# Origin of the Laplace Force Applied to a Current-Carrying Wire Immersed in a Magnetic Field

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The macroscopic force (called the Laplace force) acting on a wire carrying an electric current placed in a magnetic field is a consequence of the Lorentz force acting on each charge inside the wire. Typically, the Laplace force is explained as a *magnetic* force resulting from the interaction of the moving charges with the external magnetic field. Such an interpretation, however, is too simplistic and does not take into account all the interactions between the various charge populations inside the wire. This leads to a series of paradoxes that might hinder the understanding of this subject. For instance, a magnetic force cannot do any work, while a current-carrying wire in a magnetic field represents the paradigm to understand the working principle of an electric motor. Here, we will solve this and other inconsistencies by showing, with simple arguments comprehensible to undergraduate students, that the Laplace force is instead an *electrostatic* force.

The study of the forces acting on electrically neutral wires carrying electric currents has represented a cornerstone in the development of modern electrodynamics.[1] The pioneering work carried on in this field by André-Marie Ampère is of paramount importance in the history of science and technology and represents a fundamental step in the process of bridging the gap between electricity and magnetism and reaching the awareness that these phenomena are just different manifestations of the same fundamental laws of nature.[2]

Ampère's work anticipated the modern interpretation of electrodynamics in terms of electric and magnetic fields, which arose in the theories of Michael Faraday and later received a full mathematical description by Lord Kelvin and James Clerk Maxwell.[2] Actually, in the 1865 formulation of his field equations, Maxwell related the magnetic forces to macroscopic electric currents.[3] At that time, it was not evident that currents were connected to the movement of charged particles inside a conductor. Joseph J. Thomson was the first to attempt to derive the expression of the magnetic force acting on a moving charged object from Maxwell's field equations but included an incorrect factor of ½ in his formula. Finally, in 1895, Hendrik Lorentz derived the modern form that carries his name and includes both the electric and the magnetic contributions to the electromagnetic force on a point charge.[2] In the International System of Units, the Lorentz force assumes the following expression:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}, \tag{1}$$

with $q$ and $\boldsymbol{v}$ being the charge and velocity of the particle, and $\boldsymbol{E}$ and $\boldsymbol{B}$ the electric and magnetic field, respectively.

However, describing the complex charge dynamics in real materials by just considering the Lorentz force resulting from the interaction with externally applied electric and magnetic fields is often inadequate. Typically, the time and spatial response of the charges of a many-body system is reproduced by more complex models that need to take into account the reciprocal actions of all the charges within the material. Examples are the Boltzmann equation, which results from the semiclassical electron dynamics in a crystalline solid,[4] or the equations describing charge (and mass) transport in electro-magneto-hydrodynamics.[5]

Here, we will show that the interpretation of the Laplace force as a *magnetic* force, typically found in textbooks, is misleading. For a better understanding of such a phenomenon, one should consider the internal distribution of the charges inside the wire induced by the applied magnetic field. In the following, we will show that the Laplace force needs to be considered as an electric force. This conclusion solves a series of paradoxes and provides a better comprehension of this issue, which is not as straightforward as it might seem.

## Magnetic force on the charges moving inside the wire

Let us first evaluate the magnetic force acting on the ensemble of charges moving inside a conductor immersed in a magnetic field $\boldsymbol{B}$. We can assume that all the particles moving inside the wire carry the same charge $q$. The following conclusions can be generalized in a straightforward manner if more than one population of charged particles contribute to the total current. As usual, we will also consider a wire in steady-state conditions, i.e., with a charge density not depending on time.

The total magnetic force $\boldsymbol{F}_{\mathrm{mag}}$ on the ensemble of the moving charges is obtained by adding the individual magnetic forces acting on each particle[6]:

$$\boldsymbol{F}_{\mathrm{mag}} = \Sigma_i\, q\boldsymbol{v}_i \times \boldsymbol{B}, \tag{2}$$

where $\boldsymbol{v}_i$ is the velocity of the $i$th particle inside the wire. By applying the continuum approximation and integrating over the wire volume $V$, one obtains

$$\boldsymbol{F}_{\mathrm{mag}} = \int qn\langle\boldsymbol{v}\rangle\times\boldsymbol{B}\,dV = \int \boldsymbol{J}\times\boldsymbol{B}\,dV, \tag{3}$$

with $n$ and $\langle\boldsymbol{v}\rangle$ being the density of carriers and their average velocity in the volume $V$, respectively, and $\boldsymbol{J} = qn\langle\boldsymbol{v}\rangle$ the corresponding current density. We can then explicitly express $dV$ as $dV = d\boldsymbol{\ell}\cdot d\boldsymbol{S}$, where $d\boldsymbol{\ell}$ is the length element aligned parallel to the wire axis and $d\boldsymbol{S}$ the cross-section surface element. Assuming that the cross section of the wire is so small that the magnetic field $\boldsymbol{B}$ does not significantly change across it, we can finally transform Eq. (3) as follows:

$$\boldsymbol{F}_{\mathrm{mag}} = \int(\boldsymbol{J}\times\boldsymbol{B})(d\boldsymbol{\ell}\cdot d\boldsymbol{S}) = \int\boldsymbol{J}\cdot d\boldsymbol{S}\int d\boldsymbol{\ell}\times\boldsymbol{B}. \tag{4}$$

To derive this expression, we have implicitly assumed that $\boldsymbol{J}$ and $d\boldsymbol{\ell}$ are parallel as a consequence of the small cross section of the wire. The first integral in the last member of Eq. (4) is the flux of the current density $\boldsymbol{J}$ across the section of the wire

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and corresponds, by definition, to the current intensity $I$. We thus obtain the following textbook expression[6]:

$$\boldsymbol{F}_{\mathrm{mag}} = I \int d\boldsymbol{\ell} \times \boldsymbol{B}. \tag{5}$$

It would seem reasonable to associate $\boldsymbol{F}_{\mathrm{mag}}$ with the Laplace force $\boldsymbol{F}_{\mathrm{L}}$, i.e., with the macroscopic force acting on the wire. A plethora of experiments,[1,2] among which many can be performed with simple instruments available in high school level laboratory classes, have indeed confirmed the empirical validity of this assumption, and a common trend in textbooks[7,8] consists in presenting the passages leading to Eq. (5) as a "demonstration" that the Laplace force directly stems from magnetic forces acting on the moving charge carriers inside the wire.

We would like to stress, however, that such an interpretation is incorrect and misleading for two reasons (at least). First, it is not consistent with the wire being in a stationary state. Second, a magnetic force cannot do any work,[6] while Laplace forces acting on a moving coil in which an electric current is forced are exploited in electric motors to generate mechanical power. In the following, we will demonstrate that the Laplace force originates instead from the *electrostatic* interactions between charged particles inside the wire, which removes any limitation on the possibility of doing work.

## The Hall effect

If we assume that there are no forces acting on the moving charges other than the magnetic one discussed in the previous section, we need to conclude that the carriers will be forced to move in the direction of such forces with an average drift velocity $\langle \boldsymbol{v}_\perp \rangle$ *perpendicular* to the wire axis, generating a current density $\boldsymbol{J}_\perp = qn\langle \boldsymbol{v}_\perp \rangle$. $\boldsymbol{J}_\perp$ drives charges to the surface of the wire, where they accumulate. By indicating the surface charge density as $\sigma$ and the unit vector normal to wire surface as $\boldsymbol{u}_{\mathrm{n}}$, since the charges cannot escape through the wire surface, charge conservation leads to

$$\frac{\partial \sigma}{\partial t} = \boldsymbol{J}_\perp \cdot \boldsymbol{u}_n \neq 0. \tag{6}$$

As anticipated in the previous section, this expression violates the assumption of the wire being in a steady state, leading to the first inconsistency outlined above. Actually, the surface charges create an electric field $\boldsymbol{E}_{\mathrm{H}}$ that will rapidly grow to eventually reach a value that will prevent any further charge accumulation, restoring the steady state, with $\boldsymbol{J}_\perp = 0$ and $\langle \boldsymbol{v}_\perp \rangle = 0$. This condition is reached when the average force applied to the moving charges vanishes:

$$q\boldsymbol{E}_{\mathrm{H}} + q\langle \boldsymbol{v} \rangle \times \boldsymbol{B} = 0, \tag{7}$$

which corresponds to

$$\boldsymbol{E}_{\mathrm{H}} = -\langle \boldsymbol{v} \rangle \times \boldsymbol{B} = -\frac{\boldsymbol{J} \times \boldsymbol{B}}{qn}. \tag{8}$$

The presence of the electric field $\boldsymbol{E}_{\mathrm{H}}$ perpendicular to the wire axis gives origin to the so-called *Hall effect,* consisting in the generation of a voltage difference $V_{\mathrm{H}}$ (the Hall voltage) across a current-carrying wire immersed in a magnetic field.[9] In the simple geometry outlined in Fig. 1,

$$V_{\mathrm{H}} = \frac{IB}{qnt}. \tag{9}$$

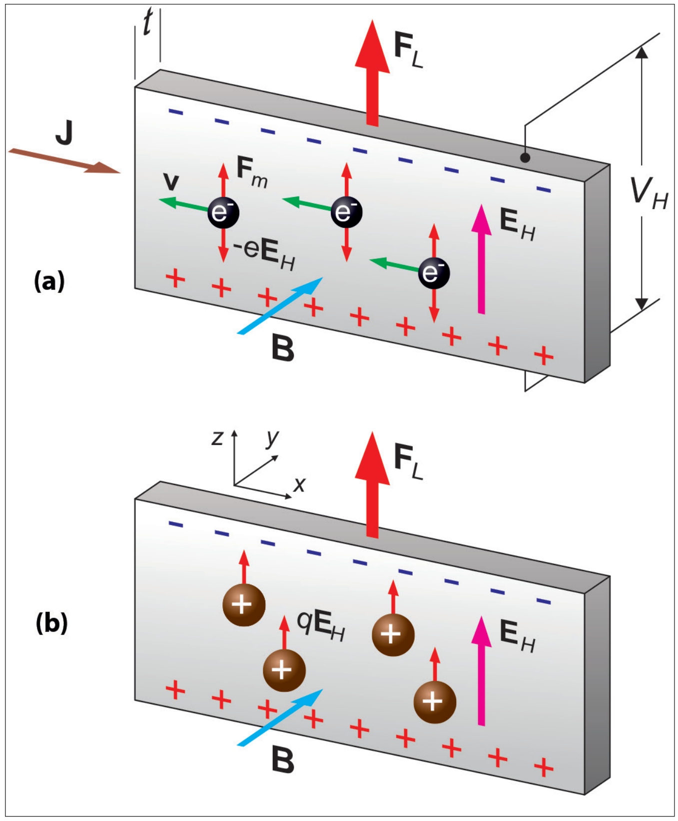


**Fig. 1. (a) Illustration of the origin of the transverse electric field $\boldsymbol{E}_{\mathrm{H}}$ generating the Hall voltage $V_{\mathrm{H}}$ across a wire carrying a current with density $\boldsymbol{J}$ in a field $\boldsymbol{B}$. This phenomenon is due to the charges pushed to the edges of the wire by the magnetic force $\boldsymbol{F}_{\mathrm{m}}$ acting on each carrier (electrons in this case). In this example, the cross section of the wire is rectangular (with thickness $t$), $\boldsymbol{B}$ is uniform and parallel to the $y$ direction, perpendicular to the wire surface. The total force $\boldsymbol{F}_{\mathrm{L}}$ applied to the wire is parallel to the $z$-axis. Note that $\boldsymbol{F}_{\mathrm{m}}$ is, on the average, compensated by the electric force $-e\boldsymbol{E}_{\mathrm{H}}$. (b) Illustration of the forces acting on the background charges. In this case, no magnetic force is present, and only the uncompensated electric force $q\boldsymbol{E}_{\mathrm{H}}$ is active.**

Here, we will not examine any further all the consequences and applications of the Hall effect. We just want to stress that the Laplace force experienced by the wire cannot be traced back to the magnetic force acting on the moving charges, since the latter is totally compensated by the electrostatic force $\boldsymbol{F}_{\mathrm{el}}$ resulting from the interaction with the charges accumulated on the wire surface:

$$\boldsymbol{F}_{\mathrm{el}} = -\boldsymbol{F}_{\mathrm{mag}}. \tag{10}$$

## Origin of the Laplace force

To understand the origin of the Laplace force, we should recall the action–reaction law of Newton's dynamics: the *magnetic* force $\boldsymbol{F}_{\mathrm{mag}}$ on the moving charges interacting with $\boldsymbol{B}$ is accompanied by a force $-\boldsymbol{F}_{\mathrm{mag}}$ acting on the sources of the magnetic field. Conversely, the *static* charges accumulated on

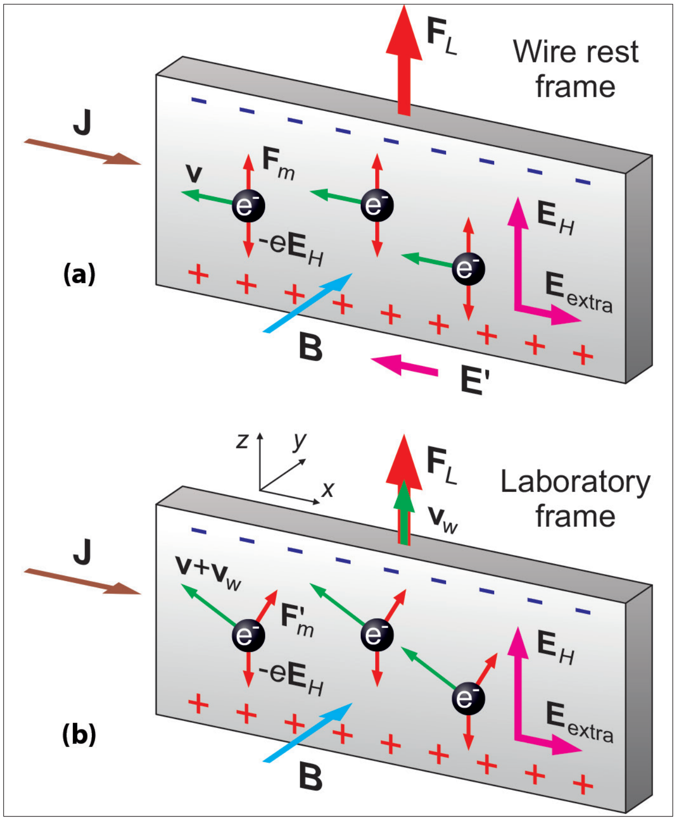


**Fig. 2. Current-carrying wire moving in a magnetic field. In this example, the wire moves with velocity $\boldsymbol{v}_w$ parallel to $\boldsymbol{F}_L$. The electromagnetic forces are either described in (a) the wire rest frame or (b) the laboratory frame. In both cases, the electric field $\boldsymbol{E}_H$ responsible for the Hall effect is the same as in Fig. 1. However, to maintain the same current density $\boldsymbol{J}$ that would flow in the wire at rest (as in Fig. 1), an extra electric field $\boldsymbol{E}_{extra}$ is required from the generator (not shown) that forces the electric current through the wire.**

the wire surface experience an *electrostatic* force $-\boldsymbol{F}_{el} = \boldsymbol{F}_{mag}$ as a consequence of the force $\boldsymbol{F}_{el} = -\boldsymbol{F}_{mag}$ [see Eq. (10)] they exert on the *moving* ones. The surface charges are held in place by electrostatic interactions with background static charges in the volume of the wire, which prevent surface charges from escaping in the vacuum.[4] Therefore, the total force acting on the background charges, corresponding to force acting on the wire as a whole and to the Laplace force $\boldsymbol{F}_L$, is *electrostatic*, despite the fact that it assumes the same expression as $\boldsymbol{F}_{mag}$: $-\boldsymbol{F}_{el} = \boldsymbol{F}_{mag}$. This conclusion on the nature of the Laplace force solves the second contradiction pointed out at the end of the "Magnetic force on the charges moving inside the wire" section: an electrostatic force can do work, justifying the fact that an electric motor can indeed exploit Laplace forces to deliver mechanical power.

## Work done by the Laplace force

Arguments similar to those presented in the previous section, valid when the wire is fixed in the laboratory frame, were already partially discussed in Ref. 10. Here, however, we would like to better illustrate the phenomena occurring when the wire is moving. This case can be explained in the following way to students familiar with relativistic electromagnetic field transformation between reference frames. A wire moving in a magnetic field $\boldsymbol{B}$ with velocity $\boldsymbol{v}_w$ experiences in its rest frame [see Fig. 2(a)] an additional electric field $\boldsymbol{E}'$, which, in the small velocity limit $v_w \ll c$, is equal to[11]

$$\boldsymbol{E}' = \boldsymbol{v}_w \times \boldsymbol{B}. \tag{11}$$

The component of $\boldsymbol{E}'$ *perpendicular* to the wire axis is compensated by the Hall field $\boldsymbol{E}'_H$ created by charges accumulated at the wire surface. However, to maintain the same current density $\boldsymbol{J}$ that would flow in the wire at rest, an extra electric field $\boldsymbol{E}_{extra}$ is required from the generator supplying current to the wire:

$$\boldsymbol{E}_{extra} = -\boldsymbol{E}'_{||}, \tag{12}$$

$\boldsymbol{E}'_{||}$ being the projection of $\boldsymbol{E}'$ on $\boldsymbol{J}$. To create this additional field, the generator is required to deliver an additional power $W_{extra}$ to the wire [see also Eq. (4)]:

$$\begin{aligned} W_{extra} &= \int \boldsymbol{J}\cdot\boldsymbol{E}_{extra}\,dV = \int(-\boldsymbol{J}\cdot\boldsymbol{E}'_{||})\,dV = \int(-\boldsymbol{J}\cdot\boldsymbol{E}')\,dV \\ &= \int[-\boldsymbol{J}\cdot(\boldsymbol{v}_w\times\boldsymbol{B})](d\boldsymbol{\ell}\cdot d\boldsymbol{S}) \\ &= \int[\boldsymbol{v}_w\cdot(\boldsymbol{J}\times\boldsymbol{B})](d\boldsymbol{\ell}\cdot d\boldsymbol{S}) = \boldsymbol{v}_w\cdot\int(\boldsymbol{J}\times\boldsymbol{B})(d\boldsymbol{\ell}\cdot d\boldsymbol{S}) \\ &= \boldsymbol{v}_w\cdot\boldsymbol{F}_{mag} = \boldsymbol{v}_w\cdot\boldsymbol{F}_L. \end{aligned} \tag{13}$$

This expression for $W_{extra}$ coincides with the mechanical power associated with $\boldsymbol{F}_L$. As is to be expected, such a power is provided by the generator supplying the current $I$ necessary to maintain, through $\boldsymbol{F}_{mag}$, the surface charge distribution responsible for the Laplace force itself.

The same result can also be obtained by describing this phenomenon in the laboratory frame [see Fig. 2(b)]. In this case, the (average) velocity of the moving charges is equal to $\langle\boldsymbol{v}\rangle + \boldsymbol{v}_w$, with $\langle\boldsymbol{v}\rangle = \boldsymbol{J}/(qn)$, and on each charge is applied an (average) magnetic force $\boldsymbol{F}'_m$ equal to

$$\boldsymbol{F}'_m = q\,(\langle\boldsymbol{v}\rangle + \boldsymbol{v}_w) \times \boldsymbol{B}. \tag{14}$$

In the steady state, the component of $\boldsymbol{F}'_m$ perpendicular to the wire axis is compensated by the electrostatic force $q\boldsymbol{E}'_H$ due to the Hall field. Instead, to maintain the same current density $\boldsymbol{J}$ that would flow in the wire at rest, the component of $\boldsymbol{F}'_m$ parallel to the axis needs to be compensated by the electrostatic force $q\boldsymbol{E}_{extra}$ exerted on each charge $q$ by an additional field $\boldsymbol{E}_{extra}$ equal to the projection of $\boldsymbol{F}'_m$ on $\boldsymbol{J}$, which, $\boldsymbol{J}$ being parallel to $\langle\boldsymbol{v}\rangle$, coincides with the projection of $q\boldsymbol{v}_w \times \boldsymbol{B}$ on $\boldsymbol{J}$. We can thus conclude that the extra power required from the generator would then be

$$W_{extra} = \int \boldsymbol{J}\cdot\boldsymbol{E}_{extra}\,dV = \int[-\boldsymbol{J}\cdot(\boldsymbol{v}_w\times\boldsymbol{B})](d\boldsymbol{\ell}\cdot d\boldsymbol{S}) = \boldsymbol{v}_w\cdot\boldsymbol{F}_L. \tag{15}$$

As expected, this expression reproduces the result already derived in Eq. (13). Note that the magnetic forces acting on the charges that are either moving or fixed within the wire do not do any mechanical work.

## Conclusions

In summary, we have discussed the nature of the Laplace force acting on a current-carrying wire immersed in a magnetic field. The conventional interpretation of this phenomenon in terms of a *magnetic* force is not compatible with the wire being in a stationary current regime and with the possibility of doing mechanical work. Instead, we have clarified the *electrostatic* nature of the Laplace force. In this way, we solve all inconsistencies and provide a better understanding of the origin of such phenomenon.

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