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The ω-Borel invariant for representations into $\mathrm{SL}(n, \mathbb{C}_\omega)$

Alessio Savini

Abstract. Let Γ be the fundamental group of a complete hyperbolic 3-manifold M with toric cusps. By following [3] we define the ω-Borel invariant $\beta_n^\omega(\rho_\omega)$ associated to a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n, \mathbb{C}_\omega)$, where $\mathbb{C}_\omega$ is a field introduced by [18] which can be constructed as a quotient of a suitable subset of $\mathbb{C}^{\mathbb{N}}$ with the data of a non-principal ultrafilter ω on $\mathbb{N}$ and a real divergent sequence λ_l such that $\lambda_l \geq 1$.

Since a sequence of ω-bounded representations ρ_l into $\mathrm{SL}(n, \mathbb{C})$ determines a representation ρ_ω into $\mathrm{SL}(n, \mathbb{C}_\omega)$, for $n = 2$ we study the relation between the invariant $\beta_2^\omega(\rho_\omega)$ and the sequence of Borel invariants $\beta_2(\rho_l)$. We conclude by showing that if a sequence of representations $\rho_l\colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ induces a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ which determines a reducible action on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ with non-trivial length function, then it holds $\beta_2^\omega(\rho_\omega) = 0$.

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1. Introduction

Given a finitely generated group Γ, the character variety $X(\Gamma, \mathrm{SL}(n, \mathbb{C}))$ is an algebraic variety obtained as GIT-quotient of the representation variety $R(\Gamma, \mathrm{SL}(n, \mathbb{C}))$ by the conjugation action of $\mathrm{SL}(n, \mathbb{C})$. When Γ is the fundamental group of a complete hyperbolic 3-dimensional manifold M with toric cusps, it is possible to attach to every equivalence class of representations a suitable invariant called Borel invariant. Indeed, in [3] the authors prove that the Borel class $\beta(n)$, already introduced and studied in [13], is a generator for the cohomology group $H^3_{cb}(P\,\mathrm{SL}(n, \mathbb{C}))$. Thus, given a representation $\rho\colon \Gamma \to \mathrm{PSL}(n, \mathbb{C})$, we can construct a class into $H^3_b(\Gamma)$ by pulling back $\beta(n)$ along ρ^*_b and then evaluate this new class on a fundamental class $[N, \partial N] \in H^3(N, \partial N)$. Here N is a compact core of M. When $n = 2$ this invariant is exactly the volume of the representation defined as the integral of the pullback of the standard volume form $\omega_{\mathbb{H}^3}$ along any pseudo-developing map D, as written both in [10] and in [11] (see for instance [15] for a proof of the equivalence). The Borel invariant of a representation $\rho\colon \Gamma \to \mathrm{SL}(n, \mathbb{C})$ is the Borel invariant of the induced representation into $\mathrm{PSL}(n, \mathbb{C})$. Moreover, since this invariant remains unchanged under conjugation, we have a well-defined function on the character variety $X(\Gamma, \mathrm{SL}(n, \mathbb{C}))$, called Borel function, which is continuous with respect to the topology of the pointwise convergence.

Inspired by the work of Thurston about the compactification of the Teichmuller space for a closed surface of genus g exposed in [22] and generalizing the constructions for algebraic curves appeared in [9], in [16] J. Morgan and P. Shalen proposed a new way to compactify a generic algebraic variety V given a generating set $\mathcal{F}$ for the algebra of regular functions $\mathbb{C}[V]$. This particular method applied to the character variety $X(\Gamma, \mathrm{SL}(2, \mathbb{C}))$ allows to interpret the ideal points of the compactification as projective length functions of isometric Γ-actions on real trees which are constructed as Bass–Serre trees associated to $\mathrm{SL}(2, \mathbb{K}_v)$, where $\mathbb{K}_v$ is a suitable valued field (see [21]). A more geometric approach based on Gromov–Hausdorff convergence was suggested by both [1] and [20]. Lately [18] extended this intepretation to the more general case of $X(\Gamma, \mathrm{SL}(n, \mathbb{C}))$ by viewing an ideal point as a projective vectorial length function relative to an isometric action, this time on a Euclidean building of type A_{n-1}. The method suggested by [18] to obtain the Euclidean building and its isometric Γ-action is based on asymptotic cones and it reminds the ones already exposed both in [1] and in [20].

In the attempt to link all these ideas, one could naturally ask if it is possible to extend continuously the Borel function to the ideal points of the compactification of $X(\Gamma, \mathrm{SL}(n, \mathbb{C}))$. Going further, one could be interested in studying the possible values attained at ideal points and trying to formulate a rigidity result, which would generalize [3, Theorem 1].

The aim of this paper is to make a small step towards this direction by defining a numerical invariant, the ω-Borel invariant, associated to a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n, \mathbb{C}_\omega)$, where $\mathbb{C}_\omega$ is a field obtained as a quotient of a suitable subset of $\mathbb{C}^{\mathbb{N}}$ by an equivalence relation which depends on a non-principal ultrafilter ω on $\mathbb{N}$ and a real divergent sequence λ_l with $\lambda_l \geq 1$. The motivation of this definition relies on the interpretation of the limit action of Γ on the Euclidean bulding of type A_{n-1} as a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n, \mathbb{C}_\omega)$, as proved in [18, Theorem 5.2].

The first section is dedicated to preliminary definitions, in particular we recall the definition of the field $\mathbb{C}_\omega$ and the notion of bounded cohomology of locally compact groups. In the second section we give the definition of the ω-Borel cohomology class $\beta^\omega(n)$, which is an element of $H^3_b(\mathrm{SL}^\delta(n, \mathbb{C}_\omega))$. In the last section we define the ω-Borel invariant $\beta^\omega_n(\rho_\omega)$ for a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n, \mathbb{C}_\omega)$ and we describe some of its properties. In particular we focus our attention on the case $n = 2$. We show that if a sequence of representations $\rho_l\colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ induces a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ which determines a reducible action on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ with non-trivial length function, then it holds $\beta^\omega_2(\rho_\omega) = 0$.

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2. Preliminary definitions

2.1. The field $\mathbb{C}_\omega$. For more details regarding the definitions and the results contained in this section we refer to [18, Section 3.3]. We start by recalling the notion of ultrafilter and some fundamental properties that we are going to exploit lately.

Definition 2.1. An *ultrafilter* ω on a set X is a family of subsets of X which satisfies the following conditions.

- The empty set is not contained in ω, that is $\varnothing \notin \omega$.
- If $A \subset B$ and $A \in \omega$, then $B \in \omega$.
- Given a collection $A_1, \ldots, A_n$ such that $A_i \in \omega$ for every $i = 1, \ldots, n$, then $A_1 \cap \cdots \cap A_n \in \omega$.
- Given $A_1, \ldots, A_n$ such that $A_1 \sqcup \cdots \sqcup A_n = X$, there exists exactly one $i_0 \in \{1, \ldots, n\}$ so that $A_{i_0} \in \omega$.

An ultrafilter is *principal* and centered at $x \in X$ if for every set $A \in \omega$ it holds $x \in A$. Otherwise we say that the ultrafilter is *non-principal*.

The importance of ultrafilters relies on their power to force convergence of sequences of points in a topological space X by selecting a suitable limit point. For the sake of clarity we first need to introduce the following

Definition 2.2. Let X be a topological space and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of points in X. Fix an ultrafilter ω on the set of natural numbers $\mathbb{N}$. We say that the sequence *ω-converges* to x_0 if for every open neighborhood U of x_0 we have $\{k \in \mathbb{N} \colon x_k \in U\} \in \omega$.

A priori a sequence may admit no limit or several limits if the topology of the space X does not have good properties. To guarantee the existence and the uniqueness of the limit we need a compact Hausdorff space. Indeed, it holds

Proposition 2.3. *Let X be a topological space which is compact and Hausdorff. Then, for any ultrafilter ω on $\mathbb{N}$ and any sequence $(x_k)_{k\in\mathbb{N}}$ of points in X, there exists a unique point $x_0 \in X$ such that*

$$\omega\text{-}\lim_{k\to\infty} x_k = x_0.$$

Another remarkable property of ultrafilters is the compatibility with continuous functions between topological spaces.

Proposition 2.4. *Let $f: X \to Y$ be a continuous function between two compact Hausdorff spaces. Let ω be an ultrafilter on $\mathbb{N}$. For any sequence $(x_k)_{k\in\mathbb{N}}$ of points in X we have*

$$\omega\text{-}\lim_{k\to\infty} f(x_k) = f(\omega\text{-}\lim_{k\to\infty} x_k).$$

We are now ready to describe the construction of the field $\mathbb{C}_\omega$. Let ω be a non-principal ultrafilter on $\mathbb{N}$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a real sequence that diverges to infinity and such that $\lambda_k \geq 1$ for every k. We define

$$\mathbb{C}_\omega = \{(a_k) \in \mathbb{C}^{\mathbb{N}} \mid \text{there exists } C > 0 \text{ such that } |a_k|^{\frac{1}{\lambda_k}} < C \text{ for all } k\}/\sim_\omega$$

where $(a_k)_{k\in\mathbb{N}} \sim_\omega (b_k)_{k\in\mathbb{N}}$ if and only if $\omega\text{-}\lim_{k\to\infty} |a_k - b_k|^{\frac{1}{\lambda_k}} = 0$. It is easy to verify that the operations of pointwise sum and pointwise multiplication defined over $\mathbb{C}^{\mathbb{N}}$ are compatible with the equivalence relation $\sim_\omega$. Thus they define two operations of sum and multiplication over $\mathbb{C}_\omega$, which make $\mathbb{C}_\omega$ a field. There is a natural field embedding of $\mathbb{C}$ into $\mathbb{C}_\omega$ given by the constant sequences.

If we denote by a_ω the equivalence class $[(a_k)]$ of the sequence $(a_k)_{k\in\mathbb{N}}$, the function

$$|a_\omega|^\omega := \omega\text{-}\lim_{k\to\infty} |a_k|^{\frac{1}{\lambda_k}}$$

is an ultrametric absolute value on $\mathbb{C}_\omega$, that is it satisfies

$$|a_\omega + b_\omega|^\omega \leq \max\{|a_\omega|^\omega, |b_\omega|^\omega\}$$

for every pair $a_\omega, b_\omega \in \mathbb{C}_\omega$. It is worth noticing the elements of $\mathbb{C}$, seen as the subfield of constant sequences, have all norm equal to 1.

Definition 2.5. The ultrametric field $(\mathbb{C}_\omega, |\cdot|^\omega)$ is called the *asymptotic cone* of $(\mathbb{C}, |\cdot|)$ with respect to the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the ultrafilter ω.

If we consider the distance induced by the absolute value $|\cdot|^\omega$ and we endow $\mathbb{C}_\omega$ with the metric topology, we obtain a topological field which is complete (see [18, Remark 3.10]), but it is not locally compact.

Proposition 2.6. *The field $\mathbb{C}_\omega$ is not locally compact with respect to the metric topology induced by the absolute value $|\cdot|^\omega$.*

Proof. Since $\mathbb{C}_\omega$ is a normed space, local compactness can be checked by verifying the compactness of the unit closed ball. Hence, it suffices to show that the closed ball

$$\overline{B}_1(0) := \{a_\omega \in \mathbb{C}_\omega \mid |a_\omega|^\omega \leq 1\}$$

is not compact. We are going to show that it is not sequentially compact. Consider the sequence $(n)_{n\in\mathbb{N}}$ where each element n has to be thought of as an element of $\mathbb{C}_\omega$ thanks to the standard embedding given by constant sequences. Given two different elements n and m it is clear that their distance in $\mathbb{C}_\omega$ is always equal to 1, indeed

$$|n-m|^\omega = \omega\text{-}\lim_{k\to\infty} |n-m|^{\frac{1}{\lambda_k}} = 1.$$

Hence it cannot exist a subsequence of $(n)_{n\in\mathbb{N}}$ which converges, as desired. $\square$

The construction exposed above can be repeated, rather than for a field, for every m-dimensional normed vector space $(V, \|\cdot\|)$ over $\mathbb{C}$. More precisely, we define

$$V_\omega := \{(v_k) \in V^{\mathbb{N}} \mid \text{there exists } C > 0 \text{ such that } \|v_k\|^{\frac{1}{\lambda_k}} < C \text{ for all } k\}/\sim_\omega,$$

where $(v_k)_{k\in\mathbb{N}}$ and $(u_k)_{k\in\mathbb{N}}$ are equivalent if and only if $\omega\text{-}\lim_{k\to\infty} \|u_k - v_k\|^{\frac{1}{\lambda_k}} = 0$. Let v_ω be the equivalence class determined by $(v_k)_{k\in\mathbb{N}}$. It is possible to endow V_ω with a structure of m-dimensional $\mathbb{C}_\omega$-vector space by considering the operations induced by pointwise sum and by pointwise scalar multiplication. As before, we have a well-defined norm $\|\cdot\|^\omega$ given by

$$\|v_\omega\|^\omega := \omega\text{-}\lim_{k\to\infty} \|v_k\|^{\frac{1}{\lambda_k}}.$$

Definition 2.7. The $\mathbb{C}_\omega$-vector space $(V_\omega, \|\cdot\|^\omega)$ is the *asymptotic cone* of the vector space $(V, \|\cdot\|)$ with respect to the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the ultrafilter ω.

We now focus our attention on the set of complex square matrices of order n, namely $M(n, \mathbb{C})$. If we choose as norm over $M(n, \mathbb{C})$ the standard matrix norm, we can apply the construction above to the normed vector space $(M(n, \mathbb{C}), \|\cdot\|)$. In this particular case we are able to enrich the structure of $M(n, \mathbb{C})_\omega$ by considering a multiplication. Indeed, the classic multiplication rows-by-columns is compatible with $\sim_\omega$ and hence it defines a structure of $\mathbb{C}_\omega$-algebra on $M(n, \mathbb{C})_\omega$.

Definition 2.8. The normed algebra $(M(n, \mathbb{C})_\omega, \|\cdot\|^\omega)$ is called the *asymptotic cone* of the algebra $(M(n, \mathbb{C}), \|\cdot\|)$ with respect to the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the ultrafilter ω.

Definition 2.9. A sequence $(g_k) \in \mathrm{GL}(n, \mathbb{C})^{\mathbb{N}}$ is *ω-bounded* if

$$\text{there exists } C > 0 \text{ such that } \|g_k\|^{\frac{1}{\lambda_k}}, \|g_k^{-1}\|^{\frac{1}{\lambda_k}} < C \text{ for all k.}$$

The previous condition implies that the sequence $(g_k)_{k\in\mathbb{N}}$ defines an element of $M(n,\mathbb{C})_\omega$ which admits a multiplicative inverse. We denote by $\mathrm{GL}(n,\mathbb{C})_\omega$ the set of all the invertible elements of $M(n,\mathbb{C})_\omega$. This is a group with respect to the multiplication rows-by-columns. We denote by $\mathrm{SL}(n,\mathbb{C})_\omega$ the subgroup

$$\mathrm{SL}(n,\mathbb{C})_\omega := \{g_\omega \in \mathrm{GL}(n,\mathbb{C})_\omega \mid \text{there exists } (g_k)_{k\in\mathbb{N}} \in g_\omega \text{ such that } \det(g_k) = 1, \text{for all } k\}.$$

Since we can also consider the normed algebra $(M(n,\mathbb{C}_\omega), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the standard supremum norm with respect to $|\cdot|^\omega$, it is natural to ask whether this algebra is isomorphic to $M(n,\mathbb{C})_\omega$ as normed algebra. The answer is given by [18, Corollary 3.18], which states that there is a natural isomorphism as normed $\mathbb{C}_\omega$-algebras between $M(n,\mathbb{C})_\omega$ and $M(n,\mathbb{C}_\omega)$. Moreover this isomorphism induces an isomorphism of groups between $\mathrm{SL}(n,\mathbb{C})_\omega$ and $\mathrm{SL}(n,\mathbb{C}_\omega)$.

We conclude this section by introducing the space $\mathbb{P}^1(\mathbb{C})_\omega$. In order to do this, we first need to recall the construction of the asymptotic cone of $\mathbb{H}^3$.

Definition 2.10. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of basepoints in $\mathbb{H}^3$. Consider the space

$$C_\omega(\mathbb{H}^3, d/\lambda_k, x_k) := \{(y_k) \in (\mathbb{H}^3)^{\mathbb{N}} \mid \text{there exists } C > 0 \text{ such that } d(x_k, y_k) < C\lambda_k \text{ for all } k\}/\sim_\omega,$$

where $(y_k)_{k\in\mathbb{N}} \sim_\omega (y'_k)_{k\in\mathbb{N}}$ if and only if $\omega\text{-}\lim_{k\to\infty} d(y_k, y'_k)/\lambda_k = 0$. Denote by y_ω the equivalence class of the sequence $(y_k)_{k\in\mathbb{N}}$. If we define

$$d_\omega(y_\omega, y'_\omega) = \omega\text{-}\lim_{k\to\infty} d(y_k, y'_k)/\lambda_k$$

we get a metric and the metric space $(C_\omega(\mathbb{H}^3, d/\lambda_k, x_k), d_\omega)$ is the *asymptotic cone* with respect to the ultrafilter ω, the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the sequence of basepoints $(x_k)_{k\in\mathbb{N}}$.

Assume to fix the origin O of the Poincaré model of $\mathbb{H}^3$ as the constant sequence of basepoints for the asymptotic cone construction. It should be clear that there exists a natural surjection

$$\pi\colon \mathbb{P}^1(\mathbb{C})^{\mathbb{N}} \longrightarrow \partial_\infty C_\omega(\mathbb{H}^3, d/\lambda_k, O)$$

defined as it follows. Thinking of $\mathbb{P}^1(\mathbb{C})$ as the boundary at infinity of $\mathbb{H}^3$, a sequence of points $(\xi_k) \in \mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$ determines in a unique way a sequence of geodesic rays $(c_k)_{k\in\mathbb{N}}$ starting from O and ending at $(\xi_k)_{k\in\mathbb{N}}$. These rays allows us to define a geodesic ray $c_\omega\colon [0,\infty) \to C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ given by $c_\omega(t) := [c_k(\lambda_k t)]$. Hence, we can define $\pi((\xi_k)_{k\in\mathbb{N}}) := c_\omega(\infty)$. The space $\mathbb{P}^1(\mathbb{C})_\omega$ is the quotient of $\mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$ by the equivalence relation induced by the surjection π. In this way $\mathbb{P}^1(\mathbb{C})_\omega$ is clearly identified with boundary at infinity of $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ and hence inherits in a natural way an action of $\mathrm{SL}(2,\mathbb{C})_\omega$ given by $[h_k].[\xi_k] := [h_k.\xi_k]$. This action is well defined because the action of $\mathrm{SL}(2,\mathbb{C})_\omega$ on $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ is well defined (see [18, Proposition 3.20]). Moreover, since the Bass–Serre tree $\Delta^{\mathrm{BS}}(\mathrm{SL}(2,\mathbb{C}_\omega))$ associated to $\mathrm{SL}(2,\mathbb{C}_\omega)$ is naturally isometric to $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$, as shown in [18, Proposition 3.21], the space $\mathbb{P}^1(\mathbb{C})_\omega$ can be identified also with $\mathbb{P}^1(\mathbb{C}_\omega)$ and this identification is compatible with the actions of $\mathrm{SL}(2,\mathbb{C})_\omega$ and $\mathrm{SL}(2,\mathbb{C}_\omega)$, respectively.

2.2. Bounded cohomology of locally compact groups. From now until the end of this section we denote by G a locally compact group. We endow $\mathbb{R}$ with the structure of a trivial normed G-module, where the considered norm is the standard Euclidean one. The space of bounded continuous functions is

$$C^n_{cb}(G,\mathbb{R}) := C_{cb}(G^{n+1},\mathbb{R}) = \{f\colon G^{n+1} \to \mathbb{R} \mid f \text{ is continuous and } \|f\|_\infty < \infty\}$$

where the supremum norm is defined as

$$\|f\|_\infty := \sup_{g_0,\dots,g_n\in G} |f(g_0,\dots,g_n)|$$

and $C^n_{cb}(G,\mathbb{R})$ is endowed with the following G-module structure

$$(g.f)(g_0,\dots,g_n) := f(g^{-1}g_0,\dots,g^{-1}g_n)$$

for every element $g \in G$ and every function $f \in C^n_{cb}(G,\mathbb{R})$ (here the notation $g.f$ stands for the action of the element g on f). We denote by δ_n the homogeneous boundary operator of degree n, namely

$$\delta_n\colon C^n_{cb}(G,\mathbb{R}) \to C^{n+1}_{cb}(G,\mathbb{R}),$$

$$\delta_n f(g_0,\dots,g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0,\dots,\hat{g}_i,\dots,g_{n+1}),$$

where the notation $\hat{g}_i$ indicates that the element g_i has been omitted.

There is a natural embedding of $\mathbb{R}$ into $C^0_{cb}(G,\mathbb{R})$ given by the constant functions on G. This allows us to consider the following chain complex of G-modules

$$0 \longrightarrow \mathbb{R} \longrightarrow C^0_{cb}(G,\mathbb{R}) \xrightarrow{\delta_0} C^1_{cb}(G,\mathbb{R}) \xrightarrow{\delta_1} \cdots$$

and thanks to the compatibility of δ_n with respect to the G-action, we can consider the submodules of G-invariant vectors

$$0 \longrightarrow C^0_{cb}(G,\mathbb{R})^G \xrightarrow{\delta_0} C^1_{cb}(G,\mathbb{R})^G \xrightarrow{\delta_1} C^2_{cb}(G,\mathbb{R})^G \xrightarrow{\delta_2} \cdots .$$

Like in any other chain complex, we define the set of the n^{th}*-bounded continuous cocycles* as

$$Z^n_{cb}(G,\mathbb{R})^G := \ker(\delta_n\colon C^n_{cb}(G,\mathbb{R})^G \longrightarrow C^{n+1}_{cb}(G,\mathbb{R})^G)$$

and the set of the n^{th}*-bounded continuous coboundaries*

$$B^n_{cb}(G,\mathbb{R})^G := \operatorname{im}(\delta_{n-1}\colon C^{n-1}_{cb}(G,\mathbb{R})^G \longrightarrow C^n_{cb}(G,\mathbb{R})^G)$$

and

$$B^0_{cb}(G,\mathbb{R}) := 0.$$

Definition 2.11. The *continuous bounded cohomology* in degree n of G with real coefficients is the space

$$H^n_{cb}(G) := H^n_{cb}(G,\mathbb{R}) = \frac{Z^n_{cb}(G,\mathbb{R})^G}{B^n_{cb}(G,\mathbb{R})^G},$$

with the quotient seminorm

$$\|[f]\|_\infty := \inf \|f\|_\infty,$$

where the infimum is taken over all the possible representatives of $[f]$.

It is possible to gain information about the bounded cohomology of G also by studying suitable spaces on which G acts. More precisely, let X be a measurable space on which G acts measurably, that is the action map $\theta\colon G \times X \to X$ is measurable (G is equipped with the σ-algebra of the Haar measurable sets). We set

$$\mathcal{B}^\infty(X^n,\mathbb{R}) := \{f\colon X^n \to \mathbb{R} \mid f \text{ is measurable and } \sup_{x\in X^n} |f(x)| < \infty\},$$

and we endow it with the structure of Banach G-module given by

$$(g.f)(x_1,\dots,x_n) := f(g^{-1}.x_1,\dots,g^{-1}.x_n),$$

for every $g \in G$ and every $f \in \mathcal{B}^\infty(X^n,\mathbb{R})$. If $\delta_n\colon \mathcal{B}^\infty(X^n,\mathbb{R}) \to \mathcal{B}^\infty(X^{n+1},\mathbb{R})$ is the standard homogeneous coboundary operator, for $n \geq 1$ and $\delta_0\colon \mathbb{R} \to \mathcal{B}^\infty(X,\mathbb{R})$ is the inclusion given by constant functions, we get a cochain complex $(\mathcal{B}^\infty(X^\bullet,\mathbb{R}),\delta_\bullet)$. We denote by $\mathcal{B}^\infty_{\text{alt}}(X^{n+1},\mathbb{R})$ the Banach G-submodule of alternating cochains, that is the set of elements satisfying

$$f(x_{\sigma(0)},\dots,x_{\sigma(n)}) = \operatorname{sgn}(\sigma) f(x_0,\dots,x_n),$$

for every permutation $\sigma \in S_{n+1}$.

Definition 2.12. Let E be a Banach G-module. The *continuous submodule* of E is defined by

$$\mathcal{C}E := \{v \in E \mid \lim_{g \to e} \|g.v - v\| = 0\}.$$

A *resolution* of E is an exact complex $(E^\bullet, \partial_\bullet)$ of Banach G-modules such that $E^0 = E$ and $E^n = 0$ for every $n \leq -1$.

$$0 \longrightarrow E \xrightarrow{\partial_0} E^1 \xrightarrow{\partial_1} E^2 \xrightarrow{\partial_2} \cdots .$$

We say that $(E^\bullet, \partial_\bullet)$ is a *strong resolution* if the continuous subcomplex $(\mathcal{C}E^\bullet, \partial_\bullet)$ admits a contracting homotopy, that is a sequence of maps $h_n \colon \mathcal{C}E^{n+1} \to \mathcal{C}E^n$ such that $\|h_n\| \leq 1$ and $h_{n+1} \circ \partial_n + \partial_n \circ h_{n-1} = \mathrm{id}_{E^n}$ for all $n \in \mathbb{N}$.

In [5, Proposition 2.1] the authors prove that the complex of bounded measurable functions $(\mathcal{B}^\infty(X^\bullet, \mathbb{R}), \delta_\bullet)$ is a strong resolution of $\mathbb{R}$. Since the homology of any strong resolution of the trivial Banach G-module $\mathbb{R}$ maps in a natural way to the continuous bounded cohomology of G by [7, Proposition 1.5.2.], there exists a canonical map

$$\mathfrak{c}^\bullet \colon H^\bullet(B^\infty(X^{\bullet+1}, \mathbb{R})^G) \longrightarrow H^\bullet_{cb}(G).$$

More precisely, every bounded measurable G-invariant cocycle $f \colon X^{n+1} \to \mathbb{R}$ determines canonically a class $\mathfrak{c}^n[f] \in H^n_{cb}(G)$. The same result holds for the subcomplex $(\mathcal{B}^\infty_{\mathrm{alt}}(X^\bullet, \mathbb{R}), \delta_\bullet)$ of alternating cochains.

3. The ω-Borel cocycle

3.1. The cocycle Vol^ω. From now until the end of the paper we will consider the spaces $\mathbb{P}^1(\mathbb{C})_\omega$ and $\mathbb{P}^1(\mathbb{C}_\omega)$ identified, hence we will refer to any of these two as they were the same space. The same will be done also for the groups $\mathrm{SL}(n, \mathbb{C})_\omega$ and $\mathrm{SL}(n, \mathbb{C}_\omega)$. Moreover, to avoid a heavy notation we are going to refer to any sequence $(x_l)_{l \in \mathbb{N}}$ by dropping the parenthesis every time that we are considering the sequence itself instead of any of its single term.

In this section we are going to construct a generalization of the hyperbolic volume function which will live on $\mathbb{P}^1(\mathbb{C}_\omega)^4$. This generalization will reveal the fundamental tool to define the ω-Borel cocycle.

Before starting, we want to underline a delicate point. Since we want to exploit the properties of the standard Borel cocycle, one could try to define the new function Vol^ω simply by taking the ω-limit of the volumes, that is $\mathrm{Vol}^\omega(x^0_\omega, \ldots, x^3_\omega) = \omega\text{-}\lim_{l \to \infty} \mathrm{Vol}(x^0_l, \ldots, x^3_l)$, where x^i_l is any representative of x^i_ω. Unfortunately this definition is not correct. Indeed, if we suppose to have 3 points that coincide, say $x^0_\omega = x^1_\omega = x^2_\omega$, different choices of representatives lead to different values of the ω-limit of their volumes. Hence, we need to be careful.

Let $\mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$ be the space of 4-tuples of distinct points on $\mathbb{P}^1(\mathbb{C}_\omega)$. As in the standard case, there is a natural cross ratio function

$$\mathrm{cr}_\omega\colon \mathbb{P}^1(\mathbb{C}_\omega)^{(4)} \longrightarrow \mathbb{C}_\omega \setminus \{0,1\}, \quad \mathrm{cr}_\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) = \frac{(x_\omega^0 - x_\omega^2)(x_\omega^1 - x_\omega^3)}{(x_\omega^0 - x_\omega^3)(x_\omega^1 - x_\omega^2)},$$

which is well defined by its purely algebraic nature. Every x_ω^i may be considered in $\mathbb{C}_\omega$ or equal to ∞. If we define the Bloch–Wigner function by

$$D_2\colon \mathbb{C} \longrightarrow \mathbb{R}, \quad D_2(z) := \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|,$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function, by still denoting D_2 its continuous extension on $\mathbb{P}^1(\mathbb{C})$, we can formulate the following

Definition 3.1. The *ω-Bloch–Wigner function* is given by

$$\begin{aligned} &D_2^\omega\colon \mathbb{C}_\omega \cup \{\infty\} \longrightarrow \mathbb{R}, \\ &D_2^\omega(x_\omega) := \omega\text{-}\lim_{l\to\infty} D_2(x_l) \quad \text{for } x_\omega \in \mathbb{C}_\omega, \\ &D_2^\omega(\infty) := 0. \end{aligned}$$

where x_l is any representative of the equivalence class x_ω.

Lemma 3.2. *If x_l and y_l are two sequences representing the same element in $\mathbb{C}_\omega$, then*

$$\omega\text{-}\lim_{l\to\infty} D_2(x_l) = \omega\text{-}\lim_{l\to\infty} D_2(y_l).$$

Proof. Since $\mathbb{P}^1(\mathbb{C})$ is compact and $\omega\text{-}\lim_{l\to\infty} |x_l - y_l|^{\frac{1}{\lambda_l}} = 0$, both sequences x_l and y_l will converge to the same limit in $\mathbb{C} \cup \{\infty\}$. Denote by ξ this point. As a consequence of Proposition 2.4 and by the continuity of D_2 we have

$$\omega\text{-}\lim_{l\to\infty} D_2(x_l) = D_2(\omega\text{-}\lim_{l\to\infty} x_l) = D_2(\xi) = D_2(\omega\text{-}\lim_{l\to\infty} y_l) = \omega\text{-}\lim_{l\to\infty} D_2(y_l),$$

as claimed. □

The previous lemma guarantees that the definition of the ω-Bloch–Wigner function is correct since it does not depend on the choice of the representative of the class x_ω.

Definition 3.3. The *ω-volume function* for a 4-tuple of points $(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^4$ is defined as

$$\begin{aligned} &\mathrm{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \\ &\quad = \begin{cases} D_2^\omega(\mathrm{cr}_\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3)) & \text{if } (x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^{(4)}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 3.4. We are going to denote by Vol the composition $D_2 \circ \mathrm{cr}$, where D_2 is the standard Bloch–Wigner function and cr is the cross ratio on $\mathbb{P}^1(\mathbb{C})$. Fix a 4-tuple $(x_\omega^0, \dots, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^4$ of distinct points. Thanks to the natural identification between $\mathbb{P}^1(\mathbb{C}_\omega)$ and $\mathbb{P}^1(\mathbb{C})_\omega$, we can think of each x_ω^i as the class of a sequence x_l^i of points in $\mathbb{P}^1(\mathbb{C})$. Now, it easy to see that

$$\mathrm{cr}_\omega(x_\omega^0, \dots, x_\omega^3) = [\mathrm{cr}(x_l^0, \dots, x_l^3)]$$

in $\mathbb{C}_\omega$ (if the x_ω^i are all distinct, also the terms of the sequences x_l^i are distinct ω-almost every $l \in \mathbb{N}$). By exploiting the previous identity, we can rewrite the definition of Vol^ω as follows:

$$\begin{aligned}\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) &= D_2^\omega(\mathrm{cr}_\omega(x_\omega^0, \dots, x_\omega^3)) \\ &= \omega\text{-}\lim_{l\to\infty} D_2(\mathrm{cr}(x_l^0, \dots, x_l^3)) \\ &= \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3);\end{aligned}$$

and this is completely independent of the choice of representatives $x_l^0, \dots, x_l^3$. Hence Vol^ω coincides with the ω-limit of the standard volumes $\mathrm{Vol}(x_l^0, \dots, x_l^3)$ on a 4-tuple $(x_\omega^0, \dots, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$, where x_l^i is any representative for x_ω^i. Even though we have already underlined that this is not true on the whole space $\mathbb{P}^1(\mathbb{C}_\omega)^4$, we can always choose suitable representatives for x_ω^i such that

$$\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3).$$

Proposition 3.5. *The function* Vol^ω *is a bounded, alternating,* $\mathrm{GL}(2, \mathbb{C}_\omega)$*-invariant cocycle.*

Proof. Most of the properties we stated follow directly from the properties of the standard volume function Vol. We are going to show $\mathrm{GL}(2, \mathbb{C}_\omega)$-invariance, for instance. From now until the end of the proof we are going to pick suitable representative sequences for points in $\mathbb{P}^1(\mathbb{C}_\omega)$ such that

$$\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3).$$

Let $g_\omega \in \mathrm{GL}(2, \mathbb{C}_\omega)$. We want to show that $g_\omega.\mathrm{Vol}^\omega = \mathrm{Vol}^\omega$.

$$\begin{aligned}g_\omega.\mathrm{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) &= \mathrm{Vol}^\omega(g_\omega^{-1}.x_\omega^0, \dots, g_\omega^{-1}.x_\omega^3) \\ &= \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(g_l^{-1}.x_l^0, \dots, g_l^{-1}.x_l^3)\end{aligned}$$

and thanks to the equivariance of the classic volume function we get

$$\omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(g_l^{-1}.x_l^0, \dots, g_l^{-1}.x_l^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3) = \mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3),$$

as required. The strategy to prove the alternating property and the cocycle property of Vol^ω is the same as above and we omit it.

Finally, the boundedness is obvious since the ω-Bloch–Wigner is nothing more than the ω-limit of a sequence of real values all bounded by ν_3 on $\mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$ and it coincides with 0 on the complementary. Here ν_3 is the volume of a regular ideal hyperbolic tetrahedron in $\mathbb{H}^3$. □

3.2. The cocycle B_n^ω. In order to define the ω-Borel invariant for a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n, \mathbb{C}_\omega)$, we first need to define the ω-Borel cocycle. We are going to follow the same construction exposed in [3, Section 3]. Let $\mathfrak{S}_k^\omega(m)$ be the following space

$$\mathfrak{S}_k^\omega(m) := \{(x_\omega^0, \dots, x_\omega^k) \in (\mathbb{C}_\omega^m)^{k+1} | \langle x_\omega^0, \dots x_\omega^k \rangle = \mathbb{C}_\omega^m\} / \mathrm{GL}(m, \mathbb{C}_\omega)$$

where $\mathrm{GL}(m, \mathbb{C}_\omega)$ acts on $(k+1)$-tuples of vectors by the diagonal action and $\langle x_\omega^0, \dots, x_\omega^k \rangle$ is the $\mathbb{C}_\omega$-linear space generated by $x_\omega^0, \dots, x_\omega^k$. It obvious that if $k < m-1$ the space defined above is empty. For every m-dimensional vector space V over $\mathbb{C}_\omega$ and any $(k+1)$-tuple of spanning vectors $(x_\omega^0, \dots, x_\omega^k) \in V^{k+1}$, we choose an isomorphism $V \to \mathbb{C}_\omega^m$. Since any two different choices of isomorphisms are related by an element $g_\omega \in \mathrm{GL}(m, \mathbb{C}_\omega)$, we get a well defined element of $\mathfrak{S}_k^\omega(m)$ which will be denoted by $[V; (x_\omega^0, \dots, x_\omega^k)]$. For

$$\mathfrak{S}_k^\omega := \bigsqcup_{m \geq 0} \mathfrak{S}_k^\omega(m) = \mathfrak{S}_k^\omega(0) \sqcup \dots \sqcup \mathfrak{S}_k^\omega(k+1)$$

we have two different face maps $\varepsilon_i^{(k)}, \eta_i^{(k)}\colon \mathfrak{S}_k^\omega \to \mathfrak{S}_{k-1}^\omega$ given by

$$\varepsilon_i^{(k)}[\mathbb{C}_\omega^m; (x_\omega^0, \dots, x_\omega^k)] := [\langle x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k \rangle; (x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k)],$$

$$\eta_i^{(k)}[\mathbb{C}_\omega^m; (x_\omega^0, \dots, x_\omega^k)] := [\mathbb{C}_\omega^m / \langle x_\omega^i \rangle; (x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k)].$$

Since these maps satisfy the same relations as in [3], that is for all $0 \leq i < j \leq k$

$$\varepsilon_j^{(k-1)} \varepsilon_i^{(k)} = \varepsilon_i^{(k-1)} \varepsilon_{j+1}^{(k)},$$

$$\eta_j^{(k-1)} \eta_i^{(k)} = \eta_i^{(k-1)} \eta_{j+1}^{(k)},$$

$$\eta_j^{(k-1)} \varepsilon_i^{(k)} = \varepsilon_i^{(k-1)} \eta_{j+1}^{(k)},$$

we can define a boundary operator

$$D_k\colon \mathbb{Z}[\mathfrak{S}_k^\omega] \longrightarrow \mathbb{Z}[\mathfrak{S}_{k-1}^\omega], \quad D_k(\sigma) := \sum_{i=0}^{k} (-1)^i (\varepsilon_i^{(k)}(\sigma) - \eta_i^{(k)}(\sigma)),$$

where $\mathbb{Z}[\mathfrak{S}_k^\omega]$ is the free abelian group generated by $\mathfrak{S}_k^\omega$ and it is equal to 0 for $k \leq -1$. We still denote by $\varepsilon_i^{(k)}$ and $\eta_i^{(k)}$ the linear extensions of face maps to $\mathbb{Z}[\mathfrak{S}_k^\omega]$. In this way we have constructed a chain complex $(\mathbb{Z}[\mathfrak{S}_\bullet^\omega], D_\bullet)$. With the purpose of dualizing this complex, we recall that we have a natural action of the symmetric group S_{k+1} on $\mathfrak{S}_k^\omega$, hence we can define

$$\mathbb{R}_{\mathrm{alt}}(\mathfrak{S}_k^\omega) := \{f\colon \mathfrak{S}_k^\omega \longrightarrow \mathbb{R} \mid f \text{ is alternating with respect to the } S_{k+1}\text{-action}\}$$

and we can define D_k^* as the dual of $D_k \otimes id_{\mathbb{R}}$. The construction above produces a cochain complex $(\mathbb{R}_{\mathrm{alt}}(\mathfrak{S}_\bullet^\omega), D_\bullet^*)$.

We are going now to define a cocycle living in $\mathbb{R}_{\mathrm{alt}}(\mathfrak{S}_3^\omega)$ which will be used to construct the ω-Borel cocycle. Since the ω-volume function Vol^ω introduced in the previous section can be thought of as defined on $(\mathbb{C}_\omega^2 \setminus \{0\})^4$, it is extendable to

$$\mathrm{Vol}^\omega\colon \mathfrak{S}_3^\omega \longrightarrow \mathbb{R}$$

where we set $\mathrm{Vol}^\omega|\mathfrak{S}_3^\omega(m)$ to be identically zero if $m \neq 2$ and

$$\mathrm{Vol}^\omega[\mathbb{C}_\omega^2; (v_\omega^0, \dots, v_\omega^3)] := \begin{cases} \mathrm{Vol}^\omega(v_\omega^0, \dots, v_\omega^3) & \text{if each } v_\omega^i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the compatibilty of the ω-limit with respect to finite sums, the following result should be clear.

Proposition 3.6. *The function* $\mathrm{Vol}^\omega \in \mathbb{R}_{\mathrm{alt}}(\mathfrak{S}_3^\omega)$ *is a cocycle, that is* $D_4^*(\mathrm{Vol}^\omega) = 0$.

Since the proof of this proposition is the same as [3, Lemma 8, Lemma 9] we omit it. In order to define the ω-Borel cocyle we are going to introduce the spaces of affine flags in $\mathbb{C}_\omega^n$. A complete flag F_ω in $\mathbb{C}_\omega^n$ is a sequence of linear subspaces

$$F_\omega^0 \subset F_\omega^1 \subset \cdots \subset F_\omega^n$$

such that every F_ω^i has dimension i as $\mathbb{C}_\omega$-vector space. An affine flag (F_ω, v_ω) is a complete flag F_ω together with an n-tuple of vectors $v_\omega = (v_\omega^1, \dots, v_\omega^n) \in (\mathbb{C}_\omega^n)^n$ such that

$$F_\omega^i = \mathbb{C}_\omega v_\omega^i + F_\omega^{i-1}, \quad i \geq 1.$$

It is clear that the group $\mathrm{GL}(n,\mathbb{C}_\omega)$ acts naturally on the space of flags $\mathcal{F}(n,\mathbb{C}_\omega)$ and on the space of affine flags $\mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)$ of $\mathbb{C}_\omega^n$. Let $\mathbb{Z}[\mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1}]$ be the abelian group generated by $\mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1}$ and let ∂_k be the standard boundary map induced by the face maps $\varepsilon_i^{(k)}\colon \mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1} \to \mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^k$ consisting in dropping the i^{th}-component for $1 \leq k \leq n-1$. Moreover set $\partial_0\colon \mathbb{Z}[\mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)] \to 0$. We are ready now to define

$$T_k\colon (\mathbb{Z}[\mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^k], \partial_k) \longrightarrow (\mathbb{Z}[\mathfrak{S}_k^\omega], D_k)$$

which will enable us to construct a morphism between the dual of the complexes above (more precisely on their alternating versions). Given a multi-index $\mathbf{J} \in \{0, 1, \ldots, n-1\}^{k+1}$, we start by defining

$$\tau_{\mathbf{J}} \colon \mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1} \longrightarrow \mathfrak{S}_k^\omega$$

as the function

$$\tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \ldots, (F_{k,\omega}, v_{k,\omega})) := \left[\frac{\langle F_{0,\omega}^{j_0+1}, \ldots, F_{k,\omega}^{j_k+1} \rangle}{\langle F_{0,\omega}^{j_0}, \ldots, F_{k,\omega}^{j_k} \rangle}; (v_{0,\omega}^{j_0+1}, \ldots, v_{k,\omega}^{j_k+1}) \right]$$

and finally

$$\begin{aligned} &T_k((F_{0,\omega}, v_{0,\omega}), \ldots, (F_{k,\omega}, v_{k,\omega})) \\ &\quad := \sum_{\mathbf{J} \in \{0,\ldots,n-1\}^{k+1}} \tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \ldots, (F_{k,\omega}, v_{k,\omega})). \end{aligned}$$

If we now recall that there exists a natural action of S_{k+1} on $\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1}$ and dualize the complex considered so far, we get the cocomplex of alternating cochains $(\mathbb{R}_{\mathrm{alt}}(\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1}), \partial_k^*)$ (here ∂_k^* is the dual of $\partial_k \otimes id_{\mathbb{R}}$). By denoting T_k^* the dual map of $T_k \otimes id_{\mathbb{R}}$, the same proof of [3, Lemma 11] guarantees that T_k^* is a morphism a complexes taking values in $(\mathbb{R}_{\mathrm{alt}}(\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1}))^{\mathrm{GL}(n,\mathbb{C}_\omega)}$.

Definition 3.7. We define the *ω-Borel function* of degree n as

$$\begin{aligned} &B_n^\omega((F_{0,\omega}, v_{0,\omega}), \ldots, (F_{3,\omega}, v_{3,\omega})) \\ &\quad := T_3^*(\mathrm{Vol}^\omega \\ &\quad = \sum_{\mathbf{J} \in \{0,\ldots,n-1\}^4} \mathrm{Vol}^\omega \left[\frac{\langle F_{0,\omega}^{j_0+1}, \ldots, F_{3,\omega}^{j_3+1} \rangle}{\langle F_{0,\omega}^{j_0}, \ldots, F_{3,\omega}^{j_3} \rangle}; (v_{0,\omega}^{j_0+1}, \ldots, v_{3,\omega}^{j_3+1}) \right]. \end{aligned}$$

Using the same approach of [3] it is straghtfoward to prove that

Proposition 3.8. *The function B_n^ω is a bounded, alternating, strict* $\mathrm{GL}(n, \mathbb{C}_\omega)$*-invariant cocycle on the space $\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^4$ of* 4*-tuples of affine flags which naturally descends to the space $\mathscr{F}(n, \mathbb{C}_\omega)^4$ of* 4*-tuples of flags. Moreover, for every* 4*-tuple of flags $(F_{0,\omega}, \ldots, F_{3,\omega}) \in \mathscr{F}(n, \mathbb{C}_\omega)^4$ we have the following bound*

$$|B_n^\omega(F_{0,\omega}, \ldots, F_{3,\omega})| \leq \frac{n(n^2-1)}{6} \nu_3.$$

We want now to use [5, Proposition 2.1] in order to obtain the desired cohomology class. Before doing this we need to underline a delicate point in the discussion.

By Proposition 2.6 the field $\mathbb{C}_\omega$ is not locally compact with respect to the topology induced by the ultrametric absolute value. In particular the group $\mathrm{SL}(n, \mathbb{C}_\omega)$ cannot be locally compact with respect to the topology inherited by $M(n, \mathbb{C}_\omega)$ seen as $\mathbb{C}_\omega^{n^2}$. Hence it is meaningless to refer to the Haar measure or to the Haar σ-algebra for $\mathrm{SL}(n, \mathbb{C}_\omega)$. In order to overcome these difficulties, we are going to consider $\mathrm{SL}^\delta(n, \mathbb{C}_\omega)$, that is the group $\mathrm{SL}(n, \mathbb{C}_\omega)$ endowed with the discrete topology. The same for $\mathrm{GL}^\delta(n, \mathbb{C}_\omega)$. Moreover, in order to apply correctly [5, Proposition 2.1], we are going to consider the discrete σ-algebra on both $\mathfrak{S}_k^\omega$ and $\mathscr{F}(n, \mathbb{C}_\omega)$.

Recall that $\mathfrak{S}_k^\omega(n)$ is a space on which the symmetric group S_{k+1} acts naturally. Let $\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_k^\omega)$ be the Banach space of bounded alternating Borel functions on $\mathfrak{S}_k^\omega$. The restriction of D_k^* gives us back a complex of Banach spaces $(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_\bullet^\omega), D_\bullet^*)$.

By restricting the map T_k^* to the subcomplexes of bounded Borel functions and by applying [5, Proposition 2.1] to $(\mathcal{B}^\infty_{\mathrm{alt}}(\mathscr{F}(n, \mathbb{C}_\omega)^{\bullet+1}), \partial_\bullet)$, we get a map

$$S_\omega^k(n) \colon H^k(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_\bullet^\omega)) \longrightarrow H_b^k(\mathrm{GL}^\delta(n, \mathbb{C}_\omega)).$$

Definition 3.9. With the notation above, we define the *ω-Borel cohomology class of degree n* as

$$\beta^\omega(n) := S_\omega^3(n)(\mathrm{Vol}^\omega) = \mathfrak{c}^3[B_n^\omega],$$

where $\mathfrak{c}^3 \colon H^3(\mathcal{B}^\infty_{\mathrm{alt}}(\mathscr{F}(n, \mathbb{C}_\omega)^{\bullet+1})^{\mathrm{GL}(n,\mathbb{C}_\omega)}) \to H_b^3(\mathrm{GL}^\delta(n, \mathbb{C}_\omega))$ is the canonical map of [5, Proposition 2.1].

Remark 3.10. We have the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}_\omega^\times & \longrightarrow & \mathrm{GL}(n, \mathbb{C}_\omega) & \longrightarrow & \mathrm{PGL}(n, \mathbb{C}_\omega) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \downarrow \cong & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & \mathrm{SL}(n, \mathbb{C}_\omega) & \longrightarrow & \mathrm{PSL}(n, \mathbb{C}_\omega) & \longrightarrow & 1
\end{array}$$

where $\mathbb{C}_\omega^\times$ is the group of invertible elements of $\mathbb{C}_\omega$ and μ_n is the group of the n-th roots of unity. Since these groups are both amenable, by functoriality of bounded cohomology it is possible to conclude that $H_b^3(\mathrm{GL}^\delta(n, \mathbb{C}_\omega)) \cong H_b^3(\mathrm{SL}^\delta(n, \mathbb{C}_\omega))$. In particular, we are going to think of the class $\beta^\omega(n)$ as an element of both $H_b^3(\mathrm{GL}^\delta(n, \mathbb{C}_\omega))$ and $H_b^3(\mathrm{SL}^\delta(n, \mathbb{C}_\omega))$.

4. The ω-Borel invariant for a representation ρ_ω

Let Γ be the fundamental group of a complete hyperbolic 3-manifold M with toric cusps. This means that we can decompose the manifold M as $M = N \cup \bigcup_{i=1}^h C_i$, where N is any compact core of M and for every $i = 1, \ldots, h$ the component C_i is a cuspidal neighborhood diffeomorphic to $T_i \times (0, \infty)$, where T_i

is a torus whose fundamental group corresponds to a suitable abelian parabolic subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Our aim is to define a numerical invariant associated to any representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n,\mathbb{C}_\omega)$. Let $i\colon (M,\varnothing) \to (M, M\setminus N)$ be the natural inclusion map. Since the fundamental group of the boundary ∂N is abelian, hence amenable, it can be proved that the maps $i_b^*\colon H_b^k(M, M\setminus N) \to H_b^k(M)$ induced at the level of bounded cohomology groups are isometric isomorphisms for $k \geq 2$ (see [2]). Moreover, it holds $H_b^k(M, M\setminus N) \cong H_b^k(N,\partial N)$ by homotopy invariance of bounded cohomology. If we denote by c the canonical comparison map $c\colon H_b^k(N,\partial N) \to H^k(N,\partial N)$, we can consider the composition

$$H_b^3(\mathrm{SL}^\delta(n,\mathbb{C}_\omega)) \xrightarrow{(\rho_\omega)_b^*} H_b^3(\Gamma) \cong H_b^3(M) \xrightarrow{(i_b^*)^{-1}} H_b^3(N,\partial N) \xrightarrow{c} H^3(N,\partial N),$$

where the isomorphism that appears in this composition holds since M is aspherical. By choosing a fundamental class $[N,\partial N]$ for $H_3(N,\partial N)$ we are ready to give the following

Definition 4.1. The *ω-Borel invariant* associated to a representation

$$\rho_\omega\colon \Gamma \longrightarrow \mathrm{SL}(n,\mathbb{C}_\omega)$$

is given by

$$\beta_n^\omega(\rho_\omega) := \langle (c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)\beta^\omega(n), [N,\partial N]\rangle,$$

where the brackets $\langle\cdot,\cdot\rangle$ indicate the Kronecker pairing.

Remark 4.2. The previous definition is indipendent of the choice of the compact core N. Moreover, it can be easily extended to any lattice of $\mathrm{PSL}(2,\mathbb{C})$.

We are going to generalize some of the classic results valid for the standard Borel invariant. The proofs are identical to the ones exposed in [3]. Before starting, we recall the existence of natural transfer maps

$$H_b^\bullet(\Gamma) \xrightarrow{\mathrm{trans}_\Gamma} H_{cb}^\bullet(\mathrm{PSL}(2,\mathbb{C})), \quad H^\bullet(N,\partial N) \xrightarrow{\tau_{\mathrm{DR}}} H_c^\bullet(\mathrm{PSL}(2,\mathbb{C})),$$

where $H_c^\bullet(\mathrm{PSL}(2,\mathbb{C}))$ denotes the continuous cohomology groups of $\mathrm{PSL}(2,\mathbb{C})$. We remind the reader that the continuous cohomology groups of a locally compact group G are constructed as the continuous bounded cohomology groups just by dropping the requirement of boundedness of cochains.

The transfer maps are defined as it follows. Let V_k be the set $C_b((\mathbb{H}^3)^{k+1},\mathbb{R})$ of real bounded continuous functions on $(k+1)$-tuples of points of $\mathbb{H}^3$. With the standard homogeneous boundary operators and the structure of Banach $\mathrm{PSL}(2,\mathbb{C})$-module given by

$$(g.f)(x^0,\dots,x^n) := f(g^{-1}x^0,\dots,g^{-1}x^n),$$

$$\|f\|_\infty = \sup_{x^0,\dots,x^n\in\mathbb{H}^3} |f(x^0,\dots,x^n)|,$$

for every $f \in C_b((\mathbb{H}^3)^{n+1}, \mathbb{R})$ and $g \in \mathrm{PSL}(2,\mathbb{C})$, we get a complex $V_\bullet = C_b((\mathbb{H}^3)^{\bullet+1}, \mathbb{R})$ of Banach $\mathrm{PSL}(2,\mathbb{C})$-modules that allows us to compute the continuous bounded cohomology of $\mathrm{PSL}(2,\mathbb{C})$. More precisely, it holds

$$H^k(V_\bullet^{\mathrm{PSL}(2,\mathbb{C})}) \cong H^k_{cb}(\mathrm{PSL}(2,\mathbb{C}))$$

for every $k \geq 0$. Moreover, by substituting $\mathrm{PSL}(2,\mathbb{C})$ with Γ, we have in an analogous way that

$$H^k(V_\bullet^\Gamma) \cong H^k_b(\Gamma)$$

for every $k \geq 0$. The previous considerations allow us to define the map

$$\mathrm{trans}_\Gamma \colon V_k^\Gamma \to V_k^{\mathrm{PSL}(2,\mathbb{C})},$$
$$\mathrm{trans}_\Gamma(c)(x_0, \ldots, x_n) := \int_{\Gamma \backslash \mathrm{PSL}(2,\mathbb{C})} c(\bar{g}x_0, \ldots, \bar{g}x_n) d\mu(\bar{g}),$$

where c is any Γ-invariant element of V_k and μ is any invariant probability measure on $\Gamma \backslash \mathrm{PSL}(2,\mathbb{C})$. Here $\bar{g}$ stands for the equivalence class of g into $\Gamma \backslash \mathrm{PSL}(2,\mathbb{C})$.

$\mathrm{trans}_\Gamma(c)$ is $\mathrm{PSL}(2,\mathbb{C})$-equivariant and trans_Γ commutes with the coboundary operator. Therefore we get a well-defined map

$$\mathrm{trans}_\Gamma \colon H^\bullet_b(\Gamma) \longrightarrow H^\bullet_{cb}(\mathrm{PSL}(2,\mathbb{C})).$$

We now pass to the description of the map τ_{DR}. If $\pi \colon \mathbb{H}^3 \to M = \Gamma \backslash \mathbb{H}^3$ is the natural covering projection, we set $U := \pi^{-1}(M \setminus N)$. Recall that the relative cohomology group $H^k(N, \partial N)$ is isomorphic to the cohomology group $H^k(\Omega^\bullet(\mathbb{H}^3, U)^\Gamma)$ of the Γ-invariant differential forms on $\mathbb{H}^3$ which vanishes on U. Since, by Van Est isomorphism we have that $H^k_c(\mathrm{PSL}(2,\mathbb{C}), \mathbb{R}) \cong \Omega^k(\mathbb{H}^3)^{\mathrm{PSL}(2,\mathbb{C})}$, we define

$$\tau_{\mathrm{DR}} \colon \Omega^k(\mathbb{H}^3, U)^\Gamma \longrightarrow \Omega^k(\mathbb{H}^3)^{\mathrm{PSL}(2,\mathbb{C})}, \quad \tau_{\mathrm{DR}}(\alpha) := \int_{\Gamma \backslash \mathrm{PSL}(2,\mathbb{C})} \bar{g}^* \alpha \, d\mu(\bar{g}),$$

where μ and $\bar{g}$ are the same as before. The map τ_{DR} commutes with the coboundary operators inducing a map

$$\begin{aligned}\tau_{\mathrm{DR}} \colon H^k(N, \partial N) &\cong H^k(\Omega^\bullet(\mathbb{H}^3, U)^\Gamma) \\ &\longrightarrow H^k(\Omega^\bullet(\mathbb{H}^3)^{\mathrm{PSL}(2,\mathbb{C})}) \cong H^k_c(\mathrm{PSL}(2,\mathbb{C})).\end{aligned}$$

For a more detailed description of the above maps we suggest to the reader to check [4, Section 3.2].

Proposition 4.3. *For $k \geq 2$ the diagram*

$$\begin{array}{ccc} H^k(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}^\omega_\bullet)) & \xrightarrow{S^k_\omega(n+1)} & H^k_b(\mathrm{GL}^\delta(n+1,\mathbb{C}_\omega)) \\ & \searrow^{S^k_\omega(n)} & \downarrow \\ & & H^k_b(\mathrm{GL}^\delta(n,\mathbb{C}_\omega)) \end{array}$$

commutes. The vertical arrow is induced by the left corner injection

$$\mathrm{GL}(n,\mathbb{C}_\omega) \longrightarrow \mathrm{GL}(n+1,\mathbb{C}_\omega).$$

In particular we have that $\beta^\omega(n+1)$ restricts to $\beta^\omega(n)$.

Proof. Let $i_n\colon \mathbb{C}^n_\omega \to \mathbb{C}^{n+1}_\omega$ be the injection $i_n(x^1_\omega,\ldots,x^n_\omega) := (x^1_\omega,\ldots,x^n_\omega,0)$. By an abuse of notation we define

$$i_n\colon \mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega) \to \mathcal{F}_{\mathrm{aff}}(n+1,\mathbb{C}_\omega)$$

as $i_n((F_\omega,v_\omega)) = (\widetilde{F}_\omega,\tilde{v}_\omega)$ where for $0 \leq j \leq n$ we have $\widetilde{F}^j_\omega = i_n(F^j_\omega)$, $\tilde{v}^j_\omega = i_n(v^j_\omega)$ and $\tilde{v}^{n+1}_\omega = e_{n+1}$. If we set $\mathbf{J} \in \{0,\ldots,n\}^{k+1}$ and $I = \{i : 0 \leq i \leq k$ such that $j_i = n\}$, it is easy to verify that if $I = \varnothing$ this implies $\mathbf{J} \in \{0,\ldots,n-1\}^{k+1}$ and

$$\tau_{\mathbf{J}}(i_n(F_{0,\omega},v_{0,\omega}),\ldots,i_n(F_{k,\omega},v_{k,\omega})) = \tau_{\mathbf{J}}((F_{0,\omega},v_{0,\omega}),\ldots,(F_{k,\omega},v_{k,\omega}))$$

while if $I \neq \varnothing$, then

$$\tau_{\mathbf{J}}(i_n(F_{0,\omega},v_{0,\omega}),\ldots,i_n(F_{k,\omega},v_{k,\omega})) = [\mathbb{C}_\omega; (\delta^I_0,\ldots,\delta^I_k)],$$

where $\delta^I_i = [e_{n+1}]$ if $i \in I$ and 0 otherwise. The previous considerations imply that i_n induces a commutative diagram of complexes

$$\begin{array}{ccc} \mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}^\omega_k) & \xrightarrow{T^*_k} & \mathcal{B}^\infty_{\mathrm{alt}}(\mathcal{F}_{\mathrm{aff}}(n+1,\mathbb{C}_\omega)^{k+1}) \\ & \searrow^{T^*_k} & \downarrow i^*_n \\ & & \mathcal{B}^\infty_{\mathrm{alt}}(\mathcal{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1}) \end{array}$$

and since the map i^*_n implements the restriction in bounded cohomology, the commutativity of the diagram which appears in the statement follows. In particular, by focusing our attention on the case of $k = 3$ we get

$$i^*_n(B^\omega_{n+1}) = i^*_n \circ T^*_3(\mathrm{Vol}^\omega) = T^*_3(\mathrm{Vol}^\omega) = B^\omega_n$$

as claimed. □

Proposition 4.4. *For any representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(n, \mathbb{C}_\omega)$ the composition*

$$H_b^3(\mathrm{SL}^\delta(n, \mathbb{C}_\omega)) \longrightarrow H_b^3(\Gamma) \xrightarrow{\mathrm{trans}_\Gamma} H_{cb}^3(\mathrm{PSL}(2, \mathbb{C}))$$

maps $\beta^\omega(n)$ to $\frac{\beta_n^\omega(\rho_\omega)}{\mathrm{Vol}(M)}\beta(2)$. In particular, it holds the following bound

$$|\beta_n^\omega(\rho_\omega)| \leq \frac{n(n^2-1)}{6}\mathrm{Vol}(M),$$

as in the classic case.

Proof. Recall that we have the following commutative diagram

$$\begin{array}{ccc}
H_b^3(\mathrm{SL}^\delta(n, \mathbb{C}_\omega)) & & \\
\Big\downarrow{\scriptstyle (\rho_\omega)_b^*} & & \\
H_b^3(\Gamma) & \xrightarrow{\mathrm{trans}_\Gamma} & \\
\Big\downarrow{\scriptstyle \cong} & & \\
H_b^3(N, \partial N) & & H_{cb}^3(\mathrm{PSL}(2, \mathbb{C})) \\
\Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle c} \\
H^3(N, \partial N) & \xrightarrow{\tau_{\mathrm{DR}}} & H_c^3(\mathrm{PSL}(2, \mathbb{C})).
\end{array}$$

Since $H_{cb}^3(\mathrm{PSL}(2, \mathbb{C})) \cong \mathbb{R}$, there exists a suitable $\lambda \in \mathbb{R}$ such that

$$\mathrm{trans}_\Gamma \circ (\rho_\omega)_b^*(\beta^\omega(n)) = \lambda\beta(2).$$

Hence by composing both sides with the comparison map c, we obtain

$$c \circ \mathrm{trans}_\Gamma \circ (\rho_\omega)_b^*(\beta^\omega(n)) = c(\lambda\beta(2)) = \lambda(c\beta(2)) = \lambda\beta(2).$$

If we pick up $\omega_{N,\partial N} \in H^3(N, \partial N)$ in such a way that its evaluation on the fundamental class $[N, \partial N]$ gives us back $\mathrm{Vol}(M)$, we have that $\tau_{\mathrm{DR}}(\omega_{N,\partial N}) = \beta(2)$. In particular

$$\tau_{\mathrm{DR}}(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*(\beta^\omega(n))) = \lambda\tau_{\mathrm{DR}}(\omega_{N,\partial N})$$

and by injectivity of the map τ_{DR} in top degree we get

$$(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)(\beta^\omega(n)) = \lambda\omega_{N,\partial N}.$$

If we evaluate both sides on the fundamental class, we obtain

$$\begin{aligned}
\beta_n^\omega(\rho_\omega) &= \langle (c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)(\beta^\omega(n)), [N, \partial N]\rangle \\
&= \langle \lambda\omega_{N,\partial N}, [N, \partial N]\rangle \\
&= \lambda\mathrm{Vol}(M).
\end{aligned}$$

At the same time it holds

$$|\lambda| = \frac{\|\mathrm{trans}_\Gamma \circ (\rho_\omega)^*_b \beta^\omega(n)\|}{\|\beta(2)\|} \leq \frac{n(n^2-1)}{6},$$

from which it follows

$$|\beta^\omega_n(\rho_\omega)| \leq \frac{n(n^2-1)}{6}\mathrm{Vol}(M),$$

as claimed. □

Recall that there is a natural inclusion of fields of $\mathbb{C}$ into $\mathbb{C}_\omega$ given by constant sequences. In particular we have natural embeddings of $\mathbb{C}^m$ into $\mathbb{C}^m_\omega$ and of $\mathrm{SL}(n,\mathbb{C})$ into $\mathrm{SL}(n,\mathbb{C}_\omega)$. Since every representation $\rho\colon\Gamma \to \mathrm{SL}(n,\mathbb{C})$ determines a representation $\hat{\rho}$ into $\mathrm{SL}(n,\mathbb{C}_\omega)$ by composing it with the previous embedding, it is quite natural to ask which is the relation between $\beta^\omega_n(\hat{\rho})$ and $\beta_n(\rho)$. We have the following

Proposition 4.5. *Let* $\rho\colon\Gamma \to \mathrm{SL}(n,\mathbb{C})$ *be a representation. If we denote by* $\hat{\rho}\colon\Gamma \to \mathrm{SL}(n,\mathbb{C}_\omega)$ *the representation obtained by composing* ρ *with the natural embedding of* $\mathrm{SL}(n,\mathbb{C})$ *into* $\mathrm{SL}(n,\mathbb{C}_\omega)$*, we have*

$$\beta^\omega_n(\hat{\rho}) = \beta_n(\rho).$$

Proof. We are going to prove that the cohomology class $\beta^\omega(n)$ restricts naturally to the class $\beta(n)$. Let $j\colon\mathrm{SL}(n,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C}_\omega)$ be the natural embedding. By endowing both spaces with the discrete topology, we have a continuous morphism of groups that induces a map

$$j^*_b\colon H^3_b(\mathrm{SL}^\delta(n,\mathbb{C}_\omega)) \longrightarrow H^3_b(\mathrm{SL}^\delta(n,\mathbb{C})).$$

We want to prove that $j^*_b(\beta^\omega(n)) = \beta(n)$. From this it will follow

$$\begin{aligned}
\beta^\omega_n(\hat{\rho}) &= \langle (c \circ (i^*_b)^{-1} \circ \hat{\rho}^*_b)\beta^\omega(n), [N,\partial N]\rangle \\
&= \langle (c \circ (i^*_b)^{-1} \circ (j \circ \rho)^*_b)\beta^\omega(n), [N,\partial N]\rangle \\
&= \langle (c \circ (i^*_b)^{-1} \circ \rho^*_b \circ j^*_b)\beta^\omega(n), [N,\partial N]\rangle \\
&= \langle (c \circ (i^*_b)^{-1} \circ \rho^*_b)\beta(n), [N,\partial N]\rangle \\
&= \beta_n(\rho).
\end{aligned}$$

Similarly to what we have done for the field $\mathbb{C}_\omega$, we define the configuration space

$$\mathfrak{S}_k(m) := \{(x^0,\dots,x^k) \in (\mathbb{C}^m)^{k+1} \mid \langle x^0,\dots,x^k\rangle = \mathbb{C}^m\}/\,\mathrm{GL}(m,\mathbb{C}).$$

for every $k \geq m-1$. This family of spaces is exactly the family introduced by [3]. There exists a natural family of maps given by

$$\hat{j}_k(m)\colon \mathfrak{S}_k(m) \to \mathfrak{S}_k^\omega(m), \quad \hat{j}_k(m)[\mathbb{C}^m; (v^0, \ldots, v^k)] := [\mathbb{C}_\omega^m; (v^0, \ldots, v^k)],$$

where each vector v^i which appears on the right-hand side of the equation is thought of as an element of $\mathbb{C}_\omega^m$. This function is well-defined because $v^0, \ldots, v^k$ are generators also for $\mathbb{C}_\omega^m$ as a $\mathbb{C}_\omega$-vector space and the identifications induced via conjugation by $\mathrm{GL}(m, \mathbb{C})$ are respected. By denoting

$$\hat{j}_k := \hat{j}_k(0) \sqcup \hat{j}_k(1) \sqcup \cdots \sqcup \hat{j}_k(k+1),$$

we get the following commutative diagram

$$\begin{array}{ccc} H^3(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}^\omega_\bullet)) & \xrightarrow{S^3_\omega(n)} & H^3_b(\mathrm{SL}^\delta(n, \mathbb{C}_\omega)) \\ {\scriptstyle H^3(\hat{j}^*_\bullet)}\downarrow & & \downarrow{\scriptstyle j^*_b} \\ H^3(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_\bullet)) & \xrightarrow{S^3(n)} & H^3_b(\mathrm{SL}^\delta(n, \mathbb{C})), \end{array}$$

where $\hat{j}^*_\bullet$ are the maps induced by $\hat{j}_\bullet$ on the Borel cochains. We will prove that $\mathrm{Vol} = \mathrm{Vol}^\omega \circ \hat{j}_3$, that is $H^3(\hat{j}^*_\bullet)[\mathrm{Vol}^\omega] = [\mathrm{Vol}]$. Let $m \in \{0, \ldots, 4\}$. It is clear that $\mathrm{Vol} = \mathrm{Vol}^\omega \circ \hat{j}_3(m)$ for $m \neq 2$ because both sides are equal to zero. Let now consider $[\mathbb{C}^2; (v^0, \ldots, v^3)] \in \mathfrak{S}_3(2)$. If any of these vectors is 0 both functions evaluated on the 4-tuple give us back 0. Hence, we can suppose that each v^i is different from 0. If the vectors $v^0, \ldots, v^3$ are in general position into $\mathbb{C}^2$, they still remain in general position into $\mathbb{C}^2_\omega$. Thus

$$\begin{aligned} \mathrm{Vol}^\omega \circ \hat{j}_3(2)[\mathbb{C}^2; (v^0, \ldots, v^3)] &= \mathrm{Vol}^\omega[\mathbb{C}^2_\omega; (v^0, \ldots, v^3)] \\ &= \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(v^0, \ldots, v^3) \\ &= \mathrm{Vol}(v^0, \ldots, v^3) \\ &= \mathrm{Vol}[\mathbb{C}^2; (v^0, \ldots, v^3)]. \end{aligned}$$

In the same way if $(v^0, \ldots, v^3)$ are not in general position into $\mathbb{C}^2$, they will not be in general position into $\mathbb{C}^2_\omega$ either, so both $\mathrm{Vol}^\omega \circ \hat{j}_3(2)$ and Vol will evaluate to be zero, as desired. □

We want now to express $\beta^\omega_n(\rho_\omega)$ in terms of boundary maps. Recall that the complement of N is M is given by a finite union $\bigcup_{i=1}^h C_i$ of cuspidal neighborhoods. For every $i = 1, \ldots, h$ the fundamental group $\pi_1(C_i) = H_i$ is an abelian parabolic subgroup of $\mathrm{PSL}(2, \mathbb{C})$, hence it has a unique fixed point ξ_i in $\mathbb{P}^1(\mathbb{C})$. We define the set

$$\mathcal{C}(\Gamma) := \bigcup_{i=1}^h \Gamma.\xi_i.$$

Definition 4.6. If $\Gamma = \pi_1(M)$ as above, given a representation

$$\rho_\omega \colon \Gamma \longrightarrow \mathrm{SL}(n, \mathbb{C}_\omega),$$

a *decoration* for ρ_ω is a map

$$\varphi_\omega \colon \mathcal{C}(\Gamma) \longrightarrow \mathcal{F}(n, \mathbb{C}_\omega)$$

that is equivariant with respect to ρ_ω.

Recall now that the cocycle B_n^ω is a strict cocycle, as in the standard case. Hence the class $(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)\beta^\omega(n)$ can be represented in $H_b^3(\Gamma)$ by $\varphi_\omega^*(B_n^\omega)$, where φ_ω is a decoration for ρ_ω (we refer to [5, Corollary 2.7] for this result about the pullback of strict cocycles along boundary maps). In order to realize the corresponding cocycle in $H_b^3(N, \partial N)$, we identify the universal cover $\widetilde{N}$ of N with $\mathbb{H}^3$ minus a set of Γ-equivariant horoballs, each one centered at an element $\xi \in \mathcal{C}(\Gamma)$. We define a map $p \colon \widetilde{N} \to \mathcal{C}(\Gamma)$ in two steps. We first send each horospherical section to the corresponding element. Then, for the interior of $\widetilde{N}$, we map a fundamental domain to a choosen $\xi_0 \in \mathcal{C}(\Gamma)$ and we extend equivariantly. In this way, any bounded Γ-invariant cocycle $c \colon \mathcal{C}(\Gamma) \to \mathbb{R}$ determines a relative cocycle on $(N, \partial N)$ as it follows

$$\{\sigma \colon \Delta^3 \to \widetilde{N}\} \longmapsto c(p(\sigma(e_0)), \ldots, p(\sigma(e_3))).$$

If τ is a relative triangulation of $(N, \partial N)$ and $\tilde{\tau}$ is the lifted triangulation of a fundamental domain in $(\widetilde{N}, \partial\widetilde{N})$, the ω-Borel invariant $\beta_n^\omega(\rho_\omega)$ can be computed by the following formula

$$\beta_n^\omega(\rho_\omega) = \sum_{\tilde{\sigma} \in \tilde{\tau}} B_n^\omega(\varphi_\omega(p(\tilde{\sigma}(e_0))), \varphi_\omega(p(\tilde{\sigma}(e_1))), \varphi_\omega(p(\tilde{\sigma}(e_2))), \varphi_\omega(p(\tilde{\sigma}(e_3))))$$

where $\tilde{\sigma}$ is a lifted copy of the simplex $\sigma \in \tau$.

5. The case $n = 2$ and properties of the invariant $\beta_2^\omega(\rho_\omega)$

In this section we are going to focus our attention on the case of representations into $\mathrm{SL}(2, \mathbb{C}_\omega)$. Suppose to have a sequence of representations $\rho_l \colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ that determines a representation $\rho_\omega \colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$. A sequence of decorations φ_l for ρ_l produces in a natural way a decoration φ_ω. Indeed it suffices to compose the standard projection $\pi \colon \mathbb{P}^1(\mathbb{C})^{\mathbb{N}} \to \mathbb{P}^1(\mathbb{C})_\omega \cong \mathbb{P}^1(\mathbb{C}_\omega)$ with the product map $\prod \varphi_l \colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$. We say that a decoration is *non-degenerate* if for every $\xi_0, \ldots, \xi_3 \in \mathcal{C}(\Gamma)$ we have that the 4-tuple $(\varphi_\omega(\xi_0), \ldots, \varphi_\omega(\xi_3))$ contains at least

3 distinct points. If the decoration φ_ω is non-degenerate we have

$$
\begin{aligned}
&\beta_2^\omega(\rho_\omega) \\
&\quad = \sum_{\tilde{\sigma} \in \tilde{\tau}} B_2^\omega(\varphi_\omega(p(\tilde{\sigma}(e_0))), \varphi_\omega(p(\tilde{\sigma}(e_1))), \varphi_\omega(p(\tilde{\sigma}(e_2))), \varphi_\omega(p(\tilde{\sigma}(e_3)))) \\
&\quad = \omega\text{-}\lim_{l \to \infty} \sum_{\tilde{\sigma} \in \tilde{\tau}} B_2(\varphi_l(p(\tilde{\sigma}(e_0))), \varphi_l(p(\tilde{\sigma}(e_1))), \varphi_l(p(\tilde{\sigma}(e_2))), \varphi_l(p(\tilde{\sigma}(e_3)))) \\
&\quad = \omega\text{-}\lim_{l \to \infty} \beta_2(\rho_l),
\end{aligned}
$$

where the last equality is obtained by applying Corollary 2.7 of [5]. The third equality exploits the non-degenerancy of the decoration φ_ω. Hence we get

Proposition 5.1. *Let $\rho_l \colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ be a sequence of representations with decorations φ_l. Let $\rho_\omega \colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ be the representation associated to the sequence ρ_l. If the decoration φ_ω produced by the sequence φ_l is non-degenerate, we have*

$$\beta_2^\omega(\rho_\omega) = \omega\text{-}\lim_{l \to \infty} \beta_2(\rho_l).$$

Corollary 5.2. *Let $\rho_l \colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ be a sequence of representations with decorations φ_l. Let $\rho_\omega \colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ be the representation associated to the sequence ρ_l. Suppose $\beta_2^\omega(\rho_\omega) = \mathrm{Vol}(M)$. If the decoration φ_ω produced by the sequence φ_l is non-degenerate, there must exist a sequence $g_l \in \mathrm{SL}(2, \mathbb{C})$ and a representation $\rho_\infty \colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ such that*

$$\omega\text{-}\lim_{l \to \infty} g_l \rho_l(\gamma) g_l^{-1} = \rho_\infty(\gamma).$$

Proof. Thanks to the assumption of non-degenerancy, by applying Proposition 5.1 we desume that $\omega\text{-}\lim_{l \to \infty} \beta_2(\rho_l) = \mathrm{Vol}(M)$. The statement now follows directly by [12, Theorem 1.1]. □

Remark 5.3. The representation ρ_∞ which appears in the previous corollary as limit of the sequence ρ_l has to be a lift of the standard lattice embedding $i \colon \Gamma \to \mathrm{PSL}(2, \mathbb{C})$.

Assume that a sequence of representations $\rho_l \colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ diverges to a ideal point of the character variety $X(\Gamma, \mathrm{SL}(2, \mathbb{C}))$ and let $\rho_\omega \colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ be the representation associated to the sequence. Recall that the identification between $\mathrm{SL}(2, \mathbb{C}_\omega)$ and $\mathrm{SL}(2, \mathbb{C})_\omega$ implies that the representation ρ_ω produces in a natural way an isometric action of Γ on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$. We are going to restrict our attention to reducible actions with non-trivial length function. We first recall the following

Definition 5.4. Let $\mathcal{T}$ be a real tree on which Γ acts via isometries. We say that the action is *reducible* if one of the following holds:

- the action of Γ admits a global fixed point;
- there exists an end $\varepsilon \in \partial_\infty \mathcal{T}$ fixed by Γ;
- there exists a Γ-invariant line $L \subset \mathcal{T}$.

Proposition 5.5. *Let $\rho_l : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ be a sequence of representations and suppose it determines a representation $\rho_\omega : \Gamma \to \mathrm{SL}(2,\mathbb{C}_\omega)$ such that the isometric action induced by ρ_ω on $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ has non-trivial length function. If the action is reducible then $\beta_2^\omega(\rho_\omega) = 0$.*

Proof. Since the length function associated to the action induced by ρ_ω is non-trivial then the action does not admit a global fixed point. Moreover, since the action is reducible, it must admit either a fixed end or an invariant line. Suppose that there exists an end fixed by Γ. By [18, Proposition 3.20] the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ is naturally identified with the Bass–Serre tree $\Delta^{\mathrm{BS}}(\mathrm{SL}(2,\mathbb{C}_\omega))$ associated to $\mathrm{SL}(2,\mathbb{C}_\omega)$. Hence, there must exist an end of $\Delta^{\mathrm{BS}}(\mathrm{SL}(2,\mathbb{C}_\omega))$ fixed by the representation ρ_ω. Thus the image $\rho_\omega(\Gamma)$ is a subgroup of a suitable Borel subgroup N_ω of $\mathrm{SL}(2,\mathbb{C}_\omega)$ and hence it is solvable, so amenable by [23, Corollary 4.1.7]. This implies that the map $(\rho_\omega)_b^* = 0$ from which we conclude $\beta_2^\omega(\rho_\omega) = 0$.

Suppose now that the action of Γ admits an invariant line. This time the image $\rho_\omega(\Gamma)$ is isomorphic to a subgroup of $\mathrm{Isom}(\mathbb{R})$. Being $\mathrm{Isom}(\mathbb{R})$ the semidirect group of the two amenable groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{R}$, it is amenable by [23, Proposition 4.1.6]. As before, $(\rho_\omega)_b^* = 0$, hence $\beta_2^\omega(\rho_\omega) = 0$. □

Remark 5.6. Another way to prove Proposition 5.5 is by using decorations. Indeed, if the action determined by ρ_ω admits a fixed end $\varepsilon_\omega \in \partial_\infty \Delta^{\mathrm{BS}}(\mathrm{SL}(2,\mathbb{C}_\omega))$ and since the boundary at infinity can be identified with $\mathbb{P}^1(\mathbb{C}_\omega)$, then the map $\varphi_\omega(\xi) = \varepsilon_\omega$ for $\xi \in \mathcal{C}(\Gamma)$ is a decoration and trivially it results $\beta_2^\omega(\rho_\omega) = 0$.

In the same way if the action admits an invariant line L_ω, we denote by ε_ω^1 and ε_ω^2 the ends of the line L_ω. For every $\xi \in \mathcal{C}(\Gamma)$ we can choose either ε_ω^1 or ε_ω^2 as the image of ξ for the decoration φ_ω. This implies that every possible choice produces a decoration for ρ_ω such that it results $\beta_2^\omega(\rho_\omega) = 0$.

Let $S = \{\gamma_1, \ldots, \gamma_s\}$ be a generating set for the group Γ. Recall that if a sequence of representations $\rho_l : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ diverges in the character variety $X(\Gamma, \mathrm{SL}(2,\mathbb{C}))$ to an ideal point of the Morgan–Shalen compactification, then the real sequence

$$\lambda_l := \inf_{x \in \mathbb{H}^3} \sqrt{\sum_{i=1}^{s} d(\rho_l(\gamma_i)x, x)}$$

is positive and divergent. As written in [18, Theorem 5.2], for any non-principal ultrafilter ω on $\mathbb{N}$, by fixing $(\lambda_l)_{l\in N}$ as scaling sequence, we can construct in a natural way a representation $\rho_\omega\colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ via the representations ρ_l.

Corollary 5.7. *Let $\rho_l\colon \Gamma \to \mathrm{SL}(2, \mathbb{C})$ be a sequence of representations diverging to an ideal point of the Morgan–Shalen compactification of the character variety $X(\Gamma, \mathrm{SL}(2, \mathbb{C}))$. Let $\rho_\omega\colon \Gamma \to \mathrm{SL}(2, \mathbb{C}_\omega)$ be the natural representation determined by the sequence $(\rho_l)_{l\in\mathbb{N}}$. If the representation is reducible, then $\beta_2^\omega(\rho_\omega) = 0$.*

Proof. It follows directly from Proposition 5.5 by obsverving that the ρ_ω has non-trivial length function since it is associated to diverging sequence of representations. □

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Alessio Savini, Department of Mathematics, University of Bologna,
Piazza di Porta San Donato 5, 40126 Bologna, Italy

e-mail: alessio.savini5@unibo.it