A direct computation of a certain family of integrals

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Abstract

Purpose – The authors propose a rather elementary method to compute a family of integrals on the half line, involving positive powers of sin x and negative powers of x, depending on the integer parameters $n \geq q \geq 1$.
Design/methodology/approach – Combinatorics, sine and cosine integral functions.
Findings – The authors prove an explicit formula to evaluate sinc-type integrals.
Originality/value – The proof is not present in the current literature, and it could be of interest for a large audience.

Keywords Integral, Sinc function, SinIntegral and CosIntegral functions
Paper type Research paper

In this note, let $n \geq q \geq 1$ be any two given integers. The symbol $\lfloor . \rfloor$ will stand, as usual, for the integer part. We consider the family of integrals

$$I_{n,q} = \int_0^\infty \frac{(\sin x)^n}{x^q} dx.$$

Theorem 1. The following formulae hold

(i) If $n + q$ is even, then

$$I_{n,q} = \frac{(-1)^{\frac{q-n}{2}} \pi}{2^n (q-1)!} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} (n-2k)^{q-1}.$$

(ii) If $n + q$ is odd and $q \geq 2$, then

$$I_{n,q} = \frac{(-1)^{\frac{q-n+1}{2}}}{2^{n-1} (q-1)!} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} (n-2k)^{q-1} \log(n-2k).$$

The formulae above are recorded in the Wolfram MathWorld web page titled *Sinc Function* [1], which refers to the result as "amazing" and "spectacular". However, the web page omits the proof, citing a 20-year-old online paper that seems not to be available any longer. Nor the proof is reported anywhere else, to the best of our knowledge. Nonetheless, particular instances of $I_{n,q}$ are discussed in several textbooks, typically by means of complex analysis tools (see, e.g. Ref. [2]).

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The remaining of the paper is devoted to our proof of Theorem 1. To this end, for $m \geq 0$, let

$$P_m(x) = \sum_{k=0}^{m} \frac{x^k}{k!}$$

denote the Maclaurin polynomial of e^x of order m. We agree to set $P_{-1} = 0$. Let $Q(x)$ be the Maclaurin polynomial of $(\sin x)^n$ of order $q-2$, with $Q = 0$ if $q = 1$. Since $(\sin x)^n$ has a zero of order n at $x = 0$, it follows that $Q(x) \equiv 0$ for all $n \geq q \geq 1$. On the other hand, as

$$(\sin x)^n = \frac{1}{(2i)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{i(n-2k)x},$$

we immediately conclude that

$$Q(x) = \frac{1}{(2i)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} P_{q-2}(i(n-2k)x) = 0. \tag{1}$$

Subtracting the two sums, we obtain

$$\begin{aligned} I_{n,q} = {} & \frac{1}{(2i)^n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} \int_0^\infty \frac{e^{i(n-2k)x} - P_{q-2}(i(n-2k)x)}{x^q}\, dx \\ & + \frac{(-1)^n}{(2i)^n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} \int_0^\infty \frac{e^{-i(n-2k)x} - P_{q-2}(-i(n-2k)x)}{x^q}\, dx. \end{aligned} \tag{2}$$

Remark 2. From (1), we also deduce that the equality

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} (n-2k)^{q-1} = 0, \tag{3}$$

holds for every $n > q \geq 2$, whenever $n + q$ is odd.

We now start from formula (2) but considering the integral on (ε, ∞) and only at the end we will take the limit $\varepsilon \to 0$. This allows us to move the integral inside the sum. In what follows $\omega(\varepsilon)$ will denote a generic function of ε, vanishing at 0 as $\varepsilon \to 0$. Moreover, for $\alpha \neq 0$, let us define

$$\mathsf{E}_\varepsilon(\alpha) = \int_\varepsilon^\infty \frac{e^{i\alpha x}}{x}\, dx.$$

Lemma 3. For every $q \geq 1$, every $\varepsilon > 0$ and every $\alpha \neq 0$, we have

$$\int_\varepsilon^\infty \frac{e^{i\alpha x} - P_{q-2}(i\alpha x)}{x^q}\, dx = c_q \alpha^{q-1} + \frac{(i\alpha)^{q-1}}{(q-1)!} \mathsf{E}_\varepsilon(\alpha) + \omega(\varepsilon),$$

where $c_q = \frac{i^{q-1}}{(q-1)!} \sum_{k=0}^{q-2} \frac{1}{k+1}$ for $q \geq 2$ and $c_1 = 0$.

Proof: The proof goes by induction on q. If $q = 1$, equality holds with $\omega(\varepsilon) = 0$. Then, we prove the formula for $q + 1$, assuming it true for $q \geq 1$. Since $P'_{q-1} = P_{q-2}$, an integration by parts yields

$$\int_\varepsilon^\infty \frac{e^{i\alpha x}-P_{q-1}(i\alpha x)}{x^{q+1}}\,dx = \frac{e^{i\alpha\varepsilon}-P_{q-1}(i\alpha\varepsilon)}{q\varepsilon^q} + \frac{i\alpha}{q}\int_\varepsilon^\infty \frac{e^{i\alpha x}-P_{q-2}(i\alpha x)}{x^q}\,dx.$$

By the inductive hypothesis,

$$\frac{i\alpha}{q}\int_\varepsilon^\infty \frac{e^{i\alpha x}-P_{q-2}(i\alpha x)}{x^q}dx = \frac{ic_q}{q}\alpha^q + \frac{(i\alpha)^q}{q!}\mathsf{E}_\varepsilon(\alpha) + \omega_q(\varepsilon),$$

for some function ω_q vanishing at 0. Noting that

$$\varpi_q(\varepsilon) = -\frac{(i\alpha)^q}{q!\,q} + \frac{e^{i\alpha\varepsilon}-P_{q-1}(i\alpha\varepsilon)}{q\varepsilon^q} \to 0 \quad \text{as } \varepsilon\to 0,$$

we end up with the equality

$$\int_\varepsilon^\infty \frac{e^{i\alpha x}-P_{q-1}(i\alpha x)}{x^{q+1}}\,dx = \left[\frac{i^q}{q!q}+\frac{ic_q}{q}\right]\alpha^q + \frac{(i\alpha)^q}{q!}\mathsf{E}_\varepsilon(\alpha) + \omega_q(\varepsilon) + \varpi_q(\varepsilon).$$

The final observation that $\frac{i^q}{q!q}+\frac{ic_q}{q} = c_{q+1}$ completes the proof. □

Proof of Theorem 1 for the case $n+q$ even. Substituting the expression given by Lemma 3 into (2) and noting that

$$\mathsf{E}_\varepsilon(n-2k) - \mathsf{E}_\varepsilon(-(n-2k)) = 2i\mathsf{Si}((n-2k)\varepsilon),$$

where

$$\mathsf{Si}(t) = \int_t^\infty \frac{\sin x}{x}\,dx$$

is the *SinIntegral* function, we obtain

$$I_{n,q} = \frac{(-1)^{\frac{q-n}{2}}}{2^{n-1}(q-1)!}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\binom{n}{k}(n-2k)^{q-1}\mathsf{Si}((n-2k)\varepsilon) + \omega(\varepsilon).$$

Since

$$\mathsf{Si}((n-2k)\varepsilon) \to \mathsf{Si}(0) = \frac{\pi}{2} \quad \text{as } \varepsilon\to 0,$$

the result follows. □

Proof of Theorem 1 for the case $n+q$ odd. Again, we substitute the expression given by Lemma 3 into (2). Using (3) and noting that

$$\mathsf{E}_\varepsilon(n-2k) + \mathsf{E}_\varepsilon(-(n-2k)) = 2\mathsf{Ci}((n-2k)\varepsilon),$$

where

$$\mathsf{Ci}(t) = \int_t^\infty \frac{\cos x}{x}dx$$

is the *CosIntegral* function, we obtain

$$I_{n,q} = \frac{(-1)^{\frac{q-n-1}{2}}}{2^{n-1}(q-1)!} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} (n-2k)^{q-1} \mathsf{Ci}((n-2k)\varepsilon) + \omega(\varepsilon).$$

By a further use of (3), we can replace $\mathsf{Ci}((n-2k)\varepsilon)$ with

$$\mathsf{Ci}((n-2k)\varepsilon) - \mathsf{Ci}(\varepsilon) \to -\log(n-2k) \quad \text{as} \ \ \varepsilon \to 0,$$

and a final limit $\varepsilon \to 0$ completes the argument. □

References

[1] Weisstein ES. Sinc function. Available from: https://mathworld.wolfram.com/SincFunction.html.

[2] Ahlfors LV. Complex analysis. New York: McGraw-Hill; 1978.

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