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Invariant measures for a stochastic nonlinear and damped 2D Schrödinger equation

Zdzisław Brzeźniak[1], **Benedetta Ferrario**[2,*] **and Margherita Zanella**[3]

[1] Department of Mathematics, University of York, Heslington, York YO105DD, United Kingdom
[2] Dipartimento di Scienze Economiche e Aziendali, Università di Pavia, 27100 Pavia, Italy
[3] Dipartimento di Matematica 'Francesco Brioschi', Politecnico di Milano, Via Bonardi 13, 20133 Milano, Italy

E-mail: benedetta.ferrario@unipv.it

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Abstract

We consider a stochastic nonlinear defocusing Schrödinger equation with zero-order linear damping, where the stochastic forcing term is given by a combination of a linear multiplicative noise in the Stratonovich form and a nonlinear noise in the Itô form. We work at the same time on compact Riemannian manifolds without boundary and on relatively compact smooth domains with either the Dirichlet or the Neumann boundary conditions, always in dimension two. We construct a martingale solution using a modified Faedo–Galerkin's method, following Brzeźniak *et al* (2019 *Probab. Theory Relat. Fields* **174** 1273–338). Then by means of the Strichartz estimates deduced from Blair *et al* (2008 *Proc. Am. Math. Soc.* **136** 247–56) but modified for our stochastic setting we show the pathwise uniqueness of solutions. Finally, we prove the existence of an invariant measure by means of a version of the Krylov–Bogoliubov method, which involves the weak topology, as proposed by Maslowski and Seidler (1999 *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **10**

* Author to whom any correspondence should be addressed.

69–78). This is the first result of this type for stochastic nonlinear Schrödinger equation (NLS) on compact Riemannian manifolds without boundary and on relatively compact smooth domains even for an additive noise. Some remarks on the uniqueness in a particular case are provided as well.

Keywords: nonlinear Schrödinger equation, multiplicative noise, Galerkin approximation, pathwise uniqueness, sequential weak Feller, tightness, invariant measure

Mathematics Subject Classification numbers: 35Q55, 35R60, 60H30, 60G10, 60H15

Contents

1. Introduction

Let us consider the following nonlinear damped stochastic Schrödinger equation

$$\begin{aligned} \mathrm{d}u(t) = &-[\mathrm{i}Au(t) + \mathrm{i}F(u(t)) + \beta u(t)]\,\mathrm{d}t - \mathrm{i}Bu(t)\circ \mathrm{d}W(t) \\ &- \mathrm{i}G(u(t))\,\mathrm{d}\mathbf{W}(t), \qquad t>0. \end{aligned} \tag{1.1}$$

Here A is a linear self-adjoint non-negative operator, β is a damping constant (usually considered not negative), B is a linear bounded operator; F and G are nonlinear terms. Moreover W and $\mathbf{W}$ are two independent Wiener processes; the first stochastic differential is in the Stratonovich form and the other one is in the Itô form.

A basic example of the operator A is the negative Laplace–Beltrami operator $-\Delta_g$ on a compact Riemannian manifold (M,g) without boundary; this appears in some previous papers on the nonlinear Schrödinger equation, see for instance [BGT04, BMi14, BHW19]. However, we can deal as well with the negative Laplacian $-\Delta$ on a relatively compact smooth domain in $\mathbb{R}^d$ with Neumann or Dirichlet boundary conditions. We will consider F to be a power-type defocusing nonlinearity.

The nonlinear Schrödinger equation occurs as a basic model in many areas of physics: hydrodynamics, plasma physics, nonlinear optics, molecular biology. It describes the propagation of waves media with both nonlinear and dispersive responses, see, e.g. [SS99] where many physical models are discussed. A lot of attention has been paid recently to the influence of a noise on the dynamics described by the equation: the noise acts as a random potential to incorporate spatial and temporal fluctuations of certain parameters in a physical model. Typically the noise depends on the solution itself and, according to the physical situation one aims at describing, the Itô or the Stratonovich forms of the noise can be taken into account. These different types of stochastic differential lead in fact to different properties of solutions. For instance, in the multiplicative noise model the mass is preserved only when the noise is taken in the Stratonovich sense of a particular type. This provides the same property as for the deterministic equation. However, with both a Stratonovich and an Itô noise, the energy is not conserved any more in general. For this reason, when addressing the problem of the existence of an invariant measure, some dissipative terms (e.g.: our model with $\beta>0$) are usually involved in the stochastic case.

The question of the existence and/or the uniqueness of solutions for nonlinear Schrödinger equations with additive or linear multiplicative noise was previously addressed in the $\mathbb{R}^d$-case by De Bouard and Debussche [dBD99, dBD03], Barbu *et al* [BRZ14, BRZ16, BRZ17], Cui *et al* [CHS19] and Hornung [Hor18b, Hor18a]. In the case of compact two dimensional Riemannian manifolds there are results by Brzeźniak and Millet [BMi14] and by Brzeźniak *et al* [BHW19].

So far, the existence of an invariant measure has been obtained for this equation with a damping term and an additive Itô noise, i.e. $\beta>0$, $B=0$ and G constant in equation (1.1). In this framework, in the papers by Kim [K06] and by Ekren *et al* [EKZ17] the result is proved in the full space $\mathbb{R}^d$, $d\geqslant 1$. Debussche and Odasso [DO05] obtain instead the result on a bounded one-dimensional domain (dealing with the cubic focusing Schrödinger equation) and solved the corresponding uniqueness problem too. The recent paper [BFZ22] proves the uniqueness of the invariant measure in the large damping regime in $\mathbb{R}^d$ for $d=2,3$. Some numerical approximations of invariant measures can be found in the book by Hong and Wang [HW]. The aim of our paper is to generalise the previously cited papers by considering a more general stochastic forcing term: we consider a linear multiplicative Stratonovich noise B, which conserves the H-norm, and a nonlinear Itô noise G. When we reduce to the case of a pure additive noise we get the existence of an invariant measure when $\beta > 0$. This finding is in line with [K06, EKZ17] that obtain the same conclusions working on the full space. We emphasise here that, as far as we know, our result is the first one providing the existence of an invariant measure in the case of a two-dimensional compact Riemannian manifold without boundary and of a relatively compact smooth domain with either the Dirichlet or the Neumann boundary conditions.

The paper is structured as follows. In the first part of the paper we shall construct a martingale solution of problem (1.1) in the energy space $V = \mathscr{D}(A^{\frac{1}{2}})$ by a modified Faedo–Galerkin's approximation, following the lines of a previous paper by the first author and collaborators [BHW19]. However here we generalise that setting by dealing with a random initial data and more general diffusion terms. One should mention here that a very recent paper [Hor20] provides another generalisation [BHW19] in the direction of stochastic NLS equations on unbounded domains and non-compact manifolds.

In the second part of the paper we shall prove the pathwise uniqueness of the solutions. Hence the existence and the uniqueness of a strong solution will follow. This result is based on further regularity properties of the martingale solutions, which are obtained by means of the Strichartz estimates in dimension two. Although the proof of our existence and uniqueness result follows the lines of the proof of the analogous results in [BHW19], we emphasise here that we allow the initial data to be random.

As far as the existence of an invariant measure is concerned, in the last part of the paper we proceed differently from [K06, EKZ17]. Following the proof of the existence of a martingale solution we prove that the corresponding Markov semigroup is sequential weak Feller in the energy space V. Moreover we show a tightness result in the space V equipped with the weak topology, when the damping coefficient is sufficiently large. In this case a new condition involving the strengths of the two noises will appear. With these two latter properties we prove the existence of at least one invariant measure, by means of the method introduced by Maslowski and Seidler [MS99], as a version of the classical Krylov–Bogoliubov technique reset with weak topologies. This method has been successful to prove existence of invariant measures for other stochastic partial differential equations (SPDE's), as the stochastic nonlinear beam and wave equations [BOS16], the Navier–Stokes equations in unbounded domains [BMO17, BF19], the stochastic Landau–Lifshitz–Bloch equation [BGL20], the stochastic damped Euler equation [BF20].

The paper is organised as follows. In section 2 we present notations and main assumptions. In section 3 we state our main results. In section 4 we collect some compactness results. Section 5 deals with the existence of martingale solutions. Pathwise uniqueness is proved in section 6. The two last sections 7 and 8 are concerned with the sequential weak Feller property and the existence of invariant measures. In section 9 we consider the particular case of multiplicative noise, where there is also uniqueness of the invariant measure. In appendix A we recall some facts about Laplacian-type operators on manifolds and on bounded domains with Dirichlet/Neumann boundary conditions and derive the needed Strichartz estimates; this is very different from the setting in $\mathbb{R}^d$ considered in many papers. In appendices B and C we collect the proofs of some results. In appendix D we present the infinite dimensional version of the Yamada–Watanabe theorem. In appendix E we prove weak measurability of the norm function, needed to prove the results of section 7. In section F we prove a technical lemma that we need in section 4.

2. Mathematical setting and assumptions

In this section, we fix the notation, explain the assumptions and formulate the framework for our problem. Let (X, Σ, μ_X) be a σ-finite measure space endowed with the metric ρ such that the corresponding Borel σ-field $\mathscr{B}(X)$ is contained in Σ. Let $D \subseteq X$. Our framework covers the following cases[4]:

- $X = \mathbb{R}^2$ with the Euclidean distance ρ and the Lebesgue measure μ_X and $D = \mathscr{O}$ is a relatively compact smooth, i.e. with C^∞ boundary, domain of $\mathbb{R}^2$.
- $D = X \equiv M$ is a two-dimensional compact Riemannian manifold without boundary with the geodesic distance ρ and the canonical volume measure μ_X on X.

By $L^q(D)$, for $q \in [1,\infty]$, we denote the space of equivalence classes of $\mathbb{C}$-valued q-Lebesgue integrable functions. We abbreviate $L^q := L^q(D)$. For $q \in [1,\infty]$, let $q' := \frac{q}{q-1} \in [1,\infty]$ be the conjugate exponent. We further abbreviate $H := L^2$. This is a complex Hilbert space with inner product $(u,v)_H = \int_D u\bar{v}\,\mathrm{d}\mu_X$. However, we often interpret H as a real Hilbert space with the inner product $\mathrm{Re}(u,v)_H$. They are different but in one-to-one correspondence: $(u,v)_H = \mathrm{Re}(u,v)_H + i\,\mathrm{Re}(u,iv)_H$. These inner products introduce the same norms and hence both spaces are topologically equivalent.

By $H^{s,q}(D)$ we denote the fractional Sobolev space of regularity $s \in \mathbb{R}$ and integrability $q \in (1,\infty)$. We abbreviate $H^{s,q} := H^{s,q}(D)$ and we shortly write $H^s := H^{s,2}$. For a definition of these spaces see appendix A.

In the sequel, given two Banach spaces E and F, we denote by $\mathscr{L}(E,F)$ the space of all linear bounded operators $B : E \to F$ and abbreviate $\mathscr{L}(E) := \mathscr{L}(E,E)$. Furthermore, we write $E \hookrightarrow F$, if E is continuously embedded in F, i.e. $E \subset F$ with natural embedding $j \in \mathscr{L}(E,F)$. For a Hilbert space H and a Banach space E, $\gamma(H,E)$ denotes the spaces of γ-radonifying operators from H to E. If E is a Hilbert space, this is indeed the space of Hilbert–Schmidt operators from H to E. The space $C^{1,2}([0,T] \times E, F)$ consists of all functions $\Phi\colon [0,T] \times E \to F$ such that $\Phi(\cdot,x) \in C^1([0,T],F)$ for every $x \in E$ and $\Phi(t,\cdot) \in C^2(E,F)$ for every $t \in [0,T]$. Given the Hilbert space H, $C_w([0,T];H)$ stands for the space of all continuous functions from the interval $[0,T]$ to the space H endowed with the weak topology (see appendix E for more details).

If functions $a,b \geqslant 0$ satisfy the inequality $a \leqslant C(A)b$ with a constant $C(A) > 0$ depending on the expression A, we write $a \lesssim_A b$; for a generic constant we put no subscript. If we have $a \lesssim_A b$ and $b \lesssim_A a$, we write $a \simeq_A b$.

2.1. Assumptions on the operator A

Assumption 2.1. The operator A that appears in equation (1.1) is a Laplacian-type operator. We consider A to be as one of the following:

(i) the negative Laplace–Beltrami operator $-\Delta_g$ on a compact two-dimensional Riemannian manifold (M,g) without boundary, equipped with a Lipschitz metric g; in this case μ_X is the canonical volume measure;
(ii) the negative Laplacian with Dirichlet boundary conditions $-\Delta_{\mathrm{D}}$ on a smooth, i.e. C^∞, relatively compact domain $\mathscr{O}$ of $\mathbb{R}^2$;
(iii) the negative Laplacian with Neumann boundary conditions $-\Delta_{\mathrm{N}}$ on a smooth, i.e. C^∞, relatively compact domain $\mathscr{O}$ of $\mathbb{R}^2$.

[4] From now on we will denote by $\mathscr{O}$ a subset of $\mathbb{R}^2$ and by M a two-dimensional manifold. We will use the letter D when we need to deal with the two cases above at the same time.

Some classical results, see e.g. [Ou09], ensure that the operator A, in any of the forms given in assumption 2.1, is a non-negative self-adjoint operator on H. We denote by $\mathscr{D}(A)$ its domain. We set $V := \mathscr{D}(A^{\frac{1}{2}})$ and note that V is a Hilbert space when equipped with the inner product

$$(u,v)_V := (u,v)_H + \left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\right)_H, \qquad u,v \in V.$$

We call it the *energy space* and we call the induced norm $\|\cdot\|_V$ the *energy norm* associated to A. For a characterisation of the energy spaces associated to the operators that appear in assumption 2.1, see remark A.5 and proposition A.6(i). We denote the dual space of V by V^* and abbreviate the duality with $\langle\cdot,\cdot\rangle$, where the complex conjugation is taken over the second variable of the duality. Note that (V,H,V^*) is a Gelfand triple, i.e.

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*.$$

Notice that, thanks to the geometry of the domain D, the condition $\alpha \in (1,\infty)$ ensures that the embedding $V \subset L^{\alpha+1}$ is compact (and hence bounded/continuous). Hence, since $(L^{\alpha+1})^* = L^{\frac{\alpha+1}{\alpha}}$, we can extend the $V - V^*$ duality $\langle\cdot,\cdot\rangle$ to the couple $L^{\alpha+1} - L^{\frac{\alpha+1}{\alpha}}$.

Let us point out that we also have the compact (and hence bounded/continuous) embedding

$$V \subset H. \tag{2.1}$$

It can be proved, see e.g. [BHW19, lemma 2.3(a)], that there exists a non-negative self-adjoint operator $\hat{A}$ on V^* with $\mathscr{D}(\hat{A}) = V$ and $\hat{A} = A$ on $\mathscr{D}(A)$. In most cases where this does not cause ambiguity or confusion, we also use the notation A for $\hat{A}$.

In the following lemma we introduce the operator S and state some properties of it. S will play the role of an auxiliary operator to cover the different cases we consider in an unified framework.

Lemma 2.2. *Given A as in assumption 2.1, there exists an operator S on H such that*

(i) S is strictly positive and self-adjoint. S commutes with A and satisfies $\mathscr{D}(S^k) \hookrightarrow V$ for sufficiently large k. Moreover, S satisfies the upper Gaussian estimate i.e. for all $t > 0$ there is a measurable function $p(t,\cdot,\cdot) : D \times D \to \mathbb{R}$ with

$$\left(e^{-tS}f\right)(x) = \int_D p(t,x,y) f(y)\, \mu_X(dy), \quad t > 0, \quad a.e.\ x \in D,$$

for all $f \in H$ and with constants $c, C > 0$ and $m \geqslant 2$

$$|p(t,x,y)| \leqslant \frac{C}{\mu_X(B(x,t^{\frac{1}{m}}))} \exp\left(-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right), \tag{2.2}$$

for all $t > 0$ and a.e. $(x,y) \in D \times D$.

(ii) S has compact resolvent. In particular, there is an orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of H and a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$ such that

$$Sx = \sum_{n=1}^{\infty} \lambda_n (x,h_n)_H h_n, \quad x \in \mathscr{D}(S) = \left\{x \in H : \sum_{n=1}^{\infty} \lambda_n^2 |(x,h_n)_H|^2 < \infty\right\}. \tag{2.3}$$

Proof. For $A = -\Delta_g$ we choose $S := I - \Delta_g$, for $A = -\Delta_D$ we choose $S = A = -\Delta_D$, for $A = -\Delta_N$ we fix $\varepsilon > 0$ and choose $S = \varepsilon I - \Delta_N$. For these choices of S all the statements of the lemma are verified: for a proof see [BHW19, sections 3.2 and 3.3, remark 2.2(b) and lemma 2.3(c)] and the therein references. □

Remark 2.3. The operator S plays a crucial role in the construction of our Galerkin approximations. The Gaussian estimate (2.2) is used in the proof of proposition 5.2 where spectral multiplier theorems for S are employed (for further details one can consult [Ou09, chapter 7]).

2.2. Assumptions on the nonlinear term F

We continue with the assumptions on the nonlinear term F of our problem. We deal with power-type defocusing nonlinearities.

Assumption 2.4. Assume that $\alpha \in (1,\infty)$ and set

$$F(u) := |u|^{\alpha-1}u.$$

The function F satisfies a set of properties that we summarise in the following lemma (for a proof see e.g. [Caz03, proposition 3.25 and remark 3.16] and [BHW19, proposition 3.1]). It is important to recall that the embedding $V \subset L^{\alpha+1}$ is compact (and hence bounded/continuous). Therefore we have

$$V \hookrightarrow L^{\alpha+1} \hookrightarrow L^2 \equiv \left(L^2\right)^* \hookrightarrow L^{\frac{\alpha+1}{\alpha}} \hookrightarrow V^*, \tag{2.4}$$

where the first and the last embeddings are compact, while all other embeddings are simply continuous.

Lemma 2.5. *Let $\alpha \in (1,\infty)$.*

(i) The map $F: L^{\alpha+1} \to L^{\frac{\alpha+1}{\alpha}}$ satisfies, for any $u,v \in L^{\alpha+1}$

$$\begin{aligned} \|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}} &= \|u\|_{L^{\alpha+1}}^{\alpha}, \\ \|F(u)-F(v)\|_{L^{\frac{\alpha+1}{\alpha}}} &\lesssim \left(\|u\|_{L^{\alpha+1}}+\|v\|_{L^{\alpha+1}}\right)^{\alpha-1}\|u-v\|_{L^{\alpha+1}}. \end{aligned} \tag{2.5}$$

Moreover, $F: V \to V^*$, $F(0)=0$ and

$$\mathrm{Re}\langle \mathrm{i}u, F(u)\rangle = 0, \quad u \in L^{\alpha+1}, \tag{2.6}$$

$$\langle F(u), u\rangle = \|u\|_{L^{\alpha+1}}^{\alpha+1}, \quad u \in L^{\alpha+1}. \tag{2.7}$$

(ii) The map $F: L^{\alpha+1} \to L^{\frac{\alpha+1}{\alpha}}$ is continuously Fréchet differentiable with

$$\|F'[u]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}} \leqslant \alpha\|u\|_{L^{\alpha+1}}^{\alpha-1}, \quad u \in L^{\alpha+1}. \tag{2.8}$$

(iii) The map F is defocusing, that is it admits the real non-negative antiderivative[5] $\hat{F}: L^{\alpha+1} \to \mathbb{R}$ given by

$$\hat{F}(u) = \frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}}^{\alpha+1}. \tag{2.9}$$

[5] We recall that, if there exists a Fréchet differentiable map $\hat{F}: L^{\alpha+1} \to \mathbb{R}$ with $\hat{F}'[u]h = \mathrm{Re}\langle F(u),h\rangle$, for every $u,h \in L^{\alpha+1}$, $\hat{F}$ is called the antiderivative of F.

Remark 2.6. It follows from (2.5) and (2.9) that

$$\|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}} = \|u\|_{L^{\alpha+1}}^{\alpha} \lesssim \left[\hat{F}(u)\right]^{\frac{\alpha}{\alpha+1}}, \qquad u \in L^{\alpha+1}. \tag{2.10}$$

Moreover, it follows from (2.8) and (2.9) that

$$\|F'[u]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}} \leqslant \alpha\|u\|_{L^{\alpha+1}}^{\alpha-1} = \alpha(\alpha+1)^{\frac{\alpha-1}{\alpha+1}}\left[\hat{F}(u)\right]^{\frac{\alpha-1}{\alpha+1}}. \tag{2.11}$$

2.3. *Assumptions on the stochastic terms*

For a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and a measurable space $(E,\mathscr{E})$, the law of a random variable $\xi:\Omega\to E$ will be denoted by $\mathrm{Law}_{\mathbb{P}}(\xi)$.

Assumption 2.7. We assume the following.

(i) Let $(\Omega,\mathscr{F},\mathbb{P},\mathbb{F})$, where $\mathbb{F}=\left(\mathscr{F}_t\right)_{t\geqslant 0}$, be a filtered probability space satisfying the usual conditions, Y_1 and Y_2 two separable real Hilbert spaces, with orthonormal bases $(f_m)_{m\in\mathbb{N}}$ and $(e_m)_{m\in\mathbb{N}}$ respectively, and W and $\mathbf{W}$ two independent, Y_1, respectively Y_2, canonical cylindrical $\mathbb{F}$-Wiener processes.

(ii) Let $B:H\to\gamma(Y_1,H)$ be a linear operator and set $B_m u := B(u)f_m$ for $u\in H$ and $m\in\mathbb{N}$. Additionally, we assume that $B_m\in\mathscr{L}(H)$ is self-adjoint for every $m\in\mathbb{N}$ and the following stronger assumption, needed to make sense of the Stratonovich correction terms, is satisfied

$$\sum_{m=1}^{\infty}\|B_m\|_{\mathscr{L}(H)}^2 < \infty. \tag{2.12}$$

Moreover we assume that $B_m\in\mathscr{L}(V)$ and $B_m\in\mathscr{L}(L^{\alpha+1})$ for all $m\in\mathbb{N}$ and

$$\sum_{m=1}^{\infty}\|B_m\|_{\mathscr{L}(V)}^2 < \infty, \tag{2.13}$$

$$\sum_{m=1}^{\infty}\|B_m\|_{\mathscr{L}(L^{\alpha+1})}^2 < \infty. \tag{2.14}$$

(iii) Let $G:H\to\gamma(Y_2,H)$ be Lipschitz continuous, i.e.

$$\exists\, L_G>0: \quad \|G(u_1)-G(u_2)\|_{\gamma(Y_2,H)} \leqslant L_G\|u_1-u_2\|_H \qquad \forall u_1,u_2\in H. \tag{2.15}$$

Moreover the following 'restrictions' of G, i.e.

$$G:V\to\gamma(Y_2,V) \text{ and } G:L^{\alpha+1}\to\gamma\left(Y_2,L^{\alpha+1}\right)$$

are measurable (see [BR21, section 2] for a reasonably thorough discussion of this issue) and of at most linear growth, i.e. for some non negative constants $C_2,\tilde{C}_2,C_3,\tilde{C}_3$, the following inequalities hold

$$\|G(u)\|_{\gamma(Y_2,V)} \leqslant C_2+\tilde{C}_2\|u\|_V \qquad \forall u\in V, \tag{2.16}$$

and

$$\|G(u)\|_{\gamma(Y_2,L^{\alpha+1})} \leqslant C_3 + \tilde{C}_3\|u\|_{L^{\alpha+1}} \qquad \forall u \in L^{\alpha+1}. \tag{2.17}$$

Finally, we also assume the following weak continuity assumption of the diffusion coefficient G: for every $m \in \mathbb{N}$ the map

$$H \ni \varphi \mapsto G(\varphi)\, e_m \in H$$

extends uniquely to a continuous map from V^* to V^*, i.e.

$$\text{the map } V^* \ni \varphi \mapsto G(\varphi)\, e_m \in V^* \text{ is continuous, } m \in \mathbb{N}. \tag{2.18}$$

Remark 2.8. It is well known that the Lipschitz assumption (2.15) implies the following linear growth condition. There exist positive constants $C_1, \tilde{C}_1$ such that

$$\|G(u)\|_{\gamma(Y_2,H)} \leqslant C_1 + \tilde{C}_1\|u\|_H \qquad \forall u \in H. \tag{2.19}$$

We mention this obvious fact since we explicitly use this estimate (2.19) in many computations in the following sections.

Remark 2.9. By assumption 2.7 (i), we can represent the Wiener processes as

$$W(t) = \sum_{m=1}^{\infty} f_m \beta_m(t) \qquad\qquad \mathbf{W}(t) = \sum_{m=1}^{\infty} e_m \boldsymbol{\beta}_m(t)$$

for two sequences of independent standard real Wiener processes $\{\beta_m\}_m$ and $\{\boldsymbol{\beta}_m\}_m$.

Remark 2.10. The estimates (2.12)–(2.14) imply

$$B \in \mathscr{L}(H, \gamma(Y_1, H)), \quad B \in \mathscr{L}(V, \gamma(Y_1, V)), \quad B \in \mathscr{L}\left(L^{\alpha+1}, \gamma\left(Y_1, L^{\alpha+1}\right)\right).$$

Remark 2.11. The property (2.18) will be exploited in the proof of lemma 5.9 given in appendix C. The corresponding property for B is not needed since the analogue of lemma 5.9 for B can be proved exploiting the selfadjointness of the operators B_m, $m \in \mathbb{N}$. For more details see [BHW19, lemma 6.3, step 4].

Example 2.12. Examples of operator B satisfying the required properties can be found in [BHW19, section 3.5]. The self-adjointness of B is crucial there, see [BHW19, remark 3.7].

Concerning the second operator G, in the case it is linear (and examples of such are the same as for B) we do not require it to be self-adjoint. An example of a nonlinear operator G can be constructed as done in [BF17, example 2.3] and [FZ18, section 2.3]. For any $m \in \mathbb{N}$, let $G(u)e_m := c_m \sigma(u) h_m$, with $c_m \in \mathbb{R}$ such that

$$\sum_{m=1}^{\infty} c_m^2 \|h_m\|_V^2 < \infty$$

and, for a fixed and given $k \in V$,

$$\sigma : V^* \ni u \mapsto \frac{\langle u, k\rangle^2}{1 + \langle u, k\rangle^2} \in \mathbb{R}.$$

It can be easily verified that this operator satisfies assumption 2.7(iii).

3. Statement of the main results

This section is devoted to the statements of our main results. The first result concerns the existence of a unique strong solution to (1.1) for a random initial data. The second result concerns the existence of an invariant measure.

We rewrite equation (1.1) in the Itô form. We have, see e.g. [BE00, NT04],

$$\begin{aligned} -\mathrm{i}Bu(t)\circ \mathrm{d}W(t) &= -\mathrm{i}Bu(t)\,\mathrm{d}W(t)+\frac{1}{2}\sum_{m=1}^{\infty}-\mathrm{i}B'[u](-\mathrm{i}B(u(t))f_m)f_m\,\mathrm{d}t \\ &= -\mathrm{i}Bu(t)\,\mathrm{d}W(t)-\frac{1}{2}\sum_{m=1}^{\infty}B(Bu(t)f_m)f_m\,\mathrm{d}t \\ &= -\mathrm{i}Bu(t)\,\mathrm{d}W(t)-\frac{1}{2}\sum_{m=1}^{\infty}B_m^2u(t)\,\mathrm{d}t. \end{aligned}$$

Hence, equation (1.1) will be understood in the following Itô form

$$\begin{aligned} \mathrm{d}u(t) = &-[\mathrm{i}Au(t)+\mathrm{i}F(u(t))+\beta u(t)-b(u(t))]\,\mathrm{d}t \\ &-\mathrm{i}Bu(t)\,\mathrm{d}W(t)-\mathrm{i}G(u(t))\,\mathrm{d}\mathbf{W}(t), \qquad t>0, \end{aligned} \tag{3.1}$$

where

$$b(u) := -\frac{1}{2}\sum_{m=1}^{\infty}B_m^2u, \qquad u\in H, \tag{3.2}$$

is the Stratonovich correction term. Notice that from assumptions (2.12) and (2.13) we infer that

$$b\in\mathscr{L}(H)\cap\mathscr{L}(V)\cap\mathscr{L}\left(L^{\alpha+1}\right),$$

i.e. b is a linear bounded operator in H as well as in V and $L^{\alpha+1}$.

We recall that the deterministic unforced nonlinear Schrödinger equation, i.e. equation (3.1) with $\beta=0$, $G=0$ and $B=0$, as a consequence of its Hamiltonian structure, has two invariant quantities: the mass $\|u\|_H^2$ and the energy $\mathcal{E}(u)$, which is defined as

$$\mathscr{E}(u) := \frac{1}{2}\|A^{\frac{1}{2}}u\|_H^2+\hat{F}(u), \qquad u\in V. \tag{3.3}$$

Note that $\hat{F}(u)$, hence $\mathscr{E}(u)$ too, is well defined for $u\in V$ thanks to the embedding $V\hookrightarrow L^{\alpha+1}$, see (2.4), and the form of $\hat{F}$, see (2.9).

In general, in the presence of stochastic forcing, these quantities are no longer conserved. But, when the noise is of purely Stratonovich form, in the form we consider here, and if there is no dissipation, i.e. $\beta=0$ and $G=0$, one has conservation of mass but not conservation of energy. This is the case studied in [BHW19] and it is a particular case of our framework. In the more general setting we consider in this work, neither the mass or energy are preserved. Nevertheless, as quite classical in the stochastic case, we can still use these functionals to prove the existence of solutions with values in the energy space V.

The following definition is given under assumptions 2.1, 2.4 and 2.7.

Definition 3.1. Let μ be a Borel probability measure on the energy space V with

$$\int \left(\|x\|_H^2 + \mathscr{E}(x)\right) \mathrm{d}\mu(x) < \infty. \tag{3.4}$$

A *martingale solution* of the equation (3.1) with the initial data μ is a system

$$\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{W}}, \tilde{\mathbb{F}}, u\right)$$

consisting of

- a filtered probability space $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}}\right)$, satisfying the usual conditions, i.e. the filtration $\tilde{\mathbb{F}} = \left(\tilde{\mathscr{F}}_t\right)_{t\in[0,\infty)}$ is right-continuous and such that all $\tilde{\mathbb{P}}$-null, i.e. $\tilde{\mathbb{P}}$-negligible, sets of $\mathscr{F}$ are elements of $\tilde{\mathscr{F}}_0$;
- two independent Y_1-cylindrical, resp. Y_2-cylindrical, Wiener processes $\tilde{W}$ and $\tilde{\mathbf{W}}$ on $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}\right)$;
- a H-valued continuous and $\tilde{\mathbb{F}}$-adapted process u with $\tilde{\mathbb{P}}$-almost all paths in $C_w([0,\infty), V)$, fulfilling the initial condition

$$\tilde{\mathbb{P}}(u(0)) = \mu$$

and such that, for every $T > 0$,

$$\tilde{\mathbb{E}}\left(\|u\|^2_{L^\infty(0,T;H)} + \|\mathscr{E}(u)\|_{L^\infty(0,T)}\right) < \infty \tag{3.5}$$

and for every $t \geqslant 0$ the equality

$$u(t) = u(0) - \int_0^t [\mathrm{i}Au(s) + \mathrm{i}F(u(s)) + \beta u(s) - b(u(s))]\,\mathrm{d}s \\ - \mathrm{i}\int_0^t Bu(s)\,\mathrm{d}\tilde{W}(s) - \mathrm{i}\int_0^t G(u(s))\,\mathrm{d}\tilde{\mathbf{W}}(s), \tag{3.6}$$

holds $\tilde{\mathbb{P}}$-almost surely in V^*.

Remark 3.2. Let us notice that the four deterministic Bochner V^*-valued integrals that appear in (3.6) make sense. First, we notice that, since by (2.1) the embedding $V \hookrightarrow H$ is compact, the (weak)-continuity in V implies the (strong)-continuity in H, hence u has $\tilde{\mathbb{P}}$-a.s. paths in $C([0,T], H)$. In addition, since by (2.4), the embedding $H \hookrightarrow V^*$ is continuous, the (strong)-continuity in H implies the (strong)-continuity in V^*. Therefore u has $\tilde{\mathbb{P}}$-a.s. paths in $C([0,T], V^*)$.

Second, we have that $u \in L^\infty(0,T;V)$ and u is Bochner integrable in V, in view of the following argument:

(i) if $u \in C_w([0,T], V)$, since $[0,T]$ is compact, the range of u is a compact subset of V_w. Since by the Banach–Steinhaus theorem, compact sets in weak topology are strongly, i.e. norm bounded, we infer that the range of u is a (norm) bounded subset of V;

(ii) if $u \in C_w([0,T], V)$, then u is $\mathscr{B}([0,T])/\mathscr{B}(V_w)$)-measurable. On the other hand, see argument after (2.8) in [BF19], $\mathscr{B}(V_w) = \mathscr{B}(V)$, see also [Zi03, theorem 7.19] and [Ed79] for more general claims. Therefore, function $u : [0,T] \to V$ is $\mathscr{B}([0,T])/\mathscr{B}(V)$-measurable.

(iii) It follows from items (i) and (ii), that if $u \in C([0,T],V_w)$ then $u:[0,T]\to V$ is measurable and bounded. Hence in particular, (the equivalence class of) u belongs to $L^\infty(0,T;V)$ and u is Bochner integrable in V, see [DU77, section II.2, p.50].

A special attention should be paid to the V^*-valued Bochner integral $\int_0^T \mathrm{i}F(u(s))\,\mathrm{d}s$. This integral exists because if $x\in C([0,T];V_w)$ then by the compactness of the embedding $V\hookrightarrow L^{\alpha+1}$ we infer that $x\in C([0,T];L^{\alpha+1})$ and therefore, by part (i) of lemma 2.5, $F\circ x\in C([0,T];L^{\frac{\alpha+1}{\alpha}})$. Thus, the integral $\int_0^T F(x(s))\,\mathrm{d}s$ exists in $L^{\frac{\alpha+1}{\alpha}}$ in the Riemann sense. Thus, by (2.4), we infer that the integral $\int_0^T F(x(s))\,\mathrm{d}s$ exists in V^* in the Riemann, and not only Bochner, sense.

For what concerns the stochastic integrals that appear in (3.6), let us emphasise that in order for them to be well defined it would be enough to require, instead of (3.5), that

$$\tilde{\mathbb{E}}\left[\int_0^T \|u(t)\|_H^2\,\mathrm{d}t\right]<\infty, \text{for every } T>0.$$

Definition 3.3. Assume that

$$(\Omega,\mathscr{F},\mathbb{P},W,\mathbf{W},\mathbb{F})$$

is a system consisting of

- a filtered probability space $\left(\Omega,\mathscr{F},\mathbb{P},\mathbb{F}\right)$, satisfying the usual conditions, i.e. the filtration $\mathbb{F}=\left(\mathscr{F}_t\right)_{t\in[0,\infty)}$ is right-continuous and such that all $\mathbb{P}$-null, i.e. $\mathbb{P}$-negligible, sets of $\mathscr{F}$ are elements of $\mathscr{F}_0$;
- two independent Y_1-cylindrical, resp. Y_2-cylindrical, Wiener processes W and $\mathbf{W}$ on $\left(\Omega,\mathscr{F},\mathbb{P}\right)$.

Let $u_0:\Omega\to V$ be a $\mathscr{F}_0/\mathscr{B}(V)$ measurable function such that with

$$\mathbb{E}\left(\|u_0\|_H^2+\mathscr{E}(u_0)\right)<\infty.$$

A *strong solution* of the equation (3.1) with the initial data u_0 is an H-valued continuous and $\mathbb{F}$-adapted process u with $\mathbb{P}$-almost all paths in $C_w([0,\infty),V)$ and such that, for every $T>0$, condition (3.5) holds and, and for every $t\geqslant 0$ the equality

$$\begin{aligned}u(t)=u_0&-\int_0^t\left[\mathrm{i}Au(s)+\mathrm{i}F(u(s))+\beta u(s)-b(u(s))\right]\mathrm{d}s\\&-\mathrm{i}\int_0^t Bu(s)\,\mathrm{d}W(s)-\mathrm{i}\int_0^t G(u(s))\,\mathrm{d}\mathbf{W}(s),\end{aligned}$$

holds $\mathbb{P}$-almost surely in V^*.

The following is a summary of the first of our main results. For a more detailed statements see theorems 5.1 and 6.5.

Theorem 3.4. *Fix $r\in[1,\infty)$. Under the assumptions 2.1, 2.4 and 2.7, for every Borel probability measure μ on V whose $r(\alpha+1)$th moment is finite, i.e.*

$$\int \|x\|_V^{r(\alpha+1)}\,\mathrm{d}\mu(x)<\infty$$

the following assertion hold true.

(i) *There exists a martingale solution to equation* (1.1) *with the initial data* μ *such that, for every* $T > 0$,

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\|u(t)\|_H^{2r} + \sup_{t\in[0,T]}\mathcal{E}(u(t))^r\right] < \infty. \tag{3.7}$$

In particular

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\|u(t)\|_V^{2r}\right] < \infty. \tag{3.8}$$

(ii) *In addition, if* $r \geqslant 2$, *then the solutions fulfilling* (3.8) *are pathwise unique.*

The above result implies the existence of a unique strong solution, see theorem 6.6.

Remark 3.5. Let us notice that inequality (3.8) is an immediate consequence of the previous estimate (3.7) since

$$\|u\|_V^{2r} \lesssim_r \|u\|_H^{2r} + \mathscr{E}(u)^r.$$

Remark 3.6. (i) When $r = 1$, the assumption on the initial data in theorem 3.4 is

$$\int \|x\|_V^{\alpha+1}\,\mathrm{d}\mu(x) < \infty \tag{3.9}$$

which is stronger than the assumption (3.4) appearing in the definition 3.1. In fact

$$\int \left(\|x\|_H^2 + \mathscr{E}(x)\right)\mathrm{d}\mu(x) \lesssim \int \left(\|x\|_V^2 + \|x\|_V^{\alpha+1}\right)\mathrm{d}\mu(x) \lesssim 1 + \int \|x\|_V^{\alpha+1}\,\mathrm{d}\mu(x).$$

We need assumption (3.9) because of our construction of a martingale solution by means of the finite-dimensional Galerkin approximation. Indeed in the finite-dimensional Galerkin approximation we will need uniform estimates of the power-type nonlinearity, which hold in the Hilbert spaces H and V but not in the Lebesgue space $L^{\alpha+1}$, see (5.4).
Similarly, to gain the additional $L^r(\widetilde{\Omega})$-integrability, with $r > 1$, of the solution process the condition on the initial datum has to be strengthened requiring its $r(\alpha+1)$th moment to be bounded, see (5.18).

(ii) On the other hand, if the initial datum $u_0 \in V$ is deterministic, theorem 3.4 ensures the existence of a unique strong solution fulfilling (3.7) and (3.8) for any finite $r \geqslant 1$.

To study the existence of an invariant measure for equation (3.1) we work with deterministic initial data $u_0 \in V$. We are thus in the situation described in remark 3.6(ii) and we deal with the unique strong solution to (3.1) fulfilling (3.7) and (3.8). Given the (non-random) initial datum $u_0 \in V$, we denote by $\{u(t;u_0)\}_{t\geqslant 0}$ this unique strong solution. We define the family of operators $\{P_t\}_{t\geqslant 0}$ by

$$P_t\phi(u_0) = \mathbb{E}[\phi(u(t);u_0)] \tag{3.10}$$

and prove that this is a Markov semigroup, see section 7, which is sequential weak Feller in V. Then we say that a Borel probability measure π on V is an invariant measure for equation (1.1) iff

$$\int_V P_t\phi\,\mathrm{d}\pi = \int_V \phi\,\mathrm{d}\pi$$

for all $t \geqslant 0$ and all bounded functions $\phi : V \to \mathbb{R}$ which are sequentially continuous with respect to the weak topology on V.

The following is our second main result.

Theorem 3.7. *Under the assumptions 2.1, 2.4 and 2.7 there exists at least one invariant measure for equation (1.1) provided*

$$\beta > \max\left(\tilde{C}_1^2 + \tilde{C}_2^2 + \|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}, \frac{\alpha+1}{2}\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))} + \alpha\tilde{C}_3^2\right). \tag{3.11}$$

Remark 3.8.

(i) Statement (i) of theorem 3.4 holds in a more general setting, see remark 5.10. Since our focus is on the existence of an invariant measure, we restrict our analysis to the less general setting. This is done also in [BHW19] to get the uniqueness of solutions.

(ii) For a discussion concerning the regularity assumptions we impose on the domain $\mathscr{O}$ in the case of assumption 2.1(ii)–(iii), see remark 5.11.

(iii) The condition (3.11) on β does not depend on the coefficients C_1, C_2 and C_3 characterising the second noise term. When $B = 0$ and $\tilde{C}_1 = \tilde{C}_2 = \tilde{C}_3 = 0$, that is when we consider a multiplicative noise $G(u)\,\mathrm{d}\tilde{\mathbf{W}}$ with bounded covariance, the condition (3.11) reduces to $\beta > 0$. Therefore, in the particular case of additive noise (i.e. $B = 0$ and G independent of u), we recover the same condition $\beta > 0$ as in the previous papers [K06, EKZ17].

(iv) In condition (3.11) there is the constant $\tilde{C}_1$ (which somehow measures the intensity of the noise G in the H-norm) but not the analogue for the noise B, that is the term $\|B\|^2_{\mathscr{L}(H,\gamma(Y_1,H))}$. This asymmetry is due to the fact that the Stratonovich noise B (in the absence of damping) preserves the H-norm, whereas the noise G does not. In other words, the correction term b of the Stratonovich noise cancels with the term $\|B\|^2_{\mathscr{L}(H,\gamma(Y_1,H))}$ acting as a sort of damping term which perfectly balance the intensity of the noise B in the H-norm.

A similar reasoning can be done to explain also why we have the different constants α and $\frac{\alpha+1}{2}$ multiplying the terms $\tilde{C}_3^2$ and $\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}$ respectively. We have $\frac{\alpha+1}{2} < \alpha$: the correction term b provides part of the dissipation in the $L^{\alpha+1}$ norm.

For the purely multiplicative noise, the uniqueness of an invariant measure will be given in corollary 9.1.

4. Compactness and tightness criteria

This section is devoted to recalling the compactness results, which will be used in section 5 to obtain a martingale solution as limit of the Faedo–Galerkin approximation and in section 7 to prove the continuous dependence of the solutions on the initial data.

Let $\alpha > 1$ and A be chosen according to assumption 2.1. We consider the Banach spaces $C([0,T];V^*)$ and $L^{\alpha+1}(0,T;L^{\alpha+1})$, and the locally convex space $C_w([0,T];V)$. So we define the space

$$Z_T = C([0,T];V^*) \cap L^{\alpha+1}\left(0,T;L^{\alpha+1}\right) \cap C_w([0,T];V)$$

with the topology $\mathscr{T}_T$ given by the supremum of the corresponding topologies. By $\mathscr{B}(Z_T)$ we denote the associated Borel σ-field, i.e. the σ-field generated by the open sets in the locally convex topology $\mathscr{T}_T$ of Z_T.

We also define a corresponding space of functions defined on the whole half-line $[0,\infty)$; first we consider the following locally convex topological spaces:

- $C([0,\infty);V^*)$ with metric $d(u,v)=\sum_{N=1}^{\infty}\frac{1}{2^N}\frac{\|u-v\|_{C([0,N];V^*)}}{1+\|u-v\|_{C([0,N];V^*)}}$;
- $L^{\alpha+1}_{\mathrm{loc}}([0,\infty);L^{\alpha+1})$ with metric $d(u,v)=\sum_{N=1}^{\infty}\frac{1}{2^N}\frac{\|u-v\|_{L^{\alpha+1}(0,N;L^{\alpha+1})}}{1+\|u-v\|_{L^{\alpha+1}(0,N;L^{\alpha+1})}}$;
- $C_w([0,\infty);V)$ with the topology being generated by the family of semi-norms $\|u\|_{N,\phi}=\sup_{0\leqslant t\leqslant N}|\langle u(t),\phi\rangle|, N\in\mathbb{N},\phi\in V^*$.

And we define the space

$$Z_\infty=C([0,\infty);V^*)\cap L^{\alpha+1}_{\mathrm{loc}}([0,\infty);L^{\alpha+1})\cap C_w([0,\infty);V).$$

It is a locally convex topological space with the topology $\mathscr{T}_\infty$ given by the supremum of the corresponding topologies.

In the next proposition, we give a criterion for compactness in Z_∞.

Proposition 4.1. *Let $(r_N)_{N=1}^{\infty}$ be a sequence of positive numbers and K be a subset of Z_∞ such that for every $N\in\mathbb{N}$,*

(a) $\sup_{u\in K}\|u\|_{L^\infty(0,N;V)}\leqslant r_N$;
(b) K is equicontinuous in $C([0,N];V^*)$, i.e.

$$\lim_{\delta\to0}\sup_{u\in K}\sup_{s,t\in[0,N]:|t-s|\leqslant\delta}\|u(t)-u(s)\|_{V^*}=0.$$

Then, K is relatively compact in Z_∞.

Proof. The proof is a minor modification of the proof of [BHW19, proposition 4.2]. Here one uses lemma F.1 which is the restatement of [BHW19, lemma 4.1] on the time interval $[0,\infty)$. □

Now we want to obtain a criterion for tightness in Z_∞. Therefore, we introduce the Aldous condition, working in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $\mathbb{F}:=\{\mathcal{F}_t\}_{t\in[0,\infty)}$ satisfying the usual conditions.

Definition 4.2. We say that a sequence $(X_n)_{n\in\mathbb{N}}$ of continuous $\mathbb{F}$-adapted stochastic processes taking values in a Banach space E satisfies the Aldous condition $[A]$ if and only if for all $T>0$, $\varepsilon>0$ and $\eta>0$ there is $\delta>0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-valued stopping times with $\tau_n\leqslant T$, one has

$$\sup_{n\in\mathbb{N}}\sup_{0<\theta\leqslant\delta}\mathbb{P}\{\|X_n((\tau_n+\theta)\wedge T)-X_n(\tau_n)\|_E\geqslant\eta\}\leqslant\varepsilon.$$

The following lemma, which generalises [Mo13, lemma A.7], gives us a useful consequence of the Aldous condition $[A]$.

Lemma 4.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted stochastic processes in a Banach space E, which satisfies the Aldous condition $[A]$. Then, for every $\varepsilon>0$, there exists a measurable subset $A_\varepsilon\subset C([0,\infty),E)$ such that*

$$\inf_{n\in\mathbb{N}}\mathrm{Law}_{\mathbb{P}}(X_n)(A_\varepsilon)\geqslant1-\varepsilon,$$

and, for every $N \in \mathbb{N}$,

$$\lim_{\delta \to 0} \sup_{u \in A_\varepsilon} \sup_{s,t \in [0,N]: |t-s| \leqslant \delta} \|u(t) - u(s)\|_E = 0.$$

Proof. The proof of [Mo13, lemma A.7] can be easily adapted to the present situation. □

The deterministic compactness result in proposition 4.1 and lemma 4.3 can be used to get the following criterion for tightness in Z_∞.

Proposition 4.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous adapted V^*-valued processes satisfying the Aldous condition $[A]$ in V^* and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|^2_{L^\infty(0,T;V)}\right] < \infty, \text{ for every } T > 0.$$

Then the sequence $(\mathrm{Law}_{\mathbb{P}}(X_n))_{n\in\mathbb{N}}$ is tight in Z_∞, i.e. for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset Z_\infty$ with

$$\inf_{n\in\mathbb{N}} \mathrm{Law}_{\mathbb{P}}(X_n)(K_\varepsilon) \geqslant 1 - \varepsilon.$$

Proof. Let us choose and fix $\varepsilon > 0$. Let us set $c := \sum_{N=1}^\infty \frac{1}{N^2}$ and let us define a sequence $(r_N)_{N=1}^\infty$ by

$$r_N := \left(\frac{2c}{\varepsilon} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|^2_{L^\infty(0,N;V)}\right]\right)^{\frac{1}{2}}.$$

Set

$$B_N := \left\{\|X_n\|_{L^\infty(0,N;V)} \leqslant N r_N\right\}.$$

Then, by the Chebyshev inequality we obtain, for every $N \in \mathbb{N}$,

$$\mathbb{P}(B_N^c) \leqslant \frac{1}{N^2 r_N^2} \mathbb{E}\left[\|X_n\|^2_{L^\infty(0,N;V)}\right] \leqslant \frac{\varepsilon}{2cN^2}.$$

Set

$$B := \left\{u \in Z_\infty : \|u\|_{L^\infty(0,N;V)} \leqslant N r_N, \text{ for every } N \in \mathbb{N}\right\} = \bigcap_{N\in\mathbb{N}} B_N.$$

We have

$$\mathbb{P}(B^c) \leqslant \sum_{N\in\mathbb{N}} \mathbb{P}(B_N^c) \leqslant \frac{\varepsilon}{2}.$$

By lemma 4.3, we can use the Aldous condition $[A]$ to get a Borel subset A of $C([0,\infty);V^*)$ such that

$$\mathrm{Law}_{\mathbb{P}}(X_n)(A) \geqslant 1 - \frac{\varepsilon}{2}, \quad n \in \mathbb{N},$$
$$\lim_{\delta\to 0} \sup_{u\in A} \sup_{s,t\in[0,N]:|t-s|\leqslant\delta} \|u(t) - u(s)\|_{V^*} = 0 \text{ for every } N \in \mathbb{N}.$$

We define $K := \overline{A \cap B}$. By proposition 4.1 this set K is compact in Z_∞. Moreover for all $n \in \mathbb{N}$ we have

$$\mathrm{Law}_{\mathbb{P}}(X_n)(K) \geqslant \mathrm{Law}_{\mathbb{P}}(X_n)(A \cap B) \geqslant \mathrm{Law}_{\mathbb{P}}(X_n)(A) - \mathrm{Law}_{\mathbb{P}}(X_n)(B^c) \geqslant 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon.$$

This completes the proof. □

In metric spaces, one can apply Prokhorov theorem (see [Pa67], theorem II.6.7) and Skorohod theorem (see [B99, theorem 6.7]) to obtain convergence from tightness. Since the space Z_∞ is a locally convex space, we use the following generalisation to nonmetric spaces.

Proposition 4.5 (Skorohod–Jakubowski). *Let $\mathcal{X}$ be a topological space such that there is a sequence of continuous functions $f_m : \mathcal{X} \to \mathbb{C}$ that separates points of $\mathcal{X}$. Let $\mathcal{A}$ be the σ-algebra generated by $(f_m)_m$. Then, we have the following assertions:*

(a) Every compact set $K \subset \mathcal{X}$ is metrisable.
(b) Let $(\mu_n)_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $(\mathcal{X}, \mathcal{A})$. Then, there are a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$, random variables X_k (for $k \in \mathbb{N}$) and X on a common probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with $\mathrm{Law}_{\tilde{\mathbb{P}}}(X_k) = \mu_{n_k}$ for $k \in \mathbb{N}$, and $X_k \to X$ $\tilde{\mathbb{P}}$-almost surely for $k \to \infty$.

We stated proposition 4.5 in the form of [BO11]; see also [Jak98], where it was first used to construct martingale solutions for stochastic geometric wave equations. We apply this result to get the final result of this section.

Corollary 4.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted V^*-valued processes satisfying the Aldous condition [A] in V^* and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|^2_{L^\infty(0,T;V)}\right] < \infty, \textit{for every } T > 0.$$

Then, there are a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ and random variables $\tilde{X}_k, \tilde{X}$ for $k \in \mathbb{N}$ on another probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{X}_k) = \mathrm{Law}_{\mathbb{P}}(X_{n_k})$ for $k \in \mathbb{N}$, and $\tilde{X}_k \to \tilde{X}$ $\tilde{\mathbb{P}}$-almost surely in Z_∞ for $k \to \infty$.

Proof. This proof is also a minor modification of the proof of [BHW19, corollary 4.7].

Let us recall that Z_∞ is a locally convex space. Therefore, the assertion follows by an application of propositions 4.4 and 4.5 if for each of the spaces in the definition of Z_∞ we find a sequence $f_m : Z_\infty \to \mathbb{R}$ of continuous functions separating points which generates the Borel σ-field. The separable Fréchet spaces $C([0,\infty);V^*)$ and $L^{\alpha+1}_{\mathrm{loc}}(0,\infty;L^{\alpha+1})$ have this property.

Let $\{h_m : m \in \mathbb{N}\}$ be a dense subset of V^*. Then, we define the countable set $F := \{f_{m,t} : m \in \mathbb{N}, t \in [0,\infty) \cap \mathbb{Q}\}$ of functionals on $C_w([0,\infty);V)$ by

$$f_{m,t}(u) := \langle u(t), h_m \rangle, \;\; u \in C_w([0,\infty);V),$$

for $m \in \mathbb{N}$, $t \in [0,\infty) \cap \mathbb{Q}$ and $u \in C_w([0,\infty);V)$.

The set F separates points, since for $u, v \in C_w([0,\infty);V)$ with $f_{m,t}(u) = f_{m,t}(v)$ for all $m \in N$ and $t \in [0,\infty) \cap \mathbb{Q}$, we get $\langle u, h_m \rangle = \langle v, h_m \rangle$ on $[0,\infty)$ for all $m \in \mathbb{N}$ by continuous continuation and therefore $u = v$ on $[0,\infty)$.

Furthermore, the density of $\{h_m : m \in \mathbb{N}\}$ and the definition of the locally convex topology yield that $(f_{m,t})_{m\in\mathbb{N}, t\in[0,\infty)\cap\mathbb{Q}}$ generate the Borel σ-algebra on $C_w([0,\infty);V)$. □

5. Existence of a martingale solution

In this section we prove the existence of at least one martingale solution, see definition 3.1. In this way we prove part (i) of theorem 3.4; moreover we provide an estimate over the time interval $[0,\infty)$.

5.1. Statement of the existence result

Keeping in mind the definition of the energy functional $\mathcal{E}$ given in (3.3), we state the main result of this section.

Theorem 5.1. *Fix $r \in [1,\infty)$ and let μ be a Borel probability measure on V whose $r(\alpha+1)$th moment is finite. Under assumptions 2.1, 2.4, 2.7, there exists a martingale solution $\left(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}},\tilde{W},\tilde{\mathbf{W}},\tilde{\mathbb{F}},u\right)$ of equation (1.1) with the initial data μ which satisfies*

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\|u(t)\|_H^{2r}+\sup_{t\in[0,T]}\mathscr{E}(u(t))^r\right]<\infty,\ \text{for every } T>0. \tag{5.1}$$

Hence, in particular,

$$\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|u(t)\|_V^{2r}<\infty,\ \text{for every } T>0. \tag{5.2}$$

Moreover, if β satisfies condition (3.11), then

$$\sup_{t\geqslant 0}\tilde{\mathbb{E}}\|u(t)\|_V^2<\infty. \tag{5.3}$$

The proof of the existence part is based on a technique already used in [BMo13, BHW19]. We present the basic steps: the estimates on the Galerkin approximation and its convergence to a process which is a martingale solution. To be more precise, in proposition 5.6 we prove that there exists a unique global solution of the approximated problem and we obtain the *a priori* estimates in the space H. In proposition 5.7 we obtain the *a priori* estimates for the energy functional and prove the Aldous condition. These results lead to corollary 5.8 which, together with the Aldous condition, implies the tightness. In section 5.3 we prove the convergence of the Galerkin approximations to the martingale solution of our problem. The estimate (5.3) will be proved first for the Galerkin approximation in proposition 5.7 and then it will hold for the limit too.

More general assumptions can be considered only to prove the existence of a martingale solutions; see remark 5.10 at the end of this section.

5.2. *The Galerkin approximation and* a priori *estimates*

In this section we introduce the Galerkin approximation. We prove the well-posedness of the approximated equation and the uniform estimates for the solutions, that are sufficient to apply corollary 4.6.

Let us recall that the operator S was introduced in lemma 2.2. By the functional calculus we define the operators $P_n : H \to H$ by $P_n := \mathbf{1}_{(0,2^{n+1})}(S)$ for $n \in \mathbb{N}_0$. Since S has the representation (2.3), we observe that P_n is the orthogonal projection from H to $H_n := \operatorname{span}\left\{h_m : m \in \mathbb{N}, \lambda_m < 2^{n+1}\right\}$ and

$$P_n x = \sum_{\lambda_m<2^{n+1}} (x,h_m)_H h_m, \qquad x \in H.$$

Note that we have $h_m \in \bigcap_{k\in\mathbb{N}} \mathscr{D}(S^k)$ for $m \in \mathbb{N}$. Since by lemma 2.3(i) $\mathscr{D}(S^k) \hookrightarrow V$ for some $k \in \mathbb{N}$, we infer that H_n is a closed subspace of V for $n \in \mathbb{N}$. In particular, H_n is a closed subspace of V^*. The fact that the operators S and A commute implies that P_n and $A^{\frac{1}{2}}$ commute. Thus we get

$$\|P_n x\|_V^2 = \|P_n x\|_H^2 + \|A^{\frac{1}{2}} P_n x\|_H^2 = \|P_n x\|_H^2 + \|P_n A^{\frac{1}{2}} x\|_H^2 \leqslant \|x\|_V^2, \quad x \in V.$$

Moreover,

$$\|P_n x\|_H \leqslant \|x\|_H, \quad x \in H \qquad \text{and} \qquad \|P_n x\|_{V^*} \leqslant \|x\|_{V^*}, \quad x \in V^*,$$

and, recalling (2.4), (2.9) and (3.3),

$$\begin{aligned}\mathscr{E}(P_n x) &= \tfrac{1}{2}\|A^{\frac{1}{2}} P_n x\|_H^2 + \tfrac{1}{\alpha+1}\|P_n x\|_{L^{\alpha+1}}^{\alpha+1} \lesssim_\alpha \|P_n A^{\frac{1}{2}} x\|_H^2 + \|P_n x\|_V^{\alpha+1} \\ &\leqslant \|x\|_V^2 + \|x\|_V^{\alpha+1} \lesssim 1 + \|x\|_V^{\alpha+1}, \quad x \in V.\end{aligned} \tag{5.4}$$

We also have

$$\begin{aligned}&\lim_{n\to\infty} \|P_n x - x\|_V = 0 \\ &\lim_{n\to\infty} \|P_n x - x\|_{L^{\alpha+1}} \leqslant C \lim_{n\to\infty} \|P_n x - x\|_V = 0.\end{aligned}$$

By density, we can extend P_n to an operator $P_n : V^* \to H_n$ with $\|P_n\|_{V^* \to V^*} \leqslant 1$ and

$$\langle v, P_n v\rangle \in \mathbb{R}, \qquad \langle v, P_n w\rangle = (P_n v, w)_H, \qquad v \in V^*, \quad w \in V. \tag{5.5}$$

Unfortunately, the operators P_n, $n \in \mathbb{N}$, are not uniformly bounded from $L^{\alpha+1}$ to $L^{\alpha+1}$. This property is crucial in the proof of the *a priori* estimates of the stochastic terms. To overcome this deficit, in the following proposition we construct the sequence $(S_n)_{n\in\mathbb{N}}$ which enjoys the needed properties.

Proposition 5.2. *There exists a sequence* $(S_n)_{n\in\mathbb{N}}$ *of self-adjoint operators* $S_n : H \to H_n$ *for* $n \in \mathbb{N}$ *such that* $S_n\psi \to \psi$ *in* V *for* $n \to \infty$ *and* $\psi \in V$ *and the uniform norm estimates*

$$\sup_{n\in\mathbb{N}} \|S_n\|_{\mathscr{L}(H)} \leqslant 1, \quad \sup_{n\in\mathbb{N}} \|S_n\|_{\mathscr{L}(V)} \leqslant 1, \quad \sup_{n\in\mathbb{N}} \|S_n\|_{\mathscr{L}(L^{\alpha+1})} < \infty \tag{5.6}$$

hold.

Remark 5.3. Somehow, S_n represents a smoothed version of the indicator function $p_n := \mathbf{1}_{(0,2^{n+1})}$ used to define the operator P_n. This allows to use the spectral multiplier theorems to prove the uniform $L^{\alpha+1}$-boundedness of the sequence $(S_n)_{n\in\mathbb{N}_0}$. In [BHW19] the same result is proved by means of the abstract Littlewood–Paley theory rather than spectral multipliers theorems. Our proof follows the lines of [BHM20, proposition 10] and [Hor18b, lemma 3.14 and remark 4.15] with the difference that here we use the classical estimate from Ouhabaz [Ou09] instead of the results from Kunstmann and Uhl [KU15].

Proof of proposition 5.2. We take a function $\rho \in C_c^\infty(0,\infty)$ with $\text{supp}(\rho) \in \left[\frac{1}{2}, 2\right]$ and $\sum_{m\in\mathbb{Z}} \rho(2^{-m}t) = 1$ for all $t > 0$. For a fixed $n \in \mathbb{N}_0$ we introduce the function

$$s_n : (0,\infty) \to \mathbb{C}, \qquad s_n(\lambda) := \sum_{m=-\infty}^{n} \rho\left(2^{-m}\lambda\right)$$

and we see that

$$s_n(\lambda) = \begin{cases} 1 & \lambda \in (0, 2^n) \\ \rho(2^{-n}\lambda) & \lambda \in [2^n, 2^{n+1}) \\ 0 & \lambda \geqslant 2^{n+1}. \end{cases}$$

We define $S_n := s_n(S)$ via the functional calculus for self-adjoint operators. In particular, by lemma 2.2(ii), we have the representation

$$S_n x = \sum_{\lambda_m < 2^n} (x, h_m)_H h_m + \sum_{\lambda_m \in [2^n, 2^{n+1})} \rho(2^{-n}\lambda_m)(x, h_m)_H h_m, \qquad x \in H,$$

from which immediately follows that the range of S_n is contained in H_n. Since s_n is real-valued and bounded by 1, the operator S_n is self-adjoint with $\|S_n\|_{\mathscr{L}(H)} \leqslant 1$. Moreover, since by lemma 2.2(i), S_n and A commute, we obtain $\|S_n\|_{\mathscr{L}(V)} \leqslant 1$ and $S_n\psi \to \psi$ for all $\psi \in V$, by the convergence property of the functional calculus.

Finally, the uniform estimate in $L^{\alpha+1}$ is a consequence of the spectral multiplier theorem [Ou09, theorem 7.23] and the Marcinkiewicz interpolation theorem. It is sufficient to show, see [Ou09, equation (7.69)], that s_n satisfies

$$\sup_{\lambda>0} |\lambda^k s_n^{(k)}(\lambda)| < \infty, \qquad k = 0, 1, 2.$$

We have

$$\sup_{\lambda>0} |\lambda^k s_n^{(k)}(\lambda)| = \sup_{\lambda \in [2^n, 2^{n+1})} |\lambda^k s_n^{(k)}(\lambda)| = \sup_{\lambda \in [2^n, 2^{n+1})} |\lambda^k \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \rho(2^{-n}\lambda)| \leqslant 2^k \sup_{\lambda>0} |\rho^{(k)}(\lambda)| < \infty,$$

for all $k \in \mathbb{N}$. This completes the proof of proposition 5.2. □

Let u_0 be an $\mathscr{F}_0$-measurable V-valued random variable such that $\mathrm{Law}_{\mathbb{P}}(u(0)) = \mu$ on $\mathscr{B}(V)$. Then, since $2r$th moment of the Borel probability measure μ on V is finite, we infer that the $2r$th moment of u_0 is finite.

Using the operators P_n and S_n, $n \in \mathbb{N}$, we approximate our original problem (1.1) by the stochastic differential equation in H_n given by

$$\begin{cases} \mathrm{d}u_n(t) = -[\mathrm{i}Au_n(t) + \mathrm{i}P_n F(u_n(t)) + \beta u_n(t)]\,\mathrm{d}t - \mathrm{i}S_n B(S_n u_n(t)) \circ \mathrm{d}W(t) \\ \qquad\qquad - \mathrm{i}S_n G(S_n u_n(t))\,\mathrm{d}\mathbf{W}(t) \\ u_n(0) = P_n u_0. \end{cases}$$

With the Stratonovich correction term

$$b_n := -\frac{1}{2} \sum_{m=1}^{\infty} (S_n B_m S_n)^2,$$

the approximated problem can be written in the following Itô form

$$\begin{cases} \mathrm{d}u_n(t) = -[\mathrm{i}Au_n(t) + \mathrm{i}P_n F(u_n(t)) + \beta u_n(t) - b_n(u_n(t))]\,\mathrm{d}t \\ \qquad\qquad - \mathrm{i}S_n B(S_n u_n(t))\,\mathrm{d}W(t) - \mathrm{i}S_n G(S_n u_n(t))\,\mathrm{d}\mathbf{W}(t) \\ u_n(0) = P_n u_0. \end{cases} \tag{5.7}$$

By the well known theory of finite dimensional stochastic differential equations with locally Lipschitz coefficients, we get a local well-posedness result for (5.7).

Proposition 5.4. *Suppose assumptions 2.1, 2.4, 2.7 hold. Assume that $r \in [1,\infty)$. If u_0 is an $\mathscr{F}_0$-measurable H-valued random variable with finite $2r$th moment, then for each $n \in \mathbb{N}$ there is a unique local solution u_n of (5.7) with continuous paths in H_n and maximal existence time τ_n, which is a blow-up time in the sense that*

$$\limsup_{t \nearrow \tau_n(\omega)} \|u_n(t,\omega)\|_{H_n} = \infty$$

for almost all $\omega \in \Omega$ with $\tau_n(\omega) < \infty$.

Now we introduce a technical lemma used in many instances for *a priori* estimates, see [BHW19, lemma 5.6].

Lemma 5.5. *Let $r \in [1,\infty)$, $\varepsilon > 0$, $T > 0$ and $f \in L^r(\Omega, L^\infty(0,T))$. Then,*

$$\|f\|_{L^r(\Omega,L^2(0,t))} \leqslant \varepsilon \|f\|_{L^r(\Omega,L^\infty(0,t))} + \frac{1}{4\varepsilon}\int_0^t \|f\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s, \qquad t \in [0,T].$$

We prove *a priori* estimates in the space H so to get global existence. We work on any bounded time interval.

Proposition 5.6. *Suppose assumptions 2.1, 2.4 and 2.7 hold. Assume that $r \in [1,\infty)$. If u_0 is an $\mathscr{F}_0$-measurable H-valued random variable with $2r$th moment finite, then for each $n \in \mathbb{N}$ there exists a unique global solution u_n of (5.7) with continuous paths in H_n. Moreover, for every finite $T > 0$,*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^{2r}\right] < \infty. \tag{5.8}$$

Proof. *Step 1:* we fix $n \in \mathbb{N}$ and take the unique maximal solution (u_n, τ_n) from proposition 5.4. We prove that the solution is global appealing to the Khasmiskii's test for non explosion, see [Kah80]. Let us introduce a sequence $\{\tau_{n,k}\}_{k\in\mathbb{N}}$ of stopping times defined by

$$\tau_{n,k} := \inf\{t \geqslant 0 : \|u_n(t)\|_{H_n} \geqslant k\}, \qquad k \in \mathbb{N}.$$

In order to prove that $\tau_n = +\infty$ $\mathbb{P}$-a.s. it is sufficient to find a Liapunov function $\mathscr{V} : H \to \mathbb{R}$ satisfying

$$\mathscr{V} \geqslant 0 \qquad \text{on} \quad H, \tag{5.9}$$

$$a_k := \inf\{\mathscr{V}(v) : \|v\|_H \geqslant k\} \to \infty, \qquad \text{as} \quad k \to \infty,$$

$$\mathbb{E}[\mathscr{V}(u_n(0))] < \infty \tag{5.10}$$

such that

$$\mathbb{E}[\mathscr{V}(u_n(t\wedge\tau_{n,k}))] \leqslant \mathbb{E}[\mathscr{V}(u_n(0))] + C\int_0^t (1 + \mathbb{E}[\mathscr{V}(u_n(s\wedge\tau_{n,k}))])\,\mathrm{d}s \tag{5.11}$$

for a constant $C < \infty$ and all $t \geqslant 0$ and $k \in \mathbb{N}$. The idea is the following: once such a function $\mathscr{V}$ is found, by the Gronwall's lemma we infer

$$\mathbb{E}[\mathscr{V}(u_n(t\wedge\tau_{n,k})] \leqslant \mathrm{e}^{Ct}(1 + \mathbb{E}[\mathscr{V}(u_n(0))]), \quad t \geqslant 0,$$

which implies

$$\mathbb{P}(\tau_{n,k} < t) \leqslant \frac{1}{a_k}\mathbb{E}\left[\mathbb{1}_{\{\tau_{n,k}<t\}}\mathscr{V}(u_n(t\wedge\tau_{n,k}))\right] \leqslant \frac{1}{a_k}\mathrm{e}^{Ct}(1 + \mathbb{E}[\mathscr{V}(u_n(0))]).$$

Passing to the limit, we get

$$\lim_{k\to\infty}\mathbb{P}(\tau_{n,k}<t)\leqslant \mathrm{e}^{Ct}(1+\mathbb{E}[\mathscr{V}(u(0))])\lim_{k\to\infty}\frac{1}{a_k}=0,$$

for every fixed $t\geqslant 0$. Therefore $\mathbb{P}(\tau_n<t)=\lim_{k\to\infty}\mathbb{P}(\tau_{n,k}<t)=0$ for every fixed $t\geqslant 0$, which means $\mathbb{P}(\tau_n=+\infty)=1$.

Set

$$\mathscr{V}(v):=\|v\|_H^2,\qquad v\in H_n. \tag{5.12}$$

The function $\mathscr{V}\in C^2(H)$, $\mathscr{V}$ is uniformly continuous on bounded sets and satisfies (5.9) and (5.10). Moreover, $\mathbb{E}[\mathscr{V}(P_nu_0)]<\infty$ is equivalent to $\mathbb{E}\left[\|P_nu_0\|_H^2\right]<\infty$.

In order to derive inequality (5.11) we use the Itô formula. The function $\mathscr{V}:H_n\to\mathbb{R}$ defined in (5.12) is twice continuously Fréchet-differentiable with

$$\mathscr{V}'[v]h_1=2\operatorname{Re}(v,h_1)_H,\qquad \mathscr{V}''[v][h_1,h_2]=2\operatorname{Re}(h_1,h_2)_H,$$

for $v,h_1,h_2\in H_n$. We look for estimates for $\|u_n\|_H^2$. The Itô formula for this real process involves only real quantities, expressed by means of the real part of the complex scalar product in H.

For a fixed $v\in H_n$ and $m\in\mathbb{N}$, we have some basic relationships:

$$\begin{aligned}
\operatorname{Re}(v,-\mathrm{i}Av)_H&=\operatorname{Re}\left[\mathrm{i}\|A^{\frac12}v\|_H^2\right]=0,\\
\operatorname{Re}(v,-\mathrm{i}P_nF(v))_H&=\operatorname{Re}\langle \mathrm{i}v,F(v)\rangle=0,\\
2\operatorname{Re}(v,b_n(v))_H&=-\sum_{m=1}^\infty\operatorname{Re}\left(v,(S_nB_mS_n)^2v\right)_H=-\sum_{m=1}^\infty\|S_nB_mS_nv\|_H^2,\\
\operatorname{Re}(v,-\mathrm{i}S_nB(S_nv)f_m)_H&=\operatorname{Re}\left[\mathrm{i}(v,S_nB_mS_nv)_H\right]=0,
\end{aligned}$$

where we used (5.5) and (2.6) for the second term and the fact that the operator $S_nB_mS_n$ is self-adjoint for the third and four terms.

Therefore, by the Itô formula we get, for $t\geqslant 0$,

$$\begin{aligned}
\|u_n(t\wedge\tau_{n,k})\|_H^2&=\|P_nu_0\|_H^2-2\beta\int_0^{t\wedge\tau_{n,k}}\|u_n(s)\|_H^2\mathrm{d}s+\int_0^{t\wedge\tau_{n,k}}\|S_nG(S_nu_n(s))\|_{\gamma(Y_2,H)}^2\mathrm{d}s\\
&\quad+2\int_0^{t\wedge\tau_{n,k}}\operatorname{Re}(u_n(s),-\mathrm{i}S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s))_H.
\end{aligned}$$

To estimate the term in the right hand side (RHS) of the above equation we introduce the stochastic process

$$\alpha_k(t):=\mathbf{1}_{(t<\tau_{n,k})}.$$

and we notice that

$$\int_0^{t\wedge\tau_{n,k}}\|u_n(s)\|_H^2\mathrm{d}s=\int_0^t\|\alpha_k(s)u_n(s)\|_H^2\mathrm{d}s\leqslant\int_0^t\|u_n(s\wedge\tau_{n,k})\|_H^2\mathrm{d}s.$$

Therefore, by proposition 5.2 and (2.19) we get

$$\|S_nG(S_nu_n(s))\|^2_{\gamma(Y_2,H)} \leqslant 2C_1^2 + 2\tilde{C}_1^2\|u_n(s)\|^2_H.$$

So we obtain

$$\|u_n(t\wedge\tau_{n,k})\|^2_H \leqslant \|P_nu_0\|^2_H + 2\int_0^t \left(C_1^2 + \left(-\beta+\tilde{C}_1^2\right)\|u_n(s\wedge\tau_{n,k})\|^2_H\right)\mathrm{d}s \\ + 2\int_0^{t\wedge\tau_{n,k}} \mathrm{Re}\left(u_n(s), -\mathrm{i}S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s)\right)_H, \qquad t\geqslant 0.$$

Taking now the expected value on both sides, we obtain

$$\mathbb{E}\left[\|u_n(t\wedge\tau_{n,k})\|^2_H\right] \leqslant \mathbb{E}\left[\|P_nu_0\|^2_H\right] + 2\int_0^t \left(C_1^2 + \left(-\beta+\tilde{C}_1^2\right)\mathbb{E}\left[\|u_n(s\wedge\tau_{n,k})\|^2_H\right]\right)\mathrm{d}s.$$

This proves (5.11) and so we conclude the proof of the global existence of the solution.

Step 2: we now prove estimate (5.8). Let us fix $T>0$. We start from equality

$$\|u_n(t)\|^2_H = \|P_nu_0\|^2_H - 2\beta\int_0^t \|u_n(s)\|^2_H\,\mathrm{d}s + \int_0^t \|S_nG(S_nu_n(s))\|^2_{\gamma(Y_2,H)}\,\mathrm{d}s \\ + 2\int_0^t \mathrm{Re}\left(u_n(s), -\mathrm{i}S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s)\right)_H, \text{for } t\in[0,T],$$

and we apply the $L^r(\Omega, L^\infty(0,T))$-norm to this identity. From proposition 5.2 and (2.19) we immediately get

$$\left\|\int_0^\cdot \|S_nG(S_nu_n(s))\|^2_{\gamma(Y_2,H)}\,\mathrm{d}s\right\|_{L^r(\Omega,L^\infty(0,T))} \leqslant 2C_1^2T + 2\tilde{C}_1^2\left\|\int_0^T \|u_n(s)\|^2_H\,\mathrm{d}s\right\|_{L^r(\Omega)}.$$

The Minkowski inequality yields

$$\left\|\int_0^T \|u_n(s)\|^2_H\,\mathrm{d}s\right\|_{L^r(\Omega)} \leqslant \left\|\int_0^T \sup_{r\in[0,s]}\|u_n(r)\|^2_H\,\mathrm{d}s\right\|_{L^r(\Omega)} \\ \leqslant \int_0^T \left\|\sup_{r\in[0,s]}\|u_n(r)\|^2_H\right\|_{L^r(\Omega)}\mathrm{d}s = \int_0^T \left\|\|u_n\|^2_H\right\|_{L^r(\Omega,L^\infty(0,s))}\mathrm{d}s.$$

By means of the Burkholder–Davis–Gundy and the Young inequalities, lemma 5.5 and (2.19) we obtain, for some constant C (depending on r)

$$\left\|\int_0^\cdot \mathrm{Re}\left(u_n(s), -\mathrm{i}S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s)\right)_H\right\|_{L^r(\Omega,L^\infty(0,T))} \\ \leqslant C\left\|\left(\int_0^T\sum_{m=1}^\infty \left(u_n(s), -\mathrm{i}S_nG(S_nu_n(s))e_m\right)^2_H\mathrm{d}s\right)^{1/2}\right\|_{L^r(\Omega)} \\ \leqslant C\left\|\|G(S_nu_n)\|_{\gamma(Y_2,H)}\|u_n\|_H\right\|_{L^r(\Omega,L^2(0,T))} \\ \leqslant C\left\|\left(C_1+\tilde{C}_1\|u_n\|_H\right)\|u_n\|_H\right\|_{L^r(\Omega,L^2(0,T))}$$

$$\lesssim \left\| 1 + \|u_n\|_H^2 \right\|_{L^r(\Omega, L^2(0,T))}$$
$$\leqslant 1 + T + \left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^2(0,T))}$$
$$\leqslant 1 + T + \varepsilon \left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^\infty(0,T))} + \frac{1}{4\varepsilon} \int_0^T \left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^\infty(0,s))} \mathrm{d}s,$$

for any $\varepsilon > 0$.

Collecting the above estimates, we obtain for any n

$$\left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^\infty(0,T))} \leqslant \left\| \|u_0\|_H^2 \right\|_{L^r(\Omega)} + C + CT + + \varepsilon C \left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^\infty(0,T))} + \left(2|\beta| + \frac{C}{4\varepsilon} \right) \int_0^T \left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^\infty(0,s))} \mathrm{d}s.$$

If we choose $\varepsilon > 0$ small enough, we can apply the Gronwall's lemma and we get that there exists a positive constant C, independent of n (but depending on T and other parameters) such that

$$\left\| \|u_n\|_H^2 \right\|_{L^r(\Omega, L^\infty(0,T))} \leqslant C,$$

for any $n \in \mathbb{N}$. □

The next goal is to find uniform energy estimates for the global solutions of the equation (5.7). Recall that the nonlinearity F has a real antiderivative denoted by $\hat{F}$ and the energy functional $\mathscr{E}$ is defined in formula (3.3).

The next proposition is the key step to show that we can apply corollary 4.6 to the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the equation (5.7).

Proposition 5.7. *Fix $r \in [1, \infty)$ and let u_0 be an $\mathscr{F}_0$-measurable V-valued random variable whose $r(\alpha+1)$th moment is finite. Under assumptions 2.1, 2.4 and 2.7 the following assertions hold.*

(a) For every finite $T > 0$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(t)\|_H^{2r} + \sup_{t\in[0,T]} \mathscr{E}(u_n(t))^r \right] < \infty.$$

(b) The sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition [A] in V^*.
(c) If in addition β satisfies condition (3.11), then

$$\sup_{n\in\mathbb{N}} \sup_{t\geqslant 0} \mathbb{E}\|u_n(t)\|_V^2 < \infty. \tag{5.13}$$

(ad a). Let us fix $T > 0$. Thanks to (5.8) it is enough to prove the estimate for the energy. By lemma 2.5, the restriction of the energy $\mathscr{E} : H_n \to \mathbb{R}$ is twice continuously Fréchet-differentiable with

$$\mathscr{E}'[v]h_1 = \mathrm{Re}\langle Av + F(v), h_1\rangle,$$
$$\mathscr{E}''[v][h_1, h_2] = \mathrm{Re}\left(A^{\frac{1}{2}}h_1, A^{\frac{1}{2}}h_2\right)_H + \mathrm{Re}\langle F'[v]h_2, h_1\rangle,$$

for $v, h_1, h_2 \in H_n$. We look for estimates on $\mathscr{E}(u_n)$.

Notice that

$$\operatorname{Re}\left[\langle Av,-\mathrm{i}P_nF(v)\rangle+\langle F(v),-\mathrm{i}Av\rangle\right]=\operatorname{Re}\left[-\langle Av,\mathrm{i}F(v)\rangle+\overline{\langle Av,\mathrm{i}F(v)\rangle}\right]=0,$$
$$\operatorname{Re}(Av,-\mathrm{i}Av)_H=\operatorname{Re}\left[\mathrm{i}\|Av\|_H^2\right]=0,$$

for all $v\in H_n$, and we can use (5.5) for

$$\operatorname{Re}\langle F(v),-\mathrm{i}P_nF(v)\rangle=\operatorname{Re}\left[\mathrm{i}\langle F(v),P_nF(v)\rangle\right]=0.$$

Therefore, the Itô formula leads to the identity

$$\begin{aligned}
\mathscr{E}(u_n(t))&=\mathscr{E}(P_nu_0)+\int_0^t\operatorname{Re}\langle Au_n(s)+F(u_n(s)),b_n(u_n(s))-\beta u_n(s)\rangle\,\mathrm{d}s\\
&\quad+\int_0^t\operatorname{Re}\langle Au_n(s)+F(u_n(s)),-\mathrm{i}(S_nBS_nu_n(s))\,\mathrm{d}W(s)\rangle\\
&\quad+\int_0^t\operatorname{Re}\langle Au_n(s)+F(u_n(s)),-\mathrm{i}S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s)\rangle\\
&\quad+\frac12\int_0^t\|A^{\frac12}S_nBS_nu_n(s)\|^2_{\gamma(Y_1,H)}\,\mathrm{d}s+\frac12\int_0^t\|A^{\frac12}S_nGS_nu_n(s)\|^2_{\gamma(Y_2,H)}\,\mathrm{d}s\\
&\quad+\frac12\int_0^t\sum_{m=1}^\infty\operatorname{Re}\langle F'[u_n(s)](S_nBS_nu_n(s))f_m,(S_nBS_nu_n(s))f_m\rangle\,\mathrm{d}s\\
&\quad+\frac12\int_0^t\sum_{m=1}^\infty\operatorname{Re}\langle F'[u_n(s)](S_nG(S_nu_n(s))e_m),S_nG(S_nu_n(s))e_m\rangle\,\mathrm{d}s,
\end{aligned}\tag{5.14}$$

almost surely for all $t\in[0,T]$.

Let us introduce the short notation

$$Z(u):=\|u\|_H^2+2\mathscr{E}(u)=\|u\|_V^2+2\hat{F}(u),\quad u\in V.\tag{5.15}$$

We will estimate the various terms that appear in the RHS of (5.14) in the $L^r(\Omega,L^\infty(0,T))$-norm. To slightly simplify the proof we will neglect here the dissipation term by assuming $\beta\geqslant0$. The case $\beta<0$ can be treated as in the proof of proposition 5.7(c), see appendix B. We set $I_{1,n}(u)(t)=2\int_0^t\operatorname{Re}\langle Au(s)+F(u(s)),b_n(u(s))\rangle\,\mathrm{d}s$. We have

$$\|I_{1,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\leqslant\left\|\int_0^T\left|\sum_{m=1}^\infty\operatorname{Re}\langle Au_n(s)+F(u_n(s)),(S_nB_mS_n)^2u_n(s)\rangle\right|\mathrm{d}s\right\|_{L^r(\Omega)}.$$

Using the bound (5.6) and assumption 2.7(ii), see also remark 2.10, by means of the Young inequality we get

$$\begin{aligned}
\left|\sum_{m=1}^\infty\operatorname{Re}\langle Au_n,(S_nB_mS_n)^2u_n\rangle\right|&\leqslant\|A^{\frac12}u_n\|_H\sum_{m=1}^\infty\|A^{\frac12}(S_nB_mS_n)^2u_n)\|_H\\
&\leqslant\|A^{\frac12}u_n\|_H\|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}\|u_n\|_V\\
&\leqslant\|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}\left(\|A^{\frac12}u_n\|_H^2+\|u_n\|_H^2\right)\\
&\lesssim\|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}Z(u_n).
\end{aligned}$$

On the other hand, (5.6), assumptions 2.7(ii), (2.5) and (2.9) lead to

$$\begin{aligned}\left|\sum_{m=1}^{\infty}\mathrm{Re}\langle F(u_n),(S_nB_mS_n)^2u_n\rangle\right| &\leqslant \|F(u_n)\|_{L^{\frac{\alpha+1}{\alpha}}}\sum_{m=1}^{\infty}\|(S_nB_mS_n)^2u_n\|_{L^{\alpha+1}}\\ &\lesssim \hat{F}(u_n)\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\\ &\lesssim \|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}Z(u_n).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\|I_{1,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\\ &\quad\lesssim\left(\|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}+\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\right)\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s. \qquad (5.16)\end{aligned}$$

We set $I_{2,n}(u)(t)=2\int_0^t\mathrm{Re}\langle Au(s)+F(u(s)),-\mathrm{i}S_nB(S_nu(s))\,\mathrm{d}W(s)\rangle$. We employ the Burkholder–Davis–Gundy inequality to get

$$\|I_{2,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\leqslant 2\left\|\left(\sum_{m=1}^{\infty}|\mathrm{Re}\langle Au_n+F(u_n),-\mathrm{i}(S_nBS_nu_n)f_m\rangle|^2\right)^{\frac{1}{2}}\right\|_{L^r(\Omega,L^2(0,T))}.$$

Using the bound (5.6), assumption 2.7(ii) and the Young inequality, we estimate

$$\begin{aligned}\left(\sum_{m=1}^{\infty}|\mathrm{Re}\langle Au_n,-\mathrm{i}(S_nBS_nu_n)f_m\rangle|^2\right)^{\frac{1}{2}} &\leqslant \|A^{\frac{1}{2}}u_n\|_H\|B(S_nu_n)\|_{\gamma(Y_1,V)}\\ &\leqslant \|A^{\frac{1}{2}}u_n\|_H\|B\|_{\mathcal{L}(V,\gamma(Y_1,V))}\|u_n\|_V\\ &\leqslant \|B\|_{\mathscr{L}(V,\gamma(Y_1,V))}\left(\|A^{\frac{1}{2}}u_n\|_H^2+\|u_n\|_H^2\right)\\ &\lesssim \|B\|_{\mathscr{L}(V,\gamma(Y_1,V))}Z(u_n).\end{aligned}$$

On the other hand (5.6), assumptions 2.7(ii), (2.5) and (2.9) lead to

$$\begin{aligned}\left(\sum_{m=1}^{\infty}|\mathrm{Re}\langle F(u_n),-\mathrm{i}(S_nBS_nu_n))f_m\rangle|^2\right)^{\frac{1}{2}} &\lesssim \|B\|_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\hat{F}(u_n)\\ &\lesssim \|B\|_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}Z(u_n).\end{aligned}$$

Therefore, the latter three estimates with lemma 5.5 yield

$$\begin{aligned}\|I_{2,n}(u)\|_{L^r(\Omega,L^\infty(0,T))} &\lesssim \left(\|B\|_{\mathscr{L}(V,\gamma(Y_1,V))}+\|B\|_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\right)\|Z(u_n)\|_{L^r(\Omega,L^2(0,T))}\\ &\lesssim \varepsilon\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,T))}+\frac{1}{4\varepsilon}\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s,\end{aligned}$$

for any $\varepsilon>0$.

We set $I_{3,n}(u)(t) = 2\int_0^t \mathrm{Re}\langle Au(s)+F(u(s)), -\mathrm{i}S_nG(S_nu(s))\,\mathrm{d}\mathbf{W}(s)\rangle$. Also for this stochastic integral we employ the Burkholder–Davis–Gundy's inequality to get

$$\|I_{3,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))} \leqslant 2\left\|\left(\sum_{m=1}^{\infty}|\mathrm{Re}\langle Au_n+F(u_n), -\mathrm{i}S_nG(S_nu_n))e_m\rangle|^2\right)^{\frac{1}{2}}\right\|_{L^r(\Omega,L^2(0,T))}.$$

Using the bound (5.6) and assumption 2.7(iii), exploiting the Young inequality we obtain

$$\begin{aligned}\left(\sum_{m=1}^{\infty}|\mathrm{Re}\langle Au_n, -\mathrm{i}(S_nG(S_nu_n))e_m\rangle|^2\right)^{\frac{1}{2}} &\lesssim \|A^{\frac{1}{2}}u_n\|_H\|G(S_nu_n)\|_{\gamma(Y_2,V)}\\ &\lesssim \|A^{\frac{1}{2}}u_n\|_H\left(C_2+\tilde{C}_2\|u_n\|_V\right)\\ &\lesssim 1+Z(u_n).\end{aligned}$$

Moreover, (5.6), assumptions 2.7(iii), (2.5), (2.9), (2.10) and the Young inequality lead to

$$\begin{aligned}\left(\sum_{m=1}^{\infty}|\mathrm{Re}\langle F(u_n), -\mathrm{i}S_nG(S_nu_n)e_m\rangle|^2\right)^{\frac{1}{2}} &\leqslant \|F(u_n)\|_{L^{\frac{\alpha+1}{\alpha}}}\|G(S_nu_n)\|_{\gamma(Y_2,L^{\alpha+1})}\\ &\lesssim \|u_n\|_{L^{\alpha+1}}^{\alpha}\left(C_3+\tilde{C}_3\|u_n\|_{L^{\alpha+1}}\right)\\ &\lesssim \left[\hat{F}(u_n)\right]^{\frac{\alpha}{\alpha+1}}+\hat{F}(u_n)\\ &\lesssim 1+\hat{F}(u_n)\\ &\lesssim 1+Z(u_n).\end{aligned}$$

Therefore, collecting the previous estimates, by means of lemma 5.5 we obtain

$$\begin{aligned}\|I_{3,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))} &\lesssim \|1+Z(u_n)\|_{L^r(\Omega,L^2(0,T))}\\ &\lesssim 1+\varepsilon\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,T))}+\frac{1}{4\varepsilon}\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s,\end{aligned}$$

for any $\varepsilon>0$.

We set $I_{4,n}(u)(t) = \int_0^t\|A^{\frac{1}{2}}S_nBS_nu(s)\|^2_{\gamma(Y_1,H)}\,\mathrm{d}s$. Exploiting remark 2.10 and bound (5.6), we easily obtain

$$\|I_{4,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))} \lesssim \|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s.$$

We set $I_{5,n}(u)(t) = \int_0^t\|A^{\frac{1}{2}}S_nG(S_nu(s))\|^2_{\gamma(Y_2,H)}\,\mathrm{d}s$. Similarly, estimate (2.16) and bound (5.6) yield

$$\|A^{\frac{1}{2}}S_nG(S_nu_n)\|^2_{\gamma(Y_2,H)} \leqslant \left(C_2+\tilde{C}_2\|u_n\|_V\right)^2 \lesssim 1+Z(u_n).$$

Hence

$$\|I_{5,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))} \lesssim 1+\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s.$$

We set $I_{6,n}(u)(t)=\int_0^t\sum_{m=1}^{\infty}\mathrm{Re}\langle F'[u(s)]\,(S_nB(S_nu(s))f_m)\,,S_nB(S_nu(s))f_m\rangle\,\mathrm{d}s$. From (2.8), remark 2.10, (5.6) and (2.9) we get

$$\begin{aligned}\left|\sum_{m=1}^{\infty}\mathrm{Re}\langle F'[u_n]\,(S_nBS_nu_n)f_m,(S_nBS_nu_n)f_m\rangle\right| &\lesssim \|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\|u_n\|^{\alpha+1}_{L^{\alpha+1}}\\ &\lesssim \|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\hat{F}(u_n)\,.\end{aligned}$$

Hence,

$$\|I_{6,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\lesssim\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s.$$

We set $I_{7,n}(u)(t)=\int_0^t\sum_{m=1}^{\infty}\mathrm{Re}\langle F'[u(s)]\,(S_nG(S_nu(s))e_m)\,,S_nG(S_nu(s))e_m\rangle\,\mathrm{d}s$. By means of (2.11), (2.17), (5.6) and the Young inequality we get

$$\begin{aligned}\left|\sum_{m=1}^{\infty}\mathrm{Re}\langle F'[u_n]\,(S_nG(S_nu_n)\,e_m)\,,S_nG(S_nu_n)\,e_m\rangle\right| &\leqslant \|F'[u_n]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}}\sum_{m=1}^{\infty}\|S_nG(S_nu_n)\,e_m\|^2_{L^{\alpha+1}}\\ &\lesssim \hat{F}(u_n)^{\frac{\alpha-1}{\alpha+1}}\|G(S_nu_n)\|^2_{\gamma(Y_2,L^{\alpha+1})}\\ &\leqslant \hat{F}(u_n)^{\frac{\alpha-1}{\alpha+1}}\left(C_3+\tilde{C}_3\|u_n\|_{L^{\alpha+1}}\right)^2\\ &\lesssim \hat{F}(u_n)^{\frac{\alpha-1}{\alpha+1}}+\hat{F}(u_n)\\ &\lesssim 1+\hat{F}(u_n)\leqslant 1+Z(u_n)\,.\end{aligned}$$

Therefore,

$$\|I_{7,n}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\lesssim 1+\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s. \tag{5.17}$$

We now go back to (5.14); using (5.16) and (5.17) we finally obtain

$$\begin{aligned}\|\mathscr{E}(u_n)\|_{L^r(\Omega,L^\infty(0,T))} &\leqslant \|\mathscr{E}(P_nu_0)\|_{L^r(\Omega)}+C+C\int_0^T\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s\\ &\quad+2\varepsilon\|Z(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\\ &\leqslant C\left(1+\|\|u_0\|_V^{\alpha+1}\|_{L^r(\Omega)}\right)+C+C\int_0^T 2\|\mathscr{E}(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s\\ &\quad+4\varepsilon\|\mathscr{E}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}+(CT+2\varepsilon)\left\|\|u_n\|_H^2\right\|_{L^r(\Omega,L^\infty(0,T))},\end{aligned} \tag{5.18}$$

for some positive constant C, independent of n. We have estimated the initial data thanks to (5.4). If we choose ε sufficiently small and bear in mind the *a priori* estimate (5.8) of proposition 5.6, we get

$$\|\mathscr{E}(u_n)\|_{L^r(\Omega,L^\infty(0,T))}\lesssim 1+\int_0^T\|\mathscr{E}(u_n)\|_{L^r(\Omega,L^\infty(0,s))}\,\mathrm{d}s.$$

By the Gronwall lemma we deduce the assertion of proposition 5.7, part (a). □

(ad b). We choose and fix $T>0$. Let us now prove the Aldous condition. The proof of this part is an extension of the proof of part (b) in [BHW19, proposition 5.7]. We will provide just the main steps. We have

$$\begin{aligned}
u_n(t)-P_nu_0 &= -\mathrm{i}\int_0^t Au_n(s)\,\mathrm{d}s-\mathrm{i}\int_0^t P_nF(u_n(s))\,\mathrm{d}s+\int_0^t b_n(u_n(s))\,\mathrm{d}s-\beta\int_0^t u_n(s)\,\mathrm{d}s\\
&\quad -\mathrm{i}\int_0^t S_nB(S_nu_n(s))\,\mathrm{d}W(s)-\mathrm{i}\int_0^t S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s)\\
&=: J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t)+J_6(t),
\end{aligned}$$

in H_n almost surely for all $t\in[0,T]$ and therefore

$$\|u_n((\tau_n+\theta)\wedge T)-u_n(\tau_n)\|_{V^*}\leqslant\sum_{k=1}^{6}\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}$$

for each sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times and $\theta>0$. Hence, we get

$$\mathbb{P}\{\|u_n((\tau_n+\theta)\wedge T)-u_n(\tau_n)\|_{V^*}\geqslant\eta\}\leqslant\sum_{k=1}^{6}\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}\geqslant\frac{\eta}{6}\right\},\tag{5.19}$$

for a fixed $\eta>0$. We aim to apply the Chebyshev inequality and estimate the expected value of each term in the sum. Proceeding as in the proof of part (b) in [BHW19, proposition 5.7] one obtains the following estimates:

$$\mathbb{E}\|J_1((\tau_n+\theta)\wedge T)-J_1(\tau_n)\|_{V^*}\leqslant\theta K_1,$$

$$\mathbb{E}\|J_2((\tau_n+\theta)\wedge T)-J_2(\tau_n)\|_{V^*}\leqslant\theta K_2,$$

$$\mathbb{E}\|J_3((\tau_n+\theta)\wedge T)-J_3(\tau_n)\|_{V^*}\leqslant\theta K_3,$$

$$\mathbb{E}\|J_5((\tau_n+\theta)\wedge T)-J_5(\tau_n)\|_{V^*}^2\leqslant\theta K_5.$$

We use part (a) to estimate

$$\begin{aligned}
\mathbb{E}\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|_{V^*} &\leqslant\beta\mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T}u_n(s)\,\mathrm{d}s\right\|_H\\
&\leqslant\beta\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|u_n(s)\|_H\,\mathrm{d}s\lesssim\theta\beta\mathbb{E}\left[\sup_{s\in[0,T]}\|u_n(s)\|_H\right]\leqslant\theta K_4.
\end{aligned}$$

The Itô isometry and (2.19) yield

$$
\begin{aligned}
\mathbb{E}\|J_6((\tau_n+\theta)\wedge T)-J_6(\tau_n)\|_{V^*}^2 &\leqslant \mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T} S_nG(S_nu_n(s))\,\mathrm{d}\mathbf{W}(s)\right\|_H^2\\
&= \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|S_nG(S_nu_n(s))\|_{\gamma(Y,H)}^2\,\mathrm{d}s\\
&\leqslant \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\left(2C_1^2+2\tilde{C}_1^2\|u_n(s)\|_H^2\right)\mathrm{d}s\\
&\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\left(1+\|u_n(s)\|_H^2\right)\mathrm{d}s\\
&\lesssim \theta+\theta\mathbb{E}\left[\sup_{s\in[0,T]}\|u_n(s)\|_H^2\right]\lesssim \theta K_6.
\end{aligned}
$$

By the Chebyschev inequality, we obtain for any given $\eta>0$

$$
\begin{aligned}
&\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}\geqslant\frac{\eta}{6}\right\}\\
&\quad\leqslant\frac{6}{\eta}\mathbb{E}\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}\leqslant\frac{6K_k\theta}{\eta}\text{for } k\in\{1,2,3,4\} \qquad (5.20)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}\geqslant\frac{\eta}{6}\right\}\\
&\quad\leqslant\frac{36}{\eta^2}\mathbb{E}\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}^2\leqslant\frac{36K_k\theta}{\eta^2}\text{for } k\in\{5,6\}. \qquad (5.21)
\end{aligned}
$$

Let us fix $\varepsilon>0$ and $\eta>0$. Due to estimates (5.20) and (5.21) we can choose $\delta_1,\dots,\delta_6>0$ such that

$$
\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}\geqslant\frac{\eta}{6}\right\}\leqslant\frac{\varepsilon}{6}
$$

for $0<\theta\leqslant\delta_k$ and $k=1,\dots,6$. With $\delta:=\min\{\delta_1,\dots,\delta_6\}$, using (5.19) we get

$$
\mathbb{P}\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{V^*}\geqslant\eta\}\leqslant\varepsilon
$$

for all $n\in\mathbb{N}$, $k=1,\dots,6$ and $0<\theta\leqslant\delta$ and therefore, the Aldous condition $[A]$ holds in V^*. □

(ad c). This point has some similarities with point (a) proved above. We prove it in appendix B. □

As an immediate consequence of propositions 5.6 and 5.7 and the fact that $\|u\|_V^{2r}\lesssim_r\|u\|_H^{2r}+\mathscr{E}(u)^r$ we obtain

Corollary 5.8. *Fix $r\in[1,\infty)$ and let u_0 be an $\mathscr{F}_0$-measurable V-valued random variable with finite $r(\alpha+1)$th moment. Then, under assumptions 2.1, 2.4, 2.7, the following bound holds*

$$
\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|u_n(t)\|_V^{2r}\right]<\infty,\quad \textit{for every } T>0. \qquad (5.22)
$$

5.3. Convergence. Proof of the first part of theorem 5.1

In this section we prove part (i) of theorem 5.1, that is the existence of a martingale solution of (1.1) which satisfies conditions (5.1) and (5.2).

We construct a solution to equation (1.1) by a suitable limiting process in the Galerkin equation (5.7), exploiting the results of the previous sections. Proposition 5.7(b) and corollary 5.8 provide the tightness to pass to the limit. One proceeds as in [BHW19, BMo13]. We will just provide the main steps of the proof and refer to these papers for more details.

Let us recall from section 4 the definition of the space Z_∞:

$$Z_\infty = C([0,\infty);V^*)\cap L^{\alpha+1}_{\text{loc}}([0,\infty);L^{\alpha+1})\cap C_w([0,\infty);V).$$

Proposition 5.7(b) and corollary 5.8 provide the *a priori* estimates on the Galerkin approximation sequence; hence, this is tight in Z_∞ thanks to proposition 4.4. Then by means of corollary 4.6 we get the convergence. More precisely, there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, a probability space $(\hat{\Omega},\hat{\mathscr{F}},\hat{\mathbb{P}})$ and random variables $v_k, v:\hat{\Omega}\to Z_\infty$ with $\text{Law}_{\hat{\mathbb{P}}}(v_k) = \text{Law}_{\mathbb{P}}(u_{n_k})$ such that

$$v_k \to v \quad \hat{\mathbb{P}}-\text{a.s.} \quad \text{in } Z_\infty \text{ for } k\to\infty. \tag{5.23}$$

Moreover, arguing as in the proof of [BHW19, proposition 6.1(b)], we infer that $v_k \in C([0,\infty),H_k)$ $\hat{\mathbb{P}}$-a.s. and

$$\sup_{k\in\mathbb{N}}\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|v_k(t)\|_H^{2r}+\sup_{t\in[0,T]}\mathcal{E}(v_k(t))^r\right]<\infty,\quad \text{for any } T>0,$$

from which, keeping in mind that $\|\cdot\|_V^{2r}\lesssim_r \|\cdot\|_H^{2r}+\mathscr{E}(\cdot)^r$, we also infer

$$\sup_{k\in\mathbb{N}}\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|v_k(t)\|_V^{2r}\right]<\infty,\quad \text{for any } T>0.$$

Let us remark that we also get

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|v(t)\|_H^{2r}+\sup_{t\in[0,T]}\mathcal{E}(v(t))^r\right]\leqslant \liminf_k \hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|v_k(t)\|_H^{2r}+\sup_{t\in[0,T]}\mathcal{E}(v_k(t))^r\right]<\infty.$$

Hence, by remark 3.5,

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|v(t)\|_V^{2r}\right]<\infty,\quad \text{for any } T>0. \tag{5.24}$$

The last two inequalities prove respectively inequalities (5.1) and (5.2).

Since each v_k has the same law as u_{n_k}, it is a martingale solution to equation (5.7): expanding the arguments in [BHW19, lemma 6.3] one can easily prove that each process $N_n:\hat{\Omega}\times[0,\infty)\to H_n$ defined by

$$N_n(t) = -v_n(t)+P_n u_0+\int_0^t\left[-\mathrm{i}Av_n(s)-\mathrm{i}P_nF(v_n(s))+b_n(v_n(s))-\beta v_n(s)\right]\mathrm{d}s$$

for $n\in\mathbb{N}$ and $t\in[0,\infty)$ is an H-valued continuous square integrable martingale w.r.t. the filtration $\hat{\mathscr{F}}_{n,t} := \sigma(v_n(s): s\leqslant t)$. As far as its quadratic variation process is concerned, in order to exploit classical results presented in a real Hilbert space setting, see e.g. [DPZ92], we work with the real inner product in H (and H_n). This means that the quadratic variation process $\langle\langle N_n\rangle\rangle$ is defined through the property that for any $\psi,\phi\in H_n$ the process

$$\mathrm{Re}\left(N_n(t),\psi\right)_H \mathrm{Re}\left(N_n(t),\phi\right)_H - \mathrm{Re}\left(\langle\langle N_n\rangle\rangle_t\psi,\phi\right)_H, \qquad t\in[0,\infty),$$

is a martingale. Therefore we find that

$$\begin{aligned}\langle\langle N_n\rangle\rangle_t\psi &= \sum_{m=1}^{\infty}\int_0^t \mathrm{i}S_nB_mS_nv_n(s)\,\mathrm{Re}\left(\mathrm{i}S_nB_mS_nv_n(s),\psi\right)_H \mathrm{d}s\\ &\quad+\sum_{m=1}^{\infty}\int_0^t \mathrm{i}S_nG(S_nv_n(s))\,e_m\,\mathrm{Re}\left(\mathrm{i}S_nG(S_nv_n(s))\,e_m,\psi\right)_H \mathrm{d}s\end{aligned}$$

for all $\psi\in H$ and $t\in[0,\infty)$. The martingale property can be rephrased as

$$\hat{\mathbb{E}}\left[\mathrm{Re}\left(N_n(t)-N_n(s),\psi\right)_H h\left(v_n|_{[0,s]}\right)\right]=0 \tag{5.25}$$

and

$$\begin{aligned}&\hat{\mathbb{E}}\Bigg[\Bigg(\mathrm{Re}\left(N_n(t),\psi\right)_H\mathrm{Re}\left(N_n(t),\varphi\right)_H-\mathrm{Re}\left(N_n(s),\psi\right)_H\mathrm{Re}\left(N_n(s),\varphi\right)_H\\ &-\sum_{m=1}^{\infty}\int_s^t\mathrm{Re}\left(\mathrm{i}S_nB_mS_nv_n(r),\psi\right)_H\mathrm{Re}\left(\mathrm{i}S_nB_mS_nv_n(r),\varphi\right)_H\mathrm{d}r\\ &-\sum_{m=1}^{\infty}\int_s^t\mathrm{Re}\left(\mathrm{i}S_nG(S_nv_n(r))\,e_m,\psi\right)_H\mathrm{Re}\left(\mathrm{i}S_nG(S_nv_n(r))\,e_m,\varphi\right)_H\mathrm{d}r\Bigg)h\left(v_n|_{[0,s]}\right)\Bigg]=0,\end{aligned} \tag{5.26}$$

for all $0<s<t<\infty$, $\psi,\varphi\in H$ and bounded continuous functions h on $C([0,\infty),H_n)$.

It is useful at this point to introduce the following notation. Let $\iota: V\hookrightarrow H$ be the usual embedding, $\iota^*: H\to V$ its Hilbert-space-adjoint, i.e. $\left(\iota u,v\right)_H=\left(u,\iota^*v\right)_V$ for $u\in V$ and $v\in H$. Further, we set $L:=(\iota^*)': V^*\to H$ as the dual operator of ι^* with respect to the Gelfand triple $V\hookrightarrow H\eqsim H^*\hookrightarrow V^*$.

Let us introduce the process

$$N(t):=-v(t)+u_0+\int_0^t\left[-\mathrm{i}Av(s)-\mathrm{i}F(v(s))+b(v(s))-\beta v(s)\right]\mathrm{d}s,\quad t\in[0,T],$$

which has V^*-valued continuous paths. Moreover, $N(t)\in L^2(\hat{\Omega},V^*)$.

We now use the martingale property of N_n for $n\in\mathbb{N}$ and a passage to the limit in (5.25) and (5.26) to show that LN is an H-valued continuous square integrable martingale with respect to the filtration $\hat{\mathbb{F}}=\left(\hat{\mathscr{F}}_t\right)_{t\in[0,T]}$, where $\hat{\mathscr{F}}_t:=\sigma(v(s):s\leqslant t)$, with quadratic variation given by

$$\langle\langle LN\rangle\rangle_t\zeta=\sum_{m=1}^{\infty}\int_0^t\left[\mathrm{i}LB_mv(s)\,\mathrm{Re}\left(\mathrm{i}LB_mv(s),\zeta\right)_H+\mathrm{i}LG(v(s))\,e_m\,\mathrm{Re}\left(\mathrm{i}LG(v(s))\,e_m,\zeta\right)_H\right]\mathrm{d}s, \tag{5.27}$$

for all $\zeta \in H$. We just provide the main steps of the limiting process (for a detailed proof see [BHW19]) and the computations of what is new.

Taking the limit as $n \to \infty$ in (5.25), for $\psi \in V$, we obtain[6]

$$\hat{\mathbb{E}}\left[\langle N(t) - N(s), \psi\rangle h\left(v|_{[0,s]}\right)\right] = 0. \tag{5.28}$$

Then we take the limit as $n \to \infty$ in (5.26), for $\phi, \psi \in V$, and obtain

$$\begin{aligned}\hat{\mathbb{E}}\Bigg[\Bigg(&\mathrm{Re}\langle N(t),\psi\rangle \mathrm{Re}\langle N(t),\varphi\rangle - \mathrm{Re}\langle N(s),\psi\rangle \mathrm{Re}\langle N(s),\varphi\rangle \\ &-\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\langle B_m v(r),\psi\rangle \mathrm{Re}\langle B_m v(r),\varphi\rangle\,\mathrm{d}r \\ &-\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\langle Gv(r)e_m,\psi\rangle \mathrm{Re}\langle Gv(r)e_m,\varphi\rangle\,\mathrm{d}r\Bigg) h(v|_{[0,s]})\Bigg] = 0. \end{aligned}\tag{5.29}$$

To prove the convergence of the first three terms in (5.26) one proceeds as in [BHW19, lemma 6.4 steps 3–4]. The convergence of the fourth term is proved in the following lemma whose proof is postponed to appendix C.

Lemma 5.9. *Under assumption 2.7(iii), for all* $0 \leqslant s \leqslant t < \infty$, $\psi, \varphi \in V$, *h a bounded continuous function on* $C([0,\infty); V^*)$, *we have*

$$\begin{aligned}&\lim_{n\to\infty}\hat{\mathbb{E}}\left[\left(\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\left(S_n G\left(S_n v_n(r)\right)e_m,\psi\right)_H \mathrm{Re}\left(S_n G\left(S_n v_n(r)\right)e_m,\varphi\right)_H \mathrm{d}r\right) h\left(v|_{[0,s]}\right)\right] \\ &= \hat{\mathbb{E}}\left[\left(\sum_{m=1}^{\infty}\mathrm{Re}\langle Gv(r)e_m,\psi\rangle \mathrm{Re}\langle Gv(r)e_m,\varphi\rangle\,\mathrm{d}r\right) h\left(v|_{[0,s]}\right)\right].\end{aligned}$$

Now let $\eta, \zeta \in H$. Then $i^*\eta, i^*\zeta \in V$ and for every z in V^* we have $\mathrm{Re}\left(Lz,\eta\right)_H = \mathrm{Re}\langle z, i^*\eta\rangle$. Thus, from (5.28) and (5.29) we deduce

$$\hat{\mathbb{E}}\left[\left(LN(t) - LN(s), \psi\right)_H h\left(v|_{[0,s]}\right)\right] = 0. \tag{5.30}$$

and

$$\begin{aligned}\hat{\mathbb{E}}\Bigg[\Bigg(&\mathrm{Re}\left(LN(t),\eta\right)_H \mathrm{Re}\left(LN(t),\zeta\right)_H - \mathrm{Re}\left(LN(s),\eta\right)_H \mathrm{Re}\left(LN(s),\zeta\right)_H \\ &-\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\left(LB_m v(r),\eta\right)_H \mathrm{Re}\left(LB_m v(r),\zeta\right)_H \mathrm{d}r \\ &-\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\left(LG(v(r)e_m),\eta\right)_H \mathrm{Re}\left(LG(v(r)e_m),\zeta\right)_H \mathrm{d}r\Bigg) h(v|_{[0,s]})\Bigg] = 0. \end{aligned}\tag{5.31}$$

Hence, from (5.30) and (5.31), we infer that LN is a continuous, square integrable martingale in H with respect to $\hat{\mathscr{F}}_t := \sigma\left(v(s) : s \leqslant t\right)$ and quadratic variation given by (5.27).

[6] For the proof see [BHW19, lemma 6.2, lemma 6.4 steps 1–2]. Here, in addition, we have to consider the convergence of the damping term but the needed estimates can be obtained rather easily.

Therefore, with the usual martingale representation theorem, see [DPZ92, theorem 8.2], we can conclude that there exist two cylindrical Wiener processes $\tilde{W}$ and $\tilde{\mathbf{W}}$ on Y defined on a probability space $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}\right) = \left(\hat{\Omega} \times \hat{\hat{\Omega}}, \hat{\mathscr{F}} \otimes \hat{\hat{\mathscr{F}}}, \hat{\mathbb{P}} \otimes \hat{\hat{\mathbb{P}}}\right)$ with

$$LN(t) = \int_0^t \mathrm{i}LB(v(s))\,\mathrm{d}\tilde{W}(s) + \int_0^t \mathrm{i}LG(v(s))\,\mathrm{d}\tilde{\mathbf{W}}(s), \tag{5.32}$$

for $t \in [0, \infty)$. Thanks to (5.24), the estimates

$$\begin{aligned} \|Bv\|^2_{L^2([0,T]\times\Omega,\gamma(Y_1,V^*))} &= \mathbb{E}\int_0^T \sum_{m=1}^{\infty} \|B_m v(s)\|^2_{V^*}\,\mathrm{d}s \lesssim \mathbb{E}\int_0^T \sum_{m=1}^{\infty} \|B_m v(s)\|^2_V \mathrm{d}s \\ &\leqslant \mathbb{E}\int_0^T \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathscr{L}(V)}\right) \|v(s)\|^2_V \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^T \|v(s)\|^2_V \mathrm{d}s \lesssim \|v\|^2_{L^2(\Omega, L^\infty(0,T;V))} < +\infty \end{aligned}$$

and

$$\begin{aligned} \|G(v)\|^2_{L^2([0,T]\times\Omega,\gamma(Y_2,V^*))} &= \mathbb{E}\int_0^T \sum_{m=1}^{\infty} \|G(v(s))e_m\|^2_{V^*}\,\mathrm{d}s \lesssim \mathbb{E}\int_0^T \sum_{m=1}^{\infty} \|G(v(s))e_m\|^2_V \mathrm{d}s \\ &= \mathbb{E}\int_0^T \|G(v(s))\|^2_{\gamma(Y_2,V)}\,\mathrm{d}s \lesssim 1 + \|v\|^2_{L^2(\Omega,L^\infty(0,T;V))} < \infty, \end{aligned}$$

yield that LN in (5.32) is a continuous martingale in H and, using the continuity of the operator L, we get

$$\begin{aligned} &\int_0^t \mathrm{i}LB(v(s))\,\mathrm{d}\tilde{W}(s) + \int_0^t \mathrm{i}LB(v(s))\,\mathrm{d}\tilde{\mathbf{W}}(s) \\ &\quad = L\left(\int_0^t \mathrm{i}B(v(s))\,\mathrm{d}\tilde{W}(s) + \int_0^t \mathrm{i}G(v(s))\,\mathrm{d}\tilde{\mathbf{W}}(s)\right), \end{aligned}$$

for all $t \in [0, T]$. The definition of N and the injectivity of L yield the equality

$$\begin{aligned} &\int_0^t \mathrm{i}Bv(s)\,\mathrm{d}\tilde{W}(s) + \int_0^t \mathrm{i}G(v(s))\,\mathrm{d}\tilde{\mathbf{W}}(s) \\ &\quad = -v(t) + u_0 + \int_0^t [-\mathrm{i}Av(s) - \mathrm{i}F(v(s)) + b(v(s))]\,\mathrm{d}s \end{aligned}$$

in V^* for $t \in [0, \infty)$.

The estimates for properties (5.1) and (5.2) and the weak continuity of the paths of v in V have already been shown at the beginning of the proof. Moreover, in view of (the beginning of) remark 3.2, we have that v is a H-valued continuous process. Hence, the system $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, v\right)$ is a martingale solution of equation (1.1) with the initial data μ, that satisfies (5.1) and (5.2).

It remains to prove lemma 5.9. This is done in appendix C.

5.4. Proof of the second part of theorem 5.1

Inequality (5.3) is a consequence of the same inequality for the Galerkin approximation (5.13), which is inherited by the limit.

Remark 5.10. Theorem 5.1 holds in a more general setting. It is sufficient for A to satisfy [BHW19, assumption 2.1] and assume (X, Σ, μ_X) to be a σ-finite measure space with metric ρ satisfying the doubling property and D to be an open bounded subset of X with $\mu_X(D) < \infty$. Moreover, it is sufficient for the nonlinear term F to satisfy [BHW19, assumptions 2.4 and 2.6(i)]. This kind of assumptions ensure the compactness of the embedding $V \subset H$ which is the crucial ingredient to prove the existence of a solution by means of a tightness argument, see section 5. In this general framework one can work in any space dimension provided a suitable condition on α is taken into account, see [BHW19, assumption 2.1(iv)]. Moreover, this framework allows A to be also a fractional power of Laplacian type operators considered so far; for more details see [BHW19, section 3.4].

One can easily check that the computations that lead to the existence of a martingale solution, as stated in theorem 5.1, hold true in this more general setting.

Remark 5.11. In light of remark 5.10, when one works under assumptions 2.1(ii) or (iii), the regularity assumptions on the domain $\mathcal{O}$ are as follows. To ensure the existence of a martingale solution, see theorem 5.1, it is sufficient for the domain $\mathcal{O}$ to be a bounded open subset of $\mathbb{R}^2$ in the case of assumption 2.1(ii) and a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary in the case of assumption 2.1(iii); the same holds in dimension d for suitable α depending on d. In fact, to prove the existence of a martingale solution we just exploit the fact that the embedding $V \subset H$ is continuous and compact. With the above mentioned regularity assumption on the domain, the continuity of the embeddings in the cases (ii)–(iii) follows by Leoni [Le17, theorem 11.23 and exercise 11.26]: roughly speaking, the regularity we require on the domain ensures that $\mathcal{O}$ is an extension domain, see also Leoni [Le17, exercises 12.11 and 12.14]. The compactness of the embedding is instead ensured by the boundedness of $\mathcal{O}$.

We emphasise that to prove the pathwise uniqueness of solutions and the existence of invariant measures, an additional regularity on the domain is required. For more details see remark 6.7.

6. Pathwise uniqueness

In this section we study the pathwise uniqueness for solutions to (1.1). We work under assumptions 2.1, 2.4 and 2.7. Here it is crucial that the spatial dimension is 2, whereas the result of existence of martingale solutions can be obtained in a more general setting. Indeed, by means of the Strichartz estimates we prove that any martingale solution fulfilling (5.2) enjoys more regularity. The deterministic and stochastic Strichartz estimates will be presented in appendix A.3; they are based on the results of Blair *et al*, see [BSS08]. These estimates allow us to work at the same time with the Laplace–Beltrami operator on a two-dimensional compact Riemannian manifold M and the realisation of the negative Laplace operator with Dirichlet or Neumann boundary conditions on a smooth relatively compact domain $\mathcal{O} \subset \mathbb{R}^2$. This is a first difference with respect to [BHW19] where the authors start from the Strichartz estimates due to Bernicot and Samoyeau, see [BS17] and [BHW19, lemmas B.3 and B.4]. Moreover, differently to [BHW19], in order to prove the pathwise uniqueness we cannot work pathwise since our noise is not conservative. We address this issue by appealing to a classical argument contained in [Sc97].

As usual, when we write L^q, $H^{s,q}$ without specifying the domain, we mean either $L^q(M)$, $H^{s,q}(M)$ or $L^q(\mathscr{O})$, $H^{s,q}(\mathscr{O})$.

Lemma 6.1. *Let $\vartheta \in (0,1)$. Then F maps the space V into $\mathscr{D}(A^{\frac{\vartheta}{2}})$ and*

$$\|F(u)\|_{\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)} \lesssim \|u\|_V^\alpha, \qquad u \in V. \tag{6.1}$$

Proof. For the case of (M,g) a compact manifold without boundary equipped with a Lipschitz metric g and $-A$ equal to the Laplace–Beltrami operator we refer to [BHW19, lemma 7.1]. Hence we have to prove (6.1) when $-A$ is the Laplace operator with either Dirichlet or Neumann boundary conditions on a smooth relatively compact subset $\mathscr{O}$ of $\mathbb{R}^2$.

Let us fix $\vartheta \in (0,1)$ and choose $s \in (1,2)$ such that

$$\vartheta < \frac{2(s-1)}{s}. \tag{6.2}$$

We start by proving that

$$\|F(u)\|_{W^{1,s}(\mathscr{O})} \lesssim \|u\|_{H^1(\mathscr{O})}^\alpha, \;\; u \in H^1(\mathscr{O}). \tag{6.3}$$

In order to prove (6.3) we compute the weak derivative of $F(u)$:

$$\nabla F(u) = \left(\frac{\alpha-1}{2}\right)|u|^{\alpha-3}(\bar{u}\nabla u + u\nabla\bar{u})\,u + |u|^{\alpha-1}\nabla u, \qquad \text{for an arbitrary } u \in H^1(\mathscr{O}).$$

The Hölder inequality and the Sobolev embedding $H^1(\mathscr{O}) \subset L^q(\mathscr{O})$ with $q = \frac{2s(\alpha-1)}{2-s}$, see proposition A.1(i), yield

$$\|\nabla F(u)\|_{L^s(\mathscr{O})} \lesssim \||u|^{\alpha-1}\|_{L^{\frac{2s}{2-s}}(\mathscr{O})}\|\nabla u\|_{L^2(\mathscr{O})} \lesssim \|u\|_{L^q(\mathscr{O})}^{\alpha-1}\|\nabla u\|_{L^2(\mathscr{O})} \lesssim \|u\|_{H^1(\mathscr{O})}^\alpha.$$

Similarly, the Sobolev embedding $H^1(\mathscr{O}) \subset L^{s\alpha}(\mathscr{O})$, yields

$$\|F(u)\|_{L^s(\mathscr{O})} \simeq \|u\|_{L^{s\alpha}(\mathscr{O})}^\alpha \lesssim \|u\|_{H^1(\mathscr{O})}^\alpha, \;\; u \in H^1(\mathscr{O})$$

and thus (6.3) immediately follows. By the choice of s in (6.2), proposition A.1(iii) ensures that the Sobolev embedding $H^{1,s}(\mathscr{O}) \subset H^\vartheta(\mathscr{O})$ holds and thus

$$\|F(u)\|_{H^\vartheta(\mathscr{O})} \lesssim \|u\|_{H^1(\mathscr{O})}^\alpha. \tag{6.4}$$

Observe now that, when on $\mathscr{O}$ we consider the Neumann boundary conditions, thanks to (A.2), $H^\vartheta = \mathscr{D}(A_N^{\frac{\vartheta}{2}})$ and thus from (6.4), (6.1) immediately follows.

In the case of the Dirichlet boundary conditions, from (A.1) and (6.4), (6.1) immediately follows when $\vartheta \in \left(0,\frac{1}{2}\right)$. When $\vartheta \in \left(\frac{1}{2},1\right)$, (6.1) is obtained by a density argument: for $\vartheta \in \left(\frac{1}{2},1\right)$, by (A.1), $\mathscr{D}(A_D^{\frac{\vartheta}{2}}) = H_0^\vartheta$, where H_0^ϑ is the closure of C_0^∞ w.r.t. the norm $\|\cdot\|_{H^{\vartheta,2}}$. One can easily verify that F maps C_0^∞ in itself and thus, by (6.4) we deduce (6.1) when $\vartheta \in \left(\frac{1}{2},1\right)$. □

We now reformulate problem (1.1) in the mild form to show additional regularity properties of solutions to (1.1) that satisfies (5.2).

Proposition 6.2. *Fix $r \in [1,\infty)$ and let μ be a Borel probability measure on V whose $r(\alpha+1)$th moment is finite.*

Let $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}},\tilde{W},\tilde{\mathbf{W}},\tilde{\mathbb{F}},u)$ *be a martingale solution to* (1.1) *with* $\mathrm{Law}_{\tilde{\mathbb{P}}}(u(0))=\mu$ *on* $\mathscr{B}(V)$ *which satisfies the condition* (5.2). *If* $2<p,q<\infty$ *satisfy the following admissibility condition*

$$\frac{2}{p}+\frac{2}{q}=1 \tag{6.5}$$

and $\vartheta\in\left(\frac{4}{3p},1\right)$, *then the following hold.*

- *For any* $T>0$,

$$u\in L^{2r/\alpha}\left(\tilde{\Omega};Y_T^\vartheta\right), \tag{6.6}$$

where Y_T^ϑ *is a Banach space defined as*

$$Y_T^\vartheta=L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)\cap C\left([0,T];\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right), \tag{6.7}$$

endowed with a norm

$$\|\cdot\|_{Y_T^\vartheta}=\|\cdot\|_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)}+\|\cdot\|_{L^\infty\left(0,T;\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)}.$$

In particular,

$$u\in C\left([0,\infty);\mathscr{D}(A^{\frac{\vartheta}{2}})\right),\qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{6.8}$$

- *For every* $t\in[0,\infty)$ *the equality*

$$\begin{aligned}\mathrm{i}u(t)&=\mathrm{i}\mathrm{e}^{-\mathrm{i}tA}u_0+\int_0^t\mathrm{e}^{-\mathrm{i}(t-\tau)A}F(u(\tau))\,\mathrm{d}\tau\\&\quad+\mathrm{i}\int_0^t\mathrm{e}^{-\mathrm{i}(t-\tau)A}b(u(\tau))\,\mathrm{d}\tau-\mathrm{i}\beta\int_0^t\mathrm{e}^{-\mathrm{i}(t-\tau)A}u(\tau)\,\mathrm{d}\tau\\&\quad+\int_0^t\mathrm{e}^{-\mathrm{i}(t-\tau)A}B(u(\tau))\,\mathrm{d}\tilde{W}(\tau)+\int_0^t\mathrm{e}^{-\mathrm{i}(t-\tau)A}G(u(\tau))\,\mathrm{d}\tilde{\mathbf{W}}(\tau)\end{aligned} \tag{6.9}$$

is satisfied $\tilde{\mathbb{P}}$*-a.s. in* $\mathscr{D}(A^{\frac{\vartheta}{2}})$.

Proof. Let $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}},\tilde{W},\tilde{\mathbf{W}},\tilde{\mathbb{F}},u)$ be a martingale solution to (1.1) given in theorem 5.1 such that it has the regularity (5.2). Let us at first show that equality (6.9) makes sense in $\mathscr{D}(A^{-\frac{3}{2}})$. Let us notice that for $\sigma=-\frac{3}{2}$, the group $(\mathrm{e}^{-\mathrm{i}tA})_{t\geqslant0}$ on L^2 extends to a C_0-group $(T_\sigma(t))_{t\geqslant0}$ on $\mathscr{D}(A^{-\sigma})$ with the generator $-\mathrm{i}A_\sigma$, where $\mathscr{D}(A_\sigma)=\mathscr{D}(A^{-\sigma+1})$, i.e. $-\mathrm{i}A_\sigma$ is a suitable extension of $-\mathrm{i}A$. To keep the notation simple we will denote this semigroup by $(\mathrm{e}^{-\mathrm{i}tA})_{t\geqslant0}$.

Let us choose and fix $t\in(0,\infty)$. We apply the Itô formula, see [BvNVW08, theorem 2.4], to the process $\mathrm{i}f(\tau,u(t-\tau))$, $\tau\in[0,t]$, where f is the function defined as

$$f:[0,t]\times\mathscr{D}\left(A^{-\frac{1}{2}}\right)\ni(\tau,x)\mapsto\mathrm{e}^{-\mathrm{i}(t-\tau)A}x\in\mathscr{D}\left(A^{-\frac{3}{2}}\right),$$

where we recall that $\mathscr{D}(A^{-\frac{1}{2}})=V^*$. Obviously, f is of $C^{1,2}$-class and since it follows from the assumptions that we can apply theorem 2.4 of [BvNVW08], we deduce that $\tilde{\mathbb{P}}$-a.s.

$$
\begin{aligned}
\mathrm{i}u(t) = \mathrm{i}\mathrm{e}^{-\mathrm{i}tA}u(0) + \int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)A}F(u(\tau))\,\mathrm{d}\tau \\
+\mathrm{i}\int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)A}b(u(\tau))\,\mathrm{d}\tau - \mathrm{i}\beta\int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)A}u(\tau)\,\mathrm{d}\tau \\
+\int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)A}B(u(\tau))\,\mathrm{d}\tilde{W}(\tau) + \int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)A}G(u(\tau))\,\mathrm{d}\tilde{\mathbf{W}}(\tau) \text{ in } \mathscr{D}\left(A^{-\frac{3}{2}}\right).
\end{aligned}
$$

We now use the Strichartz estimates from lemma A.10 to improve the regularity of the solution. Let us consider two Strichartz pairs: $(\infty,2)$ and (p,q) with $p,q\in(2,\infty)$ such that $\frac{2}{p}+\frac{2}{q}=1$. Moreover, we choose and fix a $\vartheta\in(\frac{4}{3p},1)$.

Let $T>0$ and let Y_T^ϑ be the Banach space defined by (6.7) for this choice of Strichartz pairs. Notice that, by assumption, the exponent $\frac{\vartheta}{2}-\frac{2}{3p}$ is positive. By estimating the terms in the RHS of the above expression, we will prove (6.6), (6.8) and that identity (6.9) holds a.s. in $\mathscr{D}(A^{\frac{\vartheta}{2}})$.

Thanks to the homogeneous Strichartz estimate (A.5) we get

$$
\begin{aligned}
\|\mathrm{e}^{-\mathrm{i}tA}u(0)\|_{L^{2r/\alpha}(\tilde{\Omega},Y_T^\vartheta)} &\lesssim_T \|u(0)\|_{L^{2r/\alpha}(\tilde{\Omega},(\mathscr{D}(A^{\frac{\vartheta}{2}}))} \lesssim \|u(0)\|_{L^{2r/\alpha}(\tilde{\Omega},V)} \\
&\lesssim \|u(0)\|_{L^{2r}(\tilde{\Omega},V)} \lesssim \|u(0)\|_{L^{r(\alpha+1)}(\tilde{\Omega},V)} < \infty,
\end{aligned}
$$

where in the second to last inequality we exploited the embedding $V\subset D(A^{\frac{\vartheta}{2}})$ and in the last two inequalities we used the embeddings $L^{r(\alpha+1)}\subset L^{2r}\subset L^{2r/\alpha}$ since $2r>2r/\alpha$ and $r(\alpha+1)>2r$ being $\alpha>1$ by assumption 2.4. Moreover, $\|u(0)\|_{L^{r(\alpha+1)}(\tilde{\Omega},V)}<\infty$, since $\mathrm{Law}_{\tilde{\mathbb{P}}}(u(0))=\mu$ and μ has finite $r(\alpha+1)$th moment by assumptions.

The inhomogeneous Strichartz estimate (A.6), lemma 6.1 and (5.2) yield

$$
\begin{aligned}
\left\|\int_0^\cdot \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A}F(u(\tau))\,\mathrm{d}\tau\right\|_{L^{2r/\alpha}(\tilde{\Omega};Y_T^\vartheta)} &\lesssim_T \|F(u)\|_{L^{2r/\alpha}\left(\tilde{\Omega},L^1\left(0,T;\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)\right)} \\
&\lesssim \|u\|^\alpha_{L^{2r}\left(\tilde{\Omega},L^\alpha(0,T;V)\right)} \lesssim \|u\|^\alpha_{L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right)} < \infty.
\end{aligned}
$$

Similarly, since we assume $\vartheta<1$, from the inhomogeneous Strichartz estimate (A.6) and (5.2) we infer that

$$
\left\|\int_0^\cdot \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A}u(\tau)\,\mathrm{d}\tau\right\|_{L^{2r/\alpha}(\tilde{\Omega};Y_T^\vartheta)} \lesssim \|u\|_{L^{2r/\alpha}\left(\tilde{\Omega},L^1\left(0,T;\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)\right)} \lesssim \|u\|_{L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right)} < \infty.
$$

The Itô correction term can be estimated as follows. Recalling (3.2), by the inhomogeneous Strichartz estimate (A.6), and (2.13) and (5.2) we infer that

$$
\begin{aligned}
\left\|\int_0^\cdot \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A}b(u(\tau))\,\mathrm{d}\tau\right\|_{L^{2r/\alpha}(\tilde{\Omega},Y_T^\vartheta)} &\lesssim \|b(u)\|_{L^{2r/\alpha}\left(\tilde{\Omega},L^1\left(0,T;\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)\right)} \\
&\lesssim \|b(u)\|_{L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right)} \lesssim \|u\|_{L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right)} < \infty.
\end{aligned}
$$

For what concerns the stochastic convolution term involving the operator B, by means of the stochastic Strichartz estimate (A.7), and assumptions (2.13) and (5.2), we obtain

$$\begin{aligned}\left\|\int_0^{\cdot} \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A} B(u(\tau))\, \mathrm{d}\tilde{W}(\tau)\right\|_{L^{2r/\alpha}\left(\tilde{\Omega}, Y_T^{\vartheta}\right)} &\lesssim \|B(u)\|_{L^{2r/\alpha}\left(\tilde{\Omega}, L^2\left(0,T;\gamma\left(Y_1,\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)\right)\right)} \\ &\lesssim \|B(u)\|_{L^{2r/\alpha}(\tilde{\Omega},L^2(0,T;\gamma(Y_1,V)))} \\ &\lesssim \|u\|_{L^{2r/\alpha}\left(\tilde{\Omega},L^2(0,T;V)\right)} \lesssim \|u\|_{L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right)} < \infty,\end{aligned}$$

when $\frac{2r}{\alpha} \geqslant 1$. For smaller values we first estimate with the moment of order 1 and then conclude as above. The same will be done for the next stochastic integral. The stochastic Strichartz estimate (A.7), and assumptions (2.16) and (5.2) yield

$$\begin{aligned}&\left\|\int_0^{\cdot} \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A} G(u(\tau))\, \mathrm{d}\tilde{\mathbf{W}}(\tau)\right\|_{L^{2r/\alpha}\left(\tilde{\Omega}, Y_T^{\vartheta}\right)} \\ &\quad\lesssim \|G(u)\|_{L^{2r/\alpha}\left(\tilde{\Omega}, L^2\left(0,T;\gamma\left(Y_2,\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)\right)\right)} \\ &\quad\lesssim \|G(u)\|_{L^{2r/\alpha}(\tilde{\Omega},L^2(0,T;\gamma(Y_2,V)))} = \mathbb{E}\left[\left(\int_0^T \|G(u(\tau))\|^2_{\gamma(Y_2,V)}\, \mathrm{d}\tau\right)^{r/\alpha}\right] \\ &\quad\lesssim 1 + \|u\|_{L^{2r/\alpha}\left(\tilde{\Omega},L^2(0,T;V)\right)} \lesssim 1 + \|u\|_{L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right)} < \infty.\end{aligned}$$

Thus the mild equation (6.9) holds $\tilde{\mathbb{P}}$-a.s. in $\mathscr{D}(A^{\frac{\vartheta}{2}})$ for every $t \in [0,\infty)$. Therefore, thanks to the pathwise continuity of the deterministic and stochastic integrals, we get (6.6) and (6.8). □

Remark 6.3. Let us note the following difference between our result and proposition 7.2 in [BHW19]. Here in (5.2) we assume the V-regularity of the solution while in the paper [BHW19] only H^s-regularity was assumed, see assumption (7.6) therein. We have to make a stronger assumption here because the Strichartz estimates for a boundaryless manifold are stronger than the Strichartz estimates for a bounded domain with smooth boundary, see remark A.9. However, this stronger assumption is fully sufficient for our purposes.

We are now ready to prove the pathwise uniqueness of the martingale solutions to (1.1) satisfying condition (5.2). We will need the following result which exploits the gain of regularity of solutions proved in proposition 6.2.

Lemma 6.4. *Assume $r \in [1,\infty)$ and let μ be a Borel probability measure on V whose $r(\alpha+1)$th moment is finite.*

Let $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{W}}, \tilde{\mathbb{F}}, u\right)$ be a martingale solution to (1.1) with $\mathrm{Law}_{\tilde{\mathbb{P}}}(u(0)) = \mu$ on V such that (5.2) holds. Then the trajectories of the process h defined by

$$h(s) := \|u(s)\|_{L^\infty}^{\alpha-1}, \qquad s \in [0,\infty),$$

belong to $L^1_{\mathrm{loc}}([0,\infty))$, $\tilde{\mathbb{P}}$-a.s.

Proof of lemma 6.4. Step 1. Let us assume that $\alpha \in (1,3]$. We choose $p,q \in (2,\infty)$ satisfying the admissibility condition (6.5). Note that in this case $p > \alpha - 1$.

Since $1 - \frac{2}{3p} > \frac{4}{3p}$ we can choose a number $\vartheta \in (1 - \frac{2}{3p}, 1) \cap (\frac{4}{3p}, 1)$. Thanks to proposition 6.2, for any $T > 0$, $u \in L^p(0,T;\mathscr{D}(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}))$ $\tilde{\mathbb{P}}$-a.s. Moreover, by proposition A.1(ii), definition

A.3 and theorem A.4, which ensures that $\mathscr{D}(A_{B_q}^{\frac{\vartheta}{2}-\frac{2}{3p}}) \subset H^{\vartheta-\frac{4}{3p},q}(\mathscr{O})$, for $B=D,N$, and proposition A.6(i)–(ii), we infer that $\mathscr{D}(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}) \hookrightarrow L^\infty$. In fact, this embedding holds true because $\vartheta-\frac{4}{3p}-\frac{2}{q}=\vartheta-1+\frac{2}{3p}$ so that $\vartheta-\frac{4}{3p}-\frac{2}{q}>0 \iff \vartheta>1-\frac{2}{3p}$. Since $p>\alpha-1$, by applying the Hölder inequality in time we see that the process h satisfies, $\tilde{\mathbb{P}}$-a.s.

$$\|h\|_{L^1(0,T)} \lesssim \|u\|^{\alpha-1}_{L^{\alpha-1}\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} \lesssim \|u\|^{\alpha-1}_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)}. \tag{6.10}$$

Step 2. Let us assume that $\alpha>3$. Then we set $p:=\alpha-1$. Since $p>2$ we can find $q>2$ such that the admissibility condition (6.5) holds. As in step 1, we choose a number $\vartheta\in(1-\frac{2}{3p},1)\cap(\frac{4}{3p},1)$ and observe that $\vartheta-\frac{4}{3p}-\frac{2}{q}>0$. Therefore, we have $\mathscr{D}(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}) \hookrightarrow L^\infty$. Hence, we get the following version of estimate (6.10)

$$\|h\|_{L^1(0,T)} \lesssim \|u\|^{\alpha-1}_{L^{\alpha-1}\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} = \|u\|^{p}_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)}. \tag{6.11}$$

Step 3. We conclude that, for any $T>0$, $h\in L^1(0,T)$ $\tilde{\mathbb{P}}$-a.s. as a consequence of inequalities (6.10) and (6.11) combined with the moment estimate (6.6). □

Now we are ready to prove the pathwise uniqueness. This holds for any martingale solution u, as in definition 3.1 and enjoying (5.2), since we have proved that, under suitable conditions on the initial distribution, for these solutions there is the additional regularity $u\in L^{\alpha-1}_{\mathrm{loc}}([0,\infty);L^\infty)$ $\tilde{\mathbb{P}}$-*a.s.* and $u\in L^4(\tilde{\Omega};L^4(0,T;H))$, for every $T>0$, at least.

Theorem 6.5. *Let assumptions 2.1, 2.4, 2.7 be in force. Assume $r\in[2,\infty)$. Let μ be a Borel probability measure on V whose $r(\alpha+1)$th moment is finite.*

Let $\left(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}},\tilde{W},\tilde{\mathbf{W}},\tilde{\mathbb{F}},u_i\right)$, $i=1,2$ be two martingale solutions to (1.1) with random initial data of law μ and both satisfying condition (5.2). Then

(i) these solutions to equation (1.1) are pathwise unique, i.e.

$$\tilde{\mathbb{P}}\big(u_1(t)=u_2(t) \text{ for all } t\in[0,\infty)\big)=1;$$

(ii) martingale solutions to equation (1.1) are unique in law, i.e. if $\left(\widetilde{\Omega}_i,\widetilde{\mathscr{F}}_i,\widetilde{\mathbb{P}}_i,\widetilde{W}_i,\widetilde{\mathbf{W}}_i,\widetilde{\mathbb{F}}_i,u_i\right)$, $i=1,2$ be two martingale solutions to (1.1) with random initial data of law μ and both satisfying condition (5.2), then

$$\mathrm{Law}_{\widetilde{\mathbb{P}}_1}(u_1)=\mathrm{Law}_{\widetilde{\mathbb{P}}_2}(u_2) \text{ on } Z_\infty.$$

Proof. Let us first deal with assertion (ii). In view of assertion (i), it is a consequence of [On04, theorem 2], the second assertion; see also theorem 12.1 therein.

Let us next deal with assertion (i). Since the noise is not conservative we can not work pathwise as in [BHW19]. We prove the uniqueness of the solution by means of a rather classical argument, see [Sc97]. Take two solutions $\left(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}},\tilde{W},\tilde{\mathbf{W}},\tilde{\mathbb{F}},u_i\right)$, $i=1,2$, with the same initial data μ on V which satisfy

$$u_i\in L^{2r}\left(\tilde{\Omega},L^\infty(0,T;V)\right) \text{ for } i=1,2 \text{ and } T>0. \tag{6.12}$$

Define $v := u_1 - u_2$. This difference satisfies

$$\begin{cases} \mathrm{d}v(t) = -\left[\mathrm{i}Av(t) + \mathrm{i}\left(F(u_1(t)) - F(u_2(t))\right) + \beta v(t)\right]\mathrm{d}t + \\ \qquad -\mathrm{i}B(v(t)) \circ \mathrm{d}\tilde{W}(t) - \mathrm{i}\left[G(u_1(t)) - G(u_2(t))\right]\mathrm{d}\tilde{\mathbf{W}}(t) \\ v(0) = 0. \end{cases}$$

We use the Itô formula to compute $\mathrm{d}\left(\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\right)$, by choosing a process ψ as

$$\psi(t) := 2\left[\|u_1(t)\|_{L^\infty}^{\alpha-1} + \|u_2(t)\|_{L^\infty}^{\alpha-1} - \beta\right] + L_G, \qquad t \in [0,\infty), \tag{6.13}$$

with L_G the Lipschitz constant given in (2.15). Thanks to lemma 6.4 we have that $\psi \in L^1_{\mathrm{loc}}(0,\infty)$, $\tilde{\mathbb{P}}$-a.s. For $t \in [0,\infty)$, we have

$$\mathrm{d}\left(\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\right) = -\psi(t)\,\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\,\mathrm{d}t + \mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\,\mathrm{d}\|v(t)\|_H^2, \tag{6.14}$$

where

$$\mathrm{d}\|v(t)\|_H^2 = 2\mathrm{Re}\langle v(t), [G(u_1(t)) - G(u_2(t))]\,\mathrm{d}\tilde{\mathbf{W}}(t)\rangle + \Big[2\mathrm{Re}\langle v(t), -\mathrm{i}F(u_1(t)) + \mathrm{i}F(u_2(t))\rangle \\ -2\beta\|v(t)\|_H^2 + \|G(u_1(t)) - G(u_2(t))\|_{\gamma(Y_2,H)}^2\Big]\,\mathrm{d}t.$$

By means of inequality (2.15) we have the following estimate

$$\|G(u_1(t)) - G(u_2(t))\|_{\gamma(Y_2,H)}^2 \leqslant L_G\|v(t)\|_H^2, \ \ t \geqslant 0. \tag{6.15}$$

The inequality

$$|F(z_1) - F(z_2)| \lesssim \left(|z_1|^{\alpha-1} + |z_2|^{\alpha-1}\right)|z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{C},$$

and the Hölder inequality yield

$$\mathrm{Re}\langle v(t), -\mathrm{i}F(u_1(t)) + \mathrm{i}F(u_2(t))\rangle \lesssim \|v(t)\|_H^2\left[\|u_1(t)\|_{L^\infty}^{\alpha-1} + \|u_2(t)\|_{L^\infty}^{\alpha-1}\right], \ \ t \geqslant 0. \tag{6.16}$$

By means of (6.15) and (6.16) we estimate (6.14) as follows

$$\begin{aligned} \mathrm{d}\left(\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\right) &\lesssim -\psi(t)\,\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\,\mathrm{d}t \\ &\quad + \mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\Big[2\|v(t)\|_H^2\left[\|u_1(t)\|_{L^\infty}^{\alpha-1} + \|u_2(t)\|_{L^\infty}^{\alpha-1}\right]\mathrm{d}t - 2\beta\|v(t)\|_H^2\,\mathrm{d}t \\ &\quad + L_G\|v(t)\|_H^2\,\mathrm{d}t + 2\mathrm{Re}\langle v(t), [G(u_1(t)) - G(u_2(t))]\,\mathrm{d}\tilde{\mathbf{W}}(t)\rangle\Big] \\ &= \mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\left[-\psi(t) + 2\left[\|u_1(t)\|_{L^\infty}^{\alpha-1} + \|u_2(t)\|_{L^\infty}^{\alpha-1} - \beta\right] + L_G\right]\mathrm{d}t \\ &\quad + 2\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\mathrm{Re}\langle v(t), [G(u_1(t)) - G(u_2(t))]\,\mathrm{d}\tilde{\mathbf{W}}(t)\rangle. \end{aligned}$$

Therefore, recalling (6.13), we obtain

$$\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2 \lesssim 2\int_0^t \mathrm{e}^{-\int_0^r \psi(s)\,\mathrm{d}s}\mathrm{Re}\langle v(r), [G(u_1(r)) - G(u_2(r))]\,\mathrm{d}\tilde{\mathbf{W}}(r)\rangle. \tag{6.17}$$

Let us observe that the RHS of (6.17) is a square integrable martingale. Indeed, using inequality (2.15) we get

$$\mathbb{E}\left[\int_0^t \mathrm{e}^{-2\int_0^r \psi(s)\,\mathrm{d}s}\|v(r)\|_H^2\|G(u_1(r))-G(u_2(r))\|_{\gamma(Y_2,H)}^2\,\mathrm{d}r\right]$$
$$\lesssim \mathrm{e}^{4\beta t}L_G^2\mathbb{E}\left[\int_0^t \|v(r)\|_H^4\mathrm{d}r\right] \lesssim \|v\|^4_{L^4\left(\tilde\Omega;L^4(0,T;H)\right)} \lesssim \|v\|^4_{L^{2r}\left(\tilde\Omega;L^\infty(0,T;V)\right)},$$

which is finite thanks to (6.12) because $r \geqslant 2$. Therefore, by taking the expected value in both sides of (6.17) we get

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2\right] \leqslant 0, \qquad \forall\, t\in[0,\infty).$$

Thus, in particular, for any $t\in[0,\infty)$

$$\mathrm{e}^{-\int_0^t \psi(s)\,\mathrm{d}s}\|v(t)\|_H^2 = 0, \qquad \tilde{\mathbb{P}}\text{-}a.s.$$

Therefore, if we take a sequence $\{t_k\}_{k=1}^\infty$, which is dense in $[0,\infty)$, we have

$$\tilde{\mathbb{P}}(\|v(t_k)\|_H = 0 \text{ for all } k\in\mathbb{N}) = 1.$$

Since by definition 3.1 both processes u_1 and u_2 are H-valued continuous, we deduce that

$$\tilde{\mathbb{P}}(u_1(t) = u_2(t) \text{ for all } t\in[0,\infty)) = 1$$

and this concludes the proof. □

The pathwise uniqueness and the existence of martingale solutions imply the existence of strong solutions, see e.g. [On04, theorem 2] and [Kun13, theorem 5.3 and corollary 5.4]. In section D we have formulated a suitable modification of the above two results. The following result dealing with a generic time interval $[t_0,\infty)$ is thus a direct consequence of theorems D.1, 5.1 and 6.5.

Before we formulate this result it convenient to introduce additional notation analogous to (6.7), i.e.

$$Y^\vartheta_{[t_0,\infty)} = L^p_{\mathrm{loc}}([t_0,\infty);\mathscr{D}(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}))\cap C([t_0,\infty);\mathscr{D}(A^{\frac{\vartheta}{2}}))$$

and, for $T>0$,

$$Y^\vartheta_{[t_0,T]} = L^p\left([t_0,T];\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)\cap C\left([t_0,T];\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right).$$

Theorem 6.6. *Let assumptions 2.1, 2.4 and 2.7 hold. Fix $r\in[2,\infty)$. Assume that $t_0\in[0,\infty)$ and u_{t_0} is an $\mathscr{F}_{t_0}$-measurable Borel V-valued random variable with finite $r(\alpha+1)$th moment. Then the following assertions are satisfied.*

(1) There exists a unique strong solution $u=(u(t):t\in[t_0,\infty))$ to equation (1.1) such that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\|u(t)\|_H^{2r} + \sup_{t\in[t_0,T]}\mathscr{E}(u(t))^r\right] < \infty, \text{ for every } T\geqslant t_0,$$

(and hence)

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\|u(t)\|_V^{2r}\right] < \infty, \text{ for every } T\geqslant t_0. \tag{6.18}$$

(2) *If ϑ is as in proposition 6.2, then this solution has $\mathbb{P}$-a.s. paths in $C([t_0,\infty);H\cap\mathscr{D}(A^{\frac{\vartheta}{2}}))\cap C_w([t_0,\infty);V)\cap Y^{\vartheta}_{[t_0,\infty)}$ and for every $T\geqslant t_0$,*

$$u\in L^{2r/\alpha}\left(\Omega;Y^{\vartheta}_{[t_0,T]}\right).$$

(3) *Moreover, for every $t\in[t_0,\infty)$ the following equality in V^**

$$u(t)=u_{t_0}-\int_{t_0}^{t}\left[\mathrm{i}Au(s)+\mathrm{i}F(u(s))+\beta u(s)-b(u(s))\right]\mathrm{d}s$$
$$-\mathrm{i}\int_{t_0}^{t}Bu(s)\,\mathrm{d}W(s)-\mathrm{i}\int_{t_0}^{t}G(u(s))\,\mathrm{d}\mathbf{W}(s)$$

holds $\mathbb{P}$-almost surely.

Remark 6.7. When one works under assumptions 2.1(ii) or (iii), the regularity assumptions on the domain $\mathscr{O}$ that ensure the uniqueness of the solution (and, consequently, the existence of an invariant measure too) are as follows. One needs to require that $\mathscr{O}$ has C^∞ (smooth) boundary and is relatively compact. The relative compactness of the domain is required to apply the needed Strichartz estimates, whereas the smoothness of the boundary is needed both for the Strichartz estimates to hold, and for the definition of the fractional Sobolev spaces (for this point see also remark A.2.) Notice that, to infer just the existence of a martingale solution, less regularity on the domain is required, see remark 5.11.

7. Sequential weak Feller property

Consider the family of operators $\{P_t\}_{t\geqslant 0}$ defined in (3.10). Our aim is now to prove the sequential weak Feller property in V at any fixed time t and that $\{P_t\}_{t\geqslant 0}$ is a Markov semigroup. These are part of the ingredients to prove existence of invariant measures. The sequential weak Feller property relies on an argument of continuous dependence of the solution on the initial data. The Markov property depends on the pathwise uniqueness.

Let us now recall the following fundamental definition. A function $\phi: V\to\mathbb{R}$ is sequentially continuous w.r.t. the weak topology on V (we write $\phi\in SC(V_w)$) if $\phi(x_k)\to\phi(x)$ when $x_n\rightharpoonup x$ in V. Using the subindex $SC_b(V_w)$ we add the property of boundedness. We recall that the following inclusions hold

$$C_b(V_w)\subset SC_b(V_w)\subset C_b(V_n).$$

Here V_w denotes V equipped with the weak topology and V_n denotes V equipped with the strong (norm) topology.

Let us also notice that because V is a separable space, the weak Borel and the (strong) Borel σ-fields on it are equal, i.e. $\mathscr{B}(V_n)=\mathscr{B}(V_w)$, see, e.g. [Ed77].

For $x\in V$, by $u(\cdot;x)=\{u(t;x):t\geqslant 0\}$ we denote the unique strong solution with the deterministic initial condition x, defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$. Bearing in mind the remark 3.6(ii), the unique strong solution of problem 1.1 with deterministic initial data enjoys property (6.18) for every finite $r\geqslant 1$.

It is known, see [On05, corollary 23], that the transition function is jointly measurable, that is for any Borel subset Γ of V the map

$$V\times[0,\infty)\ni(x,t)\mapsto\mathbb{P}\{u(t;x)\in\Gamma\}\in\mathbb{R}$$

is measurable.

We define the family of operators P_t, for any $t>0$,

$$(P_t\phi)(x)=\mathbb{E}[\phi(u(t;x))],\quad x\in V. \tag{7.1}$$

If $\phi:V\to\mathbb{R}$ is a bounded and Borel measurable function, then the same holds for $P_t\phi$.

We first provide the following result of continuous dependence on the initial data. For $s\geqslant 0$ let $u(t;s,x)$, $t\geqslant s$, denote the solution of equation (1.1) when the initial value at time s is x. According to the previous notation used so far we have $u(t;x)=u(t;0,x)$, $t\geqslant 0$.

Theorem 7.1. *Let assumptions 2.1, 2.4 and 2.7 hold. Assume $r\in[2,\infty)$ and $t_0\in[0,\infty)$. Let $(x_k)_k$ and x be $\mathscr{F}_{t_0}$-measurable Borel V-valued random variables with finite $r(\alpha+1)$th moments. If*

$$\sup_k\mathbb{E}\|x_k\|_V^{r(\alpha+1)}<\infty,\qquad \mathbb{E}\|x\|_V^{r(\alpha+1)}<\infty \tag{7.2}$$

and $\mathbb{P}$-a.s. x_k weakly converges to x in V, then

$$\lim_{k\to\infty}\mathbb{E}\phi(u(t;t_0,x_k))=\mathbb{E}\phi(u(t;t_0,x)) \tag{7.3}$$

for any $t\in(t_0,\infty)$ and any $\phi\in SC_b(V_w)$.

Proof. Let us choose and fix $t>t_0\geqslant 0$ and $\phi\in SC_b(V_w)$. By theorem 6.6 for each initial data there exists a unique solution to equation (1.1). Moreover we obtain the uniform estimate

$$\sup_{k\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[t_0,T]}\|u(t;t_0,x_k)\|_V^{2r}\right]<\infty,\ \text{ for every } T\geqslant t_0 \tag{7.4}$$

and the Aldous condition as in inequality (5.22) in corollary 5.8 and proposition 5.7 part (b). The only difference with respect to the Galerkin approximation sequence is on the initial data, but assumption (7.2) is a uniform estimate on them leading to (7.4). Therefore we deduce that the sequence $\left(\mathrm{Law}_{\mathbb{P}}(u(\cdot;t_0,x_k))\right)_{k=1}^{\infty}$ is tight in the space $Z_{[t_0,\infty)}=C([t_0,\infty);V^*)\cap L^{\alpha+1}_{\mathrm{loc}}([t_0,\infty);L^{\alpha+1})\cap C_w([t_0,\infty);V)$. Hence corollary 4.6 applies, i.e. there exists a subsequence $\{u(\cdot;t_0,x_{n_k})\}_k$ such that on a new probability space $(\tilde\Omega,\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ there exist $Z_{[t_0,\infty]}$-valued random variables $\{\tilde u_k\}_{k\in\mathbb{N}}$ and $\tilde u$ with $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde u_k)=\mathrm{Law}_{\mathbb{P}}(u(\cdot;t_0,x_{n_k}))$ for any $k\in\mathbb{N}$ such that $\tilde u_k\to\tilde u$ $\tilde{\mathbb{P}}$-almost surely in $Z_{[t_0,\infty)}$, as $k\to\infty$.

Since, as a consequence of corollary E.3, the function

$$Z_{[t_0,T]}\ni u\mapsto\sup_{t\in[t_0,T]}\|u(t)\|_V\in[0,\infty)$$

is well defined and $\mathcal{B}(Z_{[t_0,T]})$-measurable, and $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde u_k)=\mathrm{Law}_{\mathbb{P}}(u(\cdot;t_0,x_{n_k}))$ for any $k\in\mathbb{N}$, the sequence $\{\tilde u_k\}_{k\in\mathbb{N}}$ satisfies the same estimates as the original sequence, in particular

$$\sup_{k\in\mathbb{N}}\tilde{\mathbb{E}}\left[\sup_{t\in[t_0,T]}\|\tilde u_k(t)\|_V^{2r}\right]<\infty,\ \text{ for every } T\geqslant t_0.$$

By repeating the proof of theorem 5.1 given in subsections 5.3 and 5.4, the system $\left(\tilde\Omega,\tilde{\mathbb{F}},\tilde{\mathbb{P}},\tilde u\right)$ is also a martingale solution of equation (1.1) with the initial value x at time t_0 and one has

$$\tilde{\mathbb{E}}\left[\sup_{t\in[t_0,T]}\|\tilde u(t)\|_V^{2r}\right]\lesssim\liminf_k\tilde{\mathbb{E}}\left[\sup_{t\in[t_0,T]}\|\tilde u_k(t)\|_V^{2r}\right]<\infty,\ \text{ for every } T\geqslant t_0. \tag{7.5}$$

In particular, because $\tilde{u}_k$ converges to $\tilde{u}$ in $C_w([t_0,\infty);V)$, we deduce that

$$\tilde{u}_k(t) \to \tilde{u}(t) \text{ weakly in } V, \ \tilde{\mathbb{P}}\text{-a.s.}.$$

Hence, since function $\phi : V \to \mathbb{R}$ is bounded and sequentially weak continuous, by the Lebesgue dominated convergence theorem we infer that $\tilde{\mathbb{E}}[\phi(\tilde{u}_k(t))] \to \tilde{\mathbb{E}}[\phi(\tilde{u}(t))]$. Since $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{u}_k) = \mathrm{Law}_{\mathbb{P}}(u(\cdot;t_0,x_{n_k}))$ for any $k \in \mathbb{N}$, we infer that

$$\lim_{k\to\infty} \mathbb{E}[\phi(u(t;t_0,x_{n_k}))] = \tilde{\mathbb{E}}[\phi(\tilde{u}(t))].$$

Since the system $\left(\tilde{\Omega},\tilde{\mathbb{F}},\tilde{\mathbb{P}},\tilde{u}\right)$ is also a martingale solution of equation (1.1) with the initial value x at time t_0 and since by part (ii) in theorem 6.5 the solution of (1.1) is unique in law, i.e.

$$\text{the processes } u(\cdot;t_0,x) \text{ and } \tilde{u} \text{ have the same law on the space } Z_{[t_0,\infty)},$$

(notice that the assumptions of theorem 6.5 are satisfied in virtue of (7.5) and the same estimate plainly holds for u) we infer that

$$\tilde{\mathbb{E}}[\phi(\tilde{u}(t))] = \mathbb{E}[\phi(u(t))].$$

Summing up, we proved that

$$\lim_{k\to\infty} \mathbb{E}[\phi(u(t;t_0,x_{n_k}))] = \mathbb{E}\phi(u(t;t_0,x)).$$

Finally, using the standard sub-subsequence argument, we infer that the whole sequence $\mathbb{E}[\phi(u(t;t_0,x_k))]$ is convergent and (7.3) holds. This completes the proof of theorem 7.1. □

It easily follows the sequential weak Feller property in V, that is

$$P_t : SC_b(V_w) \to SC_b(V_w), \text{for any } t > 0.$$

Corollary 7.2. *Let assumptions 2.1, 2.4 and 2.7 hold. For any $t > 0$, if $\phi \in SC_b(V_w)$ then $P_t\phi \in SC_b(V_w)$.*

Proof. Let us fix any $t > 0$ and $\phi \in SC_b(V_w)$. We have to prove that, given a sequence $(x_k)_k \subset V$ which converges weakly in V to x, the sequence $P_t\phi(x_k)$ converges to $P_t\phi(x)$.

By the weak convergence we get the uniform estimate

$$\sup_k \|x_k\|_V < \infty;$$

hence the sequence of deterministic initial data fulfils the assumptions of theorem 7.1 on the time interval $[0,T]$, i.e. we set $t_0 = 0$. Therefore (7.3) holds true; bearing in mind the definition (7.1) of the operator P_t we conclude the proof. □

Now we consider the Markov property.

Proposition 7.3. *Let assumptions 2.1, 2.4 and 2.7 hold. For every $\phi \in SC_b(V_w)$, $x \in V$ and $t,s > 0$ we have*

$$\mathbb{E}[\phi(u(t+s;x))\,|\mathcal{F}_s] = (P_t\phi)(u(s;x)) \qquad \mathbb{P}\text{-}a.s. \tag{7.6}$$

Proof. The proof is classical when the solution is a continuous process taking values in a separable Banach space, endowed with the strong topology; see, e.g. [DPZ92, theorem 9.14]. Hence we highlight only the differences when dealing with the weak topology in V.

By the pathwise uniqueness we know that for all $t,s>0$

$$u(t+s;0,x)=u(t+s;s,u(s;0,x)) \qquad a.s.$$

Set $\eta=u(s;0,x)$; the identity (7.6) can be written as

$$\mathbb{E}[\phi(u(t+s;s,\eta))|\mathcal{F}_s]=(P_t\phi)(\eta) \qquad \mathbb{P}\text{-a.s.} \tag{7.7}$$

We notice that given any deterministic initial data $x\in V$, theorem 3.4 gives that $\mathbb{E}\|u(s;0,x)\|_V^{2r}<\infty$ for any finite $r\geqslant 1$. Let us choose $r=\alpha+1$.

Hence it is enough to show that equality (7.7) holds for an arbitrary $2(\alpha+1)$-integrable $\mathcal{F}_s$-measurable random variable η.

Let us first suppose that η is a simple random variable of the form $\sum_{j=1}^N x_j\mathbb{1}_{\Gamma_j}$ with $x_j\in V$ and a partition $\Gamma_1,\ldots,\Gamma_N\subset\mathcal{F}_s$. Then (7.7) is proved as usual by noticing that $u(t+s;s,\eta)=\sum_{j=1}^N u(t+s;s,x_j)\mathbb{1}_{\Gamma_j}$; indeed, $\mathbb{P}$-a.s. we have the following relationships

$$\begin{aligned}\mathbb{E}[\phi(u(t+s;s,\eta))|\mathcal{F}_s]&=\sum_{j=1}^N\mathbb{E}\left[\mathbb{1}_{\Gamma_j}\phi(u(t+s;s,x_j))|\mathcal{F}_s\right]\\&=\sum_{j=1}^N\mathbb{1}_{\Gamma_j}\mathbb{E}[\phi(u(t+s;s,x_j))|\mathcal{F}_s]=\sum_{j=1}^N\mathbb{1}_{\Gamma_j}\mathbb{E}(\phi(u(t+s;s,x_j)))\\&=\sum_{j=1}^N\mathbb{1}_{\Gamma_j}\mathbb{E}(\phi(u(t;0,x_j)))=\sum_{j=1}^N\mathbb{1}_{\Gamma_j}(P_t\phi)(x_j)=(P_t\phi)(\eta).\end{aligned}$$

We used that $u(t+s;s,x_j)$ is independent of $\mathcal{F}_s$ and that $u(t+s;s,x_j)$ and $u(t;0,x_j)$ have the same law.

Otherwise, for general $\eta\in L^{2(\alpha+1)}(\Omega)$ there exists a sequence of simple random variables η_n with $\lim_{n\to\infty}\eta_n=\eta$ in $L^{2(\alpha+1)}(\Omega)$ and $\mathbb{P}$-a.s. (in the strong topology of V, hence weak too); moreover

$$\sup_n\mathbb{E}\|\eta_n\|_V^{2(\alpha+1)}\leqslant\mathbb{E}\|\eta\|_V^{2(\alpha+1)}<\infty. \tag{7.8}$$

We checked before that

$$\mathbb{E}[\phi(u(t+s;s,\eta_n))|\mathcal{F}_s]=(P_t\phi)(\eta_n) \qquad \mathbb{P}\text{-a.s.} \tag{7.9}$$

Thanks to (7.8) we can proceed as in theorem 7.1 on the time interval $[s,s+t]$ in order to deal with the conditional expectation and pass to the limit in the left hand side (lhs) of (7.9). Thus we have proved that the lhs of (7.9) converges to the lhs of (7.6) as $n\to\infty$.

As far as the convergence of the RHS of (7.9) is concerned, we know from corollary 7.2 that $P_t\phi\in SC_b(V_w)$; since $\mathbb{P}$-a.s. η_n converges to η weakly in V, we obtain that $P_t\phi(\eta_n)\to P_t\phi(\eta)$, $\mathbb{P}$-a.s. □

Taking the mathematical expectation in (7.6), we deduce that the family $\{P_t\}_{t\geqslant 0}$ is a Markov semigroup, namely $P_{t+s}=P_tP_s$ for any $s,t>0$.

8. Existence of an invariant measure

Given the sequential weak Feller Markov semigroup on the separable Hilbert space V, we can define an invariant measure π for equation (1.1) as a Borel probability measure on V such that, for any time $t \geqslant 0$,

$$\int_V P_t \phi \, \mathrm{d}\pi = \int_V \phi \, \mathrm{d}\pi, \qquad \forall \phi \in SC_b(V_w). \tag{8.1}$$

Let us recall a result of Maslowski–Seidler [MS99] about the existence of an invariant measure. This is a modification of the Krylov–Bogoliubov technique, usually presented in the setting of strong topologies, see, e.g. [BK37] and [DPZ96].

Theorem 8.1. *Assume that*

(i) the semigroup $\{P_t\}_{t\geqslant 0}$ is sequential weak Feller in V;
(ii) for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\sup_{T\geqslant 1} \frac{1}{T} \int_0^T \mathbb{P}(\|u(t;0)\|_V > R_\varepsilon) \, \mathrm{d}t < \varepsilon.$$

Then, there exists at least one invariant measure for equation (1.1).

Hence we get our main result on invariant measures as defined by (8.1).

Theorem 8.2. *Let assumptions 2.1, 2.4 and 2.7 hold. If condition (3.11) is fulfilled, then there exists at least one invariant measure π for equation (1.1) and $\pi(V) = 1$.*

Proof. The proof is based on theorem 8.1. The sequential weak Feller property has been proved before in corollary 7.2. For the tightness it is enough to recall (5.3) and the Chebyshev inequality, so

$$\mathbb{P}(\|u(t;0)\|_V > R) \leqslant \frac{\mathbb{E}\|u(t;0)\|_V^2}{R^2} \leqslant \frac{C}{R^2}, \qquad \text{for all } t \geqslant 0,$$

where the constant C is independent of t. Hence we have verified that the two assumptions of theorem 8.1 are fulfilled. □

9. Existence and uniqueness of the invariant measure with purely multiplicative noise

Assume that the coefficients characterising the operator G are such that $C_1 = 0$; this implies that $G(0) = 0$, see (2.19). Hence the zero process is a solution of equation (1.1), or equivalently δ_0 is an invariant measure. Let us prove that this is the unique invariant measure if

$$\beta > \tfrac{1}{2}\tilde{C}_1^2.$$

This is our result

Theorem 9.1. *Let assumptions 2.1, 2.4 hold and assumption 2.7 holds with $C_1 = 0$, that is*

$$\|G(u)\|_{\gamma(Y_2,H)} \leqslant \tilde{C}_1 \|u\|_H \qquad \forall u \in H. \tag{9.1}$$

If

$$\beta > \frac{1}{2}\tilde{C}_1^2,$$

then there exists a unique invariant measure for equation (1.1) given by $\pi = \delta_0$.

The proof is based on an auxiliary result

Lemma 9.2. *Under the assumptions of theorem 9.1, there exists a constant* $\lambda > 0$ *such that, if* u *is a solution to (1.1), then the process* $\{e^{\lambda t}\|u(t)\|_H^2\}_{t\geqslant 0}$ *is a non-negative continuous supermartingale.*

Proof. Let u be the unique solution to equation (1.1) starting from $u_0 \in V$. We apply the Itô formula to the process $g(r,u(r))$ with $g(r,x) := e^{\lambda r}\|x\|_H^2$, for $r \in [s,t]$, since we know that the paths are in $C([0,T];H)$. We have

$$\mathrm{d}\left(e^{\lambda t}\|u(t)\|_H^2\right) = \lambda e^{\lambda t}\|u(t)\|_H^2 + e^{\lambda t}\mathrm{d}\|u(t)\|_H^2.$$

By the same computations done in the proof of proposition 5.6 we get

$$\mathrm{d}\|u(t)\|_H^2 + 2\beta\|u(t)\|_H^2\mathrm{d}t = \|G(u(t))\|_{\gamma(Y_2,H)}^2\mathrm{d}t + 2\,\mathrm{Re}\,(u(t), -\mathrm{i}G(u(t))\,\mathrm{d}\mathbf{W}(t))_H.$$

Hence, using (9.1)

$$\mathrm{d}\left(e^{\lambda t}\|u(t)\|_H^2\right) \leqslant \left(\lambda - 2\beta + \tilde{C}_1^2\right)e^{\lambda t}\|u(t)\|_H^2 + 2\,\mathrm{Re}\,(u(t), -\mathrm{i}G(u(t))\,\mathrm{d}\mathbf{W}(t))_H.$$

Taking the conditional expected value on both sides, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[e^{\lambda t}\|u(t)\|_H^2|\mathscr{F}_s\right] \leqslant \left(\lambda - 2\beta + \tilde{C}_1^2\right)\mathbb{E}\left[e^{\lambda t}\|u(t)\|_H^2|\mathscr{F}_s\right], \quad s < t,$$

so that

$$\mathbb{E}\left[e^{\lambda t}\|u(t)\|_H^2|\mathscr{F}_s\right] \leqslant e^{\left(\lambda-2\beta+\tilde{C}_1^2\right)(t-s)}e^{\lambda s}\|u(s)\|_H^2, \quad s < t.$$

If we now choose $\lambda > 0$ such that $\lambda - 2\beta + \tilde{C}_1^2 < 0$ we obtain

$$\mathbb{E}\left[e^{\lambda t}\|u(t)\|_H^2|\mathscr{F}_s\right] \leqslant e^{\lambda s}\|u(s)\|_H^2 \qquad \forall\, s < t.$$

□

Let us now prove theorem 9.1.

Proof of theorem 9.1. For the (unique) solution of problem (1.1), we put in evidence the initial datum $u_0 \in V$ by writing $u(\cdot;u_0)$. Lemma 9.2 yields

$$\mathbb{E}\|u(t;u_0)\|_H^2 \leqslant e^{-\lambda t}\|u_0\|_H^2, \qquad \forall\, t \geqslant 0.$$

Proceeding as in [BMS05, Proof of theorem 1.4] one then shows by means of the Borel–Cantelli lemma that, for every $\bar{\lambda} \in (0,\lambda)$, there exists a $\mathbb{P}$-a.s. finite function $t_0 : \Omega \to [0,\infty]$ such that

$$\|u(t;u_0)\|_H^2 \leqslant e^{-\bar{\lambda}t}\|u_0\|_H^2, \qquad \forall\, t \geqslant t_0, \quad \mathbb{P}\text{-}a.s.$$

Hence

$$\|u(t;u_0)\|_H \to 0, \quad \text{as } t \to \infty, \quad \mathbb{P}\text{-}a.s.$$

Now take any function $\phi: V \to \mathbb{R}$ which is continuous with respect to the H-norm; write $\phi \in C(V_H)$. By the above we have

$$\lim_{t\to+\infty} \phi(u(t;u_0)) = \phi(0)$$

for any initial data u_0. Moreover this function ϕ belongs to $SC_b(V_w)$, because the embedding $V \subset H$ is compact, so that any sequence weakly convergent in V is strongly convergent in H.

Now let π be any invariant measure. Then we have

$$\int_V P_t\phi \, \mathrm{d}\pi = \int_V \phi \, \mathrm{d}\pi \qquad \forall \phi \in SC_b(V_w), \quad t \geqslant 0.$$

Taking $\phi \in C_b(V_H)$, by the dominated convergence theorem the lhs converges to $\phi(0)$ as $t \to +\infty$. This implies that

$$\phi(0) = \int_V \phi \, \mathrm{d}\pi \qquad \forall \phi \in C_b(V_H). \tag{9.2}$$

We can conclude that $\pi = \delta_0$ if this equality holds for any $\phi \in SC_b(V_w)$. In fact, take $\phi(u) = \mathrm{e}^{\mathrm{i}\langle u,h\rangle}$, $h \in V^*$, so to get that the integral defines the characteristic function and this is enough to determine the measure.

Now we show by approximation that (9.2) holds for any $\phi \in SC_b(V_w)$. Given $u \in V$ and $\epsilon > 0$ define $u_\epsilon = (I + \epsilon A)^{-1}u$ so that

$$\lim_{\epsilon\to 0} \|u_\epsilon - u\|_V = 0 \tag{9.3}$$

and

$$\|u_\epsilon\|_V \leqslant C(\epsilon) \|u\|_H$$

where the constant $C(\epsilon)$ is not bounded as $\epsilon \to 0$.

Now take any $\phi \in SC_b(V_w) \subset C_b(V)$ and define

$$\phi_\epsilon(u) = \phi\left((I + \epsilon A)^{-1} u\right).$$

It is clear $\phi_\epsilon(u) \to \phi(u)$ for any $u \in V$. Moreover from the previous arguments we have that $\phi_\epsilon \in C_b(V_H)$ and therefore we can write the identity

$$\int_V \phi_\epsilon \, \mathrm{d}\pi = \phi(0).$$

Now passing in the limit as $\epsilon \to 0$ in the lhs, by means of the dominated convergence theorem and (9.3) we get

$$\int_V \phi \, \mathrm{d}\pi = \phi(0) \qquad \forall \phi \in SC_b(V_w).$$

□

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Laplacian-type operators on manifolds and on bounded domains with Dirichlet/Neumann boundary conditions and Strichartz estimates

In section 6 we need some results about Sobolev spaces on two-dimensional manifolds and on bounded domains of $\mathbb{R}^2$ with either Dirichlet or Neumann boundary conditions. We collect them here. Then we derive the Strichartz estimates employed in section 6.

A.1. Dirichlet and Neumann Laplacians on bounded domains and Sobolev spaces

The present section is devoted to recall some basic facts about Sobolev spaces on bounded domains of $\mathbb{R}^2$ and their connection with the fractional domains of the realisation of the Laplace operator with Dirichlet and Neumann boundary conditions on L^q spaces, $q \in (1,\infty)$. We recall also some Sobolev embedding theorems.

Let $\mathcal{O}$ be a bounded smooth domain of $\mathbb{R}^2$. For any $s \in \mathbb{R}$ and $q \in (1,\infty)$, the Sobolev space $H^{s,q}(\mathcal{O})$ is defined as the restriction of $H^{s,q}(\mathbb{R}^2)$, see [Tr78, definition 2.3.1], to $\mathcal{O}$, see [Tr78, definition 4.2.1(1)]. For $q=2$ we write $H^s(\mathcal{O}) := H^{s,2}(\mathcal{O})$. When s is a natural number the space $H^{s,q}(\mathcal{O})$ coincides with the Sobolev space $W^{s,q}(\mathcal{O})$, see [Tr78, remark 2.3.1(2*)]. We denote by $H_0^{s,q}(\mathcal{O})$ the competition of $C_0^\infty(\mathcal{O})$ (set of smooth functions defined over $\mathcal{O}$ with compact support) in $H^{s,q}(\mathcal{O})$, see [Tr78, definition 4.2.1(2)].

In the following proposition we list some embedding properties of the Sobolev spaces.

Proposition A.1. *Let $\mathcal{O}$ be a bounded smooth domain of $\mathbb{R}^2$, then*

(i) for $2 \leqslant q < \infty$, the embedding $H^1(\mathcal{O}) \subset L^q(\mathcal{O})$, is continuous and compact.
(ii) for $1 < q < \infty$ and $s > \frac{2}{q}$, $H^{s,q}(\mathcal{O}) \subset L^\infty(\mathcal{O})$,
(iii) for $1 < p \leqslant q < \infty$, $0 < t < s < 1$ and $s - \frac{2}{p} \geqslant t - \frac{2}{q}$, $H^{s,p}(\mathcal{O}) \subset H^{t,q}(\mathcal{O})$.

Proof. (i) See [Le17, theorem 11.23 and exercise 11.26].
(ii) See [Tr78, theorem 4.6.1(e)].
(iii) See [Tr78, remark 2.8.1(2) and theorem 4.2.2(1)].

□

Remark A.2. Since we always consider the case $|s| < 2$, where 2 is the dimension of the space, it would be enough to assume $\mathcal{O}$ bounded and C^2: see [Tr78, remark 4.2.2 (2)] for the relation between the regularity assumptions on the domain and the range of the exponent s. Smoothness of the boundary is in any case necessary for the Strichartz estimates we consider, to hold.

Let us now turn to the characterisation of the domains of the Dirichlet and Neumann Laplacian. Let $-A_{\mathrm{D}}$ and $-A_{\mathrm{N}}$ be, respectively, the realisation of the Laplace operator in $L^2(\mathscr{O})$ with zero Dirichlet and zero Neumann boundary conditions, with domains

$$\mathscr{D}(A_{\mathrm{D}}) = \left\{f \in H^2(\mathscr{O}) \ : \gamma_{|\partial\mathscr{O}} f = 0\right\},$$
$$\mathscr{D}(A_{\mathrm{N}}) = \left\{f \in H^2(\mathscr{O}) \ : \gamma_{|\partial\mathscr{O}} \partial_\nu f = 0\right\},$$

where by $\gamma_{|\partial\mathscr{O}}$ we denote the trace operator and by ν the outward normal unit vector to $\partial\mathscr{O}$. It is well known, see e.g. [Te97], that both the Dirichlet and the Neumann Laplacian are self-adjoint positive operators on $L^2(\mathscr{O})$. By means of the functional calculus for self-adjoint operators, see e.g. [Ze95], the powers A_{D}^s and A_{N}^s of the operators A_{D} and A_{N}, for every $s \in \mathbb{R}$, are then well defined and self-adjoint. Thus one can introduce the spaces $\mathscr{D}(A_{\mathrm{D}}^{\frac{s}{2}})$ and $\mathscr{D}(A_{\mathrm{N}}^{\frac{s}{2}})$, for every $s \in \mathbb{R}$ in accordance with the spectral theorem.

To derive the needed Strichartz estimates we have to consider the realisations of Dirichlet and Neumann Laplacian on Banach spaces $L^q(\mathscr{O})$, $q \in (1,\infty)$. For this part we mainly refer to [Gr16] and to therein references. The domains of the realisations of the Dirichlet and Neumann Laplacian in $L^q(\mathscr{O})$, denoted hereafter by A_{D_q} and A_{N_q} respectively, are

$$\mathscr{D}\left(A_{\mathrm{D}_q}\right) = \left\{f \in H^{2,q}(\mathscr{O}) \ : \gamma_{|\partial\mathscr{O}} f = 0\right\},$$
$$\mathscr{D}\left(A_{\mathrm{N}_q}\right) = \left\{f \in H^{2,q}(\mathscr{O}) \ : \gamma_{|\partial\mathscr{O}} \partial_\nu f = 0\right\}.$$

Definition A.3. Let $s \in (0,2)$ and $q \in (1,\infty)$. Define the spaces

$$H_{\mathrm{D}}^{s,q}(\mathscr{O}) := \left\{f \in H^{s,q}(\mathscr{O}) : \gamma_{|\partial\mathscr{O}} f = 0 \text{ if } s > \frac{1}{q}\right\},$$
$$H_{\mathrm{N}}^{s,q}(\mathscr{O}) = \left\{f \in H^{s,q}(\mathscr{O}) : \gamma_{|\partial\mathscr{O}} \partial_\nu f = 0 \ \text{ if } s > 1 + \frac{1}{q}\right\}.$$

Theorem A.4. *Let $q \in (1,\infty)$, then*

(i) for $s \in (0,2) \setminus \{\frac{1}{q}\}$, $H_{\mathrm{D}}^{s,q}(\mathscr{O}) = \mathscr{D}((A_{\mathrm{D}_q})^{\frac{s}{2}})$,
(ii) for $s \in (0,2) \setminus \{1 + \frac{1}{q}\}$, $H_{\mathrm{N}}^{s,q}(\mathscr{O}) = \mathscr{D}((A_{\mathrm{N}_q})^{\frac{s}{2}})$.

Proof. See [Gr16, theorem 2.2]. □

Remark A.5. Theorem A.4 and [Tr78, theorem 4.3.2 (1)] yield, in particular,

$$\mathscr{D}(A_{\mathrm{D}}^{\frac{s}{2}}) = \begin{cases} H^s \text{ for } s \in \left(0, \frac{1}{2}\right), \\ H_0^s \text{ for } s \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{A.1}$$

and

$$\mathscr{D}(A_{\mathrm{N}}^{\frac{s}{2}}) = H^s, \qquad \text{for } s \in (0,1]. \tag{A.2}$$

A.2. Laplace–Beltrami operators on compact Riemannian manifolds and Sobolev spaces

In the present section we recall some results about Sobolev spaces on manifolds and their connection with the fractional domains of the Laplace–Beltrami operator.

We consider (M,g), a compact Riemannian manifold without boundary of dimension two. By $-A = \Delta_g$ we denote the Laplace–Beltrami operator on $L^2(M)$. Theorem 3.5 in [Stz83] states that the restriction of $\left(e^{-tA}\right)_{t\geqslant 0}$ to $L^2(M)\cap L^q(M)$ extends to a strongly continuous semigroup on $L^q(M)$, $q\in[1,\infty)$. The infinitesimal generator of such a semigroup, denoted by $-A_{g,q}=\Delta_{g,q}$, is called the Laplace–Beltrami operator on $L^q(M)$. With this extended semigroup one can define the fractional powers of the operator $-A_{g,q}$. For our needs it is sufficient to recall the characterisation of the fractional domains of the Laplace–Beltrami operator on $L^q(M)$, in terms of Sobolev spaces.

Proposition A.6. *Let (M, g) be a compact Riemannian manifold without boundary of dimension two. Let $s\geqslant 0$ and $q\in(1,\infty)$. The fractional Sobolev space $H^{s,q}(M)$ defined as*

$$H^{s,q}(M) := \left\{ f\in L^q(M) : \|f\|_{H^{s,q}(M)} := \left(\sum_{i\in I} \|(\Psi_i f)\circ\kappa_i^{-1}\|^q_{H^{s,q}(\mathbb{R}^d)}\right)^{\frac{1}{q}} < \infty \right\},$$

where $\mathscr{A} := (U_i,\kappa_i)_{i\in I}$ is an atlas of M and $(\Psi_i)_{i\in I}$ a partition of unity subordinate to $\mathscr{A}$, has the following properties:

(i) $H^{s,q}=\mathscr{D}((-\Delta_{g,q})^{\frac{s}{2}})$.
(ii) for $s>\frac{2}{q}$, we have $H^{s,q}(M)\hookrightarrow L^\infty(M)$,
(iii) let $s\geqslant 0$ and $q\in(1,\infty)$. Suppose $q\in[2,\frac{2}{(1-s)_+})$ or $q=\frac{2}{1-s}$ if $s<1$. Then, the embedding $H^s(M)\hookrightarrow L^q(M)$ is continuous.
If $0<s\leqslant 1$ as well as $q\in[1,\frac{2}{(1-s)_+})$, the embedding $H^s(M)\hookrightarrow L^q(M)$ is compact.
(iv) *For $s,s_0,s_1\geqslant 0$ and $p,p_0,p_1\in(1,\infty)$ and $\theta\in(0,1)$ with*

$$s=(1-\theta)s_0+\theta s_1,\qquad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},$$

we have $[H^{s_0,p_0}(M),H^{s_1,p_1}(M)]_\theta = H^{s,p}(M)$.

Proof. Statement (i) follows from [Tr92, chapter 7] and the results of [Stz83]. For the other statements we refer to [BHW19, proposition B.2]. □

Remark A.7. (i) It is known, see [Tr92], that for $k\in\mathbb{N}_0$ and $q\in[1,\infty)$, $H^{k,q}(M)=W^{k,q}(M)$, where $W^{k,q}(M)$ is the classical Sobolev space defined via covariant derivatives.
(ii) For $q=2$, we write $H^s(M):=H^{s,2}(M)$.

A.3. Strichartz estimates

In this section we derive the Strichartz estimates that we need for the proof of uniqueness in section 6.

Throughout this section the operator A can be either the Laplace–Beltrami operator $-\Delta_g$ on a two-dimensional compact Riemannian manifold (M,g) without boundary, equipped with a Lipschitz metric g, or the negative Laplace operator with Dirichlet or Neumann boundary conditions on a smooth relatively compact domain $\mathscr{O}\subset\mathbb{R}^2$.

By A_q we mean the realisation of the above mentioned operators on the L^q space, see sections A.1 and A.2. As usual, if not specified, by L^q we mean either $L^q(M)$ or $L^q(\mathscr{O})$ and, for simplicity we write A instead of A_2.

When the operator A is of the type described above, for every $s \geqslant 0$ and $q \in (1,\infty)$, $(Id + A_q)^{-s}$ defines an isomorphism from L^q to $\mathscr{D}(A_q^s)$ and it holds that

$$\|f\|_{\mathscr{D}(A_q^s)} \simeq \|v\|_{L^q}, \quad \text{for } f = (I + A_q)^{-s} v. \tag{A.3}$$

In the next lemma we recall the deterministic homogeneous Strichartz estimate from a recent paper by Blair, Smith and Sogge, see [BSS08, theorem 1.1], stated here in the form more suitable for our needs.

Lemma A.8. *Let $-A$ be either the Laplace–Beltrami operator on a two-dimensional compact Riemannian manifold (M, g) without boundary equipped with a Lipschitz metric g, or the realisation of the negative Laplace operator with Dirichlet or Neumann boundary conditions on a smooth relatively compact domain $\mathscr{O} \subset \mathbb{R}^2$. Assume that (p, q) is a Strichartz pair of real numbers, i.e. $2 \leqslant p, q \leqslant \infty$ and*

$$\frac{2}{p} + \frac{2}{q} = 1, \qquad (p,q) \neq (2,\infty).$$

Then the following Strichartz estimate holds for every $x \in \mathscr{D}(A^{\frac{2}{3p}})$

$$\|e^{-i\cdot A}x\|_{L^p([0,T];L^q)} \lesssim_T \|x\|_{\mathscr{D}\left(A^{\frac{2}{3p}}\right)} \tag{A.4}$$

Let us notice that when $p = \infty$, then $q = 2$ and the inequality (A.4) becomes the classical one

$$\|e^{-i\cdot A}x\|_{L^\infty([0,T];L^2)} \lesssim_T \|x\|_{L^2}.$$

Remark A.9. In the case where (M, g) is a boundaryless manifold with $g \in C^\infty$, the estimate (A.4) holds with $s = \frac{1}{2p}$ instead of $s = \frac{2}{3p}$, see the paper [BGT04] by Burq, Gérard and Tzvetkov. In particular, the Strichartz estimates for a boundaryless manifold are stronger than the Strichartz estimates for a bounded domain with smooth boundary.

From lemma A.8 we can deduce the following Strichartz estimates for the deterministic and stochastic convolutions.

Lemma A.10. *Assume that $T > 0$. In the situation of lemma A.8, we take $\vartheta \in \left[\frac{4}{3p}, 1\right]$ and $r \in (1,\infty)$.*

(i) We have the homogeneous Strichartz estimate

$$\|e^{-i\cdot A}x\|_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} \lesssim_T \|x\|_{\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)}, \qquad \text{for } x \in \mathscr{D}\left(A^{\frac{\vartheta}{2}}\right), \tag{A.5}$$

and the inhomogeneous Strichartz estimate

$$\left\|\int_0^{\cdot} e^{-i(\cdot-\tau)A} f(\tau)\, d\tau\right\|_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} \lesssim_T \|f\|_{L^1\left(0,T,\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)}, \tag{A.6}$$

for $f \in L^1(0,T;\mathscr{D}(A^{\frac{\vartheta}{2}}))$.

(ii) *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, Y be a separable real Hilbert space, W a Y-canonical cylindrical Wiener processes adapted to a filtration* $\mathbb{F}$ *satisfying the usual conditions. We have the stochastic Strichartz estimate*

$$\begin{aligned}
&\left\| \int_0^{\cdot} \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A} \xi(\tau)\,\mathrm{d}W(\tau) \right\|_{L^r\left(\Omega, L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)\right)} \\
&\lesssim_T \|\xi\|_{L^r\left(\Omega;L^2\left(0,T;\gamma\left(Y,\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)\right)\right)}
\end{aligned} \tag{A.7}$$

for all adapted processes $\xi \in L^r(\Omega; L^2(0,T;\gamma(Y,\mathscr{D}(A^{\frac{\vartheta}{2}}))))$.

Proof. (i) Estimate (A.5) follows from (A.3) and (A.4) that yield

$$\begin{aligned}
\|\mathrm{e}^{-\mathrm{i}\cdot A} x\|_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} &\simeq \|(Id + A_q)^{\frac{\vartheta}{2}-\frac{2}{3p}} \mathrm{e}^{-\mathrm{i}\cdot A} x\|_{L^p(0,T;L^q)} \\
&\simeq \|\mathrm{e}^{-\mathrm{i}\cdot A} (Id + A_q)^{\frac{\vartheta}{2}-\frac{2}{3p}} x\|_{L^p(0,T;L^q)} \\
&\leqslant \|(Id + A_q)^{\frac{\vartheta}{2}-\frac{2}{3p}} x\|_{\mathscr{D}\left(A^{\frac{2}{3p}}\right)} \simeq \|x\|_{\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)}.
\end{aligned} \tag{A.8}$$

The proof of estimates (A.6) follows the lines of the proof of [BGT04, corollary 2.1], see also the proof of [BMi14, lemma 3.2]. The lhs in (A.6) reads

$$\begin{aligned}
I &:= \left\| \int_0^T F_\tau \,\mathrm{d}\tau \right\|_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)}, \\
F_\tau(t) &:= \mathbf{1}_{[\tau,T]}(t)\, \mathrm{e}^{-\mathrm{i}(t-\tau)A} f(\tau), \ t \in [0,T].
\end{aligned}$$

Let us observe that by estimate (A.5) (with C_T being the constant)

$$\begin{aligned}
&\|F_\tau\|^p_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} = \int_0^T \|F_\tau(t)\|^p_{\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)} \mathrm{d}t \\
&= \int_0^T \|\mathbf{1}_{[\tau,T]}(t)\, \mathrm{e}^{-\mathrm{i}(t-\tau)A} f(\tau)\|^p_{\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)} \mathrm{d}t = \int_\tau^T \|\mathrm{e}^{-\mathrm{i}(t-\tau)A} f(\tau)\|^p_{\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)} \mathrm{d}t \\
&= \int_0^{T-\tau} \|\mathrm{e}^{-\mathrm{i}sA} f(\tau)\|^p_{\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)} \mathrm{d}s \leqslant \int_0^T \|\mathrm{e}^{-\mathrm{i}sA} f(\tau)\|^p_{\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)} \mathrm{d}s \\
&\leqslant C_T^p \|f(\tau)\|^p_{\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)}.
\end{aligned}$$

Therefore the Minkowski inequality yield

$$I \leqslant \int_0^T \|F_\tau\|_{L^p\left(0,T;\mathscr{D}\left(A_q^{\frac{\vartheta}{2}-\frac{2}{3p}}\right)\right)} \mathrm{d}\tau$$
$$\leqslant C_T \int_0^T \|f(\tau)\|_{\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)} \mathrm{d}\tau = C_T \|f\|_{L^1\left(0,T,\mathscr{D}\left(A^{\frac{\vartheta}{2}}\right)\right)}.$$

(ii) When $r = p$ [BMi14, theorem 3.10] and (A.4) yield

$$\left\| \int_0^{\cdot} \mathrm{e}^{-\mathrm{i}(\cdot-\tau)A} \xi(\tau)\, \mathrm{d}W(\tau) \right\|_{L^r(\Omega,L^p(0,T;L^q))} \lesssim_T \|\xi\|_{L^r\left(\Omega;L^2\left(0,T;\gamma\left(Y,\mathscr{D}\left(A^{\frac{2}{3p}}\right)\right)\right)\right)},$$

and reasoning as in (A.8) one obtains (A.7) with $r = p$. For the case $r \neq p$ the result follows from [Hor18b, corollary 2.2]: our parameter θ is the same parameter θ that appears in that result, the parameter μ that appears there is equal to $\frac{4}{3p}$ in our case.

□

Appendix B. Proof of proposition 5.7(c)

Proof of proposition 5.7(c). This proof has some similarities with that of proposition 5.7(a). However, here we look for a uniform estimate on the unbounded time interval $[0, +\infty)$.

We use the auxiliary process $Z(u)$ defined in (5.15):

$$Z(u) = \|u\|_H^2 + 2\mathscr{E}(u) \equiv \|u\|_V^2 + 2\hat{F}(u).$$

We will prove that

$$\sup_{t\geqslant 0} \mathbb{E}\left[Z(u_n(t))\right] < \infty, \tag{B.1}$$

from which, estimate (5.3) immediately follows. In order to prove (B.1) let us deal separately with the quantities $\mathbb{E}\left[\|u_n\|_H^2\right]$ and $\mathbb{E}\left[\mathscr{E}(u_n)\right]$.

Applying the Itô formula to the squared H-norm of u_n (compare with the computations done in the proof of proposition 5.6) we obtain, almost surely for all $t \geqslant 0$

$$\|u_n(t)\|_H^2 = \|P_n u_0\|_H^2 - 2\beta \int_0^t \|u_n(s)\|_H^2 \mathrm{d}s + \int_0^t \|G(u_n(s))\|_{\gamma(Y_2,H)}^2 \mathrm{d}s + N_n(t),$$

where $N_n(t) = 2\int_0^t \mathrm{Re}\left(u_n(s), -\mathrm{i}G(u_n(s))\,\mathrm{d}\mathbf{W}(s)\right)_H$ is a martingale. Taking the expected values on both sides we obtain

$$\mathbb{E}\left[\|u_n(t)\|_H^2\right] = \mathbb{E}\left[\|P_n u_0\|_H^2\right] - 2\beta \int_0^t \mathbb{E}\left[\|u_n(s)\|_H^2\right] \mathrm{d}s + \int_0^t \mathbb{E}\left[\|G(u_n(s))\|_{\gamma(Y_2,H)}^2\right] \mathrm{d}s;$$

we write the above equation in the differential form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\|u_n(t)\|_H^2\right] = -2\beta\mathbb{E}\left[\|u_n(t)\|_H^2\right] + \mathbb{E}\left[\|G(u_n(t))\|_{\gamma(Y_2,H)}^2\right].$$

From assumption 2.7(iii) we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\|u_n(t)\|_H^2\right] \leqslant 2C_1^2 + 2\left(\tilde{C}_1^2 - \beta\right)\mathbb{E}\left[\|u_n(t)\|_H^2\right]. \tag{B.2}$$

We now apply the It ô formula to the energy functional $\mathscr{E}$ (compare also with computations done in the proof of in proposition 5.7) and obtain that, almost surely for all $t \geqslant 0$,

$$\begin{aligned}
\mathscr{E}(u_n(t)) = \mathscr{E}(P_n u_0) &+ \int_0^t \mathrm{Re}\langle A u_n(s) + F(u_n(s)), b(u_n(s)) - \beta u_n(s)\rangle\,\mathrm{d}s + M(t) \\
&+ \frac{1}{2}\int_0^t \|A^{\frac{1}{2}} B u_n(s)\|_{\gamma(Y_1,H)}^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|A^{\frac{1}{2}} G(u_n(s))\|_{\gamma(Y_2,H)}^2\,\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \sum_{m=1}^{\infty} \mathrm{Re}\langle F'[u_n(s)](B(u_n(s))f_m), B(u_n(s))f_m\rangle\,\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \sum_{m=1}^{\infty} \mathrm{Re}\langle F'[u_n(s)](G(u_n(s))e_m), G(u_n(s))e_m\rangle\,\mathrm{d}s,
\end{aligned}$$

where

$$\begin{aligned}
M(t) = &\int_0^t \mathrm{Re}\langle A u_n(s) + F(u_n(s)), -\mathrm{i}B(u_n(s))\,\mathrm{d}W(s)\rangle \\
&+ \int_0^t \mathrm{Re}\langle A u_n(s) + F(u_n(s)), -\mathrm{i}G(u_n(s))\,\mathrm{d}\mathbf{W}(s)\rangle
\end{aligned}$$

is the sum of two martingales. As above, we take the expected value on both sides of the above equality and we write the equation in its differential form as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[2\mathscr{E}(u_n(t))] = &\,2\mathbb{E}[\mathrm{Re}\langle A u_n(t) + F(u_n(t)), b(u_n(t)) - \beta u_n(t)\rangle] \\
&+ \mathbb{E}\left[\|A^{\frac{1}{2}} B u_n(t)\|_{\gamma(Y_1,H)}^2\right] + \mathbb{E}\left[\|A^{\frac{1}{2}} G u_n(t)\|_{\gamma(Y_2,H)}^2\right] \\
&+ \mathbb{E}\left[\sum_{m=1}^{\infty} \mathrm{Re}\langle F'[u_n(t)](B(u_n(t))f_m), B(u_n(t))f_m\rangle\right] \\
&+ \mathbb{E}\left[\sum_{m=1}^{\infty} \mathrm{Re}\langle F'[u_n(t)](G(u_n(t))e_m), G(u_n(t))e_m\rangle\right].
\end{aligned}$$

We now estimate the RHS of the above equality. Recalling (3.2) we have

$$\begin{aligned}
2\mathbb{E}[\mathrm{Re}\langle A u_n, b(u_n)\rangle] &\leqslant \mathbb{E}\left[\|A^{\frac{1}{2}} u_n\|_H \|B\|_{\mathscr{L}(V,\gamma(Y_1,V))}^2 \|u_n\|_V\right] \\
&\leqslant \|B\|_{\mathscr{L}(V,\gamma(Y_1,V))}^2 \mathbb{E}\left[\|u_n\|_V^2\right]
\end{aligned} \tag{B.3}$$

and, thanks to (2.5) and (2.9), we obtain

$$\begin{aligned}2\mathbb{E}\left[\operatorname{Re}\langle F(u_n), b(u_n)\rangle\right] &\leqslant \mathbb{E}\left[\|F(u_n)\|_{L^{\frac{\alpha+1}{\alpha}}}\sum_{m=1}^{\infty}\|B_m^2 u_n\|_{L^{\alpha+1}}\right]\\ &\leqslant \|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\mathbb{E}\left[\|u_n\|^{\alpha+1}_{L^{\alpha+1}}\right]\\ &= (\alpha+1)\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\mathbb{E}\left[\hat{F}(u_n)\right].\end{aligned}$$

We exploit (2.7) and (2.9) to get

$$\begin{aligned}2\mathbb{E}\left[\operatorname{Re}\langle Au_n + F(u_n), -\beta u_n\rangle\right] &= -2\beta\mathbb{E}\left[\|A^{\frac{1}{2}}u_n\|_H^2\right] - 2\beta\mathbb{E}\left[\|u_n\|^{\alpha+1}_{L^{\alpha+1}}\right]\\ &= -2\beta\mathbb{E}\left[\|A^{\frac{1}{2}}u_n\|_H^2\right] - 2\beta(\alpha+1)\mathbb{E}\left[\hat{F}(u_n)\right].\end{aligned}$$

We have

$$\mathbb{E}\left[\|A^{\frac{1}{2}}Bu_n\|^2_{\gamma(Y_1,H)}\right] \leqslant \|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}\mathbb{E}\left[\|u_n\|_V^2\right]$$

and, from (2.16),

$$\mathbb{E}\left[\|A^{\frac{1}{2}}G(u_n)\|^2_{\gamma(Y_2,H)}\right] \leqslant 2\left(C_2^2 + \tilde{C}_2^2\mathbb{E}\left[\|u_n\|_V^2\right]\right).$$

From (2.8), (2.9) and remark 2.10 we obtain

$$\begin{aligned}\mathbb{E}\left[\sum_{m=1}^{\infty}\operatorname{Re}\langle F'[u_n]((Bu_n)f_m), (Bu_n)f_m\rangle\right] &\leqslant \mathbb{E}\left[\|F'[u_n]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}}\|B(u_n)\|^2_{\gamma(Y_1,L^{\alpha+1})}\right]\\ &\leqslant \alpha\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\mathbb{E}\left[\|u_n\|^{\alpha+1}_{L^{\alpha+1}}\right]\\ &= \alpha(\alpha+1)\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}\mathbb{E}\left[\hat{F}(u_n)\right].\end{aligned}$$

Finally, from (2.8), (2.9), (2.11) and (2.17), we obtain

$$\begin{aligned}&\mathbb{E}\sum_{m=1}^{\infty}\operatorname{Re}\langle F'[u_n](G(u_n)e_m), G(u_n)e_m\rangle\\ &\quad\leqslant \mathbb{E}\left[\|F'[u_n]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}}\|G(u_n)\|^2_{\gamma(Y_1,L^{\alpha+1})}\right]\\ &\quad\leqslant \alpha\mathbb{E}\left[\|u_n\|^{\alpha-1}_{L^{\alpha+1}}2\left(C_3^2 + \tilde{C}_3^2\|u_n\|^2_{L^{\alpha+1}}\right)\right]\\ &\quad= 2\alpha C_3^2(\alpha+1)^{\frac{\alpha-1}{\alpha+1}}\mathbb{E}\left[\left(\hat{F}(u_n)\right)^{\frac{\alpha-1}{\alpha+1}}\right] + 2\alpha\tilde{C}_3^2(\alpha+1)\mathbb{E}\left[\hat{F}(u_n)\right]\\ &\quad\leqslant \frac{2}{\alpha+1}\left(\frac{\alpha-1}{\varepsilon(\alpha+1)}\right)^{\frac{\alpha-1}{2}}\left(2\alpha C_3^2\right)^{\frac{\alpha+1}{2}} + \left(\varepsilon + 2\tilde{C}_3^2\alpha\right)(\alpha+1)\mathbb{E}\left[\hat{F}(u_n)\right],\end{aligned}\tag{B.4}$$

where in the last estimate we exploited the Young inequality

$$2\alpha C_3^2\left((\alpha+1)\hat{F}(u_n)\right)^{\frac{\alpha-1}{\alpha+1}} \leqslant \varepsilon(\alpha+1)\hat{F}(u_n)+\frac{2}{\alpha+1}\left(\frac{\alpha-1}{\varepsilon(\alpha+1)}\right)^{\frac{\alpha-1}{2}}\left(2\alpha C_3^2\right)^{\frac{\alpha+1}{2}},$$

for any $\varepsilon>0$.

Collecting estimates (B.3) and (B.4) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[2\mathscr{E}(u_n(t))\right] \leqslant{}& 2C_2^2+\frac{2}{\alpha+1}\left(\frac{\alpha-1}{\varepsilon(\alpha+1)}\right)^{\frac{\alpha-1}{2}}\left(2\alpha C_3^2\right)^{\frac{\alpha+1}{2}}-2\beta\|A^{\frac{1}{2}}u_n(t)\|_H^2\\&+2\left(\tilde{C}_2^2+\|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}\right)\mathbb{E}\left[\|u_n(t)\|_V^2\right]\\&+(\alpha+1)\left((\alpha+1)\|B\|^2_{\mathscr{L}\left(L^{\alpha+1},\gamma\left(Y_1,L^{\alpha+1}\right)\right)}+\varepsilon+2\alpha\tilde{C}_3^2-2\beta\right)\mathbb{E}\left[\hat{F}(u_n(t))\right].\end{aligned} \tag{B.5}$$

Recalling the definition of $Z(u_n)$, we now take the sum in both sides of inequalities (B.2) and (B.5)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[Z(u_n(t))\right] \leqslant{}& 2C_1^2+2C_2^2+\frac{2}{\alpha+1}\left(\frac{\alpha-1}{\varepsilon(\alpha+1)}\right)^{\frac{\alpha-1}{2}}\left(2\alpha C_3^2\right)^{\frac{\alpha+1}{2}}\\&+2\left(\tilde{C}_1^2-\beta+\tilde{C}_2^2+\|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}\right)\mathbb{E}\left[\|u_n(t)\|_V^2\right]\\&+(\alpha+1)\left((\alpha+1)\|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))}+\varepsilon+2\alpha\tilde{C}_3^2-2\beta\right)\mathbb{E}\left[\hat{F}(u_n(t))\right].\end{aligned}$$

Now we assume (3.11); then for a suitable choice of ε both coefficients in front of the norms are negative. Therefore there exist two positive constants C_4 and C_5 independent of n such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[Z(u_n(t))\right] \leqslant C_4 - C_5\mathbb{E}\left[Z(u_n(t))\right].$$

From Gronwall's inequality we infer

$$\mathbb{E}\left[Z(u_n(t))\right] \leqslant \mathbb{E}\left[Z(u_n(0))\right]\mathrm{e}^{-C_5 t}+\frac{C_4}{C_5}\left(1-\mathrm{e}^{-C_5 t}\right),$$

so

$$\sup_{n\in\mathbb{N}}\sup_{t\geqslant 0}\mathbb{E}\left[Z(u_n(t))\right]<\infty.$$

□

Appendix C. Proof of lemma 5.9

Proof of lemma 5.9. Let $0\leqslant s\leqslant t<\infty$ and $\psi,\varphi\in V$. We define

$$g_n(t,s) := h\left(v_n|_{[0,s]}\right)\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\left(S_nG(S_nv_n(r))e_m,\psi\right)_H \mathrm{Re}\left(S_nG(S_nv_n(r))e_m,\varphi\right)_H \mathrm{d}r,$$

$$g(t,s) := h\left(v|_{[0,s]}\right)\sum_{m=1}^{\infty}\int_s^t \mathrm{Re}\langle G(v(r))e_m,\psi\rangle \mathrm{Re}\langle G(v(r))e_m,\varphi\rangle \,\mathrm{d}r.$$

We will prove that the functions $\{g_n\}_{n\in\mathbb{N}}$ are uniformly integrable and converge $\hat{\mathbb{P}}$-a.s. to g.
• **$\hat{\mathbb{P}}$-a.s. convergence.** Because of $h(v_n|_{[0,s]}) \to h(v|_{[0,s]})$ $\hat{\mathbb{P}}$-a.s. and the continuity of the inner product $L^2([s,t]\times\mathbb{N})$, the convergence

$$\mathrm{Re}(S_nG(S_nv_n)e_m,\psi)_H \to \mathrm{Re}\langle G(v)e_m,\psi\rangle$$

$\hat{\mathbb{P}}$-a.s. in $L^2([s,t]\times\mathbb{N})$ already implies $g_n(t,s)\to g(t,s)$ $\hat{\mathbb{P}}$-a.s. Therefore, it is sufficient to prove

$$\lim_{n\to\infty}\|\mathrm{Re}(S_nG(S_nv_n)e_\cdot,\psi)_H - \mathrm{Re}\langle G(v)e_\cdot,\psi\rangle\|_{L^2([s,t]\times\mathbb{N})} = 0 \qquad \hat{\mathbb{P}}\text{-}a.s.$$

We estimate

$$\begin{aligned}
&\|\mathrm{Re}(S_nG(S_nv_n)e_\cdot,\psi)_H - \mathrm{Re}\langle G(v)e_\cdot,\psi\rangle\|_{L^2([s,t]\times\mathbb{N})}\\
&\quad\leqslant \|\mathrm{Re}(G(S_nv_n)e_\cdot,(S_n-I)\psi)_H\|_{L^2([s,t]\times\mathbb{N})} + \|\mathrm{Re}(G(S_nv_n)e_\cdot - G(v_n)e_\cdot,\psi)_H\|_{L^2([s,t]\times\mathbb{N})}\\
&\qquad + \|\mathrm{Re}\langle G(v_n)e_\cdot - G(v)e_\cdot,\psi\rangle\|_{L^2([s,t]\times\mathbb{N})}\\
&\quad =: I_1(n) + I_2(n) + I_3(n).
\end{aligned}$$

We work pathwise. By means of (2.19) and (5.6) we estimate

$$\begin{aligned}
\|I_1(n)\|_{L^2([s,t]\times\mathbb{N})} &= \|\mathrm{Re}(G(S_nv_n)e_\cdot,(S_n-I)\psi)_H\|_{L^2([s,t]\times\mathbb{N})}\\
&\leqslant \left(\int_s^t \|GS_nv_n(r)\|^2_{\gamma(Y_2,H)}\,\mathrm{d}r\right)^{1/2}\|(S_n-I)\psi\|_H\\
&\lesssim \left(1+\sup_{n\in\mathbb{N}}\|v_n\|^2_{L^\infty(0,T;H)}\right)^{1/2}\|(S_n-I)\psi\|_V.
\end{aligned}$$

Bearing in mind, from proposition 5.6, the boundedness of the sequence $(v_n)_n$ in $L^\infty(0,T;H)$, for any $T>0$, we get the convergence to zero as $n\to\infty$ as a consequence of proposition 5.2.

Using (2.15) we estimate

$$\begin{aligned}
\|I_2(n)\|_{L^2([s,t]\times\mathbb{N})} &= \|\mathrm{Re}(G(S_nv_n)e_\cdot - G(v_n)e_\cdot,\psi)_H\|_{L^2([s,t]\times\mathbb{N})}\\
&\leqslant \left(\int_s^t\sum_{m=1}^{\infty}\|G(S_nv_n(r))e_m - G(v_n(r))e_m\|^2_H\,\mathrm{d}r\right)^{\frac12}\|\psi\|_H\\
&= \left(\int_s^t \|G(S_nv_n(r)) - G(v_n(r))\|^2_{\gamma(Y_2,H)}\,\mathrm{d}r\right)^{\frac12}\|\psi\|_H\\
&\leqslant L_G\left(\int_s^t\|(S_n-I)v_n(r)\|^2_H\,\mathrm{d}r\right)^{\frac12}\|\psi\|_H\\
&\lesssim \|\psi\|_V\|S_n-I\|_{\mathscr{L}(V)}\sup_{n\in\mathbb{N}}\|v_n\|_{L^\infty(0,T,V)}.
\end{aligned}$$

Recalling corollary 5.8, about the boundedness of the sequence $(v_n)_n$ in $L^\infty(0,T;V)$, the convergence to zero, as $n\to\infty$, follows again from proposition 5.2.

The convergence to zero, as $n \to \infty$, of the last term

$$\|I_3(n)\|_{L^2([s,t]\times\mathbb{N})} = \|\operatorname{Re}\langle G(v_n)e_\cdot - G(v)e_\cdot, \psi\rangle\|_{L^2([s,t]\times\mathbb{N})}$$

follows as a consequence of the continuity of the norm $L^2([s,t]\times\mathbb{N})$, assumption (2.18) and (5.23).

• **Uniform integrability.** It is sufficient to show that, for some $r>1$,

$$\sup_{n\geqslant 1}\hat{\mathbb{E}}\left[|g_n(t,s)|^r\right] < \infty, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Let $r>1$, we estimate

$$\begin{aligned}
\hat{\mathbb{E}}\left[|g_n(t,s)|^r\right] &\leqslant \hat{\mathbb{E}}\Big[\|\operatorname{Re}\langle S_nG(S_nv_n)e_\cdot,\psi\rangle\|^r_{L^2([s,t]\times\mathbb{N})} \\
&\qquad \times\|\operatorname{Re}\langle S_nG(S_nv_n)e_\cdot,\varphi\rangle\|^r_{L^2([s,t]\times\mathbb{N})}\left|h\left(v_n|_{[0,s]}\right)\right|^r\Big] \\
&\leqslant \hat{\mathbb{E}}\left[\left(\int_s^t\|G(S_nv_n(r))\|^2_{\gamma(Y_2,H)}\,\mathrm{d}r\right)^{\frac{r}{2}}\right]\|\psi\|^r_V\|\varphi\|^r_V\|h\|^r_\infty \\
&\lesssim \left(1+\sup_{n\geqslant 1}\hat{\mathbb{E}}\|v_n\|^r_{L^\infty(0,T;H)}\right)\|\psi\|^r_V\|\varphi\|^r_V\|h\|^r_\infty,
\end{aligned}$$

which is finite thanks to proposition 5.6.

Using Vitali's theorem, we finally obtain

$$\lim_{n\to\infty}\hat{\mathbb{E}}\left[g_n(t,s)\right] = \hat{\mathbb{E}}\left[g(t,s)\right], \qquad 0 \leqslant s \leqslant t \leqslant T,$$

which concludes the proof. □

Appendix D. Yamada–Watanabe theorem for stochastic evolution equations

The infinite dimensional version of the Yamada–Watanabe theorem has a long history. As far as we are aware, the first time Yamada–Watanabe theorem was mentioned in the infinite-dimensional setting of stochastic evolution equations (SEEs) was a paper by Brzeźniak and and Gątarek [BG99] about stochastic reaction diffusion equations. In that paper the classical version of the Yamada–Watanabe theorem from [IW89] has been used but no details were provided. A proper formulation for mild solutions to SEEs and a detailed proof have been first given by Ondreját in [On04]. Later on Kunze [Kun13] formulated and proved a similar result in a framework of weak solutions to SEEs. Let us point out, see also [BHW19, section 4], that in case when the pathwise uniqueness holds, another avenue of proving the existence of strong solutions, not by the Prokhorov–Skorokhod theorems, is possible. Namely, one can use the Gyöngy and Krylov lemma, see [GK96, lemma 1], to prove that the approximations converge in probability and that the limit process is a strong solution. This approach has been recently used by Crisan *et al* [CFH19] but it still required the use of the Skorokhod embedding theorem. As it was observed in [BHW22], it would be of interest to see if this approach works for the class of stochastic NLS studied in the present paper.

Returning to the topic of an infinite dimensional version of the Yamada–Watanabe theorem let us emphasise that the present formulation differs from the formulations from [On04, theorem 2] and [Kun13, theorem 5.3 and corollary 5.4] since we consider only solutions with a given initial law.

The pathwise uniqueness and the existence of martingale solutions imply the existence of strong solutions, see e.g. [On04, theorem 2] and [Kun13, theorem 5.3 and corollary 5.4].

Theorem D.1. *Assume that assumptions 2.1, 2.4 and 2.7 are satisfied. Assume that $r \in [1,\infty)$ and that μ is a Borel probability measure on V whose $2r$th moment is finite. If*

(i) there exists a martingale solutions *to equation (1.1;) with the initial data μ.*
(ii) pathwise-uniqueness of solutions to equation (1.1) holds, i.e. if two systems

$$\left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{W}}, \tilde{\mathbb{F}}, u_1\right) \text{ and } \left(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{W}}, \tilde{\mathbb{F}}, u_2\right)$$

are martingale solutions *of the equation (1.1) with the initial data μ, i.e. such that*

$$\mathrm{Law}_{\tilde{\mathbb{P}}}(u_i(0)) = \mu \text{ on } \mathscr{B}(V), \quad i = 1,2,$$

then

$$\tilde{\mathbb{P}}(u_1(t) = u_2(t)) = 1 \text{ for all } t \geqslant 0,$$

then there exists a strong solution to equation (1.1) with the initial data μ.

Appendix E. Weak measurability of the norm function

Let us begin by fixing some notation (we follow [BHW19]). For a Banach space X and $r > 0$ we denote

$$\mathbb{B}_X(0,r) := \{u \in X : \|u\|_X \leqslant r\}.$$

The set $\mathbb{B}_X(0,r)$ endowed with the weak topology will be denoted by $\mathbb{B}_X^w(0,r)$. The space X endowed with the weak topology τ_w will be denoted by X_w, see [Ru91, definition 3.11]. Let us recall, see e.g. [Ru91, Thorem 3.12], that $\mathbb{B}_X(0,r)$ is closed in X_w.

If the dual X^* of a Banach space X is separable, then, see [Br10, theorem 3.29], the space $\mathbb{B}_X^w(0,r)$ is metrisable and a metric is given by

$$q(x,y) = \sum_{k=1}^{\infty} 2^{-k} |\langle x-y, x_k^*\rangle|, \qquad x,y \in \mathbb{B}_X(0,r),$$

where $(x_k^*)_{k\in\mathbb{N}}$ is a dense sequence in $\mathbb{B}_{X^*}(0,1)$. If X is also separable, then the set $C([0,T];\mathbb{B}_X^w(0,r))$ is a complete separable metric space with metric

$$\rho(u,v) := \sup_{t\in[0,T]} q(u(t),v(t)), \quad u,v \in C([0,T];\mathbb{B}_X^w(0,r)).$$

Recall that, if X is a Banach space and $T > 0$, we define

$$C_w([0,T];X) := \{u : [0,T] \to X : \text{for all } x^* \in X^*, \\ [0,T] \ni t \mapsto \langle u(t), x^*\rangle \in \mathbb{C} \text{ is continuous}\}.$$

The vector space $C_w([0,T];X)$ is endowed with the locally convex topology induced by the family $\mathscr{P}$ of seminorms given by

$$\mathscr{P} := \{p_{x^*} : x^* \in X^*\},$$
$$p_{x^*}(u) := \sup_{t\in[0,T]} |\langle u(t), x^*\rangle|, \quad u \in C_w([0,T];X).$$

We can also define in a classical way the space $C([0,T];X_w)$ and we have the following result.

Proposition E.1. *Assume that X is a separable Banach space. Let $r>0$ and $T>0$. Then for a function $u:[0,T]\to X$ the following two conditions*

(i) $u\in C([0,T];X_w)$
(ii) $u\in C_w([0,T];X)$

are equivalent, and the following two conditions

(iii) $u\in C([0,T];\mathbb{B}_X^w(0,r))$
(iv) $u\in C_w([0,T];X)$ *and* $\sup_{t\in[0,T]}\|u(t)\|_X\leqslant r$

are equivalent.

Note that A.2 from [BHW19] implies sequential closedness of the set $C([0,T];\mathbb{B}_X^w(0,r))$ in $C_w([0,T];X)$. The following result, whose proof is similar to the proof of [Ru91, theorem 3.12], shows that the set $C([0,T];\mathbb{B}_X^w(0,r))$ is a closed subset of $C_w([0,T];X)$.

Lemma E.2. *Assume that X is a Banach space. The set $C([0,T];\mathbb{B}_X^w(0,r))$ is closed in $C_w([0,T];X)$.*

Proof. We prove that the complement set is open. Let us choose and fix $a\in C_w([0,T];X)\setminus C([0,T];\mathbb{B}_X^w(0,r))$. Then we can find $t_0\in[0,T]$ such that

$$a(t_0)\in X\setminus\mathbb{B}_X(0,r).$$

Since the set $A=\{a(t_0)\}$ is compact in X and the set $\mathbb{B}_X(0,r)$ is closed in X, by the separation theorem, i.e. [Ru91, theorem 3.4], we can find $\gamma_1,\gamma_2\in\mathbb{R}$ and $\lambda^*\in X^*$ such that

$$\mathrm{Re}\langle a(t_0),\lambda^*\rangle<\gamma_1<\gamma_2<\mathrm{Re}\langle y,\lambda^*\rangle,\ \text{ for every } y\in\mathbb{B}_X(0,r). \tag{E.1}$$

Next we define a set G by

$$G:=\{u\in C_w([0,T];X):\mathrm{Re}\langle u(t_0),\lambda^*\rangle<\gamma_1\}.$$

By the definition of the topology in $C_w([0,T];X)$, G is its open subset. Moreover, in view of (E.1) we infer that

$$a\in G,\quad G\cap C([0,T];\mathbb{B}_X^w(0,r))=\emptyset.$$

This concludes the proof. □

Let us now formulate the main result of this section.

Corollary E.3. *Assume that X is a separable Banach space. Then the function*

$$\|\cdot\|_T:C_w([0,T];X)\ni u\mapsto\sup_{t\in[0,T]}\|u(t)\|_X\in[0,\infty)$$

is well defined, lower semicontinuous and (hence) Borel-measurable, i.e. the function $\|\cdot\|_T$ is $\mathscr{B}\big(C_w([0,T];X)\big)/\mathscr{B}(\mathbb{R})$-measurable.

Proof. Let us choose and fix $u \in C_w([0,T];X)$. Then, by proposition E.1, $u \in C([0,T];X_w)$. Since $[0,T]$ is a compact topological space, the range of u is a compact subset of X_w. Hence it is also weakly bounded and therefore, by [Ru91, theorem 3.18], it is also bounded, i.e. bounded w.r.t. the original norm topology, in X. Hence $\|u\|_T \in [0,\infty)$ what proves the first part of the result.

To prove the lower semicontinuity, we choose $r > 0$ and we need to prove that the set $\{u \in C([0,T];X_w) : \|u\|_T \leqslant r\}$ is closed in $C([0,T];X_w)$. Note that by proposition E.1

$$\{u \in C([0,T];X_w) : \|u\|_T \leqslant r\} = C([0,T];\mathbb{B}_X^{\mathrm{w}}(0,r)).$$

Thus, by lemma E.2 we obtain the lower semicontinuity. This also shows the Borel measurability, completing the proof. □

Appendix F. A technical lemma

The following result provides a criterion for convergence of a sequence in $C([0,\infty),\mathbb{B}_V^w(0,r))$. We need this result in the proof of proposition 4.1.

Lemma F.1. *Let $(r_N)_{N=1}^{\infty}$ be a sequence of positive numbers and $(u_n)_{n\in\mathbb{N}} \subset L^{\infty}_{\mathrm{loc}}([0,\infty);V)$ be a sequence with the properties*

(a) for every $N \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,N;V)} \leqslant r_N$,
(b) for every $N \in \mathbb{N}$, $u_n \to u$ in $C([0,N];V^*)$ for $n \to \infty$.

Then, for every $n \in \mathbb{N}$, $u_n, u \in C_w([0,\infty);V)$. Moreover, for every $N \in \mathbb{N}$ and every $n \in \mathbb{N}$, $u_n, u \in C([0,N],\mathbb{B}_V^w(0,r_N))$ and

$$u_n \to u \text{ in } C([0,N],\mathbb{B}_V^w(0,r_N)) \text{ as } n \to \infty.$$

Proof. This proof is a minor modification of the proof of [BHW19, lemma 4.1]. The Strauss-Lemma (see [Te01, chapter 3, lemma 1.4]) and the assumptions guarantee that for every $n \in \mathbb{N}$

$$u_n \in C([0,\infty);V^*) \cap L^{\infty}_{\mathrm{loc}}([0,\infty);V) \subset C_w([0,\infty);V)$$

and, for every $N \in \mathbb{N}$ and every $n \in \mathbb{N}$,

$$\sup_{t\in[0,N]} \|u_n(t)\|_V \leqslant r_N.$$

Hence, we infer that $u_n \in C([0,N],\mathbb{B}_V^w(0,r_N))$ for all $n \in \mathbb{N}$ and $N \in \mathbb{N}$.

Let us now choose and fix $N \in \mathbb{N}$. Then, for every $h \in V$

$$\sup_{s\in[0,N]} |\langle u_n(s) - u(s), h\rangle| \leqslant \|u_n - u\|_{C([0,N],V^*)} \|h\|_V \to 0, \qquad n \to \infty.$$

Hence by assumption (a) and the Banach–Alaoglu theorem we find a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $v \in L^\infty(0,N;V)$ such that

$$u_{n_k} \rightharpoonup^* v \text{in } L^\infty(0,N;V).$$

Hence, by the uniqueness of the weak star limit in $L^\infty(0,N;V^*)$, we conclude $u = v \in L^\infty(0,N;V)$ with $\|u\|_{L^\infty(0,N;V)} \leqslant r_N$.

Let $\varepsilon > 0$ and $h \in V^*$. By the density of V in V^*, we choose $h_\varepsilon \in V$ with $\|h - h_\varepsilon\|_{V^*} \leqslant \frac{\varepsilon}{4r}$ and obtain for large $n \in \mathbb{N}$ and all $s \in [0,N]$

$$\begin{aligned} |\langle u_n(s) - u(s), h\rangle| &\leqslant |\langle u_n(s) - u(s), h - h_\varepsilon\rangle| + |\langle u_n(s) - u(s), h_\varepsilon\rangle| \\ &\leqslant \|u_n(s) - u(s)\|_V \|h - h_\varepsilon\|_{V^*} + |\langle u_n(s) - u(s), h_\varepsilon\rangle| \\ &\leqslant 2r\frac{\varepsilon}{4r} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $\sup_{s\in[0,N]} |\langle u_n(s) - u(s), h\rangle| \to 0$ as $n \to \infty$. By the arbitrariness of $h \in V^*$, we infer that $u_n \to u$ in $C_w([0,N];V)$. Hence by [BHW19, lemma A.2] we obtain the assertion. The proof of lemma F.1 is complete. □

ORCID iD

Zdzisław Brzeźniak https://orcid.org/0000-0001-8731-6523

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