)\Big|_{t=0}$$

on a dense domain $D(G) \subseteq \mathrm{Pol}(\mathbb{G})$. From it one can reconstruct the distribution of the process as a convolution exponential

$$\phi_t = \exp_*(tG) := \epsilon + \sum_{n=1}^{\infty} \frac{t^n}{n!} G^{*n} \qquad t > 0,$$

a semigroup on Pol($\mathbb{G}$) by

$$S_t a = \phi_t * a \qquad a \in \mathrm{Pol}(\mathbb{G}), \quad t > 0$$

and its formal generator $L : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ as $L(a) := G * a$. Then one checks that the semigroup extends to a strongly continuous, translation invariant Markovian semigropup on the C*-algebra $C(\mathbb{G})$ and that its generator is the closure of L. Moreover, the distribution and the generating functional can be written directly in terms of the semigroup and its generator

$$\phi_t = \epsilon \circ S_t, \qquad G = \epsilon \circ L.$$

The KMS-symmetry of the semigroup of a Lévy process can checked using the generating functional as follows

Theorem 10.3 *Let* $\{S_t : t > 0\}$ *be the Markovian semigroup of a Lévy process* $\{j_{s,t} : 0 \le s \le t\}$ *on compact quantum group* $\mathbb{G}$*. The following properties are then equivalent*

(i) the semigroup is $(\sigma^h, -1)$*-KMS symmetric*
(ii) the generating functional is invariant under the action of the unitary antipode

$$G = G \circ R$$

on the Hopf-algebra* Pol($\mathbb{G}$).

If the above conditions are verified then one can proceed to construct the Dirichlet form associated to the Lévy process. The differential structure of these Dirichlet forms and the generating functional can be described in terms of the Schürmann cocycle (see [30]) but we do not pursue it here.

Rather, we prefer to conclude this section with examples of Dirichlet forms on a class of compact quantum groups whose spectrum has been completely determined with application to the approximation properties treated in a previous section.

10.2.1 Free Orthogonal Quantum Groups

The universal C*-algebra $C_u(O_N^+)$ of the free orthogonal quantum group of Wang O_N^+, $N \ge 2$, is generated by a set of N^2 self-adjoint elements $\{v_{jk} : j, k = 1, \ldots, N\}$ subject to the relations which ensure that the matrix $[v_{jk}]$ is unitary

$$\sum_{l=1}^N v_{lj} v_{lk} = \delta_{jk} = \sum_{l=1}^N v_{jl} v_{kl}$$

and where a coproduct is defined as $\Delta v_{jk} := \sum_{l=1}^N v_{lj} \otimes v_{lk}$. The Haar state is a trace which is faithful on the Hopf algebra but not on $C_u(O_N^+)$ so that the Lévy semigroup is considered on the *reduced* C*-algebra $C_r(O_N^+)$, defined by the GNS representation of the Haar state. The set of equivalence classes of irreducible, unitary co-representations is indexed by $\mathbb{N}$. Denoting by $\{U_s : s \in \mathbb{N}\}$ the Chebyshev polynomial on the interval $[-N, N]$ defined recursively as

$$U_0(x) = 1, \qquad U_1(x) := x, \qquad U_n(x) = xU_{n-1}(x) - U_{n-2} \qquad n \geq 2,$$

a generating functional is then defined by

$$G(u^n_{jk}) := \delta_{jk}\frac{U'_n(N)}{U_n(N)} \qquad j, k = 1, \dots, U_n(N), \quad n \in \mathbb{N}.$$

It can be proved that the associated Dirichlet form has *discrete spectrum* whose eigenvectors are the coefficients u^n_{jk} of the irreducible, unitary co-representations and such that the corresponding eigenvalues and multiplicities are

$$\lambda_n := \frac{U'_n(N)}{U_n(N)}, \qquad m_n := (U_n(N))^2.$$

By the results of a previous section, this implies that the von Neumann algebras $L^\infty(C^*_r(O_N^+), \tau)$ generated by the GNS representation of the Haar trace states, all have the Haagerup Property. In particular, however, since for $N = 2$ one has $\lambda_n = \frac{n(n+2)}{6}$ and $m_n = (n+1)^2$, it results that $L^\infty(C^*_r(O_2^+), \tau)$ is amenable. The amenability of the free orthogonal quantum groups have been proved for the first time by Brannan [14].

10.2.2 Property (T) of Locally Compact Quantum Groups and Boundedness of Dirichlet Forms

We conclude this exposition describing succinctly a recent result of Skalski and Viselter [102] connecting the Property (T) of the von Neumann algebra of a quantum group to the boundedness of translation invariant Dirichlet forms. Their framework is more general than the one treated in this section as they consider the *locally compact quantum groups* $\mathbb{G} = (M, \Delta, \varphi_H)$, in the von Neumann algebra setting, of Kustermans and Vaes [73].

The main difference with respect to the Woronowicz theory of compact quantum groups is that the Haar weight φ_H (in general no more a state) is, together with the coproduct operation Δ, part of the structure of a locally compact quantum group $\mathbb{G}$. This causes a lack of certain common, dense, natural domain for generators, generating functionals and quadratic forms so that a subtler analysis is required.

The von Neumann algebra M (resp. its standard space $L^2(M)$) is often indicated as $L^\infty(\mathbb{G})$ (resp. $L^2(\mathbb{G})$) or as $L^\infty(\mathbb{G}, \varphi_H)$ (resp. $L^2(\mathbb{G}, \varphi_H)$) to emphasize the reference to the chosen Haar weight.

From the point of view of potential theory, the unboundedness of the Plancherel weight necessitates of the extension of the theory of Dirichlet forms with respect to weights on von Neumann algebras, developed by Goldstein and Lindsay in [61] (and amended in [102, Appendix]). We do not describe the details of this theory here but we just notice that in case the Plancherel weight φ_H is a trace we may use the theory illustrated in Sect. 4.

The following result, obtained in [102, Theorem 4.6], characterizes the Property (T) of von Neumann algebras of separable locally compact quantum groups (defined in [57] for discrete quantum groups and for general locally compact ones in [51]) in terms of a spectral property of the completely Dirichlet forms.

Theorem 10.4 *Let $\mathbb{G}$ be a locally compact quantum group such that $L^2(\mathbb{G}, \varphi_H)$ is separable. Then the following properties are equivalent*

(i) the von Neumann algebra $L^\infty(\mathbb{G}, \varphi_H)$ has the property (T)
(ii) any translation invariant completely Dirichlet form on $L^2(\mathbb{G}, \varphi_H)$ is bounded.

As in the compact case, the translation invariance of the Dirichlet form may be expressed as the invariance of the associated generating functional with respect to the unitary antipode.

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Contact Interactions and Gamma Convergence

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Abstract We study contact interactions, a generalization of Albeverio's point interactions. There are two types of contact interactions, weak and strong; the last type occurs only in a three particle system. Strong contact leads to systems that have an infinite number of bound states with eigenvalues that decrease with a scaling law. We prove that in both the strong and the weak contact cases the hamiltonians are strong resolvent limits of hamiltonians with potentials with support that vanishes with a given scaling law while the L^1 norm remains constant. In the weak contact case, the approximating hamiltonians must have a zero energy resonance. As applications we describe Bose-Einstein condensation in the low and high density regimes, the Fermi sea in solid state physics and the ground state of Nelson's polaron.

Keywords Contact interactions · Gamma convergence

1 Introduction

This is a contribution to the volume dedicated to Sergio Albeverio on the occasion of his 80th birthday. Sergio is a very close friend, an exceptional scientist and an extraordinary gentle person always ready and willing to help. This and his innate curiosity has resulted in a very large number of collaborations on an even larger number, if possible, of different aspect of Mathematical Physics and Analysis.

Sergio has given many very important contributions to science, as stressed in this volume. His work on Feynman integrals and on singular stochastic equation has opened new ways in these important fields. Among his impressive scientific production stands out his work on point interaction, initiated with Høegh-Krohn; its importance is testified among other by the two"voluminous" books, *Solvable Models in Quantum Mechanics* [2] (this book has become a "bible" in the field) and later

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Singular Perturbations of Differential Operators [3], a "must" for researchers in the field. Later Sergio and collaborators (mostly of the Russian school) obtained very interesting results on the Schrödinger equation with potentials that are singular on lower dimensional manifolds.

My contribution to this volume in honor of Sergio is on a subject, *contact interactions*, strictly related to point interactions. I pursue the research of Sergio by considering a wider class of operators defined by boundary conditions on a lower dimensional manifold; now the manifold may be three dimensional and we consider contact between two or three bodies.

Point interaction may be considered as contact with a zero-dimensional manifold. We consider two types of contact, strong and weak. In the case of contact in a three particle system the weak contact can be of a particle with the other two or simultaneous of the three particles. In the case of Bose-Einstein condensates they correspond respectively to the low and high density cases. We describe also shortly an application to atomic physics (Fermi sea) and to the Nelson problem (contact interaction of a particle with a quantized zero mass field).

2 Contact Interactions

Contact interactions in $\mathbb{R}^3$ are defined *formally* by imposing that the wave function in the domain of the hamiltonian satisfies the boundary conditions

$$\phi(X) = \frac{C_{i,j}}{|x_i - x_j|} + D_{i,j} \quad i \neq j$$

at the *coincidence manifold* Γ

$$\Gamma \equiv \cup_{i,j}\Gamma_{i,j} \qquad \Gamma_{i,j} \equiv x_i - x_j = 0, \quad i \neq j \quad x_i \in \mathbb{R}^3.$$

These conditions were later used by Skorniakov and Ter-Martirosian [35] in their analysis of three body scattering within the Faddaev formalism.

It is easy to see, integrating by parts twice, that *formally* $C \neq 0$ corresponds to an attractive potential $-C_{i,j}\delta(x_i - x_j)$, $C_{i,j} > 0$ and $D \neq 0$ corresponds to an attractive potential with radial derivative proportional to $-D_{i,j}\delta(x_i - x_j)$. Since if $C_{i,j} \neq 0$ the function is not in L^2 (and the δ is not a bona-fide potential), the definition is only formal. On the other hand it would suffice an extra factor $\frac{c}{\log(|x_i - x_j|)}$. This suggests that given $C_{i,j}$ there are infinitely many "solutions" and one must select "the right one". Even the right one *is not in the domain of the free hamiltonian*; *solution of the Schrödinger equation* is only meant in a weak sense, after averaging with a smooth function and integrating by parts. The equation holds therefore *in the sense of quadratic forms*. Quadratic forms techniques play an important role. From a mathematical point of view, the resulting operators are *self-adjoint extensions* of the symmetric operator $\widehat{H}_0$, the free hamiltonian restricted to functions with support away from Γ.

We remark that already in 1935 Bethe and Peirels [9], in order to model the very short range nuclear forces, introduced potentials "that vanish except at the origin". A formal force localized in a point was introduced earlier by Fermi.

A particular case was given a mathematical status as "point interactions" in the work of Sergio [1]. In $\mathbb{R}^3$ this interaction requires the presence of a zero energy resonance.

In Theoretical Physics the interest in the subject was renewed by recent advances in the theoretical formulation of low energy physics with very short range interactions and by the flourishing of research on ultra-cold atoms interacting through potentials of very short range [8].

In three dimensions we call *strong contact* the case $D_{i,j} = 0$ and *weak contact* the case $C_{i,j} = 0$. We shall prove that these two types of contact lead to *independent and complementary effects*. The effect is independent and complementary to that due a smooth interaction. In three dimensions two particles can have only weak contact.

In a three particle system one can have mutual weak contact of the three particles. We will see that in order to solve this case one must consider the (separate) strong contact of one particle with the other two and the result is infinitely many bound states. In two dimensions there is only one type of contact. To treat the case of mutual contact between three particles one must consider a stronger contact and this leads again to an infinity of bound states. In one dimension contact interactions in a system of three spin $\frac{1}{2}$ particles can lead to infinite number of bound states (the "Fermi sea").

The formalism we develop allows us to prove that strong contact interactions in $\mathbb{R}^3$ are limits *in strong resolvent sense* when $\epsilon \to 0$ of interactions through a two-body potential of class C^1 which scales (in the three-dimensional case) as

$$V_\epsilon(|x|) = \epsilon^{-3} V\left(\frac{|x|}{\epsilon}\right),$$

for strong contact. For weak contact the scaling is

$$V_\epsilon(|x|) = \epsilon^{-2} V\left(\frac{|x|}{\epsilon}\right),$$

and there must be a zero energy resonance. We will see that this requirement is of topological origin. This same ϵ^{-2} scaling occurs in two dimensions and there is no zero energy resonance. In one dimension the scaling is

$$V_\epsilon(|x|) = \epsilon^{-1} V\left(\frac{|x|}{\epsilon}\right).$$

We analyze first in $\mathbb{R}^3$ the case of three particles one of which is in strong contact with the other two. This analysis will be also useful in the case of mutual weak contact of three particles. Later we will treat the case of dimension two and dimension one.

Remark 1 Warning: We are using the common (but wrong) notation "particle" to indicate a wave function. A particle is an observable and therefore a density matrix (positive trace-class operator with trace one). We will come back later to this point; we remark here that a cubic local interaction term for a wave function is the strong form of a contact interaction between two densities.

3 Mathematical Formulation; The Krein Map

From a mathematical point of view the problem of zero range interaction was first analyzed by Pavlov [33] who investigated the self-adjoint extensions defined by the condition of finite value at the boundary $\Gamma_{i,j}$ (weak contact). For this case Shondin [34] followed a scheme for self-adjiont extension led out by Yu Shirokov. Later the problem was analyzed by Makarov [28], Makarov and Melezdik [29].

Contact interactions were analyzed in [25, 26] from the point of view of self-adjoint extensions. We analyze the same problem from point of view of quadratic forms [4, 22].

Consider first the (separate) strong contact between one particle and two identical particles. Since the particles are identical we consider only one pair, but we shall see the presence of a third particle is needed for the procedure we follow. This contact interaction is to be considered as limit of interactions of very short range; we will prove that the limit is in strong resolvent sense.

The quadratic form of the free hamiltonian is a strictly positive operator on the space of absolutely continuous functions and it takes value $+\infty$ on the complement. The delta distribution defines on the space of continuous functions a negative *bounded* quadratic form which not is weakly closed. To "extract" a self-adjoint operator from their sum we use an invertible map $\mathcal{K}$ that lifts the system to a space of more singular functions (so that by duality the potential term is less singular). We call $\mathcal{K}$ "Krein map" and $\mathcal{M}$ "Minlos space" the image space. We will explain later the genesis of these names. The map $\mathcal{K}$ *acts differently* on the free hamiltonian (an operator) and on the potential (a quadratic form). The map is "fractioning" and "mixing" in precise sense; this are the same properties which are commonly required in the study of *composite materials* in applied mathematics. They have the role of a *magnifying glass*.

Recall that we are considering a system of three particles one of which is in strong contact with the other two (that we assume to be identical so that the contact is with a density matrix). The map is induced by the operator $H_0^{-\frac{1}{2}}$, where H_0 is the free hamiltonian of the three particle system. We shall see that this procedure is similar to the one followed in [6, 18] for the construction of self-adjoint extensions of positive operators; this explains the name "Krein" for the map. But we stay on the side of quadratic forms [22].

We call "Minlos space" $\mathcal{M}$ the image space since the idea of using this space came while reading [25, 26]. The map is *mixing* (it entangles all coordinates) and *fractioning* (the target space contains more singular functions). The map is useful also in the study of weak contact of two particles since it helps separating the short distance and the long distance properties of the interaction (weak contact requires the presence of a zero energy resonance). Later we will study the weak contact in a three particle system.

In the new space $\mathcal{M}$ both the kinetic energy and the potential are symmetric operators. Both are unbounded, one below and one above. In $\mathcal{M}$ the kinetic energy is $\sqrt{H_0}$. The potential term is

$$-C(H_0)^{-\frac{1}{2}}\delta(x)(H_0)^{-\frac{1}{2}}.$$

Since the delta function commutes with H_0 (as easily seen taking Fourier transformation) this operator can also be written $-C\delta(x)H_0^{-2}$. The procedure we follow is therefore similar to that of [6, 18] (but on the side of quadratic forms [22]).

In $\mathcal{M}$, in position space the interaction term is $-C\frac{1}{|x|} + B$, where B is a positive operator with smooth kernel that vanishes on the diagonal. In the following we neglect the operator B. This is justified because we will prove that strong, weak contact and smooth interactions lead to *independent* and *complementary* effects. The proof makes use of the Konno-Kuroda formulation of the Birman-Schwinger formula for the perturbation of the resolvent.

In $\mathcal{M}$ kinetic and potential energy operators have opposites sign and the same singularity on functions supported in a neighborhood of the origin. Therefore their sum is a quasi-homogeneous operator [14] and if it is not positive it is represented in $\mathcal{M}$ by a continuous ordered family of self-adjoint operators.

It is proved in [14] that there is a first threshold C_1 (which depends on the masses and on the strength of the interaction) such that for $C < C_1$ the form is positive. Therefore it corresponds uniquely to a self-adjoint operator. There is a second threshold C_2 such that for $C_1 \geq C \geq C_2$ there is in $\mathcal{M}$ a one-parameter family of self-adjoint operators bounded below with one bound state. For $C \geq C_2$ there is a one-paramter family of self-adjoint operators unbounded below each with an infinite number of bound states that diverge linearly to $-\infty$.
We remark that in [25] the main tool in the proofs is a Mellin transform. The existence of infinitely many self-adjoint operators when $C \geq C_2$ is proved in [26] solving the equation $H = H^*$. This gives also the structure of the wave functions of the bond states in $\mathcal{M}$.

4 Going Back to "Physical" Space; Gamma Convergence

Coming back to "physical space", the continuous family of operators in $\mathcal{M}$ gives a continuous ordered family of quadratic forms bounded below but only *weakly closed* (both are consequences of the change in metric topology). Notice that returning to physical space *is not inversion of the Krein map* since the entire closed quadratic form is regarded as an operator.

Recall that the Krein map is *fractioning* because the image is a space of more singular functions and *mixing* because it is not diagonal in the channels of the kinetic energy. This suggests to make use of Gamma convergence [13], a variational method for the study of finely structured materials that was introduced more that sixty years ago by Buttazzo and De Giorgi.

Gamma convergence is a minimization procedure that selects an extremum out of a family of convex functionals. We consider the particular case of quadratic forms. Recall [13] that the Gamma limit of a sequence of *strictly convex weakly closed quadratic forms* F_n in a topological space Y is the *unique* weakly closed quadratic form F such that for any subsequence the following holds

$$\forall y \in Y,\ \ y_n \to y,\ \ F(y) = \liminf_{n\to\infty} F(y_n);$$
$$\forall x \in Y \exists \{x_n\} \to x\ \ F(x) \geq \limsup_{n\to\infty} F_n(x_n)$$

The first condition means that F provides an asymptotic common lower bound for the F_n; the second condition says that this lower bound is optimal.

The condition for the existence of the Gamma-limit for a sequence of strictly convex quadratic forms is that the sequence be contained in a compact set of a space Y; in the present case Y has the Frechet topology given by Sobolev semi-norms. Compactness of bounded sets is assured *by the absence of zero energy resonances* (as in the strong contact case), strict convexity is due to the fact that since the interaction is invariant under rotations we can restrict attention to s-waves. Therefore there is a minimizing (Palais-Smale) sequence.

We conclude that for strong contact there is a privileged quadratic form (the Gamma limit); being minimal and strongly convex, it is strongly closed [18] and represents a self-adjoint operator, the hamiltonian of our system. If $C_1 \leq C < C_2$ this operator has a bound state, if $C \geq C_2$ it has an Efimov sequence of bound states. The presence of the first few members of the series have been reported in low energy nuclear physics where one considers strong contact interactions (very short range potentials) between particles. .

The Efimov effect is also present in the case of mutual weak contact of three equal mass particles, e.g. in the Bose-Einstein condensate at high density. At lower density separate weak contact interactions prevail and one has only one bound state.

A similar effect (but with a different rate) is present for a system of three spin $\frac{1}{2}$ particles which satisfy the Pauli equation and move on a lattice with Y-shaped vertices (where the interaction takes place). In this case, in the Efimov effect the eigenvalues scale logarithmically. The Pauli principle lead to the *Fermi sea* (occupation of all bound states).

Remark 2 One may wonder what is the role of the other quadratic forms that are only weakly closed in physical space. They correspond to other boundary condition, e.g. due to a strong magnetic field at the boundary. So far we have used the free hamiltonian to define the Krein map but we can use e.g. the magnetic hamiltonian. The interaction takes place at the boundary and the infimum now is a different quadratic form, and therefore a new self-adjoint extension is promoted to "physical" hamiltonian.

5 Convergence of Approximations

We prove now that the hamiltonian we have described is the limit in strong resolvent sense of the hamiltonians with the approximate *negative* two-body potentials $V^{\epsilon}(|x)$ which scale as

$$V_{\epsilon}(|x|) = \epsilon^{-3} V\left(\frac{|x|}{\epsilon}\right).$$

Remark that Gamma convergence is stable under continuous perturbations. Since the potentials are negative and have constant L^1 norm, the ϵ-sequence of hamiltonians is a *decreasing sequence* of quadratic forms bounded below by the hamiltonian of contact interactions.

The hamiltonian of strong contact has no zero energy resonances and also the ϵ-dependent approximations have no zero energy resonances. Therefore the sequence and its limit belong to a compact subset for the topology induced by the Sobolev semi-norms. Any such sequence bounded below admits a convergent subsequence. The difference between the contact interaction and the approximate sequence converges to zero weakly. Since the hamiltonians are strictly decreasing the limit point is unique. Therefore when $\epsilon \to 0$ the approximate hamiltonians Gamma converge to the hamiltonian of strong contact.

Gamma convergence implies strong resolvent convergence [13] (but not quadratic form convergence), therefore the resolvents of the ϵ-dependent hamiltonians converge in strong convergence sense to the resolvent of the contact interaction. Strong resolvent convergence implies convergence of spectra and of the Wave operators.

We add a few remarks.

Remark 3 Notice that no rate of convergence is available. Notice also that gamma convergence *does not imply convergence of quadratic forms* and therefore even weak convergence of the hamiltonians. The operator and its domain are obtained by Gamma convergence, a variational technique. In this lies the strength of Gamma convergence.

Remark 4 The convergence we have proved is in the strong resolvent sense. The quadratic forms converge only in a sub-domain; in the complement the quadratic forms may diverge. Recall that the resolvent of a self-adjoint operator A is $\frac{1}{A-z}$ for z not in the spectrum of A. Therefore if $A_n \to A$ in strong resolvent it is only required that the quadratic form of A_n converges to the quadratic form of A *on a large enough domain*.

Remark 5 Contact interactions are entirely determined by the behavior of the wave functions at the boundary (compare with electrostatics; here the Krein map plays the role of the Weyl map between potential and charges in electrostatics). For this reason it natural to call the Minlos space *space of charges* [12]. There is a natural connection between strong contact interactions and the theory of Boundary Triples [10]; the difference with existing literature is that the boundary is of dimension 3 and it is *internal*.

6 Weak Contact

We consider now the case of weak contact. Since the weak potential has the same scaling properties under dilations as the kinetic energy there can be at most as many weak contacts as particles. The tightest configuration is the *simultaneous* weak contact of three particles.

We shall consider first the case of two particles. Now the boundary conditions require that functions in the domain take a finite value at the boundary. This corresponds to a boundary potential whose "gradient" has the singularity of a delta distribution.

In the study of weak contact we can proceed as in the case of strong contact and introduce the Minlos space. The Krein map in the case of two particles in weak contact is induced by the operator $H_0^{-\frac{1}{2}}$. This corresponds to a smoothing of the interaction *but there is no mixing*. In $\mathcal{M}$ the kinetic energy is represented by the operator $H_0^{\frac{1}{2}}$ and the potential has $\log(|x_i - x_j|)$ singularity at the coincidence manifold. The hamiltonian is covariant under dilation.

In physical space one has a weakly closed quadratic form bounded below and with a $\frac{1}{|x_i - x_j|}$ behavior at large distances. It is therefore strongly closed [19] and corresponds to a self-adjoint operator bounded below and with a zero energy resonance. Also in this case the weak contact hamiltonian is the limit in strong resolvent sense of the hamiltonian with the approximation potentials V^ϵ but a zero energy resonance must be *subtracted away* before on can use compactness to prove the existence of the limit. This explains why in the case of weak contact *the approximating potentials must have a zero energy resonance* (to be subtracted away).

Remark 6 The case of a weak interaction *in a two particle system* is discussed in [1] using methods of functional analysis in the case when one of the particles has infinite mass. This particle may be considered at a fixed point (point interaction). The presence of a zero energy resonance implies a singularity of the resolvent at zero momentum and this requires an accurate and difficult estimate of the zero energy limit in the B.K.S. formula for the difference of two resolvents [1]. Using the Krein map simplifies much the analysis since it acts differently on the kinetic and potential parts and allows therefore to treat separately the singularity at contact and the long-distance behavior. In [1], this analysis is presented for the weak contact interaction of a particle with a fixed point (a particle of infinite mass) but the same analysis can be done for the case of weak contact interaction of two particles in the reference frame of the barycenter.

7 Bose-Einstein Condensate; The Dilute Case

The Bose-Einstein condensate is a gas of identical bosons in weak contact. As well known the theory of the low density Bose-Einstein condensate requires the presence of a zero energy (Feshbach) resonance. We distinguish two cases: we call *low density* the case of separate weak contact of one of the particles with the other two and *high density* the case of mutual weak contact.

In the low density case there are two zero energy resonances and therefore at zero energy the resolvent is the inverse of a 2×2 matrix with zeroes on the diagonal. As a result if the interaction is strong enough, the hamiltonian has a negative eigenvalue; one has therefore a three-body bound state. This bound state is stable since the separation of one of the two particles that interact produces a system which has positive energy (and one resonance). Notice that there is no interaction between two of the particles. One can say that the interaction takes place between one wave function and the density of another particle.

Since the particles are identical, taking the scalar product with the conjugate of the wave function of the interacting particle and integrating by part the kinetic term, one obtains an energy form which is the sum of a free part and the integral over the product of the two densities. This quartic term has as coefficient the intensity of the Gross-Pitaevskii coupling.

We have already remarked that the world “particle” is misleading: wave function would be better. A particle is represented by a density matrix, and therefore the interaction is local and takes place between two densities.

The energy functional is obtained by taking the scalar product with a wave function and integrating by parts the kinetic term. Therefore the energy functional contains a term which is linear in the density and a quadratic term. The bound states is a critical point of this functional.

8 The High Density Case; The New Ground State

Consider now the high density case. The three wave functions are now in simultaneous weak contact. The interaction is represented, as before, taking the limit for $\epsilon \to 0$ in the hamiltonian

$$H_{int} = H_0 + \sum_{i \neq j \neq k} \frac{1}{\epsilon^2} V \left(\frac{|x_i - x_k|}{\epsilon} \right) .$$

In the Birman-Schwinger formula the contribution to the resolvent coming from the terms that contain only two of the three potentials is the same as in the case of separate interactions. The contribution represents weak contact interaction of two particles in presence of a non interacting particle; this provides a bound state. If the density is higher the contribution that corresponds to mutual weak contact becomes dominant.

In the contribution of the factors that depend on the product of all three potentials one can take away the factor ϵ^{-2} from one of the potentials and attribute a further factor ϵ^{-1} to the remaining two (the way we distribute the ϵ is an artifice devoid of physical meaning). This leads to consider separate *strong contact interaction* of a wave function with a pair. It also leads to an Efimov sequence of bound states. The lowest energy state (the ground state) is now the lowest Efimov state. If the particles are identical there is only one ground state Ω_s. Therefore the ground state of the high densiity Bose-Einstein gas is $\otimes_{k=1}^{N} \Omega_s^k$, where k is the index of distinct triplets in joint weak contact. *It is not related* to the ground state $\otimes_{k=1}^{N} \Omega_w^k$ of the diluted Bose-Einstein gas.

Also here the energy functional is obtained by taking the scalar product with a wave function and integrating by parts the kinetic term; it has still the form

$$D \int (\nabla\phi, \nabla\phi) - C \int |\phi^4(x)| \mathrm{d}^3x \,, \qquad D, C > 0 \,,$$

but the ratio $\frac{C}{D}$ is larger than in the previous case and the system has an infinite number of (Efimov) bound states (it should be possible to use Morse theory to see the difference in the number of critical point as a function of the coefficient of the potential term).

In both cases if the "particles" are identical the ground state of the gas is symmetric; if there are $3N$ wave functions the ground state is the tensor product of the ground states of the three wave functions system, but this ground state is different in the two cases. In fact one can "unify" the procedure and consider simultaneously the limits $\epsilon \to 0$ and $N \to \infty$ (notice that for one degree of freedom the form domain of the number operator is compact in Fock space). We will not analyze further here this problem.

Remark 7 In [8] one considers a system of N "particles" with an interaction of support $\frac{1}{N}$ (we consider only the case $\beta = 1$ in their notation). The fact that the range of the interaction depends on the number of particles is unpleasant. Setting in the interaction term $\frac{1}{N} = \epsilon$ the system represents N triples of particles that in the limit $\epsilon \to 0$ are *in weak contact*. One can take $\epsilon \to 0$ and $N \to \infty$ *independently*; this justifies the use of the de Finetti formalism and the reference to singular potentials.. The limit $\epsilon \to 0$ exists and to first order in ϵ is the ground state of triplets of wave functions in weak contact. We already remarked that a particle (an observable), while usually described in Quantum Mechanics by a wave function, is rather a density matrix.

Remark 8 The semiclassical limit. It is easy to verify that weak contact for (gaussian) coherent states implies Coulomb interaction between the barycenters. The free Schrödinger hamiltonian in $\mathbb{R}^3$ corresponds in the coherent states formalisms to the free classical hamiltonian. Therefore a quantum system of three "particles" in mutual weak contact corresponds to the newtonian three-body problem and the Efimov bound states correspond to the periodic solutions of the classical three-body problem. The same holds in $\mathbb{R}^2$ for the planar three-body problem.

9 Strong and Weak Contact Are Independent and Complementary

In three dimensions for $N \geq 3$ contact interactions and weak-contact interactions contribute *separately* and *independently* to the spectral properties and to the boundary conditions at the contact manifold. Both are independent from the contribution of regular potentials. We will only outline the proof.

For an unified presentation (which includes also the proof that the addition of a regular potential does not change the picture) it is convenient to use a symmetric presentation due to Kato and Konno-Kuroda [20] (who generalize previous work by Krein and Birman) for hamiltonians that can be written in the form

$$H = H_0 + H_{int} \qquad H_{int} = B^* A$$

where $B, \ A$ are densely defined closed operators with $D(A) \cap D(B) \subset D(H_0)$ and such that, for every z in the resolvent set of H_0, the operator $A \frac{1}{H_0+z} B^*$ has a bounded extension, denoted by $Q(z)$. Since we consider the case of attractive forces, and therefore negative potentials, it is convenient to denote by $-V_k(|y|)$ the two body potentials. The particle's coordinates are $x_k \in \mathbb{R}^3$, $k = 1, 2, 3$.

We take the interaction potential to be of class C^1 and set

$$V^\epsilon(X) = \sum_{i \neq j} [V_1^\epsilon(|x_i - x_j|) + V_2^\epsilon(|x_i - x_j|) + V_3^\epsilon(|x_i - x_j|)]$$

where V_1 and V_2 are negative and V_3 is a regular potential. For each pair of indices i, j we define $V_1^\epsilon(|y|)) = \frac{1}{\epsilon^3} V_1(\frac{|y|}{\epsilon})$ and $V_2^\epsilon(|y|) = \frac{1}{\epsilon^2} V_2(\frac{|y|}{\epsilon})$. We leave V_3 unscaled.

We define $B^\epsilon = A^\epsilon = \sqrt{-V^\epsilon}$. For $\epsilon > 0$ using Krein's resolvent formula one can give explicitly the operator B^ϵ as *convergent power series* of products of the free resolvent $R_0(z)$, $Re(z) > 0$ and the square roots of the sum of potentials V_k^ϵ, $k = 1, 2, 3$.

One has then for the resolvent $R(z) \equiv \frac{1}{H+z}$ the following form [20]

$$R(z) - R_0(z) = [R_0(z) B^\epsilon][1 - Q^\epsilon(z)]^{-1}[B^\epsilon R_0(z)] \qquad z > 0$$

with

$$R_0(z) = \frac{1}{H_0 + z}\,, \qquad Q^\epsilon(z) = B^\epsilon \frac{1}{H_0 + z} B^\epsilon\,.$$

We approximate the zero range hamiltonian with the one parameter family of hamiltonians

$$H_\epsilon = H_0 + + \sum_{m,n} V^\epsilon(|x_n - x_m|)\,, \quad n \neq m,\ x_m \in R^3\,.$$

The potential is the sum of three terms

$$V^\epsilon(|y|) = \sum_{i=1}^{3} V_i^\epsilon, \quad V_1^\epsilon(|y|) = \frac{1}{\epsilon^3} V_1\left(\frac{|y|}{\epsilon}\right), V_2^\epsilon(|y|) = \frac{1}{\epsilon^2} V_2\left(\frac{|y|}{\epsilon}\right),$$

(we omit the index m, n). All potentials are of class C^1. The potential V_3 is unscaled.

Define

$$U^\epsilon(|y|) = V_2^\epsilon + V_3\,.$$

If $\epsilon > 0$ the Born series converges and the resolvent can be cast in the Konno-Kuroda form [20], where the operator B is given as (convergent) power series of convolutions of the potential U^ϵ and V_1^ϵ with the resolvent of H_0. In general

$$\sqrt{V_1^\epsilon(|y|) + U^\epsilon(|y|)} \neq \sqrt{V_1^\epsilon(|y|)} + \sqrt{U^\epsilon(|y|)}$$

and in the Konno-Kuroda formula for the resolvent of the operator H_ϵ one loses separation between the two potentials V_1^ϵ and U^ϵ. Notice however that, if V_1^ϵ and U^ϵ are of class C^1, the L^1 norm of U^ϵ vanishes as $\epsilon \to 0$ uniformly on the support of V_1^ϵ. By the Cauchy inequality one has

$$\lim_{\epsilon \to 0} \|\sqrt{V_1^\epsilon(y)}.\sqrt{U^\epsilon(y)}\|^1 = 0\,.$$

Therefore, if the limit exists the strong and weak contact interactions act independently.

Remark 9 One can give a similar proof that regular (Rollnik class) potentials give contributions that are independent and complementary to those due to contact contributions, weak or strong.

10 The Two-Dimensional Case

In two dimensions there is only one type of contact interaction. It is described by a distributional potential $\delta(|x_i - x_j|),\ x_j \in \mathbb{R}^2$ at the coincidence manifold. It is the limit of the interaction through two-body potentials that scale as

$$V^{\epsilon}(|y|) = \frac{1}{\epsilon^2} V\left(\frac{|y|}{\epsilon}\right).$$

Consider the contact interaction of a particle with two identical particles. The Krein map and Minlos space are defined in the same way and again in $\mathcal{M}$ the free hamiltonian H_0 is represented by $H_0^{\frac{1}{2}}$ and the potentials differ from $-\frac{C}{|x_i - x_j|}, \quad C > 0$. If the interaction is strong enough in physical space there is a bound state. Since there are no zero energy resonances the mapping properties of the Wave operator in physical space are $L^p \to L^q$ for $1 < p \leq q < \infty$. This result has been obtained also for regular potentials in [16]; indeed the result can be obtained easily without using the Krein map.

More interesting is the case of *simultaneous* pairwise contact interaction of three bosons. It is represented by the potential $-C \sum \delta(|x_i - x_k), \quad C > 0$. Again in $\mathcal{M}$ if C is large enough (depending on the mass of the particles) there is a one parameter family of self-adjoint extensions each with a sequence of bound states with eigenvalues that diverge linearly. In physical space Gamma convergence provides a self-adjoint hamiltonian with an Efimov sequence of bound states. Since there is only one type of contact interaction, in the Bose-Einstein gas there is no difference between low and high density (and interaction has no zero energy resonances). The three particle system is the limit for $\epsilon \to 0$ of a system of three particles that interact through potentials that scale as $V^{\epsilon}(|x|) = \frac{1}{\epsilon^2} V(\frac{|x|}{\epsilon})$. The Bose-Einstein gas is the union of *triples of particles* which satisfy the cubic non linear equation; since there are no zero energy resonances the equation is cubic but is not of Gross-Pitaevskii type.

11 The Fermi Sea

Consider now the motion of the conduction electrons in a crystal. Conduction electrons are spin $\frac{1}{2}$ "particles"; the eigenfunctions satisfy the Pauli equation, a linear equation with hamiltonian $H_P = i\sigma \cdot \nabla + mI$, where $\sigma_k, \quad k = 1, 2, 3$, are the Pauli matrices. Their motion is constrained due to the joint action of the three neighboring nuclei. As a result the motion is restricted to a small neighborhood of a graph with Y shaped vertices; this is verified by pictures obtained with an electron microscope [5]. The restriction to a neighborhood of the vertex is due to the combined action of the three nuclei, the restriction to a neighborhood of the edges is due to two atoms. Estimates are in [37]. The potential at the vertex can be approximated *formally* by a delta function (strong contact in one dimension). This approximation does not change the spectrum since strong contact and interaction through a regular potential lead to complementary and independent effects.

Since the equation is linear homogeneous one can use as coordinates the difference of the coordinates of the particles. This replaces the equation with an equation in which each of the three spinors has a strong contact with the other two.

Notice that the fact that electrons have spin $\frac{1}{2}$ is important since electrons satisfy the Fermi-Dirac statistics and interaction is not possible for identical spin orientation. Since spinors transform under rotations according to the $SU(2)$ group, to describe a pair of spinors it is sometimes convenient to use the Dirac formulation of the Pauli equation; in this formulation the spinors have four components and the rotation group is implemented by a unitary transformation. The Pauli equation becomes then the Dirac equation. Since the observables are represented by bilinear forms we prefer to make use of the Pauli equation.

The interaction is attractive (from the point of view of the nucleus we consider) because after the interaction the electron is forced to move closer to the nucleus. We are in the setting described previously but the Pauli hamiltonian H_P is not positive and its square root cannot be defined (and its inverse used to define the Krein map).

Recall that our procedure leaves invariant the free part of the hamiltonian and modifies the "interaction potential" in such a way to obtain a self-adjoint operator for the sum. We can define as free hamiltonian the positive part of the Pauli hamiltonian, i.e. the Salpeter hamiltonian $H_S = \sqrt{H_P^2}$ (positive square root). We therefore define the Krein map using the operator H_S. The Krein map is again mixing and fractioning.

In $\mathcal{M}$ the kinetic energy of the three particle system is a (pseudo-)differential operator of order $\frac{1}{2}$. The image of the delta potential is the convolution of V with the resolvent of H_S; in position space this term differs from $-C\sqrt{\frac{1}{|x|}}$, $x \in R$, by a bounded positive operator. In $\mathcal{M}$ the hamiltonian is an almost homogeneous operator: the degree of the differential operator is $\frac{1}{2}$ and it is equal to the degree of singularity of the negative potential at the origin. Therefore [14, 23] there are constants $C_1 < C_2$ such that for $C \geq C_1$ in the sector of total angular momentum zero of $\mathcal{M}$ there is a one-parameter family of self-adjoint operators (since the potential is invariant under rotations and permutation invariant, only this sector is affected). For $C \geq C_2$ each member of the family has an infinite number of (negative) bound states that are asymptotically proportional to $-\sqrt{n}$. The constants C_1, C_2 depend on the mass of the particles and on the strength of the interaction. If the masses of the two particles in strong contact are zero one has $C_2 = 0$. C_2 is an increasing function of these masses and a decreasing function of the coupling constant. We assume that $C \geq C_2$. In our case this is due to the fact that the electron has a small mass in units in which the delta potential represents, in the semiclassical limit, Coulomb interaction of the electrons with the nuclei. Therefore in $\mathcal{M}$ there is a one parameter family of self-adjoint operators unbounded below and each with an infinite number of bound states. The relation between the Pauli hamiltonian and the Salpeter hamiltonian ($|H_P| = H_S$) indicates that the nth (negative) eigenvalue scales in $\mathcal{M}$ as $\sqrt{n}$.

Returning to physical space one has an ordered family of weakly closed forms bounded below. Gamma convergence selects the infimum. This form can be closed [18] and defines a self-adjoint operator bounded below; if $C \geq C_2$ it has an Efimov spectrum. The eigenvalues scale as $\frac{1}{\log n}$, they have the asymptotic behavior $\frac{c_n}{|x|} \frac{1}{\log^2(n|x|)}$ and are therefore very extended along the edges. Gamma convergence implies strong resolvent convergence of the approximating hamiltonians with potentials that scale when $\epsilon \to 0$ as $V^\epsilon(|y|) = \frac{1}{\epsilon^2} V(\frac{|y|}{\epsilon})$.

Recall that electrons are identical spin $\frac{1}{2}$ particles and satisfy the Pauli exclusion principle (no more than two electrons can be in the same bound state). *The occupied states are the Fermi sea.* The energies of these bound state converge to zero as $n \to \infty$ with a $\frac{1}{\log n}$ law. Since the spectrum accumulates at zero, a very small electric field is enough to "extract" the electrons (superconductivity). Due to the law of decay of the energies these states "remain in phase" under time evolution. For very large values of n the electrons have practically no binding energy and the have a "Dirac spectrum". Since the energy is then the absolute value of the momentum, the wave functions of these states remain in phase under translation along the edges.

We have so far considered only one vertex of the graph and we have taken the edges to be infinitely extended. Notice that while the interaction takes place at the boundary, the eigenfunctions are extended along the edges. The periodic structure of the crystal identifies two edges and requires a smooth connection along the edge. For this is again important that for most of the electrons in the Fermi sea the energy has practically the same value of the linear momentum and eigenfunctions of different eigenstates remain in phase under translations along the edges.

For a periodic lattice we must take into account that spinors transform under rotations according to the $SU(2)$ group and in presence of a magnetic field there may be a spin-orbit coupling. We must also consider that the electrons are Fermi particles and their wave function is anti-symmetric, but the phase may change in a complete loop around the border of a cell. This is the common explanation for the presence of an index.

Recall now that the Fermi sea is the collection of occupied states. This is a discrete set, but since the eigenvalues decrease with a $\frac{c}{\log n}$ law, if the crystal is large enough a large number of states is occupied and "almost all" of them have negligible binding energy. The Fermi sea (or Fermi band, i.e. an interval of energy with macroscopic population) can be very densely populated and can appear as a continuum. If it is added to the spectrum of the crystal of the nuclei the bands may overlap and *the gaps in the spectrum may become closed.* The electrons at the surface of the Fermi sea have practically no binding energy. This justifies the "Dirac-like" behavior of their spectra.

Remark 10 If there is a relevant amount of impurities (or if crystal is random) more levels are added, therefore there are more states that can be occupied (a Hilbert hotel). If the density of impurities is sufficiently high the "highest" occupied state has a finite negative energy. Therefore a small electric field is not sufficient to "extract an electron" and the conductivity of the sample decreases sharply. At a semiclassical level this is described by a negative potential that slows down or blocks diffusion.

In presence of a smooth magnetic field one has still a Fermi sea but the wave functions are different. Diamagnetic inequalities imply that one has only a smooth modification of the minimal form. If the field is very intense and concentrated on the boundary one has a new hamiltonian as Gamma limit. We have seen that on coherent states contact interaction is Coulomb interaction.

If the magnetic field is smooth at this scale the motion of electrons *on the surface of the Fermi sea* is seen as classical motion of particles which satisfy the laws of *classical* electrodynamics [32]. At the semiclassical level in presence of very strong electromagnetic fields the Fermi surface can have a non smooth structure and the description of dynamics may require a refined analysis [32]. Also in one dimension the semiclassical limit of the dynamics on the surface of the Fermi sea is the classical motion on the surface of the Fermi sea studied in detail by Novikov and Maltsev [30, 32] (see also [38]).

12 The Nelson Polaron

The Nelson polaron is the ground state of the system of one particle interacting with a second-quantized zero-mass field. Second quantization can be thought as Weyl quantization for a system with an infinite number of particles. Lebesgue measure is substituted by a measure on function space (Gauss measure in the Bose case). Very roughly speaking in second quantization a wave function f is substituted with a scalar field $\Psi(f) = a(f) + a^*(\bar{f})$ where $a(f)$ (resp. $a^*(\bar{f})$) destroys (resp. creates) a particle with wave function $f \in L^2(\mathbb{R}^3)$. Both terms are linear in f.

In the Bose case the field satisfies the (non relativistic) commutation relations $[\Psi(\bar{f}), \Psi(g)] = (f, g)$. One defines the *Fock representation* by postulating the existence of a vector Ω (the "vacuum") such that $a(f)\Omega = 0$ for all f in the Hilbert space. *Fock space* is the space generated by repeated action of the $a^*(f)$ on Ω (this justifies the name "creation operators")

A problem in the quantum theory of interacting quantum fields with coupling contact g is to find an irreducible representation (not necessary Fock) in which the interaction is realized by operator-valued distribution. We shall use the formalism of second quantization and denote by $a(k)$ (resp. $a^*(k)$) the annichilation (resp. creation) of a zero mass particle "of momentum k" (we omit the more precise definition).

In the following we shall consider the strong contact interaction of a particle of mass m with a non relativistic field of zero mass particles in the second quantization formulation for the field. This system is called *polaronic* and the ground state is the *polaron* [31] . One must pay attention to the fact that for zero mass particles there are infinitely *inequivalent* representations of the canonical commutation relations. A vector of finite energy in the Hilbert space may contain an infinity of zero mass particles with smaller and smaller momentum (this is known as *infrared problem*).

We denote by $\hat{H}$ the limit hamiltonian. It describes the contact interaction of the massive particle with the two mass zero particles. The ground state of the system is called *polaron*. To find the structure of the polaron we will "partially dequantize" the field by choosing properly the state *of two of the zero mass particles* (and therefore the representation of the canonical commutation relations since the zero mass particles are identical).

Let $\Phi(x)$ be the ground state of the system (a particle of mass m at the point x in separate weak contact with two zero mass particles). To find the structure of the ground state of the entire system we fiber the second quantization space of the zero mass particles *choosing as parameter the position of the particle of mass m*. We choose the representation by defining annihilation operators

$$A_x(y) = a(y) - \Phi(x)$$

For each value of x the (distribution valued) operators $A_x(y)$ satisfies the same canonical commutation relations as the operators $a(y)$ but the two representations are *inequivalent*. Different values of the position of the particle of mass m correspond to a different "infrared behaviors" of the mass zero field. If one writes the Hamiltonian as a function of the field $A(y)$ one obtains

$$H = \hat{H} + \int \omega(p) A_x^*(p) A_x(p) \mathrm{d}p\,,$$

where $\hat{H}$ is the hamiltonian that describes the contact interaction of the massive particle with two identical zero mass particles. In the Theoretical Physics literature this operation goes under the name of "completing the square" and the particle of positive mass is now "dressed" with the a particles. To minimize this, one has to choose for every x the vacuum and therefore the Fock representation for $A_x(y)$. Therefore, the a particles are described in an x-dependent representation defined by

$$a_x(y) = A(y) + \Phi(x)\,,$$

where the $A(p)$ is in the Fock representation.
There is no coupling. The ground state of the system has a cloud of mass zero a-particles. The cloud *depends on the coordinate of the heavy particle* [24, 31]. This is known as *the infrared problem.*

Remark 11 We remark that the use of perturbation theory in this context leads to serious problems. One can approximate the interaction by using the two-body potential $V^\epsilon = \frac{1}{\epsilon^3} V(\frac{|x_i - x|}{\epsilon})$ where $V \in C^1$:

$$H^\epsilon = H_0 + \int V^\epsilon(x - y_1)\Psi(y_1)\mathrm{d}y_1 + \int V^\epsilon(x - y_2)\Psi(y_2)\mathrm{d}y_2,$$
$$H_0 = -\frac{1}{2m}\Delta_x + \int \omega(p) a^*(p) a(p) \mathrm{d}p,$$

where $\omega(p) = |p|^2$ and the $a(k)$ satisfy the canonical commutation relations.

Again the interaction is linear, one can select the representation in which the ground state is a product state. But the representation of the canonical commutation relations chosen depends on ϵ and convergence must be understood in a very weak sense.

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On the Hamiltonian for Three Bosons with Point Interactions

Rodolfo Figari and Alessandro Teta

Abstract We briefly summarize the most relevant steps in the search of rigorous results about the properties of quantum systems made of three bosons interacting with zero-range forces. We also describe recent attempts to solve the unboundedness problem of point-interaction Hamiltonians for a three-boson system, keeping unaltered the spectrum structure at low energies.

Keywords Point interactions · Three-body problem

1 Introduction

In the unbounded scientific production of Sergio Albeverio the unboundedness of the zero-range Hamiltonians for three bosons and the Efimov effect play a very special role. First of all, as he pointed out in [2], because of the connection of the unboundedness problem with the existence of a non-trivial self-interacting relativistic quantum field theory, which was one of the main interest of his early scientific career. On the other hand, the peculiar structure of the spectrum at low negative energies of such Hamiltonians (suggesting the existence of the so called Efimov trimers) was for him a challenge to analize rigorously the peculiar discrete scaling of the eigenvalues of zero-range multi-particle Hamiltonians. The physics and mathematical-physics literature on the three-boson quantum system and the Efimov effect is nowadays so extensive that we cannot claim that the reader will find in this contribution a thorough summary of the subject and a comprehensive list of references. We will try first to describe the different attempts made to solve the unboundedness problem in a rigorous way and we will mention some clever suggestions about the solution of the

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problem which were never followed up with final results. On the basis of those suggestions, we tried recently to define zero-range Hamiltonians for the system of three bosons either breaking the rotational symmetry or adding three-particle contributions. We outline here methodology and results of this work-in-progress, examining in particular the effect of the three-particle interaction, suggested in the past by Minlos, Faddeev, Albeverio, Høegh-Krohn and Wu, on the spectral structure of the Hamiltonians.

Regarding attempts relying on rotational symmetry breaking, see Basti et al. [9].

We want first to give an outline of how the quantum three-body problem appeared in the physical literature.

2 The Thomas Paper

In 1935, on G.E. Uhlenbeck's suggestion, L. H. Thomas investigated the interaction between a neutron and a proton in order to analize the structure of Tritium nucleus [40]. He made the assumption of negligible interaction between the two neutrons and examined how short the neutron-proton interaction could be. He proved that the energy of the system was not bounded below for shorter and shorter interaction range. His conclusions are clearly stated in the abstract of the paper: "... We conclude that: either two neutrons repel one another by an amount not negligible compared with the attraction between a neutron and a proton; or that the wave function cannot be symmetrical in their positions; or else that the interaction between a neutron and a proton is not confined within a relative distance very small compared with 10^{-13} cm".

Notice that few years later the connection between spin and statistics was finally clarified. It then became clear that the wave function of the two neutrons could not be symmetrical under the exchange of their positions. Nevertheless, Thomas result indicated that zero-range interactions, while perfectly defined for a two-particle system, allowing a finite energy for the ground state, seem to give Hamiltonians unbounded from below in the three-boson case. Since then, this effect of "falling to the center" of the quantum three-body system with zero-range interactions is known as Thomas effect (or Thomas collapse).

In the paper, Thomas studied, in the center of mass reference frame, the eigenvalue problem for the free Hamiltonian outside a small region around the origin, where the three particles occupy the same position. He found that for negative energy values it was possible to exhibit a square integrable solution showing singularities on the planes where the two neutron positions coincide. The solution was successively used as a test function to show that the energy of the ground state of the system was unbounded from below if the range of the interaction (a sum of two-body potentials or a singular boundary condition) was made shorter and shorter.

In particular, let $\mathbf{x}_1, \mathbf{x}_2$ the position of the neutrons and $\mathbf{x}_3$ the position of the proton. In the center of mass reference frame let $\mathbf{s}_i = \mathbf{x}_i - \mathbf{x}_3$ for $i = 1, 2$ be the relative coordinates of the neutrons with respect to the proton. The free Hamiltonian eigenvalue equation then reads

$$-\left(\frac{\hbar}{4\pi^2 m}\right)(\Delta_{\mathbf{s}_1}+\Delta_{\mathbf{s}_2}+\nabla_{\mathbf{s}_1}\cdot\nabla_{\mathbf{s}_2})\Psi(\mathbf{s}_1,\mathbf{s}_2)+\mu^2\Psi(\mathbf{s}_1,\mathbf{s}_2)=0 \quad \mu>0. \tag{1}$$

Thomas found that a singular solution of (1) is given by

$$\Psi(\mathbf{s}_1,\mathbf{s}_2)=\frac{1}{s^2}K_0(\eta s)\left[\frac{\frac{\pi}{2}-\arctan\xi_1}{\xi_1(1+\xi_1^2)}+\frac{\frac{\pi}{2}-\arctan\xi_2}{\xi_2(1+\xi_2^2)}\right] \tag{2}$$

where

$$s^2=|\mathbf{s}_1|^2+|\mathbf{s}_2|^2-\mathbf{s}_1\cdot\mathbf{s}_2;\quad \xi_1=\frac{|\mathbf{s}_1|}{|\mathbf{s}_1-2\mathbf{s}_2|};\quad \xi_2=\frac{|\mathbf{s}_2|}{|\mathbf{s}_2-2\mathbf{s}_1|}, \tag{3}$$

$K_0(x)$ is the zero-th Macdonald function of integer order (see, e.g. [21]) and $\eta=\left(\frac{4\pi^2 m}{\hbar}\right)^{1/2}\mu$.

We list few important features of the solution.

The behaviour of the solution close to the coincidence plane $\{\mathbf{s}_1=0\}$ when $|\mathbf{s}_2|>0$ is

$$\Psi(\mathbf{s}_1,\mathbf{s}_2)\approx\frac{1}{|\mathbf{s}_1|}\left(\frac{\pi}{\sqrt{3}}\frac{1}{|\mathbf{s}_2|}K_0(\eta|\mathbf{s}_2|)\right). \tag{4}$$

Symmetrically, when $|\mathbf{s}_1|\cong 0$ e $|\mathbf{s}_2|>0$

$$\Psi(\mathbf{s}_1,\mathbf{s}_2)\approx\frac{1}{|\mathbf{s}_2|}\left(\frac{\pi}{\sqrt{3}}\frac{1}{|\mathbf{s}_1|}K_0(\eta|\mathbf{s}_1|)\right). \tag{5}$$

Moreover, the following scale transformation send eigenvectors in eigenvectors with different μ

$$\Psi_\lambda(\mathbf{s}_1,\mathbf{s}_2)\equiv\lambda^3\Psi(\lambda\mathbf{s}_1,\lambda\mathbf{s}_2)=\frac{\lambda}{s^2}K_0(\lambda\eta s)\left[\frac{\frac{\pi}{2}-\arctan\xi_1}{\xi_1(1+\xi_1^2)}+\frac{\frac{\pi}{2}-\arctan\xi_2}{\xi_2(1+\xi_2^2)}\right] \tag{6}$$

with $\lambda>0$ and $\|\Psi_\lambda\|_2=\|\Psi\|_2$.

Notice that the singularity shown by the solution close to the coincidence planes are the same of the potential of a density charge distributed on the planes. As we will see this behaviour is typical of the functions in the domain of the zero-range interaction Hamiltonians suggested by Ter-Martirosian and Skorniakov [39] and Danilov [17] many years after the work of L. H. Thomas.

3 Zero-Range Interaction Hamiltonians

It is well known that for system of quantum particles in $\mathbb{R}^3$ at low temperature one has that the thermal wavelength

$$\lambda = \frac{h}{p} \simeq \hbar \sqrt{\frac{2\pi}{mk_B T}}$$

is much larger than the range of the two-body interaction. Under these conditions the system exhibits a universal behavior, i.e., relevant observables do not depend on the details of the two-body interaction, but only on a single physical parameter known as the scattering length

$$a := \lim_{|\mathbf{k}|\to 0} f(\mathbf{k}, \mathbf{k}')$$

where $f(\mathbf{k}, \mathbf{k}')$ is the scattering amplitude of the two-body scattering process. In this regime it is natural to expect that the qualitative behavior of the system of particles in $\mathbb{R}^3$ is well described by the (formal) effective Hamiltonian with zero-range interactions

$$\text{“}H = -\sum_i \frac{1}{2m_i} \Delta_{\mathbf{x}_i} + \sum_{i<j} a_{ij} \delta(\mathbf{x}_i - \mathbf{x}_j)\text{”} \tag{7}$$

where m_i is the mass of the i-th particle and a_{ij} is the scattering length associated to the scattering process between the particles i and j. Hamiltonians of the type (7) are widely used in the physical literature to investigate, e.g., the behaviour of ultra-cold gases (see, e.g. [10–13, 26, 37, 42, 43]).

From the mathematical point of view the first step is to give a rigorous meaning to (7). We define the Hamiltonian for the system of n particles with two-body zero-range interactions as a non trivial self-adjoint (s.a.) extension in $L^2(\mathbb{R}^{3n})$ of the operator

$$-\sum_i \frac{1}{2m_i} \Delta_{\mathbf{x}_i} \tag{8}$$

defined on H^2-functions vanishing on each hyperplane $\{\mathbf{x}_i = \mathbf{x}_j\}$.

The complete characterization of such Hamiltonians can be obtained in the simple case $n = 2$. Indeed, in the center of mass reference frame, denoting with $\mathbf{x}$ the relative coordinate, one has to consider the operator in $L^2(\mathbb{R}^3)$

$$\tilde{h}_0 = -\frac{1}{2m_{12}} \Delta_{\mathbf{x}}, \quad m_{12} = \frac{m_1 m_2}{m_1 + m_2} \tag{9}$$

defined on H^2-functions vanishing at the origin. Such operator has defect indices $(1, 1)$ and the one-parameter family of s.a. extensions h_α, $\alpha \in \mathbb{R}$, can be explicitly constructed. Roughly speaking, h_α acts as the free Hamiltonian outside the origin and $\psi \in D(h_\alpha)$ satisfies the (singular) boundary condition at the origin

$$\psi(\mathbf{x}) = \frac{q}{|\mathbf{x}|} + \alpha q + o(1) \quad \text{for} \quad |x| \to 0 \tag{10}$$

where q is a constant depending on ψ and $\alpha = a^{-1}$. Moreover, the resolvent $(h_\alpha - z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, can be explicitly computed and all the spectral properties of h_α characterized (see [1, 5]).

For systems of n particles, with $n > 2$, the construction is much more difficult because of the presence of infinite dimensional defect spaces (see, e.g. [3, 4, 7–9, 14–16, 18, 20, 22, 24, 25, 27–36, 41]). To simplify notation, we describe the problem, and some previous attempts to solve it, in the case of three identical bosons with masses $1/2$, in the center of mass reference frame.

Let $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3 = -\mathbf{x}_1 - \mathbf{x}_2$ be the cartesian coordinates of the particles. Let us introduce the Jacobi coordinates

$$\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_3, \quad \mathbf{y} = \frac{1}{2}(\mathbf{x}_2 + \mathbf{x}_3) - \mathbf{x}_1 \tag{11}$$

with inverse given by $\mathbf{x}_1 = -\frac{2}{3}\mathbf{y}$, $\mathbf{x}_2 = \frac{1}{2}\mathbf{x} + \frac{1}{3}\mathbf{y}$, $\mathbf{x}_3 = -\frac{1}{2}\mathbf{x} + \frac{1}{3}\mathbf{y}$. Due to the symmetry constraint, the Hilbert space of states is

$$L^2_s(\mathbb{R}^6) = \left\{ \psi \in L^2(\mathbb{R}^6) \text{ s.t. } \psi(\mathbf{x}, \mathbf{y}) = \psi(-\mathbf{x}, \mathbf{y}) = \psi\left(\frac{1}{2}\mathbf{x} + \mathbf{y}, \frac{3}{4}\mathbf{x} - \frac{1}{2}\mathbf{y}\right) \right\} \tag{12}$$

and the formal Hamiltonian reads

$$\text{“}H = -\Delta_{\mathbf{x}} - \frac{3}{4}\Delta_{\mathbf{y}} + a\delta(\mathbf{x}) + a\delta(\mathbf{y} - \mathbf{x}/2) + a\delta(\mathbf{y} + \mathbf{x}/2)\text{”}, \tag{13}$$

i.e., H is a perturbation of the free dynamics in $\mathbb{R}^6$ supported by the three-dimensional hyperplanes

$$\Sigma = \{\mathbf{x} = 0\} \cup \{\mathbf{y} - \mathbf{x}/2 = 0\} \cup \{\mathbf{y} + \mathbf{x}/2 = 0\}. \tag{14}$$

According to our mathematical definition, a s.a. Hamiltonian in $L^2_s(\mathbb{R}^6)$ corresponding to the formal operator H is a non trivial s.a. extension of the operator

$$\tilde{H}_0 = -\Delta_{\mathbf{x}} - \frac{3}{4}\Delta_{\mathbf{y}}, \quad D(\tilde{H}_0) = \left\{ \psi \in L^2_s(\mathbb{R}^6) \text{ s.t. } \psi \in H^2(\mathbb{R}^6), \ \psi\big|_\Sigma = 0 \right\}. \tag{15}$$

As we already mentioned the defect indices are now infinite and the problem arises of how to choose and characterize a class of s.a. extensions with the right physical properties.

An apparently reasonable choice based on the analogy with the case $n = 2$ is due to Ter-Martirosian and Skorniakov [39]. Indeed, they defined an operator H_α acting as the free Hamiltonian outside the hyperplanes and satisfying a boundary condition at the hyperplanes. Specifically, they impose

$$\psi(\mathbf{x},\mathbf{y}) = \frac{\xi(\mathbf{y})}{|\mathbf{x}|} + \alpha\,\xi(\mathbf{y}) + o(1), \quad \text{for } \; |\mathbf{x}| \to 0 \;\; \text{and} \;\; \mathbf{y} \neq 0 \tag{16}$$

where ξ is a function depending on ψ. Due to the bosonic symmetry, the same conditions for $|\mathbf{y} - \mathbf{x}/2| \to 0$ and $|\mathbf{y} + \mathbf{x}/2| \to 0$ have to be satisfied.

We note that the first term in the right-hand side of (16) coincides with the first term of the asymptotic expansion of the potential produced by the charge density ξ distributed on $\{\mathbf{x} = 0\}$. Therefore, an equivalent way to describe a wave function ψ in the domain of H_α is the following. Any $\psi \in D(H_\alpha)$ can be decomposed as

$$\psi = w^\lambda + \mathcal{G}^\lambda \xi, \quad w^\lambda \in H^2(\mathbb{R}^6) \tag{17}$$

where $\lambda > 0$ and

$$\widehat{\mathcal{G}^\lambda \xi}(\mathbf{k},\mathbf{p}) = \sqrt{\frac{2}{\pi}}\; \frac{\hat{\xi}(\mathbf{p}) + \hat{\xi}(\mathbf{k} - \frac{1}{2}\mathbf{p}) + \hat{\xi}(-\mathbf{k} - \frac{1}{2}\mathbf{p})}{|\mathbf{k}|^2 + \frac{3}{4}|\mathbf{p}|^2 + \lambda}. \tag{18}$$

Note that the function $\mathcal{G}^\lambda \xi(\mathbf{x},\mathbf{y})$ has the asymptotic behaviour

$$\mathcal{G}^\lambda \xi(\mathbf{x},\mathbf{y}) = \frac{\xi(\mathbf{y})}{|\mathbf{x}|} - \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}} \big(T^\lambda \hat{\xi}\,\big)(\mathbf{p}) + o(1) \tag{19}$$

for $|\mathbf{x}| \to 0$ and $\mathbf{y} \neq 0$, where

$$\big(T^\lambda \hat{\xi}\,\big)(\mathbf{p}) := \sqrt{\frac{3}{4}|\mathbf{p}|^2 + \lambda}\;\, \hat{\xi}(\mathbf{p}) - \frac{1}{\pi^2} \int d\mathbf{p}' \, \frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}|^2 + |\mathbf{p}'|^2 + \mathbf{p}\cdot\mathbf{p}' + \lambda}. \tag{20}$$

We will refer to the first and second term in (20) respectively as the diagonal and the non-diagonal part of T^λ.

Therefore the boundary condition (16) is now rewritten as

$$\alpha\,\hat{\xi}(\mathbf{p}) + \big(T\,\hat{\xi}\,\big)(\mathbf{p}) \;=\; (w^\lambda\big|_{\mathbf{x}=0})^\wedge(\mathbf{p}). \tag{21}$$

There is an ambiguity in the above definition since the domain of the symmetric and unbounded operator T^λ in $L^2(\mathbb{R}^3)$ is not specified. As a first attempt one can choose

$$D(T^\lambda) = \{\hat{\xi} \in L^2(\mathbb{R}^3) \;\;|\;\; \int d\mathbf{k}\, |\mathbf{k}|^2 |\hat{\xi}(\mathbf{k})|^2 < \infty\} \equiv \{\hat{\xi} \mid \xi \in H^1(\mathbb{R}^3)\}. \tag{22}$$

Note that for $\hat{\xi} \in D(T^\lambda)$ both terms in the r.h.s. of (20) belong to $L^2(\mathbb{R}^3)$ (see, e.g. [22]).

As a matter of fact, the operator H_α defined in this way is symmetric but not s.a. and it turns out that its s.a. extensions are all unbounded from below. This fact, first noted by Danilov [17], was rigorously analyzed by Minlos and Faddeev in [33, 34].

4 Minlos and Faddeev Contributions

In the first of their seminal contributions (see [33]) Minlos and Faddeev consider a system of three bosons and approach the general mathematical problem to give a meaning to the formal Hamiltonian with zero-range interactions. Working in momentum space and using the theory of s.a. extensions of semibounded operators developed by Birman [38], they obtain the abstract characterization of all the s.a. extensions of the operator (15).

Then they observe that the s.a. extensions of the operator H_α in $L^2_s(\mathbb{R}^6)$ introduced by Ter-Martirosian and Skornyakov are in one-to-one correspondence with the s.a. extensions of the operator T^λ in $L^2(\mathbb{R}^3)$. Such operator T^λ defined on $D(T^\lambda)$ has defect indices (1, 1) and they find that a s.a. extension T^λ_β, $\beta \in \mathbb{R}$, of T^λ is defined on

$$\begin{aligned} D(T^\lambda_\beta) = \Big\{ & \hat{\xi} \in L^2(\mathbb{R}^3) \mid \hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2,\ \hat{\xi}_1 \in D(T^\lambda), \\ & \hat{\xi}_2(\mathbf{k}) = \frac{c}{|\mathbf{k}|^2+1}\Big(\beta \sin\big(s_0 \log|\mathbf{k}|\big) + \cos\big(s_0 \log|\mathbf{k}|\big)\Big)\Big\} \end{aligned} \tag{23}$$

where c is an arbitrary constant and s_0 is the positive solution of the equation

$$1 - \frac{8}{\sqrt{3}}\,\frac{\sinh\frac{\pi s}{6}}{s\,\cosh\frac{\pi s}{2}} = 0. \tag{24}$$

One can observe that both the diagonal and the non diagonal parts of T^λ diverge on functions with the asymptotic behaviour of $\hat{\xi}_2$ in (23) for $|\mathbf{k}| \to \infty$, but their sum remains finite.

Given the s.a. operator T^λ_β, $D(T^\lambda_\beta)$, one obtains the s.a. extension $H_{\alpha,\beta}$ (also called Ter-Martirosian, Skornyakov Hamiltonian) of H_α

$$\begin{aligned} D(H_{\alpha,\beta}) = \Big\{ & \psi \in L^2_s(\mathbb{R}^6) \mid \psi = w^\lambda + \mathcal{G}^\lambda \xi,\ w^\lambda \in H^2(\mathbb{R}^6),\ \hat{\xi} \in D(T^\lambda_\beta), \\ & \alpha\,\hat{\xi}(\mathbf{p}) + \big(T^\lambda \hat{\xi}\,\big)(\mathbf{p}) \;=\; (w^\lambda\big|_{\mathbf{x}=0})^\wedge(\mathbf{p})\Big\}, \end{aligned} \tag{25}$$

$$(H_{\alpha,\beta} + \lambda)\psi = (H_0 + \lambda) w^\lambda, \tag{26}$$

where

$$H_0 = -\Delta_{\mathbf{x}} - \frac{3}{4}\Delta_{\mathbf{y}}, \quad D(H_0) = H^2(\mathbb{R}^6). \tag{27}$$

Roughly speaking, β parametrizes a further boundary condition satisfied at the triple coincidence point $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = 0$. Therefore, it can be considered as the strength of a sort of an additional three-body force acting on the particles when all their positions coincide.

The authors conclude claiming that some further results on the spectrum of the Hamiltonian $H_{\alpha,\beta}$ hold. In particular, they affirm that $H_{\alpha,\beta}$ has the unphysical instability property already noted by Danilov, i.e., that there exists an infinite sequence of negative eigenvalues

$$E_n \to -\infty, \quad \text{as} \quad n \to \infty.$$

The rigorous proof of this fact is contained in their second paper on the subject and it will be described below.

At the end of the paper one finds an interesting remark on the possibility to define a modified Hamiltonian satisfying the stability property, i.e., bounded from below. The authors say: " We note that this last result (i.e., the instability property) somewhat discredits our chosen extension, since probably only semibounded energy operators are of interest in nonrelativistic quantum mechanics. It seems to us that there must exist among the other extensions of the operator $\tilde{H}_0$ semibounded extensions which have all the properties of the model of Ter-Martirosian and Skornyakov that are good from the physical point of view, namely the properties of locality and of the correct character of the continuous spectrum. The authors therefore indicate a strategy to solve the instability problem, i.e., they suggest to replace the constant α in (21) (or equivalently in (16)) with the operator α_M in the Fourier space defined by

$$(\alpha_M \hat{\xi})(\mathbf{p}) = \alpha \hat{\xi}(\mathbf{p}) + (K\hat{\xi})(\mathbf{p}) \tag{28}$$

where $\alpha \in \mathbb{R}$ and K is a convolution operator with a kernel $K(\mathbf{p} - \mathbf{p}')$ having the asymptotic behavior

$$K(\mathbf{p}) \sim \frac{\gamma}{|\mathbf{p}|^2}, \quad \text{for} \quad |\mathbf{p}| \to \infty \tag{29}$$

with the constant γ satisfying

$$\gamma > \frac{1}{\pi^3}\Big(\frac{4\pi}{3\sqrt{3}} - 1\Big). \tag{30}$$

Unfortunately, they conclude: "A detailed development of this point of view is not presented here because of lack of space."

We believe that such suggestion is interesting and we find it rather strange that the idea has never been developed in the literature. We also observe that it is not so evident that the replacement of the constant α with the operator α_M defined in (28)–(30) produces a semibounded Hamiltonian and, moreover, it is not clear the physical meaning of such replacement. We shall come back to this point in the next section.

Here we continue the analysis of the contribution of Minlos and Faddeev discussing the content of their second paper on the subject [34], where the authors show that the Hamiltonian $H_{\alpha,\beta}$ has an infinite number of eigenvalues accumulating both at zero and at $-\infty$. We give here a slightly different, and more elementary, proof than the one given in [34]. To simplify the notation, we consider only the case $\alpha = 0$. We recall that $\alpha = 0$ corresponds to a two-body interaction with zero-energy resonance (see, e.g. [5]).

Taking into account definitions (25) and (26), an eigenvector of $H_{0,\beta}$ associated to the negative eigenvalue $E = -\mu$, $\mu > 0$, has the form $\mathcal{G}^{\mu}\xi$, where $\hat{\xi} \in D(T_\beta)$ is a solution of the equation

$$\sqrt{\frac{3}{4}|\mathbf{p}|^2+\mu}\;\hat{\xi}(\mathbf{p}) - \frac{1}{\pi^2}\int \mathrm{d}\mathbf{p}'\,\frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mathbf{p}\cdot\mathbf{p}'+\mu} = 0. \tag{31}$$

We shall compute the rotationally invariant solutions $\hat{\xi} = \hat{\xi}(|\mathbf{p}|)$ of (31). Performing the angular integration in the second term of (31), one obtains the equation

$$\sqrt{\frac{3}{4}p^2+\mu}\;p\,\hat{\xi}(p) - \frac{2}{\pi}\int_0^\infty \mathrm{d}p'\,p'\hat{\xi}(p')\,\log\frac{p^2+p'^2+pp'+\mu}{p^2+p'^2-pp'+\mu} = 0. \tag{32}$$

Let us introduce a change of the independent variable

$$p = \frac{2\sqrt{\mu}}{\sqrt{3}}\sinh x, \quad x = \log\left(\frac{\sqrt{3}p}{2\sqrt{\mu}} + \sqrt{\frac{3p^2}{4\mu}+1}\right) \tag{33}$$

and define

$$\theta(x) = \begin{cases} \mu\sinh x\cosh x\,\hat{\xi}\left(\frac{2\sqrt{\mu}}{\sqrt{3}}\sinh x\right) & \text{for } x \geq 0 \\ -\theta(-x) & \text{for } x < 0 \end{cases} \tag{34}$$

so that

$$\hat{\xi}(p) = \frac{2}{\sqrt{3}}\,\frac{\theta\left[\log\left(\frac{\sqrt{3}p}{2\sqrt{\mu}} + \sqrt{\frac{3p^2}{4\mu}+1}\right)\right]}{p\sqrt{\frac{3}{4}p^4+\mu}}. \tag{35}$$

The first term in (32) in the new coordinates is

$$\sqrt{\frac{3}{4}p^2+\mu}\;p\,\hat{\xi}(p) = \sqrt{\mu\sinh^2 x+\mu}\;\frac{2\sqrt{\mu}}{\sqrt{3}}\sinh x\,\hat{\xi}\left(\frac{2\sqrt{\mu}}{\sqrt{3}}\sinh x\right)$$
$$= \frac{2}{\sqrt{3}}\,\theta(x). \tag{36}$$

The other term in (32) in the new coordinates reads

$$
\begin{aligned}
&-\frac{2}{\pi}\int_0^\infty dp'\, p'\hat{\xi}(p')\, \log\frac{p^2+p'^2+pp'+\mu}{p^2+p'^2-pp'+\mu}\\
&=-\frac{8}{3\pi}\int_0^\infty dy\,\theta(y)\log\frac{\sinh^2 x+\sinh^2 y+\sinh x\sinh y+3/4}{\sinh^2 x+\sinh^2 y-\sinh x\sinh y+3/4}\\
&=-\frac{8}{3\pi}\int_0^\infty dy\,\theta(y)\log\frac{(2\cosh(x+y)-1)\,(2\cosh(x-y)+1)}{(2\cosh(x+y)+1)\,(2\cosh(x-y)-1)}\\
&=-\frac{8}{3\pi}\int_{-\infty}^{+\infty} dy\,\theta(y)\log\frac{2\cosh(x-y)+1}{2\cosh(x-y)-1}
\end{aligned}
\tag{37}
$$

where in the last line we have used the extension $\theta(x)=-\theta(-x)$ for $x<0$. Therefore, equation (32) for $\hat{\xi}(p)$ is transformed into the following convolution equation for $\theta(x)$

$$
\theta(x)-\frac{4}{\sqrt{3}\pi}\int_{-\infty}^{+\infty} dy\,\theta(y)\,\log\frac{2\cosh(x-y)+1}{2\cosh(x-y)-1}=0. \tag{38}
$$

Finally, we compute the Fourier transform (see [21], p. 36) and we arrive at the equation for $\hat{\theta}$

$$
\left(1-\frac{8}{\sqrt{3}}\,\frac{\sinh\frac{\pi}{6}s}{s\,\cosh\frac{\pi}{2}s}\right)\hat{\theta}(s)=0. \tag{39}
$$

Denote by $g(s)$ the function in parenthesis in (39). It is easy to see that g is even, monotone increasing for $s>0$ and $g(s)\to 1$ for $s\to+\infty$. Moreover, $g(0)=1-\frac{4\pi}{3\sqrt{3}}<0$ and we conclude that the equation $g(s)=0$ has two solutions $s=\pm s_0$, with $s_0>0$. Since $\hat{\theta}$ is an odd function, the solution of (39) reads

$$
\hat{\theta}(s)=\delta(s-s_0)-\delta(s+s_0) \tag{40}
$$

apart from a multiplicative constant and therefore

$$
\theta(x)=\sin s_0 x. \tag{41}
$$

From (35) we obtain the solution of equation (32) for any $\mu>0$

$$
\hat{\xi}_\mu(p)=\frac{\sin\left[s_0\log\left(\frac{\sqrt{3}p}{2\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}\sqrt{\frac{3}{4}p^2+\mu}\right)\right]}{p\sqrt{\frac{3}{4}p^2+\mu}}\,. \tag{42}
$$

We note that the solution (42) belongs to $L^2(\mathbb{R}^3)$ but it does not belong to $D(T^\lambda)$ because of the behavior $O(p^{-2})$ for $p \to \infty$. On the other hand we can find suitable values of μ such that the solution belongs to $D(T^\lambda_\beta)$.

Indeed, denoting $\varepsilon = \frac{4\mu}{3}\, p^{-2}$, we have

$$\begin{aligned}
\hat{\xi}_\mu(p) &= \frac{\sin\left\{s_0\left[\log\frac{\sqrt{3}p}{\sqrt{\mu}} + \log\left(1+\frac{\varepsilon/2}{1+\sqrt{1+\varepsilon}}\right)\right]\right\}}{p^2\sqrt{1+\varepsilon}} \\
&= \frac{\sin\left(s_0\log p + \frac{s_0}{2}\log\frac{3}{\mu}\right)}{p^2\sqrt{1+\varepsilon}} + \eta_1(p) \\
&= \cos\left(\frac{s_0}{2}\log\frac{3}{\mu}\right)\frac{\sin\left(s_0\log p\right)}{p^2+1} + \sin\left(\frac{s_0}{2}\log\frac{3}{\mu}\right)\frac{\cos\left(s_0\log p\right)}{p^2+1} + \eta_2(p)
\end{aligned} \tag{43}$$

where $\eta_1, \eta_2 \in D(T^\lambda)$. According to (23), in order to have $\hat{\xi}_\mu \in D(T^\lambda_\beta)$ we impose the condition

$$\cos\left(\frac{s_0}{2}\log\frac{3}{\mu}\right) = \beta\sin\left(\frac{s_0}{2}\log\frac{3}{\mu}\right). \tag{44}$$

Condition (44) is satisfied if and only if μ is equal to

$$\mu_n = 3\,e^{-\frac{2}{s_0}\cot^{-1}\beta}\,e^{\frac{2\pi}{s_0}n}, \quad n \in \mathbb{Z}. \tag{45}$$

Thus we obtain an infinite sequence of negative eigenvalues

$$E_n = -\mu_n, \quad n \in \mathbb{Z} \tag{46}$$

with corresponding eigenvectors $\mathcal{G}^{\mu_n}\xi_{\mu_n}$, where $\hat{\xi}_{\mu_n}$ is given by (42).

We stress that the model Hamiltonian $H_{0,\beta}$ exhibits the Efimov effect, i.e., there exists an infinite sequence of eigenvalues $E_n \to 0$ for $n \to -\infty$ satisfying the (exact) geometrical law

$$\frac{E_n}{E_{n+1}} = e^{-\frac{2\pi}{s_0}}. \tag{47}$$

On the other hand, one also has $E_n \to -\infty$ for $n \to +\infty$, corresponding to the instability property known as Thomas effect.

5 Regularized Zero-Range Interactions

In this section we propose a model Hamiltonian for three bosons with zero-range interactions regularized around the triple coincidence point $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = 0$ in such a way to avoid the Thomas effect, i.e., with a spectrum bounded from below.

We follow the proposal contained in [4] which, in turn, essentially coincides with the already mentioned suggestion discussed at the end of [33].

In the first part of [4] the authors announce an interesting mathematical result on the Efimov effect. They consider a three-particle system with Hamiltonian H, with two-body, spherically symmetric, short range potentials such that at least two of the two-body subsystems have zero-energy resonances (i.e. infinite scattering length). They claim that H has infinitely many spherically symmetric bound states with energy $E_n \to 0$ such that

$$\lim_{n\to\infty} \frac{E_n}{E_{n+1}} = \mathrm{e}^{-\frac{2\pi}{\Sigma}} \tag{48}$$

where $\sigma > 0$ depends only on the mass ratios (and coincides with s_0 in (47) in the case of three identical bosons). The result should follow "from a detailed study of the asymptotic behavior of the action of the scaling group in the spaces of two and three-body Hamiltonians restricted to functions invariant under the natural action of $SO(3)$", where the scaling group is given by

$$H \to H_\varepsilon = \frac{1}{\varepsilon^2} U_\varepsilon H U_\varepsilon^{-1}, \quad (U_\varepsilon \psi)(\mathbf{x}) = \frac{1}{\varepsilon^{3/2}} \psi\Big(\frac{\mathbf{x}}{\varepsilon}\Big), \quad \varepsilon > 0. \tag{49}$$

Roughly speaking, for $\varepsilon \to 0$ the rescaled Hamiltonian should converge to the zero-range model $H_{0,\beta}$. Since for $H_{0,\beta}$ the Efimov effect, together with the property $\frac{E_n}{E_{n+1}} = \mathrm{e}^{-2\pi/\sigma}$, is explicitly verified, one should infer the result for H. Unfortunately, this program has not been realized and it remains as a challenging open problem.

In the last part of the paper the authors add an interesting remark for our aim concerning the construction of a reasonable three-body Hamiltonian with zero-range interactions which is bounded from below.

Indeed, they claim that one can consider a Hamiltonian with zero-range interactions where the zero-range force between two particles depends on the position of the third one. If such three-body force is suitably chosen then the Hamiltonian is bounded from below.

The proposed recipe can be rephrased in the following way: in the boundary condition (16) one replaces the constant α with a position-dependent term

$$\alpha_A(\mathbf{y}) = \alpha + \frac{\delta}{|\mathbf{y}|}, \quad \alpha, \delta \in \mathbb{R}. \tag{50}$$

In the case of equal masses, they affirm that for

$$\delta > \frac{2}{\pi^2}\Big(\frac{4\pi}{3\sqrt{3}} - 1\Big) \tag{51}$$

the corresponding zero-range Hamiltonian is bounded from below. Also the proof of this statement is postponed to a forthcoming paper which has never been published.

Let us briefly comment on the above proposal. The replacement of the constant α with a function $\alpha(\mathbf{y})$, with $\alpha(\mathbf{y}) \to \infty$ for $|\mathbf{y}| \to 0$, has a reasonable physical meaning. It means that when the positions of the three particles coincide, i.e., for $\mathbf{x} = \mathbf{y} = 0$, the two-body interactions are switched off ($\alpha = \infty$ means no interaction). In this way one compensates the tendency of the three interacting particles to "fall in the center". On the other hand, the specific choice of the function $\alpha_A(\mathbf{y})$ is not explained in the paper but one can imagine that such a function allows some explicit computations (as in the case of the choice (29) of the operator K in [33]).

It is also natural to compare the two proposals contained in [4, 33]. It is immediate to realize that the two proposals essentially coincide in the sense that one is the Fourier transform of the other, i.e.,

$$(\alpha_A \xi)^{\wedge}(\mathbf{p}) = (\alpha_M \hat{\xi})(\mathbf{p}), \quad \text{if} \quad \delta = 2\pi^2 \gamma. \tag{52}$$

It is also important to stress that only the asymptotic behavior of $K(\mathbf{p})$ for $|\mathbf{p}| \to \infty$ (see (29)) is relevant to obtain a lower bounded Hamiltonian, as it is correctly pointed out in [33]. Correspondingly, it must be sufficient to require only the asymptotic behavior $\delta|\mathbf{y}|^{-1} + O(1)$ for $|\mathbf{y}| \to 0$ for the position-dependent strength of the interaction.

In conclusion, following the (common) idea proposed in [4, 33], we introduce the Hamiltonian $H_{\tilde{\alpha}}$ characterized by the boundary condition

$$\psi(\mathbf{x}, \mathbf{y}) = \frac{\xi(\mathbf{y})}{|\mathbf{x}|} + \tilde{\alpha}(|\mathbf{y}|)\, \xi(\mathbf{y}) + o(1), \quad \text{for} \;\; |\mathbf{x}| \to 0 \;\; \text{and} \;\; \mathbf{y} \neq 0 \tag{53}$$

where

$$\tilde{\alpha} \; : \; \mathbb{R}_+ \to \mathbb{R}, \quad \tilde{\alpha}(r) = \alpha + \frac{\delta}{r} \chi^{\ell}(r) \tag{54}$$

with

$$\alpha \in \mathbb{R}, \quad \delta > 0, \quad \ell > 0, \quad \chi^{\ell}(r) = \begin{cases} 1 & r \leq \ell \\ 0 & r > \ell. \end{cases} \tag{55}$$

More precisely, we define the Hamiltonian as follows

$$D(H_{\tilde{\alpha}}) = \Big\{ \psi \in L^2_s(\mathbb{R}^6) \mid \psi = w^\lambda + \mathcal{G}^\lambda \xi, \; w^\lambda \in H^2(\mathbb{R}^6), \; \xi \in H^1(\mathbb{R}^3),$$

$$(\tilde{\alpha}\, \xi)^{\wedge}(\mathbf{p}) + \big(T^\lambda \hat{\xi}\,\big)(\mathbf{p}) \; = \; \big(w^\lambda\big|_{\mathbf{x}=0}\big)^{\wedge}(\mathbf{p}) \Big\}, \tag{56}$$

$$(H_{\tilde{\alpha}} + \lambda)\psi = (H_0 + \lambda) w^\lambda. \tag{57}$$

In [6] we proved the following result.

Proposition 1 *Let $\delta > \delta_0$, where*

$$\delta_0 = \frac{\sqrt{3}}{\pi}\left(\frac{4\pi}{3\sqrt{3}} - 1\right). \tag{58}$$

Then for any $\ell \in (0, +\infty]$ *and* $\alpha \in \mathbb{R}$ *the operator* (56) *and* (57) *is s.a. and bounded from below.*

The result shows that it is sufficient to add a three-body force with an arbitrary small (but different from zero) range to avoid the collapse. We stress that it would be interesting to prove that boundedness from below is preserved taking "some suitable limit $\ell \to 0$".

The proof is based on the analysis of the quadratic form associated to $H_{\tilde{\alpha}}$. Taking into account of (56) and (57), by an explicit computation for $\psi \in D(H_{\tilde{\alpha}})$ one obtains

$$\begin{aligned}(\psi, (H_{\tilde{\alpha}} + \lambda)\psi) &= (\psi, (H_0 + \lambda)w^\lambda) = (w^\lambda, (H_0 + \lambda)w^\lambda) + (\mathcal{G}^\lambda \xi, (H_0 + \lambda)w^\lambda)\\ &= (w^\lambda, (H_0 + \lambda)w^\lambda) + 12\pi \left[(\xi, \tilde{\alpha}\,\xi) + (\hat{\xi}, T^\lambda \hat{\xi})\right].\end{aligned} \tag{59}$$

Hence we define the quadratic form

$$F_{\tilde{\alpha}}(\psi) = (w^\lambda, (H_0 + \lambda)w^\lambda) - \lambda\|\psi\|^2 + 12\,\pi\,\Phi^\lambda_{\tilde{\alpha}}(\xi) \tag{60}$$

where

$$\Phi^\lambda_{\tilde{\alpha}}(\xi) = \int \mathrm{d}\mathbf{y}\,\tilde{\alpha}(|\mathbf{y}|)|\xi(\mathbf{y})|^2 + \int \mathrm{d}\mathbf{p}\,\overline{\hat{\xi}(\mathbf{p})}(T^\lambda \hat{\xi})(\mathbf{p}) \tag{61}$$

and

$$D(F_{\tilde{\alpha}}) = \left\{\psi \in L^2_s(\mathbb{R}^6) \mid \psi = w^\lambda + \mathcal{G}^\lambda \xi,\ w^\lambda \in H^1(\mathbb{R}^6),\ \hat{\xi} \in H^{1/2}(\mathbb{R}^3)\right\}. \tag{62}$$

The proof proceeds taking such a quadratic form as starting point and proving that it is closed and bounded from below. Therefore it uniquely defines a s.a. and bounded from below operator which coincides with (56) and (57).

6 On the Negative Eigenvalues

In this section we consider the eigenvalue problem for $H_{\tilde{\alpha}}$ in the special case

$$\alpha = 0, \qquad \ell = +\infty. \tag{63}$$

As we already noticed in the case of the TMS Hamiltonian, an eigenvector associated to the negative eigenvalue $E = -\mu$, $\mu > 0$, has the form $\mathcal{G}^\mu \xi$, where $\xi \in H^1(\mathbb{R}^3)$ is a solution of the equation

$$\frac{\delta}{2\pi^2}\int d\mathbf{p}'\,\frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|^2}+\sqrt{\frac{3}{4}|\mathbf{p}|^2+\mu}\;\hat{\xi}(\mathbf{p})-\frac{1}{\pi^2}\int d\mathbf{p}'\,\frac{\hat{\xi}(\mathbf{p}')}{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mathbf{p}\cdot\mathbf{p}'+\mu}=0. \tag{64}$$

where the first term in (64) is the Fourier transform of $\delta|\mathbf{y}|^{-1}\xi(\mathbf{y})$.

Proceeding as in Sect. 2, we consider the rotationally invariant case $\hat{\xi}=\hat{\xi}(|\mathbf{p}|)$. Performing the angular integration one obtains the equation

$$\begin{aligned}&\frac{\delta}{\pi}\int_0^\infty dp'\,p'\hat{\xi}(p')\log\frac{p+p'}{|p-p'|}+\sqrt{\frac{3}{4}p^2+\mu}\;p\,\hat{\xi}(p)\\&\quad-\frac{2}{\pi}\int_0^\infty dp'\,p'\hat{\xi}(p')\,\log\frac{p^2+p'^2+pp'+\mu}{p^2+p'^2-pp'+\mu}=0.\end{aligned} \tag{65}$$

In the following we prove that for $\delta>\delta_0$ there are no solutions of (65) and therefore the Hamiltonian has no negative eigenvalues corresponding to rotationally invariant solutions of (64). The main point of the proof is that the l.h.s. of (65) can be diagonalized by the same change of coordinates used in section 2 for the TMS Hamiltonian.

Proposition 1 *Let $\delta>\delta_0$. Then equation (65) has only the trivial solution.*

Proof Using (33) and (34), for the first term in (65) we have

$$\begin{aligned}\frac{\delta}{\pi}\int_0^\infty dp'\,p'\hat{\xi}(p')\log\frac{p+p'}{|p-p'|}&=\frac{4\delta}{3\pi}\int_0^\infty dy\,\theta(y)\log\left|\frac{\sinh x+\sinh y}{\sinh x-\sinh y}\right|\\&=\frac{4\delta}{3\pi}\int_0^\infty dy\,\theta(y)\log\left|\frac{\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}}{\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}}\right|\\&=\frac{4\delta}{3\pi}\int_0^\infty dy\,\theta(y)\log\left|\frac{\cosh\frac{x-y}{2}}{\sinh\frac{x-y}{2}}\right|+\frac{4\delta}{3\pi}\int_0^\infty dy\,\theta(y)\log\left|\frac{\sinh\frac{x+y}{2}}{\cosh\frac{x+y}{2}}\right|\\&=\frac{4\delta}{3\pi}\int_{-\infty}^{+\infty}dy\,\theta(y)\log\left|\coth\frac{x-y}{2}\right|\end{aligned} \tag{66}$$

where in the last line we have used the extension $\theta(x)=-\theta(-x)$ for $x<0$. Hence, by (36), (37) and (66), in the new variables equation (65) reads

$$\theta(x) + \frac{2\delta}{\sqrt{3}\pi} \int_{-\infty}^{+\infty} dy\, \theta(y) \log \left| \coth \frac{x-y}{2} \right|$$
$$- \frac{4}{\sqrt{3}\pi} \int_{-\infty}^{+\infty} dy\, \theta(y)\, \log \frac{2\cosh(x-y)+1}{2\cosh(x-y)-1} = 0. \tag{67}$$

We note that

$$\frac{1}{\sqrt{2\pi}} \int dx\, e^{-isx} \log \left| \coth \frac{x}{2} \right| = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx\, \cos sx\, \log\Big(\coth \frac{x}{2}\Big)$$
$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx\, \cos sx\, \log\big(1 + e^{-x}\big) - \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx\, \cos sx\, \log\big(1 - e^{-x}\big)\,. \tag{68}$$

Then we use ([23], p. 582), and (39) to write the equation for the Fourier transform of θ

$$\left(1 + 2\, \frac{\delta \sinh \frac{\pi}{2} s - 4 \sinh \frac{\pi}{6} s}{\sqrt{3}\, s \cosh \frac{\pi}{2} s}\right) \hat{\theta}(s) = 0. \tag{69}$$

It remains to show that the function in parenthesis in (69) is strictly positive for $\delta > \delta_0$. Note that the function is even and then we can consider $s \geq 0$. For $\delta > \delta_0$ it is positive in $s = 0$. Moreover, if we denote

$$F(s) = \sqrt{3}\, s \cosh \frac{\pi}{2} s + 2\delta \sinh \frac{\pi}{2} s - 8 \sinh \frac{\pi}{6} s, \tag{70}$$

for $\delta > \delta_0$ we have

$$F'(s) = (\sqrt{3} + \pi\delta) \cosh \frac{\pi}{2} s + \frac{\sqrt{3}\,\pi}{2}\, s \sinh \frac{\pi}{2} s - \frac{4\pi}{3} \cosh \frac{\pi}{6} s$$
$$> (\sqrt{3} + \pi\delta_0) \cosh \frac{\pi}{2} s - \frac{4\pi}{3} \cosh \frac{\pi}{6} s$$
$$\geq \left(\sqrt{3} + \pi\delta_0 - \frac{4\pi}{3}\right) \cosh \frac{\pi}{6} s = 0 \tag{71}$$

so that $F(s) > 0$ for any s. It follows that equation (69), and then equation (65), has only the trivial solution and this conclude the proof of the proposition. □

7 Acknowledgements by Way of Conclusion

We have many reasons for being grateful to Sergio Albeverio, mainly for his lasting friendship and for all the affection and help he has been giving us for so many years. Among all these good reasons, one concerns the support he provided to our scientific activity.

As it is very well known, he was one of the main players in the scientific project aimed to explicitly construct models in relativistic quantum field theory describing interacting bosons, a project that never reached a final result in four dimensional space-time. The formal non-relativistic limits of the relativistic models under study in constructive quantum field theory (see [19] for the only rigorous result in the investigation of such limits) describe particles in three dimensions interacting via zero-range forces. This was the reason why, together with many other scientific interests, he got involved in the theory of point interaction Hamiltonians.

In the early nineties he suggested to one of us (A.T., young postdoc in Bochum at that time) to investigate further the spectral structure of the point interaction Hamiltonian for a three-boson system, following the suggestions given in [4]. With some delay, together with a group of other younger italian researchers, we figured out better those suggestions and started working in this direction.

In this contribution we outlined the history of the main achievements in this research field and we sketched our first results.

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Mathematical Theory of Feynman Path Integrals

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Abstract Sergio Albeverio's outstanding contributions to the mathematical theory of Feynman path integrals are reviewed. Connections with quantum mechanics and quantum field theory are discussed.

Keywords Infinite dimensional integration · Feynman path integrals · Quantum mechanics

AMS Classification 35C15 · 28C20 · 81Q05

1 Introduction

Feynman path integrals are ubiquitous in theoretical physics [1–6]. Since their first introduction in R. Feynman's pioneering work [7], they have gained an important role in the formulation of quantum theory providing on the one hand an intuitive quantization procedure and on the other hand a powerful computational tool. It is commonly believed that Feynman path integrals are in fact just an *heuristic* tool, without a sound mathematical definition. This paper has the aim to disprove this misconception and show how Feynman's idea and the attempts to give them a sound mathematical foundation have lead to the development of a blooming area of mathematics where infinite dimensional analysis and mathematical physics meet and find a fruitful interplay.

Dedication It is a honor and a pleasure to contribute to this volume dedicated to Sergio Albeverio's 80st birthday. Sergio's work contains fundamental contributions to the mathematical theory of Feynman path integrals and, more generally, to several areas at the cutting edge between stochastics, infinite dimensional analysis and mathematical physics. I had the opportunity to meet Sergio for the first time in 2002, when a fruitful scientific collaboration started. Sergio has always been an inexhaustible source of inspiration and encouragement and I would like to dedicate this paper to him as a little sign of gratitude.

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Feynman path integrals were introduced in 1942 [8] when R. Feynman, developing an intuition by Dirac [9], proposed an alternative Lagrangian formulation of time evolution in quantum mechanics. According to Feynman's proposal, the solution of Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(x) \\ \psi(0,x) = \psi_0(x), \qquad \psi_0 \in C_0^\infty(\mathbb{R}^d) \end{cases} \tag{1}$$

describing the time evolution of the state ψ of a non-relativistic particle moving in a force field associated to a potential V, should be given as a *sum over all possible histories* of the system, i.e. by an heuristic integral of the following form:

$$\psi(t,x) = \text{“}C^{-1}\int_\Gamma e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))d\gamma\text{”} \tag{2}$$

In (2) Γ denotes a set of paths $\gamma : [0,t] \to \mathbb{R}^d$ with fixed end point $\gamma(t) = x$, $d\gamma$ stands for a Lebesgue-type measure on Γ while the function $S : \Gamma \to \mathbb{R}$ appearing in the integral is the classical action functional of the system, namely

$$S_t(\gamma) = S^0(\gamma) - \int_0^t V(\gamma(s))ds, \quad S^0(\gamma) = \frac{m}{2}\int_0^t |\dot{\gamma}(s)|^2 ds,$$

and C plays the role of a normalization constant. In (1) and (2) the positive constants m and $\hbar$ denote the mass of the particle and the Planck constant respectively. A formal derivation of (2) can be provided in terms of Trotter product formula [10], which yields the unitary group $U(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ generated by the quantum mechanical Hamiltonian operator[1] in terms of the strong limit in $L^2(\mathbb{R}^d)$

$$U(t)\psi = e^{-\frac{it}{\hbar}(\frac{-\Delta}{2m}+V)}\psi_0 = \lim_{n\to\infty}\left(e^{\frac{it\Delta}{2m\hbar n}}e^{-\frac{it}{n}V}\right)^n\psi_0 \tag{3}$$

By taking a subsequence the convergence holds for almost everywhere $x \in \mathbb{R}^d$ and by using the representation formula

$$e^{\frac{it\Delta}{2\hbar}}\psi_0(x) = \int_{\mathbb{R}^d}\frac{e^{\frac{im(x-y)^2}{2\hbar t}}}{(2i\pi m^{-1}\hbar t)^{d/2}}\psi_0(y)dy, \quad x \in \mathbb{R}^d$$

[1] More precisely $H : D(H) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given on smooth compactly supported vectors $\psi \in C_0^\infty(\mathbb{R}^d)$ by $H\psi(x) = -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x)$. Under suitable condition on the potential $V : \mathbb{R}^d \to \mathbb{R}$ (see, e.g. [11]) the operator H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ and its closure is the unique self-adjoint extension which generates by Stone theorem a strongly continuous 1-parameter group $U(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, formally written $U(t) = e^{-\frac{i}{\hbar}Ht}$.

we obtain

$$U(t)\psi(x)=\lim_{n\to\infty}\int_{\mathbb{R}^{dn}}\frac{e^{\frac{i}{\hbar}\sum_{j=1}^{n}\left(\frac{m(x_j-x_{j-1})^2}{2(t/n)^2}-V(x_j)\right)\frac{t}{n}}}{(2\pi i m^{-1}\hbar t/n)^{nd/2}}\psi_0(x_0)dx_0\dots dx_{n-1}. \quad (4)$$

Fixed $n\in\mathbb{N}$ and considering the finite dimensional vector space H_n of piecewise linear paths of the form

$$\gamma_n(s):=x_j+\frac{(x_{j+1}-x_j)}{t/n}(s-jt/n),\quad s\in\left[\frac{jt}{n},\frac{(j+1)t}{n}\right),$$

it is possible to look at the right hand side of (4) as the finite dimensional approximation of Feynman formula (2) where the space Γ of all paths is replaced by H_n:

$$\frac{\int_{H_n}e^{\frac{i}{\hbar}S_t(\gamma_n)}\psi_0(\gamma_{(0)})d\gamma_n}{\int_{H_n}e^{\frac{i}{\hbar}S^0(\gamma_n)}d\gamma_n}=\int_{\mathbb{R}^{dn}}\frac{e^{\frac{i}{\hbar}\sum_{j=1}^{n}\frac{m(x_j-x_{j-1})^2}{2t/n}}e^{-\frac{i}{\hbar}\frac{t}{n}V(\tilde{x}_j)}}{(2\pi i m^{-1}\hbar t/n)^{nd/2}}\psi_0(x_0)dx_0\dots dx_{n-1}, \quad (5)$$

where $\tilde{x}_j\in[x_j,x_{j+1}]$. Hence, from this point of view Feynman formula is just a mnemonic tool which resembles an approximation procedure. Nevertheless it is undoubtedly fascinating. Indeed, it creates a bridge between the classical Lagrangian description of the physical world and the quantum one, reintroducing in quantum mechanics the concept of trajectory, which had been banned by the original formulation of quantum theory. It provides a quantization method, allowing (at least heuristically) to associate a quantum evolution to any classical Lagrangian. Moreover it makes simple and intuitive the study of the semiclassical limit of quantum mechanics, i.e. the study of the detailed asymptotic behaviour of the solution of (1) in the limit $\hbar\downarrow 0$. Indeed, according to an heuristic application of the classical stationary phase method [12, 13] to (2), when $\hbar\downarrow 0$ the main contributions to the integral should come from those paths γ_c such that $\delta S(\gamma_c)=0$. These, by Hamilton's least action principle are exactly the classical orbits of the system. However, Feynman's formula (2) does not have a well defined mathematical meaning. Indeed, neither the normalization constant C not the infinite dimensional Lebesgue measure $d\gamma$ are well defined. Feynman himself was conscious of these problems, as it is possible to infere from his own words [8]

> …one feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus.

which convey clearly the lack of a sound mathematical theory underlying his techniques. Nevertheless, Feynman extended its formula to more general quantum systems, including relativistic fields [14], and producing a powerful heuristic calculus which works even when rigorous arguments fail [15]. The challenge to provide a mathematical definition of Feynman integrals was soon taken up by mathematicians.

2 An Alternative Integration Theory

Already in 1949, after attending a Feynman's lecture at Cornell University, Marc Kac realized that Feynman's ideas admit a rigorous formalization by replacing Schrödinger equation (1) with the heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x), \\ u(0,x) = u_0(x). \end{cases}$$

Indeed, in this case, under suitable assumptions on the potential V and the initial datum u_0 [16–18], it is possible to prove the following representation for the solution, given in terms of an integral over the space of continuous path with respect to the Wiener probability measure

$$u(t,x) = \int_{C_t} u_0(\gamma(t)+x)e^{-\int_0^t V(\gamma(s)+x)ds} dW(\gamma).$$

This functional integral representation for the solution of the heat equation is the celebrated *Feynman-Kac formula*, which nowadays can be regarded as the first and most famous example of an extensively developed theory [19, 20] connecting parabolic equations associated to second order elliptic operators

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}Tr[\sigma(x)\sigma^*(x)D_x^2 u(t,x)] + \langle b(x), D_x u(t,x)\rangle + V(x)u(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{6}$$

to stochastic Markov processes $X^x = (X_t^x)_{t\geq 0}$, solutions of the stochastic differential equations

$$\begin{cases} dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \\ X(0) = x, \quad x \in \mathbb{R}^d, \end{cases} \tag{7}$$

with $(W_t)_{t\geq 0}$ being a d-dimensional Wiener process. Indeed, if $\sigma : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz maps the solution of (6) can be represented by the following formula (see [19, 20])

$$u(t,x) = \mathbb{E}\left[u(0, X_t^x)e^{-\int_0^t V(X_s^x)ds}\right], \quad t \geq 0, x \in \mathbb{R}^d. \tag{8}$$

In 1960 Cameron [21] pointed out that it is impossible to provide for the solution of the Schrödinger equation a representation similar to (8), given in terms of a Lebesgue -type integral over the space of paths $\gamma : [0,t] \to \mathbb{R}^d$ with respect to a suitable *Feynman complex measure*. In order to gain a deeper insight into this no-go result, it is useful to look at the proof of Feynman-Kac formula [6] and check which arguments fail when one replaces the heat equation with the Schödinger equation.

Let us consider the C_0-semigroup $T^\alpha(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), t \geq 0$ generated by the self-adjoint operator $A = -\alpha\Delta : D(A) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, with $\alpha \in \mathbb{C}$ is a complex non-vanishing constant such that $\mathrm{Re}(\alpha) \geq 0$ and $D(A) = \{\psi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \|k\|^4 |\hat{\psi}(k)|^2 dk < \infty\}$, where $\hat{\psi}$ stands for the Fourier transform of $\psi \in L^2(\mathbb{R}^d)$. Let us denote $M_V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the multiplication operator defined on the vectors $u \in C_0^\infty(\mathbb{R})$ by $M_V u(x) = \alpha V(x) u(x)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is a continuous bounded function. Let $A + M_V : D(A) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator sum and $T_V^\alpha(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the associated semigroup, written formally as $T_V^\alpha(t) = e^{-\alpha t(-\Delta+V)}$. By the Trotter product formula [11, 22], the perturbed semigroup is given by the strong $L^2(\mathbb{R})$-limit

$$T_V^\alpha(t)u = \lim_{n\to\infty} \left(e^{\frac{t}{n}\alpha\Delta} e^{-\alpha M_V \frac{t}{n}} \right)^n u, \quad u \in L^2(\mathbb{R}).$$

By taking a subsequence and using the explicit representation formula for the action of the semigroup $T^\alpha(t)$

$$T_t^\alpha u(x) = \int_{\mathbb{R}^d} K_t^\alpha(x, y) u(y) dy, \quad K_t^\alpha(x, y) = \frac{e^{-\frac{|x-y|^2}{2\alpha t}}}{(2\pi\alpha t)^{d/2}}, \tag{9}$$

we obtain for almost every $x \in \mathbb{R}^d$:

$$\begin{aligned} T_V^\alpha(t)u(x) &= \lim_{n\to\infty} \left(e^{-\alpha\frac{t}{n}(-\Delta)} e^{-\alpha M_V \frac{t}{n}} \right)^n u(x) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{dn}} u(x_0) e^{-\alpha\frac{t}{n}\sum_{j=1}^n V(x_j)} \Pi_{j=0}^{n-1} K_{t/n}^\alpha(x_j, x_{j+1}) dx_j \end{aligned} \tag{10}$$

where $x_n \equiv x$. If $\alpha = 1$ (or, more generally, if $\alpha \in \mathbb{R}^+$) then T_t^α is the heat semigroup and its kernel $K_t^\alpha(\cdot, \cdot)$ is the density of a Gaussian probability measure on $\mathbb{R}^d$. In this case Kolmogorov theorem [23] allows to interpret the last line of Eq. (10) as an integral on the space $(\mathbb{R}^d)^{[0,t]}$ with respect to a σ-additive probability measure. However, this interpretation is impossible if the constant α is not real and positive and has a non vanishing imaginary part. Indeed, let us consider the algebra $\mathcal{A}$ of the *cylinder sets* of the form

$$E_{J;B_1,\dots,B_n} := \{\gamma \in (\mathbb{R}^d)^{[0,t]} : \gamma(t_1) \in B_1, \dots, \gamma(t_n) \in B_n\},$$

with $n \in \mathbb{N}$, $J = \{t_1, \dots, t_n\} \subset [0, t]$ and $B_1, \dots, B_n$ belonging to the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Let μ^α be the finitely additive (complex) measure on $\mathcal{A}$ defined as:

$$\mu^\alpha(E_{J;B_1,\dots,B_n}) = \int_{B_1} \dots \int_{B_n} K^\alpha_{t_n-t_{n-1}}(x_{n-1},x_n)\dots$$
$$\dots K^\alpha_{t_2-t_1}(x_1,x_2)K^\alpha_{t_1}(x,x_1)dx_1\dots dx_n. \tag{11}$$

As remarked by Cameron [21], if $Im(\alpha) \neq 0$ then it is impossible to construct a σ-additive complex measure on the σ-algebra generated by $\mathcal{A}$ which extends μ^α, since its total variation would be infinite, even when restricted to bounded sets. Nowadays Cameron's pioneering result can be considered as a particular case of a theorem by Thomas [24] generalizing Kolmogorov theorem to the case of projective systems of complex (instead of probability) measures (see also [25] for a discussion of these issues in the framework of the mathematical Feynman path integration). Remarkably, Cameron's no-go result triggered a huge research activity on the mathematical definition of Feynman path integrals; starting from the 60s till today different approaches have been proposed. It fact a common feature to all of them is the replacement of the concept of Lebesgue type integral with respect to a σ-additive measure with the more general concept of linear continuos functional on a domain of *integrable functions*. This idea has been systematically implemented only rather recently in [25], where a general theory of *projective systems of functionals*, alternative to the projective systems of measures [26], has been developed. When applied to the particular case of Feynman path integration, the theory reduces to the construction of a linear continuous functional $L : D(L) \to \mathbb{C}$ defined on a domain $D(L)$ containing *cylinder functions*, i.e. the functions $f : (\mathbb{R}^d)^{[0,t]} \to \mathbb{C}$ of the form

$$f(\omega) := F(\omega(t_1),\dots,\omega(t_n)) \tag{12}$$

for some $n \in \mathbb{N}$, $t_1,\dots,t_n \in [0,t]$ and a Borel function F on $\mathbb{R}^d \times \dots \times \mathbb{R}^d$. The action of the functional L on a function f of the form (12) must be given by a (finite dimensional) integral of the form:

$$L(f) = \int_{\mathbb{R}^d\times\dots\times\mathbb{R}^d} F(x_1,\dots,x_n)K^\alpha_{t_n-t_{n-1}}(x_{n-1},x_n)\dots K^\alpha_{t_2-t_1}(x_1,x_2)$$
$$K^\alpha_{t_1}(x,x_1)dx_1\dots dx_n. \tag{13}$$

In fact most of the different approaches to the mathematical definition of Feynman path integrals fit into this context and differ essentially for the class of functions that can be integrated, i.e. the domain of the functional. We mention briefly only the main techniques proposed and refer to [27, 28] and references therein for an extended discussion. The first approach proposed by Cameron himself and further developed within the framework of stochastic analysis is the *analytic continuation of Wiener integrals* [21, 29–33]. A different proposal relies on the implementation of an infinite dimensional distribution theory, first introduced by C. Dewitt-Morette and later extensively developed in the framework of white noise calculus [34–40]. Particular efforts have also been devoted to the so-called *sequential approach*. On the one hand a systematic study of the convergence of the sequence (3) and (4) has been conducted [41, 42], on the other hand the convergence of the finite dimensional approximations of Feynman heuristic formula along particular sequences of finite dimensional subspaces of paths in the spirit of formula (5) has been extensively studied [43–51]. Other approaches relies, e.g., on the construction of complex Poisson measures [52–54] or on nonstandard analysis [55], but they haven't been extensively developed yet.

3 Infinite Dimensional Fresnel Integrals

In the 60s a couple of papers by Itô [56, 57] proposed a different approach, leading to the development of the theory of *infinite dimensional Fresnel integrals*. The main idea is a generalization of the classical Parseval equality

$$\int_{\mathbb{R}^n} \phi(x) f(x) dx = \int_{\mathbb{R}^n} \hat{\phi}(x) \hat{f}(x) dx, \tag{14}$$

where $\hat{\phi}$ and $\hat{f}$ denote the Fourier transform of the Schwartz test functions $\phi, f \in S(\mathbb{R}^n)$. Equality (14) can be extended to the case where $\phi(x) = (2\pi i\epsilon)^{-n/2} e^{\frac{i}{2\epsilon}\|x\|^2}$, where $\epsilon \in \mathbb{R}^+$ is a positive constant, while $f : \mathbb{R}^n \to \mathbb{C}$ is the Fourier transform of a complex bounded Borel measure μ_f on $\mathbb{R}^n$

$$f(x) = \int_{\mathbb{R}^n} e^{ikx} d\mu_f(k), \quad x \in \mathbb{R}^n.$$

In this case (14) assumes the following form

$$\int_{\mathbb{R}^n} \frac{e^{\frac{i}{2\epsilon}\|x\|^2}}{(2\pi i\epsilon)^{n/2}} f(x) dx = \int_{\mathbb{R}^n} e^{-\frac{i\epsilon}{2}\|x\|^2} d\mu_f(k). \tag{15}$$

where the left hand side has to be interpreted as an improper Riemann integral. The latter equality is the starting point for the generalization of Fresnel integrals to the case where $\mathbb{R}^n$ is replaced by a real separable infinite dimensional Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. Indeed, while the left hand side is no longer meaningful for infinite dimensional Hilbert spaces due to the lack of a reasonable Lebesgue measure dx, the right hand side still makes sense within Lebesgue traditional integration theory since it is the integral of a bounded continuous function with respect to a complex Borel measure with finite total variation.

Given a real separable Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$, let us consider the set $\mathcal{F}(\mathcal{H})$ of functions $f : \mathcal{H} \to \mathbb{C}$ of the form

$$f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y), \quad x \in \mathcal{H}, \tag{16}$$

for some complex Borel measure μ_f on $\mathcal{H}$ with finite total variation. Besides being a linear space, $\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions with unity, where the multiplication is simply defined as $fg(x) := f(x)g(x)$, $x \in \mathcal{H}$ while the norm $\|f\|_{\mathcal{F}}$ of $f \in \mathcal{F}(\mathcal{H})$ is defined as the total variation of the associated via (16) measure μ_f, i.e. $\|f\|_{\mathcal{F}} := |\mu_f|$.

The *infinite dimensional Fresnel integral* is the linear functional $I_F : \mathcal{F}(\mathcal{H}) \to \mathbb{C}$ defined on $f \in \mathcal{F}(\mathcal{H})$ of the form (16) as

$$I_F(f) := \int_{\mathcal{H}} e^{-\frac{i\epsilon}{2}\langle x,x\rangle} d\mu_f(x). \tag{17}$$

It is also denoted with the symbol $\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\epsilon}\|x\|^2} f(x)dx$.

By construction, the functional $I_F : \mathcal{F}(\mathcal{H}) \to \mathbb{C}$ enjoys some remarkable properties. First of all it is straightforward to prove the continuity in the $\mathcal{F}(\mathcal{H})$-norm since

$$\left| \int_{\mathcal{H}} e^{-\frac{i\epsilon}{2}\|x\|^2} d\mu_f(x) \right| \leq |\mu_f| = \|f\|_{\mathcal{F}}$$

Moreover it is invariant under Euclidean transformations on $\mathcal{H}$ and a Fubini-type theorem holds [27].

Proposition 1 *Let the group of Euclidean transformations $E(\mathcal{H})$ be the group of transformations $x \to Ox + a$, where $a \in \mathcal{H}$ and O is an orthogonal transformation of $\mathcal{H}$ onto $\mathcal{H}$. Then the space of Fresnel integrable functions $\mathcal{F}(\mathcal{H})$ is invariant under $E(\mathcal{H})$, and $E(\mathcal{H})$ is in fact a group of isometries of $\mathcal{F}(\mathcal{H})$. Moreover the infinite dimensional Fresnel integral is invariant under the transformations in $E(\mathcal{H})$.*

Proposition 2 *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the orthogonal sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For $f(x) \in \mathcal{F}(\mathcal{H})$ set $f(x_1, x_2) = f(x_1 \oplus x_2)$ with $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$. Then for fixed x_2, $f(x_1, x_2)$ is in $\mathcal{F}(\mathcal{H}_1)$ and*

$$g(x_2) = \widetilde{\int_{\mathcal{H}_1}} e^{\frac{i}{2\epsilon}\|x_1\|^2} f(x_1, x_2)dx_1 \tag{18}$$

is in $\mathcal{F}(\mathcal{H}_2)$. Moreover

$$\widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2\epsilon}\|x_2\|^2} g(x_2)dx_2 = \widetilde{\int_{\mathcal{H}_2}} e^{\frac{i}{2\epsilon}\|x_2\|^2} \left(\widetilde{\int_{\mathcal{H}_1}} e^{\frac{i}{2\epsilon}\|x_1\|^2} f(x_1, x_2)dx_1 \right) dx_2$$
$$= \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\epsilon}\|x\|^2} f(x)dx \ .$$

A systematic implementation of the theory of infinite dimensional Fresnel integrals as long as their applications to the mathematical definition of Feynman path integrals has been presented in the first edition of the pioneering book [27] and in the fundamental paper [58] by Sergio Albeverio and Raphael Høegh-Krohn. The main application of this new infinite dimensional integral is the representation of the solution of Schrödinger equation (1). Let us consider the Cameron-Martin Hilbert space $\mathcal{H}_t$, i.e. the space of absolutely continuous paths $\gamma : [0, t] \to \mathbb{R}^d$ with fixed end point $\gamma(t) = 0$ and weak derivative $\dot{\gamma} \in L^2(\mathbb{R}^d)$, endowed with the inner product

$$\langle \gamma, \eta \rangle = \int_0^t \dot{\gamma}(s)\dot{\eta}(s)ds, \quad \gamma, \eta \in \mathcal{H}_t.$$

Under suitable conditions on the potential V and the initial datum ψ_0, the solution of the Schrödinger equation (1) admits a representation in terms of an infinite dimensional Fresnel integral on the Cameron-Martin space $\mathcal{H}_t$ with parameter $\epsilon = \hbar$.

Theorem 1 *Let V and ψ_0 be Fourier transforms of bounded complex measures in $\mathbb{R}^d$ and let $\mathcal{H}_t$ be the Cameron-Martin Hilbert space. Then the map $f : \mathcal{H}_t \to \mathbb{C}$*

$$f(\gamma) = e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}\psi_0(\gamma(0) + x) \tag{19}$$

is in $\mathcal{F}(\mathcal{H}_t)$ and the solution of the Schrödinger equation (1) is given by the infinite dimensional Fresnel integral

$$\psi(t, x) = \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\|\gamma\|^2} f(\gamma)d\gamma. \tag{20}$$

In [27] the definition (17) of infinite dimensional Fresnel integral is generalized to the case where the phase function $\phi(x) = \frac{\|x\|^2}{2}$ is replaced by a quadratic form

$$x \mapsto \langle x, Bx \rangle, \quad x \in D(B) \tag{21}$$

associated to a densely defined symmetric linear operator $B : D(B) \subset \mathcal{H} \to \mathcal{H}$. Let us assume that there exists a dense subspace $D \subset \mathcal{H}$ containing the range of B and a symmetric bilinear form $\Delta(x, y)$ on $D \times D$ such that $Im(\Delta(x, x)) \leq 0$ and

$$\Delta(x, By) = \langle x, y \rangle, \quad \forall x \in D, y \in D(B). \tag{22}$$

Further, let us assume that D is a Banach space, endowed with a norm $\| \ \|_D$ stronger than the norm $\| \ \|$ of the Hilbert space $\mathcal{H}$, i.e. there exist a $c > 0$ such that $\|x\| \le c\|x\|_D$ for all $x \in D$, and we get the chain of continuous embeddings

$$D \subset \mathcal{H} \subset D^*.$$

Further, for any fixed $x \in D$ the map $y \mapsto \Delta(x, y)$ is an element of D^*. This actually gives by (22) a mapping from D into D^* which can be considered a left inverse of B. Let us eventually define the space $\mathcal{F}(D^*)$ of maps $f : D^* \to \mathbb{C}$ of the form

$$f(x) = \int_D e^{i\langle x,y\rangle} d\mu_f(y), \quad x \in D^*, \tag{23}$$

where $\langle x, y\rangle$ stands for the dual pairing between $x \in D^*$ and $y \in D$ and μ_f is a complex bounded Borel measure on D.
The *Fresnel integral with respect to* Δ is the linear functional on $\mathcal{F}(D^*)$ defined on $f \in \mathcal{F}(D^*)$ of the form (23) as:

$$\int^{\Delta} e^{\frac{i}{2}\langle x,Bx\rangle} f(x)dx := \int_D e^{-i\Delta(x,x)} d\mu_f(x). \tag{24}$$

Extension of these results to the case of polynomially growing phase functions have been developed in [59–61], where a path integral representation of high-order heat-type equations is also proved.

Besides the Schrödinger equation with harmonic oscillator potential, the main applications of this generalization of definition (17) can be found in field theory. Already in the 1976 edition of the book [27] a construction of the Feynman path integral for the relativistic quantum boson field is presented, both in the case of free field and in the presence of a regularized interaction term, constructed out of a map $V \in \mathcal{F}(\mathbb{R})$. These pioneering results haven't been developed later as they deserved, since they provided the first construction of the functional integral for the bosonic field in real time, regardless any Euclidean approach or analytic continuation procedure.

Another interesting application of the Fresnel integral can be found in Chern-Simons theory. In a fundamental paper published in 1989 [62] Witten conjectured that there should be a connection between topological field theories based on the so-called Chern-Simons action and topological invariants. Let M be a smooth 3-dimensional oriented manifold without boundary, and consider a compact Lie group G with its Lie algebra g endowed with a Ad-invariant inner product $\langle\, , \,\rangle$. Witten's calculations are based on a path integral formulation of Chern-Simons theory

$$I(f) = \int_{\Gamma} e^{i\Phi(A)} f(A) dA \tag{25}$$

where the integration is performed on a space Γ of g-valued connection 1-forms A on M, while the phase function Φ is the Chern-Simons action:

$$\Phi(A) = \frac{k}{2\pi} \int_{M} \Big(\langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \Big),$$

where k is a non-zero real constant, $[A, A]$ is the 2-form whose value on a pair of vectors (X, Y) is $2[A(X), A(Y)]$ and $\langle A \wedge B \rangle$, for a g-valued 1-form A and a $g-$valued 2-form B is the 3-form whose value on (X, Y, Z) is given by the skew-symmetrized form of $\langle A(X), B(Y, Z) \rangle$. According to Witten's conjecture the integral $I(f)$, for suitable f, should give topological invariants. The first contribution to the rigorous mathematical definition of integral (25) was presented by Albeverio and Schäfer in 1995 [63]. These ideas were later developed by Leukert and Schäfer and by Albeverio and Sengupta [64–66]. Recent results have been obtained by Albeverio and Mitoma [67].

4 Infinite Dimensional Oscillatory Integrals

One of the main issues present in the theory of infinite dimensional Fresnel integrals is the restriction on the class of integrable functions, i.e. the domain of the functional. Indeed, in the application to the mathematical definition of Feynman path integrals, the function f defined on the Cameron Martin space $\mathcal{H}_t$ by (19) must belong to $\mathcal{F}(\mathcal{H}_t)$. In order to fulfill this requirement, both the initial datum and the potential V are required to belong to $\mathcal{F}(\mathbb{R}^d)$, hence they are continuous and bounded. This condition is rather restrictive from a physical point of view, since it excludes most of the physically relevant potentials. In addition, aiming to the functional integral construction of non trivial integrating quantum field theories, the discussion of polynomially growing potentials would be of fundamental importance. It is thus important to provide an alternative definition of Fresnel integrals in infinite dimensions which allows to enlarge the domain of the functionals defined in the previous section. The first step to accomplish this task was done in the 80s by Elworthy and Truman [68]. Their idea was extensively developed later by Albeverio and Brzezniak [69–71], leading to the definition of *infinite dimensional oscillatory integrals*.

Oscillatory integrals on $\mathbb{R}^n$ are object of the following form

$$\int_{\mathbb{R}^n} e^{\frac{i}{\epsilon}\Phi(x)} f(x) dx, \tag{26}$$

where $\epsilon \in \mathbb{R} \setminus \{0\}$ is a real parameter, $\Phi : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{C}$ are Borel functions. Φ is usually called *phase function*, in the case where it is quadratic form the integrals (26) are called *Fresnel integrals*, while if $\Phi(x) = x^3$, $x \in \mathbb{R}$, they are called *Airy integrals* and find interesting applications in optics. A systematic study of these objects from a mathematical point of view, as long as the investigation of their asymptotic behavior when $\epsilon \downarrow 0$, has been developed by Hörmander [72, 73] and by Duistermaat [12]. Remarkably, integrals of the form (26) can be defined even in the case where the function f is not summable, hence the integral (26) does not make sense within Lebesgue integration theory. The following definition goes back to Hörmander (see also [68]) and relies on the construction of a sequence of regularized (hence absolutely convergent) integrals.

Definition 1 Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel function and $\Phi : \mathbb{R}^n \to \mathbb{R}$ a phase function. If for each Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the integrals

$$I_\delta(f, \phi) := \int_{\mathbb{R}^n} e^{i\frac{\Phi(x)}{\epsilon}} f(x)\phi(\delta x)dx$$

exist for all $\epsilon > 0$, $\delta > 0$ and $\lim_{\delta\to 0} I_\delta(f, \phi)$ exists and is independent of ϕ, then the limit is called the oscillatory integral of f with respect to Φ and denoted by

$$\int^o_{\mathbb{R}^n} e^{i\frac{\Phi(x)}{\epsilon}} f(x)dx \equiv I^{\frac{\Phi}{\epsilon}}(f). \tag{27}$$

According to the definition above it is possible, e.g., to define the oscillatory integral $\int^o_{\mathbb{R}^n} e^{i\frac{\Phi(x)}{\epsilon}} f(x)dx$ in the case where both Φ and f have arbitrary polynomial growth at infinity [72, 73].

The generalization of this integration technique to the case where $\mathbb{R}^n$ is replaced by a real separable infinite dimensional Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ relies on an approximation procedure. Indeed, an *infinite dimensional oscillatory integral* is defined as the limit of sequences of finite dimensional approximations (suitably normalized). We present here the definition in [69], where the particular case of a quadratic phase function $\Phi : \mathcal{H} \to \mathbb{R}$ is considered.

Definition 2 A function $f : \mathcal{H} \to \mathbb{C}$ is said to be Fresnel integrable if for any sequence $\{P_n\}_n$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to \mathbb{I}$ strongly as $n \to \infty$ ($\mathbb{I}$ being the identity operator in $\mathcal{H}$), the oscillatory integrals

$$\int^o_{P_n\mathcal{H}} e^{i\frac{\|P_nx\|^2}{2\epsilon}} f(P_nx)d(P_nx),$$

are well defined (in the sense of Definition 1) and the limit

$$\lim_{n\to\infty}(2\pi i\epsilon)^{-n/2}\int\limits_{P_n\mathcal{H}}^{o} e^{i\frac{\|P_nx\|^2}{2\epsilon}} f(P_nx)d(P_nx) \tag{28}$$

exists and is independent of the sequence $\{P_n\}_n$. In this case the limit is called infinite dimensional oscillatory integral of f and is denoted by

$$\widetilde{\int\limits_{\mathcal{H}}} e^{i\frac{\|x\|^2}{2\epsilon}} f(x)dx.$$

A complete characterization of the set of Fresnel integrable function is still an open problem, even in finite dimension, however it is possible to present interesting subclasses. In particular, it is possible to prove that if $f \in \mathcal{F}(\mathcal{H})$, i.e. it is of the form (16) then f is Fresnel integrable and

$$\widetilde{\int\limits_{\mathcal{H}}} e^{i\frac{\|x\|^2}{2\epsilon}} f(x)dx = \int\limits_{\mathcal{H}} e^{-i\epsilon\frac{\|x\|^2}{2}} d\mu_f(x).$$

This result shows that the infinite dimensional oscillatory integral (28) is actually an extension of the infinite dimensional Fresnel integral (17). Moreover, a stronger result can be proved [68, 69]

Theorem 2 *Let $L : \mathcal{H} \to \mathcal{H}$ be a selfadjoint trace-class operator, such that $I - L$ is invertible and let $f \in \mathcal{F}(\mathcal{H})$. Then the function $g : \mathcal{H} \to \mathbb{C}$ defined by*

$$g(x) = e^{-\frac{i}{2\epsilon}\langle x,Lx\rangle} f(x), \quad x \in \mathcal{H} \tag{29}$$

is Fresnel integrable and its infinite dimensional Fresnel integral is given by the following Parseval-type equality:

$$\widetilde{\int\limits_{\mathcal{H}}} e^{\frac{i}{2\epsilon}\langle x,(I-L)x\rangle} f(x)dx = (\det(I-L))^{-1/2}\int\limits_{\mathcal{H}} e^{-\frac{i\epsilon}{2}\langle x,(I-L)^{-1}x\rangle} d\mu_f(x) \tag{30}$$

where $\det(I-L)$ *is the Fredholm determinant of the operator $(I-L)$ (that is the product of the eigenvalues of $(I-L)$) and μ_f is the complex bounded Borel measure on $\mathcal{H}$ related to f by (16).*

These results appeared for the first time in the 80s and triggered a huge research activity in the 90s, leading to the rigorous mathematical definition and the study of a wider class of Feynman path integrals representations. It is worthwhile to mention the application to the Schrödinger equation with anharmonic oscillator potential of the form $V(x) = \frac{1}{2}xA^2x + \lambda|x|^4$, where $x \in \mathbb{R}^d$, A is $d \times d$ symmetric positive definite matrix and $\lambda > 0$ is a coupling constant. In fact it is possible to provide a well defined mathematical construction of the Feynman path integral representation

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{i}{\hbar}\int_0^t \frac{\dot\gamma(s)^2}{2}ds - \frac{i}{\hbar}\int_0^t [\frac{1}{2}\gamma(s)A^2\gamma(s)+\lambda|\gamma(s)|^4]ds}\phi(\gamma(0))d\gamma, \tag{31}$$

as the analytic continuation (in the parameter λ, from $\lambda < 0$ to $\lambda \geq 0$) of an infinite dimensional oscillatory integral on the Cameron Martin Space $\mathcal{H}_t$ [74–76].

This result relies on a generalization of Parseval-type equality (30) to the case where the quadratic phase function is replaced by a fourth order polynomial. The main idea is a deformation of the integration contour in the complex plane that allows to compute a finite dimensional oscillatory integral with polynomial phase function

$$\Phi(x) = \frac{1}{2}\|x\|^2 + \lambda\|x\|^4, \quad x \in \mathbb{R}^n \tag{32}$$

in terms of a Gaussian integral:

$$(2\pi i\epsilon)^{-n/2}\int^{o}_{\mathbb{R}^n} e^{\frac{i}{2\epsilon}\|x\|^2+i\lambda\|x\|^4}dx = (2\pi\epsilon)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{1}{2\epsilon}\|x\|^2-i\lambda\|x\|^4}dx = \mathbb{E}[e^{-i\lambda\epsilon^2\|x\|^4}], \tag{33}$$

where $\epsilon, \lambda > 0$ and the symbol $\mathbb{E}$ on the right hand side denotes the expectation with respect to the standard gaussian measure on $\mathbb{R}^n$. This rather elementary equality is the key tool for the construction of infinite dimensional oscillatory integrals with polynomial phase functions of the form (32) in the case $\mathbb{R}^n$ is replaced by in infinite dimensional real separable Hilbert space $\mathcal{H}$. Indeed, when the dimension n of the integration domain is allowed to converge to ∞, the left hand side of (33) is no longer meaningful because of the presence of the Lebesgue measure dx and of the normalization constant $(2\pi i\epsilon)^{-n/2}$. On the other hand, the right hand side of (33) is still meaningful in an infinite dimensional setting since Gaussian measures on infinite dimensional spaces do exist, contrary to the case of Lebesgue measure. For the detailed development of the theory of infinite dimensional oscillatory integrals with polynomial phase we refer to the original papers [76], where the theory of abstract Wiener spaces [77–79] plays a fundamental role in the extension of formula (33) to an infinite dimensional setting. When applied to the mathematical definition of heuristic Feynman formula (31), it allows to compute an infinite dimensional oscillatory integral on the Cameron-Martin Hilbert space $\mathcal{H}_t$ in terms of a Gaussian integral over the Banach space C_t of continuous path $\omega : [0,t] \to \mathbb{R}^d$ with respect to the Wiener measure $\mathbb{P}$. Let us consider on $L^2(\mathbb{R}^d)$ the anharmonic oscillator Hamiltonian operator H of the form:

$$H = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}xA^2x + \lambda C(x,x,x,x), \tag{34}$$

where C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$, A is a positive symmetric $d \times d$ matrix, $\lambda \geq 0$ a positive constant. We shall denote with A_i, $i = 1,\ldots,d$ the eigenvalues of the matrix A. It is well known [11] that H is

essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and generates a strongly continuous 1-parameter group of unitary operators $U(t) = e^{-\frac{i}{\hbar}Ht}$.
Let $(\mathcal{H}_t, \langle\, ,\, \rangle)$ denote the Hilbert space of absolutely continuous paths $\gamma : [0, t] \to \mathbb{R}^d$ with square integrable weak derivative $\int_0^t \dot\gamma(\tau)^2 d\tau < \infty$, fixed initial point $\gamma(0) = 0$ and inner product $\langle\gamma_1, \gamma_2\rangle = \int_0^t \dot\gamma_1(s)\dot\gamma_2(s)ds$. The cylindrical Gaussian measure on $\mathcal{H}_t$ with covariance operator the identity extends to a σ-additive measure on the Wiener space $C_t = \{\omega \in C([0, t]; \mathbb{R}^d) \mid \gamma(0) = 0\}$: the Wiener measure $\mathbb{P}$.

Theorem 3 *Let us assume that $\lambda \leq 0$, and that for each $i = 1, \ldots, d$ the following inequalities are satisfied*

$$A_i t < \frac{\pi}{2}, \quad 1 - A_i \tan(A_i t) > 0. \tag{35}$$

Let $\phi_1, \phi_2 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$, i.e.

$$\phi_2(x) = \int_{\mathbb{R}^d} e^{ix\cdot k} d\mu_2(k), \quad \phi_1(x) = (2\pi i\hbar)^{-d/2} e^{\frac{i}{2\hbar}|x|^2} \int_{\mathbb{R}^d} e^{ix\cdot k} d\mu_1(k).$$

Assume in addition that the measures μ_1, μ_2 satisfy the following assumption:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4}xA^{-1}\tan(At)x} e^{(y+\cos(At)^{-1}x)(1-A\tan(At))^{-1}(y+\cos(At)^{-1}x)} d|\mu_2|(x)d|\mu_1|(y) < \infty$$

where $|\mu|$ stands for the total variation measure of μ. Then the infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathbb{R}^d\times\mathcal{H}_t}} \bar\phi_1(x) e^{\frac{i}{2\hbar}\int_0^t \dot\gamma(\tau)^2 d\tau} e^{-\frac{i}{2\hbar}\int_0^t (\gamma(\tau)+x)A^2(\gamma(\tau)+x)d\tau}$$
$$e^{-\frac{i\lambda}{\hbar}\int_0^t C(\gamma(\tau)+x,\gamma(\tau)+x,\gamma(\tau)+x,\gamma(\tau)+x)d\tau} \phi_2(\gamma(t)+x)dxd\gamma \tag{36}$$

is well defined and is equal to

$$(i)^{d/2} \int_{\mathbb{R}^d\times C_t} e^{i\frac{\lambda}{\hbar}\int_0^t C(\sqrt{\hbar}\omega(\tau)+x,\sqrt{\hbar}\omega(\tau)+x,\sqrt{\hbar}\omega(\tau)+x,\sqrt{\hbar}\omega(\tau)+x)d\tau}$$
$$e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(\tau)+x)A^2(\sqrt{\hbar}\omega(\tau)+x)d\tau} \bar\phi(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t) + e^{i\pi/4}x)d\mathbb{P}(\omega)dx. \tag{37}$$

Moreover the absolutely convergent integral (37) is an analytic function of the complex variable λ if $Im(\lambda) > 0$, and continuous in $Im(\lambda) = 0$. In particular, when $\lambda \geq 0$ it represents the scalar product between ϕ_1 and the solution of the Schrödinger equation (1) with Hamiltonian (34) and initial datum ϕ_2.

Similar techniques allow to study the case where the parameter λ attains negative values and the quantum Hamiltonian is no longer essentially-selfadjoint [80].
A further generalization of the Definition 2 was proposed by Albeverio and Brzezniak in [71]. It involves a different choice of the normalization constant appearing in the finite dimensional approximations (28) and finds an interesting application in the construction of the Feynman path integral representation of the solution of (1) in the case where a constant magnetic field is present.

The result of Theorem 2, i.e. the Parseval-type equality (30), holds under the assumption that the phase function $\Phi : \mathcal{H} \to \mathbb{R}$ is a quadratic form $\Phi(x) = \langle x, (I - L)x\rangle$ associated to a self-adjoint *trace class* operator $L : \mathcal{H} \to \mathcal{H}$, in such a way that the Fredholm determinant $\det(I - L)$ is well defined. In order to cope with the cases where this condition is not fulfilled, the definition of *class p oscillatory integral* was introduced [71]. Let $p \in \mathbb{N}$ and consider the Schatten class $\mathcal{T}_p(\mathcal{H})$ of bounded linear operators L in $\mathcal{H}$ such that [81]

$$\|L\|_p = (\mathrm{Tr}(L^*L)^{p/2})^{1/p} < +\infty.$$

In fact $(\mathcal{T}_p(\mathcal{H}), \|\cdot\|_p)$ is a Banach space. For any $p \in \mathbb{N}$ it is possible to define the regularized Fredholm determinant $\det_{(p)} : I + \mathcal{T}_p(\mathcal{H}) \to \mathbb{R}$ as:

$$\det_{(p)}(I + L) := \det\left((I + L)\exp\sum_{j=1}^{p-1}\frac{(-1)^j}{j}L^j\right), \quad L \in \mathcal{T}_p(\mathcal{H}),$$

where det denotes the usual Fredholm determinant, which is well defined since the operator $(I + L)\exp\sum_{j=1}^{p-1}\frac{(-1)^j}{j}L^j - I$ is trace class provided that $L \in \mathcal{T}_p(\mathcal{H})$ [81]. In particular $\det_{(2)}$ is called Carleman determinant.
For $p \in \mathbb{N}$, $p \geq 2$, $L \in \mathcal{T}_1(\mathcal{H})$, let us define the normalized quadratic form $N_p : \mathcal{H} \to \mathbb{C}$:

$$N_p(L)(x) := \langle x, Lx\rangle - i\hbar\mathrm{Tr}\sum_{j=1}^{p-1}\frac{L^j}{j}, \quad x \in \mathcal{H}. \tag{38}$$

Definition 3 Let $p \in \mathbb{N}$, $p \geq 2$, $L : \mathcal{H} \to \mathcal{H}$ a bounded linear operator in $\mathcal{H}$, $f : \mathcal{H} \to \mathbb{C}$ a Borel measurable function. If for any sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to I$ strongly as $n \to \infty$ (I being the identity operator in $\mathcal{H}$) the finite dimensional oscillatory integrals

$$\widetilde{\int_{P_n\mathcal{H}}^{o}} e^{\frac{i}{2\hbar}|x|^2} e^{-\frac{i}{2\hbar}N_p(P_nLP_n)(P_nx)} f(P_nx)d(P_nx), \tag{39}$$

are well defined and the limit

$$\lim_{n\to\infty} \widetilde{\int_{P_n\mathcal{H}}^{\circ}} e^{\frac{i}{2\hbar}|x|^2} e^{-\frac{i}{2\hbar}N_p(P_nLP_n)(P_nx)} f(P_nx)d(P_nx) \tag{40}$$

exists and is independent on the sequence $\{P_n\}$, then the limit is called the *class p normalized oscillatory integral* of the function f with respect to the operator L and it is denoted

$$\mathcal{I}_{p,L}(f) = \widetilde{\int_{\mathcal{H}}^{p}} e^{\frac{i}{2\hbar}|x|^2} e^{-\frac{i}{2\hbar}(x,Lx)} f(x)dx.$$

If L is not a trace class operator, then the quadratic form (38) is not well defined, nevertheless expression (39) still makes sense thanks to the fact that all the functions under the integral are restricted to finite dimensional subspaces. In this setting the following generalization of Parseval-type equality (30) can be proved, valid for functions $f \in \mathcal{F}(\mathcal{H})$ and for self-adjoint operators $L \in \mathcal{T}_p(\mathcal{H})$ such that $\det_{(p)}(I-L) \neq 0$:

$$\widetilde{\int_{\mathcal{H}}^{p}} e^{\frac{i}{2\hbar}|x|^2} e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x)dx = [\det_{(p)}(I-L)]^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}x\rangle} d\mu_f(x). \tag{41}$$

This result finds application in the construction of the Feynman path integral representation of the solution of the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = \frac{1}{2}(-i\hbar\nabla + a(x))^2\psi + V_0(x)\psi \\ \psi(0,x) = \phi(x), \quad x \in \mathbb{R}^d, t > 0 \end{cases} \tag{42}$$

describing the dynamics of a nonrelativistic charged quantum particle (with unitary charge and mass) moving in under the action of an external magnetic field. In (42) $V_0 \in \mathcal{F}(\mathbb{R}^d)$ is a scalar potential while $a(x) = Cx$ is a linear vector potential, with C an anti-self-adjoint linear operator in $\mathbb{R}^d$. Let us consider the operator $L : \mathcal{H}_t \to \mathcal{H}_t$ defined on the Cameron-Martin Hilbert space $\mathcal{H}_t$ by

$$\langle L\gamma_1, \gamma_2\rangle = \int_0^t (C\gamma_1(\tau)\cdot\dot{\gamma}_2(\tau) + C\gamma_2(\tau)\cdot\dot{\gamma}_1(\tau))\,d\tau, \quad \gamma_1,\gamma_2 \in \mathcal{H}_t$$

It is possible to prove (see [71] for details) that L is self-adjoint and belongs to the Hilbert-Schmidt class $\mathcal{T}_2(\mathcal{H}_t)$ and $\det_{(p)}(I-L) \neq 0$ if $\sin(t\sqrt{C^*C}) \neq 0$. By assuming that the initial datum ϕ belongs to $\mathcal{F}(\mathbb{R}^d)$, the heuristic Feynman path integral representation for the solution of the Schrödinger equation (42)

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{i}{2\hbar}\int_0^t \dot{\gamma}(\tau)^2 d\tau - \frac{i}{\hbar}\int_0^t C(\gamma(\tau)+x)\dot{\gamma}(\tau)d\tau - \frac{i}{\hbar}\int_0^t V_0(\gamma(\tau)+x)d\tau} \phi(\gamma(0)+x)d\gamma$$

can be rigorously mathematically realized in terms of the class-2 normalized integral:

$$\psi(t,x) = \widetilde{\int_{\mathcal{H}_t}}^{2} e^{\frac{i}{2\hbar}\int_0^t \dot\gamma(\tau)^2 d\tau + \frac{i}{2\hbar}(L\gamma,\gamma)} e^{-\frac{i}{\hbar}\int_0^t V_0(\gamma(\tau)+x)d\tau} \phi(\gamma(t)+x) d\gamma$$

For recent results on the case on non-constant magnetic field see [82].

5 Further Applications and Open Issues

This section eventually presents some further applications of the theory developed so far, as well as some open problems.

5.1 The Stationary Phase Method an the Study of the Semiclassical Limit of Quantum Mechanics

A systematic implementation of an infinite dimensional version of the classical stationary phase method [12, 13] is presented in [58], allowing for the study of the detailed asymptotic behavior of the Fresnel-Feynman integral (20) in the limit when $\hbar \downarrow 0$. Let $I : \mathbb{R}^+ \to \mathbb{C}$ be the map defined by

$$I(\hbar) := \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\|x\|^2} e^{-\frac{i}{\hbar}V(x)} g(x) dx,$$

where $\mathcal{H}$ is a real separable Hilbert space and $V, g \in \mathcal{F}(\mathcal{H})$, i.e.

$$V(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu(y), \quad g(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\nu(y).$$

By assuming that the measures μ, ν admit finite moments of any order and if the phase function $\Phi(x) = \frac{1}{2}\|x\|^2 - V(x)$, $x \in \mathcal{H}$, has only a finite number of non-degenerate critical points, then it is possible to prove [58, 71] that I is a C^∞ function of the variable $\hbar$ and to compute explicitly all terms of its asymptotic expansion for $\hbar \downarrow 0$. These depend only on the derivatives of V and g at the critical points. Furthermore, under additional assumptions on the moments of μ and ν it is even possible to prove the Borel summability of the asymptotic expansion [83]. When applied to the representation of the solution of Schrödinger equation, i.e. to formula (20), these results provide an alternative derivation of Maslov's classical results on the semiclassical asymptotics for the solution of Schrödinger equation [84]. Another interesting application of these techniques can be found in the study of the semiclassical asymptotic for the trace of the Schrödinger group [85–88]. These results actually provide a rigorous mathematical realization of the heuristic Gutzwiller trace formula [89], which is particularly relevant in the theory of quantum chaos. For analogous results in the case of the heat equation see [90, 91].

5.2 Phase Space Feynman Path Integrals

Even if Feynman's original aim was to provide a Lagrangian formulation of quantum mechanics, soon in the physical literature Hamiltonian versions of formula (2) appeared (see, e.g. [1]). *Phase space Feynman path integrals* representations are heuristic formulas of the following form

$$\text{“}\psi(t,x) = \text{const} \int_{q(t)=x} e^{\frac{i}{\hbar}S_t(q,p)}\phi(q(0))dq\,dp\text{”}, \tag{43}$$

where the integral is meant on the space of paths $(q, p) : [0, t] \to \mathbb{R}^d \times \mathbb{R}^d$ in the phase space of the system ($q : [0, t] \to \mathbb{R}^d$ is the path in configuration space and $p : [0, t] \to \mathbb{R}^d$ is the path in momentum space). S_t is the action functional in the Hamiltonian formulation:

$$S_t(q,p) = \int_0^t (\dot{q}(\tau)p(\tau) - H(q(\tau), p(\tau)))d\tau,$$

(H being the classical Hamiltonian of the system). Different mathematical definitions of formula (43) have been proposed [33, 35]. In [92] the theory of infinite dimensional oscillatory integrals is applied. Let us consider the Hilbert space $\mathcal{H} = \mathcal{H}_t \times L_t$, with $\mathcal{H}_t$ the Cameron-Martin space and $L_t = L^2([0, t])$, endowed with the natural inner product

$$\langle (q_1, p_1), (q_2, p_2)\rangle = \int_0^t \dot{q}_1(\tau)\dot{q}_2(\tau)d\tau + \int_0^t p_1(\tau)p_2(\tau)d\tau, \quad (q_1, p_1), (q_2, p_2) \in \mathcal{H}.$$

Let $B : \mathcal{H} \to \mathcal{H}$ be the operator defined by

$$B(q,p)(\tau) = \left(\int_t^\tau p(u)du, \dot{q}(\tau) - p(\tau) \right), \tag{44}$$

with the corresponding quadratic form

$$\langle (q_1, p_1), B(q_2, p_2)\rangle = \int_0^t \dot{q}_1(\tau)p_2(\tau)d\tau + \int_0^t p_1(\tau)\dot{q}_1(\tau)d\tau - \int_0^t p_1(\tau)p_2(\tau)d\tau.$$

Let us consider the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V_1(x)\psi(t,x) + V_2(-i\hbar\nabla)\psi(t,x) \\ \psi(t,x) = \phi(x) \end{cases} \tag{45}$$

with V_1, V_2 bounded functions. Given the function $f : \mathcal{H} \to \mathbb{C}$ of the form

$$f(q,p) := \phi(x+q(0))e^{-\frac{i}{\hbar}\int_0^t V(q(s)+x,p(s))ds}, \quad (q,p) \in \mathcal{H},$$

it is possible to prove (see [92] for details) that for any sequence $\{P_n\}$ of finite dimensional projectors converging strongly to the identity the limit

$$\lim_{n\to\infty}(\det(P_n B P_n))^{1/2} \widetilde{\int_{P_n\mathcal{H}}^{o}} e^{\frac{i}{2\hbar}\langle P_n(q,p), BP_n(q,p)\rangle} f \circ P_n(q,p) dP_n(q,p)$$

exists and is independent of $\{P_n\}$. Further, it provides the solution of (45).

5.3 The Stochastic Schrödinger Equation

Infinite dimensional oscillatory integrals can find interesting applications also in the quantum theory of open systems, in particular in the theory of continuous quantum measurements. The continuous time evolution of the state of a quantum particle described by the Schrödinger equation (1) is valid if the system is isolated. However, if it interacts with an external environment, in particular if the system is submitted to the measurement of one of its observables and interacts with the measuring apparatus, then the time evolution is no longer continuous because of the so-called *collapse of the wave function*, which is actually a random and discontinuous transition. Different mathematical description of this phenomena have been proposed and nowadays the theory of quantum measurements is an active area of research where physics, mathematics as well as philosophy of science find a fruitful interplay. In this framework a class of stochastic Schrödinger equations have been proposed. In particular, Belavkin-Schrödinger equation [93]

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}R^2\psi(t,x)dt + \sqrt{\lambda}R\psi(t,x)dW(t) \\ \psi(0,x) = \phi(x) \quad (t,x) \in [0,T]\times\mathbb{R}^d \end{cases} \tag{46}$$

describes the time evolution of the state of a quantum particle submitted to the measurement of one of its (M-dimensional vector) observables, described by the selfadjoint operator R on $L^2(\mathbb{R}^d)$. In Eq. (46) the symbol H stands for the quantum mechanical Hamiltonian operator, W is an $M-$dimensional Brownian motion and dW denotes the Itô differential, $\lambda > 0$ is a coupling constant which is proportional to the accuracy of the measurement. In the case of position measurement, i.e. if $R = X$, the position operator, equation (46) becomes

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}x^2\psi(t,x)dt + \sqrt{\lambda}x\psi(t,x)dW(t) \\ \psi(0,x) = \phi(x) \quad (t,x) \in [0,T]\times\mathbb{R}^d, \end{cases} \tag{47}$$

while in the case of momentum measurement, i.e. $R = -i\hbar\nabla$, we obtain:

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt + \frac{\lambda\hbar^2}{2}\Delta\psi(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi(t,x)dW(t) \\ \\ \psi(0,x) = \psi_0(x) \quad (t,x) \in [0,T]\times\mathbb{R}^d. \end{cases} \tag{48}$$

One of the first heuristic Feynman path integral formulas describing selective dynamics of a particle whose position is continuously observed was proposed by Mensky [94, 95], who suggested that the state of the particle at time t if the observed trajectory is the path $\omega(s)_{s\in[0,t]}$ should be given by the "restricted path integrals"

$$\psi(t,x,\omega) = \text{“}\int_{\{\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} e^{-\lambda\int_0^t(\gamma(s)-\omega(s))^2ds}\phi(\gamma(0))D\gamma\text{”}, \tag{49}$$

where $\phi \in L^2(\mathbb{R}^d)$ is the initial state of the system, S_t is the action functional and $\lambda > 0$ a real positive parameter. In fact, according to formula (49) as an effect of the correction term $e^{-\lambda\int_0^t(\gamma(s)-\omega(s))^2ds}$ due to the measurement, the paths γ giving the main contribution to the integral (49) should be those closer to the observed trajectory ω.

The first construction of the solution of Belavkin equation (46) in terms of infinite dimensional oscillatory integral was proposed by Albeverio, Kolokoltsov and Smolyanov in [96, 97] and by Truman and Zastawniak [98] (see also [54]). These results were further developed by Albeverio, Guatteri and Mazzucchi in [99] in the case of Belavkin position equation (47) and later for Belavkin momentum equation by means of phase-space Feynman path integrals [100] (see Sect. 5.2). The main tool to accomplish this task is a generalization of Parseval type equality (30) to the case of complex valued phase functions. This allows to handle the cases where the Feynman formula is required to describe a dynamics which is no longer unitary, as in the case of quantum open systems (see [101]).

Given a real separable Hilbert space $\mathcal{H}$, a vector $y \in \mathcal{H}$, a map $f \in \mathcal{F}(\mathcal{H}_t)$ and two self-adjoint trace class operators L_1, L_2 such that $(I + L_1)$ is invertible and L_2 is nonnegative, it is possible to prove the following result

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle} e^{\langle y,x\rangle} f(x)dx = \det(I+L)^{-1/2}\int_{\mathcal{H}} e^{\frac{-i\hbar}{2}\langle k-iy,(I+L)^{-1}(k-iy)\rangle} d\mu_f(k), \tag{50}$$

where $L = L_1 + iL_2$.

The left hand side of (50) is defined as the limit of sequences of finite dimensional approximations as in (28), while the right hand side is an absolutely convergent integral with respect to the finite complex Borel measure μ_f which is related to the map f via (16). Eventually, $\det(I + L)$ denotes the Fredholm determinant of the operator $(I + L)$.

By considering Schrödinger-Belavkin position equation (47), it is possible to prove (see [99] for details) that there exists a strong solution which admits a infinite dimensional oscillatory integral representation of the form

$$\begin{aligned}\psi(t,x) &= \widetilde{\int} e^{\frac{i}{\hbar}S_t(\gamma)-\lambda\int_0^t(\gamma(s)+x)^2ds} e^{\int_0^t\sqrt{\lambda}(\gamma(s)+x)dW(s)ds}\phi(\gamma(0)+x)d\gamma \\ &= e^{-\lambda|x|^2t+\sqrt{\lambda}x\cdot\omega(t)}\widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} e^{-2\lambda\hbar\int_0^t x\cdot\gamma(s)ds} \\ &\quad\cdot e^{-i\int_0^t V(x+\gamma(s))ds}\phi(\gamma(0)+x)d\gamma\end{aligned}$$

where $\mathcal{H}_t$ is the Cameron-Martin Hilbert space, $l \in \mathcal{H}_t$, $l(s) = \sqrt{\lambda}\int_s^t \omega(\tau)d\tau$ and $L = iL_2$, with

$$L_2 : \mathcal{H}_t \to \mathcal{H}_t, \quad (\gamma_1, L\gamma_2) = -2i\lambda\hbar\int_0^t \gamma_1(s)\gamma_2(s)ds.$$

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Gauge Theories in Low Dimensions: Reminiscences of Work with Sergio Albeverio

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Abstract This is an expository account of the author's works influenced by Sergio Albeverio. Much of it focuses on gauge theories in two and three dimensions.

Keywords Gauge theories · Quantum fields · Gaussian measure · Yang-Mills theory · Chern-Simons theory · White noise

AMS Classification: 81T13 · 58J28

1 Introduction

This paper is a personal look back at some of the work I have done with Sergio Albeverio and work that was heavily influenced by him. Though written as a personal recollection, I hope the article will be of some interest to new researchers in the field who might be interested in the background to the results of some of that work. I do not intend this to be a standard mathematical paper with precisely stated theorems and rigorous proofs; indeed most ideas are stated in the way we typically 'understand' or view mathematical results, which is quite different from the way we formally state them. Technical accounts are available in the references.

This article is focused on two and three-dimensional gauge theories, but Sergio Albeverio's influence on me is seen in other published works, such as [2, 9], as well as unpublished work.

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2 The Chern-Simons Functional Integral

In August 1995, I arrived at the Ruhr University in Bochum to work as a Humboldt Fellow in Professor Sergio Albeverio's research group. The year 1995–1996 was a very productive year for me and had a permanent impact on my interests and work, with Sergio's influence playing a determinative role for decades to come. That year I was also influenced by interactions with many other researchers in his group, including Claas Becker, Alexei Daletskii, Astrid Hilbert, Pavel Kurasov, Konstantin Makarov, Barbara Rüdiger, Victoria Steblovskaya, and numerous others.

On my first day at Bochum, Sergio introduced me to the problem of giving a mathematically rigorous definition of the Chern-Simons functional integral. I had just completed a work with my colleagues at LSU on infinite-dimensional distributions, and Sergio suggested we try using some of the technology from white noise distribution theory to construct the Chern-Simons integral. He shared with me work he had done with his former student Schäfer [7], using Fresnel integrals to define Chern-Simons functional integrals, and pointed to a paper by Fröhlich and King [26].

2.1 The Chern-Simons Action

For our discussion here we will work over three-dimensional Euclidean space $\mathbb{R}^3$ and a compact Lie group G whose Lie algebra $L(G)$ is equipped with an Ad-invariant metric $\langle\cdot,\cdot\rangle$. For our purposes here, a *connection* A over $\mathbb{R}^3$ is a smooth 1-form on $\mathbb{R}^3$ with values in the Lie algebra $L(G)$ of a compact Lie group. We take the *Chern-Simons action* to be given by

$$\mathrm{CS}(A) = \int_{\mathbb{R}^3} \mathrm{cs}(A), \tag{1}$$

where

$$\mathrm{cs}(A) = \langle A \wedge dA\rangle + \frac{1}{3}\langle A \wedge [A, A]\rangle. \tag{2}$$

The notation needs some unpacking. Let A and C be $L(G)$-valued 1-forms, and B an $L(G)$-valued 2-form. For convenience let $\{E_j\}$ be a basis of $L(G)$; then

$$[A, C](X, Y) = [A(X), C(Y)] - [C(Y), A(X)], \tag{3}$$

for all tangent vectors X, Y to $\mathbb{R}^3$, and

$$\langle A \wedge B\rangle = \sum_{j,k} A_j \wedge B_k \langle E_j, E_k\rangle, \tag{4}$$

where we have used the basis $\{E_j\}$ to write $A = \sum_j A_j E_j$ and $B = \sum_k B_k E_k$.

2.2 Some Background

What is the origin of the formula (2)? Let us take a historical detour with this as our initial and final point.

The classical Gauss-Bonnet formula says that for a geodesic triangle on a surface, the integral of the curvature over the triangle is equal to the excess of the sum of the angles over π. This result is a consequence of Gauss's 1825 definition of surface curvature [27]. Gauss defined the curvature of a piece of a surface $\Sigma \subset \mathbb{R}^3$ as the area of the image of this piece on the unit sphere S^2 under a smooth map $\nu \colon \Sigma \to S^2$, which associates to each point p a unit normal $\nu(p)$ to the surface at p; this is the immediate counterpart of the curvature of an arc in $\mathbb{R}^2$. Combined with Thomas Harriot's observation about the area of a geodesic triangle on S^2 being the excess of the sum of the angle of the triangle over π, this leads to the Gauss-Bonnet formula. A formal proof uses an application of Stokes' theorem, with the curvature being viewed as the differential of an angle-valued 1-form (a kind of 'potential' if one thinks of the curvature as somehow representing 'field strength'). Triangulating a surface and adding up the angle-excesses over all the triangles leads to the Gauss-Bonnet formula: the integral of the curvature of a compact oriented closed surface is 2π times the Euler characteristic of the surface.

The natural challenge from the Gauss-Bonnet formula is to generalize it to higher dimensions. This led eventually to Chern's generalization [14, 15] of the Gauss-Bonnet formula (with earlier work of Allendoerfer and Weil [12]): the integral of the Pfaffian of the curvature form on an even dimensional compact oriented manifold is a multiple of the Euler characteristic of the manifold. In Chern's proof the the strategy of finding a potential for the curvature appears in a highly sophisticated form.The role of the potential is played by $\mathrm{cs}(A)$:

$$d\,\mathrm{cs}(A) = \langle F^A \wedge F^A \rangle, \tag{5}$$

where F^A is the curvature 2-form

$$F^A = \mathrm{d}A + [A \wedge A]. \tag{6}$$

The Chern-Weil form $\langle F^A \wedge F^A \rangle$ is invariant under *gauge transformations*

$$A \mapsto g^{-1}Ag + g^{-1}\mathrm{d}g,$$

where $g \colon \mathbb{R}^3 \to G$ is any smooth mapping. However, $\mathrm{cs}(A)$ lacks this symmetry. As in the case of the Gauss-Bonnet formula, the 'potential' for the surface curvature involves an 'angle' variable. In the more sophisticated setting, the potential for the Chern-Simons form $\mathrm{cs}(A)$ lives not on the base manifold but on a bundle over this base manifold.

For an expository account with more details on the Chern-Simons action from the point of view of a classical dynamical system, see [48] and references therein.

2.3 Functional Integrals

In the Feynman path integral approach to quantum mechanics, time evolution of the state of a system is described by integrating against quantities of the form

$$\int_{\mathcal{C}} e^{\frac{i}{\hbar} S(\gamma)} \, \mathcal{D}\gamma,$$

where γ is a path describing the evolution of the system, from an initial state to a final, and $S(\cdot)$ is the action function that governs the classical dynamics of the system. The integration is over the infinite-dimensional space $\mathcal{C}$ of all possible paths γ. The classical dynamics emerges from the quantum as a limit, for small values of the Planck's constant $\hbar$ (scaled by 2π), in which case the integral focuses on the extremals of the classical action S. Although enormously useful, both conceptually and in some cases computationally, Feynman integrals are notoriously difficult to define rigorously. Even when rigorously definable, the rigorous version isn't always of much use. Nonetheless, it is a mathematical challenge to give meaning to such infinite-dimensional integrals.

Witten's paper [50] linked Chern-Simons functional integrals

$$\int_{\mathcal{A}} e^{\frac{ik}{4\pi} \mathrm{CS}(A)} f(A) \, \mathcal{D}A, \tag{7}$$

to the Jones polynomial in knot theory. (See also the short lecture monograph by Atiyah [11].) Following this work, there was an explosion of interest in both the Chern-Simons functional integral and knot theory. The integral (7) is, of course, a formal object lacking a precise mathematical definition. While quantum physics shows that such ill-defined mathematical quantities or entities can be enormously useful and deep, it is a mathematical challenge to give rigorous meaning to such integrals. For more on such integrals see the book by Albeverio et al. [6].

2.4 White Noise Distributions in Brief

The key notion we used in [8] to define integrals of the type (7) is the notion of an infinite-dimensional distribution. Thus, instead of (7) being an actual integral we view it as the evaluation of a distribution on a 'test function' f. The realization of such infinite-dimensional distributions in terms of a Gaussian measure background is the subject of white noise distribution theory, developed by Hida, Kuo, Potthoff, Streit, and many others. A self-contained and readable account is available in the book by Kuo [33]. Here we will discuss just the general ideas.

Let H_0 be a real separable Hilbert space. In the theory of Abstract Wiener spaces one constructs a Banach space B, in which H_0 is densely embedded, and a Gaussian measure on the space B. This theory was founded by Gross [28] and provides a standard framework for infinite-dimensional analysis (Kuo's lecture note [32] was the first expository introduction to the fundamental notions and results in this subject). The white noise framework is different, in that instead of the Banach space B a *nuclear space* $\mathcal{H} \subset H_0$ is used, with the Gaussian measure μ being defined on the dual space $\mathcal{H}'$. For our purposes, let us just think of $\mathcal{H}$ as a topological vector space, dense as a subset of H_0, and μ as the Borel measure on $\mathcal{H}'$ specified by the requirement that

$$\int_{\mathcal{H}'} e^{\lambda\phi(x)}\, d\mu(\phi) = e^{\frac{\lambda^2}{2}|x|_0^2}, \tag{8}$$

where $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$. The case of interest is where the topology on $\mathcal{H}$ comes from a family of inner-products $\langle\cdot,\cdot\rangle_p$, for $p \in \{0, 1, \ldots\}$. This structure leads to a topological vector space $[\mathcal{H}]$, and its dual $[\mathcal{H}]'$, where

$$[\mathcal{H}] \subset L^2(\mathcal{H}', \mu)$$

is viewed as the space of infinite-dimensional test functions over the space $\mathcal{H}'$, and $[\mathcal{H}]'$ the corresponding space of distributions. A useful class of test functions are those of the form

$$\phi \mapsto e^{\phi(a)+i\phi(b)},$$

for any fixed $a, b \in \mathcal{H}$. Using the complexification $\mathcal{H}_c$, we can write this function as $e^{(a+ib,\cdot)}$.

It is very useful to understand a distribution $\Phi \in [\mathcal{H}]'$ through its *S-transform* $S\Phi$, which is a function on the complexification $\mathcal{H}_c$ of $\mathcal{H}$:

$$S\Phi(z) = \Phi\left(e^{(z,\cdot)}\right) \quad \text{for all } z \in \mathcal{H}_c. \tag{9}$$

A fundamental theorem in white noise distribution theory, due originally to Potthoff and Streit [40] provides simple conditions for a function on $\mathcal{H}_c$ to be the S-transform of a distribution $\Phi \in [\mathcal{H}]'$. We used this result to construct a rigorous form of the Chern-Simons functional integral in [8]. Briefly put, we used for f in (7) an exponential test function and, through reasonable but formal computation, worked out what the S-transform of the Chern-Simons distribution should be. Then it was straightforward to check that this function does satisfy the criteria for being the S-transform of a distribution in $\mathcal{H}'$. The Hilbert space H_0 in this case is the space of Lie-algebra-valued 1-forms $A_0 dx_0 + A_1 dx_1$, where gauge fixing is used to eliminate one of the components of a 1-form on $\mathbb{R}^3$.

Although this method constructs a rigorous candidate for the Chern-Simons integral it does not provide an evaluation of the integral on functions of geometric interest, such as traces of holonomies around loops. Atle Hahn, who was Sergio's doctoral student in the late 1990s, took up this task and wrote several papers, including [1, 30], connecting our construction, and other more sophisticated constructions, to topological invariants of knots and links.

3 Yang-Mills in 2 Dimensions

Sergio's influence on my work goes further back to well before I met him. While a graduate student at Cornell in the late 1980s, I came across several preprints [3–5] by Sergio and coauthors, which constructed random variables, with values in Lie groups, that could describe stochastic parallel-transport. In those days hard copy preprints, in this case from BiBoS, were available at the Cornell Mathematics Library (White Hall) and at other research libraries. I was fascinated by the elegance of the constructions and tantalized by the possible implications. I went on to construct the Yang-Mills measure for gauge fields over S^2 [41, 42] by a different method but the ideas in the papers of Sergio and coauthors remained a permanent influence on me.

The literature on 2-dimensional Yang-Mills theory has grown too vast to even attempt a brief review here. Instead I will cite just a few papers and sketch the most basic ideas related to work I am familiar with.

3.1 Some Background

In Heisenberg's theory [31] of the nucleon, the neutron and proton can be viewed as different states of the same particle or system. This is analogous to the electron with two spin states, but in the case of the nucleon it is not the spin but something else which, following Wigner, is called 'isospin.' There is a process by which the neutron transitions to the proton state through interaction with an external field, thereby emitting a boson W^- that is an excitation of this field. This is analogous to a particle, say a proton, transitioning from one spin state to another, by emitting a photon, an excitation of the electromagnetic field. The photon has zero rest mass and travels at the speed of light. The gauge boson, on the other hand, is massive and the dynamics of the gauge field is governed by equations that resemble but are not the same as Maxwell's equations. The equations governing the gauge field are the Yang-Mills equations. These ideas were discovered and developed by Yang and Mills [54]. (For simplicity, we omit here the role of the Higgs field.)

The gauge field potential is described by a 1-form A on spacetime M with values in the Lie algebra $L(G)$ of the gauge symmetry group G of the system (it is $SU(2)$ for a two-state system like the nucleon model). The field strength is given by

$$F^A = \mathrm{d}A + A \wedge A,$$

where the notation is based on taking G to be a matrix group. Finally, the dynamics is obtained as extremal of the *Yang-Mills action*

$$S_{\rm YM}(A) = -\int_M \mathrm{Tr}(F^A \wedge *F^A), \tag{10}$$

where $*F^A$ is the Hodge dual of F^A. Briefly put, the idea is that, if M is Riemannian (instead of Lorenztian), the Yang-Mills action is like the L^2-norm of F^A, and k is a physical constant. The physics is invariant under *gauge transformations*

$$A \mapsto g^{-1}Ag + g^{-1}dg, \tag{11}$$

where $g\colon M \to G$ is a any smooth mapping.

More generally, at least from the mathematical viewpoint, a gauge field is described by a connection on a bundle over spacetime. The relationship between gauge theory and the mathematical formalism of connections on bundles was made explicit in the work of Wu and Yang [53].

3.2 The Yang-Mills Measure

We work with a spacetime Σ that is a Riemannian manifold, as opposed to the physical Lorentzian one. Let $\mathcal{A}$ be the space of $L(G)$-valued 1-forms on Σ, and $\mathcal{G}_o$ the space of smooth maps $\Sigma \to G$, with value e at a fixed basepoint $o \in \Sigma$. In the approach of constructive quantum field theory, the object of fundamental interest is the measure $\mu^T_{\rm YM}$ given by

$$\mathrm{d}\mu^T_{\rm YM}(A) = \frac{1}{Z}\mathrm{e}^{-\frac{1}{2T}S_{\rm YM}(A)}\, DA, \tag{12}$$

and is to be taken on the quotient space $\mathcal{A}/\mathcal{G}_o$. Once we have this measure, we are interested in expectation values of the type

$$\int_{\mathcal{A}/\mathcal{G}_o} f\big(h_{c_1}(A), \ldots, h_{c_k}(A)\big)\,\mathrm{d}\mu^T_{\rm YM}(A), \tag{13}$$

where $h_c(A)$ is the holonomy of A around a loop c, and f is any function of interest on G^k. Typically, f is a product of traces. In this case, we should think of the Yang-Mills measure as living on the full quotient

$$\mathcal{M} = \mathcal{A}/\mathcal{G},$$

where $\mathcal{G}$ is the space of all gauge transformations.

3.3 The Yang-Mills Measure for Two-Dimensional Manifolds

Now we focus on the case where the underlying manifold is a two-dimensional Riemannian manifold Σ.

A measure can be constructed that is a rigorous form of the measure μ^T_{YM}. It lives on a probability space that we could view as a kind of completion of the space $\mathcal{A}/\mathcal{G}_o$. The construction of this measure for $\Sigma = \mathbb{R}^2$, as well as computing loop expectation values, was carried out in [29] and simultaneously, in a different approach, by Driver [19]. It was Driver's approach that I pursued further in developing the theory over compact surfaces. There were other works in the area, including several papers by Fine [23–25], who approached the problem at the level of the functional integral. A major advance was made by Lévy [35], whose approach made it possible to define stochastic parallel transport for a much larger class of loops. Lévy's approach culminated in a very general theory of 'Markovian holonomy fields' [36].

On the probability space on which μ^T_{YM} is defined, there are G-valued random variables h_c associated to loops c on Σ, based at o, and the distributions of these variables can be explicitly computed.

3.4 The Yang-Mills Measure for Simplicial Complexes

For our purposes here let me focus on a discrete framework. Let G be a compact Lie group, with a given Ad-invariant metric on its Lie algebra $L(G)$. The heat kernel on G is a function $(0,\infty)\times G \to \mathbb{R} : (t,x) \mapsto Q_t(x)$ that satisfies the heat equation

$$\frac{\partial Q_t(x)}{\partial t} = \frac{1}{2}\Delta_G Q_t(x),$$

where Δ_G is the Laplacian over G, with initial condition

$$\lim_{t\downarrow 0}\int_G f(x)Q_t(x)\,\mathrm{d}x = f(e),$$

for any continuous function f on G. The heat kernel is a smooth function, and is the density of a probability measure on G. Moreover, $Q_t(xyx^{-1}) = Q_t(y)$ and $Q_t(x^{-1}) = Q_t(x)$ for all $x, y \in G$. Another important property of the heat kernel is the convolution formula:

$$\int_G Q_u(xy^{-1})Q_v(y)\,\mathrm{d}y = Q_{u+v}(x) \tag{14}$$

for any $u, v > 0$, the integration being with respect to unit-mass Haar measure on G.

Now let $\mathbb{T}$ be a two-dimensional simplicial complex; let $\mathbb{E}$ be the set of oriented edges, $\mathbb{V}$ the set of vertices, and $\mathbb{F}$ the set of oriented 2-simplices. To each face F we associate an 'area' $|F| > 0$. We pick a 'positive orientation' on each edge: $\mathbb{E}^+$ be a subset of $\mathbb{E}$ such that for every edge $e \in \mathbb{E}$, either e is in $\mathbb{E}^+$ or the reverse e^{-1} is in $\mathbb{E}^+$. Let $\mathcal{A}_{\mathbb{T}}$ be the set of all maps

$$x : \mathbb{E} \to G : e \mapsto x_e = x(e)$$

for which $x(e^{-1}) = x(e)^{-1}$ for all $e \in \mathbb{E}$. We call such a map a *connection* over $\mathbb{T}$. For a sequence $e_1 \dots e_n$ of edges e_j we define

$$x(e_1 \dots, e_n) = x(e_1) \dots x(e_n).$$

Next, for any $t > 0$, we define a measure ν_t on $\mathcal{A}_{\mathbb{T}}$ by

$$\mathrm{d}\nu_t(x) = \prod_{F \in \mathbb{F}} Q_{t|F|}\big(x(\partial F)\big)\mathrm{d}x, \tag{15}$$

where dx is the product of unit-mass Haar measures, one for each edge in $\mathbb{E}^+$. The notation ∂F is ambiguous but this makes no difference because of the properties of the heat kernel mentioned earlier. Specifically if F is the simplex specified by vertices a, b, c then we can take any initial point, say the vertex a, and take $\partial F = e_1 e_2 e_3$, where e_1 is the edge from a to b, e_2 runs from b to c, and e_3 runs from c back to a.

For a recent rendition of the theory see Lévy's paper [39]. The defining formula (15) has its origins in the paper by Driver [19] and in my papers such as [43]; my thinking was influenced by earlier works of Sergio and coauthors on 'stochastic multiplicative measures' [5]. Of course, there is also much inspiration drawn from the physics literature, especially the two papers of Witten [51, 52].

A discrete version of the Yang-Mills measure is of interest, especially because it is an exact discrete form of, not an approximation to, the continuum measure. It is given by $\mathcal{A}_{\mathbb{T}}$ normalizing nu_t:

$$\mu_{\mathrm{YM}}^{\mathbb{T},t} = \frac{1}{\nu_t(\mathcal{A}_{\mathbb{T}})}\nu_t. \tag{16}$$

The normalizer

$$Z_{\mathrm{YM}}^{T} = \nu_t(\mathcal{A}_{\mathbb{T}})$$

can be thought of as a *partition function* for this theory, viewed as a statistical mechanical system in some sense.

The relationship between this and (12) is a long story. To make just one brief comment connecting the discrete with the continuum theory, think of $\mathbb{T}$ as a triangulation of the surface Σ over which we have the Yang-Mills fields. For example, if $\mathbb{T}$ triangulates a torus, then the partition function works out to be

$$Z_{\mathrm{YM}}^T = \int_{G^2} Q_T(aba^{-1}b^{-1})\,da\,db, \tag{17}$$

where we have taken the total surface area of Σ to be 1.

The preceding theory can be developed quite generally, including for connections on *non-trivial bundles* and also for connections over surfaces with boundary components (see [43, 46]).

3.5 The Low-T Limit

During my stay at Bochum in 1995–1996, I worked with Claas Becker, who was then doing his doctoral work with Sergio, on the discrete Yang-Mills measure. We showed that when two surfaces, with boundary, are sewn together along the boundary, the corresponding Yang-Mills measures glue together, through a kind of convolution, into a Yang-Mills measure for the combined surface.

A topic of great interest was the limit of the Yang-Mills measure μ_{YM}^T as the 'temperature' T was frozen down to 0. Witten [51] showed that the partition function Z_{YM}^T leads, in the $T \downarrow 0$ limit, to the *volume of the moduli space of flat connections* over the surface, where the volume is with respect to a certain natural symplectic form in the case where Σ is orientable. A glance at formula (12) shows that it is certainly reasonable that the limiting measure would live on the flat connections, and a more thorough examination of the Yang-Mills action in terms of a moment map also shows, heuristically, why the limiting measure should be the symplectic volume measure in the orientable case. During my time in Bochum, visiting Sergio, and later, I studied these ideas [45–47] and, building on the work with Becker, was able to obtain a rigorous proof for the volume formula for the moduli space of flat connections, for closed orientable surfaces of genus > 1, and also for the behavior of the limiting measure itself. Some of this work, such as the study of Yang-Mills connections over surfaces, I did during the U-Bahn ride from Herne, where I lived, and Bochum. Many ideas developed through encouraging remarks and observations that Sergio made over lunch or dinner.

3.6 The Large-N Limit

Another limit of the Yang-Mills measure is the large-N limit for the gauge group $G = U(N)$. There is a large literature in physics and also a considerable literature in the rigorous mathematical approach. See for example, Lévy's overview [39] and the papers by Driver et al. [20–22]. Some of this work was inspired by ideas from a paper by Singer [49].

3.7 Concluding Thought

This article has focused on low-dimensional gauge theories, but Sergio Albeverio's influence on my work has continued over the years in many ways, and I hope it will do so for many more years to come.

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The Allure of Infinitesimals: Sergio Albeverio and Nonstandard Analysis

Tom Lindstrøm

Abstract I give a survey of Sergio Albeverio's work in nonstandard analysis, covering applications to operator theory, stochastic analysis, Dirichlet forms, quantum mechanics, and quantum field theory, and making an attempt at putting his contributions into the historical context of what has happened in the field before and since.

Keywords Nonstandard analysis · Stochastic analysis · Mathematical physics

AMS Subject Classification (2010) 3H05 (Primary) · 03H10 · 60H · 81T

Of the more than one thousand items in Sergio Albeverio's bibliography, I have found somewhere between twenty-five and thirty that deal with nonstandard analysis in an essential way. That is not a large percentage, but the picture changes when one realizes that two of these items are books containing substantial amounts of independent research, and it changes even more when one takes the contributions of Sergio's students and collaborators into account.

The purpose of this paper is to trace Sergio's contributions to nonstandard analysis in a broad sense, covering not only his own books and papers, but also those of his students and many associates. As nonstandard analysis is not so much a subject as a method—and a method that can be applied in all areas of mathematics that touches on the infinite—it has been a challenge to find the best structure for the exposition: Should it be organized chronologically or thematically? Having tried both, I have settled for a presentation that is primarily chronological, but where I usually—but not always—allow myself to follow a theme to the end once it is started. As Sergio has often been concentrating on different subjects in different periods, this approach seems to work reasonably well, and it has the advantage of keeping a sense of history without interrupting the thematic developments too much.

I have had to restrict myself. I do not discuss Sergio's many expository articles (such as [1–9]) although they are often very instructive, and I have made no attempt

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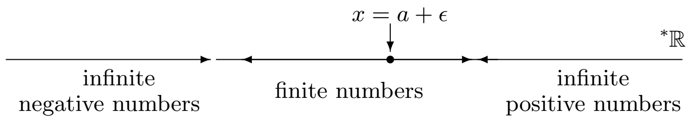

Fig. 1 The hyperfinite line ${}^*\mathbb{R}$

to connect the nonstandard papers to the rest of Sergio's production, although there are lots of overlaps, especially in areas such as point interactions, Dirichlet forms, and quantum fields. I have also had to prioritize, and I may clearly be accused of giving preferential treatment to a book [19] that I have myself contributed to, but my defense is that this book laid the foundations for most of what followed later, and that it gives a unique impression of Sergio Albeverio's and Raphael Høegh-Krohn's vision of mathematical physics in the 1980s.

The topics we shall look at span from operator theory to stochastic analysis, with Dirichlet forms as a natural meeting point. And as usual with Sergio, physics is always present—if not center stage, so at least lurking in the wings.

1 Nonstandard Analysis

Although there isn't room in this paper for a systematic introduction to nonstandard analysis, I should say a few words about the subject for those who are not familiar with it ([81] contains more or less what I would have said if I had ten extra pages, and [84] is a full introduction along the same lines).

The basic object of nonstandard analysis is a set of nonstandard reals (or *hyperreals*) ${}^*\mathbb{R}$ which extends the ordinary real line $\mathbb{R}$ by adding infinitely large and infinitely small (*infinitesimal*) numbers.[1] The extension ${}^*\mathbb{R}$ is an ordered field, and hence we can calculate with numbers in ${}^*\mathbb{R}$ and compare their size just the way we are used to from $\mathbb{R}$.

Figure 1 shows the structure of ${}^*\mathbb{R}$: We have infinite negative numbers, finite numbers, and infinite positive numbers. Each finite number x is of the form $x = a + \epsilon$ where $a \in \mathbb{R}$ and ϵ is infinitesimal. We call a the *standard part* of x and write $a = {}^\circ x$ or $a = \mathrm{st}(x)$. We shall also write $x \approx y$ to denote that x and y are two infinitely close hyperreals, i.e. that x and y differ by an infinitesimal.

What separates the hyperreal numbers from most other extensions of $\mathbb{R}$, is that sets and functions also extend: Any set $A \subseteq \mathbb{R}$ has a canonical extension to a subset ${}^*A$ of ${}^*\mathbb{R}$, and every function $f : \mathbb{R} \to \mathbb{R}$ has a canonical extension ${}^*f : {}^*\mathbb{R} \to^* \mathbb{R}$ such that ${}^*f(x) = f(x)$ for all $x \in \mathbb{R}$. These extensions preserve the defining properties of the original objects, but interpreted in a nonstandard context; e.g. will ${}^*(a, b)$ consist

[1] Actually there isn't just one set of hyperreals, but infinitely many, but for the purpose of this paper it doesn't matter much which one we choose as long as it is sufficiently rich (in technical terminology, it should be $\aleph_1$-saturated).

of all *nonstandard* numbers between a and b, and the nonstandard extension of the exponential function will satisfy $^*\exp(x + y) =^* \exp(x)^*\exp(y)$ for all $x, y \in {}^*\mathbb{R}$. As there is usually no danger of confusion, I shall drop the asterisk on the nonstandard extension of a function and write f instead of $^*f$.

The nonstandard sets and functions that arise from ordinary sets and functions in this way are (rather confusingly) referred to a *standard* objects in nonstandard parlance. They are what make nonstandard calculus possible, but they are not sufficient for more serious applications of the theory; for this, we need to extend the theory to so-called *internal* sets and functions. It would take me to far afield to give a good description of these sets and functions here; let me only say that they are the sets and functions that can be handled by the theory in a good way, just as the measurable sets and functions are the sets and functions that be handled by measure theory in a good way.

An interesting class of internal sets are the *hyperfinite sets*—these are infinite sets with most of the formal properties of finite sets. To define them, one first observes that the set $^*\mathbb{N}$ of nonstandard natural numbers consists of the ordinary natural numbers plus infinitely large elements. If $N \in {}^*\mathbb{N}$ is infinite, the set $A = \{1, 2, 3, \ldots, N\}$ is a hyperfinite set with internal cardinality N, and any other set B for which there is an *internal* bijection $\phi\colon A \to B$, is also a hyperfinite set of internal cardinality N. Hyperfinite sets occur naturally; e.g. is $T = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$ a hyperfinite set with N elements that can serve as a "hyperdiscrete" timeline.

As already mentioned, hyperfinite sets have many of the combinatorial properties of finite sets. If P is an internal function and Ω is a hyperfinite set, we may, e.g., sum P over Ω—i.e. there is a canonical way to define the sum $\sum_{\omega\in\Omega} P(\omega)$. If this sum equals 1, we may define an internal probability measure on Ω by putting $P(A) = \sum_{\omega\in A} P(\omega)$ for all internal $A \subseteq \Omega$. We can then "standardize" P by taking standard parts: $^\circ P(A) = \mathrm{st}(P(A))$.

The internal sets form an algebra, but not a σ-algebra, and hence $^\circ P$ is not a measure. It turns out, however, that the conditions of Carathéodory's extension theorem are trivially satisfied, and hence $^\circ P$ can be extended to a (complete) measure P_L. This measure is known as the *Loeb measure* of P (the Loeb measure construction was introduced by Peter Loeb in [91] and is much more general than what I have described here).

For a glimpse of what this can be used for, let $T = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$ be the hyperfinite timeline introduced above, and let Ω be the set of all *internal* functions $\omega\colon T \to \{-1, 1\}$ (think of ω as a sequence of coin tosses, one for each $t \in T$). Then Ω is a hyperfinite set with internal cardinality 2^N, and we let P be the internal counting measure on Ω. Define a process $B\colon \Omega \times T \to^* \mathbb{R}$ by

$$B(\omega, t) = \sum_{s<t} \omega(s)\sqrt{\Delta t}, \quad \text{where } \Delta t = \frac{1}{N}\,.$$

Anderson [25] showed that the standard part b of B (properly defined) is a Brownian motion on the Loeb space (Ω, P_L), and that stochastic integrals with respect to b can be recovered from hyperfinite sums

$$\sum_{s<t} X(\omega, s)\, \Delta B(\omega, s),$$

where $\Delta B(\omega, s) = B(\omega, s + \Delta t) - B(\omega, s)$ is the forward increment of B at time s. We shall return to Anderson's random walk again and again throughout this paper.

There is one more use of hyperfinite sets that I need to mention. The notion of finite dimensional vector spaces over $\mathbb{R}$ extends to a notion of hyperfinite dimensional vector spaces over ${}^*\mathbb{R}$. To get one, we may start with a hyperfinite set of basis elements $\{\mathbf{e}_n\}_{n=1}^N$ for an infinitely large $N \in {}^*\mathbb{N}$ and form all hyperfinite sums $\sum_{n=1}^N x_n \mathbf{e}_n$, $x_n \in {}^*\mathbb{R}$. A usual way to obtain such a structure, is to start with a standard Hilbert space with basis $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ and look at its nonstandard extension ${}^*H$ (there are nonstandard extensions of everything, not just ${}^*\mathbb{R}$) and cut off its basis $\{{}^*\mathbf{e}_n\}_{n\in{}^*\mathbb{N}}$ at some infinite N. In this way we get an *outer* approximation of H by something that is formally finite dimensional, and where all the techniques of linear algebra apply.

2 Sergio Albeverio's First Contribution to Nonstandard Analysis

Nonstandard analysis was invented by Abraham Robinson around 1960 [101]. He was not the first to construct an ordered field extension of the reals (see e.g. Levi-Civita [77] and Hahn [58]), but as earlier constructions did not have a way to treat transcendental functions, they could not be used for serious infinitesimal calculus in the spirit of Newton and Leibniz. The first book-length treatment of nonstandard analysis was a set of lecture notes [92] by W. A. J. Luxemburg in 1962, and the first edition of Robinson's own book [102] followed in 1966. The first breakthrough for nonstandard analysis as a research tool also came in 1966 when Bernstein and Robinson [33] solved the invariant subspace problem for polynomially compact operators.

In the summer of 1976, Edward Nelson delivered an AMS address [93] on a new axiomatic approach to nonstandard analysis called Internal Set Theory (IST). At the time, Sergio was in Oslo working with Raphael Høegh-Krohn. They both knew Nelson from Princeton and were interested in his work, but not so much for the new framework as for the applications to probability theory and mathematical physics. There was one particular example that struck their imagination—Nelson's nonstandard treatment of a problem that had previously been discussed by Berezin and Faddeev [32] and Friedman [57]: Describe all Schrödinger operators in $\mathbb{R}^d$ generated by a singular potential of the form $\lambda\delta$, where δ is the Dirac δ-function at the origin. There are no nontrivial examples for $d > 3$, but they exist for $d \leq 3$, and Nelson's thought was to use nonstandard analysis to give a description of these potentials for $d = 3$. The idea (a nonstandard version of Friedman's approach) was simple and natural: Choose an infinitesimal $\epsilon > 0$ and let $V(x) = \frac{3}{4\pi\epsilon^2}\chi_\epsilon(x)$, where χ_ϵ is the indicator function of the ball around the origin with radius ϵ. Consider operators of the form $H(\alpha) = -\Delta + \alpha V$ for standard α. Nelson finds that $H(\alpha)$ is infinitely close to the unperturbed operator $-\Delta$ except when $\alpha = -\frac{\pi^3}{3}(2n+1)^2$ for an integer n. For such α (the standard part of) $H(\alpha)$ is a nontrivial perturbation of $-\Delta$.

Sergio and Raphael soon realized that Nelson's parametrization was too coarse to give all singular perturbations of $-\Delta$. Their heuristic calculations showed that they could get *all* perturbations by instead using the parametrization $H(\lambda) = -\Delta + \lambda\chi_\epsilon$, where λ runs through all of ${}^*\mathbb{R}$, and that the nontrivial perturbations would occur when λ was of the form

$$\lambda(k, \alpha, \epsilon) = -\left(k + \frac{1}{2}\right)^2 \frac{\pi^2}{\epsilon^2} + \frac{2}{\epsilon}\alpha + \beta,$$

where k is a standard integer and α and β are two (standard) real numbers (a quick calculation will show you that Nelson's result corresponds to the situation where $\alpha = \beta = 0$). Moreover, their calculations indicated that which perturbation of $-\Delta$ they got, depended only on α and not on k and β (and hence all Nelson's perturbations are the same as they all correspond to $\alpha = 0$).

The results seemed interesting enough to publish, but Sergio and Raphael needed assistance in turning their heuristic calculations into solid nonstandard analysis, and sought the help of the logician Jens Erik Fenstad. The collaboration was successful and resulted in a joint paper [18], which in addition to treating singular perturbations from two different perspectives also contained a section on singular Sturm-Liouville problems.

The activity in Oslo was seminar-driven, and among the participants in the seminar were Bent Birkeland, Dag Normann, and myself as a beginning graduate student. Bent wrote a paper [29] which treated the singular Sturm-Liouville problem as a problem about hyperfinite difference equations, but after a while the interest of the seminar turned to the new developments in nonstandard probability theory. We studied Anderson's paper on Brownian motion and Itô integration [25], and then turned to the preprint version of Keisler's monograph [69] on infinitesimal stochastic analysis. A natural question at the time was how to extend Anderson's and Keisler's work on diffusions to martingales, and I wrote a thesis [78] on stochastic integration with respect to martingales (many of the results—and more—were discovered independently by Hoover and Perkins [66]).

3 Nonstandard Methods in Stochastic Analysis and Mathematical Physics

At some point (I am not quite sure when) Sergio, Raphael, and Jens Erik decided to write a book on nonstandard analysis. Originally, it was meant as a brief introduction with just the basic theory and a few striking applications, but it soon outgrew the original plan. After I had finished my degree and secured a postdoc position with

Keisler in Wisconsin, I was invited to join the project. Over the following years we were often asked when the book would be finished, and we would always answer “before Christmas” and be careful not to specify which Christmas we were talking about. The book [19] eventually appeared in 1986, some six or seven years after the work had started.

Nonstandard Methods in Stochastic Analysis and Mathematical Physics consists of two parts and seven chapters. The first part (called “Basic Theory” and containing the first three chapters) corresponds to some extent to the original plan; it contains the basic theory of nonstandard analysis plus some selected applications: In Chap. 1, the “chasse au canard”, a study of infinitesimal perturbations of dynamical systems due to Benoît, Callot, and Diener [30, 31, 34, 51]; in Chap. 2, a nonstandard proof of the spectral theorem for compact operators (already treated in Robinson’s book [102]); and in Chap. 3, Anderson’s nonstandard construction of Brownian motion and a few applications of Loeb measures to limit measures and measure extensions, some new, but most taken from [79].

The second part of the book is called “Selected Applications” and as it contains a mixture of original research and reports on (what was then) very recent research, I am going to spend some time on each chapter, especially as I also aim to give an account of subsequent developments where they fit in.

3.1 Chapter 4: Stochastic Analysis

This chapter begins with a quick treatment of Anderson’s version of the Itô integral. One of the big questions at the time was how to extend Anderson’s theory from Brownian motion to martingales, and Sects. 4.2–4.4 reports on the results obtained by Lindstrøm [78] and Hoover and Perkins [66]. The key to the theory is the close relationship between an internal martingale and its quadratic variation, which can be used to study both path properties and stochastic integrals (see Stroyan and Bayod’s book [107] for another exposition of the theory published about the same time).

Section 4.5 deals with stochastic differential equations, and reports on work by Keisler [69] and his student Kosciuk [76] with some minor simplifications. The basic idea is simple. Translated into a nonstandard setting, the stochastic differential equation

$$\mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + g(t, x(t))\,\mathrm{d}b(t) \tag{1}$$

becomes a stochastic difference equation

$$\Delta X(t) = f(t, X(t))\,\Delta t + g(t, B(t))\,\Delta B(t) \tag{2}$$

which, given an initial condition, obviously has an inductively defined solution. The question is when a solution of (2) can be turned into a solution of (1)? As long as the coefficients f and g are continuous in the space variable, it is not very hard to see how this can be done. The result can be extended to jointly measurable coefficients provided det g is bounded away from zero, but this is more complicated and relies on a deep inequality by Krylov. Kosciuk [76] showed that it is even possible to obtain a solution when the diffusion degenerates as long as the coefficients are continuous on the set of degeneracies, but as later pointed out in [83], the solutions tend to be so nonunique that one loses control.

Section 4.6 deals with stochastic control theory and is based on work by Nigel Cutland (see his two survey papers [37, 39] for more information), although the presentation is modified from a pathwise approach to an approach based on a nonstandard version of Girsanov's theorem. Cutland's fundamental insight was that relaxed (i.e. measure-valued) controls occur naturally as the standard part of wildly oscillating nonstandard controls, and that this can be used to obtain new existence results for optimal controls of partially observed systems.

Section 4.7 takes a quick look at Brownian motion and stochastic integration in Hilbert spaces, based on my paper [80]. The theory starts with a hyperfinite dimensional version of Anderson's random walk, and obtains a standard version of the process by taking standard parts in a norm weaker than the Hilbert space norm. Just as in the finite dimensional theory, stochastic integrals of the standard process can be obtained from hyperfinite sums.

Although infinite dimensional stochastic analysis doesn't play a big part in [19], it has later become a central area of nonstandard research. In a series of papers (neatly summed up in their book [36], see also [35, 46, 47] for later developments), Marek Capiński and Nigel Cutland developed a nonstandard approach to stochastic fluid dynamics that is formulated in terms of Anderson-like random walks on hyperfinite dimensional spaces. In another direction, Horst Osswald produced a series of papers on a nonstandard approach to Malliavin calculus, culminating in his book [98]. For other papers on nonstandard Malliavin calculus and related topics, see [44, 48–50, 88, 89], and Chap. 3 of [45].

The last section of Chap. 4 deals with white noise and Lévy Brownian motion, and is based on the Diplomarbeit [104] of Sergio's student Andreas Stoll. As Lévy Brownian motion is a multi-parameter process, we now have to replace our hyperfinite timeline by a hyperfinite lattice in ${}^*\mathbb{R}^d$:

$$\Gamma = \{(k_1\Delta t, k_2\Delta t, \ldots, k_d\Delta t) \ : \ k_i \in^* \mathbb{Z} \quad \text{and} \quad |k_i| \leq N\},$$

where Δt is infinitesimal and N is so large that $N\Delta t$ is infinite. The sample space Ω consists of all internal maps $\omega : \Gamma \to \{-1, 1\}$, and the internal probability measure P is simply the normalized counting measure on Ω. If A is an internal subset of Γ, we define

$$\chi(A) = \sum_{a\in A} \omega(a)\, \Delta t^{d/2}\ .$$

Obviously, χ is our nonstandard representation of white noise. Stoll proved that the standard part of the random field λ given by

$$\lambda(x) - \lambda(y) = k_d \sum_{a \in A} \left(\frac{1}{\|x-a\|^{(d-1)/2}} - \frac{1}{\|y-a\|^{(d-1)/2}} \right) \chi(\{a\})$$

is a Lévy Brownian motion for the right choice of the scaling parameter k_d. He also used the representation to give a Donsker type invariance principle for Lévy Brownian motion.

As was pointed out in [87], Stoll's construction is easily generalized to fractional Brownian fields; just replace Stoll's formula above by

$$\lambda(x) - \lambda(y) = k_{d,p} \sum_{a \in A} \left(\frac{1}{\|x-a\|^{(d-p)/2}} - \frac{1}{\|y-a\|^{(d-p)/2}} \right) \chi(\{a\}),$$

where $p = 2H$ is twice the Hurst exponent.

3.2 Chapter 5: Hyperfinite Dirichlet Forms and Markov Processes

This chapter consists entirely of original research that has not been published elsewhere. The inspiration for the chapter was twofold—on the one hand the deep study of (standard) Dirichlet forms and their applications that Sergio and Raphael had conducted over the previous decade, and on the other hand the need to build a solid foundation for the nonstandard study of singular perturbations (we shall take a closer look at the latter when we get to Chap. 6).

The first two sections lay the foundations for the rest by constructing a theory for bilinear forms on hyperfinite dimensional inner product spaces and describing their connection to bilinear forms on Hilbert spaces. Starting with a hyperfinite dimensional linear space H with an inner product $\langle \cdot, \cdot \rangle$, we can construct a standard Hilbert space in the following way: Let $\mathrm{Fin}(H)$ consist of the elements in H with finite norm, and define an equivalence relation on $\mathrm{Fin}(H)$ by

$$u \sim v \iff \|u - v\| \approx 0 \,.$$

If we let $^\circ u$ denote the equivalence class of u, we may introduce an inner product on $^\circ H := \mathrm{Fin}(H)/\sim$ by

$$\langle ^\circ u, ^\circ v \rangle = ^\circ \langle u, v \rangle.$$

It is not hard to check that $(^\circ H, \langle \cdot, \cdot \rangle)$ is a Hilbert space called the *nonstandard hull* of the hyperfinite space H.

The question we want to look at is this: Given an internal, nonnegative definite, symmetric, bilinear form $\mathcal{E}(\cdot, \cdot)$ on H, can we define a corresponding form E on H? It obviously suffices to define the symmetric terms $E(u, u)$ as we can get the rest by polarization.

If $\mathcal{E}$ is bounded in the sense that there is *standard* number K such that

$$\mathcal{E}(u, u) \leq K\|u\|^2$$

for all $u \in H$, the problem is easy. In this case, $u \approx v$ implies $\mathcal{E}(u, u) \approx \mathcal{E}(v, v)$, and we can just put $E(^\circ u, ^\circ u) = ^\circ \mathcal{E}(u, u)$ as it doesn't matter which representative u we choose from the equivalence class $^\circ u$.

When $\mathcal{E}$ is unbounded, there are two complications. First of all, the standard part E will now be an unbounded form, and hence only partially defined. This means that we have to determine the domain $D(E)$ of E. The second complication is that since we now may have $^\circ\mathcal{E}(u, u) \neq {^\circ\mathcal{E}}(v, v)$ for $^\circ u = {^\circ v}$, it is not clear which value to choose for $E(^\circ u, ^\circ u)$.

There is a quick fix to these problems: Just define (recall that an element x in $^\circ H$ is an equivalence class of elements in H):

$$D(E) = \{x \in {^\circ H} \mid \inf\{^\circ\mathcal{E}(u, u) \mid u \in x\} \text{ is finite}\} \tag{3}$$

and

$$E(x, x) = \inf\{^\circ\mathcal{E}(u, u) \mid u \in x\}$$

for $x \in D(E)$. I shall refer to E as the *standard part* of $\mathcal{E}$. It turns out that E is always a closed form. This is both convenient and surprising as much of the work in the standard theory goes into showing that forms are closeable.

The problem with the quick fix is that we don't have any control over how the infimum is obtained. To get control, we need to take a closer look at the nonstandard form $\mathcal{E}$. Just as in linear algebra, $\mathcal{E}$ is generated by a symmetric, linear map $A\colon H \to H$ in the sense that

$$\mathcal{E}(u, v) = \langle Au, v\rangle.$$

If $\|A\|$ is the operator norm of A (when $\mathcal{E}$ is unbounded, $\|A\|$ will be an infinitely large number), we choose an infinitesimal time increment Δt so small that $\|A\|\Delta t < 1$, and use

$$T = \{0, \Delta t, 2\Delta t, \ldots\}$$

as our timeline. Put

$$Q^{\Delta t} = I - A\Delta t,$$

where I is the identity operator, and define an internal semigroup by setting $Q^t = (Q^{\Delta t})^k$ for all $t = k\Delta t \in T$. We now define the *domain* $D(\mathcal{E})$ of the nonstandard form $\mathcal{E}$ to consist of those elements $u \in H$ such that

(i) $\mathcal{E}(u, u)$ is finite
(ii) $\mathcal{E}(Q^t u, Q^t u) \approx \mathcal{E}(u, u)$ for all infinitesimal t.

The philosophy (or rationalization) behind (ii) is that Q^t is a smoothing operator, and that the elements in the domain should be so smooth that an infinitesimal amount of smoothing doesn't change them much.

Much of the key to the theory is that the elements in $D(\mathcal{E})$ are exactly the elements in H obtaining the infimum in formula (3), i.e.

$$D(\mathcal{E}) = \{u \in H \mid {}^{\circ}\|u\| < \infty \text{ and } {}^{\circ}\mathcal{E}(u,u) = E({}^{\circ}u, {}^{\circ}u)\}$$

One may now show that the standard form E can be approximated by less singular, nonstandard objects, e.g., if $G_\alpha = (A-\alpha)^{-1}$ is the resolvent of A, we get:

$$E(x,x) = -\lim_{{}^{\circ}\alpha \to -\infty} {}^{\circ}\big(\alpha^2 \langle G_\alpha v, v\rangle + \alpha \langle v, v\rangle\big) \tag{4}$$

where v is any element in the equivalence class x. This formula will play a crucial part when we analyze singular perturbations of operators in the next chapter.

The first application of the theory above is in Sect. 5.3 where it is applied to the theory of hyperfinite Dirichlet forms and their associated Markov processes, including a study of equilibrium potentials and the proof of a nonstandard version of the Feynman-Kac formula. The Markov process generated by a hyperfinite Dirichlet form is a Markov chain with a hyperfinite state space (which may e.g. be a lattice in ${}^*\mathbb{R}^d$ with infinitesimal spacing) and a timeline with infinitesimal increments Δt. The standard part of such a Markov chain is a continuous time (standard) Markov process, and Sects. 5.4 and 5.5 studies the probabilistic and potential theoretic properties of these processes in finite and infinite dimension. The last section of the chapter sketches some applications to quantum mechanics and stochastic differential equations, but more in terms of illustrations than original research efforts.

Many years later, Sergio, in collaboration with Ruzong Fan and Frederik Herzberg, returned to the theory of hyperfinite Dirichlet forms, but as this resulted in another book [17], it deserves its own section (Sect. 4.3). Except for this book and the papers it builds on, there has unfortunately not been much done with hyperfinite Dirichlet forms. I wrote a quite speculative paper [82] on connections to diffusions on manifolds and fractals, but when I got to write "serious" papers on diffusions on fractals [85, 86], I chose not to use Dirichlet forms, and the same was the case with my student Nyberg [96, 97]. This is rather ironic as many of the subsequent standard papers used Dirichlet forms.

3.3 *Chapter 6: Topics in Differential Operators*

This chapter starts and ends with reports of already published results, but the middle three sections consist mainly of original research. As this is a book about Sergio's contributions, I'll concentrate on the middle part, but would like to say a few words about the other two sections first.

Section 6.1 deals with singular Sturm-Liouville problems of the form

$$-Y''(x) + \mu Y'(x) = \lambda Y(x), \qquad 0 \le x \le 1,$$

where μ is a Borel measure. As mentioned in Sect. 2, this problem was already treated in Sergio, Jens Erik, and Raphael's first paper on nonstandard analysis [18], but in the book we instead follow the approach by Birkeland [29] who discretized the timeline to get a hyperfinite difference equation that could be treated by linear algebra (plus a lot of estimates).

Section 6.5 reports on Leif Arkeryd's nonstandard approach to the Boltzmann equation (see his own surveys [26–28] for more information). Using nonstandard truncation techniques, Arkeryd obtained existence and uniqueness results that was in the forefront of the research at the time.

The final section of the chapter consists of some remarks on the Feynman integral from a nonstandard perspective.

Let us now turn to the central part of the chapter, Sects. 6.2–6.4, which deals with singular perturbations of operators with applications to point interactions and polymer measures. We have already taken a look at point interactions in connection with [18], but the treatment in Chap. 6 of [19] is much more ambitious and aims to develop a general framework for singular perturbations of operators. The main tool is the theory of standard parts of bilinear forms described above. As this is the heart of the book, I'll go through the arguments in some detail.

To introduce the problem, consider a nonnegative self-adjoint operator A on some L^2-space (most typically $-\Delta$ on $L^2(\mathbb{R}^d, m)$) and the closed, bilinear form E obtained by closing

$$E(f, g) = \langle Af, g\rangle$$

If C is a "small" set, we may wonder whether E (and hence A) has a perturbation supported by C, i.e. a closed form $\tilde{E}$ that is different from E, but agrees with E on all functions vanishing in a neighborhood of C. Formally, it is natural to think of such a form as given by

$$\tilde{E}(f, g) = E(f, g) - \int_C \lambda f g \, d\tilde{\rho}$$

where $\tilde{\rho}$ is a measure supported on C and λ is a function on C (we could, of course, have incorporated λ in $\tilde{\rho}$, but in many applications $\tilde{\rho}$ is a naturally given measure, and λ is the part we can adjust).

The nonstandard approach starts by replacing the original L^2-space $L^2(X, m)$ by a hyperfinite space $L^2(Y, \mu)$, and the form E by a nonstandard form $\mathcal{E}$ on $L^2(Y, \mu)$ that has E as its standard part (typically, Y is a hyperfinite lattice in ${}^*\mathbb{R}^d$, and $\mathcal{E}$ is the form generated by a hyperdiscrete Laplacian). We also replace C and $\tilde{\rho}$ by nonstandard representations B and ρ in a similar way. The problem is now to figure out when the perturbed, nonstandard form

$$\tilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v) - \sum_{x \in B} \lambda(x) u(x) v(x) \rho(x)$$

has a standard part different from E. For both physical and mathematical reasons, we need the perturbed form to be lower bounded, i.e. $\tilde{\mathcal{E}}(u, u) \geq -K\|u\|^2$ for some finite K.

If L is the operator generating $\mathcal{E}$, the operator H generating $\tilde{\mathcal{E}}$ is given by

$$Hu(x) = Lu(x) - \lambda(x)u(x)\frac{\rho(x)}{\mu(x)}.$$

The best way to control the perturbation seems to be through the resolvents, and if we let $G_\alpha = (L - \alpha)^{-1}$ be the resolvent of L, the resolvent of the perturbed operator H is given by

$$(H - \alpha)^{-1} = G_\alpha \left(I - \frac{\lambda\rho}{\mu} G_\alpha \right)^{-1} = G_\alpha \sum_{l=0}^{\infty} \left(\frac{\lambda\mu}{\rho} G_\alpha \right)^l .$$

Rearranging the terms in the Neumann series and then adding them up again (see [19] for the calculations), we end up with the expression:

$$(H - \alpha)^{-1} f(x) = G_\alpha f(x) + \hat{G}^*_\alpha \left(\frac{1}{\lambda} - G'_\alpha \right)^{-1} \hat{G}_\alpha f(x),$$

where the operator $\hat{G}\colon L^2(Y, \mu) \to L^2(B, \rho)$ and its adjoint $\hat{G}^*_\alpha\colon L^2(B, \rho) \to L^2(Y, \mu)$ are determined through

$$\hat{G}_\alpha g(x) = \sum_{y \in Y} G_\alpha(x, y) g(y) \mu(y),$$

and $G'_\alpha : L^2(B, \rho) \to L^2(B, \rho)$ is defined by

$$G'_\alpha g(x) = \sum_{y \in B} G_\alpha(x, y) g(y) \rho(y) .$$

This calculation shows that the perturbation is governed by the operator $\frac{1}{\lambda} - G'_\alpha$. If we assume that there is a standard α_0 and a standard $\epsilon > 0$ such that

$$\frac{1}{\lambda(x)} \geq \sum_{y \in B} G_{\alpha_0}(x, y) \rho(y) + \epsilon \tag{5}$$

for all $x \in B$, it follows by a simple calculation that the operator $\frac{1}{\lambda} - G'_{\alpha'}$ is positive for all $\alpha \leq \alpha_0$ and that the perturbed form $\tilde{\mathcal{E}}$ is bounded from below. This means that we can apply the theory from Chap. 5 to find the standard part of $\tilde{E}$ of $\tilde{\mathcal{E}}$. According to formula (4), it is given by

$$\tilde{E}(\tilde{f}, \tilde{f}) = -\lim_{{}^\circ\alpha \to \infty} {}^\circ\big(\alpha^2 \langle H - \alpha\rangle^{-1} f, f\rangle + \alpha\langle f, f\rangle\big)$$

where f is a nonstandard representation of the standard function $\tilde{f}$ (a so-called *lifting*; I admit details are getting a little blurred here!). Using our formulas above, this can be rewritten as

$$\begin{aligned}
&\tilde{E}(\tilde{f}, \tilde{f}) \\
&= -\lim_{{}^\circ\alpha \to \infty} {}^\circ\left(\alpha^2 \langle G_\alpha f, f\rangle + \alpha\langle f, f\rangle + \alpha^2 \left\langle \left(\frac{1}{\lambda} - G'_\alpha\right)^{-1} \hat{G}_\alpha f, \hat{G}_\alpha f \right\rangle_{L^2(B,\rho)}\right) \\
&= E(\tilde{f}, \tilde{f}) - \lim_{{}^\circ\alpha \to \infty} {}^\circ\left(\alpha^2 \left\langle \left(\frac{1}{\lambda} - G'_\alpha\right)^{-1} \hat{G}_\alpha f, \hat{G}_\alpha f \right\rangle_{L^2(B,\rho)}\right).
\end{aligned}$$

Let us return to formula (5). The sum $\sum_{y \in B} G_{\alpha_0}(x, y)\rho(y)$ is over a hyperfinite set, and can be both finite and infinite. Let us first assume that it is finite and that we can find a finite function λ satisfying (5). As α goes to $-\infty$ in the limit above, $-\alpha G_\alpha f$ approaches f and $\frac{1}{\lambda} - G'_\alpha$ approaches $\frac{1}{\lambda}$, and we may hope that the whole final term approaches $\sum_{x \in B} \lambda(x) f(x)^2 \rho(x)$.

This indeed the case, and the result in standard terms (forgetting all technical conditions) is as follows: Let E be a standard Dirichlet form with resolvent R_α. Assume that $\tilde{\rho}$ is a Borel measure on a set C and that λ is a (standard) Borel function on C such that for some $\alpha_0 \in \mathbb{R}$

$$\frac{1}{\lambda(x)} \geq \int_C R_{\alpha_0}(x, y)\, \mathrm{d}\tilde{\rho}(y) + \epsilon$$

Then the form

$$\tilde{E}(f, g) = E(f, g) - \int_C \lambda(x) f(x) g(x)\, \mathrm{d}\tilde{\rho}(x)$$

is a closed perturbation of E supported on C. Note that in this situation we have a perturbation that can be described in terms of a measure $\lambda\tilde{\rho}$ on C.

Returning to the nonstandard picture, we may ask what happens if the sum $\sum_{y \in B} G_{\alpha_0}(x, y)\rho(y)$ in formula (5) is infinitely large. By choosing $\lambda(x)$ infinitesimal, it is still possible to keep

$$\frac{1}{\lambda(x)} - \sum_{x \in B} G_{\alpha_0}(x, y)\rho(y)$$

positive and finite. The problem is that since we are interested in the limit as α goes to $-\infty$, we need to keep $\frac{1}{\lambda(x)} - \sum_{x\in B} G_\alpha(x,y)\rho(y)$ finite not only for one value of α, but for *all* finite values. And if $\sum_{x\in B} G_\alpha(x,y)\rho(y)$ is infinite and decaying, this may be seem unlikely.

Let us take a closer look. We need to keep the following quantity finite:

$$\frac{1}{\lambda(x)} - \sum_{x\in B} G_\alpha(x,y)\rho(y)$$
$$= \left(\frac{1}{\lambda(x)} - \sum_{x\in B} G_{\alpha_0}(x,y)\rho(y)\right) + \left(\sum_{x\in B} G_{\alpha_0}(x,y)\rho(y) - \sum_{x\in B} G_\alpha(x,y)\rho(y)\right) .$$

As the first term is finite by assumption, we can concentrate on the second term. By the resolvent equation

$$\sum_{x\in B} G_{\alpha_0}(x,y)\rho(y) - \sum_{x\in B} G_\alpha(x,y)\rho(y) = (\alpha_0 - \alpha)\sum_{y\in B} G_\alpha G_{\alpha_0}(x,y)\rho(y),$$

where the kernel $G_\alpha G_{\alpha_0}$ is defined by

$$G_\alpha G_{\alpha_0}(x,y) = \sum_{z\in Y} G_\alpha(x,z) G_{\alpha_0}(z,y)\mu(z) .$$

Now the point is that the kernel $G_\alpha G_{\alpha_0}$ is much less singular than the original kernel $G_\alpha(x,y)$, and hence there is good hope that $\sum_{y\in B} G_\alpha G_{\alpha_0}(x,y)\rho(y)$ is finite even if $\sum_{y\in B} G_{\alpha_0}(x,y)\rho(y)$ is infinite—and if so, our procedure may still lead to a perturbation of the original form.

Translated into standard terms (and again dropping all technical conditions), the final result is: Let E be a standard Dirichlet form with resolvent R_α on a space $L^2(X,m)$. Assume that $\tilde{\rho}$ is a Borel measure on a set C and assume that for some $\alpha_0 \in \mathbb{R}$

$$R_{\alpha_0} R_{\alpha_0}(x,y) \quad \text{is} \quad \tilde{\rho}\times\tilde{\rho} - \text{integrable.}$$

Then the form E has a closed, nontrivial perturbation supported on C. As we now may have to choose λ infinitesimal, the perturbed form can not necessarily be written as

$$\tilde{E}(f,g) = E(f,g) - \int_C \lambda(x) f(x) g(x)\, \mathrm{d}\tilde{\rho}(x)$$

in the standard universe.

To see the difference between the two results, note that if E is the form generated by $-\Delta$, the resolvent kernel $R_\alpha(x,y)$ has a singularity of order $\|x-y\|^{2-d}$ on the diagonal, while the the kernel $R_\alpha R_\alpha(x,y)$ has a singularity of order $\|x-y\|^{4-d}$ (assuming that d is large enough). This means that perturbations of the second kind (corresponding to infinitesimal λ's) usually exist two dimensions higher than perturbations of the first kind.

In Sect. 6.2, the general theory is applied to point interactions; i.e., the original form E is the closure of $E(f, g) = \langle -\frac{1}{2}\Delta f, g\rangle$ in $L^2(\mathbb{R}^d, m)$, and C is a single point. The result is as expected; perturbations exist for $d \leq 3$, but are only given by a measure for $d = 1$.

Section 6.3 deals with perturbations of the Laplacian along Brownian paths (in the nonstandard setting this means perturbations along the paths of a d-dimensional random walk moving on a lattice with infinitesimal spacing Δx). By a rather straight forward application of the general theory, we show that they exist for $d \leq 5$, but are given by measures only when $d \leq 3$. What is not so straight forward is to show that in dimension 3, we get perturbations of the form

$$E(f, g) - \int_0^1 \lambda(b(t)) f(b(t)) g(b(t))\, \mathrm{d}t$$

for all bounded functions λ. This requires some hefty estimates involving fifteen dimensional integrals.

There is a close connection between perturbations along Brownian paths and polymer measures. To see why, we apply the nonstandard Feynman-Kac formula proved in Chap. 5 to the semigroup $\tilde{Q}^t$ generated by the perturbed form $\tilde{E}$ (the nonstandard version of the Feynman-Kac formula is strong enough to deal rigorously with extremely singular potentials). The result is

$$\tilde{Q}^t f(x) \approx \tilde{E}_x \left[f(\tilde{B}(t)) \exp\left(\int_0^t \int_0^1 \lambda(\tilde{B}(r)) \tilde{\delta}(B(s) - \tilde{B}(r))\, \mathrm{d}s\, \mathrm{d}r \right)\right] .$$

Here B is the original Anderson random walk that carries the perturbation, $\tilde{B}$ is a new random walk independent of B and generated by the semigroup, $\tilde{E}_x$ is expectation with respect to the measure of $\tilde{B}$, and $\tilde{\delta}$ is the nonstandard δ-function on the d-dimensional lattice given by

$$\tilde{\delta}(x) = \begin{cases} \Delta x^{-d} & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The formula above shows that our singular perturbation theory gives us a certain control over expressions of the form

$$\exp\left(\int_0^1 \int_0^1 \lambda(\tilde{B}(r)) \tilde{\delta}(B(s) - \tilde{B}(r))\, \mathrm{d}s\, \mathrm{d}r \right) .$$

These expressions also occur in the formal definitions of polymer measures, except that λ is a negative constant and the two Brownian motions are not independent, but the same (the idea is to penalize self-intersections). Self-intersections of the same Brownian path are more singular than intersections of two independent paths, but Westwater [108–110] had shown that it is possible (at least in $d = 3$) to use the latter to control the former; the trick is to split the double integral over the square $[0, 1] \times [0, 1]$ into integrals over smaller squares that just touch the diagonal, and then control the sum of all these contributions. The (open) question is whether it is possible to do something similar in $d = 4$. As we shall see when we get to Chap. 7, this is a question that comes up naturally in quantum field theory.

At the time the book was getting finished, Sergio's student Andreas Stoll was making a more direct, nonstandard attack on self-repellent random walks and polymer measures. He starts his doctoral dissertation [103] (see also the published papers [105, 106]) with a study of local times for Brownian self-intersections of the form

$$L(x, \omega) = \sum_{t=0}^{1/2} \left(\sum_{s=1/2}^{1} \Delta x^{-d} \chi_{\{\omega : B(s,\omega) - B(t,\omega) = x\}} \Delta t \right) \Delta t,$$

where B is a d-dimensional version of Anderson's random walk and χ_A is the indicator function of A. The main tools are the nonstandard version of Kolmogorov's continuity theorem and a hyperdiscrete version of the Fourier inversion formula.

In the second part of the thesis, these results are used to make sense of the heuristic formula

$$\frac{\mathrm{d}\nu(\phi, g)}{\mathrm{d}m}(\omega) = \frac{1}{Z(\phi, g)} \exp\left(-g \int_0^1 \int_0^1 \phi(\omega(t) - \omega(s))\, \mathrm{d}s\, \mathrm{d}t \right)$$

for the density of a polymer measure against Wiener measure in dimension 2. The nonstandard approach yields among other things a Donsker-type invariance principle for Varadhan's model.

3.4 Chapter 7: Hyperfinite Lattice Models

As the title says, this chapter deals with hyperfinite lattice models for random fields and quantum fields in ${}^*\mathbb{R}^d$. In the first three sections, the lattices have standard spacing (the shortest distance between sites in the lattice is 1), and the only difference between the standard and the nonstandard models is that the nonstandard lattices have additional sites infinitely far out. These sites are important, however, as they allow us to put boundary conditions "at infinity". The first section of the chapter follows the (nonstandard) papers by Helms and Loeb [60, 61] (see also [59]) in describing how to find the semigroup that governs the evolution of the lattice system—the point here is that if we fix the configuration outside an infinitely large cube in ${}^*\mathbb{R}^d$, the (nonstandard) dynamics inside the cube is easily described, and the standard dynamics can be obtained by "taking standard parts".

Section 2 deals with equilibrium models and thermodynamical properties, and builds partly on earlier nonstandard treatments by Helms and Loeb [60], Hurd [67, 68], and Ostebee, Gambardella, and Dresden [99, 100]. The main idea here is that one can replace the rather cumbersome limit definitions of the standard theory by working directly on a hyperfinite part of the lattice. The section ends with a discussion of how phase transitions can occur in hyperfinite models although they do not occur in finite models—the clue is that a function can be differentiable in a nonstandard sense (with an infinitely large derivative) without being differentiable in the suitable standard sense.

Section 3 deals with the global Markov property of lattice fields. For fields there is a distinction between the local Markov property which deals with the interaction between a bounded set and its exterior, and the global Markov property which also deals with the interaction between two unbounded sets. The intuitive reason is that even if you separate two unbounded sets by a boundary that the interaction does not reach across, one set can still influence the other "through infinity". The nonstandard content of this section is to a large extent based on the work of Sergio's student Christoph Kessler (see [70–73, 75]). There is a jungle of conditions leading to the global Markov property, and in his thesis, Kessler helped clarify the relationship between them—in particular, he used nonstandard analysis to construct models satisfying some properties and not others.

The last two sections of the book, Sects. 7.4 and 7.5, deal with quantum fields. We are still working with lattices in ${}^*\mathbb{R}^d$, but now the spacing is infinitesimal, i.e. the distance between neighboring sites is $\delta \approx 0$. The contents of these sections were previously unpublished, but some of it must be classified as reworking of standard theory in nonstandard terms.

We first show how the free field on $\mathbb{R}^d$ can be obtained as the standard part of a hyperfinite Gaussian field; one of the advantages of this representation is that the hyperfinite field is defined pointwise and not only in a distributional sense (there are two traditional ways to treat the singularities of quantum field theory: through distributions or through lattice approximations—see Kessler's paper [74] for a nonstandard discussion of the relationship between these two approaches). Interactions are introduced as

$$U_g^\delta = \lambda_\delta \sum \delta^d g(n\delta) u_\delta(\Phi_\delta(n)),$$

where Φ_δ is the free field, g is a cut-off function (often taken to be the indicator function of a "large" set), λ_δ is a coupling constant, and the sum is over ${}^*\mathbb{Z}^d$. The function u_δ describes the interaction, and may typically be an exponential $u_\delta(y) = \exp(\alpha y)$ or a polynomial of low order. The associated probability measure is given by

$$\mathrm{d}\mu_{g,\Lambda_\delta} = \frac{\exp(-U_g^\delta)}{\int \exp(U_g^\delta)\,\mathrm{d}\mu_{0,\Lambda_\delta}}\,\mathrm{d}\mu_{0,\Lambda_\delta},$$

where $\mathrm{d}\mu_{0,\Lambda_\delta}$ is the free measure.

The challenge is twofold: On the one hand to prove that the standard part of the interacting field satisfies the axioms for Euclidean quantum fields, and on the other hand to prove that it is nontrivial, i.e. different from the free field. For exponential interactions in dimension 1 and 2, the situation was well understood through earlier (standard) work by Sergio and Raphael, and this is used a test case for the nonstandard theory. The real challenge is polynomial interactions, especially the famous (or infamous?) Φ^4_d-model of fourth degree interactions.

The final section of the book, Sect. 7.5, is called “Fields and polymers” and contains a serious attempt to get a better grip on polynomial interactions. Using Anderson’s random walk and a nonstandard representation of Poisson processes, we first construct a “Poisson field of Brownian bridges” in a very concrete way. The second step is to prove that this Poisson field is a probabilistic representation of the square of the free lattice field, and the third step is to use this representation to study interacting scalar fields (representations of this kind were first obtained by Dynkin [54, 55] in a standard context). The famous Φ^4 fields are given by interactions of the form

$$U(\Phi_\delta) = \frac{\lambda}{4}\sum_i \Psi_\delta(i)^2\delta^d + \frac{a}{2}\sum_i \Psi_\delta(i)\delta^d,$$

where Ψ_δ is the square of the free field, the sums are over the lattice, and λ and a are constants. If we use the representation above to calculate the crucial entity $E[\exp(-\Phi_\delta(g))\exp U]$, we end up with expressions of the type

$$\exp\left(-\int_0^t\int_0^{\tilde{t}} \lambda\delta(b_1(s)-b_2(s))\right)\,\mathrm{d}\tilde{s}\,\mathrm{d}s, \tag{6}$$

which are exactly the kind of expressions we got acquainted with when we looked at perturbations along Brownian paths. A major problem is that in some of these expressions, b_1 and b_2 are not independent, but the same Brownian motion, and this leads us to the complicated problems of polymer measures that we just touched on at the end of Sect. 6.4. Westwater managed to tame them when $d = 3$, but $d = 4$ is a much more singular case.

A way to avoid this problem, is to look at two interacting fields Φ_1 and Φ_2 in a $\Phi_1^2\Phi_2^2$-model. The calculations are much the same as before, but as we now have two different fields, we only get the expression in formula (6) for two independent Brownian motions b_1 and b_2. This leads to the questions we analyzed in Sect. 6.4 on perturbations of the Laplacian along Brownian paths. The main problem in this case is that in the physical dimension $d = 4$, the coupling “constant” in Chap. 6 was positive and allowed to depend on x. In the present situation, we need it to be negative and independent of x. This seems to be a quite difficult problem as we had to work extremely hard to prove that λ can be chosen constant in the much easier three dimensional case (and if it doesn’t sound hard to prove that an infinitesimal function can be chosen constant, recall that it is the infinite function $\frac{1}{\lambda(x)}$ that we really need to control).

So the book ends on an open note; we had shown that hyperfinite lattice models were an interesting setting for quantum fields, much closer to intuition than the traditional formalism, but we hadn't been able to obtain the definitive results we were aiming for (but then they have proved to be quite elusive for all kind of approaches!).

4 Later Contributions

After the completion of [19], Sergio has continued to work with nonstandard methods in a variety of subjects and often with different groups of collaborators. Much of this activity can be seen as a natural continuation of ideas and challenges from [19], and I shall try to give an exposition of the main results.

4.1 Nonstandard Constructions of Singular Traces

In the first half of the 1990s, Sergio wrote four papers [20–23] on singular traces in collaboration with Daniele Guido, Arcady Ponosov, and Sergio Scarlatti. In spirit these papers are close to Sergio's first nonstandard paper [18] with Fenstad and Høegh-Krohn in the sense that they give concrete, nonstandard descriptions of otherwise rather elusive operators, but the setting of the papers is quite different from [18].

If $\mathcal{R}$ is a von Neumann algebra (you can safely think of the case where $\mathcal{R} = B(H)$ is the algebra of all bounded operators on a Hilbert space H) and $\mathcal{R}^+$ is its cone of positive elements, a *weight* on $\mathcal{R}$ is a linear map

$$\phi\colon \mathcal{R}^+ \to [0, \infty] \, .$$

Using linearity, we can extend ϕ to its natural domain $\mathrm{Span}\{T \in \mathcal{R}^+ : \phi(T) < \infty\}$. A *trace* is a weight τ with the property $\tau(T^*T) = \tau(TT^*)$. We say that τ is *normal* if for all increasing nets $\{T_\alpha \mid \alpha \in I\}$ with $T = \sup_{\alpha\in I} T_\alpha$, we have $\phi(T) = \lim_\alpha \phi(T_\alpha)$. A classical result [53] tells us that all normal traces are proportional to the usual trace, so the question is how many nonnormal traces are there? If we define a trace τ to be *singular* if it is trivial on all operators of finite rank, it turns out that any trace on the compact operators $K(H)$ can be written uniquely as a sum $\tau = \tau_1 + \tau_2$ of a normal trace τ_1 and a singular trace τ_2, and hence we can concentrate on singular traces.

Dixmier [52] proved that nonnormal traces exist. To get an impression of his construction, we first fix a regular, slowly increasing and divergent sequence α_n of real numbers ($\alpha_n = \log(n+1)$ will do the job, but there are other possibilities). The idea is to use this divergent sequence to speed up the decay of nonsummable sequences of eigenvalues so that they become summable.

Next we choose a state (i.e. a normalized weight) on $l^\infty(\mathbb{N})$. The idea is now to define a trace τ_ϕ on $B(H)^+$ by

$$\tau_\phi(T) = \begin{cases} \phi\left(\left\{\frac{\sigma_n(T)}{\alpha_n}\right\}\right) & \text{if } T \in I(H) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\sigma_n(T) = \sum_{k=1}^{n} \mu_k(T)$, where $\mu_k(T)$ are the eigenvalues of T in decreasing order and counted with multiplicity, and $I(H)$ is the ideal of all compact operators such that the sequence $\{\sigma_n(T)/\alpha_n\}$ is bounded.

The main problem with this construction is that as we in general only have an inequality

$$\sigma_n(T+S) \le \sigma_n(T) + \sigma_n(S),$$

τ_ϕ will usually not be linear. Dixmier realized that if ϕ is 2-dilation on $l^\infty(\mathbb{N})$, i.e. $\phi(\{a_n\}) = \phi(\{a_{2n}\})$, then we also have the opposite inequality (this needs both the slow growth of α_n and the dilation property), and hence τ_ϕ is linear and a trace. As ϕ vanishes on the set c_0 of sequences converging to 0, τ_ϕ is a nonnormal trace.

So how do we get hold of 2-dilations on $l^\infty(\mathbb{N})$? It is here nonstandard analysis comes in with a very simple and elegant description. If $\{a_n\}_{n\in{}^*\mathbb{N}}$ is the nonstandard extension of a bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, then for any infinite $\omega \in {}^*\mathbb{N}$, we define

$$\phi_\omega(\{a_n\}) =^\circ \left(\frac{1}{\omega}\sum_{k=1}^{\omega} a_{2^k}\right).$$

As $\phi_\omega(\{a_n\}) - \phi_\omega(\{a_{2_n}\} = \frac{1}{\omega}\left(a_1 - a_{2^{\omega+1}}\right) \approx 0$, we see that ϕ is a 2-dilation, and hence τ_{ϕ_ω} is a nonnormal trace. More generally,

$$\phi_{k,m,n}(\{a_n\}) =^\circ \left(\frac{1}{n}\sum_{i=k+1}^{k+n} a_{(2m-1)2^{k-1}}\right)$$

is a 2-dilation for all $k, m \in \mathbb{N}$ and all infinite $n \in {}^*\mathbb{N}$. This means that we have a three-parameter family $\tau_{k.m,n}$ of associated traces (with repetitions).

So how general is this construction? It is proved in [20] that any *Dixmier trace* (i.e. any trace coming from a 2-dilation) is in the closure of the convex hull of the traces $\tau_{k,m,n}$. The proof is based on a close study of the extremal dilation invariant states. We cannot go deeper into the arguments here, but would like to say that they exploit the product structure ${}^*(\mathbb{N}\times\mathbb{N}) = {}^*\mathbb{N}\times{}^*\mathbb{N}$ of the nonstandard natural numbers in a way that is not possible in the usual approach through Stone-Čech compactifications as $\overline{\mathbb{N}\times\mathbb{N}} \neq \overline{\mathbb{N}}\times\overline{\mathbb{N}}$.

In [21] the theory is extended to another class of nonnormal traces (called *anti-Dixmier traces* as they are in a sense reflections of the Dixmier traces around the usual trace), but the results and techniques are much the same as in [20]. In [22] the emphasis has shifted. The question now is to classify those compact operators T that admit a singular trace in the sense that there is a singular trace τ with $0 < \tau(T) < \infty$. If we define $\{S_n(T)\}$ to be the sequence such that $S_n(T) - S_{n-1}(T) = \mu_n(T)$ and

$$S_0(T) = \begin{cases} 0 & \text{if } T \notin L^1(H) \\ -\mathrm{tr}(T) & \text{if } T \in L^1(H), \end{cases}$$

we say that T is *generalized eccentric* if 1 is a limit point for the sequence $\{S_{2n}(T)/S_n(T)\}$. The main theorem states that T admits a singular trace if and only if it is generalized eccentric.

The proof of this theorem is entirely standard, but again 2-dilations coming from hyperfinite sums are used to throw light on how these operators occur, and the paper ends with an interesting example of how such sums can be used to calculate a closed formula for a Dixmier trace of a concrete operator. The last paper [23] in the series deals with the same problems as [22], but the main emphasis is now on the nonstandard analysis of 2-dilation invariant states.

4.2 Quantum Fields as Flat Integrals

About ten years after the work on [19] was finished, Sergio returned to hyperfinite models of quantum fields with three papers in collaboration with Jiang-Lun Wu ([14–16], see also Wu's later paper [111]). The basic idea was to use nonstandard analysis to make rigorous sense of quantum fields as flat integrals. Intuitively, flat integrals are representations of Gaussian fields as integrals of infinite dimensional Lebesgue measure, and they have been much used as a heuristic tool by both physicists and probabilists. The only problem is that since infinite dimensional Lebesgue measure doesn't exist, flat integrals do not exist—at least not in an immediate sense.

What do exist are nonstandard Lebesgue measures on hyperfinite dimensional spaces, and in a series of papers [38, 40–43] Nigel Cutland used these to give nonstandard flat integral representations of a variety of Gaussian fields. In [15] Sergio and Jiang-Lun set out to extend these ideas to the quite singular case of Euclidean quantum fields. Their starting point is that if Λ is a bounded subset of $\mathbb{R}^d$, the free Euclidean field ϕ in Λ with mass m is heuristically given by the flat integral

$$\mathrm{d}\mu(\phi) = \kappa \exp\left\{-\frac{1}{2}\int_\Lambda \left(|\nabla\phi(x)|^2 + m^2\phi(x)\right) \mathrm{d}x\right\} \prod_{x\in\Lambda} \mathrm{d}\phi(x),$$

where $\prod_{x\in\Lambda} d\phi(x)$ is the infinite dimensional Lebesgue measure.

Working on a hyperfinite lattice approximation Λ_δ of Λ, Sergio and Jiang-Lun in [15] obtain a rigorous version of this formula

$$\Gamma(A) = \int_A \kappa \exp\left\{-\frac{1}{2}\sum_{z\in\Lambda_\delta} \left(|\nabla_\delta q_z|^2 + m^2 q_z\right) \delta^d\right\} \prod_{z\in\Lambda_d} \mathrm{d}q_z,$$

where ∇_δ is a hyperdiscrete approximation of the appropriate gradient on Λ and $\prod_{z\in\Lambda_d} dq_z$ is a (well-defined) hyperfinite dimensional Lebesgue integral. Although totally rigorous, this formula only makes sense inside the nonstandard universe, but the authors also derive a standard white noise representation of ϕ as

$$\phi(f,\omega) = \int\limits_{\Lambda} \left(-\Delta_\Lambda + m^2\right)^{-1/2} f(x)\, d\xi_x(\omega), \qquad f \in \mathcal{D}(\Lambda), \tag{7}$$

where $\{\xi_x\}$ is an independent family of one-dimensional white noises. This formula is obtained by first defining a nonstandard white noise η on the hyperfinite lattice and showing that the nonstandard lattice field Φ_δ (as defined in Sect. 7.4 of [19]) can be obtained as an integral of η. Some technical work is needed to show that the standard field is the standard part of Φ_δ in the appropriate Sobolev space. Formula (7) is then used to obtain a Cameron-Martin formula and a Schilder-type large deviation principle for the free Euclidean field on Λ.

The companion paper [14] is written primarily for a nonstandard audience (and not an audience of physicists) and extends the discussion from the free field to fields with exponential interaction. In addition to another discussion of large deviations of quantum fields, the slightly later paper [16] also extends one of Cutland's flat integral representations from l^2 to l^p, $1 \le p < \infty$.

4.3 A Return to Hyperfinite Dirichlet Forms

In 2011, Sergio, in collaboration with Ruzong Fan and Frederik Herzberg, published a book [17] entitled *Hyperfinite Dirichlet Forms and Stochastic Processes*. The work on the project had actually started more than 20 years earlier, and had resulted in a number of papers in the 1990s, mainly by Fan, but in close collaboration with Sergio.

The main difference between the theory in the new book and the one in [19] is that the forms are no longer required to be symmetric, but they do have to be *weakly coercive* in the sense that there is a finite constant C such that

$$\mathcal{E}_1(u,v) \le C\sqrt{\mathcal{E}_1(u,u)}\sqrt{\mathcal{E}_1(v,v)}.$$

This condition works as a replacement for Schwarz' inequality.

The (nonsymmetric) form $\mathcal{E}$ has a *coform* $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$ that can be used to form the *symmetric part* $\overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v) + \hat{\mathcal{E}}(u,v)$ and the *anti-symmetric part* $\mathring{\mathcal{E}}(u,v) = \mathcal{E}(u,v) - \hat{\mathcal{E}}(u,v)$ of $\mathcal{E}$. All four forms play an important part in the exposition.

One of the differences between the standard and the nonstandard theory of Dirichlet forms is that in the standard theory the domain of the form is usually assumed to be given (at least until one starts looking at examples!), while in the nonstandard theory much of the basic work goes into identifying the domain. Although the theory of weakly coercive forms is in many ways similar to the symmetric theory, there is an important difference in the description of domains: In the symmetric case, the domain is easily described in terms of the semigroup, but in the weakly coercive case it seems necessary to approach the domain via the resolvent. Rather reassuringly, it turns out that the domain of the original form $\mathcal{E}$ coincides with the domain of the much simpler, symmetric form $\overline{\mathcal{E}}$.

After the initial study of domains and standard parts of weakly coercive, hyperfinite forms, the book continues with a detailed study of potential theory and the relationship between a Dirichlet form and its associated Markov process, at the same time generalizing and simplifying the corresponding theory in [19].

The last part of the book is on the theory of hyperfinite Lévy process. As this is the topic of the next subsection, I'll leave the discussion till then.

4.4 Nonstandard Lévy Processes

In 2003, Sergio's student Frederik S. Herzberg wrote a Diplomarbeit on nonstandard Lévy processes. Unaware of Frederik's work, I was at the same time starting my own investigations into the subject. Fortunately, we approached the problem from opposite angles, and when the first papers appeared ([10, 11] by Sergio and Frederik and [89] by me—see also the corrections in [65]), they complemented each other more than they overlapped.

For a quick, intuitive understanding of how the nonstandard theory works, it is convenient to start with the definitions in [89]. Choose a hyperfinite set $A \subseteq^* \mathbb{R}^d$, an internal set $\{p_a \mid a \in A\}$ of positive numbers such that $\sum_{a\in A} p_a = 1$, and a positive infinitesimal Δt. Let X be a random walk in ${}^*\mathbb{R}^d$ with timeline $T = \{k\Delta t \mid k \in \mathbb{N}_0\}$ and transition probabilities p_a; i.e. let $X(0) = 0$ and put $P[\Delta X(t) = a \mid X(0), X(1), \ldots, X(t)] = p_a$. We call X a *hyperfinite Lévy process* if almost all paths stay finite for all finite t. This sounds like a silly, totally uncheckable condition, but it turns out that there is an equivalent, easy to verify characterization. Hyperfinite random walks have cadlag standard parts that are Lévy processes, and any Lévy process can be obtained in this way (at least in the sense that every Lévy triple (γ, C, ν) can be realized—see also [95]).

My focus in [89] is very much on the hyperfinite random walks, and the Lévy processes only enter the theory to show that it has achieved what it set out to achieve. In the first two papers [10, 11] by Sergio and Frederik, the Lévy processes are on the contrary the primary objects, and the main focus is to find good, nonstandard representations (*liftings*) preserving the properties of the original process. Typical examples are the lifting results in [10] where the hyperfinite representations live on lattices, and where the time development is divided into sequences of binomial events. Notions from adapted probability logic are used to characterize how close the nonstandard liftings are to the original processes. In [62], Frederik used these hyperfinite representations to construct an intrinsic theory for stochastic integration with respect to Lévy processes.

Lévy processes have been much used to model financial markets, and this is also a clear motivation for Sergio and Frederik. By refining the timeline and modeling the internal processes as sums of binomial increments as in [11], they achieve a model with a unique martingale measure that can be used for hedging (see also [63]). An alternative approach to finance in hyperfinite Lévy markets is presented in [90] where the focus is on minimal martingale measures.

In Chap. 5 of their book [17] with Ruzong Fan, Sergio and Frederik give a review of both approaches to the theory of nonstandard Lévy processes, and Frederik has also given an exposition in another book [64], this time in the framework of Nelson's "radically elementary probability theory" (see [94]). The notion of hyperfinite random walks seems to fit perfectly into Nelson's vision.

4.5 The Power of Loeb Measures

As nonstandard measure and probability theory developed, it soon became clear that Loeb measure spaces have many desirable qualities—e.g., they seem to be *universal* in the sense that anything that can be constructed on some measure space (no matter how exotic), can also be constructed on simple Loeb spaces. This intuitive notion of universality was formalized through several versions of probability logic, mainly developed by Jerry Keisler and his (former) students Douglas N. Hoover and Sergio Fajardo (see [56] for a systematic exposition). Along the way other notions, such as saturation and homogeneity, were added to universality.

Three of Sergio's later papers exploits the richness of Loeb spaces. The first is a joint paper with Yeneng Sun and Jiang-Lun Wu [24] published in 2007. The authors start with two hyperfinite probability spaces I and Ω and study processes $X\colon I \times \Omega \times T \to^* \mathbb{R}$, where T is a timeline. If the processes $(\omega, t) \mapsto X(i, \omega, t)$ (with i fixed) are independent martingales, they prove that the "empirical process" $(i, t) \mapsto X(i, \omega, t)$ (with ω fixed) is a martingale for almost all i. Due to measurability problems such questions are hard even to make sense of in a standard setting, but the paper exploits the fact that the Loeb measure of a product of nonstandard measures is richer than the product of the Loeb measures to circumvent these problems. The result extends to sub- and supermartingales.

The other two examples are joint papers [12, 13] with Frederik Herzberg from about the same time. The first of these deals with the optimization of functionals where the main variable is a probability measure P. The functional is evaluated by observing a given process g at a fixed set of points $t_1, t_2, \ldots, t_n$, and then integrating an expression of the type $\phi(g_{t_1}, g_{t_2}, \ldots, g_{t_n})$ against the varying measure P. In the nonstandard setting of an internal, hyperfinite probability space the existence of an optimal measure is almost trivial (it is just a finite dimensional optimization problem), but to transfer this solution to an arbitrary measure space, the full machinery of adapted probability logic is needed.

The third paper in this group [13] is more traditional in its methods, but still uses the power of Loeb measure techniques to the full. The problem is to extend the solution of the classical moment problem from $\mathbb{R}^n$ to Wiener space. Using Anderson's random walk as a representation of Brownian motion, the problem is translated into a hyperfinite dimensional setting where the nonstandard version of the original problem applies. An extra condition on the quadratic variation is needed to pull this nonstandard solution back to the classical Wiener space.

We have reached the end! I hope this little survey has not only given you new insight into the work of Sergio and his school, but also provided you with a better understanding of the power and versatility of nonstandard methods.

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Sergio's Work in Statistical Mechanics: From Quantum Particles to Geometric Stochastic Analysis

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Abstract We discuss the contribution of Sergio Albeverio to the development and study of mathematical models of interacting particle systems on discrete metric spaces, e.g. integer lattices. Our main attention is focused on the Euclidean approach to quantum statistical mechanics and stochastic analysis on infinite product manifolds.

Keywords Interacting particle system · Euclidean Gibbs measures · Stochastic dynamics

2020 Mathematics Subject Classification. 82B10 · 60J60 · 60G60 · 46G12 · 46T12

1 Introduction

It is our great pleasure to contribute to this volume dedicated to Sergio Albeverio on the occasion of his 80th birthday. This paper is our humble attempt to give insight into Sergio's work on the development and study of mathematical models of statistical mechanics of infinite particle systems. This being a vast topic, we restrict our attention to its two particular directions—the Euclidean approach to quantum statistical mechanics and stochastic analysis on infinite product manifolds. Mathematically, these two seemingly different fields require development of (stochastic) analysis on infinite dimensional spaces of special type, which in turn is based on the notion of Gibbs measures.

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More precisely, we consider a system of classical or quantum particles indexed by the vertices k of an integer lattice $\mathbb{Z}^d$.[1] A particle with position $k \in \mathbb{Z}^d$ carries an internal parameter (spin) $x_k \in X$, where X is a topological space equipped with a reference measure $\chi(\mathrm{d}x)$. We define the infinite product space

$$\mathbf{X} \equiv X^{\mathbb{Z}^d} := \left\{ x = (x_k)_{k\in\mathbb{Z}^d},\ x_k \in X,\ k \in \mathbb{Z}^d \right\}$$

endowed with the product topology. Given $\Lambda \subset \mathbb{Z}^d$, let

$$\mathbf{X} \ni x \mapsto x_\Lambda = (x_k)_{k\in\Lambda} \in X^\Lambda \tag{1.1}$$

be the natural projection of $\mathbf{X}$ onto X^Λ. Our particles are allowed to interact via a family of spin-spin potentials

$$\mathcal{U} = \left\{ U_\Lambda : X^\Lambda \to \mathbb{R},\ \Lambda \in \mathcal{B}_0(\mathbb{Z}^d) \right\}, \tag{1.2}$$

where $\mathcal{B}_0(\mathbb{Z}^d)$ is the collection of all finite subsets of $\mathbb{Z}^d$. The corresponding system of classical particles as a whole is governed by the heuristic Hamiltonian

$$H(x) = \sum_{\Lambda\in\mathcal{B}_0(\mathbb{Z}^d)} U_\Lambda(x_\Lambda). \tag{1.3}$$

In this case, X is usually a finite dimensional (e.g. Euclidean) space. In contrast to that, quantum systems are governed by operator valued Hamiltonians, see Sect. 2.1, which in turn leads to infinite dimensional one-particle spaces X. The most common class of particle systems comprises of those with pair interaction of finite range, which means that $U_\Lambda \equiv 0$ unless

$$\Lambda = \left\{ (k, j) \in \mathbb{Z}^d \times \mathbb{Z}^d : |k - j| \le R \right\}$$

for some fixed interaction radius $R < \infty$.

For the convenience of the reader and in order to fix main notations, we start by introducing the notion of Gibbs measures and outlining fundamental problems arising in their study.

According to the paradigm developed in the works of Dobrushin, Lanford and Ruelle (1968–70), equilibrium states of our system are given by Gibbs measures μ on $\mathbf{X}$ of the (heuristic) form

$$\mu(\mathrm{d}x) = \text{“}\ Z^{-1} \exp\{-H(x)\} \bigotimes_{k\in\mathbb{Z}^d} \chi(\mathrm{d}x_k)\text{”}\ .$$

[1] Without principal changes the whole theory extends from $\mathbb{Z}^d$ to infinite graphs with uniformly bounded vertex degree.

Rigorously, any such μ is a probability measure on $\mathbf{X}$ with prescribed conditional distributions (or local specifications)

$$d\mu_\Lambda(x_\Lambda|\xi) = \frac{1}{Z_\Lambda(\xi)} e^{-V_\Lambda(x_\Lambda|\xi)} dx_\Lambda, \ \xi \in \mathbf{X},$$

for an exhausting system of sets $\Lambda \in \mathcal{B}_0(\mathbb{Z}^d)$, where

$$V_\Lambda(x_\Lambda|\xi) = \sum_{\substack{\Delta \in \mathcal{B}_0(\mathbb{Z}^d): \\ \Delta \cap \Lambda \neq \emptyset}} U_\Delta(x_{\Delta\cap\Lambda}, \xi_{\Delta\cap\Lambda^c})$$

is the energy of the interaction in the volume Λ with fixed boundary condition $\xi \in \mathbf{X}$ and $Z_\Lambda(\xi) = \int_{X^\Lambda} e^{-V_\Lambda(x_\Lambda|\xi)} dx_\Lambda$. That is, μ is called a Gibbs measure (for given $\mathcal{U}$ and χ) if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equation

$$\int_{\mathbf{X}} \mathbb{E}_\Lambda f \, d\mu = \int_{\mathbf{X}} f \, d\mu$$

for each $\Lambda \in \mathcal{B}_0(\mathbb{Z}^d)$ and any bounded cylinder function $f : \mathbf{X} \to \mathbb{R}$, where

$$(\mathbb{E}_\Lambda f)(\xi) := \int_{X^\Lambda} f(x_\Lambda, \xi_{\Lambda^c}) d\mu_\Lambda(x_\Lambda|\xi).$$

So, the study of Gibbs measures is reduced to the generic problem of reconstructing a Markov random field on $\mathbf{X}$ from its local specification. This constitutes the standard Dobrushin–Lanford–Ruelle formalism, see e.g. the classical monograph [67].

We denote by $\mathcal{G}$ the set of all such measures (for fixed $\mathcal{U}$ and χ). The study of the structure of the set $\mathcal{G}$ is of a great importance. In particular, there are three fundamental questions arising here:

(**E**) *Existence*: is $\mathcal{G}$ not empty?
(**U**) *Uniqueness:* is $\mathcal{G}$ a singleton?
(**M**) *Multiplicity:* does $\mathcal{G}$ contain at least two (and hence infinitely many) elements? If the answer is positive, the system admits phase transitions.

In order to handle these problems for various types of interacting particle systems (classical or quantum, discrete or in the continuum), in modern mathematical physics there has been developed a wide variety of powerful techniques such as Dobrushin's abstract existence and uniqueness criteria for random fields [60], Ruelle's technique of (super-) stability estimates [80, 86, 87], cluster expansions universally working in perturbative regime, different types of correlation inequalities employing a particular structure of the interaction, and so on.

Apart from the classical Ising model, the most studied and well-understood system is that of one-dimensional classical anharmonic oscillators with nearest-neighbour pair interaction. However, the passage to the study of a quantum anharmonic oscillator and more general quantum systems requires X to be infinite-dimensional, which is reflected by the Euclidean approach in quantum statistical mechanics. On the other hand, the case of non-flat single-particle spaces leads to a non-trivial influence of the geometry of X on the global properties of the system.

It is important to mentioned that, along with the traditional DLR (or Markov field) formalism described above, there are two further conceptually different approaches to the study of Gibbs measures. Namely, these are the analytic and stochastic approaches, based respectively on

(IbP) characterization of $\mu \in \mathcal{G}$ via integration by parts, and
(SD) construction of the corresponding stochastic dynamics, that is, a Markov process for which μ is invariant measure, and then studying its properties.

In what follows, we will address the problems (E), (U) and (M) for two rather different models—quantum lattice systems and classical systems with compact manifolds as single spin spaces, which require the development and application of very different techniques. We will explore the interplay between all three approaches—(DLR), (IbP) and (SD), mainly referring to our joint work with Sergio. Let us point out that the goal of this paper is to outline Sergio's contribution and not to give a comprehensive review of the field.

The structure of the paper is as follows. In Sect. 2 we consider the Euclidean approach to lattice models of quantum statistical mechanics. Section 3 is devoted to the development of general stochastic analysis on infinite product manifolds and its applications to classical lattice models. Finally, in Sect. 4 we give a quick overview of some further developments rooted in the ideas and works described in Sects. 2 and 3.

2 Euclidean Gibbs Measures of Quantum Statistical Systems

2.1 Euclidean Approach: From Quantum Statistical Mechanics to Markov Fields

The systematic development of the Euclidean approach to problems of quantum statistical mechanics has been one of Sergio's favorite topics of permanent interest. Initiated in 1973–75 in [11, 12], this work received its contemporary form in the monograph [20]. Below, we will outline the main ideas of this approach.

To start with, we consider a system of quantum anharmonic oscillators described by the formal operator-valued Hamiltonian

$$\mathbb{H} := \sum_{k\in\mathbb{Z}^d}\left[-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x_k^2}+\frac{a}{2}x_k^2+V_k(x_k)\right]+\frac{1}{2}\sum_{\langle k,j\rangle\subset\mathbb{Z}^d}W_{kj}(x_k,x_j), \tag{2.1}$$

where the second sum is taken over all (unordered) pairs $\langle k, j\rangle$ in $\mathbb{Z}^d$ such that $|k-j|=1$. The potentials are continuous functions

$$V_k:\mathbb{R}\to\mathbb{R},\quad W_{kj}=W_{jk}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$$

satisfying the following growth conditions (which guarantee the stability of the whole system).

(W) There exist some constants $R\geq 2$ and $J, C\geq 0$ such that

$$|W_{kj}(q,q')|\leq\frac{1}{2}J(C+|q|^R+|q'|^R),\quad q,q'\in\mathbb{R},\ k,j\in\mathbb{Z}^d.$$

(V) There exist a continuous function $V:\mathbb{R}\to\mathbb{R}$ and constants $P>R$, $A>0$ and $B\in\mathbb{R}$, such that

$$A|q|^P+B\leq V_k(q)\leq V(q),\quad q\in\mathbb{R},\quad k\in\mathbb{Z}^d.$$

Observe that, in terms of (1.2), we have $V_k=U_{\{k\}}$ and $W_{kj}=U_{\{k,j\}}$, $k,j\in\mathbb{Z}^d$.

Each quantum anharmonic oscillator (of mass $m>0$ and rigidity $a>0$) is individually described by the Schrödinger operator

$$\mathbb{H}_k := \left[-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x_k^2}+\frac{a}{2}x_k^2\right]+V_k(x_k)=\mathbb{H}_k^{\mathrm{har}}+V_k(x_k) \tag{2.2}$$

in the (physical) Hilbert state space $\mathcal{H}_k := L^2(\mathbb{R},\mathrm{d}x_k)$. Again, the infinite volume Hamiltonian (2.1) has no rigorous mathematical meaning and is "represented" by local Hamiltonians

$$\mathbb{H}_\Lambda=\sum_{k\in\Lambda}\mathbb{H}_k+\frac{1}{2}\sum_{\langle k,j\rangle\subset\Lambda}W_{kj}(x_k,x_j)$$

acting (as self-adjoint and lower bounded operators) in the corresponding Hilbert spaces $\mathcal{H}_\Lambda := L^2(\mathbb{R}^\Lambda,\mathrm{d}x_\Lambda)$.

In quantum statistical mechanics, Gibbs states are usually defined as positive normalized functionals on proper algebras of observables satisfying the Kubo-Martin-Schwinger (KMS) thermal equilibrium condition, see [52]. For a subsystem restricted to a finite volume $\Lambda\subset\mathbb{Z}^d$ and thus described by the local Hamiltonian $\mathbb{H}_\Lambda$ in the Hilbert space $\mathcal{H}_\Lambda$, the KMS condition is formulated by means of the unitary operators $\exp(it\mathbb{H}_\Lambda)$, $t\in\mathbb{R}$. To construct the dynamics of the whole system one has to take the limit of $\exp(it\mathbb{H}_\Lambda)$ as $\Lambda\nearrow\mathbb{Z}^d$. For the quantum lattice model (2.1), the above operator limit does not make rigorous sense and consequently the KMS condition cannot be formulated.

As an alternative, Sergio Albeverio and Raphael Høegh-Krohn proposed in their pioneering works [11, 12] to use the Euclidean (or path space) approach, which is conceptually analogous to the well-known Euclidean strategy in quantum field theory (see e.g. [68, 88]); for further developments see e.g. the review articles [17, 29, 79] and monograph [20], as well as extensive bibliography therein. The main idea of the Euclidean approach is to implement a path integral representation for the local Gibbs states, using stochastic processes generated by the semi-group $\exp(-t\mathbb{H}_\Lambda)$, $t > 0$. Then, being translated into a " probabilistic language" , the quantum model (2.1) at a fixed inverse temperature $\beta = 1/T > 0$ can be interpreted as an interacting system of continuous periodic paths (i.e., loops) $\omega_k \in C(S_\beta)$ indexed by $k \in \mathbb{Z}^d$, where $S_\beta \cong [0, \beta]$ is a circle of length β. Respectively, the initial problem of giving a proper meaning to the infinite volume quantum Gibbs state G_β transforms into the problem of studying a certain Euclidean Gibbs measure μ on the loop lattice $\Omega_\beta := [C(S_\beta)]^{\mathbb{Z}^d}$ (see Sect. 2.2). The distribution of each single spin ω_k is given by the path measure of the β-periodic Gaussian process (corresponding to the one-particle Hamiltonian $\mathbb{H}_k^{\mathrm{har}}$) multiplied by a density obtained from the anharmonic potential $V_k(x_k)$ with the help of the Feynman–Kac formula. Afterwards, finite subsystems in volumes Λ are associated with conditional probability measures on $\Omega_\Lambda := C(S_\beta)^\Lambda$, which by the standard DLR theory determine the set of tempered infinite volume Gibbs measures $\mathcal{G}^{\mathrm{t}}$.

Lattice systems of the above type (classical and quantum) are commonly viewed in statistical physics as mathematical models of a crystalline substance (for more physical background, see e.g. [17, 29, 68]). A particularly strong motivation to study such systems comes from the fact that they provide a mathematically rigorous as well as physically realistic description for the important phenomenon of phase transitions (i.e., non-uniqueness of Gibbs states). So, if the potential V has several minima, in the large mass limit $m \to \infty$ the quantum system (2.1) may undergo (ferroelectric) structural phase transitions connected with the appearance of macroscopic displacements of particles for low temperatures $\beta^{-1} < \beta_{\mathrm{cr}}^{-1}(m)$. For the mathematical theory of this effect, extending to the quantum setting the two basic techniques, namely: (i) the method of reflection positivity (for $d \geq 3$) involving the so-called infrared (Gaussian) bounds on two-point correlation functions, and (ii) the Peierls energy-entropy argument (for $d \geq 2$) being a part of the Pirogov–Sinai contour method, see e.g. [73, 74, 79] resp. [30, 61]. On the other hand, quantum effects occurring in particular at small values of the particle mass $m > 0$ can suppress abnormal fluctuations (which on the physical level was discussed e.g. in [82]). Thus, in this case one might expect that $|\mathcal{G}^{\mathrm{t}}| = 1$ holding simultaneously at all temperatures $\beta > 0$, which would be the strongest uniqueness result available for the ferromagnetic system (2.1). A mathematical justification of this effect, which was a long standing open problem, has been completely settled within the Euclidean approach in [14–19]; more on this see in Sect. 2.3. For the the ground state case $\beta = \infty$, the convergence of cluster expansions w.r.t. to the small masses parameter $m > 0$ has been proved in [24, 82]. The corresponding Gibbs measures on the path space $C(\mathbb{R})$ are known as $P(\varphi)_1$-processes and can be seen as a special case of the Euclidean field theory in the space-dimension zero [65, 72].

2.2 Definition of Euclidean Gibbs Measures

Below we briefly describe the corresponding Euclidean Gibbsian formalism just for the concrete class of quantum lattice systems (2.1); for a detailed exposition and an extensive bibliography we refer the reader to [17, 20, 29].

Let $S_\beta \cong [0, \beta]$ be a circle of length $\beta > 0$ (= inverse temperature) considered as a compact Riemannian manifold with Lebesgue measure $\mathrm{d}\tau$ as a volume element. Consider the standard Banach spaces

$$L^r(S_\beta) := L^r(S_\beta \to \mathbb{R}, \mathrm{d}\tau), r \geq 1,$$
$$C^\alpha(S_\beta) := C^\alpha(S_\beta \to \mathbb{R}), \quad \alpha \geq 0,$$

of all integrable resp. Hölder-continuous functions on S_β (i.e. loops of length β) and define the single-spin space

$$X := C(S_\beta) \,(= C^0(S_\beta)).$$

Thus the configuration space $\mathbf{X} = X^{\mathbb{Z}^d}$ of our infinite volume system is identified with the space of all temperature loop sequences

$$\Omega_\beta := [C(S_\beta)]^{\mathbb{Z}^d} = \left\{ \omega = (\omega_k)_{k \in \mathbb{Z}^d} \,\middle|\, \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d}, \omega_k \in C(S_\beta) \right\}$$

over $\mathbb{Z}^d$, endowed with the product topology and the corresponding Borel σ-algebra $\mathcal{B}(\Omega_\beta)$. Let $\mathcal{P}(\Omega_\beta)$ denote the set of all probability measures on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$. Next, we define a (Fréchet-type locally convex) space of exponentially tempered configurations

$$\Omega_\beta^{\mathrm{t}} := \left\{ \omega \in \Omega_\beta \,\middle|\, ||\omega||_{-\delta} := \left[\sum_{k \in \mathbb{Z}^d} \mathrm{e}^{-\delta|k|} \, |\omega_k|^2_{C(S_\beta)} \right]^{\frac{1}{2}} < \infty, \; \delta > 0 \right\} \tag{2.3}$$

and the set of tempered measures

$$\mathcal{P}^{\mathrm{t}}(\Omega_\beta) := \left\{ \mu \in \mathcal{P}(\Omega_\beta) \,\middle|\, \mu\left(\Omega_\beta^{\mathrm{t}}\right) = 1 \right\}.$$

Heuristically, the Euclidean Gibbs measures μ corresponding to the Hamiltonian (2.1) have the representation

$$\mathrm{d}\mu(\omega) := Z^{-1}\exp\{-\mathcal{I}(\omega)\}\bigotimes_{k\in\mathbb{Z}^d}\mathrm{d}\gamma_\beta(\omega_k), \tag{2.4}$$

where Z stands for a normalization factor and

$$\mathcal{I}(\omega) := \int_{S_\beta}\left[\sum_{k\in\mathbb{Z}^d}V_k(\omega_k) + \sum_{\langle k,j\rangle\subset\mathbb{Z}^d}W_{kk'}(\omega_k,\omega_j)\right]\mathrm{d}\tau$$

can be viewed as a potential energy describing the interacting system of loops $\omega_k \in C(S_\beta)$. Here, γ_β is a centered Gaussian measure on $C(S_\beta)$ with the (finite trace) correlation operator $\mathbb{A}_\beta^{-1}$ in $L^2(S_\beta)$, where $\mathbb{A}_\beta := -m\Delta_\beta + a\mathbf{1}$ is the (shifted) Laplace–Beltrami operator on the circle S_β. This "*free*" measure γ_β is related to a single harmonic oscillator with the Hamiltonian $\mathbb{H}_k^{\mathrm{har}}$, cf. (2.2). Notably, the associated β -periodic Ornstein-Uhlenbeck process $S_\beta \ni \tau \mapsto \omega_k(\tau) \in \mathbb{R}$ has appeared first in the context of quantum statistical mechanics in the papers of Albeverio and Høegh-Krohn [11, 12]; for a detailed account of its regularity properties see [16, 29]. In full analogy with classical statistical mechanics, a rigorous meaning can be given to the measure μ by the DLR formalism, namely as a Gibbsian random field on the lattice $\mathbb{Z}^d$. However, as compared with classical lattice systems like as in (2.7), the situation with Euclidean Gibbs measures is technically more complicated since now the spin spaces $X = C(S_\beta)$ are infinite dimensional and their topological features should be taken into account carefully.

To this end, for every finite set $\Lambda \subset \mathbb{Z}^d$, we define a probability kernel π_Λ on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ as follows: for all $\Delta \in \mathcal{B}(\Omega_\beta)$ and $\xi \in \Omega_\beta$

$$\pi_\Lambda(\Delta|\xi) := Z_\Lambda^{-1}(\xi)\int_{\Omega_\Lambda}\exp\{-\mathcal{I}_\Lambda(\omega|\xi)\}\mathbf{1}_\Delta(\omega_\Lambda,\xi_{\Lambda^c})\bigotimes_{k\in\Lambda}\mathrm{d}\gamma_\beta(\omega_k) \tag{2.5}$$

(where $\mathbf{1}_\Delta$ denotes the indicator on Δ). Here $Z_\Lambda(\xi)$ is the normalization factor and

$$\mathcal{I}_\Lambda(\omega|\xi) := \int_{S_\beta}\left[\sum_{k\in\Lambda}V_k(\omega_k) + \sum_{<k,j>\subset\Lambda}W_{kj}(\omega_k,\omega_j) + \sum_{k\in\Lambda, j\in\Lambda^c}W_{kj}(\omega_k,\xi_j)\right]d\tau$$

is the interaction in the volume Λ, subject to the boundary condition $\xi_{\Lambda^c} := (\xi_j)_{j\in\Lambda^c}$ in the complement $\Lambda^c := \mathbb{Z}^d\backslash\Lambda$. Obviously, the RHS in (2.5) makes sense under the above Assumptions (V), (W) on the interaction potentials V_k, W_{kj}. An important point is the consistency property: for all $\Lambda \subset \Lambda'$, $\xi \in \Omega_\beta$ and $\Delta \in \mathcal{B}(\Omega_\beta)$

$$(\pi_{\Lambda'}\pi_\Lambda)(\Delta|\xi) := \int_{\Omega_\beta}\pi_{\Lambda'}(\mathrm{d}\omega|\xi)\pi_\Lambda(\Delta|\omega) = \pi_{\Lambda'}(\Delta|\xi).$$

A probability measure μ on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ will be called the Euclidean Gibbs measure corresponding to the quantum lattice system (2.1) (at inverse temperature $\beta > 0$) if it satisfies the *DLR* equation:

$$\mu\pi_\Lambda(\Delta) := \int_{\Omega_\beta} \mu(\mathrm{d}\omega)\pi_\Lambda(\Delta|\omega) = \mu(\Delta) \tag{2.6}$$

for all $\Lambda \in \mathcal{B}_0(\mathbb{Z}^d)$ and $\Delta \in \mathcal{B}(\Omega_\beta)$. Fixing $\beta > 0$, we will be mainly concerned with the subset $\mathcal{G}^{\mathrm{t}}_\beta$ of tempered Gibbs measures supported by $\Omega^{\mathrm{t}}_\beta$, cf. (2.3).

Note that the large-mass limit $m \to +\infty$ of model (2.1) is a classical anharmonic crystal with the potential energy

$$H_{\mathrm{cl}}(x) = \sum_{k\in\mathbb{Z}^d} \left[\frac{a}{2}x_k^2 + V_k(x_k)\right] + \frac{1}{2}\sum_{\langle k,j\rangle\subset\mathbb{Z}^d} W_{kj}(x_k, x_j), \tag{2.7}$$

$x = (x_k)_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, see ([20] Sect. 4.2). So, under strong enough geometrical conditions on the interaction, most of the results below (e.g., the existence and uniqueness Theorems 1 and 3) are actually independent of the mass m and hence applicable both in the classical and quantum cases. On the other hand, there also could occur purely quantum effects, which are impossible in the classical analog of model (2.1), like e.g. the so-called quantum stabilization described in Theorem 4.

2.3 Existence and Uniqueness Results

The theorems of this section provide us with basic information which is needed for any further investigation of the Euclidean Gibbs measures. We suppose that Assumptions (W) and (V) are fulfilled without mentioning this again in the formulations of all subsequent statements.

Theorem 1 *[20, 79, 83] For all values of $\beta > 0$, the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}_\beta$ is nonempty. Moreover, every $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ is supported by the set of Hölder loops $\bigcap_{0\le\alpha<1/2} C^\alpha(S_\beta)$ and satisfies the exponential bound*

$$\sup_{k\in\mathbb{Z}^d} \int_{\Omega_\beta} \exp\left\{\varkappa|\omega_k|^2_{C^\alpha(S_\beta)} + \lambda|\omega_k|^R_{L^R(S_\beta)}\right\}\mu(\mathrm{d}\omega) \le \Xi_{\alpha,\beta}(\varkappa, \lambda), \tag{2.8}$$

holding for all $\alpha \in [0, 1/2)$, $\lambda > 0$ and $\varkappa \in [0, \varkappa^*_\beta)$, with the upper bound $\varkappa^*_\beta > 0$ depending only on $a, m, \beta > 0$. The constant $\Xi_{\alpha,\beta}(\varkappa, \lambda)$ (calculated explicitly in terms of the interaction parameters) can be chosen the same for all $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$.

Estimate (2.8) is called *a priori* since it certainly holds for each $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ independently of the way it might be constructed. It allows to easily gain further information about regularity and support properties of the elements of $\mathcal{G}^{\mathrm{t}}_{\beta}$, like e.g. a Ruelle-type bound on their local projections (cf. [87]). Moment estimates like (2.8) are also useful for the study of Gibbs measures by means of the associated Dirichlet operators $\mathbb{H}_{\mu}$ in the spaces $L^{p}(\mu),\ p \geq 1$, which is known as the Holley–Stroock approach to equilibrium states of infinite particle systems [36, 37, 70]; see also Sect. 2.5.

Let us give short comments on the technique used to prove Theorem 1. The key idea here, stated below as Lemma 2 and successively developed for a variety of models in [20, 78, 79], is to establish certain Lyapunov-type estimates for the one-point specification kernels $\pi_k(\mathrm{d}\omega|\xi)$ subject to varying boundary conditions $\xi \in \Omega^{\mathrm{t}}_{\beta}$ (to shorten notation we just write π_k instead of $\pi_{\{k\}}$).

Lemma 2 *For any $\alpha \in [0, 1/2)$, $\lambda > 0$ and $\varkappa \in [0, \varkappa^*_{\beta})$, there exists a corresponding $\Upsilon := \Upsilon_{\alpha,\beta}(\varkappa, \lambda) > 0$ such that*

$$\int_{\Omega_\beta} \exp\left\{\varkappa|\omega_k|^2_{C^\alpha(S_\beta)} + \lambda|\omega_k|^R_{L^R(S_\beta)}\right\} \pi_k(\mathrm{d}\omega|\xi) \leq \exp\left\{\Upsilon + J \sum_{j:|k-j|=1} |\xi_j|^R_{L^R(S_\beta)}\right\} \tag{2.9}$$

for all $k \in \mathbb{Z}^d$ and $\xi \in \Omega^{\mathrm{t}}_{\beta}$.

We note that the exponential estimate (2.9) is stronger (but actually easier to check) than those usually required in Dobrushin's classical existence criterion [60]. Therefrom, using the spatial Markov property of the Gibbs specification, one concludes that there exists a constant $\Xi_{\alpha,\beta}(\varkappa, \lambda) > 0$ such that

$$\limsup_{\Lambda \nearrow \mathbb{Z}^d} \int_{\Omega_\beta} \exp\left\{\varkappa|\omega_k|^2_{C^\alpha(S_\beta)} + \lambda|\omega_k|^R_{L^R(S_\beta)}\right\} \pi_\Lambda(\mathrm{d}\omega|\xi) \leq \Xi_{\alpha,\beta}(\varkappa, \lambda), \tag{2.10}$$

uniformly for all $k \in \mathbb{Z}^d$ and $\xi \in \Omega^{\mathrm{t}}_{\beta}$. As a consequence (2.10), implies not only the existence of at least one $\mu \in \mathcal{G}^{\mathrm{t}}_{\beta}$, but also yields the uniform bounds on all points of the set $\mathcal{G}^{\mathrm{t}}_{\beta}$ and its compactness in appropriate topologies. The method obviously extends to general N-particle interactions or spin systems on irregular graphs (see e.g. [83]), which essentially improves all related existence results. Recent developments show that the method also applies to the interacting particle systems in continuum [56, 78] , so that to certain extent it can be viewed as a reasonable alternative to Ruelle's superstability estimates [80, 87].

In contrast, the uniqueness results presented below takes regard of the concrete structure of the one-particle and pair potentials. Just for simplicity, let us consider the translation invariant system (with $V_k = V$ and $W_{kj} = W$ for $|k - j| = 1$), assuming the attractive (i.e., ferromagnetic) harmonic pair interaction

$$W(x_k, x_j) := J(x_k - x_j)^2/2 \geq 0 \text{ with intensity } J > 0. \tag{2.11}$$

Theorem 3 *[41–43] Suppose that the anharmonic self-interaction admits the decomposition*

$$V = U + Q, \tag{2.12}$$

where $U \in C^2(\mathbb{R})$ is a strictly convex function and $Q \in C_{\mathrm{b}}(\mathbb{R})$ is a bounded perturbation (describing the presence of possible wells). Define

$$b := \inf_{q\in\mathbb{R}} U''(q), \operatorname{osc}(Q) := \sup_{q\in\mathbb{R}} Q(q) - \inf_{q\in\mathbb{R}} Q(q).$$

Then, for all values of the mass $m > 0$, the set $\mathcal{G}^{\mathrm{t}}_{\beta}$ is a singleton provided the following relation between the model parameters holds:

$$\frac{e^{\beta \operatorname{osc}(Q)}}{2d + J^{-1}(a^2 + b^2)} < \frac{1}{2d}.$$

The proof of Theorem 3 employes Dobrushin's uniqueness criterion for Markov fields [60]. Because of unbounded interactions, one has to use Wasserstein-type distances to control the weak dependence of single-spin conditional measures $\pi_k(\mathrm{d}\omega|\xi)$ on boundary configurations $\xi \in \Omega^{\mathrm{t}}_{\beta}$. It is well known, however, that multi-dimensional Wasserstein distances are hard (and often impossible) to estimate accurately. To overcome this technical issue, there has been first proposed in [41–43] to estimate the coefficients of Dobrushin's interdependence matrix by means of log-Sobolev inequalities proved for the measures $\pi_k(\mathrm{d}\omega|\xi)$ on the tangent Hilbert space $L^2(S_\beta)$. By this method, the uniqueness has been established for small values of the inverse temperature $\beta > 0$, but under the geometric stability conditions independent of the particle mass (and hence holding also in the quasiclassical regime $m \searrow 0$). The above result remains true if one takes a general ferromagnetic interaction $W(x_k, x_j) := w(x_k - x_j)$ given by a nonnegative convex function $w \in C^2(\mathbb{R} \to \mathbb{R})$ such that $J \le \inf_{\mathbb{R}} w''(q) \le \sup_{\mathbb{R}} w''(q) < \infty$. The uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}_{\beta}$ in quantum lattice systems (2.1) with superquadratic growth of the many-particle interaction (i.e., beyond the range of application of Dobrushin's theorem) was studied in [83].

Typical one-particle potentials satisfying (2.12) (as well as all the basic assumption V with $P = 2\mathrm{p} \ge 2$) are polynomials of even degree and with a positive leading coefficient

$$V_k(q) := P(q) = \sum_{1\le \mathrm{s} \le \mathrm{p}} b_{\mathrm{s}} q^{2\mathrm{s}} \text{ with } b_{\mathrm{p}} > 0 \text{ and } \mathrm{p} \ge 2. \tag{2.13}$$

In this case one speaks about so-called ferromagnetic $P(\varphi)$–models, which also can be looked upon as lattice discretizations of quantum $P(\varphi)$–fields [68, 88]. Due to choice of a large enough negative b_1, the potential (2.13) may have arbitrarily deep double wells. So, the corresponding lattice system (2.1) may serve as the simplest but realistic model for the appearance of phase transitions and the influence of quantum effects. Moreover, this system is technically well-suitable for the study of critical behavior, in sofar one can use various correlation inequalities employing additional symmetries of the polynomial $P(q)$. Below we state the strongest uniqueness result for scalar ferromagnetic systems, which demonstrates a purely quantum effect of suppression of the structural phase transitions by the small particle mass. Its proof its rather involved and requires, among other things, the detailed spectral analysis of the single-particle oscillators (2.2).

Theorem 4 *[15–19] For the quantum lattice model (2.1) with the harmonic interaction (2.11) and polynomial self-interaction (2.13), there exists* $m^* > 0$ *such that, for all* $m \in (0, m^*)$ *and all temperatures* $\beta > 0$, *the set* $\mathcal{G}^{\mathrm{t}}_{\beta}$ *consists of exactly one point.*

2.4 Integration by Parts Characterization

Next, we outline the so-called analytic approach to Gibbs measures, which is based on their characterization via integration by parts [(instead of the traditional one through the local specification $\{\pi_\Lambda\}$ and the DLR equation (2.6)]. Although such alternative descriptions of Gibbs measures have long been known for a number of specific models in statistical mechanics and field theory (see e.g. [68, 70]), the corresponding program for the quantum lattice systems (2.2) [(as a by-product, including their classical version (2.7)] has first been completely realized by Albeverio with coworkers in [25–29] and [36, 37, 44, 45]. This provides yet another striking example of Sergio's activity in building bridges between stochastic analysis, infinite dimensional analysis, and quantum physics.

Let us consider $\mathcal{H}_\beta := l^2(\mathbb{Z}^d \to L^2(S_\beta))$ with the scalar product $< \omega, \omega >_{\mathcal{H}_\beta} = ||\omega||^2_{\mathcal{H}_\beta} := \sum_{k\in\mathbb{Z}^d} |\omega_k|^2_{L^2(S_\beta)}$ as the tangent Hilbert space to the configuration (i.e., product) space Ω_β. We fix an orthonormal basis in $\mathcal{H}_\beta$ consisting of the vectors $h_i := \{\delta_{k-j}\varphi_n\}_{j\in\mathbb{Z}^d}$ indexed by $i = (k, n) \in \mathbb{Z}^{d+1}$, where $\{\varphi_n\}_{n\in\mathbb{Z}} \subset C^\infty(S_\beta)$ is the complete orthonormal system of eigenvectors of the operator $\mathbb{A}_\beta$ in $H := L^2(S_\beta)$, i.e., $\mathbb{A}_\beta\varphi_n = \lambda_n\varphi_n$ with $\lambda_n = (2\pi n/\beta)^2 m + a$. Recall that the Laplace–Beltrami operator $\mathbb{A}_\beta = -m\Delta_\beta + a\mathbf{1}$ was used to define the "free" Gaussian measure γ_β in Sect. 2.2.

From hereon, we additionally assume that the potentials V_k and W_{kj} are of C^1-class, that is, continuously differentiable. This allows us to define the vector field $b = (b_i)_{i\in\mathbb{Z}^{d+1}} : \Omega_\beta \to \mathbb{R}^{\mathbb{Z}^{d+1}}$ with components

$$b_i(\omega) := -(\mathbb{A}_\beta\varphi_n, \omega_k)_{L^2(S_\beta)} - (F_k(\omega), \varphi_n)_{L^2(S_\beta)}, \tag{2.14}$$

where $F_k : \Omega_\beta \to C(S_\beta)$ is the nonlinear Nemytskii-type operator acting by

$$F_k(\omega) := V_k'(\omega_k) + \sum_{j\neq k} \partial_{q'} W_{kj}(\omega_k, q')\big|_{q'=\omega_j}. \tag{2.15}$$

For each $i = (k, n) \in \mathbb{Z}^{d+1}$, we denote by $C^1_{\mathrm{dec},i}(\Omega_\beta)$ the set of all functions $f : \Omega_\beta \to \mathbb{R}$ which are bounded and continuous together with their partial derivatives

$\partial_i f$ in direction h_i and, moreover, satisfy the extra decay condition

$$\sup_{\omega \in \Omega_\beta} |f(\omega)| \cdot \left[1 + |\omega_k|_{L^2(S_\beta)} + |F_k(\omega)|_{L^2(S_\beta)}\right] < \infty.$$

Of course, $f b_i \in L^\infty(\mu)$ for any $f \in C^1_{\mathrm{dec},i}(\Omega^{\mathrm{t}}_\beta)$ and any $\mu \in \mathcal{P}(\Omega_\beta)$, even though we do not know a priori whether $b_i \in L^1(\mu)$.

For smooth interaction potentials, the initial definition (2.6) of $\mu \in \mathcal{G}_\beta$ as spatial Markov (i.e., DLR) fields will be equivalent to their characterization as differentiable measures solving the integration by parts (for short, IbP) equations

$$\partial_{h_i} \mu\,(\mathrm{d}\omega) = b_i(\omega) \cdot \mu(\mathrm{d}\omega), \, i \in \mathbb{Z}^{d+1},$$

with the so-called partial logarithmic derivatives b_i prescribed by (2.14).

Theorem 5 *[25–29] Let $\mathcal{P}_b$ denote the set of all probability measures $\mu \in \mathcal{P}(\Omega_\beta)$ which satisfy the (IbP)-formula*

$$\int_{\Omega_\beta} \partial_i f(\omega)\, \mathrm{d}\mu(\omega) = -\int_{\Omega_\beta} f(\omega) b_i(\omega)\, \mathrm{d}\mu(\omega) \tag{2.16}$$

for all test functions $f \in C^1_{\mathrm{dec},i}(\Omega_t)$ and all basis directions h_i, $i = \mathbb{Z}^{d+1}$. Then $\mathcal{G}_\beta = \mathcal{P}_b$.

Let us stress that the above mappings b_i depend only on the potentials V_k and W_{kj}, and hence are the same for all $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ associated with the heuristic Hamiltonian (2.1). In stochastic analysis, solutions μ to the (IbP)-formula (2.16) are also called symmetrizing measures. For further connections to reversible diffusion processes and Dirichlet operators in infinite dimensions see Sect. 2.5.

The most progress achieved so far in the analytic approach is related with the existence problem and a priori estimates for the associated Gibbs measures; see e.g. [25–29] for an alternative proof of Theorem 1. The key ingredient of the (IbP)-method is that according to (2.16) each $\pi_\Lambda(\mathrm{d}\omega|\xi)$ resp. $\mu \in \mathcal{G}_\beta$ might be viewed as a solution of an infinite system of first order partial differential equations (PDE's). Under reasonable assumptions on the potentials V_k and W_{kj}, the corresponding vector fields b will possess certain coercivity properties w.r.t. the tangent space $\mathcal{H}_\beta$, which then enables us to employ here an analog of the Lyapunov function methodwell-known from finite dimensional PDE's. On this way we get, in particular, the uniform moment estimates on $\pi_\Lambda(\mathrm{d}\omega|\xi)$ similar to those in (2.10), which in turn is a crucial step for proving Theorem 1. For the first time this approach has been implemented in [44, 45], however in the much simpler situation of the classical spin systems (2.7). Its extension to the quantum case performed in [25–29] requires for (highly non-trivial) technical modifications, also involving a "loop space analysis" based on the spectral properties of the elliptic operator $\mathbb{A}_\beta$. Other important and long-standing problem in infinite dimensions is to find conditions sufficient for the uniqueness of symmetrizing measures; for some particular results on this topic see [49].

2.5 Stochastic Dynamics Associated with Euclidean Gibbs Measures

The (IbP)-description of $\mu \in \mathcal{G}_\beta^{\mathrm{t}}$ provides a background for the stochastic dynamics method (also referred to in quantum physics as "stochastic quantization"), in which the Gibbs measures are treated as invariant (more precise, reversible) distributions for certain stochastic evolutions in time, see e.g. [46, 47]. Of course, some additional technical restrictions are required on the interaction in order to ensure the unique solvability of the corresponding stochastic equations in infinite dimensional spaces.

Actually, in the literature there are two complementary and deeply interrelated constructions of the corresponding stochastic dynamics.

(i) Equilibrium dynamics. Given $\mu \in \mathcal{P}_b^{\mathrm{t}}$, let us assume that its partial logarithmic derivatives b_i, $i = (k, n) \in \mathbb{Z}^{d+1}$, exist and belong to $L^2(\mu)$ (which holds for all $\mu \in \mathcal{G}_\beta^{\mathrm{t}}$ by Theorem 1). Consider a differential expression

$$\mathbb{H}_b f := -\frac{1}{2} \sum_{i \in \mathbb{Z}^{d+1}} [\partial_i^2 f + b_i \partial_i f]$$

correctly defined on smooth cylinder functions $f \in \mathcal{F}C_{\mathrm{b}}^2(\Omega_\beta)$. Each element of $\mathcal{D} := \mathcal{F}C_{\mathrm{b}}^2(\Omega_\beta)$ can be written in the form

$$f(\omega) = f_L\left((\omega_{k_1}, \varphi_{n_1})_{L^2(S_\beta)}, \ldots, (\omega_{k_L}, \varphi_{n_L})_{L^2(S_\beta)}\right)$$

with some $f_L \in C_{\mathrm{b}}^2(\mathbb{R}^L)$ and $L \in \mathbb{N}$, where $\{\varphi_n\}_{n \in \mathbb{Z}} \subset C^\infty(S_\beta)$ is a complete orthonormal system of the operator $\mathbb{A}_\beta$ in $L^2(S_\beta)$, cf. Sect. 2.4. Then, as follows from the IbP-formula (2.16), μ will be a symmetrizing measure for $\mathbb{H}_b$ in a sense that for all $f, g \in \mathcal{D}$

$$\int_{\Omega_\beta} g \cdot \mathbb{H}_b f \, \mathrm{d}\mu = \int_{\Omega_\beta} f \cdot \mathbb{H}_b g \, \mathrm{d}\mu = \frac{1}{2} \sum_{i \in \mathbb{Z}^{d+1}} \int_{\Omega_\beta} \partial_i f \cdot \partial_i g \, \mathrm{d}\mu. \tag{2.17}$$

A self-adjoint operator $(\mathbb{H}_\mu, \mathcal{D}(\mathbb{H}_\mu))$, defined as the Friedrichs extension of $(\mathbb{H}_b, \mathcal{D})$ in $L^2(\mu)$, is called the (classical) Dirichlet operator of the measure μ. Consequently, this μ will be a reversible measure for the sub-Markovian (i.e., positivity and identity preserving) semigroup – equilibrium stochastic dynamics $\mathbb{P}_t^\mu := \exp(-t\mathbb{H}_\mu)$, $t \geq 0$, i.e.,

$$(\mathbb{P}_t^\mu f, \, g)_{L^2(\mu)} = (f, \, \mathbb{P}_t^\mu g)_{L^2(\mu)}, \quad \forall f, g \in L^2(\mu), t \geq 0. \tag{2.18}$$

Moreover, using the powerful machinery of Dirichlet forms [81], one can conclude that there exists a $\Omega_\beta^{\mathrm{t}}$-valued diffusion process $x(t)$, $t \geq 0$, with the generator $(\mathbb{H}_\mu, \mathcal{D}(\mathbb{H}_\mu))$ and time-reversible initial distribution μ, which is a weak solution to the stochastic differential equation (2.23).

One of Albeverio's most impressive results here is characterization of the set of all tempered measures $\mathcal{G}_\beta^{\mathrm{t}}$ through the properties of the associated stochastic dynamics, which extends the famous theorem of Holley and Stroock initially proved in [70] for the Ising model. By [36, 37] (and, respectively [39, 40] for particle systems in the continuum), a given μ is an extreme point (or pure phase) in $\mathcal{G}_\beta^{\mathrm{t}}$ if and only if the corresponding semigroup $\mathbb{P}_t^\mu$, $t \geq 0$, is ergodic in $L^2(\mu)$, that is,

$$\lim_{t\to\infty} \left\| \mathbb{P}_t^\mu f - < f >_\mu \right\|_{L^2(\mu)} = 0, \quad \forall f \in L^2(\mu).$$

This further motivates the study of spectral properties of the Dirichlet operators $\mathbb{H}_\mu$. Under appropriate semi-dissipativity assumptions on the logarithmic derivatives b_i, the essential self-adjointness of the Dirichlet operators, as well as the presence of a spectral gap and the validity of a log-Sobolev inequality for them, was first shown in [35, 89, 90] and [42, 83] in the classical and quantum cases, respectively.

(ii) Nonequilibrium dynamics. Conversely to (i), let us start with a time-homogeneous continuous Markov process x_t, $t \geq 0$, taking values in some Polish space X. Suppose that the associated transition semigroup $\mathbb{P}_t$, $t \geq 0$, which is defined on $f \in C_{\mathrm{b}}(X)$ by

$$\mathbb{P}_t f(x) := \mathbb{E}\{f(x(t)) \,|\, x(0) = x\}, x \in X,$$

is Feller (i.e., it preserves the Banach space $C_{\mathrm{b}}(X)$). A basic (but often very difficult) problem here is to describe the sets $\mathcal{R}(X)$ and $\mathcal{I}(X)$ of all reversible and invariant distributions for x_t, $t \geq 0$, respectively. By definition, these are probability measures μ on $(X, \mathcal{B}(X))$ obeying, cf. (2.18),

$$\int_X (\mathbb{P}_t f) g \,\mathrm{d}\mu = \int_X f(\mathbb{P}_t g) \,\mathrm{d}\mu \quad \text{resp.} \tag{2.19}$$

$$\int_X \mathbb{P}_t f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu, \quad \forall f, g \in C_{\mathrm{b}}(X), t \geq 0. \tag{2.20}$$

Notably, there is a priori inclusion $\mathcal{R}(X) \subseteq \mathcal{I}(X)$. For any $\mu \in \mathcal{R}(X)$ (provided such exists), $\mathbb{P}_t$, $t \geq 0$, uniquely extends to a symmetric contraction C_0-semigroup on $L^2(\mu)$. Furthermore, this semigroup is sub-Markovian and hence contractive in all $L^p(\mu)$, $1 \leq p \leq +\infty$. Let $(\mathbb{H}, \mathcal{D}(\mathbb{H}))$ be its infinitesimal generator in $L^2(\mu)$; it is clear that $\mathbf{1} \in \mathcal{D}(\mathbb{H})$ with $\mathbb{H}\mathbf{1} = 0$. In particular, Eqs. (2.19)–(2.20) implies that μ will be symmetrizing and hence also infinitesimally invariant for $\mathbb{H}$, that is, by analogy with (2.17)

$$(\mathbb{H}f, g)_{L^2(\mu)} = (f, \mathbb{H}g)_{L^2(\mu)}, \int_X \mathbb{H}f \,\mathrm{d}\mu = 0, \forall f, g \in \mathcal{D}(\mathbb{H}).$$

For some dynamics of the gradient type, which are associated with classical or quantum particle systems of interest, one might expect that the set $\mathcal{R}(X)$ is nonempty in so far as it has to contain the related Gibbs states. The question when the sets of invariant and reversible distributions do coincide, i.e., $\mathcal{R}(X) = \mathcal{I}(X)$, was dealt with for classical lattice systems in [50, 70].

(iii) Stochastic quantization. Based on [46], we now briefly describe the main ingredients of the stochastic quantization procedure when applied to the lattice system (2.1) at a finite temperature $\beta > 0$ (whereas the case of $\beta = \infty$ was considered in [47]). Let us restrict ourselves to the case of harmonic pair interactions; in a straightforward way the method extends to (many-particle) interactions of at most quadratic growth. Concerning the one-particle potentials $V_k \in C^2(\mathbb{R} \to \mathbb{R})$, the following semi-monotonicity and polynomial growth conditions

$$(V_k'(q) - V_k'(q'))(q - q') \geq K^{-1}(q - q')^2 - L, \tag{2.21}$$

$$|V_k'(q)| \leq K(1 + |q|)^M, q, q' \in \mathbb{R}, \tag{2.22}$$

with some $K, L > 0$ and $M \geq 1$ (implying the basic Assumption (V) in Sect. 2.2) are required to hold uniformly for all $k \in \mathbb{Z}^d$.

Under the above assumptions, one can construct a Markov process $x_t = (x_{k,t})_{k\in\mathbb{Z}^d}$, $t \geq 0$, which gives the unique generalized solution (in a usual for PDE's sense) to the (so-called Langevin or Glauber) stochastic evolution equation with a drift coefficient being the logarithmic gradient b of the measures $\mu \in \mathcal{G}_\beta$. More precisely, x_t, $t \geq 0$, takes values in the weighted Banach spaces $\mathbf{X}_\delta := l^2(\mathbb{Z}^d \to C(S_\beta); e^{-\delta|k|})$ (used in (2.3) to define $\Omega_\beta^{\mathrm{t}} = \bigcap_{\delta>0}\mathbf{X}_\delta$) and satisfies the following infinite system of stochastic partial differential equations (SPDE's):

$$\frac{\partial}{\partial t} x_{k,t} = -\frac{1}{2}\mathbb{A}_\beta x_{k,t} + F_k(x_t) + \dot{w}_{k,t}, k \in \mathbb{Z}^d, t > 0, \tag{2.23}$$

where $\dot{w}_{k,t}(\tau)$ is a Gaussian white noise on $\Omega_\beta \times [0, \infty)$ (i.e., heuristically $\mathbb{E}\dot{w}_{k,t}(\tau)\dot{w}_{j,t'}(\tau') = \delta_{k-j}\delta_{t-t'}\delta_{\tau-\tau'}$). Along with the singular random forces $\dot{w}_{k,t}$, a further technical problem here is caused by the nonlinear (possibly unbounded) drift terms $F_k(x) := V_k'(x_k) + J\sum_{j:\,|k-j|=1}(x_j - x_k)$, cf. (2.15). In the classical case (where the continuous parameter $\tau \in S_\beta$ is absent), we just obtain a system of interacting Itô diffusions like those considered in Sect. 3. On the other hand, each single line in (2.23) is a parabolic reaction–diffusion equation with the periodic boundary conditions on $[0, \beta]$, driven by an additional noise $\dot{w}_{k,t}$.

In the trivial case when $W = V = 0$, the solution of (2.23), starting with initial data $\zeta \in \mathbf{X}_\delta$, is explicitly given by the Ornstein–Uhlenbeck process

$$g_{k,t} := \mathrm{e}^{-t\mathbb{A}_\beta/2}\zeta_k + \int_0^t \mathrm{e}^{-\frac{1}{2}(t-s)\mathbb{A}_\beta}\mathrm{d}w_{k,s}, \quad k \in \mathbb{Z}^d, t \geq 0. \tag{2.24}$$

Taking into account the regularity properties of the semigroup $e^{-t\mathbb{A}_\beta}$, $t \geq 0$, one can deduce from (2.24) that $g_t = (g_{k,t})_{k\in\mathbb{Z}^d}$ admits a continuous modification in the spaces of Hölder loops $C^\alpha(S_\beta)$, $\alpha \in [0, 1/2)$, and is ergodic in time with the unique invariant distribution $\otimes_{k\in\mathbb{Z}^d}\gamma_\beta(\mathrm{d}\omega_k)$ (where γ_β is the same as in (2.4)).

A standard practice then consists of replacing (2.23) with the equivalent system of integral equations

$$x_{k,t} = g_{k,t} + \int_0^t \mathrm{e}^{-\frac{1}{2}(t-s)\mathbb{A}_\beta} F_k(x_s)\,\mathrm{d}s,\ k \in \mathbb{Z}^d,\ t \geq 0, \tag{2.25}$$

whereby any process $x_t = (x_{k,t})_{k\in\mathbb{Z}^d}$ satisfying (2.25) is called mild solution to (2.23). Using finite volume approximations, it was proved in [46] that under conditions (2.21)–(2.22), for each initial data $\zeta \in \Omega^{\mathrm{t}}_\beta := \cap_{\delta>0}\mathbf{X}_\delta$ there exists a unique continuous solution $t \mapsto x_t \in \Omega^{\mathrm{t}}_\beta$ to the Cauchy problem (2.25). Moreover, for any $M \geq 1$ we have the following asymptotic bound

$$\limsup_{t\to\infty} \mathbb{E}\left\{ |x_{k,t}|^M_{C(S_\beta)} + \frac{1}{t}\int_t^{2t} |x_{k,s}|^2_{C^\alpha(S_\beta)}\mathrm{d}s \right\} \leq \Xi_M < +\infty, \tag{2.26}$$

where the constant on the right-hand side can be chosen the same for all $\zeta \in \Omega^{\mathrm{t}}_\beta$. Analogous estimates also hold for the solutions $x^\Lambda_t = (x^\Lambda_{k,t})_{k\in\Lambda} \in C(S_\beta)^\Lambda$ of the corresponding "*cut-off*" dynamics

$$\frac{\partial}{\partial t}x^\Lambda_{k,t} = -\frac{1}{2}\left[\mathbb{A}_\beta x^\Lambda_{k,t} + F_k(x^\Lambda_t)\right] + \dot{w}_{k,t},\ k \in \Lambda,\ t > 0, \tag{2.27}$$

in finite volumes $\Lambda \Subset \mathbb{Z}^d$, with fixed initial data $x_0 := \zeta$ and boundary conditions $x^\Lambda_{k,t} := \zeta_k$ for $k \in \Lambda^{\mathrm{c}}$.

As was further shown in [46],

$$(\mathbb{P}_t f)(\omega) := \mathbb{E}\{f(x_t)\,|\,x_0 = \omega\}, \quad \omega \in \Omega^{\mathrm{t}}_\beta, \tag{2.28}$$

is a Feller transition semigroup in the space $C_{\mathrm{b}}(\Omega^{\mathrm{t}}_\beta)$ of all bounded continuous functions $f : \Omega^{\mathrm{t}}_\beta \to \mathbb{R}$. Let $\mathcal{R}^{\mathrm{t}}$ (resp. $\mathcal{I}^{\mathrm{t}}$) denote the family of all tempered reversible (resp. invariant) distributions $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ for the Markov process x_t, $t \geq 0$, in the sense of (2.19)–(2.20). Then there is the following relation

$$\mathcal{G}^{\mathrm{t}}_\beta = \mathcal{R}^{\mathrm{t}} \subseteq \mathcal{I}^{\mathrm{t}},$$

with the non-trivial equivalence between the Gibbsian property and the stochastic reversibility (for its proof involving Itô's stochastic calculus and (IbP)-formulas, cf. e.g. [65, 71, 72, 77, 85]). Moreover, in our situation one can directly verify that the finite volume Gibbs measures $\pi_\Lambda(\mathrm{d}\omega|\xi)$, which were defined in (2.5), are exactly reversible distributions for the corresponding cut-off dynamics (2.27). Thus, in order to get the required information on $\mu \in \mathcal{G}_\beta^{\mathrm{t}} \subseteq \mathcal{I}^{\mathrm{t}}$, one could apply standard tools used for the long-time analysis of diffusion processes. So, the existence of invariant measures $\mu \in \mathcal{I}^{\mathrm{t}}$ is a standard consequence of (2.26) combined with Prokhorov's tightness criterion and the Bogolyubov–Krylov argument. Furthermore, by the ergodic theorem for invariant distributions, Eq. (2.26) readily implies that

$$\sup_{\mu\in\mathcal{I}^{\mathrm{t}},k\in\mathbb{Z}^d} \int\limits_{\Omega_\beta} \left[|x_{k,t}|^M_{C(S_\beta)} + |x_{k,t}|^2_{C^\alpha(S_\beta)} \mathrm{d}s \right] \mathrm{d}\mu(\omega) < \infty,$$

which agrees with the moment bounds from Theorem 1. To verify the existence of $\mu \in \mathcal{G}_\beta^{\mathrm{t}} = \mathcal{R}^{\mathrm{t}}$, it would be enough to prove the tightness in $\Omega_\beta^{\mathrm{t}}$ of the local kernels $\left\{\pi_{\beta,\Lambda}(\mathrm{d}\omega|0)\right\}_{\Lambda\Subset\mathbb{Z}^d}$ with fixed boundary condition $\xi = 0$. This later would be again a consequence of Prokhorov's criterion , but now combined with the uniform estimates on the solutions x_t^Λ, $t \geq 0$, of the finite-volume dynamics (2.27). Due to the finite range of the pair interaction, each cluster point $\mu := \lim_{\Lambda_n \nearrow \mathbb{Z}^d} \pi_{\beta,\Lambda_n}(\mathrm{d}\omega|0)$ will be surely Gibbs.

Finally, let us mention that the ergodicity problem (yielding the uniqueness of $\mu \in \mathcal{I}^{\mathrm{t}}$) for infinite stochastic systems with unbounded spins like (2.23) (except for the special cases of linear or strictly dissipative ones) is commonly recognized to be extremely difficult and so far remains mostly open. By now, there are no technical means to recover the uniqueness results of Theorems 3 and 4 for $\mu \in \mathcal{G}_\beta^{\mathrm{t}}$ by the stochastic dynamics method.

3 Stochastic Dynamics for Lattice Models with Compact Spin Spaces

3.1 Infinite Product Manifolds

On the other end of the scale is the study of classical infinite-particle systems with single-particle space X of complicated geometry. In particular, we consider the case where X is a compact Riemannian manifold (for which we will use notation M in order to distinguish it from a linear single-particle space). The existence of Gibbs measures for compact spin spaces under very general conditions on the interaction potentials is well known, see e.g. [67]. Our primary goal here is to discuss the construction of non-equilibrium stochastic dynamics associated with these measures, which we do in Sect. 3 using the stochastic differential equations techniques.

Observe that the space $\mathbf{M} = M^{\mathbb{Z}^d}$ possesses the natural structure of a Banach manifold modelled on the Banach space of bounded sequences $y = (y_k)_{k\in\mathbb{Z}^d}$, $y_k \in \mathbb{R}^{\dim M}$, with the supremum norm. However, this norm being not smooth, one gets difficulties in using the corresponding manifold structure for the purposes of stochastic analysis. The main idea of papers [4–8] is to introduce a special Riemannian-like structure on $\mathbf{M}$. On a heuristic level, the tangent bundle $T\mathbf{M}$ is the $\mathbb{Z}^d$-power of TM, that is, $T_x\mathbf{M} = \times_{k\in\mathbb{Z}^d} T_{x_k} M_k$. It is natural to consider certain Hilbert sub-bundles of $T\mathbf{M}$. For a fixed weight sequence $p = (p_k)_{k\in\mathbb{Z}_+} \in l_1^+$ we define the Hilbert space $\mathbf{T}_{p,x} \subset T_x\mathbf{M}$ with the inner product

$$(\xi, \eta)_{p,x} = \sum_{k\in\mathbb{Z}^d} p_{|k|}(\xi_k, \eta_k)_{T_{x_k}M}$$

which will play the role of a Riemannian-like structure for $\mathbf{M}$. The space $\mathbf{M}$ endowed with this structure will be denoted by $\mathbf{M}_p$. Observe that $\mathbf{M}_p$, which topologically coincides with $\mathbf{M}$, is not a Hilbert manifold in a proper sense. However, we can introduce the classes of "$\mathbf{M}_p$-differentiable" mappings. In particular, a vector filed $\beta(x) = \big(\beta_j(x)\big)_{j\in\mathbb{Z}^d}$, $\beta_j(x) \in T_{x_j}M$, is said to be $\mathbf{M}_p$-differentable if the infinite matrix $\nabla\beta(x) := \big(\nabla_k\beta_j(x)\big)_{k,j\in\mathbb{Z}^d}$ generates a bounded operator in $\mathbf{T}_{p,x}$. Here ∇_k stands for the Levi-Civita covariant derivative w.r.t. x_k. This class of vector fields will be denoted by $C^1(\mathbf{M}_p \to T\mathbf{M}_p)$. Similarly, we can introduce the spaces $C^m(\mathbf{M}_p \to T\mathbf{M}_p)$, $m = 2, 3, ...$

In this section, we give a brief review of the work devoted to the development of general stochastic analysis on infinite product manifolds, concentrating mainly on the study of (non-equilibrium) stochastic dynamics of lattice models with spin spaces given by M, cf. Sect. 3.1. Thus, we consider a system of particles governed by Hamiltonian (1.3) with $X = M$ and χ = Riemannian volume on M. The compactness of M allows us to deal with interactions of unbounded range. We will require however that the family of potentials $\mathcal{U}$, cf. (1.2), satisfies the following regularity condition: $U_\Lambda \in C^1(M^\Lambda)$, $\Lambda \in \mathcal{B}_0(\mathbb{Z}^d)$, and

$$\sum_{\Lambda\in\Omega(k)} \sup_{x\in\mathbf{M}} |U_\Lambda(x_\Lambda)| < \infty,\ \ k \in \mathbb{Z}^d, \tag{3.1}$$

where $\Omega(k)$ is the collection of all $\Lambda \in \mathcal{B}_0(\mathbb{Z}^d)$ such that $k \in \Lambda$.

In order to be able to construct and study the corresponding stochastic dynamics, we first consider the general theory of stochastic differential equations and differentiable measures on $\mathbf{M}$.

A more advanced geometric analysis on $\mathbf{M}$ has been developed in [9, 10]. In particular, in [9], the authors defined the de Rham complex over $\mathbf{M}_p$, and considered Markov processes generated by the corresponding Bochner and de Rham Laplacians. Some other questions of stochastic analysis on product manifolds, like the quasi-invariance and Gibbs structure of distributions of the stochastic dynamics and its ergodicity, were studied in [4, 5] and [8], respectively.

3.2 Stochastic Differential Equations on Product Manifolds

We consider the following system of SDEs describing (non-equilibrium) stochastic dynamics of spins x_k:

$$\mathrm{d}x_k(t) = b_k(x(t))\mathrm{d}t + \mathrm{d}w_k(t),\ k \in \mathbb{Z}^d. \tag{3.2}$$

Here

$$b_k(x) := -\nabla_k V_k(x) \in T_{x_k}M \tag{3.3}$$

with $V_k(x) := \sum_{\Lambda\in\Omega(k)} U_\Lambda(x_\Lambda)$, and $w_k(t),\ k \in \mathbb{Z}^d$, is a collection of independent Brownian motions in M.

Let us first recall that a Brownian motion in M is a Markov process $W(t)$ with generator $\frac{1}{2}\Delta$, where Δ is the Laplace-Beltrami operator. One of the possible ways to construct and understand it is as follows. Consider a smooth embedding of M into a Euclidean space $\mathbb{R}^n$. It is well known that such an embedding exists if $n \geq 2N$. Then the tangent bundle TM is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. Let $P(x) : \mathbb{R}^n \to T_xM$ be the corresponding orthogonal projection. Then $w(t)$ is a solution of the SDE

$$\mathrm{d}w(t) = P(w(t)) \circ \mathrm{d}\widetilde{w}(t)$$

in $\mathbb{R}^n$. Here $\widetilde{w}(t)$ stands for a standard Wiener process in $\mathbb{R}^n$ and $\circ$ the Stratonowich differential.

In this way, the heuristic equation

$$\mathrm{d}x(t) = b(x(t))\mathrm{d}t + \mathrm{d}w(t) \tag{3.4}$$

on M, which defines a Brownian motion with drift b (a C^1-vector field on M), can be understood as an SDE in $\mathbb{R}^n$.

More precisely, let us consider the normal bundle vM with the fibers v_xM being the orthogonal complements to the corresponding fibers T_xM in $\mathbb{R}^n$. It is known (see e.g. [63]) that there exists $r > 0$ and a neighborhood $U_r \subset vM$ of the zero section $(M, 0)$ of vM, $U_r = \{(x, v) : |v| < r\}$, which is diffeomorphic to the tubular neighborhood

$$N_r = \left\{ y \in \mathbb{R}^n : |y - x| < r,\ x \in M \right\}$$

of M in $\mathbb{R}^n$.

Let us choose some positive $r_1 < r$ and a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ with support in N_r that is equal to 1 on N_{r_1}. Next, for any mapping Φ from M into a linear space P, we define its extension $\widetilde{\Phi} : \mathbb{R}^n \to P$ as follows:

$$\widetilde{\Phi}(y) := \begin{cases} \Phi(x_y)F(y),\ y \in N_r \\ 0,\ y \notin N_r \end{cases},$$

where $\left(x_y, v_y\right)$ is the image of y in U_r.

Equation (3.4) can now be understood as the following SDE in $\mathbb{R}^n$:

$$\mathrm{d}x(t) = \widetilde{b}(x(t))\mathrm{d}t + P(x(t)) \circ \mathrm{d}w_k(t),\ x(0) \in M. \tag{3.5}$$

Let us now assume that the family of potentials $\mathcal{U}$ satisfies (in addition to (3.1)) the following condition: $U_\Lambda \in C^2(M^\Lambda)$, $\Lambda \in \mathcal{B}_0(\mathbb{Z}^d)$, and

$$\sup_{k\in\mathbb{Z}^d} \sum_{\Lambda\in\mathcal{B}_0(\mathbb{Z}^d)} |||\nabla_k U_\Lambda|||_{TM} + \sup_{k\in\mathbb{Z}^d} \sum_{\Lambda\in\mathcal{B}_0(\mathbb{Z}^d)} \sum_{j\in\mathbb{Z}^d} |||\nabla_j \nabla_k U_\Lambda|||_{TM\otimes TM} < \infty, \tag{3.6}$$

where $|||\cdot||| := \sup_{x\in\mathbf{M}} \|\cdot\|$.

Remark 6 In the case of finite range of interactions conditions (3.1) and (3.6) are obviously fulfilled.

Our main result is the following theorem.

Theorem 7 *[6] Let the family of potentials satisfy conditions (3.1) and (3.6). Then, for any $x_0 \in \mathbf{M}$, there exists a unique Markov process $x(t) \in \mathbf{M}$, $t > 0$, with initial value $x(0) = x_0$, which is a strong solution the system (3.2).*

In order to use the theory of infinite-dimensional SDE for the proof, we will rewrite this system in the form of an equation in the Hilbert space $\mathcal{H}_p \subset (\mathbb{R}^n)^{\mathbb{Z}^d}$ of $\mathbb{R}^n$ -valued sequences with the inner product

$$(\xi, \eta)_p = \sum_{k\in\mathbb{Z}^d} p_{|k|}(\xi_k, \eta_k)_{\mathbb{R}^n}$$

for some weight sequence $p = (p_s)_{s\in\mathbb{Z}_+} \in l_1$, which will be chosen later. Obviously, $\mathcal{H}_p$ contains the space of bounded sequences and, therefore, contains $\mathbf{M}$ (were we identify M with its image in $\mathbb{R}^n$).

We denote by $\mathcal{H}$ the space $\mathcal{H}_p$ with $p = (p_s = 1)_{s\in\mathbb{Z}_+}$, that is, the space of square-integrable $\mathbb{R}^n$-valued sequences.

Similar to (3.5), we extend coefficients b_k to $(\mathbb{R}^n)^{\mathbb{Z}^d}$, setting

$$\widetilde{U}_\Lambda(y) := U_\Lambda(x_y) \prod_{j\in\Lambda} F(y_j),\ y = (y_k)_{k\in\mathbb{Z}^d} \in (\mathbb{R}^n)^{\mathbb{Z}^d}$$

and defining $\widetilde{b} = (\widetilde{b}_k)_{k\in\mathbb{Z}^d}$ by formula (3.3) with $\widetilde{U}_\Lambda$ instead of U_Λ. Consider the equation

$$d\xi(t) = \widetilde{b}(\xi(t))\mathrm{d}t + \mathbf{P}(\xi(t)) \circ \mathrm{d}w(t) \tag{3.7}$$

in $\mathcal{H}_p$, where $\mathbf{P}(y)$ is generated by the block-diagonal matrix with nonzero blocks $P_{kk}(y) = P(y_k)$ and $w(t)$ is the Wiener process in $\mathcal{H}$. It is clear that $\mathbf{P}(x) \in \mathcal{S}_2(\mathcal{H}, \mathcal{H}_p)$ [(the space of Hilbert-Schmidt operators $\mathcal{H} \to \mathcal{H}_p$)] and the mapping

$$\mathcal{H}_p \ni x \mapsto \mathbf{P}(x) \in \mathcal{S}_2(\mathcal{H}, \mathcal{H}_p)$$

is C^∞ for any weight sequence $p \in l_1$.

The following result can be proved by an application of the Schur test to the matrix $r_{kj} := \sup_{y\in(\mathbb{R}^n)^{\mathbb{Z}^d}} \left\|\nabla_j \widetilde{b_k}(y)\right\|, k, j \in \mathbb{Z}^d$.

Lemma 8 *[6] There exists a weight sequence $p \in l_1$ such that the mapping*

$$\mathcal{H}_p \ni x \mapsto \widetilde{b}(x) \in \mathcal{H}_p$$

is bounded and satisfies the Lipschitz condition.

Let us fix a weight sequence $p \in l_1$ as in Lemma 8 and consider SDE (3.7) in Hilbert space $\mathcal{H}_p$. Observe that the corresponding induced topology on $\mathbf{M} \subset \mathcal{H}_p$ coincides with the product topology (see [10]). The next statement follows from the general theory of SDEs in Hilbert spaces.

Theorem 9 *[6] (1) For any $x \in \mathcal{H}_p$ there exists a unique strong solution $\xi_x(t)$, $t > 0$, of equation (3.7) with initial value x. This solution continuously depends on x in the square mean sense. (2) The process $\xi_x(t)$, $t > 0$, with initial value $x \in \mathbf{M}$ does not leave $\mathbf{M}$ a.s.*
(3) The process $\xi_x(t)$ defines the Markov semigroup

$$\mathbb{P}_t f(y) := \mathbb{E}(f(\xi_x(t))) \tag{3.8}$$

in the space $C(\mathbf{M})$ of continuous functions on $\mathbf{M}$.

Theorem 7 follows now from Theorem 9.

Remark 10 So far, we have not explicitly used the geometric structure $\mathbf{M}_p$ introduced on page 16, although it appeared implicitly as a technical tool in Theorem 9. In [10], Theorem 7 was proved under more general conditions on the coefficients guaranteeing their $\mathbf{M}_p$-differentiability. Observe that $\mathbf{M}_p$-differentiable coefficients cannot in general be extended to differentiable functions on $\mathcal{H}_p$. Thus, in order to prove the existence and uniqueness results, the authors used more " geometrical" technique of the orthonormal frame bundle of $\mathbf{M}_p$. A sufficient condition of $\mathbf{M}_p$-differentiability of a vector field $\beta(x) = \left(b_j(x)\right)_{j\in\mathbb{Z}^d}$, $b_k(x) \in T_{x_k}M$, for some weight sequence p is

$$\sup_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \sup_{x\in\mathbf{M}} \left\|\nabla_k b_j(x)\right\| < \infty,$$

which is weaker than conditions (3.1) and (3.6).

3.3 Dirichlet Forms of Differentiable Measures and Probabilistic Representations of Associated Semigroups

In this section consider Dirichlet operators associated with differentiable measures on $\mathbf{M}$ (including Gibbs measures of the class $\mathcal{G}$) and use the results of the previous section in order to construct probabilistic representations of the corresponding semigroups.

We will use the spaces $\mathcal{F}C^m(\mathbf{M}) := \bigcup_{\Lambda\in\mathcal{B}_0(\mathbb{Z}^d)} C^m(M^\Lambda)$ of m-times continuously differentiable real-valued cylinder functions on $\mathbf{M}$ and similarly defined spaces $\mathcal{F}C^m(\mathbf{M}\to T\mathbf{M})$ of cylinder vector fields.

Let μ be a probability measure on $\mathbf{M}$ differentiable in the sense that the following integration by parts formula holds true: for any $u \in \mathcal{F}C^1(\mathbf{M})$ and any vector field $\xi \in \mathcal{F}C^1(\mathbf{M}\to T\mathbf{M})$

$$\int \sum_{k\in\mathbb{Z}^d} (\nabla_k u(x), \xi_k(x))_{T_{x_k}M} \, \mathrm{d}\mu(x) = -\int b_\xi^\mu(x) u(x) \, \mathrm{d}\mu(x),$$

with some $b_\xi^\mu \in L^2(\mathbf{M}, \mu)$ (the logarithmic derivative of μ in the direction ξ). We assume that b_ξ^μ is given by

$$b_\xi^\mu(x) = \sum_{k\in\mathbb{Z}^d} ((b_k^\mu(x), \xi_k(x))_{T_{x_k}M} + div\, \xi_k(x)), \tag{3.9}$$

where $b^\mu(x) := (b_k^\mu(x)) \in C^1(\mathbf{M}_p \to T\mathbf{M}_p)$ for some weight sequence $p \in l_1^+$. We will call β^μ the (vector) logarithmic derivative of μ (cf. Sect. 2.4).

For $u, v \in \mathcal{F}C^2(\mathbf{M})$ we define the classical pre-Dirichlet form $\mathcal{E}_\mu$ associated with μ :

$$\mathcal{E}_\mu(u, v) = \frac{1}{2}\int \sum_k (\nabla_k u(x), \nabla_k v(x))_{T_{x_k}M} \, \mathrm{d}\mu(x).$$

Obviously it has a generator $\mathbb{H}_\mu$ acting in $L^2(\mathbf{M}, \mu)$ on the domain $\mathcal{F}C^2(\mathbf{M})$ as

$$\mathbb{H}_\mu u(x) = -\frac{1}{2}\sum_k \Delta_k u(x) - \frac{1}{2}\sum_k (b_k^\mu(x), \nabla_k u(x))_{T_{x_k}M}.$$

Here $\Delta_k = Tr\,\nabla_k^2$ is the corresponding Laplace-Beltrami operator.

Our goal is to construct a Markov process on $\mathbf{M}$ such that its generator coincides with $\mathbb{H}_\mu$ on $\mathcal{F}C^2(\mathbf{M})$. Such a process is sometimes called the stochastic dynamics associated with μ. One possible construction of the stochastic dynamics is given by the theory of Dirichlet forms. Indeed, the pre-Dirichlet form $\mathcal{E}_\mu$ is closable. Its closure defines the classical Dirichlet form given by μ (which will be denoted also by $\mathcal{E}_\mu$, for this concept see e.g. [33]). We can consider the semigroup $\mathbb{P}_t^\mu$ in $L^2(\mathbf{M}, \mu)$ associated with its generator and construct the corresponding process as described in [33] (cf. also Sect. 2.5(i)).

Another approach (which gives in our case better control on properties of the stochastic dynamics) is based on the SDE theory. In the case where the relevant SDE has "nice coefficients" this can be solved and the so constructed process (sometimes called "Glauber dynamics") coincides (in the sense of having the same transition semigroup) with the stochastic dynamics process (cf. also Sect. 2.5(ii), (iii)).

In our framework the corresponding process ξ can be obtained as the Brownian motion on $\mathbf{M}$ with drift b^μ, cf. Remark 10. Let us consider the corresponding Markov semigroup $\mathbb{P}_t$ in $C(\mathbf{M})$ defined by (3.8). We observe that the semigroup $\mathbb{P}_t$ can be uniquely extended to a strongly continuous semigroup $\widetilde{\mathbb{P}}_t$ of symmetric contraction operators in $L^2(\mathbf{M}, \mu)$ (see e.g. [66, pp. 27/28]). We have the following result, which follows essentially from Theorem 7 and Remark 10.

Theorem 11 *[6, 10] There exists a unique* $\mathbf{M}-$*valued Markov process* ξ_x *such that the associated semigroup* $\mathbb{P}_t$ *acts in the space* $C(\mathbf{M} \to \mathbf{R}^1)$ *of continuous functions on* $\mathbf{M}$, *and its generator* $\mathbb{H}$ *coincides with* $-\mathbb{H}_\mu$ *on* $\mathcal{F}C^2(\mathbf{M})$. *This process is given by Brownian motion on* $\mathbf{M}$ *(constructed in Remark 10) with drift* β^μ. *If* $\beta^\mu \in C^3(\mathbf{M}_p \to T\mathbf{M}_p)$, *then* $\mathbb{H}_\mu$ *is an essentially self-adjoint operator on* $\mathcal{F}C^2(\mathbf{M})$. *In this case we have* $\widetilde{\mathbb{P}}_t = \mathbb{P}_t^\mu$ *for all* $t > 0$.

We return now to the study of Gibbs measures of the class $\mathcal{G}$. The next statement shows that any $\nu \in \mathcal{G}$ can be completely characterized by its logarithmic derivative, see [2, 8] for the case of finite range interactions (cf. also Theorem 5).

Theorem 12 *[8] The following conditions are equivalent: (i) the measure* ν *belongs to the class* $\mathcal{G}$;
(ii) the measure ν *is differentiable and the components* $b_k^\nu(x)$ *of its vector logarithmic derivative are given by the formulae*

$$b_k^\nu(x) = -\nabla_k V_k(x),\ V_k(x) = \sum_{\Lambda \in \Omega(k)} U_\Lambda(x_\Lambda),$$

cf. (3.9).

In view of this result, Theorem 11 can be applied to any Gibbs measure $\mu \in \mathcal{G}$. Observe that the essential self-adjointness of $\mathbb{H}_\mu$ can be proved in this case without the additional smoothness assumption $b^\mu \in C^3(\mathbf{M}_p \to T\mathbf{M}_p)$ by employing the approximating parabolic criterion, see [8, 10].

4 Further Developments: Gibbs Measures on Random Graphs and Relationship with Configuration Space Analysis

There has been a huge number of works rooted in the research described above. The most notable direction is motivated by mathematical modelling of particle systems in continuum. It has lead to the development of the configuration space analysis, which studies analysis and geometry of the space $\Gamma(\mathfrak{X})$ of locally finite sets (configurations) $\gamma \subset \mathfrak{X}$ in a metric space $(\mathfrak{X}, \rho)$, equipped with a Poisson or Gibbs measure. Initiated in seminal papers [39, 40], the configuration space analysis has lead to a great variety of applications in mathematical physics and mathematical biology, see e.g. [64, 76].

The volume limitations of the present paper do not allow as to go any deeper in this topic. We are going however to briefly touch another (related) direction of research - the study of infinite particle spin systems on irregular (possibly random) graphs in place of integer lattices.

More precisely, we consider a system of classical particles with location points $x \in \mathfrak{X}$ forming a configuration $\gamma \in \Gamma(\mathfrak{X})$, and spins $\sigma_x \in S$. Here we customarily use the notation S instead of X for the single spin space. We suppose here that both $\mathfrak{X}$ and S are Euclidean spaces. Two spins σ_x and σ_y interact via a pair potential U_{xy} if the distance between x and y is no more than a fixed interaction radius R. In other words, x and y must be adjacent in the geometric (Gilbert) graph with the vertex set γ. In contrast to the case where γ is a regular graph, e.g. $\mathbb{Z}^d$, the number n_x of particles interacting with particle x can be unbounded in x. Our main example of a "growing" configuration γ is a typical realization of a Poisson (or Gibbs) point process π on $\mathfrak{X}$, for which n_x obeys a logarithmic bound. In the physical terminology, cf. [51], the equilibrium states of our system are quenched Gibbs states of an amorphous magnet.

For general unbounded degree graphs and unbounded spins, the question of existence of Gibbs measures was first studied in [75]. In [56] , these results were used to show that the set of Gibbs measures on S^γ is non-empty for π-distributed γ (under certain natural conditions on the growth of the interaction potentials). In addition, the authors described support of those measures and obtain uniform estimates on their exponential moments. The proof is based on exponential moment bounds for the local Gibbs specification of our model and its weak dependence on the boundary conditions (cf. Lemma 2). Such a technique is effective in dealing with spatially irregular systems, see [75]. The two fundamental tools—Ruelle's (super-) stability technique and general Dobrushin's existence and uniqueness criteria—are not directly applicable to our model (due to the unboundedness of the degree function n_x and the lack of the spatial transitivity of γ). At the same time, for our model the uniqueness problem remains open.

In [57], the problem of the multiplicity of Gibbs states on S^γ was considered for $S = \mathbb{R}$. It was proved that, for π-a.a. configurations γ, the (ferromagnetic) model admits multiple Gibbs states if the intensity of the underlying point process and the inverse temperature of the system are big enough. The main technical tools used in this work were the Wells inequality and percolation techniques.

Construction of non-equilibrium stochastic dynamics of infinite particle systems of the aforementioned type has been a long-standing problem, even in the case of a linear drift and additive noise. Due to the unboundedness of the vertex degrees of γ, the coefficients of the corresponding equations cannot be controlled in a single Hilbert or Banach space (in contrast to the situation considered in Sects. 2.5 and 3). However, under mild conditions on the density of γ (holding in particular for π-a.a. configurations γ), it is possible to construct a solution in the scale of Hilbert spaces S^γ_α of weighted sequences $\bar{q} = (q_x)_{x\in\gamma} \in S^\gamma$ such that $\sum_{x\in\gamma} |q_x|^2 e^{-\alpha|x|} < \infty,\ \alpha > 0$, in both deterministic and stochastic case, see [54, 55], respectively. The price to pay here is that, for an initial condition in S^γ_α, the solution lives in the bigger space S^γ_β, $\beta > \alpha$.

Let us remark that the development of stochastic analysis on S^γ has become important in the framework of the studies of the spaces $\Gamma(\mathfrak{X}, S)$ of configurations $\{(x, \sigma_x)\}_{x\in\gamma}$ with marks $\sigma_x \in S$ (see e.g. [22, 59]), and is motivated by a variety of applications, in particular in modeling of non-crystalline (amorphous) substances, e.g. ferrofluids and amorphous magnets, see e.g. [51, 84]. The questions of existence, uniqueness and multiplicity of Gibbs states on marked configuration spaces have been considered in [53, 58], respectively. The space $\Gamma(\mathfrak{X}, S)$ possesses a fibration-like structure over the space $\Gamma(\mathfrak{X})$ of position configurations γ, with the fibres identified with S^γ, see [56]. Thus the construction of spin dynamics of a quenched system (in S^γ) is complementary to that of the dynamics in $\Gamma(\mathfrak{X})$ (see references given in [54]). It is anticipated that (some of) these results can be combined with the approach described in the present paper allowing to build stochastic dynamics on the marked configuration space $\Gamma(\mathfrak{X}, S)$.

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Hydrodynamic Models

Benedetta Ferrario and Franco Flandoli

Abstract Sergio Albeverio's research in hydrodynamics is revised, focusing on statistical analysis of the bidimensional deterministic Euler equations and the bidimensional Navier–Stokes equations with space-time white noise. Both the subjects had influence on the activity of many researchers including the present authors. The main interactions with the recent research and some (open) future problems are described.

Keywords Euler equations · Navier–Stokes equations · Invariant measures · Generalized solutions

MSC2010: 76M35 · 35Q31 · 60H30 · 60H15 · 76D06

1 Introduction

It is a pleasure to write this contribution to celebrate the 80th birthday of Sergio Albeverio. One of his research areas concerns the statistical analysis of the motion of fluids. The equations for a homogeneous incompressible fluid are given by

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f; \quad \text{div}\, u = 0 \tag{1}$$

where $u = u(t, x)$ and $p = p(t, x)$ are the velocity vector and the (scalar) pressure, respectively, defined for $t \geq 0$ and $x \in D \subseteq \mathbb{R}^d$ ($d = 2$ or $d = 3$). Suitable initial and boundary conditions are given. For $\nu = 0$ they describe the motion of an inviscid fluid and are called Euler equations; for $\nu > 0$ they describe the motion of a viscous fluid and are called Navier–Stokes equations, and ν is the kinematic viscosity parameter.

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Sergio Albeverio gave contributions for both these equations in the two dimensional setting, by using tools of infinite dimensional stochastic analysis, also used by him in other fields as statistical mechanics or field theory. This activity was in a sense the prosecution of preliminary investigations of statistical fluid mechanics (see, e.g., [42, 51]) and opened the door to numerous more recent contributions. The next sections review part of this story.

2 Euler Equations

This is a paper honouring Sergio Albeverio's scientific work in *Stochastic Hydrodynamics*. However, in order to see the interlacement with other fields—Albeverio as a precursor, in this case—let us start from something quite different.

2.1 Wave Equation

In 2008 two outstanding papers by Nicolas Burq and Nikolay Tzvetkov appeared on Inventiones [17, 18]. They were devoted to the solvability of nonlinear wave equation on the 3D torus $\mathbb{T}^3$

$$\partial_{tt}^2 u = \Delta u - u^3 \quad \text{in } \mathbb{T}^3. \tag{2}$$

This equation was known to be well posed in regular spaces, with counterexamples to well posedness in less regular ones. Equation (2) is commonly studied in the Hilbert spaces

$$(u, \partial_t u) \in \mathcal{H}^s = H^s\left(\mathbb{T}^3\right) \times H^{s-1}\left(\mathbb{T}^3\right).$$

When $s = 1$ (more generally $s \geq 1$), well posedness is proved by energy methods, thanks to the invariance of the energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla_x u\|_{L^2}^2 + \frac{1}{2}\|\partial_t u\|_{L^2}^2 + \int_{\mathbb{T}^3} \frac{u^4}{4}\mathrm{d}x\right) = 0.$$

For $s \in \left(\frac{1}{2}, 1\right)$ well posedness has been proved by Strichartz estimates. The strategy is:

(i) first prove special regularity results (the Strichartz estimates) for the semigroup $S(t)$ associated to the linear problem

$$\partial_{tt}^2 u = \Delta u \quad \text{in } \mathbb{T}^3$$
$$u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = v_0$$

(ii) then apply them to the iteration

$$(u, \partial_t u)(t) = S(t)(u_0, v_0) - \int_0^t S(t-s)\left(0, u^3(s)\right) \mathrm{d}s.$$

Since this strategy has something in common with the exciting recent advances on singular SPDEs, let me sketch some more details. One decomposes the solution as

$$(u, \partial_t u)(t) = (\overline{u}, \partial_t \overline{u})(t) + (\widetilde{u}, \partial_t \widetilde{u})(t)$$

(singular + regular parts) where

$$(\overline{u}, \partial_t \overline{u})(t) = S(t)(u_0, v_0).$$

Then we have to solve

$$(\widetilde{u}, \partial_t \widetilde{u})(t) = -\int_0^t S(t-s)\left(0, [\widetilde{u}(s) + \overline{u}(s)]^3\right) \mathrm{d}s. \tag{3}$$

By contraction principle arguments one can prove:

Lemma 1 *Assume*

$$\int_0^T \|\overline{u}(s)\|_{L^6}^3 \, ds < \infty.$$

Then Eq. (3) has a unique local solution $(\widetilde{u}, \partial_t \widetilde{u}) \in C\left([0, T_0]; \mathcal{H}^1\right)$*, for some* $T_0 \in (0, T]$.

The question then is whether $\overline{u}$, solution of the linear wave equation, has the special integrability property required by this lemma, in spite of the fact that (u_0, v_0) is apparently unrelated with such integrability. Here come into play the celebrated Strichartz estimates, that we quote in the special case $s = 2/3$ for our present purposes:

Theorem 1

$$\|\overline{u}\|_{L^3(0,1;L^6)} \leq C\left(\|u_0\|_{H^{\frac{2}{3}}} + \|v_0\|_{H^{\frac{2}{3}-1}}\right)$$

Using this estimate one can apply the lemma. Summarizing what said until now:

- $s \geq 1$: well posed by energy methods $\left(\mathcal{H}^s = H^s\left(\mathbb{T}^3\right) \times H^{s-1}\left(\mathbb{T}^3\right)\right)$
- $s \in \left(\frac{1}{2}, 1\right)$: well posed by Strichartz estimates
- Below $s = \frac{1}{2}$ there are counterexamples to well posedness.

Now comes the contribution of Nicolas Burq and Nikolay Tzvetkov.

Theorem 2 *(Burq-Tzvetkov, Inv '08) For every $s \in \left[0, \frac{1}{2}\right]$ there are Gaussian measures μ supported on $\mathcal{H}^s$ (with $\mu\left(\mathcal{H}^{s+\epsilon}\right) = 0$ for every $\epsilon > 0$) such that the nonlinear wave equation is well posed for μ-a.e. initial condition (u_0, v_0).*

The strategy of proof is similar to what described above: prove *probabilistic Strichartz estimates* for the semigroup $S(t)$ using *Gaussian analysis (and Kolmogorov regularity theorem).*

Lemma 2 *Given $s > 0$, there are Gaussian measures μ supported on $\mathcal{H}^s$ (with $\mu\left(\mathcal{H}^{s+\epsilon}\right) = 0$ for every $\epsilon > 0$) such that $\overline{u}$ has a jointly continuous in space-time version.*

Having this result one can apply Lemma 1 above. Proving the a priori bound

$$\sup_{t\in[0,T]} \|(\widetilde{u}, \widetilde{v})\|_{\mathcal{H}^1} \le C$$

one can also deduce globality.

2.2 Hydrodynamics

What has to do all of this with hydrodynamics, and in particular with Sergio's work? After the papers by Burq and Tzvetkov (and many other researchers) it was mandatory to ask whether something similar can be proved for equations of fluid mechanics.

Remark 1 Until now the regularization mechanism of random initial conditions did not prove to be sufficiently strong for parabolic equations (see also [14]): too difficult to compete with parabolic regularization. This is why, in fluid mechanics, the attention goes to inviscid problems.

The discovery is that, in fluid mechanics, a partial analog of Burq and Tzvetkov result was already done!

Theorem 3 *(Albeverio-Cruzeiro CMP '90) There exists a probability space $(\Omega, \mathcal{F}, P)$ and a stationary process $(\omega_t)_{t\ge 0}$, with paths of class $C\left([0, T]; H^{-1-}\right)$, such that:*

(i) $(\omega_t)_{t\ge 0}$ *solves the 2D Euler equations*
(ii) *the law of ω_t is the enstrophy measure μ, for every $t \ge 0$.*

Here, the space H^{-1-} is defined as $\bigcap_{\epsilon>0} H^{-1-\epsilon}$. We shall explain this result in the rest of the section.

Remark 2 The only precursor of Burq-Tzvetkov result was Bourgain [15], on nonlinear Schrödinger equations, in 1996, also posterior to Albeverio-Cruzeiro. It is not clear whether there was an influence from 2D Euler to dispersive equations.

2.3 *The 2D Euler Equations*

Let us consider the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \text{div} u = 0. \end{aligned} \tag{4}$$

The vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ satisfies the nonlinear transport equation

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

We shall always consider the vorticity formulation. For smooth solutions, we consider energy, enstrophy and Casimir:

$$\begin{aligned} \mathcal{E}(\omega) = \text{energy} &= \frac{1}{2} \int |u(x)|^2 \, \mathrm{d}x \\ &\sim -\frac{1}{2\pi} \int \int \log|x-y| \, \omega(x) \, \omega(y) \, \mathrm{d}x \, \mathrm{d}y \\ \mathcal{S}(\omega) = \text{enstrophy} &= \int \omega^2(x) \, \mathrm{d}x \\ \text{Casimir} &= \int |\omega(x)|^p \, \mathrm{d}x. \end{aligned}$$

Existence of solutions in function spaces and for measure-vorticity is based on them. Let us review these results.

2.3.1 Existence and Uniqueness in L^∞, When $\omega_0 \in L^\infty$

This is a celebrated result of Yudovich [52]. The notion of solution, in general for solutions of class $L^\infty(0, T; L^p)$, is the classical weak one: for all $\phi \in C_c^\infty$ it is required that

$$\int \omega(t,x) \phi(x) \, \mathrm{d}x - \int \omega_0(x) \phi(x) \, \mathrm{d}x = \int_0^t \int \omega(s,x) \, u(s,x) \cdot \nabla \phi(x) \, \mathrm{d}x \mathrm{d}s$$

where

$$u(t,x) = \int K(x-y) \, \omega(t,y) \, \mathrm{d}y$$

$K(x-y)$ = Biot-Savart kernel ($\sim \frac{1}{2\pi} \frac{(x-y)^\perp}{|x-y|^2}$ at short distance). Concerning existence of solution in $L^\infty(0, T; L^p)$ when the initial condition ω_0 belongs to L^p, it is based on compactness arguments, combining the a priori bounds coming from the transport structure (plus $\text{div} u = 0$)

$$\int |\omega(t,x)|^p \, \mathrm{d}x \leq \int |\omega_0(x)|^p \, \mathrm{d}x$$

and the additional regularity of $u(t,x) = \int K(x-y)\,\omega(t,y)\,\mathrm{d}y$

$$\omega(t,\cdot) \in L^p \Rightarrow u(t,\cdot) \in W^{1,p}$$

Uniqueness, which holds only under the assumption $\omega_0 \in L^\infty$, is more special. For $p=\infty$ we do not have $u(t,\cdot) \in W^{1,\infty}$ but only

$$u \text{ log-Lipstchitz.}$$

One way to understand uniqueness is to invoke the Lagrangian formulation

$$\frac{\mathrm{d}X_t^x}{\mathrm{d}t} = u\left(t, X_t^x\right), \quad X_0^x = x$$

$$\omega\left(t, X_t^x\right) = \omega_0(x).$$

When the flow $x \mapsto X_t^x$ is sufficiently well behaved, it is a tool for uniqueness. Log-Lipstchitz condition on u is sufficient for this purpose.

Existence of Measure-Valued Solutions

When the initial condition is a positive measure $\omega_0(\mathrm{d}x)$ (or slightly more general), of class H^{-1}, namely with velocity $u_0 \in L^2$, Delort [25] proved existence of measure-valued solutions. The proof is based on a priori bound from energy conservation and a clever argument to pass to the limit in a very weak formulation, acceptable for solutions which are just measures. We give this detail since it is very relevant for the sequel. The nonlinearity

$$\int \omega(s,x)\,u(s,x)\cdot\nabla\phi(x)\,\mathrm{d}x = \int u(s,x)\cdot\nabla\phi(x)\,\omega(s,\mathrm{d}x)$$

is a priori meaningless, since $u(s,\cdot)$ is only L^2. But one can rewrite it by the so called *Schochet symmetrization* [47] (Delort used a different trick):

$$\begin{aligned} &\overset{u=K*\omega}{=} \int\int K(x,y)\cdot\nabla\phi(x)\,\omega(s,\mathrm{d}y)\,\omega(s,\mathrm{d}x) \\ &= \frac{1}{2}\int\int K(x,y)\cdot(\nabla\phi(x)-\nabla\phi(y))\,\omega(s,\mathrm{d}y)\,\omega(s,\mathrm{d}x) \end{aligned}$$

which is better because $H_\phi(x,y) := \frac{1}{2}K(x,y)\cdot(\nabla\phi(x)-\nabla\phi(y))$ is bounded, smooth outside the diagonal. In order to deal with this expression, the involved measures should not give weight to the diagonal; this fact is guaranteed by the condition $\omega \in H^{-1}$ which, intuitively speaking, excludes delta Dirac masses.

2.3.2 Point Vortex Solutions

But precisely the case of only delta Dirac concentrations of vorticity, in finite number, is acceptable and even well posed by another approach. Marchioro and Pulvirenti [44] have proved existence and uniqueness of solutions of the form

$$\omega(t, \mathrm{d}x) = \sum_{i=1}^{N} \omega_i \delta_{X_t^i}$$

for almost every point vortex initial measure $\omega_0(\mathrm{d}x) = \sum_{i=1}^{N} \omega_i \delta_{X_0^i}$, where the qualification a.e. refers to Lebesgue measure on product space of N copies of the underlying domain. These solutions belong to $H^{-1-} = \bigcap_{\epsilon>0} H^{-1-\epsilon}$ and the velocity is not of finite energy, $u \notin L^2$. This result is usually described as an ODE result:

$$\frac{\mathrm{d}X_t^i}{\mathrm{d}t} = \frac{1}{2\pi} \sum_{j \neq i} \omega_j \frac{\left(X_t^i - X_t^j\right)^{\perp}}{\left|X_t^i - X_t^j\right|^2} \quad i = 1, \ldots, N$$

This $2N$-dimensional equation has a unique solution, without collision of points,

$$\text{for } Leb_{2N}\text{-a.e. } \left(X_0^1, ..., X_0^N\right).$$

This is a first example of probabilistic result for 2D Euler equations, for distributional solutions. Point vortices are distributional solutions in the sense of Schochet: setting

$$H_\phi(x, y) := \frac{1}{2} K(x, y) \cdot (\nabla\phi(x) - \nabla\phi(y)) \quad \text{for } x \neq y$$
$$H_\phi(x, y) := 0 \quad \text{for } x = y$$

(which corresponds to $\sum_{j \neq i}$ in the ODE) and setting

$$\omega_0(\mathrm{d}x) = \sum_{i=1}^{N} \omega_i \delta_{X_t^i}$$

we have

$$\int \phi(x)\, \omega(t, \mathrm{d}x) - \int \phi(x)\, \omega_0(\mathrm{d}x)$$
$$= \int_0^t \int \int H_\phi(x, y)\, \omega(s, \mathrm{d}y)\, \omega(s, \mathrm{d}x)\, \mathrm{d}s.$$

2.3.3 Distributional Solutions

Until $\omega \in H^{-1}$ we have Delort result. For very special $\omega \in H^{-1-}$ we have point vortex result. Albeverio-Cruzeiro theorem represents another, more substantial, existence result of H^{-1-}-solutions. Recall that Theorem 3 from Albeverio and Cruzeiro [2] states the existence of a probability space $(\Omega, \mathcal{F}, P)$ and a stationary process $(\omega_t)_{t\geq 0}$, with paths of class $C\left([0,T]; H^{-1-}\right)$, such that $(\omega_t)_{t\geq 0}$ solves the 2D Euler equations. Recall also that μ (still to be defined in this review) is the law at time zero—as well as at any time. In particular, the previous result implies:

Corollary 1 *For μ-a.e. initial condition $\omega_0 \in H^{-1-}$ there is a solution of class*

$$\omega_{\cdot} \in C\left([0,T]; H^{-1-}\right).$$

2.4 *The Enstrophy Measure*

Let us see what is the measure μ. It is the law of *white noise* on $\mathbb{T}^2$, here called ω_0. White noise on $\mathbb{T}^2$ is a centered Gaussian r.v. $\omega_0 : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ (here $(\Xi, \mathcal{F}, P)$ is a probability space) such that

$$\mathbb{E}\left[\langle \omega_0, \phi\rangle \langle \omega_0, \psi\rangle\right] = \langle \phi, \psi\rangle$$

for all $\phi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. In more heuristic terms,

$$\mathbb{E}\left[\omega_0(x)\,\omega_0(y)\right] = \delta(x-y).$$

Remark 3 The solution constructed by Albeverio and Cruzeiro [2] is white noise *at every time* (similarly to KDV equations, see Oh [45], and to stochastic viscous Burgers equation of KPZ theory or stochastic Navier–Stokes equations with space-time white noise).

Recall the invariant

$$\mathcal{S}(\omega) = \text{enstrophy} = \int \omega^2(x)\,\mathrm{d}x.$$

Formally white noise measure is

$$\mu_\nu(\mathrm{d}\omega) = \frac{1}{Z_\nu} \mathrm{e}^{-\nu \mathcal{S}(\omega)} \mathrm{d}\omega \quad \text{(enstrophy measure)}$$

for any $\nu > 0$, and heuristically it is invariant for Euler equations because "dω" is invariant by the Hamiltonian structure.

Albeverio et al. [9] (and with Ribeiro de Faria in [3]) understood the following fundamental facts.

Theorem 4 *(i) The enstrophy measure is infinitesimally invariant for Euler dynamics.*

(ii) The nonlinear term

$$\langle u \cdot \nabla\omega, \phi\rangle$$

of Euler equation, for smooth cylindrical test functions ϕ, is rigorously defined by Fourier analysis, on $L^2\left(H^{-1-}, \mu_\nu\right)$

$$\langle u \cdot \nabla\omega, \phi\rangle = L^2_{\mu_\nu} - \lim_{N\to\infty} \langle B_N(\omega), \phi\rangle$$

providing a rigorous definition of the infinitesimal generator (and thus infinitesimal invariance of μ_ν) on cylindrical smooth functions.

(iii) Defined $\mathcal{E}_N(\omega)$ as a truncated Fourier kinetic energy, its limit is infinite on H^{-1-}, but the following limit exists in $L^2\left(H^{-1-}, \mu_\nu\right)$:

$$:\mathcal{E}(\omega): = L^2_{\mu_\nu} - \lim_{N\to\infty} \left(\mathcal{E}_N(\omega) - \mathbb{E}^{\mu_\nu}[\mathcal{E}_N]\right)$$

called renormalized kinetic energy.

(iv) Then the measure

$$\mu_{\nu,\gamma}(d\omega) = \frac{1}{Z_{\nu,\gamma}} \exp\left(-\gamma : \mathcal{E}(\omega) :\right) \mu_\nu(d\omega)$$

is also infinitesimally invariant for 2D Euler equations.

2.4.1 Back to Albeverio-Cruzeiro Result

Now we can fully appreciate the result of Sergio Albeverio and Ana-Bela Cruzeiro, that we restate with the additional informations given above.

Theorem 5 *(Albeverio-Cruzeiro CMP '90) There exists a probability space $(\Omega, \mathcal{F}, P)$ and a stationary process $(\omega_t)_{t\geq 0}$, with paths of class $C\left([0,T]; H^{-1-}\right)$, such that:*

(i) $(\omega_t)_{t\geq 0}$ solves the 2D Euler equations in weak form

$$\partial_t \omega + B(\omega) = 0$$
$$\langle B(\omega), \phi\rangle = L^2_{\mu_\nu} - \lim_{N\to\infty} \langle B_N(\omega), \phi\rangle$$

(ii) the law of ω_t is the enstrophy measure μ_ν, for every $t \geq 0$.

Let us shortly discuss the idea of proof. All the classical results recalled above for 2D Euler equations are based on a priori estimates due to invariance of energy, enstrophy or Casimirs. These quantities are infinite for vorticity fields of class H^{-1-}

(velocity in H^{-}). [The result for point vortices, which belong to H^{-1-}, is based on the invariance of interaction energy, which is finite.]

The invariance of μ_ν is the substitute. Let us understand the proof from the viewpoint of a priori estimates (the true proof requires an approximation scheme, Galerkin in Albeverio and Cruzeiro [2] or point vortices in later works, where similar estimates can be proved rigorously and uniformly). Let us try to use a form of Aubin-Lions lemma. *We may estimate* $\int_0^T \|\omega(t)\|_{H^{-1-\delta}}^p \, \mathrm{d}t$ *in* μ_ν*-average using stationarity*:

$$\begin{aligned}\mathbb{E}_{\mu_\nu}\left[\int_0^T \|\omega(t)\|_{H^{-1-\delta}}^p \, \mathrm{d}t\right] &= \int_0^T \mathbb{E}_{\mu_\nu}\left[\|\omega(t)\|_{H^{-1-\delta}}^p\right] \mathrm{d}t \\ &= T\mathbb{E}_{\mu_\nu}\left[\|\omega(0)\|_{H^{-1-\delta}}^p\right] \\ &= T\int \|\omega\|_{H^{-1-\delta}}^p \, \mu_\nu(\mathrm{d}\omega) < \infty.\end{aligned}$$

Compactness in time makes use of Euler equation:

$$\begin{aligned}\mathbb{E}_{\mu_\nu}\left[\int_0^T \|\partial_t \omega(t)\|_{H^{-N}} \, \mathrm{d}t\right] &\le \int_0^T \mathbb{E}_{\mu_\nu}\left[\|u(t) \cdot \nabla\omega(t)\|_{H^{-N}}\right] \mathrm{d}t \\ &\le T\int \|(K * \omega) \cdot \nabla\omega\|_{H^{-N}} \, \mu_\nu(\mathrm{d}\omega)\end{aligned}$$

and of special estimates on $\|(K * \omega) \cdot \nabla\omega\|_{H^{-N}}$ with respect to μ_ν similar to those mentioned above to give a meaning to $B(\omega)$.

2.5 Summary of Results

What described until now is an extraordinary set of results, followed by investigations about Markov uniqueness, generalizations to 2D Navier–Stokes equations with space-time noise (and improvements of the results in that case). The following is only a partial list of those papers: [1–6, 9]. In summary, what we have discussed until now is the scheme for wave equation:

$$\partial_{tt}^2 u = \Delta u - u^3 \quad \text{in } \mathbb{T}^3$$

$$(u, \partial_t u) \in \mathcal{H}^s = H^s \times H^{s-1}$$

- $s \ge 1$: well posed by energy methods
- $s \in \left(\frac{1}{2}, 1\right)$: well posed by Strichartz estimates

- $s \in \left[0, \frac{1}{2}\right]$: well posed for a.e. initial condition with respect to certain Gaussian measures.

and the analogous scheme for 2D Euler equations:

- existence and uniqueness, when $\omega_0 \in L^\infty$ (Wolibner, Yudovich)
- existence for $\omega_0 \in L^p$, $p \geq 2$ (velocity $u \in W^{1,p}$)
- existence for positive measures $\omega_0\,(\mathrm{d}x)$ of class H^{-1} (Delort) (velocity $u \in L^2$)
- existence and uniqueness for a.e. point vortex measure $\omega_0\,(\mathrm{d}x) = \sum_{i=1}^N \omega_i \delta_{X_0^i}$ (Marchioro-Pulvirenti) (velocity $u \notin L^2$)
- existence for μ_ν-a.e. $\omega_0 \in H^{-1-}$ (μ_ν described in Sect. 2.4) (Albeverio-Cruzeiro, CMP '90).

2.6 Recently Treated Questions

Recently there has been a revived interest for the set-up of distributional white noise solutions of Albeverio and Cruzeiro. The following list of questions received some positive contribution:

1. Is the meaning of the nonlinearity related to Schochet symmetrization?
2. Is a white noise solution the limit of *suitable* smoother solutions?
3. Is a white noise solution the limit of point vortices?
4. Are there generalizations to $\mathcal{L}(\omega_0) \ll \mu_\nu$?
5. Are there generalizations to stochastic Euler equations in the enstrophy measure regime?

Problem n. 2 was suggested by Nikolay Tzvetkov. We have positive answers to questions 1–4 in [32] and to 5 in [34, 35] in the case of multiplicative transport noise (in regimes different from the enstrophy measure see Brzezniak et al. [16] and references therein) and in [38] for stochastic Euler equation with additive noise and friction. Further questions of potential interest for turbulence and other aspects of fluid mechanics have been addressed in other works, like microcanonical variants of the enstrophy measure [20, 37] and the convergence of stochastic Euler equations to stochastic Navier–Stokes equations, always in the regime of the enstrophy measure [36].

On the contrary, open or partially open remain the questions:

1. Uniqueness (also in the stochastic case)
2. Is a white noise solution the limit of *arbitrary* smoother solutions?
3. When $\rho_0 := \frac{d\mathcal{L}(\omega_0)}{d\mu_\nu} \sim \delta_{\overline{\omega}_0}$ with $\overline{\omega}_0$ smooth, is $\rho_t \sim \delta_{\overline{\omega}_t}$?
4. Can we work with initial measures really *different* from the enstrophy measure?
5. Are there invariant measures with higher degree of aggregation (for turbulence theory)?

We address to Flandoli and Luo [35] for a discussion of the uniqueness problem 1 in the potentially more favourable case of stochastic Euler equations, which however still remains unsolved. Problem 4 in the case of moderate variants, like Gibbs measures including the renormalized energy or microcanonical projections have been discussed above.

Problem 5 is of fundamental importance; as remarked in [44], the enstrophy measure is not directly suitable for the description of turbulence. However, it may be a fundamental building block. In [37], it is argued that a correct description of inverse cascade turbulence could be in a regime intermediate between a microcanonical version of the enstrophy measure and the very structured regime of Onsager theory.

3 Stochastic Navier–Stokes Equations

Now we consider the contribution of Sergio Albeverio to the analysis of viscous fluids, whose motion is governed by the Navier–Stokes equations. For a homogeneous incompressible viscous fluid they are given by

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f; \quad \text{div}\, u = 0 \tag{5}$$

The parameter $\nu > 0$ is called kinematic viscosity and f is a given forcing term. Suitable initial and boundary conditions are given.

For $\nu = 0$ they reduce formally to the equations of motion of an inviscid fluid (4) considered in the previous section. There is a large literature on these equations; the main difference is about the spatial dimension. For $d = 2$ the problem is well posed and the first results on existence and uniqueness of classical and weak solutions with initial data of finite energy were found by Leray and later Ladyszenskaja, J.-L. Lions and Prodi. On the other side, for $d = 3$ there are either results on global existence of weak solutions or local existence and uniqueness of smooth solutions. Uniqueness of weak solutions and global existence of strong solutions are both challenging open problems. This is true both in the deterministic and stochastic setting.

When the forcing term is a stochastic one, existence of invariant measures is expected due to the balance between dissipation of energy given by the Laplace operator and the injection of energy given by the forcing term. The invariant (or stationary) measures are the objects investigated in the statistical analysis of turbulence. Indeed the individual solutions may give a detailed and very complicated picture of the fluid, while one might be interested in the behaviour of some global quantity related to the fluid, where the microscopic picture is replaced by the macroscopic one.

Actually in the two dimensional setting there are many results on existence and uniqueness of invariant measures, assuming that the noise is sufficiently regular in space (see, e.g., [22, 23, 31, 43, 51] and references therein). However there are few informations about these measures and no explicit expression of these invariant measures are given. Notice the difference with respect to the Euler equations, which have many explicit (at least formally) invariant measures. Actually, Gallavotti suggested to express the stationary solution of the stochastic Navier–Stokes equation by means of Girsanov theorem (see Sect. 6.1 in [39]); his formula is only a formal object. Rigorous results in that direction have been proven so far only for the stochastic *hyperviscous* Navier–Stokes equations (see [26–28]).

The only result providing an explicit expression of an invariant measure for the Navier–Stokes equations was given in the paper by Albeverio and Cruzeiro [2] already quoted. We will revise this result in the next section. Then we briefly present other results related to this one.

Before going into details, let us notice that we make a change with respect to the previous section and we work on the velocity u instead of the vorticity ω. This allows to make some remarks to the 3D case as well. With some abuse of notation we shall use the same symbols for the measure μ_ν and the nonlinear term B in both settings, since the meaning is the same (but before they were defined in terms of the vorticity, now in terms of the velocity).

Finally, as usual we write these equations in abstract form (see, e.g., [50]); this is obtained by projecting the equation for the velocity onto the subspace of periodic divergence free velocity fields, so to get rid of the pressure term. We denote by Π the projector onto this subspace and define $B(u, v) = \Pi[(u \cdot \nabla)v]$. Therefore, we get that the Euler equations (4) in abstract form are

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + B(u(t), u(t)) = 0 \tag{6}$$

and the Navier–Stokes equations are

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \nu A u(t) + B(u(t), u(t)) = f(t) \tag{7}$$

where $A = -\Delta$ (the Laplacian operator) and now f is the projection of the previous forcing term, but we use the same symbol with some abuse of notation.

3.1 The 2D Navier–Stokes Equations with Space–Time White Noise

The contribution of Sergio Albeverio to the analysis of viscous fluids started with the paper in collaboration with Albeverio and Cruzeiro [2]. They realized that the Gibbs measure of the enstrophy μ_ν given in Sect. 2.4 is invariant for the unforced Euler equations (6) as well as for a stochastic Stokes equation. This latter equation corresponds to equations (7) when f is a stochastic forcing term and the nonlinearity is neglected. It is

$$\mathrm{d}u(t) + \nu A u(t)\,\mathrm{d}t = \sqrt{2}\mathrm{d}w(t), \quad t > 0 \tag{8}$$

where w is an $\mathcal{H}^0$-cylindrical Wiener process, that is a space-time white noise (see, e.g., [22]; here $\mathcal{H}^0$ is the space of finite energy and divergence free velocity fields). Indeed it is well known that the linear stochastic Stokes equation (8) has a unique invariant measure; so they chose the covariance of the noise in such a way that this invariant measure is exactly the Gaussian measure μ_ν.

Therefore the measure μ_ν is an infinitesimally invariant measure for the stochastic Navier–Stokes equation

$$d\,u(t) + [\nu A u(t) + B(u(t), u(t))]\,\mathrm{d}t = \sqrt{2}\mathrm{d}w(t) \tag{9}$$

The fact that the noise is white also in space means that it is not too regular in space and again we have to analyze the viscous dynamics for velocity fields in the support of the measure μ_ν, which we already know are not functions but distributions.

We now need to introduce some more notations. First, we work on the torus, since we are using the invariance of the Gibbs measure of the enstrophy valid for the 2D inviscid fluid equations; in general the viscid and inviscid fluids obey different boundary conditions (velocity tangent to the boundary for an inviscid fluid and velocity vanishing on the boundary for a viscid fluid). But in the periodic setting the boundary conditions agree for the both fluids. We denote by $\mathcal{H}^s_p$ the Sobolev space of periodic divergence free vectors fields such that $A^{s/2}u \in L_p$ and by $\mathcal{B}^s_{p\,q}$ the Besov space of periodic divergence free vectors fields (see [11]); the latter can be defined as real interpolation space

$$\begin{aligned} \mathcal{B}^s_{p\,q} = (\mathcal{H}^{s_0}_p, \mathcal{H}^{s_1}_p)_{\theta,q}, \quad & s \in \mathbb{R}, 1 \le p, q \le \infty \\ & s = (1-\theta)s_0 + \theta s_1, \quad 0 < \theta < 1 \end{aligned}$$

In particular, $\mathcal{B}^s_{2\,2} =: \mathcal{H}^s$ are Hilbert spaces.

Now the enstrophy is given by $S(u) = \frac{1}{2}\|u\|^2_{\mathcal{H}^1}$ and the Gibbs measure of the enstrophy is heuristically defined as

$$\mu_\nu(\mathrm{d}u) ='' \frac{1}{Z_\nu}\mathrm{e}^{-\nu S(u)}\mathrm{d}u''$$

so that $\mu_\nu(\mathcal{H}^r) = 0$ for any $r \ge 0$ but $\mu_\nu(\mathcal{H}^r) = 1$ for any $r < 0$.

Albeverio and Cruzeiro dealt with existence of μ_ν-stationary martingale (weak) solutions to the stochastic Navier–Stokes equation (9), following the technique used for the Euler equation, already explained in the previous sections. Later on, Da Prato and Debussche in [21, 24] gave another proof of the existence result by Albeverio and Cruzeiro but also proved existence of strong solutions to the stochastic Navier–Stokes equation (9) for μ_ν-a.e. initial velocity. The pathwise uniqueness of these solutions was proved later by Albeverio and Ferrario in [7].

Let us work on *any* finite time interval $[0, T]$. Now we state the result of existence and uniqueness of solutions, which are strong from the probabilitistic point of view (see [7, 21, 24]).

Theorem 6 *Let the real parameters* $\sigma, p, q, \alpha, \beta$ *satisfy* $2 \leq p, q < \infty, 1 \leq \beta < \infty$ *and*

$$0 < \sigma < \alpha < \frac{2}{p}$$
$$\frac{1}{p} - \frac{1}{2} < \frac{\alpha}{2} - \frac{1}{\beta} < -\frac{\sigma}{2}$$

Then, given $T > 0$ *for* μ_ν*-a.e.* $u_0 \in \mathcal{B}^{-\sigma}_{pq}$ *there exists a unique solution* u^{u_0} *to the stochastic Navier–Stokes equation* (9) *with initial velocity* u_0 *such that*

$$u^{u_0} \in C([0, T]; \mathcal{B}^{-\sigma}_{pq}) \quad \mathbb{P} - a.s.$$

Moreover, for any $l \in \mathbb{N}$

$$\mathbb{E}\left(\sup_{t \in [0,T]} \|u^{u_0}(t)\|^l_{\mathcal{B}^{-\sigma}_{pq}} \right) < \infty \tag{10}$$

In few words, the technique exploits the properties of the linear Stokes problem. Instead of dealing with the stochastic equation (9) one introduces the auxiliary process $v = u - z$ solving the nonlinear random equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v, v) + B(z, v) + B(v, z) = -B(z, z) \tag{11}$$

where z is the μ_ν-stationary solution to the linear Stokes equation (8). We can analyze pathwise this equation; now $-B(z, z)$ plays the role of forcing term for this modified Navier–Stokes equation and the key tool to study the equation for v is the analysis of regularity of the term $B(z, z)$. Using functional analysis estimates, it is hard to estimate $B(z, z)$ since z is a Gaussian process with nonsmooth paths; indeed one does not know how to give a meaning to the product $(z \cdot \nabla)z$ when z is a distribution ($z \in \mathcal{H}^\sigma$ for $\sigma < 0$). However, the integral of this product with respect to the Gaussian measure μ_ν is well defined[1]:

$$\int \|B(y, y)\|^m_{\mathcal{H}^{-1-\varepsilon}} \, \mu_\nu(\mathrm{d}y) < \infty \tag{12}$$

for any $m \in \mathbb{N}$ and $\varepsilon > 0$ (see [2, 20, 21]). Hence, dealing with the stationary solution z of the stochastic Stokes equation (8) we get by stationarity

[1] This is another formulation of part (ii) in Theorem 4.

$$\mathbb{E}\int_0^T \|B(z(t), z(t))\|^m_{\mathcal{H}^{-1-\varepsilon}} \mathrm{d}t = T\int \|B(y, y)\|^m_{\mathcal{H}^{-1-\varepsilon}}\, \mu_\nu(\mathrm{d}y) < \infty.$$

This provides that $B(z, z) \in L^m(0, T; \mathcal{H}^{-1-\varepsilon})$, $\mathbb{P}$-a.s. Coming back to Eq. (11), we can expect that it has a solution more regular than z and indeed one proves (pathwise) that there exists a unique local mild solution $v \in C([0, T_*]; \mathcal{B}^{-\sigma}_{pq}) \cap L^\beta(0, T_*; B^\alpha_{pq})$ to Eq. (11); now the terms $B(v, v)$, $B(z, v)$ and $B(v, z)$ are well defined by means of paraproduct techniques (see Chemin [19]) involving Besov spaces, since v is more regular ($v \in L^\beta(0, T_*; B^\alpha_{pq})$ with $\alpha > 0$ fulfilling the assumptions of Theorem 6).

This gives the existence of a local solution $u = v + z$. The proof that this solution is globally defined in time exploits again the invariance of the measure μ_ν.

As far as pathwise uniqueness is concerned, let us consider another solution $\tilde{u}^{u_0}$ as given by the existence result. The difference $V = u^{u_0} - \tilde{u}^{u_0}$ fulfills the equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} V(t) + AV(t) = -B(u^{u_0}(t), u^{u_0}(t)) + B(\tilde{u}^{u_0}(t), \tilde{u}^{u_0}(t)), & t > 0 \\ V(0) = 0 \end{cases} \tag{13}$$

Since we proved that pathwise the r.h.s. has regularity $L^m(0, T; \mathcal{H}^{-1-\varepsilon})$ for any $m \in \mathbb{N}$ and $\varepsilon > 0$, we find that any solution V is more regular than the two solutions u^{u_0} and $\tilde{u}^{u_0}$: we have $V \in C([0, T]; \mathcal{B}^{1-\varepsilon-\frac{2}{m}}_{2m}) \cap L^m(0, T; \mathcal{H}^{1-\varepsilon})$. This is a parabolic regularity result. Being more regular it is possible that uniqueness holds. This is indeed proven, by working on Eq. (13) now written (using bilinearity) as

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) + AV(t) + B(u^{u_0}(t), V(t)) + B(V(t), \tilde{u}^{u_0}(t)) = 0$$

with $V(0) = 0$. Notice that $B(u^{u_0}(t), V(t))$ and $B(V(t), \tilde{u}^{u_0}(t))$ are well defined again thanks to the Chemin's estimates. Since $V(t) = 0$ is a solution of the latter equation, by uniqueness we get that this is the only solution. This proves pathwise uniqueness.

These are the basic steps to prove Theorem 6. A further property of the Gaussian measure μ_ν has been obtained by Debussche [24]. There is exponential convergence for the solution given in Theorem 6 in the sense that there exists a constant $\lambda > 0$ such that

$$\int \mathbb{E}|\phi(u^{u_0}(t)) - \phi_{\mu_\nu}|^2\, \mu_\nu(\mathrm{d}u_0) \le \mathrm{e}^{-\lambda t} \int \mathbb{E}|\phi(u_0) - \phi_\mu|^2\, \mu_\nu(\mathrm{d}u_0)$$

for any $t > 0$ and any $\phi \in L^2(\mu_\nu)$, where $\phi_{\mu_\nu} = \int \phi(y)\, \mu_\nu(\mathrm{d}y)$.

3.2 *Research Areas Inspired by Albeverio and Cruzeiro's Paper*

In this section we list other research lines inspired by the method and the problem studied by Albeverio and Cruzeiro [2]; we briefly present results and open problems, without claiming to be exhaustive.

1. Instead of dealing with a stochastic equation, one can approach the problem by studying the associated **Kolmogorov equation**. Flandoli and Gozzi [33] constructed a solution of the Kolmogorov equation associated to the 2D Navier–Stokes equation (9).
   If one would define a unique solution, hence a semigroup $\{P_t\}_{t\geq 0}$, it is important to characterise the infinitesimal generator K $(P_t = \mathrm{e}^{-tK})$ by finding the domain of the **Kolmogorov operator** $(K, D(K))$ in $L^2(\mu_\nu)$.
   Albeverio worked on infinite dimensional Kolmogorov operators associated with many different kinds of SPDE's. Also that one associated to the 2D Navier–Stokes equation (9) presented interesting features. Actually the Kolmogorov operator, when defined on smooth cylinder functions $\mathcal{F}\mathcal{C}_b^\infty$, can be written as $\tilde{K} = Q + L$ with Q the positive symmetric Ornstein–Uhlenbeck operator associated to the Stokes equation (8) and L the Liouville operator associated to the Euler equation (6) (see more details in [2–4]). It is a dissipative and closable operator in $L^2(\mu_\nu)$ and the infinitesimal invariance of the Gibbs measure of the enstrophy holds:
   $$\int \tilde{K} f \, \mathrm{d}\mu_\nu = 0 \quad \forall f \in \mathcal{F}\mathcal{C}_b^\infty.$$
   The question of $L^2(\mu_\nu)$-uniqueness, that is if there exists a unique C^0-semigroup in $L^2(\mu_\nu)$ whose generator extends $(\tilde{K}, \mathcal{F}\mathcal{C}_b^\infty)$, is still an open problem. Partial results were given in [1, 4, 48, 49]. For other results on Kolmogorov equation for stochastic Navier–Stokes, see e.g. [8, 10, 34, 46].
   However, for $d = 1$ Gubinelli and Perkowski [41] have constructed a domain for the infinitesimal generator associated to the stochastic Burgers equation.
2. A similar uniqueness problem holds for the **Liouville operator** $(L, \mathcal{F}\mathcal{C}_b^\infty)$ associated to the Euler equation (6): is the operator $(L, \mathcal{F}\mathcal{C}_b^\infty)$ essentially skew-self-adjoint in $L^2(\mu_\nu)$? This is an open problem, appearing already in the first paper by Albeverio and collaborators [3] (see also Sect. 2.9 in [8]).
3. In a couple of papers Bessaih and Ferrario [12, 13] studied **shell models** (including the GOY and Sabra models) along the lines presented in the previous sections. The aim was to introduce invariant measures of explicit (Gaussian) type also for the shell models of turbulence.
   Shell models are the most interesting and most popular examples of simplified phenomenological models of turbulence. This is because, although departing from reality, they capture some essential statistical properties and features of turbulent flows, such as the energy and the enstrophy cascade and the power law decay of the structure functions in some range of wave numbers, the inertial range.

In the first paper [12], general shell models are considered for which only the energy is an invariant of motion and are therefore approximation models for 3D hydrodynamics. A Gibbs measure μ_ν^E, defined by means of the energy E, is constructed and it is proved to be invariant for these shell models, both for a stochastic viscous shell model as well as for the inviscid shell model (as done in the previous sections for the 2D stochastic Navier–Stokes equations and the 2D deterministic unforced Euler equations). The support of this Gaussian measure is a Sobolev space of negative exponent and the space of finite energy initial velocity is negligible with respect to this measure. Thus, one looks for a flow with initial data of infinite energy.

Even if this is a model for 3D fluids, we can prove rigorous results since from the analytic point of view the shell models are easier than the ”true” models. Existence of a unique global solution is proved for μ_ν^E-a.e. initial data and there exists a unique stationary process whose law at any fixed time is μ_ν^E. Moreover, for the inviscid shell model, existence of a μ_ν^E-stationary process for solving it is proved.

In the second paper [13], a Gaussian measure $\mu_{\nu,\beta}$ is constructed in such a way that it is invariant for both a stochastic viscous shell model and the deterministic inviscid model. These shell models are related to the 2D hydrodynamical equations of the previous sections. The same results as before are proved and in addition the measure $\mu_{\nu,\beta}$ is the unique invariant measure. We can summarize the results by saying that for 2D shell models one can prove all the properties that one thinks are true for the 2D hydrodynamic equations but some of them are mathematically difficult to prove on the “true” model.

4. The **Gibbs measure of the energy**, defined heuristically as

$$\mu_\nu^E(\mathrm{d}u) ='' \frac{1}{Z} \mathrm{e}^{-\frac{\nu}{2}\|u\|^2_{\mathcal{H}^0}} \mathrm{d}u'',$$

is a formal invariant measure for the following stochastic Navier–Stokes equation

$$d\ u(t) + [\nu A u(t) + B(u(t))]\,\mathrm{d}t = \sqrt{2} A^{\frac{1}{2}} \mathrm{d}w(t). \tag{14}$$

Now the spatial regularity of the noise is worse than before. Notice that the support of the Gibbs measure of the energy is the space $\cap_{r<-\frac{d}{2}} \mathcal{H}^r$.

Analysis of existence of flows for this equation is a difficult problem.

Gubinelli and Jara [40] proved an existence result for the following hyperviscous version of equation (14) , defined for $\sigma > 0$,

$$d\,u(t) + [\nu A^{1+\sigma} u(t) + B(u(t))]\,\mathrm{d}t = \sqrt{2} A^{-\sigma} \mathrm{d}w(t)$$

in dimension $d = 2$.

No uniqueness results are know. Nothing is known for the Navier–Stokes equation (14) when $d = 3$.

5. Recently Ferrario and Olivera [29] have considered a Navier–Stokes equation similar to (9), where the Wiener process w is replaced by a **fractional Wiener process** w^H. This is also called cylindrical fractional white noise. This can be represented with respect to a complete orthonormal system $\{e_k\}_k$ of the space $\mathcal{H}^0$ as

$$w^H(t) = \sum_k \beta_k^H(t) e_k$$

where $\{b_k^H\}_k$ is a sequence of i.i.d. fractional Brownian processes, that is each b_k^H is a centered Gaussian process whose covariance is

$$C(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

The bigger is the Hurst parameter $H \in (0,1)$ the more regular is the fractional Brownian motion; the case $H = \frac{1}{2}$ corresponds to the Wiener process considered in the previous section. The Gibbs measure μ_ν^H is now defined (formally) as

$$\mu_\nu^H(\mathrm{d}u) ='' \frac{1}{Z_\nu^H} \mathrm{e}^{-\nu \|u\|^2_{\mathcal{H}^{2H}}} \mathrm{d}u''$$

and its support is $\cap_{r<2H-1}\mathcal{H}^r$.

In [29] there are two results: local existence and uniqueness of solutions for $\frac{7}{16} < H < \frac{1}{2}$ and global existence and uniqueness for $\frac{1}{2} < H < 1$. The strategy to solve the problem is the same as in the case $H = \frac{1}{2}$: splitting method $u = v + z$, leading to two equations, one for z (similar to equation (8)) and the other one (11) for v. Moreover there is a property analogous to (12) for the quadratic term $B(z,z)$: for any $m \in \mathbb{N}$

$$\int \|B(z,z)\|^{2m}_{\mathcal{H}^\rho} \, \mu_\nu^H(\mathrm{d}z) < \infty$$

where

$$\rho < 4H - 3 \qquad \text{if } \tfrac{1}{4} < H < \tfrac{1}{2} \tag{15}$$

$$\rho < 2(H-1) \qquad \text{if } \tfrac{1}{2} \le H < 1 \tag{16}$$

The main difference with respect to the previous case where $H = \frac{1}{2}$ is that the measure μ_ν^H is no longer an invariant mesure for the Navier–Stokes dynamics; it is invariant only for the stochastic Stokes equation

$$\mathrm{d}z(t) + \nu A z(t)\,\mathrm{d}t = \sqrt{2}\mathrm{d}w^H(t), \quad t > 0$$

but not for the Euler equation (6). Hence for the more difficult case $H < \frac{1}{2}$ there is only a partial result under the restriction that $H \in (\frac{7}{16}, \frac{1}{2})$, whereas for $H \in (\frac{1}{2}, 1)$ the problem is easier. Indeed, when $H > \frac{1}{2}$ the first good result is that $\mu_\nu^H(\mathcal{H}^0) =$

1; moreover the solution to the stochastic Navier–Stokes equation has finite energy (we deal no more with distributions!) and the problem is well-posed in the space $\mathcal{H}^\sigma$ for $\sigma \in (0, 2H - 1)$.

6. The paper by Albeverio and Cruzeiro focused the attention of other researchers to nonlinear SPDE's with space-time white noise. Difficulties arise in dimension $d \geq 2$, since the solution is not regular in space (it is a distribution) and the nonlinearity needs some care to be well defined. As far as other equations of hydrodynamics are concerned, we recall the result by Zhu and Zhu [53] on the **3D Navier–Stokes equations with space-time white noise**. Previous known results of existence of global solutions were for much more regular noise (i.e. the noise is colored in space and white in time; see, e.g. [30]). Thanks to the theory of regularity structures introduced by Martin Hairer and the paracontrolled distribution proposed by Gubinelli, Imkeller and Perkowski, in Zhu and Zhu [53] local existence and uniqueness is proved.

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On Strong Solution to the 2D Stochastic Ericksen–Leslie System: A Ginzburg–Landau Approximation Approach

Zdzisław Brzeźniak, Gabriel Deugoué, and Paul André Razafimandimby

Abstract In this manuscript, we consider a highly nonlinear and constrained stochastic PDEs modelling the dynamics of 2-dimensional nematic liquid crystals under random perturbation. This system of SPDEs is also known as the stochastic Ericksen–Leslie equations (SELEs). We discuss the existence of local strong solution to the stochastic Ericksen–Leslie equations. In particular, we study the convergence of the stochastic Ginzburg–Landau approximation of SELEs, and prove that the SELEs with initial data in $\mathsf{H}^1 \times \mathsf{H}^2$ has at least a martingale, local solution which is strong in PDEs sense.

Keywords Stochastic Ericksen–Leslie equations · Nematic liquid crystals · Local solution · Martingale solutions · Ginzburg–Landau approximation

Mathematics Subject Classification Primary: 60H15 · 37L40 · Secondary: 35R60

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1 Introduction

Nematic Liquid Crystal (NLC) is a state of matter whose properties vary between amorphous liquid and crystalline solid. Their molecules are long and thin and have no positional order (like a fluid), but they tend to align along a direction called the optical director, denoted by a unit vector **d**. This optical director can be easily distorted and aligned to form a specific pattern using an external control with intensity above a certain threshold value. This passage from one stable to another stable state, possibly with higher energy, caused by an external force or control is called the Fréedericksz transition and it plays an important role in many branches of applied sciences such as in nonlinear optics and the industry of Liquid Crystal Displays (LCDs). For more details on physical modeling of liquid crystal we refer to the books [13, 39] and the papers [15, 28].

To model the hydrodynamics of NLC most scientists use the continuum theory developed by Ericksen [15] and Leslie [28]. From this theory Lin and Liu [32] derived the most basic and simplest form of the dynamical system describing the motion of nematic liquid crystals filling a bounded region $\mathcal{O} \subset \mathbb{R}^2$. This system is given by

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \Delta \mathbf{v} + \nabla \mathrm{p} = -\operatorname{div}\ (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \text{ in } [0, T) \times \mathcal{O} \tag{1.1a}$$

$$\partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \tag{1.1b}$$

$$\operatorname{div} \mathbf{v} = 0, \quad \text{in } [0, T) \times \mathcal{O}, \tag{1.1c}$$

$$\mathbf{v} = \frac{\partial \mathbf{d}}{\partial \nu} = 0, \text{ on } [0, T) \times \partial \mathcal{O}, \tag{1.1d}$$

$$|\mathbf{d}| = 1, \text{ in } [0, T) \times \mathcal{O}, \tag{1.1e}$$

$$(\mathbf{v}(0), \mathbf{d}(0)) = (\mathbf{v}_0, \mathbf{d}_0), \text{ in } \mathcal{O}. \tag{1.1f}$$

The number $T > 0$ is a fixed real number, $\mathcal{O}$ is a bounded domain or $\mathbb{R}^2$ with smooth boundary, the vector fields $\mathbf{v} : [0, T) \times O \to \mathbb{R}^2$ and $\mathbf{d} : [0, T) \times \mathcal{O} \to \mathbb{R}^3$ represent the velocity and director fields, respectively. The function $p : [0, T) \times \mathcal{O} \to \mathbb{R}$ is the fluids pressure and $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ is the matrix defined by

$$[\nabla \mathbf{d} \odot \nabla \mathbf{d}]_{ij} = \sum_{k=1}^{3} \partial_i \mathbf{d}_k \partial_j \mathbf{d}_k, \quad i, j \in \{1, 2\}.$$

The model (1.1) is an oversimplification of a Ericksen–Leslie model of nematic liquid crystal with the one-constant simplification of the Frank–Oseen energy density

$$\frac{1}{2} |\nabla \mathbf{d}|^2.$$

However, the model still retains many mathematical and essential features of the hydrodynamic equations for nematic liquid crystals. Moreover, the mathematical analysis of the above equations is quite challenging due to the sphere constraint (1.1e), the highly nonlinear coupling term $-\operatorname{div}\ (\nabla \mathbf{d} \odot \nabla \mathbf{d})$ and the non-parabolicity of the problem which can be seen from the fact

$$\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} = -\mathbf{d} \times (\mathbf{d} \times \Delta \mathbf{d}), \text{ for } \mathbf{d} \in \mathbb{S}^2.$$

Because of these observations, the system (1.1) has been extensively studied and several important results have been obtained. In addition to the paper [32] we cited above we refer, among others, to [19, 20, 31, 33, 35, 45] for results obtained prior to 2013, and to [11, 21–24, 29, 30, 44, 46, 47] for results obtained after 2014. For detailed reviews of the literature about the mathematical theory of nematic liquid crystals and other related models, we recommend the review articles [12, 17, 34] and the recent papers [22, 30].

In this paper, we fix two numbers $T, \varepsilon > 0$ and consider in the 2D torus $\mathcal{O}$ the following stochastic system

$$d\mathbf{u} + \left[(\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p\right] dt = -\nabla \cdot (\nabla \mathbf{n} \odot \nabla \mathbf{n})dt + dW, \tag{1.2a}$$

$$\operatorname{div} \mathbf{u} = \int_{\mathcal{O}} \mathbf{u} dx = 0, \tag{1.2b}$$

$$d\mathbf{n} + (\mathbf{u} \cdot \nabla)\mathbf{n} dt = \left[\Delta \mathbf{n} - \frac{1}{\varepsilon^2}(1 - |\mathbf{n}|^2)\mathbf{n}\right] dt + (\mathbf{n} \times \mathbf{h}) \circ d\eta, \tag{1.2c}$$

$$\mathbf{u}(t = 0) = \mathbf{u}_0 \text{ and } \mathbf{n}(t = 0) = \mathbf{n}_0, \tag{1.2d}$$

where $\mathbf{u}_0 : \mathcal{O} \to \mathbb{R}^d$, $\mathbf{n}_0 : \mathcal{O} \to \mathbb{R}^3$, $\mathbf{h} : \mathcal{O} \to \mathbb{R}^3$ are given functions, W and η are respectively independent cylindrical Wiener process and standard Brownian motion, $\circ d\eta$ stands for the Stratonovich integral.

We should note that the deterministic version of (1.2), that is,

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + \nabla p = -\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \tag{1.3a}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{1.3b}$$

$$\partial_t \mathbf{d} + (\mathbf{v} \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} - \frac{1}{\varepsilon^2}(1 - |\mathbf{d}|^2)\mathbf{d}, \tag{1.3c}$$

$$\mathbf{v} = 0 \text{ and } \frac{\partial \mathbf{d}}{\partial \nu} = 0 \text{ on } \partial \mathcal{O}, \tag{1.3d}$$

$$\mathbf{v}(0) = \mathbf{v}_0 \text{ and } \mathbf{d}(0) = \mathbf{d}_0, \tag{1.3e}$$

was proposed in [32] as an approximation of the simplified Ericksen–Leslie system (1.1).

Our study in this paper is motivated by the need for a sound mathematical analysis of the effect of a stochastic external perturbation on the dynamics of nematic liquid crystals. In fact, the effect of noise on the dynamics of the optical director has been the subject of numerous theoretical and experimental studies, but there are still many questions that remain unsolved. For instance, in previous studies in physics the fluid velocity is assumed to be negligible, hence understanding of the simultaneous effects of the noise and the fluid velocity on the Fréedericksz transition is an open and challenging problem. One should notice that de Gennes and Prost [13] pointed out that the fluid velocity plays an important role in the dynamics of the optical director. The mathematical papers that we cited above have taken into account the effects of the fluid velocity, but their equations contain neither deterministic nor stochastic external forces.

Rigorous mathematical results related to models for nematic liquid under random perturbations are still very few. The unpublished manuscript [7] is the first paper to prove the existence of strong solution of the stochastic (1.2). This result was generalized in [4] to the case where the quadratic $\mathbb{1}_{|\mathbf{d}|\le 1}(1-|\mathbf{d}|^2)\mathbf{d}$ is replaced by a more general polynomial function. The paper [6] deals with weak, both in PDEs and stochastic calculus sense, solutions and the maximum principle. Very recently, Hausenblas along with the first and the third authors of the present considered in [5] the stochastic Ericksen–Leslie Equations (SELEs)

$$d\mathbf{u} + [\mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla \mathrm{p}]dt = -\operatorname{div}\ (\nabla\mathbf{n}\odot\nabla\mathbf{n})dt + dW, \text{ in } [0,T)\times\mathcal{O} \tag{1.4a}$$

$$d\mathbf{n} + \mathbf{u}\cdot\nabla\mathbf{n}dt = [\Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n}]dt + (\mathbf{n}\times\mathbf{h})\circ d\eta, \tag{1.4b}$$

$$\operatorname{div}\mathbf{u} = 0, \quad \text{in } [0,T)\times\mathcal{O}, \tag{1.4c}$$

$$|\mathbf{n}| = 1, \text{ in } [0,T)\times\mathcal{O}, \tag{1.4d}$$

$$(\mathbf{u}(0),\mathbf{n}(0)) = (\mathbf{u}_0,\mathbf{n}_0), \text{ in } \mathcal{O}. \tag{1.4e}$$

By using the Banach Fixed Point Theorem, they showed the existence of local strong solution $(\mathbf{u}_0,\mathbf{n}_0)\in \mathrm{H}^\alpha\times\mathrm{H}^{\alpha+1}$ for $\alpha > \frac{n}{2}$, where $n = 2, 3$ is the space dimension. There is also the paper [36] which seeks for a special weak solution $(\mathbf{u},\mathbf{n})$ of (1.4) with the unknown $\mathbf{n}$ being replaced by an angle θ such that $\mathbf{n} = (\cos\theta, \sin\theta)$. This model reduction considerably simplify the mathematical analysis of (1.4).

As we mentioned the model (1.3) was proposed in [32] as an approximation of the simplified Ericksen–Leslie system (1.1). It is also widely used in numerical analysis to handle the sphere constraint (1.1e) in the Ericksen–Leslie equations, see for instance [43]. Hence, a natural questions which now arises is to know whether the solutions $(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon)$ converge to a solution to the stochastic Ericksen–Leslie equations as $\varepsilon\to 0$. This question is very interesting and has been the subject of intensive studies in deterministic case. These studies have generated several important results which were published in [16, 19, 20]. These papers are only related to the convergence of smooth solutions solutions. The convergence of the weak solution remains an open questions. Note that an attempt to solve this open problem was done in [32], but it is not clear whether the limit satisfies (1.1) or not.

In the case of the stochastic case, it seems that this note is the first analysis presenting a result on the convergence of the solutions $(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon)$ to (1.2). In particular, we show that by studying the convergence the strong solutions of (1.2) we can construct martingale, local strong to (1.4) with initial data in $\mathsf{H}^1\times\mathsf{H}^2$. Note that strong solution is taken in the sense of PDEs. The result we obtain is not covered in [5] which considered the stochastic ELEs with initial data $(\mathbf{v}_0, \mathbf{d}_0) \in \mathrm{H}^\alpha \times \mathrm{H}^{\alpha+1}$ for $\alpha > \frac{n}{2}$, where $n = 2, 3$ is the space dimension. Moreover, the approaches are completely different.

Let us now close this introduction by giving the layout of this paper. In Sect. 2 we introduce the frequently used notations in this manuscript and our main results, see Theorem 2.6. The proof of this main theorem is based on careful derivation of estimates uniform in ε of the solutions $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon)$ to (1.2) and the proof of tightness of laws of these solutions on the space $C([0, T]; \mathsf{H} \times \mathsf{H}^1) \cap C_{\text{weak}}([0, T]; \mathsf{V} \times \mathsf{H}^2) \cap L^2_{weak}(0, T; D(A) \times \mathsf{H}^3)$ and passage to the limits. These steps are very technical and require long and tedious calculation. Hence, in order to save space we only sketch the main steps of the derivations of uniform estimates in Sect. 3. Also, we only outline the main ideas of the proof of the tightness and the passage to the limits in Sect. 4.

2 Notations, the Stochastic Model and Our Main Result

2.1 Notations and the Stochastic Model

Let us begin with a brief description of the functional setting.

We will use the symbol $\mathcal{O}$ to denote the 2D torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2 = \mathbb{R}^2/\sim$, where by $\sim$ we understand the standard equivalence relation on $\mathbb{R}^2$ defined by $x = (x_1, x_2) \sim y = (y_1, y_2)$ iff there exist $k \in \mathbb{Z}^2$ such that $y = x + 2\pi k$. It is well known that $\mathcal{O}$ can be equipped in a natural differentiable structure so that it becomes a compact Riemannian manifold (without boundary). Occasionally it is convenient to view $\mathcal{O}$ as the square $[0, 2\pi]^2$ with the sides identified. In particular, the Riemannian volume measure on $\mathcal{O}$ can be identified with the Lebesgue measure on $[0, 2\pi]^2$ and the Riemannian distance is equal to the following one

$$d([x]_\sim, [y]_\sim) = \sqrt{\sum_{i=1}^{2} \min\{|y_i - x_i|, \quad |y_i - x_i - 2\pi|\}^2}, \quad [x]_\sim, [y]_\sim \in \mathcal{O},$$

where the both representatives $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $[x]_\sim$, and respectively, of $[y]_\sim$ have been chosen from $[0, 2\pi)^2$.

Throughout, we will use the following notation

$$\mathcal{M} = \{d : \mathcal{O} \to \mathbb{R}^3 : |d(x)| = 1 \ \text{ Leb} - \text{a.e.}\}.$$

All the vector spaces defined on $\mathcal{O}$ can also be defined in terms of functions defined on $[0, 2\pi]^2$ satisfying appropriate compatibility conditions on the boundary

$\partial([0, 2\pi]^2) = [0, 2\pi] \times \{0, 2\pi\} \cup \{0, 2\pi\} \times [0, 2\pi]$. We follow here the presentation from [41, 42, Chap. VIII, Sect. 4], see also recent papers [2, 3]. In particular, we denote by $\mathbb{H}^k(\mathcal{O})$, for $k \in \mathbb{N}$, the Sobolev space of all vector fields defined on $\mathcal{O}$, equivalently all $\mathbb{R}^2$ valued functions defined on $[0, 2\pi]^2$ satisfying appropriate compatibility conditions on the boundary $\partial([0, 2\pi]^2)$, which are weakly differentiable up to order k and those weak derivatives are square integrable. Obviously, $\mathbb{H}^0(\mathcal{O}) = \mathbb{L}^2(\mathcal{O})$. We denote by $\mathcal{V}$ the space of all C^∞ vector fields defined on $\mathcal{O}$, equivalently all $\mathbb{R}^2$ valued functions defined on $[0, 2\pi]^2$ satisfying appropriate compatibility conditions on the boundary $\partial([0, 2\pi]^2)$, such that $\operatorname{div} u = 0$. We also put

$$\mathbb{L}_0^2 = \left\{ u \in \mathbb{L}^2(\mathcal{O}) : \int_{\mathcal{O}} u(x)\, dx = 0 \right\}. \tag{2.1}$$

Then, by H we define the closure of the space $\mathcal{V}$ in the space $\mathbb{L}_0^2$ equipped with the norm and scalar product inherited from the latter space. It is known that H is equal to the set $\left\{u \in \mathbb{L}_0^2 : \operatorname{div} u = 0\right\}$. We also put

$$\mathrm{V} = \mathbb{H}^1(\mathcal{O}) \cap \mathrm{H}, \tag{2.2}$$

equipped with the norm and scalar product inherited from the space $\mathbb{H}^1(\mathcal{O})$. It turns out that V can be equipped with another scalar product and norm defined by

$$\langle u, v\rangle_{\mathrm{V}} := \langle \nabla u, \nabla v\rangle_{L^2}, \quad u, v \in \mathrm{V}, \tag{2.3}$$

$$\|u\|_{\mathrm{V}}^2 := \langle \nabla u, \nabla u\rangle_{L^2}, \quad u \in \mathrm{V}. \tag{2.4}$$

It it known that the original norm is equivalent to the new one. We will only use the latter.

We denote by A, the Stokes operator defined by

$$\begin{aligned} \mathrm{D}(\mathrm{A}) &= H^2(\mathcal{O}) \cap \mathrm{H} \\ \mathrm{A} \colon \mathrm{D}(\mathrm{A}) &\ni u \mapsto -\Pi\,(\Delta u) \in \mathrm{H}, \end{aligned} \tag{2.5}$$

where

$$\Pi : \mathbf{L}^2(\mathcal{O}) \to \mathrm{H}$$

is the orthogonal projection called the Leray–Helmholtz projection. It is known that A is a positive, self-adjoint operator in H with its inverse A^{-1} being compact. We will use the following norm on the space $D(\mathrm{A})$:

$$|u|_{\mathrm{D(A)}}^2 := |\mathrm{A}u|_{L^2}^2.$$

Obviously D(A) is a Hilbert space endowed with that norm (and the corresponding scalar product). Moreover, it is known that

$$\mathrm{D}(\mathrm{A}^{1/2}) = \mathrm{V} \quad \text{and} \quad \langle Au, u\rangle_{\mathrm{H}} = \|u\|^2_{\mathrm{V}} = |\nabla u|^2_{L^2}, \quad u \in \mathrm{D}(\mathrm{A}). \tag{2.6}$$

It is also well known (and follows from [41, Sect. 2.2]), that Π and A commute so that for every $\theta \geq 0$,

$$\Pi : \mathrm{D}(\mathrm{A}^{\theta}) \to \mathrm{D}(\mathrm{A}^{\theta}) \text{ is a bounded linear operator.} \tag{2.7}$$

So far we have introduced mostly the functional spaces corresponding to the velocity field. Let us next introduce the spaces corresponding to the director field. By H^k, $k \in \mathbb{N}$, we will denote the Sobolev space of all functions $\mathbf{n} : \mathcal{O} \to \mathbb{R}^3$, equivalently all $\mathbb{R}^3$ valued functions defined on $[0, 2\pi]^2$ satisfying appropriate compatibility conditions on the boundary $\partial([0, 2\pi]^2)$, which are weakly differentiable up to order k and those weak derivatives are square integrable. It is well known that H^k is a Hilbert space. Let us recall that by the Sobolev embedding theorem, $\mathsf{H}^k \hookrightarrow C(\mathcal{O})$ iff $k > 1$.

We now give few assumptions and notation about the stochastic perturbations.

Assumption 2.1 Throughout this paper we are given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual hypothesis, i.e., the filtration is right-continuous and all null sets of $\mathcal{F}$ are elements of $\mathcal{F}_0$.

We introduce what we mean by a cylindrical Wiener process in the following definition.

Definition 2.2 Assume also that Assumption 2.1 is satisfied and that K is a separable Hilbert space with orthonormal basis $\{e_j : j \in \mathbb{N}\}$. By a K-cylindrical Wiener process we understand a formal series $W(t) = \sum_{j=1}^{\infty} w_j(t)e_j$, $t \geq 0$, where $w_j = (w_j(t))_{t\geq 0}$, $j \in \mathbb{N}$, is a sequence of i.i.d. standard Wiener processes defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Equivalently, see [10, Definition 4.1], a K-cylindrical Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ we understand a family $W(t)$, $t \geq 0$ of bounded linear operators from K into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(i) for all $t \geq 0$, and $k_1, k_2 \in \mathrm{K}$, $\mathbb{E}\, W(t)k_1 W(t)k_2 = t\langle k_1, k_2\rangle_{\mathrm{K}}$,
(ii) for each $k \in \mathrm{K}$, $W(t)k$, $t \geq 0$ is a real valued $\mathbb{F}$-Wiener process.

Now, by projecting the stochastic model (1.2) into the space of divergence free function we obtain the following stochastic PDEs with periodic boundary conditions:

$$d\mathbf{u} + [A\mathbf{u} + \Pi_L(\mathbf{u} \cdot \nabla\mathbf{u})]\, dt = -\Pi_L\, [\mathrm{div}(\nabla\mathbf{n} \odot \nabla\mathbf{n})]\, dt + dW \tag{2.8a}$$

$$d\mathbf{n} + (\mathbf{u} \cdot \nabla)\mathbf{n}\, dt = \left[\Delta\mathbf{n} - \frac{1}{\varepsilon^2}(1 - |\mathbf{n}|^2)\mathbf{n}\right] dt + (\mathbf{n} \times \mathbf{h}) \circ d\eta \tag{2.8b}$$

$$\mathbf{u}(t = 0) = \mathbf{u}_0 \quad \text{and} \quad \mathbf{n}(t = 0) = \mathbf{n}_0, \tag{2.8c}$$

where we assume that the initial data satisfies

$$\mathbf{n}_0 \in \mathcal{M}, \tag{2.9}$$

and $\circ\, d\eta$ denotes the Stratonovich differential.

2.2 Our Main Results

Let us start with some definitions about stopping times.

Definition 2.3 A random function $\tau : \Omega \to [0, \infty]$ is called a stopping time, see [26, Definition I.2.1], [37, Definition 4.1], [14, Sect. III.5], iff for each $t \geq 0$, the set $\{\omega \in \Omega : t < \tau(\omega)\} \in \mathcal{F}_t$ (or equivalently, $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$). A stopping time $\tau : \Omega \to [0, \infty]$ is called accessible, see [27, Sect. 2.1, p. 45], iff there exists an increasing sequence[1] of stopping times $\tau_n \colon \Omega \to [0, \infty)$ such that $\mathbb{P}$-a.s. (i) for all $n \in \mathbb{N}$, $\tau_n < \tau$; (ii) and $\lim_{n\to\infty} \tau_n = \tau$.

The sequence $(\tau_n)_{n\in\mathbb{N}}$ as above is usually called an announcing sequence for τ.

We now continue with the definition of a strong solution to (1.2), see [7] and also [4].

Definition 2.4 Assume that $\varepsilon > 0$ and $(\mathbf{u}_0, \mathbf{n}_0) \in \mathrm{V} \times \mathsf{H}^2$ satisfies the constraint condition (2.9). Assume also that Assumption 2.1 is satisfied. A process $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon) : [0, \infty) \to \mathrm{V} \times \mathsf{H}^2$ is called a strong solution to the SGL (2.8) iff

(i) the process $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon)$ is V-valued continuous and $\mathbb{F}$-progressively measurable,
(ii) there exists an $D(A) \times \mathsf{H}^3$-valued $\mathbb{F}$-progressively measurable process $(\bar{\mathbf{u}^\varepsilon}, \bar{\mathbf{n}^\varepsilon})$ such that

$$(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon) = (\bar{\mathbf{u}^\varepsilon}, \bar{\mathbf{n}^\varepsilon}) \text{ almost everywhere w.r.t. Leb} \otimes \mathbb{P};$$

and, $\mathbb{P}$ almost surely,

$$(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon) \in C([0, \infty); \mathrm{V} \times \mathsf{H}^2) \quad \text{and} \quad (\bar{\mathbf{u}^\varepsilon}, \bar{\mathbf{n}^\varepsilon}) \in L^2_{\mathrm{loc}}([0, \infty); D(A) \times \mathsf{H}^3); \tag{2.10}$$

(iii) for all $t \in [0, \infty)$,

$$|\mathbf{n}^\varepsilon(t)|_{\mathbb{L}^\infty} \leq 1, \quad \mathbb{P}\text{-almost surely},$$

(iv) for all $t \in [0, \infty)$, the following identities hold true in H and H^1 respectively, $\mathbb{P}$-almost surely,

---

[1] In the sense that for all $n \in \mathbb{N}$, $\tau_n \leq \tau_{n+1}$, $\mathbb{P}$-a.s.

$$\mathbf{u}(t) = \mathbf{u}_0 - \int_0^t \Big[A\mathbf{u} + \Pi_L(\mathbf{u}\cdot\nabla\mathbf{u})\Big]\,ds + \Pi_L\Big[\mathrm{div}(\nabla\mathbf{n}\odot\nabla\mathbf{n})\Big]dt + W(t), \tag{2.11}$$

$$\mathbf{n}(t) = \mathbf{n}_0 \int_0^t \Big[-(\mathbf{u}\cdot\nabla)\mathbf{n} + \Delta\mathbf{n} - \frac{1}{\varepsilon^2}(1-|\mathbf{n}|^2)\mathbf{n}\Big]\,ds + (\mathbf{n}\times\mathbf{h})\circ d\eta(s). \tag{2.12}$$

We now recall the following result about the existence and uniqueness of a global strong solution to (1.2), see [4, Theorem 3.17]. Note that the condition (iii) of Definition 2.4 was proved in [6, Theorem 5.1].

Theorem 2.5 *Assume that* $\mathbf{h} = h(1, 1, 1)$ *where* $h \in \mathrm{H}^2(\mathcal{O}, \mathbb{R})$. *Assume that Assumption 2.1 is satisfied. Assume that* $W = (W(t))_{t\geq 0}$ *and* $\eta = (\eta(t))_{t\geq 0}$ *are respectively* V *and* $\mathbb{R}$*-valued Wiener process defined on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. *Assume finally that* $\varepsilon \in (0, 1)$. *Then, for every* $(\mathbf{u}_0, \mathbf{n}_0) \in \mathrm{V}\times\mathsf{H}^2$ *there exists a process* $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon) : [0,\infty) \to \mathrm{V}\times\mathsf{H}^2$ *which is a unique strong solution to* (2.8).

A natural questions which now arises is to know whether the solutions $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon)$ converge to a solution to the stochastic Ericksen–Leslie equations as $\varepsilon \to 0$. This is the subject of the present paper and it seems that this note is the first analysis presenting a result on the convergence of the solutions $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon)$ to the SGL. In particular, we obtained the following result.

Theorem 2.6 *Assume that* K *is a separable Hilbert space such that the embedding* $\mathrm{K} \hookrightarrow \mathrm{V}$ *is Hilbert–Schmidt. Assume that* $\mathbf{h} = h(1, 1, 1)$ *where* $h \in \mathrm{H}^2(\mathcal{O}, \mathbb{R})$.

There exists a filtered complete probability space $(\Omega, \mathcal{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, *a finite stopping* $\tau > 0$, *a* $\mathrm{K}\times\mathbb{R}$*-cylindrical Wiener process* $(\tilde{W}, \tilde{\eta})$, $(\mathbf{u}, \mathbf{n})$, $(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon) : [0, \tau] \to \mathrm{V}\times\mathsf{H}^2$, *such that*

$$(\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon) \to (\mathbf{u}, \mathbf{n}) \text{ a.s. in } C\left([0,\tau]; D(\mathsf{A}^{\frac{\alpha-1}{2}})\times\mathsf{H}^\alpha\right)$$
$$\cap L^2\left(0,\tau; D(\mathsf{A}^{\frac{\alpha}{2}})\times\mathsf{H}^{1+\alpha}\right), \quad \alpha \in [1, 2),$$

for all $t \in [0, T]$, *a.s.* $\mathbf{n}(t) \in \mathcal{M}$

$$\mathbf{u}(t\wedge\tau) - \mathbf{u}_0 = \tilde{W}(t\wedge\tau) - \int_0^{t\wedge\tau} (A\mathbf{u} + \Pi_L[\mathbf{u}\cdot\nabla\mathbf{u} + \mathrm{div}(\nabla\mathbf{n}\odot\mathbf{n})])\,ds$$

$$\mathbf{n}(t\wedge\tau) - \mathbf{n}_0 = \int_0^{t\wedge\tau} \left(\Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n} - \mathbf{u}\cdot\nabla\mathbf{n}\right)ds + \int_0^{t\wedge\tau} (\mathbf{n}\times\mathbf{h})d\tilde{\eta}.$$

Remark 2.7 In the deterministic case, Hong [19] proved that

$$\frac{1}{2} \le |\mathbf{n}^{\varepsilon}(t,x)|_{\mathbb{R}^3} \le 1, \ \forall t \in [0,T) \text{ a.e. } x. \tag{2.13}$$

This is important to handle $\frac{1}{\varepsilon^2}(1-|\mathbf{n}^{\varepsilon}|^2)\mathbf{n}^{\varepsilon}$ when one wants to study the convergence of the deterministic Ginzburg–Landau approximation. Unfortunately, we do not know how to prove it in the stochastic case. Therefore, we will need to devise an unusual technique.

The proof of this theorem follows the standard scheme of deriving uniform a priori estimates, establishing tightness in a appropriate spaces, using the famous Jakubowski–Skorokhod representation theorem to pass to the limit. However, the steps of this scheme are quite difficult due to the non-parabolicity of the limiting equations. Moreover, these steps involve long and tedious calculations. Therefore, in order to save space we only give a sketch of the main ideas of the proof of the above theorem.

Remark 2.8 Note that the previous results obtained in [5] only give the existence of a local solution $(\mathbf{u},\mathbf{n}) : [0,\tau] \to D(A^{\frac{\alpha}{2}}) \times \mathsf{H}^{1+\alpha}$ whenever $(\mathbf{u}_0,\mathbf{n}_0) \in D(A^{\frac{\alpha}{2}}) \times \mathsf{H}^{1+\alpha}$, $\alpha > \frac{d}{2}$, $d = 2, 3$. Hence, the present note improves the results from that paper.

Throughout, we put

$$f_{\varepsilon}(\mathbf{n}) = \frac{1}{\varepsilon^2}(1-|\mathbf{n}|^2)\mathbf{n} \quad \text{and} \quad F_{\varepsilon}(\mathbf{n}) = \frac{1}{4\varepsilon^2}(1-|\mathbf{n}|^2)^2.$$

3 Ideas of the Proof of Theorem 2.6: Uniform Estimates

As mentioned the proof of Theorem 2.6 consists in deriving uniform estimates, proving tightness results and passage to the limit. In this section we concentrate on the first part, i.e. uniform estimates. In the following section we will deal with the second and third parts.

In what follows we choose and fix a separable Hilbert space K such that the embedding $\mathrm{K} \hookrightarrow \mathrm{V}$ is Hilbert–Schmidt. We also assume that assumptions of Theorem 2.5 are satisfied, i.e. we assume Assumption 2.1 and that $W = (W(t))_{t\ge 0}$ and $\eta = (\eta(t))_{t\ge 0}$ are respectively V and $\mathbb{R}$-valued Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $(\mathbf{u}_0,\mathbf{n}_0) \in \mathrm{V} \times \mathsf{H}^2$. We denote by $Q \in \mathscr{L}(\mathrm{V})$ the covariance operator of W. Here $\mathscr{L}(\mathrm{V})$ is the space of all bounded linear maps from V into itself.

In this section we also fix $\varepsilon \in (0,1]$ denote by $(\mathbf{u}^{\varepsilon},\mathbf{n}^{\varepsilon}) : [0,\infty) \to \mathrm{V} \times \mathsf{H}^2$ the unique strong solution to the SGL (2.8) guaranteed by Theorem 2.5. Since we want to prove the existence of a local solution, we fix for the remainder of this section a finite time horizon $T > 0$. In all our results below we will find estimates independent of ε. The constants will depend on both the initial data as well as on T but we will not make this dependence explicit.

The first estimates we get are given in the following lemma.

Lemma 3.1 *For any* $p \in \mathbb{N}$ *there exists a constant* $K_0(p) > 0$, *independent of* $\varepsilon \in (0, 1)$, *such that*

$$\mathbb{E} \sup_{t\in[0,T)} \left(|\mathbf{u}^\varepsilon(t)|^2_{L^2} + |\nabla \mathbf{n}^\varepsilon(t)|^2_{L^2} + |F_\varepsilon(\mathbf{n}^\varepsilon(t))|_{L^1}\right)^p \leq K_0(p), \tag{3.1}$$

$$\mathbb{E}\left(\int_0^T \left[|\nabla \mathbf{u}^\varepsilon(t)|^2 + |\Delta \mathbf{n}^\varepsilon(t) + f_\varepsilon(\mathbf{n}^\varepsilon(t))|^2_{L^2}\right] dt\right)^p \leq K_0(p). \tag{3.2}$$

Sketch of the proof of Lemma 3.1. The application of the Itô Lemma [38] to the functional $\Gamma_1(\mathbf{u}) + \Gamma_2(\mathbf{n})$, where

$$\Gamma_1(\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2_{\mathbb{L}^2} \text{ and } \Gamma_2(\mathbf{n}) = \frac{1}{2}|\nabla \mathbf{n}|^2_{\mathbb{L}^2} + \frac{1}{4\varepsilon^2}\int_{\mathcal{O}} \left[1 - |\mathbf{n}|^2\right]^2 dx,$$

the use of the fact $f_\varepsilon(\mathbf{n}^\varepsilon) \perp \mathbf{n}^\varepsilon \times \mathbf{h}$ and

$$\langle B(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) + M(\mathbf{n}^\varepsilon), \mathbf{u}^\varepsilon\rangle + \langle \mathbf{u}^\varepsilon \cdot \nabla \mathbf{n}^\varepsilon, f_\varepsilon(\mathbf{n}^\varepsilon) - \Delta \mathbf{n}^\varepsilon\rangle = 0, \tag{3.3}$$

and the use of the elementary equality

$$|\nabla(\mathbf{n}^\varepsilon \times \mathbf{h})|^2_{\mathbb{L}^2} + \langle \nabla \mathbf{n}^\varepsilon, \nabla\left([\mathbf{n}^\varepsilon \times \mathbf{h}] \times \mathbf{h}\right)\rangle = |\mathbf{n}^\varepsilon \times \nabla \mathbf{h}|^2_{\mathbb{L}^2},$$

yield the following energy equality which is the basis of the proof of the lemma:

$$\begin{aligned}\mathcal{E}[\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon](t) + 2\int_s^t \mathcal{D}[\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon](r)dr &= \mathcal{E}[\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon](s) + |Q|^2_{\mathscr{L}(\mathrm{V})}(t-s)\\ &+ 2\int_s^t \langle \nabla \mathbf{n}^\varepsilon, (\mathbf{n}^\varepsilon \times \nabla \mathbf{h})d\eta\rangle + 2\int_s^t \langle \mathbf{u}^\varepsilon, dW\rangle\\ &+ \int_s^t |(\mathbf{n}^\varepsilon \times \nabla \mathbf{h})|^2_{\mathbb{L}^2} dr,\end{aligned} \tag{3.4}$$

$$\mathcal{E}[\mathbf{u}, \mathbf{n}](t) = \int_{\mathcal{O}} \left[|\mathbf{u}(t,y)|^2 + |\nabla \mathbf{n}(t,y)|^2\right] dy + \int_{\mathcal{O}} F_\varepsilon(\mathbf{n}(t,y))dy,$$

$$\mathcal{D}[\mathbf{u}^\varepsilon, \mathbf{n}^\varepsilon](t) = |\nabla \mathbf{u}^\varepsilon(t)|^2_{L^2} + |\Delta \mathbf{n}^\varepsilon(t) + f_\varepsilon(\mathbf{n}^\varepsilon(t))|^2_{L^2}.$$

Once we have this energy estimate we can refine the approach in [6] to obtain the estimates in Lemma 3.1. □

The above lemma gives two natural and important uniform estimates, but they are not sufficient for our purpose. We need to derive uniform estimates in the space $C([0,T];\mathrm{V}\times\mathsf{H}^2)\cap L^2([0,T];D(A)\times\mathsf{H}^3)$. In order to derive such estimates let us define an important stopping time.

Let $\delta, R>0$,

$$\mathcal{E}_R[\mathbf{u},\mathbf{n}](t,x)=\int\limits_{B(x,R)}\left[|\mathbf{u}(t,y)|^2+|\nabla\mathbf{n}(t,y)|^2+F_\varepsilon(\mathbf{n}(t,y))\right]dy,$$

and define the following three $\mathbb{F}$-stopping times

$$\sigma_1^\varepsilon(R):=\sigma_1^\varepsilon(\delta,R)=\inf\left\{t\in[0,\infty):\sup_{x\in\mathcal{O}}\mathcal{E}_R[\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon](t,x)\geq\delta\right\}\wedge T,\tag{3.5}$$

$$\sigma_2^\varepsilon=\inf\left\{t\in[0,\infty):\sup_{x\in\mathcal{O}}|\mathbf{n}^\varepsilon(t,x)|\leq\frac{1}{2}\right\}\wedge T,\tag{3.6}$$

$$\sigma^\varepsilon(R)=\sigma_1^\varepsilon(R)\wedge\sigma_2^\varepsilon.\tag{3.7}$$

We will now use this stopping time to derive uniform estimates in $C([0,T];\mathrm{V}\times\mathsf{H}^2)\cap L^2([0,T];D(A)\times\mathsf{H}^3)$ for the stopped processes $(\mathbf{u}^\varepsilon(\cdot\wedge\sigma^\varepsilon(R)),\mathbf{n}^\varepsilon(\cdot\wedge\sigma^\varepsilon(R)))$ for appropriate choice of R. This is motivated by the theory from the deterministic case which shows that uniform estimates in $C([0,T];\mathrm{V}\times\mathsf{H}^2)\cap L^2([0,T];D(A)\times\mathsf{H}^3)$ hold if the energy remains small and $|\mathbf{n}(t)|_{L^\infty}$ does not enter the ball $B(0,\frac{1}{2})$, see [19, Eq. 3.3].

Hereafter, we set $\mathcal{O}_t=[0,t]\times\mathcal{O}$, $t>0$ and recall the following important lemma, see [40, Lemma 3.1]. Note that Struwe proved his result on a general compact Riemannian manifold and hence his result is valid in our case of a compact 2D torus.

Lemma 3.2 *(The Ladyzhenskaya–Struwe inequality) There exists two constants $c_0>0$ and $r_1>0$, independent of $\varepsilon\in(0,1]$, such that for every $R\in(0,r_1]$ the following inequality holds*

$$\begin{aligned}|\nabla\mathbf{n}^\varepsilon(t,x)|^4_{L^4(\mathcal{O}_t)}\leq c_0&\left(\sup_{(s,x)\in[0,t]\times\mathcal{O}}\int\limits_{B(x,R)}|\nabla\mathbf{n}^\varepsilon(s,y)|^2dy\right)\\&\times\left(|\Delta\mathbf{n}^\varepsilon|^2_{L^2(\mathcal{O}_t)}+R^{-2}|\nabla\mathbf{n}^\varepsilon|^2_{L^2(\mathcal{O}_t)}\right).\end{aligned}\tag{3.8}$$

Remark 3.3 Since $\mathbb{P}$ almost surely $(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon):[0,T]\to\mathrm{V}\times\mathsf{H}^2$ is continuous, for any $\delta\in(0,1/8c_0)$ one can find $r_0>0$ such that for any $R\leq r_0$

$$\sigma^\varepsilon(R)>0\text{ a.s.}$$

Hereafter, we set

$$R_0 = r_1 \wedge r_0, \tag{3.9}$$
$$\sigma^\varepsilon = \sigma^\varepsilon(R), \quad \text{for a fixed } R \in (0, R_0]. \tag{3.10}$$

Lemma 3.4 *Let $\delta \in (0, 1/8c_0)$, $p \in \mathbb{N}$ r_1 and r_0 as in Remark 3.3. Let $R_0 = r_1 \wedge r_0$ and for $R \in (0, R_0]$ we set $\sigma^\varepsilon = \sigma^\varepsilon(R)$. Then, there exists a constant $K_1(p) > 0$ independent of $\varepsilon \in (0, 1]$, such that*

$$\mathbb{E}\left(\int_0^{\sigma^\varepsilon} |\Delta \mathbf{n}^\varepsilon|^2_{\mathbb{L}^2} ds + \frac{1}{8\varepsilon^4}\int_0^{\sigma^\varepsilon} |1 - |\mathbf{n}^\varepsilon|^2|^2_{\mathbb{L}^2} ds + \int_0^{\sigma^\varepsilon} |\nabla |\mathbf{n}^\varepsilon|^2|^2_{\mathbb{L}^2} ds\right)^p \leq K_1(p). \tag{3.11}$$

Sketch of the proof of Lemma 3.4. The idea of the proof consists in the following three steps.

- We expand $|\Delta \mathbf{n}^\varepsilon + f_\varepsilon(\mathbf{n}^\varepsilon)|^2_{\mathbb{L}^2}$, use integration by parts and the Young inequality to obtain

$$\begin{aligned} &|\Delta \mathbf{n}^\varepsilon|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})} + \frac{1}{\varepsilon^4}|(1 - |\mathbf{n}^\varepsilon|^2)\mathbf{n}^\varepsilon|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})} + \frac{1}{\varepsilon^2}|\nabla|\mathbf{n}^\varepsilon|^2|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})} \\ &\leq |\Delta \mathbf{n}^\varepsilon + f_\varepsilon(\mathbf{n}^\varepsilon)|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})} + 4|\nabla \mathbf{n}^\varepsilon|^4_{\mathbb{L}^4(\mathcal{O}_{\sigma^\varepsilon})} + \frac{1}{8\varepsilon^4}|(1 - |\mathbf{n}^\varepsilon|^2)|^2_{L^2(\mathcal{O}_{\sigma^\varepsilon})}. \end{aligned} \tag{3.12}$$

- We use the fact $|\mathbf{n}^\varepsilon(t)|^2 \geq \frac{1}{4}$ for $t \in [0, \sigma^\varepsilon]$ to control the term containing $|(1 - |\mathbf{n}^\varepsilon|^2)\mathbf{n}^\varepsilon|^2$ (this yields the term $|1 - |\mathbf{n}^\varepsilon|^2|^2$ in the estimates (3.11)!)
- We finally use the Ladyzhenskaya–Struwe lemma and Lemma 3.1 to conclude.

□

Before proceeding further, we recall the following lemma which was proved in [18] in the case of a general domain. For the case of the torus, it is enough to observe that the $d = 2$-dimensional result follows from the $d = 1$-dimensional one. In the latter case, it follows by a simple scaling argument applied to a (large) interval $[0, L]$ with radius $R = 1$ with the centers chosen by $x_i = i$, $i = 0, \ldots, [L]$. Here $[L]$ denotes the integer part of L.

Lemma 3.5 *There exists a positive number $C > 0$ such that the following holds.*

For every $R > 0$ there exists a natural number $N_R \in \mathbb{N}$ such that $N_R \leq CR^{-2}$ and a finite set $\{x_i : i = 1, \ldots, N_R\} \subset \mathcal{O}$ such that

$$\text{for every } x \in \mathcal{O} \text{ there exists } i \in \{1, \ldots, N_R\} \text{ such that } B(x, R) \subset B(x_i, 2R) \tag{3.13}$$

Note that in particular $\mathcal{O} = \bigcup_{i=1}^{N_R} B(x_i, 2R)$.

By staying in $[0, \sigma^\varepsilon]$ and using the above covering lemma we obtain the following lemma.

Lemma 3.6 *Assume that $\delta \in (0, 1/8c_0)$, R_0 as before. Then, for every $p \in \mathbb{N}$ there exists a constant $K_2(p) > 0$ independent of $\varepsilon \in (0, 1]$ such that*

$$\mathbb{E}\exp\left(p\sup_{t\in[0,\sigma^\varepsilon]}\mathcal{E}[\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon](t)\right)\le K_2(p) \tag{3.14}$$

$$\mathbb{E}\exp\Big(p\int_0^{\sigma^\varepsilon}\Big[|\Delta\mathbf{n}^\varepsilon|^2_{\mathbb{L}^2}+|\nabla\mathbf{u}^\varepsilon|^2_{\mathbb{L}^2} + \frac{1}{8\varepsilon^4}|[1-|\mathbf{n}^\varepsilon|^2]|^2_{\mathbb{L}^2}+\frac{1}{\varepsilon^2}|\nabla|\mathbf{n}^\varepsilon|^2|^2_{\mathbb{L}^2}\Big]ds\Big)\le K_2(p), \tag{3.15}$$

$$\mathbb{E}\exp\left(p\int_0^{\sigma^\varepsilon}\int_{\mathcal{O}}(|\nabla\mathbf{n}^\varepsilon|^4+|\mathbf{u}^\varepsilon|^4)ds\right)\le K_2(p). \tag{3.16}$$

Sketch proof of Lemma 3.6. The first estimate (3.14) can be easily obtained. In fact, by covering the torus $\mathcal{O}$ by balls $B(x_k,R_0)$ we obtain

$$\sup_{t\in[0,\sigma^\varepsilon]}\mathcal{E}[\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon](t)\le\sum_{k=1}^{N_{R_0}}\sup_{(t,x)\in[0,\sigma^\varepsilon]\times B(x_k,R_0)}\mathcal{E}_{R_0}[\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon](t,x)\le N_{R_0}\delta,$$

from which we easily derive (3.14).

The proof of the second estimate (3.15) is quite long. We start using the previous estimates and the energy inequality (3.4) and derive that

$$\begin{aligned}
&|\Delta\mathbf{n}^\varepsilon|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})}+\frac{1}{8\varepsilon^4}|(1-|\mathbf{n}^\varepsilon|^2)|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})}+\frac{1}{\varepsilon^2}|\nabla|\mathbf{n}^\varepsilon|^2|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})}\\
&\le|\Delta\mathbf{n}^\varepsilon+f_\varepsilon(\mathbf{n}^\varepsilon)|^2_{\mathbb{L}^2(\mathcal{O}_{\sigma^\varepsilon})}\\
&\le K_3+\int_0^{\sigma^\varepsilon}\langle\mathbf{u}^\varepsilon,dW\rangle+\int_0^{\sigma^\varepsilon}\langle\nabla\mathbf{n}^\varepsilon,\mathbf{n}^\varepsilon\times\nabla\mathbf{h}\rangle d\eta+\frac{K_3}{R_0^2}\int_0^{\sigma^\varepsilon}|\nabla\mathbf{n}^\varepsilon|^2_{\mathbb{L}^2}ds.
\end{aligned} \tag{3.17}$$

Next, by the Itô formula and the previous exponential inequality estimate (3.14) we obtain

$$\mathbb{E}\left(\exp\left[p\int_0^{\sigma^\varepsilon}\langle\mathbf{u}^\varepsilon,dW\rangle+\int_0^{\sigma^\varepsilon}\langle\nabla\mathbf{n}^\varepsilon,\mathbf{n}^\varepsilon\times\nabla\mathbf{h}\rangle d\eta+\frac{pK_3}{R_0^2}\int_0^{\sigma^\varepsilon}|\nabla\mathbf{n}^\varepsilon|^2_{\mathbb{L}^2}ds\right]\right)\le C,$$

from which along with (3.17) we derive (3.15). □

We now can derive the following important sets of uniform estimates.

Lemma 3.7 *For every $p\in\mathbb{N}$ there exists a constant $K_3(p)>0$ such that*

$$\mathbb{E} \sup_{t\in[0,\sigma^\varepsilon]} [|\nabla \mathbf{u}^\varepsilon(t)|^2_{\mathbb{L}^2} + |\Delta \mathbf{n}^\varepsilon(t) + f_\varepsilon(\mathbf{n}^\varepsilon(t))|^2_{\mathbb{L}^2}]^p \le K_3(p), \tag{3.18}$$

$$\mathbb{E}\left[\int_0^{\sigma^\varepsilon} \left(|A\mathbf{u}^\varepsilon(s)|^2_{\mathbb{L}^2} + |\nabla[\Delta \mathbf{n}^\varepsilon(s) + f_\varepsilon(\mathbf{n}^\varepsilon(s))]|^2_{\mathbb{L}^2}\right) ds\right]^p \le K_3(p), \tag{3.19}$$

$$\mathbb{E}\left[\int_0^{\sigma^\varepsilon} \left(\left|\frac{1}{\varepsilon}(\Delta \mathbf{n}^\varepsilon(s)) + f_\varepsilon(\mathbf{n}^\varepsilon(s)) \cdot \mathbf{n}^\varepsilon(s)\right|^2_{\mathbb{L}^2}\right) ds\right]^p \le K_3(p). \tag{3.20}$$

The proof of this lemma is very similar to the proof of the following key uniform estimates.

Proposition 3.8 *Let $\delta \in (0, 1/8c_0)$, $R_0 = r_0 \wedge r_1$. Then for every $p \in \mathbb{N}$ there exists a constant $K_4(p) > 0$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,\sigma^\varepsilon]} [|A^{\frac{1}{2}} \mathbf{u}^\varepsilon(t)|^2_{\mathbb{L}^2} + |\Delta \mathbf{n}^\varepsilon(t)|^2_{\mathbb{L}^2}]^p\right) \le K_4(p), \tag{3.21}$$

$$\mathbb{E}\left[\int_0^{\sigma^\varepsilon} \left(|A\mathbf{u}^\varepsilon|^2_{\mathbb{L}^2} + |\nabla \Delta \mathbf{n}^\varepsilon|^2_{\mathbb{L}^2}\right) ds\right]^p \le K_4(p). \tag{3.22}$$

Moreover, $\sigma^\varepsilon < T$ is satisfied $\mathbb{P}$ almost surely.

To derive the above crucial uniform estimates we will need to apply the Itô formula to the functional $\Lambda : \mathrm{V} \times \mathsf{H}^2 \to [0, \infty)$ defined by

$$\Lambda(u, d) = \Lambda_1(u) + \Lambda_2(d), \tag{3.23}$$

where $\Lambda_1 : \mathsf{H}^2 \to [0, \infty)$ and $\Lambda_2 : \mathrm{V} \to [0, \infty)$ are the energy functionals defined by

$$\Lambda_1(d) = \frac{1}{2}|\Delta d|^2_{\mathbb{L}^2} \text{ and } \Lambda_2(v) = \frac{1}{2}|\nabla v|^2_{\mathbb{L}^2}, \quad v \in \mathrm{V}, d \in \mathsf{H}^2. \tag{3.24}$$

We need to establish several lemmas involving the first and second Fréchet derivatives of Λ_1 and Λ_2. Before stating and proving these lemmas we recall the formulae for the derivative of Λ_1

$$\Lambda_1'(d)[\mathbf{g}] = \langle \Delta d, \Delta \mathbf{g}\rangle \text{ and } \Lambda''(d)[\mathbf{g}, \mathbf{p}] = \langle \Delta \mathbf{g}, \Delta \mathbf{p}\rangle, \quad d, \mathbf{g}, \mathbf{p} \in \mathsf{H}^2. \tag{3.25}$$

We state the following lemma which can be proved using elementary inequalities.

Lemma 3.9 *There exists a constant α_0 such that for all $v \in D(\mathsf{A})$ and $d \in \mathsf{H}^3$*

$$\Lambda_2'(v)[-v\cdot\nabla v - \Pi_L[\mathrm{div}\,(\nabla d\odot\nabla d)]] = -\langle \mathsf{A}v, \Pi_L[\mathrm{div}\,(\nabla d\odot\nabla d)]\rangle$$
$$\leq \frac{1}{8}\left(|\nabla\Delta d|^2_{\mathbb{L}^2} + |\mathsf{A}v|^2_{\mathbb{L}^2}\right) + \alpha_0|\nabla d|^4_{\mathbb{L}^4}|\Delta d|^2_{\mathbb{L}2}. \tag{3.26}$$

Lemma 3.10 *There exists a constant* $\alpha_1 > 0$ *such that for all* $v\in D(\mathsf{A})$ *and* $d\in \mathsf{H}^2$

$$\Lambda_1'(d)[-v\cdot\nabla d] = -\langle \Delta d, \Delta(v\cdot\nabla d)\rangle$$
$$\leq \frac{1}{8}\left(|\nabla\Delta d|^2_{\mathbb{L}^2} + |\mathsf{A}v|^2_{\mathbb{L}^2}\right) + \alpha_1[|\nabla v|^2_{\mathbb{L}^2} + |\nabla d|^4_{\mathbb{L}^4}]|\Delta d|^2_{\mathbb{L}^2}. \tag{3.27}$$

Lemma 3.11 *Let* $\mathbf{h}\in\mathsf{H}^2$. *Then, there exists a constant* $\alpha_3 > 0$ *such that for all* $d\in\mathsf{H}^2$

$$\frac{1}{2}\Lambda_1'(d)[(d\times\mathbf{h})\times\mathbf{h}] + \frac{1}{2}\Lambda_1''(d)[d\times \mathrm{H}] \leq \alpha_3|\mathbf{h}|^2_{\mathsf{H}^2}\left[|\Delta d|^2 + |\nabla d|^2_{\mathbb{L}^4} + |d|^2_{\mathbb{L}^\infty}\right]. \tag{3.28}$$

Lemma 3.12 *Let* $\mathbf{h}\in\mathsf{H}^2$. *Then, there exists a constant* $\alpha_4 > 0$ *such that for all* $d\in\mathsf{H}^2$

$$|\Lambda_1'(d)[d\times\mathbf{h}]|^2 \leq \alpha_4|\mathbf{h}|^2_{\mathsf{H}^2}|\Delta d|^2_{\mathbb{L}^2}\left(|\Delta d|^2_{\mathbb{L}^2} + |\nabla d|^2_{\mathbb{L}^4} + |d|^2_{\mathbb{L}^\infty}\right). \tag{3.29}$$

One of the most difficult term to control in the application of Itô formula for $\Lambda(v,d)$ is the term involving the Ginzburg–Landau functional $f_\varepsilon(d)$. However, with skillful and careful analysis we were able to derive the following important result.

Lemma 3.13 *There exists a constant* $\alpha_2 > 0$ *such that for all* $\varepsilon\in(0,1]$ *and* $d\in\mathsf{H}^3$ *be satisfying*

$$\frac{1}{2} < |d(x)|^2 \leq 1 \textit{ for all } x\in\mathcal{O}, \tag{3.30}$$

$$\begin{aligned}\Lambda_1'(d)[f_\varepsilon(d)] &= \langle\Delta d, \Delta f_\varepsilon(d)\rangle\\ &\leq \alpha_2\left[|\nabla(\Delta d + f_\varepsilon(d))|^2_{\mathbb{L}^2} + |\Delta d + f_\varepsilon(d)|^4_{\mathbb{L}^2} + |\nabla d|^4_{\mathbb{L}^4} + |\Delta d|^2_{\mathbb{L}^2}|\nabla d|^2_{\mathbb{L}^4}\right]\\ &\quad + \frac{1}{4}|\nabla\Delta d|^2_{\mathbb{L}^2} - \frac{1}{2\varepsilon^2}|\Delta(1-|d|^2)|^2_{\mathbb{L}^2}\\ &\quad - \frac{1}{\varepsilon^2}\int_{\mathcal{O}}(1-|d|^2)\left[|\nabla^2 d|^2 + |\Delta d|^2\right]dx\end{aligned} \tag{3.31}$$

Proof Let $\varepsilon\in(0,1]$ and $d\in\mathsf{H}^3$ satisfying

$$\frac{1}{2} < |d(x)|^2 \leq 1 \text{ for all } x\in\mathcal{O}. \tag{3.32}$$

Now, in order to prove the lemma we will need the following identity which is taken from [21]

$$d \cdot \Delta^2 d = \frac{1}{2}\Delta^2|d|^2 - 4\nabla d \nabla \Delta d - 2|\nabla^2 d|^2 - |\Delta d|^2. \tag{3.33}$$

We also need the following inequality which follows from (3.30)

$$\left|\frac{1}{\varepsilon^2}(1-|d|^2)\right| = \left|\frac{1}{\varepsilon^2}(1-|d|^2)d\right|(|d|)^{-1} \le 2|f_\varepsilon(d)|. \tag{3.34}$$

With these two observations in mind we have

$$|(\Delta d, \Delta f_\varepsilon(d))| = \frac{1}{\varepsilon^2}(\Delta^2 d, (1-|d|^2)d) \tag{3.35}$$

$$= \frac{1}{\varepsilon^2}((1-|d|^2), \frac{1}{2}\Delta^2|d|^2 - 4\nabla d \nabla \Delta d - 2|\nabla^2 d|^2 - |\Delta d|^2) \tag{3.36}$$

$$= -\frac{1}{\varepsilon^2}|\Delta(1-|d|^2)|^2_{\mathbb{L}^2} - \frac{1}{\varepsilon^2}\int_{\mathcal{O}}(1-|d|^2)[|\nabla^2 d|^2 + |\Delta d|^2]dx \tag{3.37}$$

$$+ \frac{4}{\varepsilon^2}\int_{\mathcal{O}}[(1-|d|^2)\nabla d \nabla \Delta d]dx \tag{3.38}$$

$$= I_1 + I_2 + I_3. \tag{3.39}$$

It is clear that $I_1 \ge 0$, hence we do not need to worry about it. Since $(1-|d|^2) \ge 0$ we do not need to deal with I_2. Let us then estimate I_3. For doing so we use (3.34) and get

$$\begin{aligned} I_3 &\le 8\int_{\mathcal{O}}[|(1-|d|^2)||\nabla d||\nabla \Delta d|]dx \\ &\le 8\int_{\mathcal{O}}[|f_\varepsilon(d) + \Delta d - \Delta d||\nabla d||\nabla \Delta d|]dx \\ &\le 8\int_{\mathcal{O}}[|\Delta d + f_\varepsilon(d)||\nabla d||\nabla \Delta d|]dx + \int_{\mathcal{O}}[|\Delta d||\nabla d||\nabla \Delta d|]dx \\ &= J_1 + J_2. \end{aligned} \tag{3.40}$$

Using the Young, the Hölder, the Gagliardo–Nirenberg inequalities and the Young inequality in this order yields that for any $\alpha > 0$ there exists a constant $C(\alpha) > 0$ such that

$$J_1 \le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)\int_{\mathcal{O}} (|\Delta d + f_\varepsilon(d)|^2|\nabla d|^2)dx \tag{3.41}$$

$$\le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)|\Delta d + f_\varepsilon(d)|^2_{\mathbb{L}^4}|\nabla d|^2_{\mathbb{L}^4} \tag{3.42}$$

$$\le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)|\Delta d + f_\varepsilon(d)|_{\mathbb{L}^2}|\nabla(\Delta d + f_\varepsilon(d))|_{\mathbb{L}^2}|\nabla d|^2_{\mathbb{L}^4} \tag{3.43}$$

$$\le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)\Big[|\nabla(\Delta d + f_\varepsilon(d))|^2_{\mathbb{L}^2} + |\Delta d + f_\varepsilon(d)|^2_{\mathbb{L}^2}|\nabla d|^4_{\mathbb{L}^4}\Big] \tag{3.44}$$

Next, we deal with J_2 in a similar way. Using the Young, the Hölder, the Gagliardo–Nirenberg inequalities and the Young inequality in this order yields that for any $\alpha > 0$ there exists a constant $C(\alpha) > 0$ such that

$$J_2 \le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)\int_{\mathcal{O}} (|\Delta d|^2|\nabla d|^2)dx \tag{3.45}$$

$$\le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)|\Delta d|^2_{\mathbb{L}^4}|\nabla d|^2_{\mathbb{L}^4} \tag{3.46}$$

$$\le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)|\Delta d|_{\mathbb{L}^2}|\nabla\Delta d|_{\mathbb{L}^2}|\nabla d|^2_{\mathbb{L}^4} \tag{3.47}$$

$$\le \alpha|\nabla\Delta d|^2_{\mathbb{L}^2} + C(\alpha)\Big[|\Delta d|^2_{\mathbb{L}^2}|\nabla d|^4_{\mathbb{L}^4}\Big] \tag{3.48}$$

The inequality (3.31) follows from (3.39), (3.40), (3.48) and (3.44) by choosing $\alpha = \frac{1}{4}$. □

Let us sum up our findings from the above lemma in the next remark.

Remark 3.14 Let $\mathbf{h} \in \mathsf{H}^2$, $v \in D(\mathsf{A})$ and $d \in \mathsf{H}^3$ be satisfying

$$\frac{1}{2} < |d(x)|^2 \le 1 \text{ for all } x \in \mathcal{O}. \tag{3.49}$$

Let $\alpha_0, \alpha_1, \alpha_2$ be the constants from Lemmas 3.9–3.13 and let us put

$$\mathsf{R}_1(d) := \alpha_2\left[|\nabla(\Delta d + f_\varepsilon(d))|^2_{\mathbb{L}^2} + |\Delta d + f_\varepsilon(d)|^4_{\mathbb{L}^2} + |\nabla d|^4_{\mathbb{L}^4}\right], \tag{3.50}$$

$$\mathsf{R}_2(d) := \alpha_3\left[|\Delta d|^2_{\mathbb{L}^2} + |\nabla d|^2_{\mathbb{L}^4} + |d|^2_{\mathbb{L}^\infty}\right], \tag{3.51}$$

$$\mathsf{S}(v,d) := |\mathrm{A}v|^2_{\mathbb{L}^2} + |\nabla\Delta d|^2_{\mathbb{L}^2} + \left|\frac{1}{\varepsilon}\Delta(1-|d|^2)\right|^2_{\mathbb{L}^2} + \left|\frac{1}{\varepsilon}\sqrt{(1-|d|^2)}\nabla^2 d\right|^2_{\mathbb{L}^2} + \left|\frac{1}{\varepsilon}\sqrt{(1-|d|^2)}\Delta d\right|^2_{\mathbb{L}^2}, \tag{3.52}$$

$$\mathsf{N}_1(d) := [\alpha_0 + \alpha_1 + \alpha_2]|\nabla d|^4_{\mathbb{L}^4}. \tag{3.53}$$

Then, it follows from Lemmas 3.9–3.11 that

$$
\begin{aligned}
&\Lambda_2'(v)[-\mathsf{A}d - v\cdot\nabla v - \Pi_L(\mathrm{div}[\nabla d \odot \nabla d])] \\
&\quad + \Lambda_1'(d)[\Delta d + f_\varepsilon(d) - v\cdot\nabla d + \frac{1}{2}(d\times \mathbf{h})\times\mathbf{h}] + \frac{1}{2}\Lambda''[\Delta(d\times\mathbf{h})] \\
&\quad \le -\mathsf{S}(v,d) + \mathsf{R}_1(d) + |\mathbf{h}|^2_{\mathsf{H}^2}\mathsf{R}_2(d) + \mathsf{N}_1(d)\Lambda(v,d).
\end{aligned} \tag{3.54}
$$

Bearing the notation of this remark in mind, we set

$$
\Phi_1(t\wedge\sigma^\varepsilon) = \exp\left(-\int_0^{t\wedge\sigma^\varepsilon} \mathsf{N}_1(\mathbf{n}^\varepsilon(s))ds\right), \quad t\ge 0.
$$

We now state and sketch the proof of the following result.

Proposition 3.15 *For any $p\in\mathbb{N}$ there exist constants $K_5(p), K_6(p) > 0$, independent of $\varepsilon > 0$, such that*

$$
\mathbb{E}\big(\sup_{t\in[0,T]}[|\mathsf{A}^{\frac{1}{2}}\mathbf{v}(t\wedge\sigma^\varepsilon)|^2_{\mathbb{L}^2} + |\Delta\mathbf{n}(t\wedge\sigma^\varepsilon)|^2_{\mathbb{L}^2}]^p\big) \le K_5(p), \tag{3.55}
$$

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^{T\wedge\sigma^\varepsilon}\big(|\mathsf{A}\mathbf{v}|^2_{\mathbb{L}^2} + |\nabla\Delta\mathbf{n}|^2_{\mathbb{L}^2} + \frac{1}{2\varepsilon^2}|\Delta(1-|d|^2)|^2_{\mathbb{L}^2} \\
&\quad + \frac{1}{\varepsilon^2}\Big|\sqrt{(1-|d|^2)}\nabla^2 d\Big|^2_{\mathbb{L}^2} + \Big|\sqrt{(1-|d|^2)}\Delta d\Big|^2_{\mathbb{L}^2}\big)ds\Big]^p \le K_6(p).
\end{aligned} \tag{3.56}
$$

Proof The proof involves long and tedious calculation, so we will only outline the main idea. Without of loss generality we only prove the estimate for $p\in\mathbb{N}$.

We need to use the Itô's formula for several processes. We firstly apply Itô's formula to $\Lambda_2(\mathbf{u}^\varepsilon(t\wedge\sigma^\varepsilon))$ and $\Lambda_1(\mathbf{n}^\varepsilon(t\wedge\sigma^\varepsilon))$, then to $Z(t\wedge\sigma^\varepsilon)$ where

$$
Z(t\wedge\sigma^\varepsilon) = \Phi_1(t\wedge\sigma^\varepsilon)\Lambda(\mathbf{u}^\varepsilon(t\wedge\sigma^\varepsilon), \mathbf{n}^\varepsilon(t\wedge\sigma^\varepsilon)), \quad t\in[0,T],
$$

and Λ is defined in (3.23). Using (3.54), the uniform estimates in Lemmas 3.1, 3.4, 3.7 and Proposition 3.8, and the fact $\Phi_1\le 1$ we infer that for all $p\ge 1$ there exist constants $K_7 > 0$ (depending only on p) and $K_8 > 0$ which depends only on p, T, $|Q|_{\mathscr{L}(\mathrm{V})}$, $|\mathbf{h}|^2_{\mathsf{H}^2}$ and $|(\mathbf{v}_0,\mathbf{n}_0)|^{4p}_{\mathsf{V}\times\mathsf{H}^2}$ such that for all $\varepsilon\in(0,1]$

$$\mathbb{E}\sup_{s\in[0,t]}[Z(s\wedge\sigma^\varepsilon)]^p+\mathbb{E}\left[\int_0^{t\wedge\sigma^\varepsilon}\Phi_1(s)\mathsf{S}(\mathbf{u}^\varepsilon(s),\mathbf{n}^\varepsilon(s))ds\right]^p$$
$$\leq K_7\Big(\mathbb{E}[Z(0)]^p+|\mathbf{h}|_{\mathsf{H}^2}^{2p}\left(\mathbb{E}\int_0^{T\wedge\sigma^\varepsilon}\mathsf{R}_2(\mathbf{d}(s))ds\right)^p$$
$$+\mathbb{E}\left(\int_0^{t\wedge\sigma^\varepsilon}\mathsf{R}_1(\mathbf{d}(s))ds\right)^p+\mathbb{E}\sup_{s\in[0,t]}|M(s\wedge\sigma^\varepsilon)|^p\Big)$$
$$\leq K_8+K_7\mathbb{E}\sup_{s\in[0,t]}|M(s\wedge\sigma^\varepsilon)|^p,$$

where the process M is defined by

$$M(s)=\int_0^s\Phi_1(r)\Lambda_2'(\mathbf{u}^\varepsilon(r))dW(r)$$
$$+\sum_{j=1}^\infty\int_0^t\Phi_1(r)\Lambda_1'(\mathbf{n}^\varepsilon(r))[\mathbf{n}^\varepsilon(r)\times\mathbf{h}]\circ d\eta(r),\quad s\in[0,T].$$

We now use the Burkholder–Davis–Gundy, the Hölder, the Young inequalities, the uniform estimates in Lemmas 3.1, 3.7 and Proposition 3.8, and the fact $\Phi_1\leq 1$ to deduce that

$$K_7\mathbb{E}\sup_{s\in[0,t]}|M(s\wedge\sigma^\varepsilon)|^p\leq\frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}[Z(s\wedge\sigma^\varepsilon)]^p+K_8. \tag{3.57}$$

Collecting all the above estimates yield there exists a constant $K_9>0$ which depends only on p, T, $|\mathbf{h}|_{\mathsf{H}^2}^{2p}$ and $|(v_0,\mathbf{n}_0)|_{\mathsf{V}\times\mathsf{H}^2}^{4p}$ such that for all $\varepsilon\in(0,1]$

$$\mathbb{E}\sup_{s\in[0,t]}[Z(s\wedge\sigma^\varepsilon)]^p+\mathbb{E}\left[\int_0^{t\wedge\sigma^\varepsilon}\Phi_1(s)\mathsf{S}(\mathbf{u}^\varepsilon(s),\mathbf{n}^\varepsilon(s))ds\right]^p\leq K_9. \tag{3.58}$$

With this at hand, we can now estimate the $\mathbb{E}\sup_{s\in[0,T]}[\Psi(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon)(s\wedge\sigma_\epsilon)]^p$ as follows. Let

$$\Phi_1^{-1}=1/\Phi_1=\exp\left(\int_0^\cdot\mathsf{N}_1(\mathbf{n}^\varepsilon(s))ds\right).$$

Since Φ_1^{-1} is an increasing function of the time t we then obtain

$$\mathbb{E}\sup_{s\in[0,T]}[\Psi(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon)(s\wedge\sigma_\epsilon)]^p = \mathbb{E}\sup_{t\in[0,T]}\left[\Phi_1^{-1}(s\wedge\sigma_\epsilon)Z(s\wedge\sigma_\epsilon)\right]^p \\ \leq \left(\mathbb{E}|\Phi_1^{-1}(T\wedge\sigma_\epsilon)|^{2p}\mathbb{E}\sup_{s\in[0,T]}Z^{2p}(s)\right)^{\frac{1}{2}} \quad (3.59)$$

from which along with the definition of Φ_1 and the exponential estimates in Lemma 3.6 and (3.58) we derive that for any $R>0$, $p\geq 1$ there exists a constant K_{10} such that for all $\varepsilon\in[0,1)$

$$\mathbb{E}\sup_{s\in[0,T]}\left[|\nabla\mathbf{u}^\varepsilon(s\wedge\sigma_\epsilon)|^2_{\mathbb{L}^2}+|\Delta\mathbf{n}^\varepsilon(s\wedge\sigma_\epsilon)|^2_{\mathbb{L}^2}\right]^p\leq K_{10}. \quad (3.60)$$

We establish the estimate (3.56) in a similar way. This completes the proof of the proposition. □

4 Ideas of the Proof of Theorem 2.6: Tightness and Passage to the Limit

As in the previous section, in what follows we choose and fix a separable Hilbert space K such that the embedding $\mathrm{K}\hookrightarrow\mathrm{V}$ is Hilbert–Schmidt. We also assume that assumptions of Theorem 2.5 are satisfied, i.e. we assume Assumption 2.1 and that $W=(W(t))_{t\geq 0}$ and $\eta=(\eta(t))_{t\geq 0}$ are respectively V and $\mathbb{R}$ valued Wiener processes defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, and $(\mathbf{u}_0,\mathbf{n}_0)\in\mathrm{V}\times\mathsf{H}^2$. Since we want to prove the existence of a local solution, we fix for the remainder of this section a finite time horizon $T>0$. But contrary to the previous section, here we do not fix $\varepsilon\in(0,1]$ but instead consider a family $(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon)_{\varepsilon\in(0,1]}$, where $(\mathbf{u}^\varepsilon,\mathbf{n}^\varepsilon):[0,\infty)\to\mathrm{V}\times\mathsf{H}^2$ the unique strong solution to (2.8) guaranteed by Theorem 2.5.

It will convenient to introduce the following notation. Please note that we omit the superscript $^\varepsilon$.

$$f_1(t)=\mathbb{1}_{[0,\sigma^\varepsilon]}(t)\left(-B(\mathbf{u}^\varepsilon(t),\mathbf{n}^\varepsilon(t))-\Pi_L(\mathrm{div}[\nabla\mathbf{n}^\varepsilon\odot\nabla\mathbf{n}^\varepsilon])\right),\ t\in[0,T]$$
$$f_2(t)=\mathbb{1}_{[0,\sigma^\varepsilon]}(t),\ t\in[0,T],$$
$$g_1(t)=\mathbb{1}_{[0,\sigma^\varepsilon]}(t)\left(-\mathbf{u}^\varepsilon(t)\cdot\nabla\mathbf{n}^\varepsilon(t)+|\nabla\mathbf{n}^\varepsilon(t)|^2\mathbf{n}^\varepsilon(t)+\frac{1}{2}(\mathbf{n}^\varepsilon(t)\times\mathbf{h})\times\mathbf{h}\right),\ t\in[0,T],$$
$$g_2(t)=\mathbb{1}_{[0,\sigma^\varepsilon]}(t)\mathbf{n}^\varepsilon(t)\times\mathbf{h},\ t\in[0,T].$$

We then consider the following problem

$$\begin{cases} d\mathbf{u}(t) + \mathsf{A}\mathbf{u}(t) = f_1(t)dt + f_2(t)dW, t \in (0,T], & (4.1a) \\ d\mathbf{n}(t) - \Delta\mathbf{n}(t)dt = g_1(t)dt + g_2(t) \times d\eta, t \in (0,T], & (4.1b) \\ \mathbf{u}(0) = \mathbf{u}_0 \text{ and } \mathbf{n}(0) = \mathbf{n}_0. & (4.1c) \end{cases}$$

This has a unique mild solution $(\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon)$ such that $(\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon) \in X_{[0,T]} = C([0,T]; \mathrm{V} \times \mathsf{H}^2) \cap L^2(0,T; D(A) \times \mathsf{H}^3)$ almost surely. Following the idea of [8, p. 128], we can prove that

$$(\mathbf{v}^\varepsilon(t \wedge \sigma^\varepsilon), \mathbf{n}^\varepsilon(t \wedge \sigma^\varepsilon)) = (\mathbf{u}^\varepsilon(t \wedge \sigma^\varepsilon), \mathbf{n}^\varepsilon(t \wedge \sigma^\varepsilon)), \ \forall t \geq 0 \ \mathbb{P}\text{-a.s.} \tag{4.2}$$

Thanks to these observations and the uniform estimate in Lemma 3.1 and Proposition 3.8 we obtain the following global estimates

Proposition 4.1 *For any $p \in \mathbb{N}$, there exists $K_{11}(p) > 0$ independent of $\varepsilon \in (0,1]$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left[|A^{\frac{1}{2}}\mathbf{v}^\varepsilon(t)|^2_{\mathbb{L}^2} + |\mathbf{d}^\varepsilon(t)|^2_{\mathsf{H}^2}\right]^p\right) \leq K_{11}, \tag{4.3}$$

$$\mathbb{E}\left[\int_0^T \left(|A\mathbf{v}^\varepsilon|^2_{\mathbb{L}^2} + |\mathbf{d}^\varepsilon|^2_{\mathbf{h}^3}\right) ds\right]^p \leq K_{11}. \tag{4.4}$$

Thanks to this proposition we can prove that the family $(\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon)$ satisfies the following Aldous condition.

Proposition 4.2 *There exists a constant $K_{12} > 0$, independent of $\varepsilon \in (0,1)$, such that for every $\kappa > 0$ and every sequence $(\rho_n)_{n\in\mathbb{N}}$ of $(0,T]$-valued stopping times,*

$$\sup_{0\leq\theta\leq\kappa} \mathbb{E}\left(|(\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon)((\rho_n + \theta)\wedge T) - (\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon)(\rho_n)|_{\mathsf{H}\times\mathsf{H}^1}\right) \leq K_{12}\kappa. \tag{4.5}$$

Now, let us introduce the following notation

$$\mathbf{X}_T = C([0,T]; \mathsf{H} \times \mathsf{H}^1) \cap C_{\text{weak}}([0,T]; \mathrm{V} \times \mathsf{H}^2) \cap L^2_{weak}(0,T; D(A) \times \mathsf{H}^3). \tag{4.6}$$

We also put

$$\mathbf{X}^\alpha_T = C\left([0,T]; D\left(\mathsf{A}^{\frac{\alpha-1}{2}} \times \mathsf{H}^\alpha\right) \cap L^2\left(0,T; D\left(\mathsf{A}^{\frac{\alpha}{2}}\right)\right) \times \mathsf{H}^{1+\alpha}\right), \quad \alpha \in [1,2),$$
$$\mathbf{Y}_T = C([0,T]; \mathrm{V} \times \mathbb{R}).$$

The first corollary below follows from Lemma 3.1.

Corollary 4.3 *We have*

$$(1 - |\mathbf{u}^\varepsilon|^2) \to 0 \text{ in } L^2(\Omega; C([0,T); \mathbb{L}^2)).$$

The second corollary is a consequence of Lemma 3.1, Propositions 3.8 and 4.2, and [9, Corollary 3.9].

Corollary 4.4 *The family of laws of* $[(\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon); (W, \eta); \sigma^\varepsilon]$ *is tight on* $\mathbf{X}_T \times \mathbf{Y}_T \times [0, T]$.

From Corollaries 4.3 and 4.4, applying the Jakubowski–Skorokhod representation theorem, [25] (see also [9, Theorem 3.11]), we have the following result.

Proposition 4.5 *There exist a new probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, not relabeled,* $\mathbf{X}_T \times \mathbf{Y}_T \times [0, T]$*-valued sequence* $(Z^\varepsilon) := ([(\tilde{\mathbf{v}}^\varepsilon, \tilde{\mathbf{d}}^\varepsilon); (W^\varepsilon, \eta^\varepsilon); \tau^\varepsilon])$ *and* $\mathbf{X}_T \times \mathbf{Y}_T \times [0, T]$*-valued random variable* $Z := [(\mathbf{v}, \mathbf{d}); (\tilde{W}, \tilde{\eta}); \tau]$ *such that*

$$law_{\mathbf{X}_T\times\mathbf{Y}_T\times[0,T]}(Z^\varepsilon) = law_{\mathbf{X}_T\times[0,T]}([(\mathbf{v}^\varepsilon, \mathbf{d}^\varepsilon); (W, \eta); \sigma^\varepsilon]), \tag{4.7}$$

$$Z^\varepsilon \to Z \text{ in } (\mathbf{X}_T \cap \mathbf{X}_T^\alpha) \times \mathbf{Y}_T \times [0, T] \ \ \mathbb{P}\text{-a.s.}, \tag{4.8}$$

$$\mathbb{P}\text{-a.s.} \mathbb{1}_{[0,\tau^\varepsilon]} |\tilde{\mathbf{d}}^\varepsilon|^2 - 1\mathbb{1}_{[0,\tau]} \to 0 \text{ in } L^q([0, T]; \mathbb{L}^2) \forall q \in [2, \infty). \tag{4.9}$$

In order to conclude the proof of Theorem 2.6 we need to pass to the limit. Thanks to the strong convergence (4.8) the passage to the equation for the velocity $\mathbf{v}$ can be done as in [9]. The passage to the limit in the director equation needs special care. In particular, we need the convergence (4.9) and the following equivalence result which can be established as in [1].

Proposition 4.6 *Let* $\mathbf{u} \in C([0, T]; \mathrm{H}) \cap L^2(0, T; \mathrm{V})$ *and consider the problems*

$$d\mathbf{n} + \mathbf{u} \cdot \nabla \mathbf{n} = \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} + \frac{1}{2} G_{\mathbf{h}}^2(\mathbf{n}) + (\mathbf{n} \times \mathbf{h}) d\eta, \quad |\mathbf{n}| = 1 \tag{4.10}$$

and

$$\mathbf{n} \times d\mathbf{n} + \mathbf{n} \times (\mathbf{u} \cdot \nabla \mathbf{n}) = -\operatorname{div}(\mathbf{n} \times \nabla \mathbf{n}) + \frac{1}{2} \mathbf{n} \times G_{\mathbf{h}}^2(\mathbf{n}) + \mathbf{n} \times (\mathbf{n} \times \mathbf{h}) d\eta, \quad |\mathbf{n}| = 1. \tag{4.11}$$

If $\mathbf{n} \in C([0, T]; \mathsf{H}^1) \cap L^2(0, T; \mathsf{H}^2)$ *satisfies* (4.10) *then it satisfies* (4.11)*, and vice versa.*

With this proposition at hand and (4.9) we can now carry out as is done in [1] the passage to the limit in the equations for the director field $\mathbf{n}$ and conclude the proof of Theorem 2.6.

Remark 1 The proof of $\tau > 0$ a.s. is technical and very long and will be published in a separate paper. In fact, one must know the existence and regularity of the pressure P which is a delicate matter. We also must establish local energy (energy on balls) estimates which are quite long and tedious.

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Stability Properties of Mild Solutions of SPDEs Related to Pseudo Differential Equations

V. Mandrekar and B. Rüdiger

Abstract This is a review article which presents part of the contribution of Sergio Albeverio to the study of existence and uniqueness of solutions of SPDEs driven by jump processes and their stability properties. The results on stability properties obtained in Albeverio et al. (Random Oper. Stoch. Equ. 25(2):79–105, 2017 [4]) are presented in a slightly simplified and different way.

Keywords Stochastic partial differential equations · Non-Gaussian additive noise · Existence · Uniqueness · Itô formula, invariant measures for infinite-dimensional dissipative systems

Mathematics Subject Classification 60H15 · 60G51 · 37L40 · 60J76

1 Introduction

The theory of SPDEs driven by Brownian motion was studied for a long time and solutions taking values in a Hilbert space are described in [6, 9] based on previous work. Sergio Albeverio was among the first mathematicians to initiate the study of SPDEs driven by jump processes [2] with solutions in Hilbert spaces in contrast to Kallianpur and Xiong [11] who studied generalized solutions. In order to study these equations in general, Sergio et al. provided the Lévy–Itô decomposition in Banach spaces [1]. There was a previous approach by Dettweiler [7], where the stochastic integrals are defined differently from those of Itô. In [1] it was however proven that the definitions are equivalent. Starting from [1] (M-type 2 and type 2) Banach valued stochastic integrals with respect to Lévy processes and compensated Poisson random measures associated to additive processes were defined in [12, 15, 18], including the case of separable Hilbert space valued stochastic integrals. (For the theory of

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stochastic integration on Banach spaces see also [17, 21] and references there.) Following this, the article [3] establishes the basic generalization of classical work for mild solutions of SPDE's driven by Lévy processes and associated Poisson random noise. An Itô-formula was proved in this case [19] which was later generalized in [13] and further in [4]. It has interesting applications to stability of solutions of such SPDEs which originated in [13] and have been continued in [12] and by Albeverio et al. in [4].

Our project in this paper is to present first a review of the work mentioned. The results related to the stability properties obtained in [4] are presented in Sect. 4 in a simplified and slightly different way, by involving a dissipativity condition (condition i in Theorem 9). These motivated further investigations of stability properties of SPDEs with multiple invariant measures in [8], which introduces a "generalized dissipativity condition", that are not reported in this paper due to stipulated page limitations for this article.

2 Stochastic Integrals and Itô-Formula

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let H be a separable Hilbert space with norm $\|\cdot\|_H$ and scalar product $<\cdot,\cdot>_H$, which for simplicity we will often denote with $\|\cdot\|$ and $<\cdot,\cdot>$. Let $\{L_t\}_{t\geq 0}$ be an H-valued Lévy process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\mathcal{B}(H)$ denote the Borel-σ-Algebra on H. For $B \in \mathcal{B}(H)$ with $0 \notin \overline{B}$, we define

$$N((0,t] \times B) = \sum_{0<s\leq t} \mathbb{1}_B(\Delta L_s) \quad t \geq 0$$

and

$$N((0,t] \times \{0\}) = 0$$

We define

$$\begin{aligned} \beta:\ \mathcal{B}(H) &\to \mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\} \\ B &\to \beta(B) := \mathbb{E}[N((0,1] \times B)] \end{aligned}$$

Observe that the random measure $N(dt,dx)$ induced on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(H)$ is a Poisson random measure with compensator $\nu(dt,dx) := dt \otimes \beta(dx)$. We recall that on the trace σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(H \setminus \{0\})$ the Poisson random measure and its compensator are σ-finite measures, but might not be finite, since the jumps of the underlying Lévy process $\{L_t\}_{t\geq 0}$ in a time interval $[0,T]$ are numerable, but might not be finite. These are however finite on each set $[0,T] \times A$ with $A \in \mathcal{B}(H)$, $0 \notin \overline{A}$. We shall be dealing with non-Gaussian Lévy processes, i.e.

$$L_t - \int_0^t \int_{\|x\|_H \le 1} x N(dt, dx) = 0 \quad \forall t \ge 0$$

(See e.g. Proposition 3.3.8 in [12] or Sect. 4.5 in [17].)

We denote with $q(dt, dx) := N(dt, dx) - \nu(dt, dx)$ the compensated Poisson random measure associated to $N(dt, dx)$.

Let us remark that $M_\cdot := \{M_t\}_{t\ge 0}$ with $M_t := q((0, t] \cap A \times B)$ is for each $A \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(H)$ with $\beta(B) < \infty$ a $\{\mathcal{F}_t\}_{t\ge 0}$-martingale. (See e.g. Lemma 2.4.7 in [12].)

Let F be a separable Hilbert space. Let $\mathrm{Ad}(F)$ denote the space of all functions $f : \mathbb{R}_+ \times H \times \Omega \to F$ which are adapted on the enlarged space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}}) = (\Omega \times H, \mathcal{F} \times \mathcal{B}(H), (\mathcal{F}_t \times \mathcal{B}(H))_{t\ge 0}, \mathbb{P} \otimes \beta).$$

We can define the Itô-Integral of f w.r.t. the compensated Poisson random measure $q(ds, dx)$ basically starting with simple functions which are square integrable w.r.t. $\mathbb{P} \otimes \beta$ and then by density arguments for all $f \in L^2_{\mathrm{ad}}(F) := L^2(\tilde{\Omega} \times \mathbb{R}_+, \tilde{\mathcal{F}} \otimes \mathcal{B}(\mathbb{R}_+), \mathbb{P} \otimes \beta \otimes \lambda; F) \cap \mathrm{Ad}(F)$, where λ denotes the Lebesgue-measure. (See e.g. Sect. 3.5 in [12].)

The Itô-Integral of f w.r.t. the compensated Poisson random measure $q(ds, dx)$

$$Z_t := \int_0^t \int_A f(s, x) q(ds, dx) \quad t \ge 0 \tag{1}$$

is then a square integrable martingale for all $A \in \mathcal{B}(H)$ such that $0 \notin \overline{A}$.

Through stopping times the Itô-Integral (1) can be extended also to all $f \in \mathcal{K}^2_{\infty,\beta}(F)$ which denotes the linear space of all progressively measurable functions $f : \mathbb{R}_+ \times H \times \Omega \to F$ such that

$$\mathbb{P}\left(\int_0^t \int_H \|f(s, x)\|^2 q(ds, dx) < \infty \right) = 1 \quad \forall t \ge 0.$$

The Itô-Integral (1) is then a local martingale (see e.g. [12, Sect. 3.5], where this theory has been discussed for F being a separable Banach space).

Let us present the Itô-Formula for the Itô-Process $(Y_t)_{t\ge 0}$, with

$$Y_t := Z_t + \int_0^t F_s ds + \int_0^t \int_\Lambda k(s, x) N(ds, dx), \tag{2}$$

where $(Z_t)_{t\ge 0}$ is defined through Eq. (1), $F_\cdot := \{F_t\}_{t\ge 0}$ is an F-valued $\{\mathcal{F}_t\}_{t\ge 0}$ adapted process, which satisfies

$$\mathbb{P}\left(\int_0^t \|F_s\|\, ds \ < \infty\right) = 1 \quad \forall t \geq 0,$$

$\Lambda \in \mathcal{B}(H)$ is a set with $\beta(\Lambda) < \infty$, $k : \Omega \times \mathbb{R}_+ \times H \to F$ is a progressively measurable process. Moreover k is càdlàg or càglàd $\beta(dx) \otimes \mathbb{P}$–almost surely and

$$\int_0^t \int_\Lambda \|k(s,x)\|\, \nu(ds,dx) \ < \infty \quad \mathbb{P} - a.s.$$

Let $\mathcal{H} \in C_b^{1,2}(\mathbb{R}_+ \times H; F)$, the space of functions $\mathcal{H} : \mathbb{R}_+ \times H \to F$ which are differentiable in $t \in \mathbb{R}_+$ and twice Fréchét differentiable in $x \in H$, with bounded derivatives. Then similar to [10] (for the finite dimensional case) it can be proven that the following Itô-Formula holds. (See e.g. [5] or [19] for the Banach valued case.)

Theorem 1 *Let $A \in \mathcal{B}(H)$. Assume*

$$\int_0^t \int_A \|f(s,x)\| \nu(ds,dx) \ < \infty \quad \mathbb{P} - a.s.$$

or

$$\int_0^t \int_A \|f(s,x)\|^2 \nu(ds,dx) \ < \infty \quad \mathbb{P} - a.s.$$

1. We have $\mathbb{P}$–almost surely

$$\begin{aligned}
\mathcal{H}(t,Y_t) &= \mathcal{H}(0,Y_0) + \int_0^t \partial_s \mathcal{H}(s,Y_s) ds \\
&+ \int_0^t \partial_y \mathcal{H}(s,Y_s) F_s ds + \int_0^t \int_A \big(\mathcal{H}(s,Y_{s-} + f(s,x)) \\
&- \mathcal{H}(s,Y_{s-})\big) q(ds,dx) + \int_0^t \int_A \big(\mathcal{H}(s,Y_s + f(s,x)) \\
&- \mathcal{H}(s,Y_s) - \partial_y \mathcal{H}(s,Y_s) f(s,x)\big) \nu(ds,dx) \\
&+ \int_0^t \int_\Lambda \big(\mathcal{H}(s,Y_{s-} + k(s,x)) - \mathcal{H}(s,Y_{s-})\big) N(ds,dx), \quad t \geq 0,
\end{aligned}$$

where

2. *for all $t \in \mathbb{R}_+$ we have $\mathbb{P}$–almost surely*

$$\int_0^t \|\partial_s \mathcal{H}(s, Y_s)\| ds < \infty,$$

$$\int_0^t \int_A \|\mathcal{H}(s, Y_s + f(s,x)) - \mathcal{H}(s, Y_s)\|^2 \nu(ds, dx) < \infty,$$

$$\int_0^t \int_A \|\mathcal{H}(s, Y_s + f(s,x)) - \mathcal{H}(s, Y_s) - \partial_y \mathcal{H}(s, Y_s) f(s,x)\| \nu(ds, dx) < \infty,$$

$$\int_0^t \int_\Lambda \|\mathcal{H}(s, Y_{s-} + k(s,x)) - \mathcal{H}(s, Y_{s\_})\| N(ds, dx) < \infty.$$

However we are often interested in applying the Itô formula to functions $\mathcal{H}$ which are only in $C^{1,2}(\mathbb{R}_+ \times H; H)$, i.e. where the Fréchét derivatives are not necessarily bounded. Especially for stochastic models applied to physics we might be interested in taking advantage of conservation of energy of a random process and would like to compute $\|Y_t\|^2$. Remark however that $H(y) = \|y\|^2$ is of class $C^2(H; \mathbb{R})$.

Let us define

Definition 1 A continuous, non-decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is *quasi-sublinear* if there is a constant $C > 0$ such that

$$h(x+y) \leq C\big(h(x) + h(y)\big), \quad x, y \in \mathbb{R}_+,$$
$$h(xy) \leq Ch(x)h(y), \quad x, y \in \mathbb{R}_+.$$

In [13] the following was proved:

Theorem 2 *Let us assume*

(a) $\mathcal{H} \in C^{1,2}(\mathbb{R}_+ \times H; F)$ *is a function such that*

$$\|\partial_y \mathcal{H}(s, y)\| \leq h_1(\|y\|), \quad (s, y) \in \mathbb{R}_+ \times H$$
$$\|\partial_{yy} \mathcal{H}(s, y)\| \leq h_2(\|y\|), \quad (s, y) \in \mathbb{R}_+ \times H$$

for quasi-sublinear functions $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$.

(b) $f : H \times \mathbb{R}_+ \times \Omega \to F$ *is a progressively measurable process such that for all $t \in \mathbb{R}_+$ we have $\mathbb{P}$–almost surely*

$$\int_0^t \int_A \|f(s,x)\|^2 \nu(ds,dx) + \int_0^t \int_A h_1(\|f(s,x)\|)^2 \|f(s,x)\|^2 \nu(ds,dx)$$
$$+ \int_0^t \int_A h_2(\|f(s,x)\|) \|f(s,x)\|^2 \nu(ds,dx) < \infty.$$

Then the Itô-Formula 1. with 2. holds.

Remark 1 We remark that $H(y) = \|y\|^2$ is of class $C^2(H; \mathbb{R})$ and

$$H_y(y)v = 2 < y, v > \quad and \quad H_{yy}(y)(v)(w) = 2 < v, w >, \tag{3}$$

so that if for all $t \in \mathbb{R}_+$ we have $\mathbb{P}$–almost surely $\int_0^t \int_A \|f(s,x)\|^2 \nu(ds,dx) < \infty$ and $\int_0^t \int_A \|f(s,x)\|^4 \nu(ds,dx) < \infty$, then Theorem 2 can be applied to $\mathcal{H}(s,y) := H(y) = \|y\|^2$.

3 SPDEs on Hilbert Spaces

In this section we shall be studying Stochastic Partial Differential Equations (SPDEs) driven by Lévy processes. Let $(H, \|\cdot\|_H)$ be a Hilbert space and A be an infinitesimal generator of a semigroup $\{S_t, t \geq 0\}$ on H to H. This means

(i) $S_0 = I$
(ii) $S_{s+t} = S_s S_t \quad \forall s, t \geq 0$

We also assume that $\{S_t, t \geq 0\}$ is strongly continuous, i.e.

(iii) $\lim_{t \to 0} S_t x = x$ (in norm $\|\cdot\|_H$) for all $x \in H$

If $\{S_t, t \geq 0\}$ is a semigroup satisfying the above properties, we call it a "strongly continuous semigroup" (C_0-Semigroup). For such a semigroup we note that there exists $\alpha \geq 0$ and $M \geq 1$ such that the operator norm in the space $L(H)$ of bounded linear operators from H to H satisfies

$$\|S_t\|_{L(H)} \leq M e^{\alpha t} \quad t \geq 0.$$

We call the semigroup $\{S_t, t \geq 0\}$ "pseudo-contraction" semigroup if $M = 1$, "uniformly bounded semigroup" if $\alpha = 0$ and "contraction semigroup" if $M = 1$ and $\alpha = 0$.

If $t \to S_t$ is differentiable for all $x \in H$ then the semigroup $\{S_t, t \geq 0\}$ is differentiable.

Let $\{S_t\} := \{S_t, t \geq 0\}$ be a C_0-semigroup on H. The linear operator A with domain

$$\mathcal{D}(A) := \left\{ x \in H, \lim_{t \to 0^+} \frac{S_t x - x}{t} \quad exists \right\}$$

defined by

$$Ax = \lim_{t \to 0^+} \frac{S_t x - x}{t}$$

is called the infinitesimal generator (i.g.) of $\{S_t\}$.

The following facts for an i.g. A of a C_0-semigroup $\{S_t\}$ are well known (see e.g. [16]):

(1) For $x \in H$ $\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S_s x ds = S_t x$.
(2) For $x \in \mathcal{D}(A)$, $S_t x \in \mathcal{D}(A)$ and $\frac{d}{dt} S_t x = A S_t x = S_t A x$.
(3) For $x \in H$, $\int_0^t S_s x ds \in \mathcal{D}(A)$ and $A \int_0^t S_s x ds = S_t x - x$.
(4) $\mathcal{D}(A)$ is dense in H and A is a closed operator.
(5) Let $f : [0, T] \to \mathcal{D}(A)$ be a measurable function with $\int_0^T \|f(s)\|_{\mathcal{D}(A)} ds < \infty$, then $\int_0^T f(s) ds \in \mathcal{D}(A)$ and $\int_0^T A f(s) ds = A \int_0^T f(s) ds$.

We associate with A the resolvent set $\rho(A)$ as the set of complex numbers λ for which $\lambda I - A$ has bounded inverse

$$R(\lambda, A) := (\lambda I - A)^{-1} \in L(H)$$

and we call $R(\lambda, A)$, $\lambda \in \rho(A)$ the resolvent of A.

We note that $R(\lambda, A) : H \to \mathcal{D}(A)$ is one-to-one, i.e.

$$\begin{aligned} &(\lambda I - A) R(\lambda, A) x = x, \quad x \in H \\ &and\ R(\lambda, A)(\lambda I - A) x = x, \quad x \in \mathcal{D}(A), \\ &giving\ A R(\lambda, A) x = R(\lambda, A) A x, \quad x \in \mathcal{D}(A) \end{aligned}$$

Remark that $R(\lambda_1, A) R(\lambda_2, A) = R(\lambda_2, A) R(\lambda_1, A)$ for $\lambda_1, \lambda_2 \in \mathcal{P}(A)$.

Lemma 1 *Let $\{S_t\}$ be C_0-semigroup with infinitesimal generator A. Let*

$$\alpha_0 := \lim_{t \to \infty} t^{-1} ln(\|S_t\|_{L(H)}),$$

then any real number $\lambda > \alpha_0$ belongs to the resolvent set $\rho(A)$ and

$$R(\lambda, A) x = \int_0^\infty e^{-\lambda t} S_t x dt \quad x \in H$$

In addition for $x \in H$

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda, A)x - x\|_H = 0$$

Theorem 3 Hille–Yosida Theorem *Let $A : \mathcal{D}(A) \subset H \to H$ be a linear operator on a Hilbert space H. Necessary and sufficient conditions for A to generate a C_0-semigroup is*

(1) A is closed and $\overline{\mathcal{D}(A)} = H$
(2) There exists α, $M \in \mathbb{R}$ such that for $\lambda > \alpha$, $\lambda \in \rho(A)$

$$\|R(\lambda, A)^r\|_{L(H)} \le M(\lambda - \alpha)^{-r}, \quad r = 1, 2, \ldots$$

In this case $\|S_t\|_{L(H)} \le Me^{\alpha t}$, $t \ge 0$.

For $\lambda \in \rho(A)$, consider the family of operators

$$R_\lambda := \lambda R(\lambda, A).$$

Since the range $\mathcal{R}(R(\lambda, A))$ of $R(\lambda, A)$ is such that $\mathcal{R}(R(\lambda, A)) \subset \mathcal{D}(A)$, we define the "Yosida approximation" of A by

$$A_\lambda x = AR_\lambda x, \quad x \in H$$

Using $\lambda(\lambda I - A)R(\lambda, A) = \lambda I$ it is easy to prove

$$A_\lambda x = \lambda^2 R(\lambda, A) - \lambda I, \quad A_\lambda \in L(H)$$

Denote by S_t^λ the uniformly continuous semigroup

$$S_t^\lambda x = e^{tA_\lambda}x, \quad x \in H$$

Using the commutativity of the resolvent, we get $A_{\lambda_1}A_{\lambda_2} = A_{\lambda_2}A_{\lambda_1}$, and clearly

$$A_\lambda S_t^\lambda = S_t^\lambda A_\lambda$$

Theorem 4 Yosida Approximation *Let A be an infinitesimal generator of a C_0-semigroup $\{S_t\}$ on a Hilbert space H. Then*

(a) $\lim_{\lambda\to\infty} R_\lambda x = x, \quad x \in H$
(b) $\lim_{\lambda\to\infty} A_\lambda x = Ax, \quad for \quad x \in \mathcal{D}(A)$
(c) $\lim_{\lambda\to\infty} S_t^\lambda x = S_t x, \quad x \in H$

The convergence in (c) is uniform on compact subsets of $\mathbb{R}_+$ and

$$\|S_t^\lambda\|_{L(H)} \le Mexp\left(\frac{t \wedge \alpha}{\lambda - \alpha}\right)$$

with constants M, α as in Hille–Yosida Theorem.

We conclude this section by introducing a concept of solution. Let us look at the deterministic problem

$$\frac{du(t)}{dt} = Au(t), \; u(0) = x, \quad x \in H$$

Here H is a real separable Hilbert space and A is an unbounded operator generating a C_0-semigroup.

A classical solution $u : [0, T] \to H$ of the above equation will require a solution to be continuously differentiable and $u(t) \in \mathcal{D}(A)$. However,

$$u^x(t) = S_t x, \quad t \geq 0$$

is considered as a (mild) solution to the equation [16, Chap. 4].

One can consider the non-homogeneous equation

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), \; u(0) = x, \quad x \in H$$

then for $f \in L^1([0, T], H)$, Bochner integrable, one can consider the integral equation

$$u^x(t) = S_t x + \int_0^t S_{t-s} f(s, u(s))ds \tag{4}$$

A solution of (4) is called a "mild solution", if $u \in C([0, T], H)$.

Motivated by the initial work of Sergio Albeverio with Wu and Zhang [2], we continued with Sergio [3] and further in [12] to analyze mild solutions of stochastic partial differential equations (SPDEs) with Poisson noise on any filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, satisfying the usual conditions with values on a separable Hilbert space H. (For this topic see also the monograph by Peszat and Zabczyk [17] and references there.) Remark that the stochastic integral $\int_0^t S_{t-s} f(s, x) q(ds, dx)$, which appears in such SPDEs, is in general not a martingale. However similar to Doob inequalities the following Lemma holds.

Lemma 2 *[Lemma 5.1.9 [12]] Assume $\{S_t\}_{t\geq 0}$ is pseudo-contractive. Let $q(ds, dx)$ be a compensated Poisson random measure on $\mathbb{R}_+ \times E$, for some Hilbert space E, associated to a Poisson random measure N with compensator $dt \otimes \beta(dx)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For each $T \geq 0$ the following statements are valid:*

1. *There exists a constant $C > 0$ such that for each $f \in L^2_{ad}(H)$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t\int_E S_{t-s}f(s,x)q(ds,dx)\right\|^2\right]$$
$$\leq Ce^{2\alpha T}\mathbb{E}\left[\int_0^T\int_E \|f(s,x)\|^2\beta(dx)ds\right]. \tag{5}$$

2. *For all $f\in L^2_{ad}(H)$ and all $\varepsilon>0$ we have*

$$\mathbb{P}\left[\sup_{t\in[0,T]}\left\|\int_0^t\int_E S_{t-s}f(s,x)q(ds,dx)\right\|>\varepsilon\right]$$
$$\leq \frac{4e^{2\alpha T}}{\varepsilon^2}\mathbb{E}\left[\int_0^T\int_E \|f(s,x)\|^2\beta(dx)ds\right]. \tag{6}$$

where $\int_0^t S_{t-s}f(s,x)q(ds,dx)$ is well defined, if the right side is finite. $\int_0^t S_{t-s}f(s,x)q(ds,dx)$ is càdàg.

Let us assume that we are given

$$F: H\to H, \tag{7}$$

$$f: H\times H\to H. \tag{8}$$

Assume

(A) $f(u,z)$ is jointly measurable,
(B) $F(z)$ is measurable,
(C) there exist constants L_f and $L_F>0$, s.th.

$$\|F(z)-F(z')\|^2\leq L_F\|z-z'\|^2$$
$$\int_H \|f(u,z)-f(u,z')\|^2\beta(du)\leq L_f\|z-z'\|^2$$
$$for\quad all\quad z,z'\in H$$

(D)
$$\int_H \|f(u,0)\|^2\beta(du)<\infty \tag{9}$$

(E) A is the infinitesimal generator of a pseudo-contraction semigroup $\{S_t\}_{t\in[0,T]}$.

Remark that Assumptions (C) and (D) imply that there is a constant $K > 0$ such that

$$\int_H \|f(u,z)\|^2 \beta(du) \leq K(1+\|z\|^2) < \infty, \tag{10}$$

since

$$\int_H \|f(u,z)\|^2\beta(du) \leq 2\int_H \|f(u,z)-f(u,0)\|^2\beta(du) + 2\int_H \|f(u,0)\|^2\beta(du)$$
$$\leq 2max\left\{L_f, \int_H \|f(u,0)\|^2\beta(du)\right\}(1+\|z\|^2) < \infty$$

In Albeverio et al. [3, 12], we analyzed (in more generality than in Theorem 5 below) the existence and uniqueness of mild solutions of the stochastic differential equation on intervals $[0,T]$, $T > 0$, like e.g.

$$dX_t = (AX_t + F(X_t))dt + \int_H f(u, X_t)q(dt, du) \tag{11}$$

$$X_0 = \xi, \tag{12}$$

where $q(dt, du) := N(dt, du) - dt\beta(du)$ is a compensated Poisson random measure with compensator $\nu(dt, du) := dt\beta(du)$.

In other words, we looked at the solution of the integral equation

$$X_t = S_t X_0 + \int_0^t S_{t-s}F(X_s)ds + \int_0^t\int_H S_{t-s}f(u, X_s)q(ds, du) \tag{13}$$

where integrals on the r.h.s. are well defined [12].

Definition 2 A stochastic process $X_.$ is called a mild solution of (11), if for all $t \leq T$

(i) X_t is $\mathcal{F}_t$-adapted on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$,
(ii) $\{X_t, t \geq 0\}$ is jointly measurable and $\int_0^T \mathbb{E}\,\|X_t\|_H^2\,dt < \infty$,
(iii) $X_.$ satisfies (13) $\mathbb{P}$-a.s. on $[0, T]$.

Definition 3 A stochastic process $X_.$ is called a strong solution of (11), if for all $t \leq T$

(i) X_t is $\mathcal{F}_t$-adapted on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, \mathbb{P})$,
(ii) $X_.$ is càdlàg with probability one,
(iii) $X_t \in \mathcal{D}(A)$, $dt \otimes d\mathbb{P}$ a.e., $\int_0^T \|AX_t\|_H\,dt < \infty$ $\mathbb{P}$-a.s.,
(iv) $X_.$ satisfies (11) $\mathbb{P}$-a.s. on $[0, T]$.

Obviously, a strong solution $X_\cdot$ of (11) is a a mild solution of (11). The contrary is not necessarily true, since e.g. $X_t \in \mathcal{D}(A)$ might not be true. (See e.g. Sect. 2.2 in Albeverio et al. [4] where sufficient conditions for a mild solution $X_\cdot$ of (11) are listed, for $X_\cdot$ to be also a strong solution.)

Let S_T^2 be the linear space of all càdlàg, adapted processes $X_\cdot$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|X_t\|_F^2\Big] < \infty, \tag{14}$$

where we identify processes whose paths coincide almost surely. Note that, by the completeness of the filtration, adaptedness does not depend on the choice of the representative.

Lemma 3 *[Lemma 4.2.1 [12]] The linear space S_T^2, equipped with the norm*

$$\|X_\cdot\|_{S_T^2} = \mathbb{E}\Big[\sup_{t\in[0,T]} \|X_t\|^2\Big]^{1/2}, \tag{15}$$

is a Banach space.

Theorem 5 *[Theorem 5.3.1 [12]] Suppose assumptions (A)–(E) are satisfied. Then for $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $T > 0$, there exists a unique mild solution $X_\cdot^\xi$ in S_T^2 to (11) with initial condition ξ, and satisfying X_t^ξ is $\mathcal{F}_t$-measurable.*

Remark 2 For each $\xi, \eta, \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, the corresponding unique solutions $X_\cdot^\xi$ and $Y_\cdot^\eta$ to (11) in Theorem 5 satisfy

$$\mathbb{E}\left[\|X_t - Y_t\|_H^2\right] \le C(T)\mathbb{E}\left[\|\xi - \eta\|_H^2\right], \quad t \in [0, T], \tag{16}$$

for some constant $C(T)$ depending on $T > 0$. (See Sect. 5.7 in [12].)

If $X_0 \equiv x \in H$, then the corresponding solution $X_\cdot^x$ to (11) in Theorem 5 is Markov. (See Sect. 5.4 in [12].) Such solution constitutes a Markov process whose transition probabilities $p_t(x, dy) = \mathbb{P}[X_t^x \in dy]$ are measurable with respect to x. By slight abuse of notation we denote by $(p_t)_{t\ge 0}$ its transition semigroup, i.e., for each bounded measurable function $f : H \longrightarrow \mathbb{R}$, $p_t f$ is given by

$$p_t f(x) = \mathbb{E}\left[f(X_t^x)\right] = \int_H f(y) p_t(x, dy), \quad t \ge 0, \quad x \in H. \tag{17}$$

Since due to (16) the solution depend continuously on the initial condition, it can be shown that $p_t f \in C_b(H)$ for each $f \in C_b(H)$, i.e. the transition semigroup is C_b-Feller.

Let $R_n = nR(n, A)$, with $n \in \mathbb{N}$, $n \in \rho(A)$, the resolvent set of A, $R(n, A) = (nI - A)^{-1}$. The SPDE

$$dX_t = (AX_t + R_n F(X_t))dt + \int_H R_n f(u, X_t) q(dt, du) \qquad (18)$$
$$X_0 = R_n \xi(\omega).$$

obtained by Yosida Approximation of (11) has a unique strong solution $X_\cdot^{n,\xi}$ which approximates its mild solution $X_\cdot^\xi$ of (11) with initial condition $X_0^\xi = \xi$. The precise statement is given in the following Theorem:

Theorem 6 *Suppose assumptions (A)–(E) are satisfied. Then for* $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ *and* $T > 0$, *there exists a unique strong solution* $X_\cdot^{n,\xi} := \left\{X_t^{n,\xi},\ t \geq 0\right\}$ *in* S_T^2 *to (18) with initial condition* ξ, *and satisfying* $X_t^{n,\xi}$ *is* $\mathcal{F}_t$*-measurable* $\forall t \geq 0$. *Moreover,*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} \left\| X_t^{n,\xi} - X_t^\xi \right\|_H^2\right] = 0, \qquad (19)$$

where $X_\cdot^\xi := \left\{X_t^\xi,\ t \geq 0\right\}$ *is the mild solution of Eq. (11) with initial condition* ξ.

For the proof see Theorem 2.9 of Albeverio et al. [4].

Definition 4 $X_\cdot^{n,\xi}$ is called "the Yosida approximation of $X_\cdot^\xi$".

Remark 3 Let A be the infinitesimal generator of a pseudo-contraction semigroup $\{S_t\}_{t\in[0,T]}$. Assume that $X_\cdot$ is a strong solution of (11) and all the hypotheses in Theorem 2 are satisfied. Then the Itô-Formula holds and can be written in the following way:

$\mathbb{P}$–almost surely

$$\mathcal{H}(t, X_t) = \mathcal{H}(0, X_0) + \int_0^t \partial_s \mathcal{H}(s, X_s) ds + \int_0^t \mathcal{L}\mathcal{H}(s, X_s) ds$$
$$+ \int_0^t \int_A \big(\mathcal{H}(s, X_{s-} + f(s, u)) - \mathcal{H}(s, X_{s-})\big) q(ds, du)$$

with

$$\mathcal{L}\mathcal{H}(s, x) :=< \partial_x \mathcal{H}(s, x), Ax + F(x) >$$
$$+ \int_H \big(\mathcal{H}(s, x + f(s, u)) - \mathcal{H}(s, x) - < \partial_x \mathcal{H}(s, x), f(s, u) > \big) \beta(du) \qquad (20)$$

Remark 4 Assume that hypotheses (A)–(E) and all hypotheses (a) and (b) in Theorem 2 are satisfied. Then the Itô-Formula for the Yosida approximation $X^{n,\xi}_{\cdot}$ of the mild solution $X^{\xi}_{\cdot}$ of (11) holds and can be written in the following way:

$$\mathcal{H}(t, X_t^{n,\xi}) = \mathcal{H}(0, X_0^{n,\xi}) + \int_0^t \partial_s \mathcal{H}(s, X_s^{n,\xi}) ds$$
$$+ \int_0^t \mathcal{L}_n \mathcal{H}(s, X_s^{n,\xi}) ds + \int_0^t \int_A \big(\mathcal{H}(s, X_{s-}^{n,\xi} + R_n f(s,u)) - \mathcal{H}(s, X_{s-}^{n,\xi})\big) q(ds, du)$$

with

$$\mathcal{L}_n \mathcal{H}(s,x) := < \partial_x \mathcal{H}(s,x), Ax + R_n F(x) >$$
$$+ \int_H \big(\mathcal{H}(s, x + R_n f(s,u)) - \mathcal{H}(s,x) - < \partial_x \mathcal{H}(s,x), R_n f(s,u) > \big) \beta(du)$$

This follows directly from Theorem 6 and Remark 3.

In the next section we will use the following result, which was obtained in [4] as a consequence of an Itô-formula for mild solutions of SPDEs, introduced in Albeverio et al. [4] and written in terms of Yosida approximation

Theorem 7 *[Corollary 3.7 [4]] Assume conditions (A)–(E) and all the hypotheses in Theorem 2 are satisfied. Then*

$$\lim_{n\to\infty} |\mathcal{L}\mathcal{H}(s, X_s^{n,\xi}) - \mathcal{L}_n \mathcal{H}(s, X_s^{n,\xi})| = 0 \quad \mathbb{P} - a.s. \tag{21}$$

4 Some Stability Properties for Solutions of SPDEs on Hilbert Spaces

In this section we discuss how the Itô Formula in Theorem 2 was applied by Albeverio et al. [4] to establish through a Lyapunov function approach stability properties for the mild solution of (11) converging to a unique invariant measure.

Assumption We assume in the whole section that conditions (A)–(E) are satisfied.

The mathematical tools introduced in [4] have been later extended in [8] to analyze the limiting behaviour of mild solutions of SPDEs with multiple invariant measure. This will however not be discussed here, due to a problem of space.

We start to recall some definition related to the Lyapunov function approach presented in [14] as well as [4, 9, 12].

Definition 5 We say that the solution of (11) is exponentially stable in the mean square sense if there exists $c, \varepsilon > 0$ such that for all $t > 0$ and $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$

$$\mathbb{E}[\|X_t^\xi\|^2] \leq ce^{-\varepsilon t}\mathbb{E}[\|\xi\|^2] \tag{22}$$

Definition 6 Let $\mathcal{L}$ be defined as in (20). A function $\mathcal{H} \in C^2(H; \mathbb{R})$ is a Lyapunov function for the SPDE (11) if it satisfies the following conditions:

I. There exist finite constants $c_1, c_2 > 0$ such that for all $x \in H$

$$c_1\|x\|^2 \leq \mathcal{H}(x) \leq c_2\|x\|^2$$

II. There exists a constant $c_3 > 0$ such that

$$\mathcal{L}\mathcal{H}(x) \leq -c_3\mathcal{H}(x) \quad \forall x \in \mathcal{D}(A)$$

In Albeverio et al. [4] we proved the following Theorem

Theorem 8 *[4] Assume that there exists a function $\mathcal{H} \in C^2(H; \mathbb{R})$ which is a Lyapunov function for the SPDE (11) and the hypotheses (a) and (b) in Theorem 2 are satisfied. Then the mild solution of (11) is exponentially stable in the mean square sense. Moreover the constants in (22) can be chosen so that $c = \frac{c_2}{c_1}$ and $\varepsilon = c_3$.*

Remark that for the case $\mathcal{H} \in C_b^2(H; \mathbb{R})$ a proof can be found in [14, Theorem 4.2] (see also [19, Sect. 7.1] and for the Gaussian case [9, Theorem 6.4]). The results are stated there for the Yosida approximants.

Proof Since all the hypotheses of Theorem 2 are satisfied, Itô formula can be applied to the Yosida approximation.

$$e^{c_3t}\mathbb{E}[\mathcal{H}(X_t^{n,\xi}) - \mathcal{H}(R_n\xi)] = \mathbb{E}\left[\int_0^t e^{c_3s}c_3(\mathcal{H}(X_s^{n,\xi}) + \mathcal{L}_n\mathcal{H}(X_s^{n,\xi}))ds\right] \tag{23}$$

From Condition II it follows

$$c_3\mathcal{H}(X_s^{n,\xi}) + \mathcal{L}_n\mathcal{H}(X_s^{n,\xi}) \leq -\mathcal{L}\mathcal{H}(X_s^{n,\xi}) + \mathcal{L}_n\mathcal{H}(X_s^{n,\xi}) \tag{24}$$

$$e^{c_3t}\mathbb{E}[\mathcal{H}(X_t^{n,\xi}) - \mathcal{H}(R_n\xi)] \leq \mathbb{E}\left[\int_0^t e^{c_3s}(-\mathcal{L}\mathcal{H}(X_s^{n,\xi}) + \mathcal{L}_n\mathcal{H}(X_s^{n,\xi}))ds\right] \tag{25}$$

From Theorems 6 and 7 it follows $e^{c_3 t}\mathbb{E}[\mathcal{H}(X_t^\xi)] \le \mathbb{E}[\mathcal{H}(\xi)]$. Condition I implies then

$$c_1\mathbb{E}[\|X_t^\xi\|^2] \le \mathbb{E}[\mathcal{H}(X_t^\xi)] \le e^{-c_3 t}\mathbb{E}[\mathcal{H}(\xi)] \le c_2 e^{-c_3 t}\mathbb{E}[\|\xi\|^2] \tag{26}$$

and hence

$$\mathbb{E}[\|X_t^\xi\|^2] \le \frac{c_2}{c_1} e^{-c_3 t}\mathbb{E}[\|\xi\|^2] \tag{27}$$

The statement follows by choosing $c = \frac{c_2}{c_1}$ and $\varepsilon = c_3$.

Using Theorem 8 we can provide an easy proof of the following statement, known in the literature from e.g. [6, Sect. 16], [17, Chap. 11, Sect. 5].

Theorem 9 *Assume that the conditions (A)–(E) are satisfied for (11), and the following conditions hold*

(i) A satisfies the "dissipativity condition", i.e. there exists $\alpha > 0$ *such that*

$$\begin{aligned} &< Ax - Ay, x - y > + < F(x) - F(y), x - y > \\ &\le -\alpha\|x - y\|^2 \quad \forall x, y \in \mathcal{D}(A); \end{aligned} \tag{28}$$

(ii) $\varepsilon := 2\alpha - L_f > 0.$
(iii) $\forall z \in H \int_A \|f(u, z)\|^4 \beta(du) < \infty$

Then for all $\xi,\ \eta\ \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$

$$\mathbb{E}[\|X_t^\xi - X_t^\eta\|^2] \le e^{-\varepsilon t}\mathbb{E}[\|\xi - \eta\|^2] \quad \forall t > 0 \tag{29}$$

Proof The stochastic process $X_\cdot^\xi - X_\cdot^\eta$ is the mild solution of

$$\begin{aligned} d(X_t^\xi - X_t^\eta) &= A(X_t^\xi - X_t^\eta)dt + (F(X_t^\xi) - F(X_t^\eta))dt \\ &\quad + \int_H \Big(f(u, X_t^\xi) - f(u, X_t^\eta)\Big)\, q(dt, du) \end{aligned} \tag{30}$$

$$X_0^\xi - X_0^\eta = \xi - \eta. \tag{31}$$

Condition (iii) implies that all hypotheses of Theorem 2 are satisfied for $\mathcal{H}(x, y) := \|x - y\|^2$. Moreover, according to the definition of $\mathcal{L}$ in (20), we have

$$\begin{aligned} \mathcal{L}\|x - y\|^2 &:= 2 < x - y, A(x - y) > +2 < x - y, F(x) - F(y) > \\ &\quad + \int_H \|f(u, x) - f(u, y)\|^2 \beta(du) \end{aligned}$$

$$\|x - y + f(u, x) - f(u, y)\|^2 - \|x - y\|^2$$
$$- 2 < x - y, f(u, x) - f(u, y) >= \|f(u, x) - f(u, y)\|^2$$

Conditions (i) and (ii) imply that the function $\mathcal{H}(x, y) := \|x - y\|^2$ is a Lyapunov function for (30) with $c_1 = c_2 = 1$ and $c_3 = \varepsilon$. Hence $X^{\xi}_{\cdot} - X^{\eta}_{\cdot}$ is exponentially stable in the mean square sense.

We denote by p^*_t the adjoint operator to p_t defined in (17), i.e.

$$p^*_t\rho(dx) = \int_H p_t(y, dx)\rho(dy), \quad t \geq 0.$$

Recall that a probability measure π on $(H, \mathcal{B}(H))$ is called *invariant measure* for the semigroup $(p_t)_{t\geq 0}$ if and only if $p^*_t\pi = \pi$ holds for each $t \geq 0$. Let $\mathcal{P}_2(H)$ be the space of Borel probability measures ρ on $(H, \mathcal{B}(H))$ with finite second moments. Recall that $\mathcal{P}_2(H)$ is separable and complete when equipped with the *Wasserstein-2-distance*

$$\mathrm{W}_2(\rho, \widetilde{\rho}) = \inf_{G \in \mathcal{H}(\rho, \widetilde{\rho})} \left(\int_{H \times H} \|x - y\|^2_H G(dx, dy) \right)^{\frac{1}{2}}, \quad \rho, \widetilde{\rho} \in \mathcal{P}_2(H). \tag{32}$$

Here $\mathcal{H}(\rho, \widetilde{\rho})$ denotes the set of all couplings of $(\rho, \widetilde{\rho})$, i.e. Borel probability measures on $H \times H$ whose marginals are given by ρ and $\widetilde{\rho}$, respectively, see [22, Sect. 6] for a general introduction to couplings and Wasserstein distances.

As a consequence of our key stability estimate (29) we can provide, by following the proof of Theorem 4.1 in [8], a proof for the existence and uniqueness of a unique limiting distribution in the spirit of classical results such as [17, Sect. 16], [6, Chap. 11, Sect. 5], [20].

Theorem 10 *Assume that the conditions (A)–(E) are satisfied for (11), and the conditions (i)–(iii) in Theorem 9 hold. Then*

$$\mathrm{W}_2(p^*_t\rho, p^*_t\widetilde{\rho}) \leq \mathrm{W}_2(\rho, \widetilde{\rho})\mathrm{e}^{-\varepsilon t/2}, \quad t \geq 0, \tag{33}$$

holds for any $\rho, \widetilde{\rho} \in \mathcal{P}_2(H)$. In particular, the Markov process determined by (11) has a unique invariant measure π. This measure has finite second moments and it holds that

$$\mathrm{W}_2(p^*_t\rho, \pi) \leq \mathrm{W}_2(\rho, \pi)\mathrm{e}^{-\varepsilon t/2}, \quad t \geq 0, \tag{34}$$

for each $\rho \in \mathcal{P}_2(H)$.

Proof From Theorem 9 it follows

$$\mathbb{E}[\|X^x_t - X^y_t\|^2_H] \leq \mathrm{e}^{-\varepsilon t}\|x - y\|^2_H, \quad x, y \in H. \tag{35}$$

Using the definition of the Wasserstein distance, we conclude that

$$\mathrm{W}_2(p_t^*\delta_x, p_t^*\delta_y) \leq \left(\mathbb{E}[\|X_t^x - X_t^y\|_H^2]\right)^{1/2} \leq \|x-y\|_H \mathrm{e}^{-\varepsilon t/2}.$$

The latter one readily yields (33). Finally, the existence and uniqueness of an invariant measure as well as (34) can be derived from (33) combined with a standard Cauchy argument.

In [8] we introduced a "generalized dissipativity condition" and studied SPDEs with multiple invariant measures. There we developed further the methods presented in this section, which have been mainly derived from Albeverio et al. [4] in combination with the results obtained in [3, 12].

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Comment by Barbara Rüdiger My co-author and friend V. Mandrekar (Atma) passed away the 23 June 2021. A couple of days before his departure he contacted me through email to make sure the procedure for the submission of this article would be successful. The invitation to contribute to this Volume, dedicated to Sergio Albeverio, was accepted by him with enthusiasm.
Atma and Sergio had, to my feeling, a deep respect for each other and, despite the geographic distance, a solid friendship. I think that this friendship and respect is also due to common aspects they have in their character and soul: both are very generous in sharing with other scientists their original ideas. Both trust in youngsters and enjoy knowing that they can contribute to these with their own developments and ideas, as well. This way they both are friends, supporters, coaches and co-authors to many young (and in the meanwhile older) mathematicians and physists. I feel very lucky to be among them.

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Random Processes on Non-Archimedean Spaces

Witold Karwowski

Abstract The Lévy stochastic processes on p-adic numbers have been constructed by different methods. We present the construction by Albeverio and Karwowski using the Chapman–Kolmogorov equations. This method does not rely on the algebraic structure of p-adics and so it is applicable beyond the class of Lévy processes and allows to enlarge the family of the state spaces to general tree structures. We indicate the influence this results had on the subsequent research by ourselves and we point to interactions with the work of other authors.

Keywords Hierarchical spaces · Non-Archimedean · Stochastic processes · Chapman–Kolmogorov equations

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1 Introduction

I had the honour and pleasure to work with Albeverio on a number of projects. In this note I will present our work on random processes on hierarchical spaces and indicate further developments of the subject by other authors. Our adventure with hierarchical spaces began in 1989 when Sergio suggested to study random processes on p-adic numbers.

Sergio and I were not the first to work on the random processes on p-adics. There had been studies of physical phenomena related to random processes on hierarchical spaces. See for example [37] and references therein. Also in mathematics the stochastic objects related to hierarchical spaces had been studied. We refer to [18–22] as examples. In 1989 there appeared papers by Evans [18] and Brekke and Olson [17]. Evans studied local properties of Lévy processes on totally disconnected groups. He considered the p-adic unit ball as an example of such groups. Brekke

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and Olson constructed the class of symmetric processes on the field $\mathbb{Q}_p$ of p-adic numbers and discussed an application of the processes in physics. They obtained a non-exponential formula for relaxation in spin glasses. In both papers the constructions of the processes relied on algebraic properties of $\mathbb{Q}_p$. Our approach was based on the Chapman–Kolmogorov equations. The algebraic properties of $\mathbb{Q}_p$ were no longer necessary but the group structure was used to simplify the discussion. In [1, 2] we presented solutions of the Chapman–Kolmogorov equations thus obtaining translation invariant transition functions for the spherically symmetric processes on $\mathbb{Q}_p$. We also gave an explicit formula of the Dirichlet form and generator of the process together with its complete spectral description. The fact that the approach using Chapman–Kolmogorov equations did not require algebraic properties of $\mathbb{Q}_p$ suggested a possibility to modify the procedure used in [1, 2] to investigate processes without translation invariance of the transition functions and the processes on hierarchical spaces other than $\mathbb{Q}_p$.

This possibility has indeed been exploited. Additionally the results of [2] became a basis for further studies of stochastic processes on hierarchical spaces. For this reason we shall present main ideas of [2] with some details. The paper is organized as follows: In Sect. 2 we introduce the concepts of the field of p-adic numbers $\mathbb{Q}_p$ and a more general class of state spaces the rings $\mathbb{Q}_q$. Then we describe the main ideas of [2] reformulated to include the Lévy processes on $\mathbb{Q}_q$. In Sect. 3 based on [5, 13, 31] we demonstrate a modification of the technique developed in [2] to construct the processes with weighted target states. An example of further studies of random processes on p-adics using the processes introduced in [2] is presented in Sect. 4. In Sect. 5 we briefly indicate several problems addressed by different authors and their relations with the results of [2].

2 Spherically Symmetric Random Processes on $\mathbb{Q}_q$

We begin with basic information on the state spaces $\mathbb{Q}_p$ and $\mathbb{Q}_q$. Let $\mathbb{Q}$ be the field of rational numbers. For a given prime $p > 1$, any $a \in \mathbb{Q}$ can be expressed uniquely by

$$a = p^M \frac{q}{r}, \tag{1}$$

for some integer M and the integers q, r, where q and r have no common factor (except for 1) and p divides neither q nor r. For $0 \neq a \in \mathbb{Q}$ expressed by (1) we set $\|a\|_p = p^{-M}$ and $\|0\|_p = 0$. One can show that the mapping $\|\cdot\|_p$ from $\mathbb{Q}$ to $\mathbb{R}_+$, is a norm: i.e. it satisfies the following three conditions:

(a) $\|a\|_p = 0$ if and only if $a = 0$;
(b) $\|ab\|_p = \|a\|_p \|b\|_p$, $a,\ b \in \mathbb{Q}$;
(c) $\|a + b\|_p \leq \|a\|_p + \|b\|_p$, $a,\ b \in \mathbb{Q}$.

Moreover, $\|\cdot\|_p$ has a property stronger than (c):

$$\|a+b\|_p \le \max\{\|a\|_p, \|b\|_p\} \tag{2}$$

and

$$\|a+b\|_p = \max\{\|a\|_p, \|b\|_p\} \tag{3}$$

when $\|a\|_p \neq \|b\|_p$ (see [34]). We call $\|\cdot\|_p$ the p-adic norm. Condition (2) is called the non-Archimedean inequality or ultrametric property and a norm satisfying (2) is called non-Archimedean. According to Ostrowski's theorem (see e.g., [34, Theorem 1, p. 3]) the only nontrivial norms on $\mathbb{Q}$, (up to norm equivalence) are the absolute value norm $|\cdot|$ and the p-adic norms $\|\cdot\|_p$ for every prime $p > 1$. The completion of $\mathbb{Q}$ in the absolute value norm is the set $\mathbb{R}$ of real numbers. Completion of $\mathbb{Q}$ in a p-adic norm $\|\cdot\|_p$ is denoted by $\mathbb{Q}_p$ and called the p-adic numbers. Since the topology of $\mathbb{Q}_p$ is given by a non-Archimedean norm we call it a non-Archimedean space. The addition and multiplication of $\mathbb{Q}$ extends to $\mathbb{Q}_p$ so that it becomes a field (see [34, Chap. 1]). It is known that every $a \in \mathbb{Q}_p$, $a \neq 0$ has the unique representation:

$$a = \sum_{i=0}^{\infty} a_{m+i} p^{m+i}, \tag{4}$$

where m is an integer and $1 \le a_m \le p-1,\ 0 \le a_i \le p-1,\ i = m+1, m+2, \ldots$. The p-adic norm is then given by the map $a \to \|a\|_p = p^{-m}$. Expression (4) is well defined because the series on the right hand side is convergent under the norm $\|\cdot\|_p$. The representation (4) provides natural rules for adding, subtracting, multiplying and dividing p-adic numbers. Algebraically $\mathbb{Q}_p$ is a field or more precisely a local field. Topologically $\mathbb{Q}_p$ is a complete, separable, totally disconnected, locally compact normed space with cardinality of continuum (see [34]). Set $a_i = 0$ for $i < m$. Then $\sum_{i=N}^{\infty} a_i p^i = \sum_{i=0}^{\infty} a_{m+i} p^{m+i} = a$ for every $N \le m$. Let $a \in \mathbb{Q}_p$ and $M \in \mathbb{Z}$. The set $K(a, p^M) = \{x \in \mathbb{Q}_p;\ \|a-x\|_p \le p^M\}$ is called a p-adic ball of radius p^M centered at a. We mention a characteristic property of non-Archimedean metrics: if $b \in K(a, p^M)$, then $K(b, p^M) = K(a, p^M)$.

Let $q > 1$ be a (not necessary prime) integer. Similarly like in the case of $\mathbb{Q}_p$ the elements of $\mathbb{Q}_q$ are represented by

$$a = \sum_{i=0}^{\infty} a_{m+i} q^{m+i} = \sum_{i=N}^{\infty} a_i p^i = \sum_{i=0}^{\infty} a_{m+i} p^{m+i} \tag{5}$$

where $N \le m,\ a_i = 0$ for $i < m,\ 1 \le a_m \le q-1,\ \ 0 \le a_i \le q-1$ for $i = m+1, m+2, \ldots$.

With the addition and multiplication rules suggested by (5) the map $a \to \|a\|_q = q^{-m}$, satisfies:

(i) $\|a\|_q = 0$ if and only if $a = 0$;
(ii) $\|ab\|_q \le \|a\|_q \|b\|_q,\ a, b \in \mathbb{Q}_q$;
(iii) $\|a + b\|_q \le \max\{\|a\|_q, \|b\|_q\}$.

Besides, if $\|a\|_q \ne \|b\|_q$ then $\|a + b\|_q = \max\{\|a\|_q, \|b\|_q\}$.

The formula

$$\rho_q(a, b) \equiv \|a - b\|_q \tag{6}$$

defines a metric on $\mathbb{Q}_q$. Algebraically $\mathbb{Q}_q$ is an additive Abelian group. If q is not a prime number then it is a ring rather than a field. Topologically it is a complete, separable, totally disconnected, locally compact metric space with cardinality of continuum.

Let $M \in \mathbb{Z},\ a \in \mathbb{Q}_q$. The set $K(a, q^M) := \{b \in \mathbb{Q}_q, \rho_q(a, b) \le q^M\}$ will be called a ball of radius q^M centered at a. Note following properties of the balls.

(a_1) Any ball is open and compact.
(a_2) If $K(a, q^M) \cap K(b, q^M) \ne \emptyset$ then $K(a, q^M) = K(b, q^M)$.
(a_3) For any $M \in \mathbb{Z}$ there is a family $\mathcal{K}^M = \{K_i^M\}_{i\in \mathbb{N}}$ of disjoint balls $K_i^M = K(a^i, q^M)$ such that $\mathbb{Q}_q = \cup_{i=1}^{\infty} K_i^M$.

If $c \notin K(a, q^N)$ resp. $K(a, q^N) \cap K(b, q^M) = \emptyset$ then we shall write $dist_q(c, K(a, q^N)) = \rho_q(c, a)$ resp. $dist_q(K(a, q^N), K(b, q^M)) = \rho(a, b)$. We finish the characterisation of $\mathbb{Q}_q$ introducing a Borel measure on it. Let ν be a σ-additive set function defined by

$$\nu(K(a, q^M) = q^M. \tag{7}$$

ν extends uniquely to the Haar measure under the group of q-adic translations.

It turns out that the family of spherically symmetric Lévy processes on $\mathbb{Q}_p$ is identical with the class of random walks introduced in [1, 2]. Here we are going to present main steps of the construction but since it automatically extends to the ring $\mathbb{Q}_q$ of q-adic numbers ($1 < q \in \mathbb{N}$), we take $\mathbb{Q}_q$ for the state space. It will be convenient to use shorthand AK for the processes on $\mathbb{Q}_q$ constructed by the method of [1, 2]. Our aim is to construct a Markov process with $\mathbb{Q}_q$ as the state space but as the first step we solve the system of forward and backward Chapman–Kolmogorov equations to obtain a Markov chain on $\mathcal{K}^M$.

$$\dot{P}_{K_i^M K_k^M}(t) = -\tilde{a}(K_k^M) P_{K_i^M K_k^M}(t) + \sum_{j\ne k}^{\infty} \tilde{u}(K_j^M, K_k^M) P_{K_i^M K_j^M}(t), \tag{8}$$

$$\dot{P}_{K_i^M K_k^M}(t) = -\tilde{a}(K_i^M) P_{K_i^M K_k^M}(t) + \sum_{j\ne i}^{\infty} \tilde{u}(K_i^M, K_j^M) P_{K_j^M K_k^M}(t) \tag{9}$$

with

$$\tilde{a}(K_i^M) = \sum_{j \neq i}^{\infty} \tilde{u}(K_j^M, K_i^M) \tag{10}$$

for $t \geq 0$ and $i,\ j,\ k \in \boldsymbol{N}$, with the initial conditions

$$P_{K_i^M K_k^M}(0) = \delta_{ik}. \tag{11}$$

$\tilde{a}(K_j^M)$ is interpreted as intensity of the state K_j^M and $\tilde{u}(K_i^M, K_k^M)$ as the infinitesimal transition probability. In all models discussed in this work the coefficients $\tilde{a}(K_i^M)$ and $\tilde{u}(K_i^M, K_j^M)$ are chosen to be non-negative constants. It is known that under assumed properties of the coefficients there is a unique $P_{K_i^M, K_j^M}(t)$ solving (8), (9) and satisfying

$$P_{K_i^M, K_j^M}(t + \tau) = \sum_{k=1}^{\infty} P_{K_i^M, K_k^M}(t) P_{K_k^M, K_j^M}(\tau). \tag{12}$$

Thus $P_{K_i^M, K_j^M}(t)$ can be interpreted as the transition probability of a continuous time Markov chain on a state space indexed by natural numbers.

In this chapter the coefficients in Eqs. (8), (9) are specified as follows. Let $\{a(M), M \in \boldsymbol{Z}\}$ be a given sequence of non-negative numbers satisfying

$$(A1) \qquad \begin{cases} \text{(i)}\ \ a(M) \geq a(M+1); \\ \text{(ii)}\ \lim_{M \to \infty} a(M) = 0. \end{cases}$$

In this paper we shall use the shorthand "parameter sequence" for a sequence $\{a(M), M \in \boldsymbol{Z}\}$ satisfying (A1). Put

$$u(M, m) \equiv (q-1)^{-1} q^{-m+1} (a(M+m-1) - a(M+m)). \tag{13}$$

If $i \neq j$ then $dist_q(K_i^M, K_j^M) = q^{M+j_0}$ for some $j_0 \in \boldsymbol{N}$. Then we set

$$\tilde{u}(K_i^M, K_k^M) \equiv u(M, j_0). \tag{14}$$

Accordingly $\tilde{u}(K_i^M, K_k^M)$ depends only on the q-adic distance of the balls. As a consequence of (10) we obtain by direct computation.

$$\tilde{a}(K_i^M) = a(M). \tag{15}$$

Note the following simple facts:

(b_1) If $K_j \in \mathcal{K}^M$, $b \in \boldsymbol{Q}_q$, then there is $k \in \boldsymbol{N}$ so that $K_j + b = K_k$.
(b_2) $\mathcal{K}^M$ is invariant under translations in $\boldsymbol{Q}_q$.
(b_3) Given $j, k \in \boldsymbol{N}$, there is $b \in \boldsymbol{Q}_q$ such that $K_j + b = K_k$.
(b_4) For any pair $i, j \in \boldsymbol{N}$ and any $b \in \boldsymbol{Q}_q$ we have $dist_q(K_i, K_j) = dist_q(K_i + b, K_j + b)$.

This observations together with the fact that $\tilde{a}(K_j) = a(M)$ is independent of j and $\tilde{u}(K_i, K_j)$ depends only on $dist_q(K_i, K_j)$ imply that Eqs. (8) and (9) are invariant under q-adic translations in the sense that if the spheres $K_j,\ j \in \boldsymbol{N}$ are substituted by $K_j + b,\ b \in \boldsymbol{Q}_q$ then the resulting system coincide with (8) resp. (9).

By direct verification we obtain

Proposition 1 *Let $i \in \boldsymbol{N}$ be fixed and $P_{K_i,K_j}(t),\ j \in \boldsymbol{N},\ t \geq 0$ satisfy (8) resp. (9) with the initial condition $P_{K_i,K_j}(0) = \delta_{ij}$. Put $\hat{P}_{K_k,K_l}(t) = P_{K_i,K_j}(t)$ where k, l are such that $K_k = K_i + b,\ K_l = K_j + b$ then $\hat{P}_{K_k,K_l}(t)$ satisfies (8) resp. (9) and the initial conditions $\hat{P}_{K_k,K_l}(0) = \delta_{k,l}$.*

By the proposition and (b_3) we conclude that if we find a solution of (8), (9) for one value of i then we can construct complete solution of (8), (9). Moreover if $P_{K_i^M,K_j^M}$ solves (8), then it also solves (9). Let the spheres K_i be numbered so that $K_1 = K(0, q^M)$. Then it is sufficient to solve (8) with $i = 1$. We put $i = 1$ and drop the index K_1 writing P_{K_j} for P_{K_1,K_j}. For $m \in \boldsymbol{N}$ we write

$$P_{K_j^{M+m}} = \sum_i P_{K_i} \tag{16}$$

where K_j^{M+m} is a ball of radius q^{M+m} such that $K_j \subset K_j^{M+m}$ and the summation runs over $K_i \subset K_j^{M+m}$. If $j = 1$ we have $K_1^{M+m} = K(0, q^{M+m})$. Let $\mathrm{dist}_q(K_j, K_1) = q^{M+m}$. After a few steps of straightforward calculations we obtain

$$\begin{aligned} \dot{P}_{K_j} = &-(a(M) + u(M, 1))P_{K_j} + \sum_{k=1}^{m-1} (u(M, k) - u(M, k+1))\, P_{K_j^{M+k}} \\ &+ \sum_{l=0}^{\infty} (u(M, m+l) - u(M, m+l+1))\, P_{K(0,q^{M+m+l})}. \end{aligned} \tag{17}$$

We also have

$$\dot{P}_{K(0,q^M)} = -\,(a(M) + u(M, 1))\, P_{K(0,q^M)} + \sum_{i=1}^{\infty} (u(M, i) - u(M, i+1))\, P_{K(0,q^{M+i})}. \tag{18}$$

If $m = 1$ then summing Eq. (17) over k such that $\mathrm{dist}_q(K_k, K(0, q^M)) = q^{M+1}$ and adding (18) we obtain

$$\begin{aligned} \dot{P}_{K(0,q^{M+1})} = &-\,(a(M+1) + u(M+1, 1))\, P_{K(0,q^{M+1})} \\ &+ \sum_{i=1}^{\infty}(u(M+1, +i) - u(M+1, i+1))P_{K(0,q^{M+i+1})}. \end{aligned} \tag{19}$$

Remark 1 Note that if we substitute M by $M+1$, $\mathcal{K}^M \longrightarrow \mathcal{K}^{M+1}$ then (18) is substituted by the equation identical to (19). ■

Iterating this procedure we get

$$\dot{P}_{K(0,q^{M+m})} = -\left(a(M+m) + q^m u(M, m+1)\right) P_{K(0,q^{M+m})} + q^m \sum_{i=1}^{\infty}(u(M, m+i) - u(M, m+i+1))P_{K(0,q^{M+m+i})}. \tag{20}$$

Let $l \in \boldsymbol{N}_0$. By direct computations we obtain the equation

$$q\dot{P}_{K(0,q^{M+m+l})} - \dot{P}_{K(0,q^{M+m+l+1})} = (q-1)^{-1}[qa(M+m+l) - a(M+m+l+1)] \tag{21}$$

$$(qP_{K(0,q^{M+m+l})} - P_{K(0,q^{M+m+l+1})}). \tag{22}$$

Its solution with the initial condition $P_{K_1,K_j} = \delta_{1j}$ divided by q^{l+1} reads

$$qP_{K(0,q^{M+m+l})} - P_{K(0,q^{M+m+l+1})} = (q-1)\exp\left\{-(q-1)^{-1}[qa(M+m+l) - a(M+m+l+1)]t\right\}. \tag{23}$$

Summing (23) over $l = 0, 1, \ldots, k-1$ we obtain

$$P_{K(0,q^{M+m})} - q^{-k}P_{K(0,q^{M+m+k})} = \frac{q-1}{q}\sum_{l=0}^{k-1} q^{-l}\exp\left\{-(q-1)^{-1}[qa(M+m+l) - a(M+m+l+1)]t\right\} \tag{24}$$

As $k \to \infty$ we get in the limit

$$P_{K(0,q^{M+m})}(t) = \frac{q-1}{q}\sum_{l=0}^{\infty} q^{-l}\exp - \left[qa(M+m+l) - a(M+m+l+1)\right]t. \tag{25}$$

To compute $P_{K_1^M K_j^M}(t)$ note that the number of balls K_j^M with $\text{dist}_q(K_1^M, K_j^M) = q^{M+m}, m \in \boldsymbol{N}$ is $(q-1)q^{m-1}$ and the functions $P_{K_1^M K_j^M}(t)$ depend only on the distance $\text{dist}_q(K_1^M, K_j^M)$. Hence

$$
\begin{aligned}
P_{K_j^M}(t) &= (q-1)^{-1}q^{1-m}(P_{K(0,q^{M+m})} - P_{K(0,q^{M+m-1})}) \\
&= q^{-m}[q^{-1}(q-1)\sum_{i=0}^{\infty} q^{-i}\exp[-(q-1)^{-1} \\
&\quad [qa(M+m+i) - a(M+m+i+1)]t] \\
&\quad - \exp[-(q-1)^{-1}[qa(M+m-1) - a(M+m)]t]]. \qquad (26)
\end{aligned}
$$

Formula (25) for $m = 0$ together with (26) gives transition probability from the ball $K(0, q^M)$ to any ball $K^M \in \mathcal{K}^M$. In our notation $K(0, q^M) = K_1 \in \mathcal{K}^M$. Let $K_1^{M-n} = K(0, q^{M-n}) \in \mathcal{K}^{M-n}$, $n \in \boldsymbol{N}$. Then by Remark 1 $P_{K(0,q^M),K(0,q^{M+m})} = P_{K(0,q^{M-n}),K(0,q^{M+m})}$. Since $\{0\} = \bigcap_{n\in \boldsymbol{N}} K(0, q^{M-n})$ we can define

$$
P_{0,K(0,q^{M+m})} = \lim_{n\to\infty} P_{K(0,q^{M-n}),K(0,q^{M+m})}. \qquad (27)
$$

Combining (25), (26), (27) and Proposition 1 we have

$$
P_{x,K(x,q^N)}(t) = \frac{q-1}{q}\sum_{l=0}^{\infty} q^{-l}\exp -(q-1)^{-1}[qa(N+l) - a(N+l+1)]t, \qquad (28)
$$

and

$$
\begin{aligned}
P_{x,K^M}(t) &= q^{-m}[q^{-1}(q-1) \\
&\quad \sum_{i=0}^{\infty} q^{-i}\exp[-(q-1)^{-1}[qa(M+m+i) - a(M+m+i+1)]t] \\
&\quad - \exp[-(q-1)^{-1}[qa(M+m-1) - a(M+m)]t]], \qquad (29)
\end{aligned}
$$

where $x \in \boldsymbol{Q}_q$, K^M is a ball of radius q^M and $\text{dist}_q(x, K^M) = q^{M+m}$.
It can be shown by direct verifications that $P_{x,K^M}(t)$ given by (28) and (29) defines a transition probability in $\boldsymbol{Q}_q$. Hence we have

Theorem 1 *Given a parameter sequence $a(M)$, $M \in \boldsymbol{Z}$. Then there is a continuous time random process X_t, $t \geq 0$ with state space $\boldsymbol{Q}_q$ given by the transition probabilities $P_{x,A}(t)$, $t \geq 0$, $x \in \boldsymbol{Q}_q$, A a Borel subset of $\boldsymbol{Q}_q$. The transition probabilities $P_{x,A}(t)$ are determined by (28), (29).* ■

Remark 2 When passing from (25), (26) to (28), (29) we roughly speaking kept the target ball fixed and shrunk the initial ball to a point. The same trick is used in Sect. 3 for nonsymmetric processes and in [3] for the processes on trees with varying numbers of ledges. This procedure has been analysed in general setting in [38].

The transition probabilities $P_{x,A}(t)$ define a Markovian semigroup in $L^2(\boldsymbol{Q}_q, \nu)$

$$T_t f(x) = \int_{\mathbb{Q}_q} P_{x,dy}(t) f(y), \; f \in L^2(\mathbb{Q}_q, \nu). \tag{30}$$

The semigroup $(T_t, t \geq 0)$ has the representation

$$T_t = e^{-Ht} \tag{31}$$

where H is a nonnegative self-adjoint operator. H is defined by

$$(Hf)(x) = \lim_{t\downarrow 0} t^{-1} \left[f(x) - (T_t f)(x)\right] = \lim_{\downarrow 0} t^{-1} \left[f(x) - \int f(y) P_{x,dy}(t)\right] \tag{32}$$

whenever the strong limit exists. Put χ_A for the characteristic function of $A \subset \mathbb{Q}_q$. If $f = \chi_{K(a,q^M)}$ then (32) yields

$$-H\chi_{K(a,q^M)}(x) = \begin{cases} a(M), & \text{for } x \in K(a, q^M), \\ -q^{-m+1}(q-1)^{-1}\left[a(M+m-1) - a(M+m)\right], \\ \qquad \text{if } x : \mathrm{dist}_q(x, K(a, q^M)) = q^{M+m}. \end{cases} \tag{33}$$

Let D_0 stands for the linear hull spanned by the characteristic functions of all q-adic balls. D_0 is dense in $L^2(\mathbb{Q}_q, dx)$ and by linearity it is a subset of $D(H)$ the domain of H. It can be shown that D_0 is a core of H. Hence (33) defines H uniquely.

Let $K_j^M \in \mathcal{K}^M$ and K_{ij}^{M-1}, $i = 0, 1, \ldots, q-1$ the balls of radius q^{M-1} such that $\bigcup_{i=0}^{i=q-1} K_{ij}^{M-1} = K_j^M$. The spectral properties of H are described in

Theorem 2 *Let $-H$ denote the generator of strongly continuous semigroup $(T_t, \; t \geq 0)$ with the kernel defined by (28), (29). Then*

(a) For any $M \in \mathbb{Z}$ and $j \in \mathbb{N}$ there corresponds an eigenvalue of H given by

$$h_M = -(q-1)^{-1}(qa(M) - a(M+1)) \tag{34}$$

and a $(q-1)$-dimensional eigenspace spanned by vectors of the form

$$e_{M,j} = \sum_{i=0}^{q-1} b_i \chi_{K_{ij}^{M-1}}, \tag{35}$$

where $\sum_{i=0}^{i=q-1} b_i = 0$.

(b) Let $e_{M,j}^k$, $k = 1, \ldots, q-1$ be the orthonormal basis of the space spanned by the vectors (35). Then the system $\{e_{M,j}^k, M \in \mathbb{Z}, j \in \mathbb{N}, k = 1, \ldots q-1\}$ is an orthonormal basis in $L^2(\mathbb{Q}_q, dx)$.

(c) $\chi_{K_j^M}$ is an eigenvector of H iff $a(M) = 0$. Then the corresponding eigenvalue equals zero. ■

We also have

Corollary 1 *The operator H has pure point spectrum given by the numbers*

$$h_M = -(q-1)^{-1}(qa(M) - a(M+1)).$$

If all states of the process are stable, i.e. $\lim_{M\to-\infty} a(M) < \infty$ *then H is bounded. If all states are instantaneous i.e.* $\lim_{M\to-\infty} a(M) = \infty$ *then H is unbounded.* ■

Our next aim is to describe the Dirichlet form for the process corresponding to the semigroup $(T_t, t \geq 0)$.

Let $-H$ be the generator of $(T_t, t \geq 0)$. Put $D(H^{\frac{1}{2}})$ for the domain of $H^{\frac{1}{2}}$ and define

$$\mathcal{E}(f, g) = (H^{\frac{1}{2}} f, H^{\frac{1}{2}} g) \text{ for } f, g \in D(H^{\frac{1}{2}}). \tag{36}$$

Then by [23, 24] the quadratic form $\mathcal{E}$ with the domain $D[\mathcal{E}] = D(H^{\frac{1}{2}})$ is a closed symmetric Markovian form, i.e. a Dirichlet form. D_0 is a core for H and also for $H^{\frac{1}{2}}$. Hence $\mathcal{E}$ restricted to $D_0 \times D_0$ is closable and its closure is $(\mathcal{E}, D[\mathcal{E}])$. In other words the last sentence can be formulated by

(1_r) D_0 is dense in $D[\mathcal{E}]$ in the norm $(\mathcal{E}_1(.,.))^{\frac{1}{2}} = [\mathcal{E}(.,.) + (.,.)]^{\frac{1}{2}}$.

Put $C_0(\mathcal{Q}_q)$ for the space of continuous functions of bounded support. By the Weierstrass–Stone theorem.

(2_r) D_0 is dense in $C_0(\mathcal{Q}_q)$ in the supremum norm.

A subset of $D[\mathcal{E}]$ enjoying properties (1r) and (2r) is called a core for $\mathcal{E}$ and a Dirichlet form which posses a core is called regular. It is known [23, 24] that any regular Dirichlet form admits the Beurling–Deny representation:

$$\mathcal{E}(f, g) = \mathcal{E}^{(c)}(f, g) + \int\limits_{\mathcal{Q}_q \times \mathcal{Q}_q \setminus d} (f(x) - f(y))(g(x) - g(y)) J(dx, dy) + \int\limits_{\mathcal{Q}_q} f(x) g(x) k(dx). \tag{37}$$

Without going into details we only say that the first term defines the continuous part of the corresponding stochastic process, the second term jumps of the process and the third part killing the process. Since $\mathcal{Q}_q$ is totally disconnected there is no continuous path so the first term vanishes. It can bee seen that there is no killing in the class of processes we are discussing. Hence

$$\mathcal{E}(f, g) = \int\limits_{\mathcal{Q}_q \times \mathcal{Q}_q \setminus d} (f(x) - f(y))(g(x) - g(y)) J(dx, dy). \tag{38}$$

J is a symmetric Radon measure on $\mathcal{Q}_q \times \mathcal{Q}_q \setminus d$, where d stands for the diagonal. Let $K(a, q^M)$, $K(b, q^N)$ be disjoint i.e. $\text{dist}_q(K(a, q^M), K(b, q^N)) = q^n$ where $n >$

$\max\{M, N\}$. It can be shown that then

$$J(K(a, q^M), K(b, q^N)) = \frac{1}{2} q^{N+M-n+1}(q-1)^{-1}[a(n-1) - a(n)], \tag{39}$$

and (39) determines J uniquely.

Among the most extensively studied examples of the AK–random walks are the so-called **stable processes**.

Definition 1 An AK–random walk associated with parameter sequence $\{a(M), M \in \mathbb{Z}\}$ is referred to be **stable** if $a(M) = a_0 c^M$, $M \in \mathbb{Z}$ for some positive constants a_0 and $0 < c < 1$. ■

A particular class of stable processes has been studied in [39].

Let $\phi \in D_0$. For any $\alpha \neq -1$ define

$$D^\alpha \phi(x) := \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \|y\|_p^{-\alpha-1} [\phi(x - y) - \phi(x)] dy. \tag{40}$$

It turns out that for $\alpha > 0$ the operator D^α can be interpreted as the generator of an AK–random walk with the parameter sequence $\{a(M), M \in \mathbb{Z}\}$ given by

$$a(M) = \frac{p - 1}{p(1 - p^{-\alpha-1})} p^{-\alpha M}, \; M \in \mathbb{Z}. \tag{41}$$

Clearly, it is a specific class of stable random walks with $a_0 = \frac{p-1}{p-c}$.

3 $\mathbb{Q}_q$-adic Processes with Weighted Target States

In Sect. 2 we gave a construction of the class of random walks on q-adics with translation symmetric transition functions. The symmetry followed from the fact that the transition probabilities were assumed to depend only on the q-adic distance between the states. In 1993 Vilela-Mendes made a remark that the p-adic processes with weighted target states may be applicable in the study of turbulence cascades. Such processes were presented in [31]. The authors also followed Albeverio suggestion to consider the random processes on adeles. As expected the processes with weighted target states found application as a technical tool in a model of turbulent cascade in [36]. In this section we present main points of [31] and the subsequent papers [5, 13]. In contrast to [1, 2] the target states are (statistically) biased by a measure μ on $\mathbb{Q}_q$. Thus the corresponding process favors some states more than the others. The transition functions of such processes are not translation symmetric unless μ is the Haar measure. As in Sect. 2 we begin by taking $\mathcal{K}^M$ for the state space and solve the systems of Chapman–Kolmogorov Eqs. (8), (9). The coefficients $\tilde{u}(K_i^M, K_j^M)$ are defined by

$$\tilde{u}(K_i^M, K_j^M) = \mu(K_j^M)u(M, m), \tag{42}$$

where μ is a Borel measure on $\mathcal{Q}_q$, $u(M, m) \equiv a(M + m - 1) - a(M + m),\ m \in \boldsymbol{N}$ and $\mathrm{dist}_q(K_i^M, K_j^M) = q^{M+m}$.

In view of (8)–(11) $P_{K_i^M K_j^M}(t)$ is a transition function of a random process on $\mathcal{K}^M$.

In the following presentation we shall not go into the details but indicate the main steps and their analogies and differences with the symmetric case. Put

$$W_{K_k^M} \equiv -\sum_{m=1}^{\infty} u(M, m)\mu\left(K(b, q^{M+m}) \setminus K(b, q^{M+m-1})\right) = -\sum_{j \neq k} \tilde{u}(K_j^M, K_k^M), \tag{43}$$

where $b \in K_k^M$. Then by (10)

$$\tilde{a}(K_k^M) = -W_{K_k^M}. \tag{44}$$

We further define

$$\mathcal{W}_l^{K_k^M} = -\sum_{n=l}^{\infty} (u(M, n) - u(M, n+1))\, \mu(K(b, q^{M+n})),\ l \in \boldsymbol{N}. \tag{45}$$

Lemma 1 *Let $b \in K_k^M \subset K_{k'}^{M+r} \in \mathcal{K}^{M+r}$. If $r < l$ then* $\mathcal{W}_{l-r}^{K_{k'}^{M+r}} = \mathcal{W}_l^{K_k^M}$. ∎

In particular we have

$$(u(M, 1) - u(M, 2))\mu(K(b, q^{M+1})) - W_{K_k^M} = -\mathcal{W}_1^{K_k^M}. \tag{46}$$

We require that for any fixed $M \in \boldsymbol{Z}$, $W_{K_k^M}$ and $\mathcal{W}_l^{K_k^M}$ are finite.

Lemma 2 *Let $M \in \boldsymbol{Z}$ be fixed. The quantities $W_{K_k^M}$ and $\mathcal{W}_l^{K_k^M}$ are finite for all $k, l \in \boldsymbol{N}$ iff*

$$\sum_{n=1}^{\infty} a(n)\mu(K(0, q^{n+1}) \setminus K(0, q^n)) < \infty. \tag{47}$$

∎

Now, we turn to solving Eqs. (8) and (9) with initial conditions (11). Under the notation of (42) and (43), Eq. (8) become

$$\dot{P}_{K_i^M K_k^M}(t) = W_{K_k^M} P_{K_i^M K_k^M}(t) + \mu(K_k^M) \sum_{\substack{j \in \boldsymbol{N} \\ j \neq k}} u(M, m) P_{K_i^M K_j^M}(t) \tag{48}$$

with m such that $q^{M+m} = \mathrm{dist}_q(K_k^M, K_j^M)$ and the initial conditions

$$P_{K_i^M K_k^M}(0) = \delta_{ik}.$$

Similarly like in the symmetric case we obtain the formula analogous to (20).

$$\begin{aligned}\dot{P}_{K_i^M, K_j^{M+n}}(t) &= \mathcal{W}_{n+1}^{K_i^M} P_{K_i^M K_j^{M+n}}(t) \\ &+ \mu(K_j^{M+n}) \sum_{l=n+1}^{\infty} (u(M, l) - u(M, l+1)) P_{K_i^M K_j^{M+l}}(t)\end{aligned} \tag{49}$$

with the initial condition $P_{K_i^M K_j^{M+n}}(0) = 1$ if $K_i^M \subset K_j^{M+n}$ and $P_{K_i^M K_j^{M+n}}(0) = 0$ if $K_i^M \cap K_j^{M+n} = \emptyset$.

Let $\mu(K_j^{M+n}) \neq 0$ then we follow the procedure of Sect. 2 to obtain the analogue of (24)

$$\begin{aligned}&P_{K_i^M K_i^{M+n}}(t) - \frac{\mu(K_i^{M+n})}{\mu(K_i^{M+n+m+1})} P_{K_i^M K_i^{M+n+m+1}}(t) \\ &= \mu(K_i^{M+n}) \sum_{k=0}^{m} \left(\frac{1}{\mu(K_i^{M+n+k})} - \frac{1}{\mu(K_i^{M+n+k+1})} \right) e^{t \mathcal{W}_{n+k+1}^{K_i^M}}.\end{aligned} \tag{50}$$

If $\mu(\mathcal{Q}_q) = \infty$. Then the limit of (50) as $m \to \infty$ reads

$$P_{K_i^M K_i^{M+n}}(t) = \mu(K_i^{M+n}) \sum_{k=0}^{\infty} \left(\frac{1}{\mu(K_i^{M+n+k})} - \frac{1}{\mu(K_i^{M+n+k+1})} \right) e^{t \mathcal{W}_{n+k+1}^{K_i^M}}. \tag{51}$$

Taking the limit $n \to \infty$ we get $P_{K_i \mathcal{K}^M}(t) \equiv 1$.
If $\mu(\mathcal{Q}_q)$ is finite then

$$P_{K_i^M K_i^{M+n}}(t) = \mu(K_i^{M+n}) \left\{ \frac{1}{\mu(\mathcal{K}^M)} + \sum_{k=0}^{\infty} \left(\frac{P_{K_i^M \mathcal{Q}_q}(t)}{\mu(K_i^{M+n+k})} - \frac{1}{\mu(K_i^{M+n+k+1})} \right) e^{t \mathcal{W}_{n+k+1}^{K_i^M}} \right\}. \tag{52}$$

In this case the limit $n \to \infty$ yields an identity so it does not determine $P_{K_i \mathcal{Q}_q}(t)$. However direct examination shows that (52) solves Eq. (49) with the initial condition iff $P_{K_i \mathcal{Q}_q}(t) \equiv 1$.

Let c be a positive number. It follows from (45) that change of measure $\mu \to c\mu$ in (51) and (52) is equivalent to changing the time scale $t \to ct$. Hence in the case of finite measure we can assume $\mu(\mathcal{Q}_q) = 1$ without loss of generality. Then (52) reads

$$P_{K_i^M K_i^{M+n}}(t) = \mu(K_i^{M+n})\left(1 + \sum_{k=0}^{\infty}\left(\frac{1}{\mu(K_i^{M+n+k})} - \frac{1}{\mu(K_i^{M+n+k+1})}\right)e^{t\mathcal{W}_{n+k+1}^{K_i^M}}\right). \tag{53}$$

The next step will be to compute $P_{K_i^M, K_j^M}(t),\ \ i \neq j$. Then $dist_q(K_i^M, K_j^M) = q^{M+m_0}$, $m_0 \in \boldsymbol{N}$. Again we note that if $K_i^{M-n} \subset K_i^M$ for all $n \in \boldsymbol{N}_0$ then $P_{K_i^{M-n}, K_i^M}(t)$ and $P_{K_i^{M-n}, K_j^M}(t)$ is independent of n. Put $x = \lim_{n\to\infty} \bigcap_{n\to\infty} K_i^{M-n}$. This allows us to define $P_{x, K_i^M}(t) = P_{K_i^{M-n}, K_j^M}(t)$. We omit the technicalities and provide the result. If $\mu(\mathcal{Q}_q) = 1$ then

$$\begin{aligned} P_t(x, K(x, q^M)) &= \mu(K(x, q^M)) \\ &\left\{1 + \sum_{k=0}^{\infty}\left(\frac{1}{\mu(K(x, q^{M+k}))} - \frac{1}{\mu(K(x, q^{M+k+1}))}\right)e^{t\mathcal{W}_{k+1}^{K(x,q^M)}}\right\}, \end{aligned} \tag{54}$$

and if $\mathrm{dist}_q(x, y) = q^{M+n},\ n \in \boldsymbol{N}$,

$$\begin{aligned} P_t(x, K(y, q^M)) &= \mu(K(y, q^M)) \\ &\left\{1 + \sum_{k=0}^{\infty}\left(\frac{1}{\mu(K(x, q^{M+n+k}))} - \frac{1}{\mu(K(x, q^{M+n+k+1}))}\right)e^{t\mathcal{W}_{n+k+1}^{K(x,q^M)}}\right. \\ &\left. - \frac{1}{\mu(K(x, q^{M+n}))}e^{t\mathcal{W}_n^{K(x,q^M)}}\right\}; \end{aligned} \tag{55}$$

If $\mu(\mathcal{Q}_q) = \infty$,

$$P_t(x, K(x, q^M)) = \mu(K(x, q^M)) \sum_{k=0}^{\infty}\left(\frac{1}{\mu(K(x, q^{M+k}))} - \frac{1}{\mu(K(x, q^{M+k+1}))}\right)e^{t\mathcal{W}_{k+1}^{K(x,q^M)}}. \tag{56}$$

Similarly if $\mathrm{dist}_q(x, y) = q^{M+n}$ then

$$P_t(x, K(y, q^M)) = \mu(K(y, q^M))\Bigg\{ \sum_{k=0}^{\infty} \left(\frac{1}{\mu(K(x, q^{M+n+k}))} - \frac{1}{\mu(K(x, q^{M+n+k+1}))}\right) e^{t\mathcal{W}_{n+k+1}^{K(x,q^M)}} - \frac{1}{\mu(K(x, q^{M+n}))} e^{t\mathcal{W}_n^{K(x,q^M)}}\Bigg\}. \tag{57}$$

Put $\mathcal{Q}$ for the $\sigma - algebra$ of Borel sets in $\mathbb{Q}_q$. We finally obtain

Proposition 2 *$P_t(x, A)$, $t > 0$, $x \in \mathbb{Q}_q$, $A \in \mathcal{Q}_q$ as defined by (54), (55) resp. (56), (57) is a Markovian μ–symmetric transition function on the measurable space $(\mathbb{Q}_q, \mathcal{Q}_q)$.* ■

As a consequence of Proposition 2 the formula

$$(T_t u)(x) := \int_{\mathbb{Q}_q} P_t(x, dy)u(y), \quad u \in L^2(\mathbb{Q}_q, \mu) \tag{58}$$

defines a strongly continuous Markovian semigroup $(T_t, t > 0)$ in the Hilbert space $L^2(\mathbb{Q}_q, \mu)$. Put $-H$ for its generator i.e. $T_t = \exp\{-Ht\}$. Formula (32) yields

$$- H\chi_{K(a,q^M)}(x) = \begin{cases} W_{K(a,q^M)}, & \text{for } x \in K(a, q^M), \\ \mu(K(a, q^M))u(M, j), & \\ \qquad \text{if } x : \mathrm{dist}_q(K(a, q^M), x) = q^{M+j}. & \end{cases} \tag{59}$$

This formula implies that $\chi_{K(a,q^M)}$ is an eigenvector for H iff $u(M, j) = 0$ for all $j \in \mathbb{N}$ in which case $W_{K(a,q^M)} = 0$ is the eigenvalue. Then the process starting in $K(a, q^M)$ is confined to stay in $K(a, q^M)$ forever.

Given $a \in \mathbb{Q}_q$ and $M \in \mathbb{Z}$, then $K(a, q^M) = K_j^M \in \mathcal{K}^M$ for some $j \in \mathbb{N}$. Put K_{ij}^{M-1}, $i = 0, 1, \ldots, q-1$ for the balls of radius q^{M-1} such that $\bigcup_{i=0}^{i=q-1} K_{ij}^{M-1} = K_j^M$. The spectral properties of H are described in

Theorem 3 *Let $-H$ denote the generator of strongly continuous semigroup $(T_t,\ t \geq 0)$ with the kernel defined by (54), (55) resp. (56), (57). Then*

(a) For any $a \in \mathbb{Q}_q$, $M \in \mathbb{Z}$ there corresponds an eigenvalue $h_{K(a,q^M)}$ of H given by

$$h_{K(a,q^M)} = \mathcal{W}_0^{K(a,q^M)}.$$

To this eigenvalue there is a $q-1$ dimensional eigenspace spanned by vectors of the form

$$e_{K(a,q^M)} = \sum_{i=0}^{q-1} b_i \chi_{K_{ij}^{M-1}}, \ \textit{where} \ \sum_{i=0}^{q-1} b_i \mu(K_{ij}^{M-1}) = 0. \tag{60}$$

(b) *Denote by* $e^s_{K(a,q^M)}$, $s = 1, 2, \dots, q-1$ *the orthonormalized eigenvectors corresponding to* $h_{K(a,q^M)}$. *If* $\mu(\mathbb{Q}_q) = \infty$ *then the orthonormal system* $\{e^s_{K(a,q^M)}$, $a \in \mathbb{Q}_q$, $M \in \mathbb{Z}$, $s = 1, 2, \dots, q-1\}$ *is a basis for* $L^2(\mathbb{Q}_q, \mu)$. *If* $\mu(\mathbb{Q}_q) = 1$ *then the above vectors together with the constant function 1 form a basis for* $L^2(\mathbb{Q}_q, \mu)$. ∎

The structure of eigenvectors of H implies

Corollary 2 D_0 *is a core for* H.

Remark 3 In the Haar symmetric case every eigenvalue has infinite degeneracy. It is not necessarily so in the μ–symmetric case. Indeed consider two different balls of the same radius, say $K(a, q^M)$ and $K(b, q^M)$. If the process is Haar–symmetric then both $e_{K(a,q^M)}$ and $e_{K(b,q^M)}$ are eigenvectors corresponding to the same eigenvalue

$$h_{K(a,q^M)} = h_{K(b,q^M)} = h_M = (q-1)^{-1}[qa(M-1) - a(M)]. \tag{61}$$

If the process is μ–symmetric $h_{K(a,q^M)} = \mathcal{W}_0^{K(a,q^M)}$ and $h_{K(b,q^M)} = \mathcal{W}_0^{K(b,q^M)}$ are not the same in general. ∎

Since by Corollary 2 D_0 is a core for H one shows in the manner similar to that of Sect. 2 that the Dirichlet form

$$\mathcal{E}(f, g) = (H^{\frac{1}{2}} f, H^{\frac{1}{2}} g),\ f, g \in D(H^{\frac{1}{2}}) \tag{62}$$

is regular. We have

Theorem 4 *The Dirichlet form corresponding to a* μ*–symmetric process is defined by its Beurling–Deny representation*

$$\mathcal{E}(u, v) = \int\limits_{\mathbb{Q}_q \times \mathbb{Q}_q \setminus \mathbf{d}} (u(x) - u(y))(v(x) - v(y)) J(dx, dy)$$

where the measure J *is given by*

$$J(dx, dy) = \frac{1}{2}(a(n-1) - a(n))\mu(dx)\mu(dy) \tag{63}$$

and $u,\ v \in D_0 \cap L^2(\mathbb{Q}_q,\ \mu)$. ∎

We complete this section by mentioning further generalization of the construction developed in [2]. It is well known that $\mathbb{Q}_q$ can be identified with a tree which has $q+1$ ledges at every node. On the other hand $\mathbb{Q}_q$ can be looked upon as a space of numerical sequences $\{a_i\}_{i \in \mathbb{Z}}$ with the entries $a_i = 0, 1, \dots, q-1$ terminating at $i \to -\infty$. In the paper [3] we defined classes S_B of numerical sequences $\{a_i\}_{i \in \mathbb{Z}}$, $a_i \in \mathbb{Z}$ terminating at $i \to -\infty$ which correspond to the trees with varying number of ledges. We adopted the method of [2] to give explicit formulae for the transition functions of random processes on S_B, the generators with their spectral descriptions and the Dirichlet forms in terms of the Beurling–Deny formula.

4 Trace Formula

In this section we present main ideas of [4]. We give a p-adic analogue of Selberg's trace formula relating the trace of a semigroup generated by a natural elliptic operator with a sum over contributions coming from closed geodesics. The construction uses probabilistic methods to define the generator.

The Laplace–Beltrami operator as defined on a given Riemannian manifold is determined by the geometrical structure of the manifold and the boundary conditions. Its properties and in particular the trace formula for the semigroup generated by the Laplace–Beltrami operator are of interest not only as a pure mathematical problem but also in view of physical and technical applications. Adopting a probabilistic point of view we demonstrate that some basic structural properties of the trace formula carry over to $\boldsymbol{Q}_p$ which is drastically different as far as topology and metrics are concerned. Roughly speaking we rely on following procedure. The Laplace–Beltrami operator on the upper half-plane with the Poincaré metric generates a diffusion process X_t, $t \geq 0$, on the half-plane. Similarly the Laplace–Beltrami operator on a compact Riemann surface M generates a diffusion process $\widetilde{R}_t$ on this surface. The surface M results from the identification of the points $x \in F$ (where F is the fundamental domain relative to a discrete subgroup Γ of the group $SL(2, \boldsymbol{R})/\{1, -1\}$ of isometries of the upper half-plane) with the points $\gamma x,\ e \neq \gamma \in \Gamma$, where e is the unite element of Γ. Let $p_t(x, y)$ be the transition density for X_t. Define a process R_t on M by its transition density $q_t(x, y)$ given by the formula

$$q_t(x, y) = p_t(x, y) + \sum_{e \neq \gamma,\ \gamma \in \Gamma} p_t(x, \gamma y), \qquad x, y \in F.$$

It turns out that the transition density $\tilde{q}_t(x, y)$ for $\widetilde{R}_t$ is equal to $q_t(x, y)$. Moreover

$$\mathrm{Tr} e^{\Delta t} = \int_F p_t(x, x) dx + \sum_{e \neq \gamma,\ \gamma \in \Gamma} \int_F p_t(x, \gamma x) dx$$

is expressed in terms of lengths of closed geodesics, i.e., the distances between x and $\gamma x,\ e \neq \gamma,\ \gamma \in \Gamma$.

Passing to $\boldsymbol{Q}_p$ the obvious possibility, successfully explored by Yasuda in [41], is to keep with the analogy and carry on the Selberg procedure for the upper half-plane of a complex extension of $\boldsymbol{Q}_p$. However even the structure of $\boldsymbol{Q}_q$ itself offers a possibility for vaguely similar approach resulting in the trace formula in terms of lengths of closed geodesics. We begin with the $\boldsymbol{Q}_q$ framework and obtain the trace formula.

Then restricting the state space to $\mathbb{Q}_p$ we investigate connection between the trace formula and representations of the group of isometries in $\mathbb{Q}_p$. Let $a(M)$, $M \in \mathbb{Z}$ be a parameter sequence, $P_t(x, A)$ and T_t the transition function corresponding to $a(M)$, and the Markovian semigroup defined by $P_t(x, A)$ respectively. By Theorem 2 the generator $-H$ of T_t has the pure point spectrum given by (34). The transition density is defined by

$$p_t(x, y) \equiv \lim_{M \to -\infty} q^{-M} P_t(x, K(y, q^M)).$$

If $dist_q(x, y) = q^n$ then by (28), (29) and (34) we get

$$p_t(x, y) = q^{-n} \left\{ q^{-1}(q-1) \sum_{i=0}^{\infty} q^{-i} e^{-h_{n+i}t} - e^{-h_{n-1}t} \right\}$$

$$= q^{-1}(q-1) \sum_{i=n}^{\infty} q^{-i} e^{-h_i t} - q^{-n} e^{-h_{n-1}t}, \tag{64}$$

$$p_t(x, x) = q^{-1}(q-1) \sum_{i=-\infty}^{\infty} q^{-i} e^{-h_i t} = \lim_{y \to x} p_t(x, y). \tag{65}$$

It can be checked that the sum in (65) converges iff

$$\lim_{n \to \infty} \frac{a(-n)}{n} = \infty. \tag{66}$$

If for some $M_0 \in \mathbb{Z}$, $a(M_0) = 0$ and $a(M_0 - 1) > 0$, then $a(N) = 0$ and $h_N = 0$ for all $N \geq M_0$. In this case the characteristic function of $K(a, q^{M_0})$ is an eigenfunction for H with eigenvalue $h_{M_0} = 0$. If $M < M_0 - 1$ then the eigenvalues h_M are given by (34). As a consequence we have

Proposition 3 *Let for some $M_0 \in \mathbb{Z}$, $a(M_0 - 1) > 0$ and $a(M_0) = 0$ hold. Put P_a for the projector in $L^2(\mathbb{Q}_q)$ onto $L^2(K(a, q^{M_0}))$. Then H and P_a commute. The operator $-H_a := -HP_a$ is a generator of a random walk in $K(a, pq^{M_0})$.* ■

Remark 4 Note that Formulae (64), (65) covers the case when $a(M_0) = 0$, $a(M_0 - 1) > 0$. Then we put $h_i = 0$ for $i \geq M_0$ and (65) resp. (65) become

$$p_t(x, y) = q^{-1}(q-1) \sum_{i=n}^{M_0-1} q^{-i} e^{-h_i t} + q^{-M_0} - q^{-n} e^{-h_{n-1}t}, \tag{67}$$

$$p_t(x, x) = q^{-M_0} \left(q^{-1}(q-1) \sum_{l=1}^{\infty} q^l e^{-h_{M_0-l}t} + 1 \right). \tag{68}$$

The factor q^{-M_0} in (68) is a normalization but the coefficients inside the bracket are equal to the dimensions of the corresponding eigenspaces. Hence

Proposition 4 *Let for some* $M_0 \in \mathbb{Z}$*,* $a(M_0 - 1) > 0$*,* $a(M_0) = 0$ *and (66) holds then*

$$Tre^{-H_a t} = \int\limits_{K(a, q^{M_0})} p_t(x, x)\mu(dx) = \sum_{l=0}^{\infty} n_l e^{-h_{M_0 - l} t}, \tag{69}$$

where n_l *is the multiplicity of the eigenvalue* $h_{M_0 - l}$*.* ■

$\mathbb{Q}_q$ is a group under addition and any ball $K(0, q^{M_0})$ is its invariant subgroup. Put $\Gamma_{M_0} = \mathbb{Q}_q / K(0, q^{M_0})$ and identify $\Gamma_{M_0} = \{\gamma_i : i \in \mathbb{N}\}$ where γ_i are the representatives of the cosets of $K(0, q^{M_0})$.

Then we have

(i) $(\gamma_i + K(0, q^{M_0})) \cap (\gamma_j + K(0, q^{M_0})) = \phi \quad$ for $i \neq j$,
(ii) $\cup_{i=1}^{\infty} (\gamma_i + K(0, q^{M_0})) = \mathbb{Q}_q$,
(iii) Γ_{M_0} is discrete.

In view of (i)–(iii) we say that $K(0, q^{M_0})$ is a fundamental domain for $\mathbb{Q}_q$ relative to Γ_{M_0}. To simplify notations we put $M_0 = 0$, $K := K(0, 1)$, and choose Γ_0 so that $\Gamma_0 \cap K = \{0\}$. Let $x \in K$ and $u, u' \in \gamma + K$ for some $\gamma \in \Gamma_0$, $\gamma \neq 0$. Then $p_t(x, u) = p_t(x, u')$ and

$$q_t(x, y) := \sum_{\gamma \in \Gamma_0} p_t(x, \gamma + y), \qquad x, y \in K \tag{70}$$

is independent of choice of the representatives $\Gamma_0 \setminus \{0\}$. We will show in the following theorem that (70) is the transition density for a process confined to K. Roughly speaking, if at time $t = 0$ the original process started from $x \in K$ then the probability for the new process to be at time $t > 0$ at the point $y \in K$ equals the probability of the original process to be at any one of the points $\gamma + y$, $\gamma \in \Gamma_0$.

Theorem 5 *(Trace Formula) Let* $-H$ *be the generator of a random walk on* $\mathbb{Q}_q$ *defined by a parameter sequence* $a(M), M \in \mathbb{Z}$*. Put* h_M*,* $M \in \mathbb{Z}$*, for the corresponding eigenvalues and* $p_t(x, y)$ *for the transition density. Then the sequence*

$$\tilde{a}(M) = \begin{cases} 0, & M > 0, \\ a(M) - q^M a(0), & M \leq 0, \end{cases} \tag{71}$$

defines a random walk on K*. If* $-\widetilde{H}$ *is the generator of this process,* $\tilde{h}_M$*,* $M \leq 0$*, are the eigenvalues of* $\widetilde{H}$*, and* $\tilde{p}_t(x, y)$ *is the transition density, then*

$$\tilde{h}_0 = 0, \qquad \tilde{h}_M = h_M \ (M < 0), \tag{72}$$

and

$$\tilde{p}_t(x,y) = q_t(x,y) = p_t(x,y) + \sum_{m=1}^{\infty} \sum_{\substack{\gamma \in \Gamma_0 \\ \|\gamma\|_q = q^m}} p_t(x, \gamma + y), \quad x, y \in K. \tag{73}$$

Vice versa if $-\widetilde{H}$ *is the generator of a random walk on K with eigenvalues* $\tilde{h}_{-l}, l \geq 0$*, and the corresponding parameter sequence* $\tilde{a}(M)$ *is such that* $q^{-M}\,(\tilde{a}(M-1) - \tilde{a}(M)) > \eta$ *for some* $\eta > 0$ *and all* $M \leq -1$*, then any parameter sequence* $a(M), M \in \mathbb{Z}$*, satisfying*

$$a(0) < \tilde{a}(-1), \ a(M) = \tilde{a}(M) + q^M a(0), \ M \leq -1, \tag{74}$$

defines a random walk on $\mathbb{Q}_q$ *such that (72) and (73) hold.*

If moreover $\sum_{l=0}^{\infty} q^l e^{-h_{-l}t} < \infty$ *then*

$$\sum_{l=0}^{\infty} n_l e^{-h_{-l}t} = Tre^{-\widetilde{H}t} = p_t(0,0) + \sum_{m=1}^{\infty} \sum_{\substack{\gamma \in \Gamma_0 \\ \|\gamma\|_q = q^m}} p_t(0,\gamma) < \infty, \tag{75}$$

where n_l *is the multiplicity of* h_{-l}*.*

Remark 5 In the above discussion we required that $\Gamma_0 \cap K = \{0\}$. By this condition the identification of K with itself was given by identity operation $x \to x + 0$. Besides of that Formula (75) is independent on the choice of Γ_0.

Notice that (75) is independent of h_i, $i \geq 0$.

4.1 q-adic Closed Geodesics

Presently we invoke the tree formulation of $\mathbb{Q}_q$ and the concept of geodesic to obtain even closer analogy of our trace formula for $\mathbb{Q}_q$ with the Selberg trace formula for the Riemann surfaces. For basic facts about trees used here see [3] or [29].

Formula (5) suggests to identify $\mathbb{Q}_q$ with the space of sequences $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$, $\alpha_i = 0, \dots, q-1$ terminating as $i \to -\infty$. Let $\alpha \in \mathbb{Q}_q$. Put $\{\alpha\}_k \equiv \{\alpha_i\}_{i\le k}$. The nods of the tree are defined as the sets $a_N = \{\alpha\}_N$, where $\alpha \in \mathbb{Q}_q$ and $N \in \mathbb{Z}$. The pairs (a_N, a_{N+1}) are the ledges. For any $N, M \in \mathbb{Z}$, $N < M$ set $R_{N,M}(\alpha) = (a_N, a_{N+1}, \dots, a_M)$ and $R_{N,\infty}(\alpha) = (a_N, a_{N+1}, \dots)$. Then $R_{N,M}(\alpha)$ is a simple path and hence the infinite sequence $R_{N,\infty}(\alpha)$ is a geodesic ray representative of α. Put $L_{N,M}(\alpha)$ for the length of the path $R_{N,M}(\alpha)$ and define $L_{N,N+1}(\alpha) = \frac{1}{2}(q-1)q^{-(N+2)}$. Then

$$L_{N,\infty}(\alpha) = \frac{1}{2}\sum_{i=N}^{\infty}(q-1)q^{-(i+2)} = \frac{1}{2}q^{-(N+1)}.$$

Graphically one can think of the nods as one point sets in $\mathbb{R}^2$ and ledges as the segments connecting the nods. Let us make particular choice of the parameter sequence setting $a(M) = q^{-M}$. Then $h_i = (q+1)q^{-(i+1)}$ and by (64),

$$p_t(x, \gamma + x) = q^{-m}\left\{\sum_{i=0}^{\infty}\frac{(q-1)(q+1)^i}{q^{i+1}-1}\frac{1}{i!}(q^{-m}t)^i - \exp\left(-(q-1)q^{-m}t\right)\right\},$$

where $\|\gamma\|_q = q^m$ $m > 0$ and $\|x\|_q \le 1$. Hence $\{\gamma\}_{-(m+1)} = \{0\}_{-(m+1)}$, $\{\gamma\}_{-m} \ne \{0\}_{-m}$ and $\{x\}_k = \{0\}_k$ for $k < 0$. It follows that the geodesic rays $R_{-(m+1),\infty}(\gamma + x)$ and $R_{-(m+1),\infty}(x)$ begin at the same nod $\{0\}_{-(m+1)}$ but their next nods are different. Since the points x and $x + \gamma$ are identified by action of $\gamma \in \Gamma_0$ the line $R_{-(m+1),\infty}(x) \cup R_{-(m+1),\infty}(x + \gamma)$ may be interpreted as a closed geodesic. The length of it equals $2L_{-(m+1),\infty}(\gamma) = q^m$. Thus similarly as in Selberg's trace formula, each term in the sum in (75) is expressed by the distances of points which are identified under the action of Γ_0 and like in Selberg trace formula the distances equal to the length of corresponding closed geodesics.

4.2 Representations of the Group of Isometries and Trace Formula

In this section we shall discuss connection between our trace formula and the representations of the group of isometries in $\mathbb{Q}_p$. Let M_0 be an integer or ∞. We put $K(0, p^{M_0}) = \mathbb{Q}_p$ for $M_0 = \infty$. With the p-norm the ball $K(0, p^{M_0})$ is a compact space if M_0 is finite and locally compact if $M_0 = \infty$. We shall write $X := K(0, p^{M_0})$ as a topological space and $G_+ := K(0, p^{M_0})$ as an additive group. We also put $G_* := K(0, 1) \setminus K(0, p^{-1})$ as a multiplicative group. Then G_* defines a group of automorphisms $\theta_z, z \in G_*$, of G_+ by $\theta_z a = za$, $a \in G_+$.

We put G for the semidirect product of G_* and G_+ relative to θ; $G := G_* \times_\theta G_+$, i.e., $G = \{g \in [z, a] : z \in G_*, a \in G_+\}$ and $g_1g_2 = [z_1z_2, z_1a_2 + a_1]$. It is then natural to define an action of G on X by

$$gx = [z, a]x = zx + a, \qquad x \in X,\ g \in G. \tag{76}$$

One shows that G is a doubly transitive group of isometries for X.

In terminology of [25] G is thus a transitive group of motions of X. Let M_0 be finite. We put μ_* and μ_+ for the normalized restrictions of the Haar measure μ to G_* and G_+ respectively. Then μ_* and μ_+ are defined by

$$\mu_*(K(u, p^k)) = (p-1)^{-1}p^{k+1}, \qquad u \in G_*,\ k \le -1,$$
$$\mu_+(K(b, p^m)) = p^{m-M_0}, \qquad b \in G_+,\ m \le M_0.$$

If $M_0 = \infty$ we set $\mu_+ = \mu$ and μ_* as above. Finally we put $\mu_G := \mu_* \times \mu_+$ for the product measure on G. One shows that μ_G is a bi-invariant measure on G. Let us come back to the action of G on X.

The subgroup $[G_*, 0]$ leaves the point $0 \in X$ invariant. For $g = [z, a]$ the left coset relative to $[G_*, 0]$ is of the form $g[G_*, 0] = [G_*, a]$, and the action of G on X is covariant under the action of G on G/G_*; $g'[G_*, a] = [G_*, z'a + a']$, $g' = [z', a']$. Thus we can identify X with G/G_*. Put $g = [z, a] \in G$ and consider the set $\mathcal{L}_g := \{[z_1, 0]g[z_2, 0] : z_1, z_2 \in G_*\}$. Notice that $[z_1, 0]g[z_2, 0] = [z_1z_2z, z_1a]$. If $\|a\|_p = p^k$ and z_1 runs over G_* then z_1a runs over the set $L_k := \{x \in \mathbb{Q}_p : \|x\|_p = p^k\} = K(0, p^k) - K(0, p^{k-1})$. On the other hand for any $z, z_1 \in G_*$ we have $\{z_1z_2z : z_2 \in G_*\} = G_*$. Thus we have $\mathcal{L}_g = [G_*, L_k]$, and the coset $\mathcal{L}_g/G_*$ is isomorphic to L_k. We remark that L_k and hence $\mathcal{L}_g$ are independent of a provided $\|a\|_p = p^k$. Thus it follows that a function h on G is G_*-invariant iff $h(g) = h([z, a]) = h(\|a\|_p)$.

For $g \in G, f \in L^2(G, \mu_G)$ the formula $T_gf(\cdot) := f(g\,\cdot)$ defines an unitary representation of G in $L^2(G, \mu_G)$. Let $\varphi \in L^2(G, \mu_G)$ and define the operator

$$T_\varphi f(g) = \int_G \varphi(g')f(g'g)\mu_G(dg') = \int_G \varphi(g'g^{-1})f(g')\mu_G(dg'), \tag{77}$$

for $f \in L^2(G, \mu_G)$. The subspace of $L^2(G, \mu_G)$ consisting of the functions independent of $z \in G_*$ i.e., such that $f(g) = f([z, a]) = f(a)$ can be naturally identified with $L^2(X, \mu_+)$. In case $M_0 = \infty$ we put $\varphi_t(g) = \varphi_t(a) = p_t(0, a)$ where $p_t(0, a)$ is given by (68). Since $p_t(0, a)$ depends only on $\|a\|_p$ the operator T_{φ_t} defined by (77) is a positive self-adjoint operator in $L^2(\mathbb{Q}_p, \mu_+)$.

We shall now see that in the case when M_0 is finite we have an interpretation of the left hand side of the trace formula (75) in terms of a sum over eigenvalues of an operator T_φ of the form (77) thus describing a unitary representation of motions of the group of translations acting on the fundamental domain K for $\mathbb{Q}_p$, whereas the right hand side of (75) is connected with an operator T_{φ_t} acting on the whole space $L^2(\mathbb{Q}_p, \mu_+)$.

Thus in case of finite M_0 we take $\varphi(g) = \varphi(\|a\|_p) \in L^2(X, \mu_+)$. Then T_φ as given by (77) is a normal compact operator on $L^2(X, \mu_+)$. In this case there exists an orthonormal basis $\{f_k\}_{k=1}^{\infty}$ consisting of eigenfunctions of T_φ. Let λ_k be the eigenvalue corresponding to f_k. Then

$$\sum_{k=1}^{\infty} \lambda_k = \mathrm{Tr} T_\varphi = \int_X \varphi(0) d\mu_+ = \varphi(0). \tag{78}$$

If in particular $M_0 = 0$ and $\varphi(\|a\|_p) = \tilde{p}_t(0, a)$, where $\tilde{p}_t(0, a)$ is given by (73), then our trace formula (75) is expressed in the form

$$\sum_{k=1}^{\infty} \lambda_k(t) = \mathrm{Tr} T_{\tilde{p}_t} = p_t(0, 0) + \sum_{m=1}^{\infty} \sum_{\substack{\gamma \in \Gamma_0 \\ \|\gamma\|_p = p^m}} p_t(0, \gamma), \tag{79}$$

(where $\lambda_k(t)$ is the value of λ_k in (78) obtained for $\varphi(\|a\|_p) = \tilde{p}_t(0, a)$). We remark that the λ_k in (79) are degenerate i.e. as compared with (75), λ_k is equal to $e^{-\tau_l t}$ for certain l with multiplicity n_l.

5 Miscellany

In Sects. 2 and 3 we explained with some details the technics of [2] and one of its generalizations [5, 13, 31]. The research described in Sect. 4 does not extend the scope of processes derived by methods of [2] but it exploits the results of it. In this section we illustrate by a number of examples the interaction of the results of [2] with the research on random processes on hierarchical spaces.

Stochastic differential equations. Typically the stochastic differential equations in $\boldsymbol{R}^n$ contain differential of the Brownian motion. When discussing the stochastic differential equations over p-adic fields Kochubei [35], Kaneko [27] and later Kaneko and Kochubei [28] use the differential of a process obtained in [2] instead. For illustration we formulate one of the results of [27] and comment on the generalizations provided in [28]. Given $\gamma \geq 1$. Put $\mathcal{A}(\gamma)$ for the family of parametric sequences $A = \{a(M)\}_{M \in \boldsymbol{Z}}$ satisfying

$$\sum_{M=-\infty}^{\infty} a(M) p^{\gamma M} < \infty. \tag{80}$$

Let $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$ stands for the family of all right continuous, left limited sample paths $\omega : [0, T] \to \boldsymbol{Q}_p$. The random process $X(t)$ of Theorem 1 defined by parametric sequence $A \in \mathcal{A}(\gamma)$ is a $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$ valued random variable. In this section we introduce the concepts of stochastic integral with respect to the random process $X(t)$, stochastic differential equations, and discuss their solutions. We begin with the concept of stochastic integral with respect to $X(t)$ with $X(0) = 0$. For the notation and the general concepts used the reader is referred to [23, 24, 35].

Let $\{\mathcal{F}_t\}$ be a filtration such that $\mathcal{F}_t \supset \sigma[X(s)|s \leq t]$ for any t. Then $\{\mathcal{F}_t\}$ is independent of $\sigma[X(s+t) - X(t)]$ $[s > 0]$ for every $t \geq 0$. Denote by S_T the set of random variables $\phi = \sum_{i=0}^{n-1} f_i \chi_{[t_i, t_{i+1}]}$, where $\{t_i\}_{i=0}^{i=n}$ is a division $0 = t_0 < t_1 < \ldots < t_n = T$ of $[0, T]$ and each f_i is an $\{\mathcal{F}_{t_i}\}$-measurable $\mathbb{Q}_p$-valued random variable. Then for any $\phi \in S_T$ the stochastic integral with respect to $X(t)$ is defined by

$$\int_0^t \phi(s)dX(s) = \sum_{i=0}^{n-1} f_i(X(t_{i+1} \wedge t) - X(t_i \wedge t)) \text{ for } 0 \leq t \leq T. \tag{81}$$

$\{\int_0^t \phi(s)dX(s)\}_{t\in[0,T]}$ is a family of the $\mathcal{D}([0, T] \to \mathbb{Q}_p)$ valued random variables and can also be regarded as an $\{\mathcal{F}_t\}$-adapted process.

Put L^γ for the set of $\mathbb{Q}_p$-valued random variables X such that $E[\|X\|_p^\gamma] < \infty$ and denote the set of $\mathcal{F}_t$-adapted $\mathbb{Q}_p$-valued random processes regarded as continuous maps $[0, T] \to L^\gamma$ by $\mathcal{C}([0, T] \to L^\gamma)$.

Consider a function $\sigma(t, X) : [0, T] \times \mathcal{C}([0, T] \to L^\gamma) \longrightarrow \mathcal{C}([0, T] \to L^\gamma)$ for some γ. Assume:

(1) If $X \in \mathcal{C}([0, T] \to L^\gamma)$, then $\sigma(., X) \in \mathcal{C}([0, T] \to L^\gamma)$.
(2) If $s \in [0, T]$ and $X \in \mathcal{C}([0, T] \to L^\gamma)$ then $\sigma(s, X)$ is a $\mathcal{F}_s$ random variable depending only on the random variables $X(u)$, $u \in [0, s]$.
(3) There exists a constant C_T such that

$$E[\|\sigma(t, X) - \sigma(t, X')\|_p^\gamma] \leq C_T E\left[\sup_{0\leq u\leq t} \|X(u) - X'(u)\|_p^\gamma\right] \tag{82}$$

for all $X, X' \in \mathcal{C}([0, T] \to L^\gamma)$ and $t \in [0, T]$.

Let $X(t)$ be defined by $A \in \mathcal{A}(\gamma)$. If $\{Y(t)\} \in \mathcal{C}([0, T] \to L^\gamma)$ satisfies the stochastic integral equation $Y(t) = x + \int_0^t \sigma(s, Y)dX(s)$, $0 \leq t \leq T$, for some starting point $x \in \mathbb{Q}_p$, then $\{Y(t)\}$ is called a solution of the stochastic differential equation

$$dY(t) = \sigma(t, Y)dX(t), \ Y(0) = x. \tag{83}$$

Theorem 6 *If $\{X(t)\}$ is a random process defined by $A \in \mathcal{A}(\gamma)$, $\gamma \geq 1$, then the stochastic differential equation*

$$dY(t) = \sigma(t, Y)dX(t), \ Y(0) = x, \tag{84}$$

has a unique solution $\{Y(t)\}$ for every starting point $x \in \mathbb{Q}_p$.

In [28] the authors relax the continuity requirements of the coefficient in the stochastic differential equations. They introduce a more general construction of stochastic integral to admit predictable integrand with finite moment. As the result they obtain a sufficient condition for existence of a weak solution of the stochastic differential equation driven by the the process defined by the parameter sequence (41) $a(M) = \frac{p-1}{p(1-p^{-\alpha-1})} p^{-\alpha M}, \ M \in \mathbb{Z}$.

Extension of the AK technics to some hierarchical spaces. Yasuda [40] generalized the AK construction to local fields. She has also given a necessary and sufficient condition for a process to be recurrent.

The processes constructed in Sect. 3 could have been obtained from the AK processes by changing the jump measure of the Dirichlet form using the multiplicative functionals. Kaneko constructed the classes of space inhomegenous [26] and time inhomogenous [27] processes wider than that discussed in Sect. 3. His construction yields also the processes which cannot be obtained by multiplicative functionals. Another extension of the AK technics was presented in [3]. The authors constructed a class of random processes on hierarchical spaces corresponding to the trees with varying number of ledges at the nods. They obtained explicit formulas for the transition functions, the Dirichlet forms and the generators together with their complete spectral descriptions. Further extensions of the class of random processes on trees obtained by the method different from that used in AK are due to Kigami [33] and Kaneko [29]. Their starting point was to consider quadratic forms on the finite dimensional spaces of functions analogical to (60). The collections of such forms were then used to construct the Dirichlet forms in terms of nonnegative functions λ defined on the nods and regular Borel measure μ on the ends of the tree.

Stochastic processes of diffusion in $\boldsymbol{R}^1$, $\boldsymbol{R}^2$ and jumps on fractal. Yet another application of the AK processes appeared in [30, 32]. The 2-adic ball $K(0, 1) \subset \mathbb{Q}_2$ can be mapped onto the Cantor set on real line in an obvious way. Thus any AK process on $\mathbb{Q}_2$ with $a(0) = 0$ generates a random process on the Cantor set. Consequently the 2-adic jump measure J_2 determines a jump measure J_C on $\boldsymbol{R} \times \boldsymbol{R} \setminus d$, supported by $C \times C \setminus d$, where C stands for the Cantor set and d for the diagonal. Similarly, given an AK process on $K(0, 1) \subset \mathbb{Q}_q$ defined by a jump measure J_q one can map the ball $K(0, 1) \subset \mathbb{Q}_q$ onto a fractal set $\Gamma \subset \boldsymbol{R}^2$ and obtain corresponding process on Γ and the jump measure J_Γ on $\boldsymbol{R}^2 \times \boldsymbol{R}^2 \setminus d$ supported by $\Gamma \times \Gamma \setminus d$.

Put $d_H(\Gamma)$ for the Haussdorff dimension of the fractal Γ. For a class of fractals including the Cantor set, Sierpiński carpet and Sierpiński gasket it has been shown in [30, 32] that under the condition

$$\lim_{M\to\infty} \sqrt[M]{a(M)} < q^{\alpha},$$

where $\alpha = 2d_H(\Gamma)^{-1}$ the quadratic form

$$\mathcal{E}(f,g) = \int_{\boldsymbol{R}^n} \sum_{k=1,l=1}^{n} \frac{\partial f(x)}{\partial x_k}\frac{\partial g(x)}{\partial x_l} dx + \int_{\boldsymbol{R}^n \times \boldsymbol{R}^n \setminus d} (f(x) - f(y))(g(x) - g(y)) j(dx, dy) \tag{85}$$

($n = 1, 2$) can be defined on C_0^∞ and then it is closable and its closure is a regular Dirichlet form. Thus (85) defines a random process of diffusion on $\boldsymbol{R}^n$ $n = 1, 2$ and jumps on $\Gamma \subset \boldsymbol{R}^n$. If $\lim_{M\to\infty} \sqrt[M]{a(M)} > q^\alpha$, then (85) does not admit C_0^∞ in its domain. Whether under some boundary conditions on Γ (85) would define a (regular) Dirichlet form is an open question.

Infinite system of linear differential equations. As the first step in the constructions of stochastic processes presented in [2, 3, 31] we defined the Markov chains on the space $\mathcal{K}^M$ of disjoint balls. Since $\mathcal{K}^M$ is countable infinite we had to solve the infinite systems of Chapman–Kolmogorov Eqs. (8), (9). Thus without referring to the probabilistic context we had to solve the system

$$\dot{u}_i(t) = \sum_{j=0}^{\infty} a_{ij} u_j(t), \ \ i \in \boldsymbol{N}_0, \tag{86}$$

with the initial condition $u_i(0) = c_i$. Even if we knew there was a solution the effective computation would be a problem. The procedure we used to obtain the solutions relied on the hierarchical structures of the set of coefficients and their labelling. Note that since the coefficients in (86) are labelled by nonnegative integers it would be practically very difficult to recognise the hierarchical structure if there was any. Albeverio and Zhao [12] specified a class of the systems (86) which can be solved by using the generators of spherically symmetric processes on $\mathcal{K}^0$. Put $R_0 = K(0, 1)$ and $R_i = K(0, q^i) \setminus K(0, q^{i-1})$ for $i \geq 1$. Given a parametric sequence $a(M)$ with $a(0) > 0$. The matrix a_{ij} $i, j \in \boldsymbol{N}_0$ is defined as follows; $a_{00} = a(0)$. If either $i \neq 0$ or $j \neq 0$ then $a_{ij} = \sum_k \tilde{u}(K, K_k)$, where $\tilde{u}$ is defined by (14), K is any ball of radius 1 included in R_i and summation runs over $q^{j-1}(q-1)$ disjoint balls $K_k = K(a_k, 1),\ a_k \in R_j$. It is then shown that the system (86) with bounded sequence $u_i(0) = c_i$ of the initial conditions has the unique solution expressed in terms of (25), (26). The authors discussed also the corresponding heat equations in linear and nonlinear cases.

Conclusions. In this note I concentrated on the ideas and procedures presented in [2], their extensions and point interaction with various studies involving hierarchical spaces. I must admit that many interesting developments as for instance those of [10, 14, 16] are not included here. Also in retrospection I realized that the stochastic processes on hierarchical spaces were only the starting point for Sergio Albeverio engagement in the research on broad range of mathematical problems in p-adic. Neither space of this note nor my personal knowledge of the subject are adequate to give the reader a closer look at his achievements beyond the area of stochastic processes.

Let me however at least mention some directions of his research. As far as I can see most of his p-adic works were based on the concept of pseudo-differential equations. These are: the wavelet theory [7], Schrödinger type operators [8], dynamical systems [9, 11]. There is also a substantial amount of work on p-adic distributions [6]. To summarize, I still hope that this note, as limited as it is, distinctly demonstrates Sergio Albeverio mathematical intuition to recognize important problems and his talent to solve them.

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Interview to Sergio Albeverio

Barbara: Dear Sergio, we are very pleased that you agreed to do this interview for the Volume that has been dedicated to you, which is now almost finished and titled *Quantum and Stochastic Mathematical Physics. Sergio Albeverio—Adventures of a Mathematician.* As you know there were many discussions about the title: someone proposed *Adventures of a Physicist*, someone else proposed to maintain both terms, Mathematician and Physicist, other suggested to add also Philosopher. Certainly, in one title we can't introduce too many concepts; the idea of this interview came to explore something more about you and to know as much as possible about the impact you had in the scientific community. We will ask you some questions. The chairwoman will be Stefania Ugolini. The other participants are Astrid Hilbert, Elisa Mastrogiacomo and Sonia Mazzucchi. Paul Fischer will take care of technical issues. Now I give the floor to the chairwoman.

Stefania: It is a great pleasure to start with the interview. The first question is: could you give us a short summary of your life as a scientist?

Sergio: This is not an easy task. Let me try to give first a short answer: it has been both a passionate adventure (from an original strong will of knowing to the discovery again and again how little we can know) and a struggle (lot of work, trying not to forget the happy and sad sides of life outside the world of science). I certainly had a strong will that helped me in overcoming hard periods of insecurity.

In my teenage years it became clear that I was not really made for what seemed to be a perspective, from the side of my parents, namely that I would be studying at some technical school and later on taking over the small heating and plumbing firm in Lugano that my father was running; I was rather oriented towards more theoretical university studies. But finding out which subjects to study was not easy for me: when I was roughly between 12 and 15 years old I was fascinated by natural sciences, at one point it was chemistry that attracted me, but it ended soon, since the kitchen in our rather small apartment was not really appropriate for chemical, often bad smelling experiments ...; then came botany, but in dissecting flowers I developed a terrible

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hay fever and had to give up; the construction of small electric circuits and even a small radio were a bit more successful, but I discovered that it was really the physics behind them that interested me: quite decisive was the discovery of an antiquarian bookshop in Lugano, that had in a rather dark backroom, piled on the floor, several mathematics books. They were at university level (calculus, algebra) and in Spanish language. So I combined the study of them with a self learning of that language, both aspects fascinated me (and I loved Spanish language and became an enthusiastic reader of Spanish poetry).

Barbara: Do you still have those books?

Sergio: Yes! Sometimes I still consult them, to recall some formulae (and reviving the emotions of my first encounters with them when I was about 15). The fascination for mathematics those well written books gave me was reinforced by further readings, I remember particularly an article by A. Padoa on mathematical logic, in a further book purchased at the same place, "Enciclopedia delle Matematiche Elementari", as well as a book by E. Waissman, "Introduzione alla filosofia matematica", that I had hired from the Biblioteca Cantonale di Lugano. I also read some popularization books on relativity theory and non-Euclidean geometry, and soon after I was sort of proud to understand what the physics teacher at the Liceo alluded to when he called us in the Aula Magna to commemorate the departure of Einstein in April 1955. But in that year the summer was particularly hot and humid, some virus attacked the whole region, I got very sick, taken to the hospital with high fever and a very painful long-lasting pleuritis. I had to take a few months long recovery, part of it I spent at Nervi, a small town at the seaside near Genova. The beautiful nature there, the long walks, the light and wonderful sunsets helped my health but also brought me to develop other interests: I read with great enthusiasm a book on dodecaphonic music and the modern movement in arts and something happened in my development, in fact for a few years, until the end of the Liceo, I shifted completely my interests from science to arts, music, poetry, literature, languages—next to philosophy, psychology and sociology. Back in school, a few months after the new year of studies had begun, I was a very different young boy than I had been before. This change was also enhanced by a new student, Franco Beltrametti, who soon after joined our class, he was a couple of years older than me, and he was also very interested in those topics (and later, after studies of architecture, was to become a rather well-known poet of the Italian branch of the "beat generation", as Fernanda Pivano used to call it). With him I went on discovering the world of exhibitions, readings, art, cinema, theatre—at the expense of partly neglecting strict school learning. When the Liceo was over, July 1958, I found myself to be not so well-prepared for taking a decision concerning my future university studies. I was rather undecided, between choosing philosophy/psychology versus mathematics/physics. I consulted my mathematics teacher from the Liceo, an algebraic geometer whom I respected very much, Ambrogio Longhi: he reassured me by stating that he had no doubt I could successfully study mathematics/physics, but he made the error of adding "and why not also philosophy/psychology", which revived my hesitations. At this point my mother, who had a calm, intuitive, deep insight in human understanding advised me to get in contact with a woman psychologist she happened to know. So I went to her and she made me go through some Rorschach tests and, after looking at my early handwritten schoolboy writings, came to the conclusion that I could well study anything but, being more theoretically minded rather than suited for practical work, advised me to rather study mathematics/physics (where in her opinion one could more quickly reach satisfactory achievements without much practical work needed before entering a real career in areas like philosophy and psychology). So finally in the fall of 1958 I started my studies at the ETH in Zürich, in the “Abteilung Mathematik und Physik”. During all studies I continued to pursue my other interests, mainly in areas like poetry, arts, politics and societal issues, literature, languages, as well as psychology and philosophy (in the latter I had the luck to have as one of the teachers Ferdinand Gonseth, a mathematician and philosopher, founder of a new movement in philosophy of mathematics around the journal “Dialectica”).

Stefania: Could you briefly describe the moment in which you decided to dedicate to theoretical physics and started to conduct research in this field?

Sergio: For the first two years mathematicians and physicists were together, with the same basic courses. From the third year a choice between mathematics or physics as an area of specialization had to be taken, and I chose physics. In my case I think what determined the choice was my regained wish to better understand physics and nature, before going over to philosophy, the discipline I cherished most. And at that time I had the somewhat naive idea that mathematics could be added to my formation afterwards, being sort of a “self-contained discipline”. Also I was genuinely fascinated by the challenge posed by problems arising in the study of nature, whereas at that time I probably did not recognize as much challenge in natural problems posed within mathematics. So I chose theoretical physics.

At the “Seminar für Theoretische Physik” when I started the third year of studies there were Professor Markus Fierz (a former direct student of Gregor Wenzel and a coworker of Wolfgang Pauli, well-known for his work in quantum field theory) and Res Jost (who had studied Mathematics in Bern and physics in Zürich, also with Wenzel as a teacher, Jost ist well-known for his work on scattering theory and axiomatic quantum field theory). They attracted me by “living with the matter” they were teaching. I then went to Markus Fierz asking for a topic in mathematical physics in view of a master’s thesis, he said something like “you see, I have some white hair … but in mathematical physics there is David Ruelle, he has good topics to work on”, so I went to him (who was at that time Privatdozent at ETH), he gave me to study generalized Ising models (later on he gave me a paper that had just been produced in preprint form handling combinatorial aspects of such models). I worked hard and with David’s help I found a unified way to handle those models. And then knocked at the door of Fierz with the partly still just handwritten “Diplomarbeit” (master thesis). Fierz asked me to tell him shortly what I had done: I started describing the main results, that obviously he grasped immediately, and then he took over, explaining to me everything upstream, from the origins of the Ising model, and in which way it is a significant approximation of more realistic models; the whole ended up in a small, beautiful lecture, with quotations from the classics of literature (like Goethe’s *Faust*) and deep insights into the nature of science, his way of exposing and his vivid sense of the interplay between physics and mathematical formalism fascinated me. Suddenly he realized it was late and he had to leave: “What is the title of your work”? I hesitated since I hadn’t thought of it—then he took an ink pen from the pocket and wrote by hand “Generalized Ising models”. I told him that I would hand him a better typed copy, he told me to simply deliver it officially like it was, to save time to prepare my oral examinations.

After that summer I became Ph.D. student and assistant in theoretical physics, my interests shifted to quantum field theory, starting with writing up, with Martin Kummer, Fierz’s lectures on an introduction to quantum field theory and elementary particles. I enjoyed being at the “seminar of theoretical physics”, with the presence of scientists, friends that later would become colleagues, like Philippe Blanchard, Klaus Hepp, Walter Hunziker, Martin Kummer, Peter Minkowski, John Roberts, Ruedi Seiler, Walter Schneider, Robert Schrader, Walter Wyss, in addition to many visitors, including, at one point, Edward Nelson. Concerning the topic of my Ph.D. thesis I was advised to wait until Res Jost would come back from a long stay at the Institute for Advanced Studies in Princeton. Personally, I was trying to orient myself, being disappointed with the situation in that area at that time, between more physical but often only heuristic approaches to cope with divergences, various axiomatic approaches and the study of concrete but in one way or the other unsatisfactory models. Once back in Zürich, Jost proposed me to look at the problem of relations between Wightman and Haag–Kastler axiomatic approaches form the point of view of relating field operators appearing in those frameworks. This depends on controlling domains of essential self-adjointness of certain operators, and there was work by Borchers and Zimmermann concerning a sufficient condition for essential self-adjointness from which possibly one could learn some techniques. I did some work on an example showing that certain differential operators having similarity with Wick powers of relativistic free fields were not essentially self-adjoint. Jost was satisfied by the result, but he meant the original more general problem would probably take too much time of elaboration for a thesis, so he proposed another problem concerning scattering theory for a quantum mechanical multiparticle model (a prototype of models with point interactions I studied extensively later on). I liked very much to work on this concrete model and found, with Res’s help, an explicit solution, based on his important work on difference equations with meromorphic coefficients. For this solution I had to learn a lot of classical mathematics, including Riemann surfaces, Riemann–Roch theorem, algebraic geometry, finally the solution was expressed in terms of quotients of products of hyperelliptic functions, then simplified (by exploiting symmetries of the problem) to products of elliptic theta-functions. The mathematics I learned was going to enter, later on, in other work of mine, in completely different contexts, like the one of trace formulae associated to heat semigroups and the Schrödinger unitary group. Res was an excellent teacher, full of wit and quite direct in his expressions. He influenced me very much, as well as Markus, and I am very grateful to them for their generous help in difficult times. From them I learned how mathematical concepts form a unity (and I was proud of the positive judgement of Res when he wrote me in the 80ies on the work I was pursuing in infinite-dimensional analysis).

Barbara: Did you turn back to the problem he gave to you or not, later?

Sergio: Just a little. In fact in recent years I wrote papers (with Benedetta Ferrario and Minoru Yoshida) providing concrete results about an Euclidean version of the original essential self-adjointness problem: Euclidean methods have been very successful in the construction of non trivial low space-time dimensional models of relativistic quantum fields satisfying all Wightman axioms, on which I worked for many years and still I am working in various collaborations, as part of more general studies about singular partial differential equations. The original question can now be looked upon as of establishing relations between different constructions of these models. So although the original problem in four space-time dimensions is still open, solutions inspired by it are available for lower space-time-dimensional models.

Let me add a general remark: in a sense, some problems you meet in one stage of your development may come back in very different contexts, again and again. Although perhaps the original motivation might be less strong, since meanwhile the "hot problems" have changed, good problems, at the forefront of research, can still be stimulating in new contexts.

Sonia: You studied lots of problems during your career. Which topics in science impressed you the most and which one do you consider the most challenging?

Sergio: I found most challenging those problems that are in some sense natural, in the sense that they arise from some compelling context in either mathematics itself or a science like physics, biology, ecology, or even from socio-economical: and not just as opportunities for applying pre-existing mathematical methods. In mathematics natural problems can arise in a specific area, but again I find most interesting those whose solutions require joint methods of different areas …

Let me take the opportunity to mention some problems I investigated, starting from physics. Like many researchers of my generation, I spent a lot of time to understand whether a synthesis of relativity theory and quantum mechanics would be possible, at least at the level of methods and models. This goes back to work I started between London and Zürich (1968–70) and continued, especially in Princeton (1971–72), on scattering theory in a partly relativistic model, and soon later, first in Oslo (1972–77) and Naples (1973) I continued this research by looking more closely on the specific construction of relativistic quantum fields models. For this I went through the elaboration of areas of pure mathematics, including potential analysis, stochastic analysis, the theory of random fields, and stochastic partial differential equations, topics I also pursued in later years up to the present. In Oslo I started working on these topics with Raphael Høegh-Krohn (a strong collaboration that continued until his sudden death in 1988, just before he was going to be 50). Other collaborations with him included in particular the work on infinite dimensional analysis, in particular integration theory on infinite-dimensional spaces, with two aspects, the oscillatory integral side (Feynman path type integrals) and the probabilistic (Wiener-type) integrals (soon other coworkers joined us, among them Philippe Blanchard, Anne Boutet de Monvel, Zdzisław Brzeźniak, Philippe Combe, Andrei Khrennikov, Itaru Mitoma, Roger Rodriguez, Jorge Rezende, Madeleine Sirugue-Collin and Michel Sirugue, Ambar Sengupta, Oleg Smolianov, Victoria Steblovskaya, Bogusław Zegarliński; and you, Sonia, were very crucial in very numerous later developments and their final unification in a general theory of continuous infinite-dimensional integrals).

In another direction, early in Oslo with Raphael we discovered that the theory of Dirichlet forms on locally compact spaces, beautifully developed particularly by Masatoshi Fukushima, to provide a systematic framework of unification between symmetric Markov process and a large class of diffusion and jump operators, could be extended to infinite-dimensional spaces (my work with Raphael in the direction was then continued, with coworkers including Wolfhard Hansen Zhi-Ming Ma, Michael Röckner, Ludwig Streit in Bielefeld, and many other coworkers from other places and countries such as Masatoshi Fukushima, Alexei Daletskii, Hanno Gottschalk, Zbigniew Haber, Yuri Kondratiev, Yuri Kozitskii, Shigeo Kusuoka, Laura Morato, Turi Rozanov, Francesco Russo, Barbara Rüdiger, Song Shiqui, Stefania Ugolini, Minoru Yoshida, Jiang-Lun Wu …). This line of work also took up further extensions in mathematics, including semigroup theory, stochastic processes and stochastic differential equations involving other types of noises or other state spaces (e.g. manifolds, configuration spaces, non-commutative state spaces, non-Archimedean spaces, fractal spaces …). In all these developments many coworkers joined, too numerous to be mentioned separately, but let me mention a perhaps less-known component in representation theory of infinite-dimensional groups of mappings (a book with Raphael and Jean Marion, Daniel Testard and Bruno Torrésani, that appear in ’93 and has been continued also in recent work in my collaboration with Masha Gordina, Bruce Driver, and Anatoliĭ Vershik).

Let me also mention problems that arose in understanding the origins of quantum mechanics itself that sparked joint work on Ed Nelson’s stochastic mechanics which led to the discovery of what Nelson called Albeverio–Høegh-Krohn phenomena of confinement, further studied with Masatoshi Fukushima and Ludwig Streit.

The connection with stochastic mechanics also led (in work with Raphael and Philippe Blanchard) to unexpected applications to the understanding of the law of Titius–Bode for planetary orbits and other astrophysical regularity phenomena, pursued also recently by other groups (especially around Aubrey Truman in Swansea, and Jacky Cresson in Paris).

I also worked on the application of mathematics (ideas, methods and models) in other areas of science; in this I had the support of different interdisciplinary projects in Bielefeld, Bochum, Bonn (BiBoS), and in Locarno (CERFIM and ISSI). In Bonn I founded with Volker Jentsch an international Center for interdisciplinary research (IZKS) (having as main topic the investigation of complex systems and extreme events). In Trento I worked on problems of neurostochastics (mainly with Luca di Persio and Elisa Mastrogiacomo). In Mendrisio I had the great luck to work (from 1966 to 2009) in the “Accademia di Architettura” founded by Mario Botta (with projects on mathematics, architecture and urbanism).

In all these studies I was especially fascinated by how the power of abstraction of mathematics, its concepts and methods help in understanding the complementary aspects of those pairs like finite and infinite, discrete and continuous, order and disorder, that build up the complex texture of the world.

Barbara: In your opinion, can it happen that a problem that initially seems purely mathematical can later find several applications in physics?

Sergio: It is a rather rare phenomenon, but certainly it exists: for example both C. Maxwell and P. A. M. Dirac wrote their respective equations (for classical electromagnetism respectively for the motion of classical relativistic electrons), essentially for mathematical and aesthetic reasons, but those equations became the basis also of much successive physics. Another example is provided by hypergeometric functions, studied in the nineteenth century for their intrinsic mathematical interest; funding in the following century applications to the study of the quantum mechanical hydrogen atom; or even the conics studied in hellenistic times by Apollonius of Perga that found applications in Kepler–Newton's theory of planetary orbits … The creation by Norbert Wiener of a mathematical theory suitable for describing the natural phenomenon of Brownian motion is also a lucky case of a chapter of pure mathematical work motivated by natural phenomena, and becoming afterwards the basis for very numerous other applications in natural and socio-economical sciences. I like to view such discoveries as "crystals or gems of knowledge" that serve as generators of many further developments (in general such "crystals" have a lot of symmetries, and it is in my eyes a big challenge in philosophy of mathematics to investigate the deep reasons for their working so well).

Elisa: The scientific community believes that you have had a major impact on its evolution in many ways. What can you say, from your point of view, about your influence on the scientific community?

Sergio: I was very lucky to find a large number of great and enthusiastic coauthors on my path, in mathematics and the sciences: this is certainly connected with my enjoying collaborations, but I also firmly believe that research is basically not just a solitary undertaking, interesting problems are in every epoch sort of around, and, on the way of their study, you find outstretched hands joining in for their solution. I was also lucky to find many real friends among the coauthors, and sort of grew myself through the interaction with them. This was a main motor in my research, plus the inspiration I got in reading the work of some of the masters. In the present publication, at other places, it is described how the environment in Zürich, London (Imperial College, where I went to after my Ph.D., to work with Ray Streater and where, besides teaching a course on multi-particle problems in non-relativistic quantum mechanics (an area to which I had been introduced by Klaus Hepp and Walter Hunziker), I started to seriously study models of quantum fields, having to report in a series of lectures on the work by James Glimm and Arthur Jaffe that had just appeared). My stay in London ended abruptly since I rushed back to Lugano, due to the sudden worsening of the health of my parents. Whereas my mother had to be hospitalized due to frequent heart problems, my father's health deteriorated rapidly and he passed away in early summer of '68. I suffered very much, and had a very difficult period, trying at the same time to help by all means my mother, staying with her, taking up a teaching job at the local Liceo while also taking care of the small plumbing and heating enterprise that was owned by my father. My mother's death by heart failure in November that very year was a new devastating blow for me.[1] I had a terrible period of grief and depression afterwards in my hometown and I was only able to recover through the nearness and love of several persons, and in particular my future wife, Solvejg née Manzoni.

Over one year later I went with Solvejg to Princeton, where I found myself immersed in an environment of collaboration between mathematics and physics, meeting scientists like Arthur Wightman, Barry Simon, Elliot Lieb, Edward Nelson and also guests, like Raphael Høegh-Krohn (who invited me to spend a year in Oslo 1972–1973) and Gianfausto Dell'Antonio (who invited me to Naples where I spent the year 1973, the very same year Solvejg gave birth to our daughter Mielikki). In Princeton I also met Daniel Kastler who invited me to Marseille (1976–77). The two years in Princeton were decisive for my development. I first continued and brought to an end the work I had already started in Zürich under the influence of Klaus Hepp on spectral problems in non-relativistic quantum mechanics and on scattering theory in a model of scalar quantum fields interacting with (spinless) quantum particles with relativistic kinematic (Nelson–Eckmann model). In the meantime, I had become particularly interested in the study of analysis in infinite dimensions in relation with quantum fields. I followed, partly with Francesco Guerra (who had joined Princeton from Naples), a course by Edward Nelson on Euclidean quantum fields, and our exchanges with Raphael and Francesco on this also led to long-standing friendships with both of them. I described in a previous publication dedicated to Raphael how we became very close friends and coworkers, and we had a fantastic and very productive time, first in Oslo for almost four years in the period 1972–77, and then by continuous exchanges when I moved for permanent jobs to Germany. Our collaboration lasted until it was ended abruptly in 1988 due to Raphael's sudden untimely departure. Afterwards I had a long period of depression, and I recovered both thanks to my family and the intensive will to try and continue the research along the lines jointly developed with Raphael.

I had learned in particular from him to ask the students about what they wanted to do, their answers were different from epoch to epoch, mainly from topics connected, besides in some ways with mathematical physics (when I was in Bielefeld and Bochum) to biology and neural networks (when I was in Bochum), to mathematical finance and complex systems, as well as pure mathematics (in Bonn). I gave them accordingly problems motivated by their interests, keeping in mind that on one hand the best comes out when motivations are present and I was always convinced and still am that one of the most fascinating aspects of mathematics is how the different areas of applications are linked by internal ties, similar equations and methods finding applications to most disparate contexts. In this way, through the students I often also found a key to enter new areas of investigations.

[1] Let me take this opportunity to express my deep feelings of immense gratitude to my parents for having given me the possibility to follow my passion of knowledge, even though it caused so much hardship of separation and solitude for us as I had to go far away from where they lived and I had grown up.

As I mentioned before, the inner structure of my research is often led by an interplay between opposites (dialectic pairs). For example, to understand better the relation between discrete and continuum, in particular, in relation with the problem of infinities appearing in certain problems in physics, in Oslo with Raphael we started work on nonstandard analysis (that led to a book with the mathematical logicians Jens Erik Fenstad and Tom Lindstrøm) and to work on singular interactions in quantum mechanics (with two books, one with Fritz Gesztesy and Helge Holden, and, later on, one by myself with Pavel Kurasov, and a very fruitful and steady collaboration with Gianfausto Dell'Antonio, Rodolfo Figari, Alessandro Teta and their many coworkers in developing the area of singular interactions, that also included other friends in many other countries). Another main thread for me was variation of the underlying field of numbers, like quantum mechanics on *p*-adics rather than real numbers (and this led to outcomes in stochastic analysis, particularly in collaboration with Witold Karwowski and Kumi Yasuda, and the study of wavelets in other collaborations).[2]

Astrid: I would like to go back to the time, on the one hand, in Zürich and, on the other hand, in Princeton. How did you integrate these experiences? As you mentioned, the places and the interests of the people were different. How do you integrate these to become your own?

Sergio: I learned a lot through the experience with people I met at those places, and in a sense I understood myself and what I was looking for much better after having been in such places, and other ones, in many countries where I have been. I have the feeling on the other hand that it was always important for me not to lose the roots of where I come from, because I do not think that what I can express in science can be dissociated from what I feel, that is the reason why many collaborations of mine are connected with friendships; and it is also the reason why again and again I tried to maintain close connections with the places in the world that are connected with my origins, both in Southern Switzerland and in Italy. Speaking more generally, I think it is good for young people to go to other places, make new experiences, participate to

[2] There are many other areas in mathematics where I did some work but do not manage to discuss here due to limitations by time and by the very dynamics of the conversation. Let me take the opportunity to at least name some of them: classical and quantum dynamical systems, and ergodic theory; measures on spaces of mappings in relation with string theory; asymptotics (of integrals and solutions of differential equations); spectral analysis, functional analysis, partial differential equations stochastic processes on manifolds and configuration spaces; potential theory (commutative and non-commutative); spaces of generalized functions and relative analysis; Fourier integral operators and wavelet analysis; probability theory; random fields; kinetic theory and statistical mechanics; operator algebras, non-commutative geometry and quantum fields; fluid mechanics; polymer physics; topology (especially knot theory); fractal analysis; algebra (Leibniz algebras, braid groups, combinatorics; infinite-dimensional algebras and groups and their representations); non standard analysis number theory (automorphic forms, trace formulae, complex analytic methods) graphs and network theory; mathematical statistics; filter theory; variational calculus optimal deterministic and stochastic control, optimization theory; astrophysics; complex systems models in economics and finance, biology engineerings, social sciences, urbanism; epistemological and philosophical questions. In undertaking these studies I had the joy of cooperating with many scientists in very many countries, to which I express my hearty thanks; hopefully there will be other opportunities to present a bit more on at least some of these works.

international meetings. I would advise to listen as much as possible to what experts say, but at the same time keep the own autonomy of thought. In science as in society there are fashions, but they do not last long, and even though in our current society they play a larger role, one should not forget that they can, by their very nature, change very quickly.

Astrid: The time in Oslo was a time where you could identify best, thanks to Raphael Høegh-Krohn. You two did beautiful and outstanding mathematics. Your scientific relationship and your creativity had a special flame. What was its essence and what have you passed on to other collaborators?

Sergio: I am convinced that my encounter with Raphael was really exceptional, since we had at the same time a profound friendship and an exceptional concomitance of scientific interests and orientations. We were working very hard, the kind of intensity you can only have when you are young as we were. But it was also a special joy of collaboration and dreaming about the future we experienced (I wrote about this more extensively in a publication[3] dedicated to Raphael after his departure in 1988). In Oslo we were relatively isolated at that time (around the years 1972–75), the mathematics community in Norway was pretty small, in our area of work we had, before we started to get our own students, mainly contacts with few mathematicians and physicists from the countries we came from, but also mathematicians from Soviet Union, like Gelfand, Berezin, Dobrushin, Minlos, Sinai, Maslov, Pavlov, Vershik, and their coworkers. But those contacts were first only at the level of exchanging letters and manuscripts (that would take a long time to reach each other, due to the cold war that was going on. Later on, they were expanded by personal contacts and also our spectrum of contacts reached other countries in Europe and Asia).

We had to find our own way, our main interests in the years 1972–75 were in constructive field theory but at that time there were a strong competition between certain centers, mainly in the US, we had to find our problems of interest, without starting a direct competition with the strong groups at those centers. Our techniques were more probabilistic, influenced by our respective mentors and also by work of Irving Segal's school, especially by Leonard Gross and by Edward Nelson. Dirichlet forms were an example of the topics we decided to study …

Barbara: Did you start the theory of Dirichlet forms in Oslo?

Sergio: I discovered the book of Masatoshi Fukushima on Dirichlet forms (on locally compact spaces) in the library of the mathematics institute in Oslo.[4] It was so very well written and rich in details that it permitted us to quickly start extending the theory to the infinite-dimensional case, particularly interesting for what we had in

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[3] S. Albeverio: "On the Scientific Work of Raphael Høegh-Krohn", pp. 15–92 in: S. Albeverio, J. E. Fenstad, H. Holden and T. Lindstrøm (eds.): *Ideas and Methods in Mathematical Analysis, Stochastics, and Applications, in Memory of Raphael Høegh-Krohn (1938–1988)*, Vol. 1. Cambridge University Press, 1992.

(See also Vol. 2 by the same editors and publisher: *Ideas and Methods in Quantum and Statistical Physics*, 1992).

[4] I described this in an article mentioned in footnote 3.

mind to do with quantum fields. One of the first papers we wrote had "potential analysis" in the title. This was a lucky circumstance since a couple of years later a position in Bielefeld was announced in that area and I got the position (1977). Raphael and myself continued the strong collaboration also after I had left Oslo until his untimely departure (1988), with regular trips between our places and long joint stays in other places, especially in Bielefeld, Bochum, Leningrad, Marseille, Moscow, Naples, Paris, Rome, Wrocław and Warsaw. Concerning the development of the theory of Dirichlet forms we were joined by many students, colleagues and friends, especially in China, Germany, and Japan, but also e.g. in Canada, France, India, Italy, Mexico, Poland, Russia, Ukraine, the United Kingdom and the USA.

Moreover the strong relations between Dirichlet forms, martingale methods, Markov processes and more generally probability theory turned out to be a guide for the study of disordered systems (an area that I started developing also with students in Bielefeld, Bochum and Bonn, in particular Hannes Brasche and Werner Kirsch), and hydrodynamics (Margarida De Faria, continued later with Ana Bela Cruzeiro and Benedetta Ferrario), besides processes on manifold, Markov fields, quantum fields and strings (among my students in these area and at those places were Teresa Arede, Claas Becker, Hanno Gottschalk, Atle Hahn, Frederik Herzberg, Astrid Hilbert, Koichiro Iwata, Hannes Brasche, Stephan Mihalache, Sylvie Paycha, Haio Roeckle, Michael Röckner, Jörg Schäfer, Sergio Scarlatti). Let me stress that the relations between quantum fields and stochastic analysis have constituted a main thread of my work, through the years. This includes also my involvement in recent constructive approach to models of relativistic quantum fields using singular stochastic partial differential equations (stochastic quantization equations), following work initiated by Martin Hairer and Massimiliano Gubinelli (in Bonn I am working on this with Massimiliano himself, Luigi Borasi and Francesco De Vecchi, as well as in cooperations with Seiichiro Kusuoka, Song Liang, Hiroshi Kawabi and Minoru Yoshida, in Japan).

Stefania: What kind of testimony do you feel you leave to the new generations of scientists?

Sergio: As I said it is important in my opinion to be informed, to know what is going on, in particular also where the main stream goes, keeping however a certain distance from it, not to succumb to the temptation of running from one fashion to the next one. In fact, the judgement of importance of certain problems in science, and elsewhere, is to a large extent relative to the given particular historical moment. Especially nowadays we are under the pervasive influence of the media (traditional and so called social), opinions, popularizations that are often more interested in increasing their audience rather than being objective: very often small steps forward are presented as sensational events. My advise is to keep calm, work hard, seek the own path in science, develop the own judgement on the complex dynamics of research.

Also, one should not get discouraged when one is faced with a problem that one does not manage to solve; it could well be that it is not clearly formulated, so one can try to reformulate it, perhaps in different settings. And one should not be intimidated by authorities; listen to them carefully, but do not throw away your own work. It can also be the case that one simply does not yet have the right ideas to cope with the problem one has at a given moment, so sometimes it is better to put the problem aside and come back to it later looking at it from a fresh perspective. More generally, some problems can also not be ripe yet for a mathematical treatment, but later on they can get better into the focus of attention, because the mathematical and scientific community has advanced in the investigations of other problems and unexpected connections to the original problem have arisen.

Let me take as an example the study of stochastic perturbations of classical dynamical systems.

There was a time (about as Astrid Hilbert started the research that led in 1991 to her Ph.D. Thesis in Bielefeld, and later to joint publications with Axel Klar, Vassily Kolokoltsov and Edy Zehnder) where many mathematicians and mathematical physicists considered with some skepticism, as a kind of undue intrusion, the study of stochastic perturbations of classical dynamical systems, like say Newton equations or deterministic Navier–Stokes equations. Although there existed various books on a stochastic hydrodynamics, a stochastic approach was considered by many analysts at best as having only some heuristic value, the real problems of hydrodynamics, in particular understanding turbulence and instabilities being best understandable already in a pure classical setting. This opinion has now drastically changed, for instance stochastic initial conditions are accepted as natural since the support of attractors of interesting classical systems have been found to be singular, hence of the type of typical stochastic systems.

Historically such shifts of attention were quite frequent, it suffices to recall how Ludwig Boltzmann at the turn of the nineteenth century had to suffer from opposition to his ideas about kinetic equations because the existence of atoms and molecules themselves, as physical entities, was still doubted: the situation changed dramatically only a decade later, especially with Einstein and Smoluchowski theory of Brownian motion (the history is well presented e.g. in Ed. Nelson's book Dynamical Theory of Brownian motion).

In the study of the kinetic equations themselves on methods of the theory of stochastic processes have already found applications, see e.g. in work by Barbara Rüdiger and coworkers (in which I had the great pleasure to participate), and more is to be expected.

Let me add a more general comment on the rapidly expanding area of stochastic processes, as an example of the present situation. It is an area were traditionally the connections with applications has always been very strong. They started with the study of chance in gambling and early applications in social science (e.g. in N. de Condorcet work) and continued with Laplace's use of chance as a tool for taming the complexity of many particles systems, that gave origins to the methods of statistical mechanics.

It is known at least since Kolmogorov's axiomatic approach and Wiener's work on Brownian motion that, on the other hand, the theory of stochastic processes has also a deep pure mathematical side. It seems to me that it constitutes an example of an area of mathematics where connections with other areas are natural, and the mutual fertilization between theoretical aspects and applications are particularly strong. In my conversation I have already mentioned the relation of stochastic processes with classical dynamical systems, statistical mechanics, fluid dynamics, quantum mechanics and quantum field theory (the latter having nowadays a strong revival from our point of view through the connection with stochastic quantization equations and singular partial differential equations). All these connections extend to the geometric and algebraic frameworks and problems to which I also gave contributions (roughly changing the state space to have a geometric structure like a manifold rather than just being a Euclidean or linear space; or changing to a non-commutative state space; or else an algebraic state space, like for p-adic processes. The study of symmetries of systems of stochastic differential equations has also undergone strong developments, connected with Stefania Ugolini and Francesco De Vecchi and their coworkers).

Speaking of connections with other areas than mathematics and physics, let me mention that the theory of stochastic processes has interesting active applications and gets new stimulations also from areas like biology, ecology, hydrodynamics, meteorology, as well as socio-economic sciences. E.g. for the investigation of the brain in clinical patients that suffer from epilepsy (or neurodegenerative diseases like Parkinson's and Alzheimer's), models involving stochastic jump-diffusion processes have been considered. Here the conjunction of stochastic methods with those coming from the theory of complex systems has proven to be helpful. Also very interesting are inverse problems, where the characteristic of a stochastic neuronal network based on observations should be determined; such investigations require, in addition to good ways of modeling, considerable numerical and data analytic methods.

Here too the universality of mathematics makes that one can both concentrate on particular problems or try to find new methods, often profiting from what has already been obtained in other areas of applications. For this however collaboration between mathematicians and specialists of other areas is essential. Progress in all areas of knowledge is a social, communitarian enterprise, and a wonderful one!

Let me now take the opportunity to thank from the bottom of my heart my close friends Astrid, Barbara, Elisa, Sonia and Stefania for the hard work they invested in overcoming all difficulties concerning the conception and organization of the conference, and the book emanating from it, including this interview. First of all, they managed to overcome my reluctance to accept the homage, since I anticipated its realization would require too much work for them. They made everything possible for making the event and the book run smoothly and successfully, both from the scientific and human side. I am also very grateful to all lecturers, participants and contributors for expressing on this occasion their appreciation of my presence in the community of scientists. Throughout my scientific life, my work would not have been possible without the help of numerous students and coworkers I have had from many countries, let me express my deep gratitude to them.

And let me also take the opportunity to express my hearty special thanks to Solvejg and Mielikki, for all the love and support they always gave to me. Only with their help I was able to pursue my dreams in science, throughout all vicissitudes of life.

Summary of Others Interviews

We propose a brief summary of the conversations we had with Sergio Albeverio's colleagues and collaborators,[5] which give a glimpse into his multifaceted personality. These conversations took place during the long months of the pandemic and revealed to be an exciting and moving experience thanks to the participants who gladly shared their scientific and human memories.

(1) **On what occasion did you meet Sergio and in what form did you collaborate?**

Philippe Blanchard: I met Sergio more than fifty years ago at Eidgenössische Technische Hochschule (ETH) in Zurich. We both arrived in Zürich–Sergio from Ticino, the Italian part of Switzerland, I from the French side—and we worked there as de facto foreigners. At the time, ETH was still a small university with about 6000 students. In the Physics department there were no more than twenty of us. Sergio worked as an assistant in one of the courses I took. I want to mention the importance of Res Jost, an exceptional human, crucial for our scientific career and development. Not only was he an excellent scholar but also a wonderful person. Also, we were lucky to find ourselves in a very interdisciplinary department. Physics enabled Sergio to extend his mathematical visions and ideas. Physics, from time to time, achieves more than the pure, hard, and abstract mathematics and allows to extend it.

Hochstrasse 60 was un *endroi heureux et un espace extraordinaire*. The institute was located in a very narrow street that went downhill. I remember one time in the winter I parked my car there. Robert Schrader was there too. I put the handbrake on and went to the institute to discuss something with Res Jost. At some point two policemen came in and asked Jost if he knew a certain Philippe Blanchard. He pointed at me, and the policemen told us that I had done something truly terrible: my car had gone downhill and had crashed. Fortunately, nobody got hurt. However, they wanted

[5] As everyone will know, Sergio's collaborators over the years have been countless and we have not been able to include them all in this volume. However, we hope that everyone can recognize themselves in the words of those who have been interviewed.

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to confiscate my license and put me on trial. Jost looked at the policemen and told them that it did not make any sense to confiscate my license as nothing bad happened while I was actually driving. He said that my accident only happened because I did not park correctly and that, consequently, the only reasonable thing to do was to disallow me from parking. The policemen were so baffled and unprepared for this answer that they finally decided not to prosecute me.

Then Sergio went to Princeton and there too he worked on excellent projects. It was an environment full of creativity, but also pressure and competition. Finally, I also want to mention the importance of Raphael Høegh-Krohn for Sergio's career—and ours too. Sergio frequently visited Marseille where he collaborated with Raphael achieving important results.

Ludwig Streit: I first met my longtime friends Philippe Blanchard and Sergio Albeverio at ETH. Philippe was already there when I arrived, Sergio was working on his doctoral dissertation in Physics related to the 3-body problem with Markus Fierz. I arrived in Zurich particularly interested in two papers by Araki and Coester and Haag from the 1960s, which tried to give a rigorous mathematical description of the dynamics in quantum field theory (QFT) in terms of fundamental states instead of potentials. Again, no examples for their theory were found, but I was fascinated by their approach.

I remember a member of the financing committee for our first research project (at ZiF in 1975)—a poet, I believe—asking: "Why do you propose a project between mathematics and physics? One understands well that physics needs mathematics, but what can a mathematician gain from the insights of a physicist?". One of our colleagues replied that he really enjoyed working with physicists because of their intuition. Sergio was formed at ETH, in an environment dedicated to physics, which always requires a lot of imagination; there he developed his great intuition as well as his rigor. Markus Fierz was Sergio's advisor for his thesis. He was also a *homme de lettres*, fascinating, with a very distinct sense of humor, and always open. One day, while walking up from the main physics building to the lecture halls, Fierz asked me: "Do you know by any chance how one measures cosmological distances? Come to my office and I shall give you a private lecture about it". Actually, he could have given me a private lecture on a baroque author as well. I see a lot of Markus Fierz in Sergio and I know that Sergio adores his teacher.

Francesco Guerra: I met Sergio when I arrived in Princeton in September 1970. I was immediately impressed by his great humanity. In particular, he generously and spontaneously helped me and my family in the settlement. At the time he was a visiting researcher in Princeton, and I could appreciate his wide culture and his lucid understanding of the physical and mathematical aspects of the hot problems in theoretical and mathematical physics, but it was his calmness, his gentle smile and his practical sense that impressed me most. In Sergio there is a spontaneity and empathy toward all the people he comes into contact with. I also met his wife Solvejg who is deeply involved with poetry and painting. On me she had an impact of the highest human and cultural value.

At that time Edward Nelson held a high-level special course on Euclidean quantum field theory in the Mathematics Department. Sergio advised me to take the course. Nelson started with the approach of Kurt Symanzik, one of the pioneers on Euclidean methods. I attended the first lectures, then I negligently left the course, because I already knew Symanzik quite well. So, I was absent when Nelson began to develop his own highly original and powerful approach, including Markov fields, Euclidean fields and the connection with the fields in Minkowski space-time. Nelson's approach was completely new, full of deep ideas, waiting for a full development and possible applications. And I was not there. Then, I attended a talk by Nelson in Princeton in April 1971 and suddenly I understood that his ideas had a strategic value. So, I immediately started searching for the content of the lost course. Sergio's help was crucial. His personal notes of the course were a true masterpiece. He was able to capture Nelson's thinking with great efficacy and depth. It is amusing to me that I learned Nelson's Euclidean theory from Sergio's explanations and his notes. This is only an example, but my involvement with the application of Euclidean methods in constructive quantum field theory entirely started from this.

Gianfausto Dell'Antonio: I remember meeting Sergio in Princeton when he was a young researcher there. I saw him again in Naples in 1973 where he had come to teach for six months. He was working on constructive field theory. Then, there were endless encounters. Sergio is able to present the most complicated works in a simple way. We share sixty-five years of friendship. His journey has been exceptional.

Rodolfo Figari: In 1971, as part of our thesis work, Chiara Nappi and I were investigating a persistent model in Constructive Quantum Field theory. We learned that Sergio Albeverio—at the time in Princeton with E. Nelson—was working on scattering theory in a quite similar way. We wrote to him asking for technical help and he immediately sent us some very clear unpublished notes with the indication that should they not be useful, we could use the back of the manuscript pages to write notes or draw. Since then I got used to such kind of understatements by him. Invited by Gianfausto Dell'Antonio, he spent 1973 in Naples working at the Theoretical Physics Institute. It was probably the least productive year of his scientific career apart from a few papers, which he prepared together with Raphael Høegh-Krohn spending hours on the phone. The following year Sergio moved to Oslo and I followed him a few months later.

Fritz Gesztesy: I met Sergio in the late summer of 1980 in Bielefeld. I was Humboldt fellow at the University of Bielefeld and Sergio returned to Bielefeld to visit Ludwig Streit, who was sort of my Ph.D. advisor. Sergio came with a guest. Now you can almost guess who that was. Raphael Hoegh-Krohn. We all know he is one of the longtime collaborators of Sergio's. Raphael was a bear of a man—imagine the biggest Russian bear—and he instantly broke the ice by putting his arm around my shoulders, which was quite impactful because in those days I was really shy and that really changed everything. Sergio was in the room, but he was shy too and Raphael was the only one who talked at that meeting.

After the meeting was over, we agreed that I should visit Sergio in Bochum. So, a bit later I took a train to Bochum and my life changed forever. We became lifelong friends and we collaborated intensively for the next ten years. Together we wrote many papers and a book on point interactions which appeared in 1988. It was a success. The second edition appeared in 2005 and it was translated into Russian in 1991.

Sylvie Paycha: I was studying in Paris and I wanted to do a Ph.D. in something related to mathematical physics, in particular about quantum field theory, but I had no idea what it really was. I was working on analysis at that time, on Schrödinger operators as a matter of fact, and then Anne Boutet de Monvel suggested to me some people, Sergio was one of them. I did not know him nor Bochum. I inquired about Bochum and it looked like a very dull, gray, rainy and industrial place … Well, I went there and Sergio was very welcoming. I remember going into his office and him presenting a whole blackboard for one topic and another whole blackboard for another one. I could not understand a thing about either board but I was supposed to choose between them and I really did not know what to do. Eventually I chose one, it was a stochastic-probabilistic approach to string theory. He was working on that with Raphael Høegh-Krohn.

Alessandro Teta: I met Sergio in 1984 in Bielefeld (at BiBoS). In 1985 I started working on singular perturbations of the Laplacian. I participated at the wonderful conferences of Ascona and Locarno, which were democratic conferences, where everyone spoke for the same length of time. Although we have never written a work together, we have always been friends and very close scientifically.

Pavel Kurasov: I remember precisely when I met Sergio for the first time because it was at a conference in Dubna in 1987, and I remember trying to collect myself and go to speak with him. But it was Sergio who came to me and we discussed many things. For me it was a great experience. Then, I met him a second time five years later, passing by Bochum, and again he came to me. What was surprising is that he immediately recognized me. In addition, I was surprised that he was always surrounded by dozens of visitors, if not more, and I wondered why he spent so much time this way. With my current experience I cannot fathom how he managed to find time for everybody. I can say that he changed my career completely because only after collaborating with him I started to develop independent research; he taught me how to *swim*. Each time you meet him, you get dozens of new problems to work on.

Andrei Y. Khrennikov: As many Soviets, some time ago [after the Soviet Union's dissolution] I had big problems and I was about to leave science because all my friends either moved to business, died, or became billionaires. At that time, I was in China and there I met a colleague who told me: "Why don't you study Feynman integrals? It may be interesting for Sergio!". So, I wrote him a letter from Beijing. Later, after we became friends, Sergio told me "I have never read letters from people I do not know and especially if these people are asking something. Nevertheless, I read yours because it is not every day that you receive a letter from China written by a Russian". This combination—a Russian from China—seems to have helped me to become a sort of *son* in Sergio's eyes. Then when I finally met him, I found that he had been deeply studying one of my papers and that he had read all the papers we had sent him. It was amazing. After this, Sergio invited me to come for an Alexander von Humboldt fellowship and this opportunity was my way to science because otherwise I would not have been a scientist anymore, since in Russia there was no possibility. And I would like to underline that I was not alone. Sergio has really helped many, many scientists from Russia, Ukraine, and all the other republics of the former Soviet Union in that really terrible time. I don't know where he found the strength because it took huge efforts and a lot of energy on his part. But he helped us.

When I arrived in Bochum, I found such a friendly and unusual scientific school. I have many contacts, but I have never seen this special vitality in many international contexts. Sergio's *school* was unique. As I emphasized in many interviews, I was impressed by his great knowledge of science. I would compare him with Leonardo da Vinci—the Leonardo da Vinci of our times. I work with many scientists and they are brilliant in their small area of interest, but, up to now, I haven't met anybody with such a huge overview of science as a whole. I have spoken with him about Feynman integrals, about the p-adics, about many different things and he always knows everything. According to me this is the greatest part of his personality.

(2) **What is, in your opinion, the main impact that the collaboration with Sergio had on your scientific development?**

Yuri Kondratiev: Sergio belongs to the older generation of great mathematicians like Skorokhod. The role of Sergio in my scientific life is decisive and special. I would like to stress that Sergio was not only my teacher, I think of him as my symmetrical counterpart, but also a prominent example of a great scientist with broad areas of interest. I am always happy to discuss with him, not only about particular mathematical problems, but also about several aspects in psychology, philosophy, history, physics, etc. The main point with Sergio is clearly his attractive and friendly personality as well as his nice style in the discussions with colleagues and students.

One of the essential properties of Sergio is that he is always open to new ways, new programs and, contrary to many very good mathematicians, he is not concentrated only on technical questions. Some people are completely focused on one particular topic and related technical problems and do not see the general picture. Sergio has the absolutely fantastic property of always seeing symmetry and unity, of seeing deep motivations coming from physics. That is very impressive. And for me it was essential because he supported my attempt to understand the relation between infinite dimensional analysis and statistical physics. His was really a crucial influence. I can say that my scientific taste was essentially shaped by Sergio, by his relation to science as a central point of our life, not science as a job. For Sergio, science is the main content of life. It is my honour to call Sergio my teacher and friend.

Michael Röckner: I wouldn't be the scientist, probably not even the person I am today without Sergio. He had so much influence on my scientific life, but probably also on the development of my personality and character. He was just a pure inspiration from the first day. So, when I started to follow his lectures, I was immediately fascinated by the way he explained the subject and I started to work very hard, because, even if I did not understand much of what he said, I was aware of his deep knowledge. In addition, the way he presented each topic was somehow motivating to deepen my own knowledge.

I would also like to mention what I call Sergio's *sense of respect*. Sergio is an adorable and very nice person. Everybody knows this, but I really want to point out that he has respect for any piece of science made by any person in the world. So even when I was just a student, he always respected me and tried to understand my thoughts. And he did the same thing with other students too. I think this is very important, in particular for young people.

Sergio is not the type of person who thinks that Mathematics has sense only if it has consequences in applications. For Sergio, mathematics has value in itself. He always fascinated me and the other students—even the very, very young—because he would not shy from teaching us really difficult theories. Once, for instance, he gave a course on abstract potential theory, and we got lost immediately. However, he really motivated us to study such an abstract and advanced theory and he passed along the message of its great value.

Another thing I learned from him is the joy you can get from collaborating with somebody else on mathematics. Consequently, I wrote almost all of my papers with co-authors because it's so important to have feedback from colleagues.

Lastly, I have always admired Sergio for his immense networking all over the world, as a scientist, but also as a person. I have never met a scientist who has as many contacts and as many close relationships inside our community as Sergio. He is, of course, very international—he speaks six or seven languages, maybe more—and I admire the openness he approaches people with, no matter which country they are coming from.

Tom Lindstrøm: I was only 22 when I met Sergio and I didn't really have any prior experience with research and things like that. So, I can't really imagine what my mathematical life would be like without his influence. I think he has influenced me on many levels. Scientifically, I have never really got out of his shadow. I mean, so much of what I did was developing the ideas that I got from him during the first few years. On a personal level, I'm impressed by his kindness, his generosity, and his enormous breadth of knowledge. He seems to know almost everything, I have learned a lot from him. You ask him anything and he has this way of explaining to you thoughts by making them *resonate with you* and your way of thinking. I am very impressed by this quality.

Fritz Gesztesy: I also was struck and I'm still struck by his kindness. Ours is a cutthroat business and he's one of the great exceptions to that. I have already mentioned earlier that after meeting Sergio a whole new universe opened because of the international collaborators and friends around him, a huge community. Before I met him, I was fairly isolated in Graz, Austria, but after I met him I found myself every other year abroad until I finally moved to the United States. I began to *blossom*, one might say, after that initial contact with Sergio.

Ludwig Streit: After our time in Zurich, we were supposed to meet again at a conference organized by Res Jost in 1968 in Varenna, where I also met *Sergio's brother* Raphael Høegh-Krohn for the first time. Varadhan spoke on Edwards model, a problem concerning (weakly) non-intersecting random paths. In Varenna I asked Res Jost again about quantum dynamics in terms of the ground state. As a simple example: what if one does not define the dynamics of the harmonic oscillator by its ground state but, e.g. instead by the first excited state? Jost's answer was: "Good question". The problem was still open. Sergio and I met again in 1975 at a conference in Marseille where Sergio and Raphael presented their fundamental paper on dynamics in QFT using Dirichlet forms. Then, I did not understand the details of their work but, nevertheless, it answered my question. Shortly after we organized a one-year research program in Bielefeld with Sergio, Raphael, Philippe and many others attending. Here we came back to work on my initial question concerning the dynamics in QFT and I was very excited to hear that Sergio was still thinking about the problem. In particular, he suggested defining the non-relativistic quantum mechanical dynamics in terms of wave functions. This led to our paper on distorted Brownian motion. Even though this paper was nearly finished, Sergio still invited me to collaborate. I am glad that we ended up writing several sequels to this paper to which I was able to make more serious contributions. Through our work on the dynamics of quantum theory, Sergio introduced me to Dirichlet forms and their uses in quantum dynamics, giving rise to non-perturbative dynamics. The theory of Dirichlet forms provides also a link to that of Markov processes, so I was motivated to learn more about stochastic analysis—again a result of Sergio's impact. I must mention that I met my wife through Sergio: she was one of his Ph.D. students. As you can see, Sergio's influence on my life is undeniable. While I started out trying to correctly formulate QFT in Zurich, I eventually made contributions to stochastics. In particular, with others we elaborated the theory of white noise analysis guided by the insights of Takeyuki Hida. Later on, it was Freeman Dyson who suggested reformulating constructive quantum field theory in terms of white noise. This led to further collaborations with Sergio, Michael Röckner, and Jürgen Potthoff. Together, we were able to describe bosonic relativistic quantum field theory in terms of white noise. In the 2000s Sergio and Michael Röckner were able to construct Dirichlet forms based on the Varadhan–Edwards measure with associated Markov processes giving rise to certain stochastic differential equations. In such a long time many things have moved in mathematics and our paths have crossed. Each time there was yet another impulse from Sergio to move things on, not only for me but for the whole scientific community.

Philippe Blanchard: If I had to describe Sergio very briefly, I would say that he *is first and foremost a scientific explorer*. He was never scared of anything and worked in all kinds of different areas with great success. The width of his knowledge is incredible. The IAMP (International Association of Mathematical Physics) has a subject classification for theoretical physics and his work is important in all of the four subareas. Sergio is almost like a brother to me, in particular, I am the godfather of his daughter who was born in 1973.

Sergio is interested in absolutely everything scientific—for example, I am thinking about problems in Astronomy and Cosmology—but he is also fascinated by the Arts and their relationship to Science. He is a real scholar, eager to learn and discover.

Pavel Kurasov: In St. Petersburg there was the attitude to solve original problems but not in full generality, and people liked to say "the Germans will do the rest". When I was a Ph.D. student I was studying delta-interactions and that was the way Sergio and I started to collaborate. It was Sergio who taught me how to generalize; before meeting him, I would solve each concrete problem individually. He showed me that one can look at the same problem from a more general point of view. Maybe this is a more Western point of view. Sergio showed me how to work. When I was working in Bochum it was impossible to go home before Sergio; those who were in Bochum probably remember that each evening he went home with big bags full of books that he managed to read during the night. It was an honour to help him carry those books to the car. And, somehow, I was also forced to work at least as much as Sergio did. It was a great school.

Sylvie Paycha: Sergio had an enormous impact on me. I was on board for string theory, bosonic string theory, together with Sergio Scarlatti, and some of the questions underlying this thesis have guided my research since then. One of the questions is implicitly in the background; it is how to deal with infinity. In that work we adopted a stochastic approach. Since then, I've tried other approaches, but it's always the same question.

From Sergio I've also learned generosity. He's an extremely generous scientist, generous with his time and his knowledge. This quality is especially precious nowadays when everybody is in a hurry, in such a competitive field. He is one of the rare persons who Frenchies would call *humanists* for the enlightenment of his mind, generous enough to get interested in all aspects of science, well beyond the official borders of mathematics, physics, philosophy, but also architecture and biology. This broadness is now very rare because it takes time to get interested in different things that are not directly in one's way. Nowadays the pressure is so great that not many people take the time. It was not always easy for Sergio to take the time, I think, but he did and I appreciate this propensity a lot. I've learned so much from his behavior.

Andrei Y. Khrennikov: The main lesson we got from Bochum was that we learned to be brave and look at mathematics and science from a very general perspective. Since then, I have not been afraid to start something new. For example, in Sweden, I started to study the foundations of quantum mechanics and quantum information, and, step by step, I developed these subjects. Also I spent a lot of time looking at how Sergio organized his research and I tried to copy his methods even if I wasn't able to replicate all of them, because Sergio had his special know-how, in particular he knew how to speak with people. For example, I organized very big conferences on the foundations of quantum mechanics and I tried to contact people, very often high-level people, by copying Sergio. All this I learned in Bochum. I absorbed part of his personality, which is very special and which, I think, contributed to his success in creating a network of scientists. When I got money and I tried to invite people from

Russia, Belarus, Ukraine, I could speak with one or two people per day at the most. After that I was tired. What Sergio would do instead was really amazing because he could work at the same time with ten, fifteen people and on totally different topics. I had a great example in Sergio and my career in western science had a *milestone* in Bochum, this small town that maybe is not Cambridge or Oxford, but for me it was better, since it was a great town with a great scientific school.

Francesco Guerra: It was very interesting to see how Sergio moved after the introduction of Euclidean methods. It was the beginning of the use of stochastic methods in quantum field theory, which in itself has nothing stochastic about it. I was working with Barry Simon and Lon Rosen on the Glimm and Jaffe program of constructive quantum field theory, through the exploitation of the new Euclidean methods. Our results were very interesting. On the other hand, Sergio found a deep scientific understanding and collaboration with Raphael Høegh-Krohn, another exceptional person, forming a team of the highest scientific level, launched with the application of probabilistic methods to a large spectrum of problems in theoretical and mathematical physics. After leaving Princeton I began to be also involved in the study of Nelson's Stochastic Mechanics, a very intriguing theory aiming at the description of quantum mechanics in the frame of a purely probabilistic setting. Nelson theory was probably ahead of its time and not fully understood and accepted by theoretical and mathematical physicists. In particular, while working in Salerno with my brilliant student, Patrizia Ruggiero, we discovered a new very deep connection between Nelson stochastic mechanics and Euclidean Field Theory. I must say that the continuous flow of valuable information coming from the work by Albeverio and Høegh-Krohn was comforting and very encouraging for us. They built very comprehensive and general frames, as for example the Dirichlet processes, and gave interesting application to a variety of different research fields, including for example the structures emerging as a result of the planetary winds.

Alessandro Teta: Talking about relationships with Sergio means running the risk of rhetoric. He has been a source of inspiration for my scientific work. I learned a lot from him. He is a *references wizard*; he is always able to say who did what. However, his influence went even further. I met his family, the private aspects of him. When we meet, not only do we talk about mathematics but also philosophy and the fundamentals of science. His wide culture combines with his meekness, openness to dialogue and confrontation. His way of playing the role of the scientist is what inspired me; he believes science and scientists have a fundamental role in the attempt to *improve the world*.

(3) **What is, in your opinion, the main impact that Sergio had on science and the scientific community?**

Sylvie Paycha: I think his legacy is really the *interdisciplinarity* that characterizes him, that impregnates any audience who listens to him and anybody who works with him. I would say this open mindedness to interdisciplinarity will have a great impact on research in the long run. He thinks in an interdisciplinary way, bringing all the aspects of one problem together. This approach inevitably opens new paths.

Fritz Gesztesy: Sergio left his mark very broadly on quantum mechanics, constructive quantum field theory, singular perturbation theory and point interactions, infinite dimensional analysis, Dirichlet forms, stochastic processes, Feynman path integrals, statistical mechanics and also non-standard analysis. And this is just scratching the surface because, in many ways, he has been a *universalist* all his life. I guess this is due, as Sylvie said, to the humanistic nature Sergio has.

Michael Röckner: If I can use an image from physics, I would look at Sergio as an *energy operator*. First of all, his spectrum would not be bounded. Definitely, it would not be a pure point. Definitely, it would have an enormous essential part. And finally, spectra are usually subsets of complex numbers with two dimensions. I think that Sergio's spectrum is not a d-dimensional set, but an infinite dimensional set.

Of course, among the areas to which Sergio has contributed most, there are infinite dimensional analysis, Feynman path integrals, Dirichlet forms, infinite dimensional group theory, non-standard analysis, Schrödinger operators and, in particular, point perturbations. There is absolutely no question; all these areas have benefited from his work. But Sergio is too nice a person to advertise his work.

I also want to stress what we call *Sergio's family*. All of us belong to this huge family, which also comprehends many other people. When he organized one of his many conferences he would always link his family to other groups and families. Or sometimes other groups were simply sort of swallowed by his big welcoming family.

When I was a young scientist, this was very important. We had contacts with other groups like the one led by Paul Malliavin and Terry Lyons and other many very famous people who knew and liked Sergio. This was because the atmosphere at his conferences was always friendly. Sometimes, conferences can be very competitive, people can fight. This never happened with Sergio. Everybody loved to participate at his conferences because there was respect between all participants. Everybody knew everybody and if ever there was some friction somebody could always mediate. It was fantastic. Sergio's huge network was not only in Europe, but also in the United States, in Russia, in Japan and in China. He had a lot of connections also in Africa, in particular Tunisia.

Giuseppe Da Prato: This is not a simple question because Sergio has worked on many topics, so I will answer by not considering all his production, but only the one with which I have had contact. Some of his fundamental results are the theory of Dirichlet forms in infinite dimension and their application to stochastic differential equations in infinite dimension. I think he did other important things, for example in mathematical physics.

Philippe Blanchard: At the time of my studies in France, mathematicians, at least at ENS, did not consider probability theory to be a real part of mathematics. Sergio did a lot of work in explaining the importance of probability to the mathematical community. In particular, he emphasized that probability was not only benefiting from mathematical axiomatization but that mathematics itself was to be revolutionized by probabilistic ideas. That is an undeniable impact of Sergio's on the mathematical community.

I want to stress the depth of his interactions with collaborators. In a certain sense if two or three decide to work together the final result is a kind of *convolution* between them and the strength of their interaction. Due to his character, Sergio interacts very strongly. Sergio emerged from a group of researchers who produced brilliant scientific works. Additionally, I want to recall the respect and affection, the interest, the passion, the kindness and the generosity that people experience with Sergio.

Ludwig Streit: Another prominent feature of Sergio's is his passion for the interaction between Arts and Science. He wrote a lot about their relation. It is actually easier to list the subjects Sergio was not interested in than the other way around. That runs in the family. He found a wife who fits into the relation between Arts and Science.

Yuri Kondratiev: When I was a student my teacher gave me a preprint by Sergio and Raphael. This preprint was about Dirichlet forms in linear spaces, and it did completely change my understanding of the subject. I understood that infinite dimensional analysis is in some sense really dangerous, because in each point you can go in infinite directions. This work by Sergio and Raphael was a problem of zero-time quantum fields, scalar quantum fields and corresponding generators and so on. It was a beautiful combination of analysis from one side, and quantum field theory from the other. With Sergio and Michael, we did several works related to these developments of infinite dimensional analysis, but in concrete directions motivated by statistical physics and stochastic dynamics and so on. Sergio was able to translate some ideas, which are very obscure for mathematicians, from theoretical physics into mathematical language. And after that, such problems became a big part of our mathematical world, and we could work on them. Also, this beautiful possibility to translate questions from one area to another is absolutely exceptional and not at all obvious for many experts. So, I can say that one of the central advances that Sergio made in science is really the organization of concrete, deep connections between applications, precisely between theoretical physics and mathematics in a completely rigorous sense.

Pavel Kurasov: After listening to a speaker at a conference, one usually goes to them saying: "You know, I have a paper on this subject, you can read it!". However, the usual reaction from Sergio was different. He would say: "Do you know that there is a paper on this subject by this guy and that guy? You should read it!"

The second thing I learned from Sergio is to listen. When Sergio speaks usually the first impression you get is that he is completely relaxed, apparently speaking without putting serious thoughts into his words. But if you speak to him at length, then you will understand that you have to listen to his every word. He has such a particular way of speaking that is completely different from other people.

Andrei Y. Khrennikov: Sergio and Raphael's preprint was, I think, one of the first mathematical papers on Feynman integrals, which before were considered something mystical. In his career Sergio has published plentiful results and papers in different domains of science, but not once did he say "Look at what I did!". Even though many people spend their lives working on a paper of the same caliber, Sergio never boasted. He also gave a really great contribution to infinite dimensional analysis and p-adic analysis. But his name is also well known in many other domains.

Gianfausto Dell-Antonio: Sergio had a huge impact on science and what he did is exceptional. Sergio is a very open person beyond mathematics. He is a humanist, a Renaissance type personality. It should be considered a great merit that he has many faces, not just the scientific one. Sergio chose mathematical physics as a profession, he did his job like everyone else, but he is not just a mathematical physicist!

Rodolfo Figari: Peculiarities of his way of working are his calm, patience, and generosity. The number of his collaborations is enormous, an attitude that sometimes has generated criticism. In fact, he agreed to work in many sub-disciplines giving visibility and motivation to many collaborators and working in research groups with few or no links to the rest of the scientific community. Some people have found this attitude to be in conflict with the ideology of "excellence", but there is no doubt that many of Sergio's collaborators reached levels of excellence and, last but not least, that many mathematical physicists love him as a teacher.

Francesco Guerra: Sergio gave a very important contribution to science, characterized by an impact on the new generations that is bound to grow over time. There are valuable results and directions of research that will have more and more relevance. I would like to mention the important evergreen problem of the global Markov property in Euclidean field theory, which Sergio has pursued for many years. I must say that generosity is essential in the context of culture. Sergio's personality is comparable to the one of a Renaissance man, full of *magic content*. In relation with the contemporary world of research and with young people in particular, Sergio showed a remarkable generosity, which I consider a really rare and precious positive feature. He has had such an influence on research that it has certainly brought him great satisfaction. Sergio has chosen to always be himself: to always be kind to people.

Tom Lindstrøm: His main influence spreads through all his students and all the people he has collaborated with. They have been *spreading* the words, the methods, the attitude, the kindness and the willingness to engage with anyone who is interested in a subject. So, his largest contribution is through all these people and all the interactions he had with them and the ways he has been combining methods from different fields and creating new possibilities, seeing new openings that other people hadn't seen before.

(4) **Do you think that there are Theories or Topics in science that we can identify with Sergio's name?**

Francesco Guerra: Surely the possibility to give precise attribution to method and results is a very fundamental problem in the History of Science: who did what? There is a kind of international *Albeverio's school*, including, among other things, problems such as Dirichlet forms, Feynman path integrals (according to Albeverio and Høegh-Krohn approach), stochastic dynamics, strongly localized interactions. These last topics point to very important issues. For example, they are at the heart of the explanation of the slow neutron effects, studied by Enrico Fermi. The impetus that Sergio has given may have future important developments. In particular, I believe that his followers should dedicate the highest attention to the global Markov property, a topic that will gain more and more weight in the future developments of the theory. The fact that the future does depend on the past through essential elements in the present, but for some intrinsically purely stochastic unstructured external influence, seems to be at the core of any realistic rational interpretation of Nature. Let me end with a sentence that after half a century still reverberates in my mind. A seminar by Sergio in Princeton (1971) began with the sentence: "We live in a time in which everything we say must be rigorously proven". A methodological manifesto that links Sergio to the best and strictest Galilean tradition and is part of his legacy to future generations. To the young people I say: "Be consistent with yourself, as Sergio was".

Giuseppe Da Prato: I will limit myself to my point of view. I identify him as the inventor of the theory of Dirichlet forms in infinite dimension. The passage from the finite dimension to the infinite dimension is not at all obvious—actually, it is very difficult. There are some detractors, who say that this passage has no value because the results on Dirichlet forms in infinite dimension are obtained up to functions of zero capacity, which are a very large set in infinite dimension. I would say that this opinion has been proven wrong by the huge amount of work that has been done using Dirichlet forms in infinite dimension.

Ludwig Streit: This is a difficult question since I am not competent enough to assess the full and vast scope of Sergio's influence. I will thus only focus on my personal perspective. From what I could appreciate, his impact is most felt in the use of Dirichlet forms in physics and beyond. I also want to emphasize Sergio's tremendous influence on the development of the University of Bielefeld in the 1970s. With Philippe and Sergio, we founded the BiBoS Research Group in Stochastics at Bielefeld that saw the rise of very eminent mathematicians who were Ph.D. students of Sergio's. However, the list of schools that were strongly influenced by the interaction with Sergio is very very long.

Michael Röckner: Sergio was one of the founding fathers of non-standard analysis and he was also a founding father of the mathematical theory of Feynman path integrals. I think there is no doubt about it. These are the two topics I would like to put first. On a bit smaller scale, I would like to mention the paper with Raphael from 1977. This was a breakthrough paper on the theory of Dirichlet forms, also in infinite dimensions with connections to quantum field theory. The original idea there was that a Φ_2-quantum field can be identified with the path measure of a Markov process that comes from a Dirichlet form for the time-zero quantum field. This means that if you can prove the global Markov property for the Φ_2-quantum field, then it would be related to the Markov process that comes from the Dirichlet form for the time-zero quantum field. It would be exactly the path measure that has the time-zero field as invariant measure. It's unproven up to today, but it's a fantastic idea. I would definitely put this paper under the most influential papers of Sergio's in mathematics. I can also mention his results in the theory of Dirichlet forms and about the characterization of the existence of corresponding Markov processes. There are also many other things about particle systems and so on.

Yuri Kondratiev: Several directions in mathematics, mathematical physics and physics, were initiated for the first time by Sergio. For example, Sergio and Raphael were probably the first to use Feynman integral representation in quantum statistical physics. The same for the case of the so-called perturbation. This direction was initiated by Sergio and Raphael in 1975. After their paper, a lot of people worked in this direction by using this technique, but the crucial idea of representing quantum states with Feynman integral was already there. According to me, he is the father of several absolutely new research lines in mathematical physics and in pure and applied mathematics. One peculiar aspect of Sergio's activity can be very well formulated by the sentence: "I am working in applied physics. I apply physics to mathematics". To apply physics to mathematics, on a philosophical level, on the level of understanding, it is absolutely a beautiful ability of Sergio's.

Pavel Kurasov: For me the main contribution is, of course, the theory of point interactions. I just checked that actually the book by Albeverio, Gesztesy, Holden and Høegh-Krohn has been cited two thousand three hundred sixty-five times, it's impressive. This book was useful, people read it. But what was impressive was Sergio's attitude towards mathematics. He taught me that you should think about mathematics as pursuing the truth and not what you have done yourself. There are a lot of ways you can teach mathematics: you can give lectures, exercise lessons, and so on, but there are very few ways you can teach how to be a human. You have only to show your personal approach to mathematics, nothing else. This is what Sergio was doing to many of us.

Andrei Y. Khrennikov: The mathematical theory of Feynman integral is maybe his greatest contribution on the level of ideas. Forty years ago, the Feynman integral was something so tricky, which was used by physicians without any mathematical meaning. The second main contribution is non-standard analysis, and, in some way, it was very exciting for Sergio.

Gianfausto Dell'Antonio: Sergio founded the study of point interactions. Raphael and Sergio's book is fundamental. There are also important articles on non-standard analysis by Sergio and collaborators on Feynman integration and Dirichlet forms theory.

Alessandro Teta: Fundamental contributions of Sergio's are: Schrödinger's operators, stochastic analysis and quantum field theory. I would also like to mention his ability to train many young people. During the crisis of the Soviet system, Bochum was a meeting point for scientists and provided economic support as well. For many years it has been at the center of a network of great human and scientific relationships. Sergio also created the conditions for the development of science. He supported and encouraged a precise method of doing science, without excessive competition and aggression.

Rodolfo Figari: Just because he has never worked alone it is difficult to identify him with a research group or with a specific research field. One could say: quantum field theory, zero-range interactions, Feynman path integrals and Markovian random fields.

Fritz Gesztesy: Sergio characterizes himself as someone who focuses his efforts on mathematics, but he does that by building bridges between different areas of research, including applications to physics, astrophysics, biology, medicine, economics and philosophy. So, again, this makes it very difficult to single out special examples, but I guess his universal approach to sciences can be considered his principal legacy.

Sylvie Paycha: I think that it would be a shame to reduce his great impact to contributions to some specific areas. Sergio's characterizing feature is that he sprinkles all over science. The way he proposes his ideas is very delicate. He is a very sensitive person and that's also why he can work with so many people. I think he has left his footprint in so many areas—a very delicate but very deep footprint thanks to his meekness.

(5) **How do you think Sergio will impact the future and the next scientific generation?**

Pavel Kurasov: Sergio has seventy-eight registered students and two hundred three descendants. It's impressive. He has a direct registered influence on two hundred three people. Maybe if one adds another one hundred of Humboldt fellows that he supervised with all their descendants, it will be something like one thousand young, or not so young people, that he influenced directly. Because he changed us, he changed our students. This is his personal influence, and it is very important.

Luciano Tubaro: I think that Sergio's impact on the young generations is direct, not just indirect. In front of me now there are five women who are currently collaborating with him.

Giuseppe Da Prato: One of his qualities is the extraordinary ability to collaborate with many people.

Fritz Gesztesy: His future impact on generations is realized through us, his collaborators and his students. There are 36 Ph.D. students, 110 diploma theses and 20 habilitations that were written under his direction. So you could say that all of our students are already impacted by what he has instilled in us. I think that having this world of scientists around him that all feel very closely connected to him is really the lifeline for his future impact.

Rodolfo Figari: His work has left a great legacy for young researchers. Sergio's search for an encyclopedic knowledge and his multi-disciplinary gaze on science constitute a way of countering the exasperated technicalities of current scientific research.

Alessandro Teta: I hope that his example and his human and scientific skills will influence young people. The hope is that there is a generational passing of the baton. His collaborators should carry on Sergio's way of working. This is a great responsibility.

Andrei Y. Khrennikov: The main lesson I learned from Sergio is to teach our students that science is central in their life, because now for many people science has become just a joke or a business. It's not an easy thing to convey, but we should try to do it.

Philippe Blanchard: I would also mention his exceptional talent in educating students. Sergio is a good professor according to my definition, i.e., one is a good professor exactly when one stops trying to be one. A good professor has to create ties of friendship with his scholars. Sergio was able to do so and that quality of character is not given to everyone. Once his students trusted him, he started to bring forward his words of wisdom. He went on this way and he pursued an exceptional education. This is how Sergio was able to pass down his knowledge in the most efficient way. To summarize, Sergio is at the same time an *amazing teacher* and a great scientific explorer.

Michael Röckner: I would like to mention that Sergio is still influencing young people; in Bonn he is working with very young people again. The duty of his former students is to pass this tradition on to the next generation. It is not easy to do it with the same intensity Sergio has, but we have to try. There are a few essential aspects of how he managed to put together such a large family and such a large network. Maybe when life will turn normal again, we should think of having a revival of these big conferences. For instance, we used to have a conference every three years in Europe or in the United States to keep the groups together. They were very successful, but this is maybe a little bit dormant now. We have of course small conferences, but I think in the future we should try to organize some of these big ones that combine the areas; this is Sergio's legacy. On the other hand, I am not so worried about the future because *Sergio's family* is a large family, and it is still multiplying.

Ludwig Streit: Look at Sergio to understand how you can be an extremely successful scientist and, at the same time, be kind with all your collaborators, in particular the young ones.

Yuri Kondratiev: We already mentioned that Sergio works very intensively and productively with young students. And for all of us, he is a teacher. We are then extremely thankful to Sergio for our high scientific education and for having produced a lot of very nice Ph.D. students during the last several years and for continuing to teach. I tried to somehow extend the approach of Sergio to science and life to my students. So, if we are children of Sergio, they are all grandchildren of Sergio. And as they become professors, their students will continue to be part of the family. So I am sure that the tradition created by Sergio will be continuing.

Acknowledgements

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We especially thank the numerous people, who contributed to the scientific content and by highlighting Sergio's social engagement in science in their articles, interviews, and personal discussions. At this point we also gratefully mention those whose silent efforts played a central role in producing this monography, e.g. Ester Angela Guglielmi and Silvia Purpuri for proofreading of the additional material, Françou Blanchard for her work with the texts supplied by her husband, Peter Kuchling for his technical support. This is surpassed by our extreme gratefulness to Solvejg Albeverio Manzoni who donated a wonderful etching, considerably enriching this volume.

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Philippe Blanchard's Homage to Sergio Albeverio

This text is a homage to my dear friend Sergio Albeverio. Sergio's talent has manifold facets. The diversity and the difference in the level of difficulty of the treated examples perfectly reflects the variety of different mathematical and physical methods to be met in Sergio's multiform works.

Sergio is incomparable amongst mathematicians, simultaneously a perceptive physicist, and also a bit of an artist. Mastering passion and creativity, he depicts and translates reality into equations. His trace is permanent in his domains.

His capacity to work is extraordinary. He is able to establish relationships amongst manifold subjects of interest, revealing close connections between seemingly unrelated questions. This is reflected in the immensity of his written production.

Sergio has worked on many questions concerning both mathematics and physics. To speak of Sergio's contributions is to speak of the history of mathematical physics since 1960.

As Cécile Dewitt-Morette formulated: "between physics and mathematics there are bridges, which means there are architects, brick layers, and also pedestrians". Sergio is one of the best representatives of a "*savanturier*" (*savant aventurier*), "Scientist adventurer".

Beyond mathematics and physics, he is interested in all contemporary science (of his era) and by extension, through his vast knowledge, in all of science.[6]

For Plato, Descartes, and Kant, without mathematics, there will be no philosophy, and for Heisenberg, without philosophy there will be no good physics.

Science means having an idea, and hence phantasy, and proving whether this idea is right is vital. The process is democratic and indeed the notion of democratic

[6] I described on the occasion of Sergio's 60th birthday some of these aspects in more details, see pp. 37-39 in Stochastic processes physics and geometry: new interplays. A volume in honor of Sergio Albeverio, Eds. F. Gesztesy et al., Can. Math. Soc. ad AMS, 2000.

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thought emerged in Greece at the same time as mathematics. In science, facts have the power of veto.

While the Greeks were unable to overcome chance, Pascal knew how to subject chance to the laws of science. Pascal laid the foundation for the modern theory of probability rooted in geometry. Pascal holds that

Truth is that "Idea" consisting of three facets:

- Grasp Truth by enquiry
- Prove it when grasped
- Contrast it with the False [7]

According to Badiou "real life" is to live under the guidance of an "idea" (*Éloge des Mathématiques*, Flammarion, Café Voltaire, 2015).

For Sergio, "real life" is a blend of Pascal's spirit of geometry and Voltaire's art of persuasion.

[7] On peut avoir trois principaux objects dans l'étude de la vérité; l'un, de la découvrir, quand on la cherche, de la démontrer, quand on la possède, le dernier, de la discerner d'avec le faux quand on l'examine (Blaise Pascal, Géometrie I).

CV of Sergio Albeverio

I was born in Lugano (Ticino, Southern Switzerland) on January 17, 1939. My parents were Olivetta Albeverio née Brighenti (born in Rivera, Ticino), and Luigi (Gino) Albeverio (born in Luino, Italy). My mother was a tailor and then housewife, my father was a plumber, then owner of a small heating and plumbing firm.

I am married since 1970 with Solvejg Albeverio Manzoni, born in Arogno (Ticino), an artist and writer. We have a daughter, Mielikki (Aglaja, Olivetta) Albeverio, born in Lugano, Dipl. Social Sciences.

I grew up in Lugano, and after the Liceo, from 1958, I studied Mathematics and Physics at the Eidgenössische Technische Hochschule (ETH) in Zürich, ending in 1962 with a Diploma (Master) in theoretical physics, in the area of statistical mechanics (generalized Ising models) under the supervision of Markus Fierz and David Ruelle. I then continued at the "Seminar für Theoretische Physik" as Assistant of Markus Fierz and Res Jost, getting the Dr. rer. nat. (Ph.D.) under their supervision with work on mathematical physics (quantum mechanical scattering theory).

During the academic year 1967–68 I was a lecturer at Imperial College (IC) London (with joint support of the Swiss National Foundation (SNF) and IC), following invitations by Paul Matthews, Abdus Salam (Theoretical Physics) and Ray F. Streater (Applied Mathematics). For family reasons (sickness and untimely departure of my parents) I returned already in the course of spring to my own town, taking up a job as mathematics and physics teacher at the local Liceo.

This was followed in the fall of '69 by a fellowship of the SNF, first at ETH Zürich and then at Princeton University, where I then spent the years 1970–71 in the Departments of Mathematics and Physics (as researcher associated with Arthur S. Wightman and Edward Nelson).

From 1972 to 1977 I have been a visiting lecturer or professor at various universities and institutes:

- University of Oslo (1972 and 1974–1977) with Raphael Høegh-Krohn,
- University of Naples (1973), Institute for Theoretical Physics, with Gianfausto Dell'Antonio,

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- University and CNRS, Aix-Marseille (Luminy) (1976–78), with Daniel Kastler and Raymond Stora.

During this period, I also gave lectures at various other institutes, including summer schools at the University of Boulder (1971) and at the Advanced Study Institute, Denver (1973), and a winter school in Karpacz (1975). I also lectured at the IV. International Symposium on Information Theory, organized by R. L. Dobrushin (1976 in Leningrad), a conference in Torún (1976), and a Cours de III$^{\text{ème}}$ Cycle at the École Polytechnique Fédérale de Lausanne (1977). Further stays and lectures I gave in that period include a course on quantum mechanics at the Postgraduate School of the Italian Mathematical Society in Catania (1977) and a series of lectures at the University of Bielefeld and the Institute for Interdisciplinary Research (ZiF), where Ludwig Streit organized a research year in 1975–1976.

In 1977 I got tenure as Associate Professor at the Department of Mathematics, University of Bielefeld (in the section Analysis/Potential Theory). I remained associated with that University in various ways since then, including the founding, with Philippe Blanchard and Ludwig Streit, of the Research Center for Stochastic Processes, BiBoS (initially funded by a five-year project of the Volkswagenstiftung), and the membership in the scientific committee of ZiF.

In 1979 I became a full professor (Chair for Probability and Mathematical Physics) at the Ruhr-University Bochum. There I remained until 1997, when I moved to the University of Bonn as a full professor with the Chair of Probability and Mathematical Statistics, where I still working (since 2007 as Professor Emeritus).

Both in Bochum and Bonn I have been member of several Collaborative Research Centers (SFB) of the DFG and directed other collaboration projects of the DFG, and of the European Community. I have also hosted over 30 scientists of the Alexander von Humboldt Foundation, including several of their prize winners.

Moreover, I had with Hans Föllmer a long-ranging collaboration project of the Volkswagenstiftung with scientists coming from countries of the former Soviet Union (in particular Russia, Ukraine, Uzbekistan). In Bonn I was also a founding member of an Excellence Cluster in Mathematics which then gave rise in 2006 to the Hausdorff Center for Mathematics (HCM), of which I have been first a steady member, and then since 2020 an associate member. Furthermore, I founded and directed with Volker Jentsch the Interdisziplinäres Zentrum für Komplexe Systeme (IZKS) of the University of Bonn.

Besides carrying out the activities directly connected with my positions in Germany, I had also long ranging commitments as visiting professor with several universities and institutions. In particular I developed scientific collaborations and activities in various countries within Europe, and outside, including long stays in China (Beijing, Wuhan), Japan (Fukuoka, Hiroshima, Katata, Kyoto, Kyushu, Kumamoto, Osaka, Sendai, Tokyo, Nagoya, Nara), Mexico (CINVESTAV), Russia (St. Petersburg, Moscow), Saudi Arabia (KFUPM Dharhan), Tunisia (Tunis).

A particular intensive collaboration with Universities and Institutes in Italy has lead to further longer stays, especially at following places: Trento (University and CIRM), Verona (University), Trieste (SISSA), Pisa (Scuola Normale Superiore and

Centro de Giorgi), Universitá dell'Insubria (Como and Varese), Universitá di Roma (La Sapienza and Tor Vergata), Universitá degli Studi, Milano (where since 2020 I am also member of the Collegio di Dottorato).

In Switzerland from 1996 to 2009 I have been Professor and Director of the Mathematics Department at the Accademia di Architettura in Mendrisio (founded by Mario Botta); in 2015 I have organized and directed (with A. B. Cruzeiro and D. Holm) a research semester on Stochastic Geometric Mechanics at the Bernoulli Center of Mathematics, EPFL. With my region of origins, Canton Ticino, I also maintained other connections, particularly as a scientific director of the research center CERFIM in Locarno. With this center we organized several international conferences in Locarno and Ascona, on themes from stochastic processes, geometry, quantum physics, to fractals and biology.

I have given plenary talks at over 250 international conferences and workshops. My publications include over 950 articles in scientific journals, 12 monographs, and 40 volumes of proceedings. I have been/am on the editorial board of over 20 specialized journals or book series and served as scientific advisor of various research centers (in Czechoslovakia, France, Portugal, Sweden, Switzerland, United Kingdom). I have supervised over 110 Master's Theses, supervised or co-supervised over 70 Ph.D. Theses and 25 Habilitations.

Honors and Awards

- 2021: Socio Straniero della Accademia Nazionale dei Lincei (Foreign Member of Accademia dei Lincei, Rome)
- 2021: Member of Academia Europaea (London)
- 2019: Conference in Honor of S. Albeverio for his 80th birthday, Verona (publications in preparation, Springer Verlag)
- 2018: Doctor honoris causa, University of Stockholm
- 2015: Director (with A. B. Cruzeiro and D. Holm) of Research Semester on Geometric Mechanics, Variational and Stochastic Methods, Centre Interfacultaire Bernoulli (CIB), École Polytechnique Fédérale (Lausanne)
- 2011–2015: Excellence Chair Professorship in Mathematics, KFUPM (Dhahran)
- 2005: Plenary lecture for the 90th birthday of K. Itô, University of Oslo
- 2003: Prize for an interdisciplinary collaboration project on Extreme Events, University of Bonn
- 2002–2010: Long-term Professorship (part time) "per chiara fama", University of Trento, and Research Leader of Project "Neurostochastics"
- Since 2002: Listed in International Statistics Institute (ISI) (Thomson) "Highly Cited Researchers"
- 2002: Doctor honoris causa, University of Oslo (on the bicentennial of the birth of Niels H. Abel)
- 2000: St. Flour Lecture (on the 30th anniversary of St. Flour Probability Lectures)

- 2000: Conference in Honor of S. Albeverio for his 60th birthday, Max-Planck-Institute for Mathematics in the Sciences (Leipzig, 2 volumes of Proceedings on Stochastic Processes, Physics and Geometry, Eds: F. Gesztesy, H. Holden, J. Jost, S. Paycha, M. Röckner and S. Scarlatti CMS Conference Proceedings, AMS 2000)
- 1999: Plenary lecture at Wiener Memorial Symposium, East Lansing
- 1998: Nomination for Professorship "per chiara fama", University of Rome II (Tor Vergata)
- 1995: Solomon Lefshetz Memorial Lecture, AMS and Mexican Math. Society, Mexico City
- 1992: Max-Planck-Award in Mathematics (with. Z. M. Ma and M. Röckner)
- 1988: Plenary lecture at the International Congress of the Association of Mathematical Physics (ICMP), Swansea; likewise in 1986 (Marseille), 1983 (Boulder), 1981 (Berlin), 1977 (Rome)
- Over 250 invited lectures at international conferences in mathematics, physics, and applications and over 30 invited lectures in other areas of culture at various Centers including ZiF, University of Bielefeld; Centro Monte Verità, Ascona, Academia Vivarium Novum, Frascati; Centro Internazionale Insubrico "Carlo Cattaneo" e "Giulio Preti", Varese.

Links to Publications by Sergio Albeverio

- Webpage at Institute for Applied Mathematics, University of Bonn: https://wt.iam.uni-bonn.de/albeverio/publications/
- Webpage at the Hausdorff Center for Mathematics, University of Bonn: http://www.hcm.uni-bonn.de/people/profile/sergio-albeverio/
- Entry at MathSciNet: https://mathscinet.ams.org/mathscinet/search/author.html?mrauthid=24435
- Entry at Mathematics Genealogy Project: https://genealogy.math.ndsu.nodak.edu/id.php?id=23869
- Profile at ResearchGate: https://www.researchgate.net/profile/Sergio-Albeverio
- Entry at Wikipedia: https://en.wikipedia.org/wiki/Sergio_Albeverio
- Entry at Accademia dei Lincei (Roma), https://www.lincei.it/it/soci/categorie-scienze-fisiche or https://www.lincei.it/it/content/albeverio-sergio
- Entry at Academia Europaea (London), https://www.ae-info.org/ae/Member/Albeverio_Sergio

File folder with list of Sergio Albeverio's publications at Cloud service for organizing science in Nordrhein-Westfalen (Cloud-Service für den Wissenschaftsbetrieb in Nordrhein-Westfalen) sciebo:

https://uni-bonn.sciebo.de/s/z37qK7ztMo7OQtZ

It contains pdf and source files originally arranged by Timo Weiss.

File folder with Sergio Albeverio's CV at Accademia dei Lincei via Link:

https://uni-bonn.sciebo.de/s/x1MRANySz1WQ863

A. Hilbert et al. (eds.), *Quantum and Stochastic Mathematical Physics*,
Springer Proceedings in Mathematics & Statistics 377,
https://doi.org/10.1007/978-3-031-14031-0

Photos Selected by Sergio Albeverio

Photo of Sergio Albeverio during the Conference in the honor of his 80th birthday "Geometry and Invariance in Stochastic Dynamics", Verona, Italy, March 25–29, 2019

A. Hilbert et al. (eds.), *Quantum and Stochastic Mathematical Physics*,
Springer Proceedings in Mathematics & Statistics 377,
https://doi.org/10.1007/978-3-031-14031-0

Photo of Sergio Albeverio's daughter Mielikki Albeverio honoring Sergio with music during the dinner of the Conference "Geometry and Invariance in Stochastic Dynamics", Verona, Italy, March 25–29, 2019. The dinner took place at Villa San Michele in Valpolicella. Mielikki played and sang "La Tarara", from the collection "Cantares populares" by Federico Garcia Lorca, one of the poets Sergio likes most.

Non avevo purtroppo potuto partecipare all' evento di Verona e dedico alle autrici di questo bel volume per Sergio una delle mie incisioni, un po' "alchemica", intitolata "Il giullare gioca insidiosamente con il sole".

Solvejg Albeverio Manzoni

Etching by the artist Solvejg Albeverio Manzoni, for this volume