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Analysis and computations of a stochastic Cahn–Hilliard model for tumor growth with chemotaxis and variable mobility

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Abstract
In this work, we present and analyze a system of PDEs, which models tumor growth by taking into account chemotaxis, active transport, and random effects. Tumor growth may undergo erratic behaviors such as metastases that cannot be predicted simply using deterministic models. Moreover, random perturbations are evident in models accounting for therapeutic treatment in terms of therapy uncertainty or parameter identification problems. The stochasticity of the system is modeled by Wiener noises that appear in the tumor and nutrient equations. The volume fraction of the tumor is governed by a stochastic phase-field equation of Cahn–Hilliard type, and the mass density of the nutrients is modeled by a stochastic reaction-diffusion equation. We allow a variable mobility function and nonincreasing growth functions, such as logistic and Gompertzian growth. Via approximation and stochastic compactness arguments, we prove the existence of a probabilistic weak solution and, in the case of constant mobilities, the well-posedness of the model in the strong probabilistic sense. Lastly, we propose a numerical approximation based on the Galerkin finite element method in space and the semi-implicit Euler–Maruyama scheme in time. We illustrate the effects of stochastic forcings in tumor growth in several numerical simulations.

Keywords Stochastic tumor growth model · Stochastic Cahn–Hilliard system · Well-posedness · Martingale solutions · Yosida approximation · Galerkin approximation · Energy estimates · Euler–Maruyama scheme · Finite elements

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1 Introduction

Cancer is among the main global causes of death. According to [52], there were 19.3 million new cancer diagnoses and 9.96 million cancer-related deaths worldwide. By 2040, the annual number of new cancer cases is projected to reach 30.2 million, with 16.3 million fatalities attributable to cancer. Each tumor is different and depends on various characteristics. There is no guaranteed procedure for curing cancer, nor is its cause completely known. Using mathematical models to precisely depict tumor progression is the primary objective of mathematical oncology.

Mathematical research on tumor growth dynamics has been extensively developed in recent decades. One of the most frequent approaches to describe tumor dynamics is based on the so-called diffuse interface theory, for which we refer, for example, to [7, 26, 32, 55].

In this work, we consider a diffuse interface model for tumor growth, which takes into account precise biological effects due to both chemotaxis and also possible random perturbations affecting the evolution. More precisely, we consider a colony of tumor cells in an open bounded domain $D \subset \mathbb{R}^3$, e.g. representing an organ, and we focus on phenomenological characterizations to capture mesoscale and macroscale events around the tumor's evolution in the time interval $[0, T]$ for some fixed final time $T > 0$. The difference in volume fractions between tumor and healthy cells is described by the field $\varphi : [0, T] \times D \to [0, 1]$, with $\{\varphi = 1\}$ representing unmixed tumor tissue, and $\{\varphi = 0\}$ represents surrounding healthy tissue. The diffuse interface approach is based on the constitutive assumption that the tumor and healthy phases are separated by a narrow diffuse interface $\{0 < \varphi < 1\}$ in which the phase-field variable may also assume intermediate values. The further constitutive assumption is that tumor cell proliferation occurs by absorption of some nutrient (e.g. glucose) that is delivered to the cells via capillary action. The local concentration of nutrients is represented by a field $\sigma : [0, T] \times D \to [0, 1]$.

1.1 Deterministic model

Following [36], the constituents φ and σ are governed by the extended mass balance laws

$$\begin{aligned}\partial_t \varphi &= -\operatorname{div} J_\varphi + \Gamma_\varphi, \\ \partial_t \sigma &= -\operatorname{div} J_\sigma + \Gamma_\sigma,\end{aligned}$$

where J_φ, J_σ are mass fluxes and Γ_φ, Γ_σ are source functions. Typically, sink-and-source-type functions are assumed, that is, it holds $\Gamma_\sigma = -\Gamma_\varphi$ and consequently, the system is referred to as closed or conserved. However, in order to consider more general biological effects, such as apoptosis (i.e. the natural cell death of tumor cells), we proceed as in [28] and define the source functions as:

$$\Gamma_\varphi(\varphi, \sigma) = (\beta\sigma - \alpha) f(\varphi), \qquad \Gamma_\sigma(\varphi, \sigma) = -\delta\sigma f(\varphi).$$

Here, α denotes the apoptosis rate, β the tumor proliferation rate, δ the rate of nutrient consumption. In the particular case of $\beta = \delta$ and $\alpha = 0$, one is in the sink and source scenario, that is, the total mass of the tumor and nutrients are conserved. The function f is assumed to be nonnegative and bounded in the relevant domain $[0, 1]$, and normalized in the sense $f(0) = f(1) = 0$. This corresponds to the absence of nutrient uptake by the tumor mass if there is no tumor $\{\varphi = 0\}$ anyway or if the tumor is already saturated in the sense $\{\varphi = 1\}$. A relevant choice is the logistic growth function $f(\varphi) = \varphi(1 - \varphi)$, see [19, 20, 22, 23], or the Gompertz function $f(\varphi) = \varphi \log(1/\varphi)$ as in [54].

The mass fluxes are related to the underlying energy of the system, and we choose the following extended Ginzburg–Landau energy:

$$\mathcal{E}(\varphi, \sigma) = \int_\Omega \frac{\varepsilon^2}{2}|\nabla\varphi|^2 + \Psi(\varphi) + \frac{1}{2}|\sigma|^2 - \chi\varphi\sigma \,\mathrm{d}x.$$

The parameters $\varepsilon > 0$ and $\chi \geq 0$ denote the interfacial width and the chemotaxis factor. Chemotaxis describes an adhesion force between tumor cells and nutrients. Moreover, Ψ describes a double-well potential and often one considers the regular function $\Psi(\varphi) = \frac{1}{4}\varphi^2(1 - \varphi)^2$ as an approximation to the physically realistic Flory–Huggins potential; see [6]. We follow Gurtin's approach in [31] and employ a scaled mass flux for the phase-field equation, i.e., we propose

$$J_\varphi = -m_1(\varphi)\nabla\delta_\varphi\mathcal{E}(\varphi, \sigma), \qquad J_\sigma = -m_2(\sigma)\nabla\delta_\sigma\mathcal{E}(\varphi, \sigma),$$

where $\delta_\varphi\mathcal{E}$ denotes the first variation of $\mathcal{E}$ with respect to φ. Here, m_i, $i \in \{1, 2\}$, denote mobility functions that are assumed to be non-degenerative, such as $m_1(\varphi) = \varphi^2(1-\varphi)^2 + M$ for some arbitrarily small constant $M > 0$. Moreover, m_2 corresponds to the diffusivity of the nutrients.

By calculating the first variations of the Ginzburg–Landau energy, we obtain the system

$$\begin{aligned} \partial_t\varphi &= \operatorname{div}(m_1(\varphi)(\nabla\mu - \chi\nabla\sigma)) + \Gamma_\varphi(\varphi, \sigma), \\ \mu &= \Psi'(\varphi) - \varepsilon^2\Delta\varphi, \\ \partial_t\sigma &= \operatorname{div}(m_2(\sigma)(\nabla\sigma - \chi\nabla\varphi)) + \Gamma_\sigma(\varphi, \sigma). \end{aligned} \tag{1.1}$$

This is usually complemented with no-flux boundary conditions for φ, μ, and σ, as well as suitable initial data φ_0 and σ_0.

The deterministic model was studied in [24, 25] regarding the existence of weak solutions, in [30] regarding the long-time behavior, in [27, 35] regarding an optimal control problem, and in [29, 33] regarding numerical approximations. Other biological phenomena were included in the works [18, 20–22] such as ECM degradation, angiogenesis, and subdiffusive behavior.

1.2 Stochastic model

The class of deterministic models for tumor growth is certainly widely used and provides a more-than-adequate description of several biological mechanisms. However, as intuitive as the tumor-nutrient balance equations may seem, it is well established that tumor growth may undergo erratic behaviors that one cannot predict by merely using deterministic models. It is the case, for example, of metastases whose activation appears to depend on random biological signals (see [53]), and the creation of capillary networks and angiogenesis (see [43]). Moreover, random perturbations are evident in models accounting for therapeutic treatment in terms of therapy uncertainty or parameter identification problems. These considerations inevitably call for the switch to classes of models that are capable of capturing such randomness in the underlying process: in the scientific literature, models of tumor growth that account for unpredictability have been considered in [1, 3, 15, 37, 51].

Generally, stochastic partial differential equations (SPDEs) are the mathematical tools used to model physical systems subjected to the influence of internal, external, or environmental noises. As noted in [38, 39], such stochastic systems can also be used to describe models that are too complex to be described deterministically. Therefore, a statistical strategy is employed to manage the complex free energy that defines the biological features of the system in the phase-field equation. To accomplish this, a noise is added to the phase-field equation, thus considering any microscopical fluctuations that affect the evolution of the phase parameter. Moreover, the phase-field equation aims at metastable round shapes, but real-world tumor masses are not perfectly round. In a stochastic model, we introduce a type of uncertainty that results in an irregular shape for the tumor mass that is biologically more realistic. Lastly, as noted in [45], one can mimic the consequences of angiogenesis by incorporating a multiplicative noise into the reaction-diffusion equation. This kind of stochastic forcing is related to the oxygen received by malignant cells; as a result, its contribution to the overall tumor growth process may be increased.

By taking these considerations into account, we further extend the deterministic model (1.1) by introducing stochastic components to the system. We define the independent cylindrical Wiener (Gaussian) processes W_1, W_2 on the separable Hilbert spaces U_1, U_2, respectively, and end up with the stochastic system

$$\begin{aligned}
\mathrm{d}\varphi &= \operatorname{div}(m_1(\varphi)(\nabla\mu - \chi\nabla\sigma))\,\mathrm{d}t + (\beta\sigma - \alpha) f(\varphi)\,\mathrm{d}t + G_1(\varphi)\,\mathrm{d}W_1, \\
\mu &= \Psi'(\varphi) - \varepsilon^2\Delta\varphi, \\
\mathrm{d}\sigma &= \operatorname{div}(m_2(\sigma)(\nabla\sigma - \chi\nabla\varphi))\,\mathrm{d}t - \delta\sigma f(\varphi)\,\mathrm{d}t + G_2(\sigma)\,\mathrm{d}W_2,
\end{aligned} \tag{1.2}$$

where G_1 and G_2 are operators that are specified below. The role of the stochastic forcings is to account also for growth uncertainty, reflecting the fact that tumors and nutrient do not grow at a constant rate in real-world scenarios. More precisely, G_1 models typical erratic growth of the tumor volume, and is usually rendered in a multiplicative way by the choice $G_1(\varphi) = f(\varphi)$, where again f acts as a cutoff: this results in a more erratic behaviour for higher concentrations of tumor cells. As for G_2, one can be slightly more general and assume an additive structure $G_2(\sigma) = 1$, which models an environmental noise possibly influencing the nutrient growth, such as growing capillaries around the domain. We equip the system with the classical Neumann boundary data $\partial_n \varphi = \partial_n \mu = \partial_n \sigma = 0$ on ∂D and the initial data $\varphi(0) = \varphi_0$, $\sigma(0) = \sigma_0$.

The mathematical literature on the stochastic Cahn–Hilliard equation is quite developed, e.g., see the articles [8, 9, 47–49]. We also quote the works [10, 13, 14, 42] for studies on stochastic diffuse interface models that are coupled to the Navier–Stokes equations. The work [45] studied a stochastic Cahn–Hilliard equation with a coupled reaction-diffusion system and in addition to its well-posedness an optimal control problem was investigated. However, typical features of a tumor growth model, such as chemotaxis, flow, active transport, and logistic growth, were not considered.

1.3 Goals, novelties, and structure of the paper

The main goal of the present work is twofold: we focus on the stochastic system, both in terms of mathematical analysis and numerical simulations.

More precisely, we first show that the stochastic system admits a probabilistic-weak solution in a very general setting, including non-constant mobilities and positive chemotaxis coefficients. Up to our knowledge, this is the first available result in the literature of stochastic models for tumor growth dealing with both random perturbations and non-constant mobilities. The analysis is carried out in a wide generality, allowing the double-well potential to have polynomial growth at infinity, whose order depends on the presence of chemotaxis and the fact that the mobility m_1 is constant or not. Let us also stress that the presence of non-constant mobilities and the coupling between the Cahn–Hilliard equation and the reaction-diffusion equation prevent us from framing the analysis in any well-established variational theory based on monotonicity. This calls for an ad-hoc study of the system, for which we employ a two-level approximation scheme: a Galerkin-type approximation on the functional spaces and a Yosida-type approximation on the nonlinearity. A significant challenge in the analysis arises from the nonlinear coupling between the equations, as the system under consideration does not conform to standard analytical frameworks.

Secondly, we prove that in the more classical case of constant mobilities the stochastic system is well posed also in the strong probabilistic sense: this is based on a Yamada–Watanabe argument and comes down to showing pathwise uniqueness of solutions. Again, the issue is nontrivial due to the singular (non-Lipschitz) behavior of the double-well potential in the Cahn–Hilliard equation.

Eventually, we propose a numerical scheme to approximate the solutions of the stochastic system, and we provide several numerical simulations. These explicitly show the effect of noise on the tumor dynamics, according to a span of noise intensities, and confirm a more accurate description of the cell aggregation phenomenon.

Let us point out that the case of degenerate mobilities is not treated here. Although the stochastic Cahn–Hilliard equation has been analysed in [48] under a specific assumption on the noise, here the presence of the proliferation terms prevents to cover the degenerate case. This behaviour is typical in phase-field systems also in the deterministic case, unless specific structural conditions on the source are imposed: we refer to [17] for an example of a phase-field model for tumor growth with degenerate mobilities, under a different choice of the proliferation.

The work is structured as follows. In Sect. 2 we give some mathematical preliminaries, such as required compactness results and important inequalities. Moreover, we introduce the function spaces that are typically used in the variational analysis of stochastic PDEs. Well-known results in stochastic analysis such as Prokhorov's theorem and the Burkholder–Davis–Gundy inequality are briefly described. In Sect. 3, we prove the existence of probabilistic weak solutions to the stochastic system. Moreover, in the case of constant mobilities, we prove the existence and pathwise uniqueness of strong solutions to the stochastic system. In Sect. 4, we propose a fully discrete scheme for approximating the stochastic model. We show some selected numerical simulations to highlight the influence of the stochastic components.

2 Mathematical notation and preliminaries

In this section, we introduce the main mathematical setting of the work, the functional spaces that will be used, and recall some useful preliminaries on infinite-dimensional stochastic analysis.

In the following, let $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with a C^2-boundary ∂D. Furthermore, let $T > 0$ be a finite time horizon and let the space-time cylinders be denoted by $D_t = (0, t) \times D$, for every $t \in (0, T]$. We simply write $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_{D_t}$ to indicate the inner products in $L^2(D)$ and in $L^2(D_t)$, respectively, for every $t \in (0, T]$. We also use the classical symbol $(\cdot)_D$ for the space-average on D.

2.1 Functional analysis

Let X be an arbitrary Banach space. We denote its dual by X' and the duality pairing between X and its dual is written as $\langle \cdot, \cdot \rangle_X$. If X is a Hilbert space, then the scalar product is denoted by $(\cdot, \cdot)_X$. In the case of a further Hilbert space Y, we denote the space of Hilbert–Schmidt operators from X to Y by $\mathcal{L}^2(X, Y)$ consisting of operators $T \in \mathcal{L}(X, Y)$ that satisfy $\|T\|^2_{\mathcal{L}^2(X,Y)} = \sum_{k\in\mathbb{N}} \|Te_k\|^2_Y < \infty$ for an orthonormal basis $\{e_k\}_k$ of X. Analogously, we denote by $\mathcal{L}^1(X, Y)$ the space of trace-class operators from X to Y.

We set the Hilbert spaces

$$H := L^2(D), \qquad V := H^1(D), \qquad V_2 = \{u \in H^2(D) : \partial_n u = 0 \text{ a.e. on } \partial D\},$$

endowed with their natural norms and identify H with its dual through the Riesz isomorphism. We recall that this ensures that

$$V_2 \subset V \subset H \subset V' \subset V_2',$$

where all inclusions are dense and compact.

2.2 Stochastic calculus

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space. We denote the progressive σ-algebra on $\Omega \times [0, T]$ by $\mathcal{P}$, that is, the collection of subsets of $\Omega \times [0, T]$ with the indicator function being a progressively measurable process; see [46, p. 314].

Let X denote an arbitrary Banach space, and $T > 0$. The strongly measurable Bochner-integrable functions on Ω and $(0, T)$ of order $p, q \in [1, \infty]$ are denoted by $L^p(\Omega; X)$ and $L^q(0, T; X)$, respectively, with the usual convention that $p = 0$ and $q = 0$ denote the space of strongly measurable functions on Ω and $(0, T)$, respectively. In the case of progressive measurability, we use the notation $L^p_{\mathcal{P}}(\Omega; L^q(0, T; X))$. If X is separable and reflexive, we introduce an analogous notation for weakly-$*$ measurable functions

$$
\begin{aligned}
&L^p_w(\Omega; L^\infty(0, T; X^*)) \\
&\quad = \{v : \Omega \to L^\infty(0, T; X^*) \text{ weakly-}* \text{ meas. } : \mathbb{E}\|v\|^p_{L^\infty(0,T;X^*)} < \infty\} \\
&\quad = (L^{p/(p-1)}(\Omega; L^1(0, T; X)))^*,
\end{aligned}
$$

where the second equality holds due to [11, Thm. 8.20.3].

We introduce the independent cylindrical Wiener processes $W_i = (W_i(t))_{t \geq 0}$, $i \in \{1, 2\}$, on the separable Hilbert spaces U_i. We further equip the spaces with the ONBs $\{u^i_n\}_n$ for $i \in \{1, 2\}$ i.e. it holds $W_i(t) = \sum_{k=1}^\infty \beta^i_k(t) u^i_k$, $i \in \{1, 2\}$, for $\{\beta^i_k\}_k$ being a family of i.i.d. $\mathcal{F}_t$-Brownian motions. Let us recall that the series are formal, and they converge in every Hilbert-Schmidt extension $\widetilde{U}_i$ of U_i, $i = 1, 2$: in particular, W_i is a well-defined process with continuous trajectories in $\widetilde{U}_i$. Moreover, if K is a Hilbert space, for $i = 1, 2$ the stochastic integral $\int_0^\cdot G(s)\,\mathrm{d}W_i(s)$ is a well-defined continuous K-valued process for every $G \in L^2_{\mathcal{P}}(\Omega; L^2(0, T; \mathcal{L}^2(U_i, K)))$. We further recall that the Burkholder–Davis–Gundy inequality, see [5, Proposition 2.3.8], states

$$
\mathbb{E} \sup_{t \in [0,T]} \left\| \int_0^t G(s)\,\mathrm{d}W_i(s) \right\|^p_K \leq C_p \mathbb{E} \left(\int_0^T \|G(s)\|^2_{L^2(U_i, K)}\,\mathrm{d}s \right)^{p/2} \tag{2.1}
$$

for every $G \in L^2_{\mathcal{P}}(\Omega; L^2(0, T; \mathcal{L}^2(U_i, K)))$.

It is well-known that if X is a Polish space (i.e., a separable completely metrizable topological space), then every probability measure on X is tight. Furthermore, by Prokhorov's theorem, see [5, Theorem 2.6.1], a collection of probability measures on X is tight if and only if it is relatively weakly compact. Together with the Skorokhod theorem, this links the concept of weak convergence of a probability measure with that of almost sure convergence of random variables. Indeed, Skorokhod theorem states the existence of X-valued random variables $(U_n)_n$ on some probability space such that the law of $(U_n)_n$ is equal to a given weakly converging sequence of probability measures on X, and it holds $U_n(\omega) \to U_0(\omega)$ in X almost surely. This result can be generalized to the class of sub-Polish spaces, which are topological spaces that are not necessarily metrizable but retain several important properties of Polish spaces; then the result is referred to as Jakubowksi–Skorokhod theorem, see [5, Theorem 2.7.1].

2.3 Itô's formula

Let $i \in \{1, 2\}$ be fixed, let K be a Hilbert space, let $G \in L^2_{\mathcal{P}}(\Omega; L^2(0, T; \mathcal{L}^2(U_i, K)))$, let $f \in L^2_{\mathcal{P}}(\Omega; L^1(0, T; K))$, and let φ_0 be an $\mathcal{F}_0$-measurable K-valued random variable. Then the process

$$\varphi(t) = \varphi_0 + \int_0^t f(s)\,\mathrm{d}s + \int_0^t G(s)\,\mathrm{d}W_i(s) \tag{2.2}$$

is a well-defined continuous K-valued process. If the function $F\colon K \to \mathbb{R}$ is twice Fréchet-differentiable and its derivatives DF, D^2F are uniformly continuous on bounded subsets of K, then the Itô formula, see [5, Theorem 2.4.1], reads

$$\begin{aligned} F(\varphi(t)) = F(\varphi_0) &+ \int_0^t \langle DF(\varphi(s)), f(s)\rangle\,\mathrm{d}s + \int_0^t \langle DF(\varphi(s)), G(s)\,\mathrm{d}W(s)\rangle \\ &+ \frac{1}{2}\int_0^t \mathrm{Tr}\big(G(s)^* D^2F(\varphi(s))G(s)\big)\,\mathrm{d}s, \end{aligned} \tag{2.3}$$

for all $t \in [0, T]$, $\mathbb{P}$-almost surely, with the trace being defined as $\mathrm{Tr}(A) = \sum_{k=1}^{\infty} \langle Au_k^i, u_k^i\rangle$ for every $A \in \mathcal{L}^1(U_i, U_i)$. Let us recall that when the function F is the square of the K-norm, then the Itô formula holds also in more general setting, such as the classical variational triplets: in this regard, we refer the reader to [40, Thm. 4.2.5].

3 Mathematical analysis of the stochastic model

The subsequent assumptions will be in order throughout the paper.

Assumption 3.1 Let the following assumptions hold:

(A1) $\alpha, \beta, \delta, \chi \geq 0$ and $\varepsilon > 0$ are fixed;
(A2) $f \in C^{0,1}(\mathbb{R}; [0, 1])$;
(A3) $m_1, m_2 \in C^0(\mathbb{R}; [m_0, m_\infty])$ for $0 < m_0 \leq m_\infty < \infty$;
(A4) $\Psi \in C^2(\mathbb{R}; \mathbb{R}_{\geq 0})$ with $\Psi'(0) = 0$ satisfies the growth conditions

$$|\Psi'(x)|^q + |\Psi''(x)| \leq C_\Psi(1 + |\Psi(x)|)$$

and

$$\Psi''(x) \geq -C_\Psi$$

for any $x \in \mathbb{R}$, for some $C_\Psi > 0$, where

$$\begin{cases} q \in (1, 2] & \text{if either } \chi = 0 \text{ or } m_1 \text{ is constant,} \\ q = 2 & \text{if both } \chi > 0 \text{ and } m_1 \text{ is non-constant;} \end{cases}$$

(A5) for $i \in \{1, 2\}$, $G_i : H \to \mathcal{L}^2(U_i, H)$ is measurable and there is a sequence $(g_{i,k})_k \subset W^{1,\infty}(\mathbb{R})$ such that $G_i(v)u_k^i = g_{i,k}(v)$ for any $v \in H$ and $k \in \mathbb{N}$, with $C_{G_i} := \sum_{k=0}^{\infty} \|g_{i,k}\|^2_{W^{1,\infty}(\mathbb{R})} < \infty$.

(A6) φ_0, σ_0 satisfy $\varphi_0 \in V$ with $\Psi(\varphi_0) \in L^1(D)$ and $\sigma_0 \in H$.

Let us comment on the assumptions above. Foremost, we note that assumption (A2) allows both the Gompertz and logistic growth functions, as discussed in Sect. 1. Moreover, (A4) prescribes different growth conditions on Ψ, depending on χ and m_1. In particular, we note that in the case where either no chemotaxis is present or the first mobility is constant, the assumption $q \in (1, 2]$ allows every polynomial double-well potential of any growth. In the pathological case with both chemotaxis and a variable mobility in the Cahn–Hilliard equation, the situation is much more delicate and the assumption on the potential is quite strict, prescribing double-well structure with at most quadratic growth at infinity. This restriction is due to the strong coupling in the system: indeed, if $\chi > 0$ one cannot rely on a maximum-principle argument for σ, and this results in the presence of a non-bounded proliferation term in the Cahn–Hilliard equation, which cannot be treated unless using an ad-hoc technique working in the constant mobility case.

Assumption (A3) allows both constant mobilities and positive mobilities, such as $m(\varphi) = 1_{[0,1]}\varphi^2(1-\varphi)^2 + m_0$ for some small $m_0 > 0$. We note that the case of degenerating mobility, as studied in [48] for the Cahn–Hilliard equation, is not addressed here. While we are confident that the method could be extended under stricter assumptions on f, this would necessitate an additional approximation scheme for the mobility, beyond the Yosida and Galerkin approximations. As a consequence, we do not demonstrate that $\varphi \in [0, 1]$ if $\varphi_0 \in [0, 1]$, since this requires the use of degenerating mobility, as shown in [48, Theorem 2.7].

The assumption (A5) on the operators G_i is widely used in literature and ensures the Lipschitz-continuity and the linear boundedness of the operators. The assumptions (A6) on the initial data are restricted to the nonrandom case. However, we note that it can be extended to the random case, as done for example in [49] for the stochastic Cahn–Hilliard equation.

Definition 3.2 (*Martingale solution*) We call $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_t, \widehat{\mathbb{P}}, \widehat{W}_1, \widehat{W}_2, \widehat{\varphi}, \widehat{\mu}, \widehat{\sigma})$ a martingale solution to (1.2) if $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_t, \widehat{\mathbb{P}})$ is a filtered probability space satisfying the usual conditions, $\widehat{W}_1$ and $\widehat{W}_2$ are cylindrical Wiener processes on U_1 and U_2, respectively, and

$$
\begin{aligned}
&\widehat{\varphi} \in L^0_{\mathcal{P}}(\widehat{\Omega}; C^0([0, T]; H) \cap L^2(0, T; V_2)),\\
&\widehat{\mu} \in L^0_{\mathcal{P}}(\widehat{\Omega}; L^2(0, T; V)),\\
&\widehat{\sigma} \in L^0_{\mathcal{P}}(\widehat{\Omega}; C^0([0, T]; H) \cap L^2(0, T; V)),
\end{aligned}
$$

are such that $\widehat{\mu} = -\varepsilon^2 \Delta \widehat{\varphi} + \Psi'(\widehat{\varphi})$ and

$$
\begin{aligned}
(\widehat{\varphi}(t), v)_D + (m_1(\widehat{\varphi})\nabla(\widehat{\mu} - \chi\widehat{\sigma}), \nabla v)_{D_t} &= (\varphi_0, v)_D + (\beta\widehat{\sigma} - \alpha, f(\widehat{\varphi})v)_{D_t} \\
&\quad + (\textstyle\int_0^t G_1(\widehat{\varphi}(s))\mathrm{d}\widehat{W}_1(s), v)_D \\
(\widehat{\sigma}(t), v)_D + (m_2(\widehat{\sigma})\nabla(\widehat{\sigma} - \chi\widehat{\varphi}), \nabla v)_{D_t} &= (\sigma_0, v)_D - (\delta\widehat{\sigma}, f(\widehat{\varphi})v)_{D_t} \\
&\quad + (\textstyle\int_0^t G_2(\widehat{\sigma}(s))\mathrm{d}\widehat{W}_2(s), v)_D
\end{aligned}
\tag{3.1}
$$

for every $v \in V$, for every $t \in [0, T]$, $\widehat{\mathbb{P}}$-almost surely.

Definition 3.3 (*Strong solution*) We call (φ, μ, σ) a (probabilistically-) strong solution to (1.2) if

$$
\begin{aligned}
&\varphi \in L^0_{\mathcal{P}}(\Omega; C^0([0, T]; H) \cap L^2(0, T; V_2)), \\
&\mu \in L^0_{\mathcal{P}}(\Omega; L^2(0, T; V)), \\
&\sigma \in L^0_{\mathcal{P}}(\Omega; C^0([0, T]; H) \cap L^2(0, T; V)),
\end{aligned}
$$

are such that $\mu = -\varepsilon^2 \Delta \varphi + \Psi'(\varphi)$ and

$$
\begin{aligned}
(\varphi(t), v)_D + (m_1(\varphi)\nabla(\mu - \chi\sigma), \nabla v)_{D_t} &= (\varphi_0, v)_D + (\beta\sigma - \alpha, f(\varphi)v)_{D_t} \\
&\quad + (\textstyle\int_0^t G_1(\varphi(s))\mathrm{d}W_1(s), v)_D \\
(\sigma(t), v)_D + (m_2(\sigma)\nabla(\sigma - \chi\varphi), \nabla v)_{D_t} &= (\sigma_0, v)_D - (\delta\sigma, f(\varphi)v)_{D_t} \\
&\quad + (\textstyle\int_0^t G_2(\sigma(s))\mathrm{d}W_2(s), v)_D
\end{aligned}
\tag{3.2}
$$

for every $v \in V$, for every $t \in [0, T]$, $\mathbb{P}$-almost surely.

The existence of a martingale solution to the stochastic tumor growth system is stated in the next theorem. This is the main result of this work.

Theorem 3.4 (Existence of martingale solutions) *Let Assumption 3.1 hold. Then there exists a martingale solution $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_t, \widehat{\mathbb{P}}, \widehat{W}_1, \widehat{W}_2, \widehat{\varphi}, \widehat{\mu}, \widehat{\sigma})$ to the stochastic tumor system in the sense of Definition 3.2 with regularity*

$$
\begin{aligned}
&\widehat{\varphi} \in L^\ell(\widehat{\Omega}; C^0([0, T]; H)) \cap L^\ell_w(\widehat{\Omega}; L^\infty(0, T; V)) \cap L^{\ell/2}(\widehat{\Omega}; L^2(0, T; V_2)), \\
&\widehat{\mu} \in L^{\ell/2}(\widehat{\Omega}; L^2(0, T; V)) \cap L^\ell(\widehat{\Omega}; L^2(0, T; H)), \\
&\widehat{\sigma} \in L^\ell(\widehat{\Omega}; C^0([0, T]; H)) \cap L^{\ell/2}(\widehat{\Omega}; L^2(0, T; V)),
\end{aligned}
$$

for every $\ell \geq 2$, and there exists a constant c, depending only on the data $D, T, \alpha, \beta, \delta, \chi, \varepsilon, C_\Psi, q, m_0, m_\infty, C_{G_1}, C_{G_2}$, such that the following energy inequality holds:

$$
\begin{aligned}
&\widehat{\mathbb{E}\sup}_{s\in[0,T]} \|\widehat{\varphi}(s)\|_V^2 + \widehat{\mathbb{E}\sup}_{s\in[0,T]} \|\Psi(\widehat{\varphi}(s))\|_{L^1(D)} + \widehat{\mathbb{E}}\|\nabla\widehat{\mu}\|^2_{L^2(0,T;H)} \\
&\quad + \widehat{\mathbb{E}}|\widehat{\mu}\|_{L^2(0,T;H)} + \widehat{\mathbb{E}\sup}_{s\in[0,T]} \|\widehat{\sigma}(s)\|_H^2 + \widehat{\mathbb{E}}|\widehat{\sigma}\|^2_{L^2(0,T;V)} \\
&\leq c\Big(\|\varphi_0\|_V^2 + \|\Psi(\varphi_0)\|_{L^1(D)} + \|\sigma_0\|_H^2 \Big).
\end{aligned}
\tag{3.3}
$$

In the special case where the mobilities are constant and the potential is the classical fourth-order polynomial potential, we show that pathwise uniqueness holds and that the system is well-posed in the strong probabilistic sense on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with respect to the original Wiener processes W_1 and W_2.

Theorem 3.5 (Strong well-posedness) *Let Assumption 3.1 hold, assume that the mobilities m_1, m_2 are constant and that*

$$|\Psi'(x_1) - \Psi'(x_2)| \leq C_\Psi(1 + |x_1|^2 + |x_2|^2)|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}. \tag{3.4}$$

Then there exists a unique strong solution (φ, μ, σ) to the stochastic tumor system in the sense of Definition 3.3 with regularity

$$\begin{aligned}
&\varphi \in L^\ell(\Omega; C^0([0,T]; H)) \cap L^\ell_w(\Omega; L^\infty(0,T; V)) \cap L^{\ell/2}(\Omega; L^2(0,T; V_2)),\\
&\mu \in L^{\ell/2}(\Omega; L^2(0,T; V)) \cap L^\ell(\Omega; L^2(0,T; H)),\\
&\sigma \in L^\ell(\Omega; C^0([0,T]; H)) \cap L^{\ell/2}(\Omega; L^2(0,T; V)).
\end{aligned}$$

for every $\ell \geq 2$, and there exists a constant c, depending only on the data $D, T, \alpha, \beta, \delta, \chi, \varepsilon, C_\Psi, q, m_0, m_\infty, C_{G_1}, C_{G_2}$, such that the following energy inequality holds:

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} \|\varphi(s)\|_V^2 + \mathbb{E} \sup_{s\in[0,T]} \|\Psi(\varphi(s))\|_{L^1(D)} + \mathbb{E}\|\nabla\mu\|^2_{L^2(0,T;H)}\\
&\quad + \mathbb{E}\|\mu\|_{L^2(0,T;H)} + \mathbb{E} \sup_{s\in[0,T]} \|\sigma(s)\|_H^2 + \mathbb{E}\|\sigma\|^2_{L^2(0,T;V)}\\
&\quad \leq c\Big(\|\varphi_0\|_V^2 + \|\Psi(\varphi_0)\|_{L^1(D)} + \|\sigma_0\|_H^2 \Big).
\end{aligned} \tag{3.5}$$

Moreover, for two strong solutions $(\varphi_1, \mu_1, \sigma_1), (\varphi_2, \mu_2, \sigma_2)$ associated to some initial data $(\varphi_0^1, \sigma_0^1), (\varphi_0^2, \sigma_0^2)$ satisfying (A6), there exist two sequences of constants $\{c_n\}_n$ and of stopping times $\{\tau_n\}_n$, such that $\tau_n \nearrow T$ almost surely and the following continuous dependence holds:

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,\tau_n]} \|(\varphi_1 - \varphi_2)(s)\|_H^2 + \mathbb{E}\|\varphi_1 - \varphi_2\|^2_{L^2(0,\tau_n;V_2)}\\
&\quad + \mathbb{E} \sup_{s\in[0,\tau_n]} \|(\sigma_1 - \sigma_2)(s)\|_H^2 + \mathbb{E}\|\sigma_1 - \sigma_2\|^2_{L^2(0,\tau_n;V)}\\
&\quad \leq c_n\Big(\left\|\varphi_0^1 - \varphi_0^2\right\|_H^2 + \left\|\sigma_0^1 - \sigma_0^2\right\|_H^2 \Big) \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.6}$$

Note that the interpretation of the variables φ and σ would suggest that such quantities take values in the interval $[0, 1]$. However, in Theorems 3.4–3.5 one is not able to show this and the variables φ and σ can take any real value. As for the variable φ, this is due both to the presence of a regular potential (not singular) and to the non-degenerate mobility: we refer to [48] for an example with singular potential and degenerate mobility instead. For the variable σ, the issue is given by the the presence of a chemotaxis term: in this direction, we refer to [45] for an analogous stochastic model where sufficient conditions are given for the nutrient to be confined in $[0, 1]$. In our case, since the physical constraints are not met by φ and σ, the physical interpretation is recovered by using typical normalisation procedures on the variables through classical cut off functions, such as $\min\{\max\{\cdot, 0\}, 1\}$ or $\tanh(\cdot)$.

The proof of Theorem 3.4 is structured in several steps based on intermediate auxiliary results.

Step 1: We introduce an approximated problem, where we consider a smooth Yosida-type regularization of the nonlinearity Ψ', called Ψ'_λ for some arbitrary but fixed $\lambda > 0$. Our main goal lies in showing the existence of a martingale solution $(\widetilde{\varphi_\lambda}, \widetilde{\mu_\lambda}, \widetilde{\sigma_\lambda})$ to this λ-approximated problem, deriving λ-uniform estimates and passing to the limit $\lambda \to 0$. However, to achieve the existence of a martingale solution, we have to first consider a Galerkin approximation, depending on some further parameter n, to the λ-approximation. The well-posedness of the Galerkin system is discussed in this step.
Step 2: The goal in this step is deriving n-uniform estimates of the Galerkin approximation. Unfortunately, such estimates are independent of n, but might depend on λ: this is a classical scenario for nonlinear problems such as generalised gradient flows whenever the potential is not necessarily a polynomial function, as in our case. This motivates the need of two distinct parameters in the proof.
Step 3: Having derived n-uniform estimate, we can pass to the limit $n \to \infty$. This ensures the existence of a solution $(\widetilde{\varphi_\lambda}, \widetilde{\mu_\lambda}, \widetilde{\sigma_\lambda})$ to the λ-approximation.

Step 4: We derive λ-uniform estimates in suitable spaces.
Step 5: Through the theorems of Prokhorov and Skorokhod, we show here the existence of martingale solutions.

The proof of Theorem 3.5 relies instead on proving pathwise uniqueness for the system, which yields also strong-existence by a classical argument à la Yamada–Watanabe. The main point comes down then to proving the continuous dependence (3.6).

In the following we denote by c a generic constant depending solely on the structure of the problem introduced above. If the constant depends on some approximation parameter such as λ or n, we use the symbols c_λ and c_n, respectively.

3.1 Step 1: Yosida and Galerkin approximations

By (A4), it holds $\Psi''(r) \geq -C_\Psi$ for any $r \in \mathbb{R}$ and thus, the function $\gamma(r) := \Psi'(r) + C_\Psi r$ is nondecreasing and continuous. We can identify γ with a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and it satisfies $\gamma(0) = 0$. For any $\lambda > 0$, we define the Yosida approximation of γ, see [4, p. 99], by $\gamma_\lambda(r) := \frac{1}{\lambda}(r - (I + \lambda\gamma)^{-1}(r))$ for any $r \in \mathbb{R}$. We recall that γ_λ is $\frac{1}{\lambda}$-Lipschitz continuous, and it holds $\gamma_\lambda(r) \to \gamma(r)$ as $\lambda \downarrow 0$. Then, we define the approximated double-well potential $\Psi_\lambda : \mathbb{R} \to \mathbb{R}$ by $\Psi_\lambda(r) := \Psi(0) - \frac{C_\Psi}{2} r^2 + \int_0^r \gamma_\lambda(s)\,\mathrm{d}s$. In particular, it holds $\Psi'_\lambda(r) = \gamma_\lambda(r) - C_\Psi r$ for any $r \in \mathbb{R}$ and we conclude that Ψ'_λ is $\max\{\frac{1}{\lambda}, C_\Psi\}$-Lipschitz continuous. Moreover, it holds $|\Psi_\lambda(r)| \leq C_\lambda(1 + |r|^2)$ for any $r \in \mathbb{R}$ for some constant C_λ possibly depending on λ. Eventually, we recall that the definition of Yosida approximation implies that $\gamma_\lambda(x) = \gamma(J_\lambda(x))$ for all $x \in \mathbb{R}$, where $J_\lambda := (I + \lambda\gamma)^{-1} : \mathbb{R} \to \mathbb{R}$ is the resolvent of γ. Moreover, we set $\widehat{\gamma} : \mathbb{R} \to \mathbb{R}$ as $\widehat{\gamma}(x) := \Psi(0) + \int_0^x \gamma(s)\,\mathrm{d}s = \Psi(x) - \frac{C_\Psi}{2}x^2$, $x \in \mathbb{R}$, and its respective Moreau-Yosida regularisation $\widehat{\gamma_\lambda} : \mathbb{R} \to \mathbb{R}$ as $\widehat{\gamma_\lambda}(x) := \Psi(0) + \int_0^x \gamma_\lambda(s)\,\mathrm{d}s = \Psi_\lambda(x) - \frac{C_\Psi}{2}x^2$, $x \in \mathbb{R}$. We recall that by the properties of the Moreau-Yosida regularisation (see again [4]) it holds that $\widehat{\gamma}(J_\lambda(x)) \le \widehat{\gamma_\lambda}(x) \le \widehat{\gamma}(x)$ for all $x \in \mathbb{R}$. This implies by (A4), since the resolvent J_λ is non-expansive, that

$$
\begin{aligned}
|\Psi_\lambda'(x)| &\le |\gamma_\lambda(x)| + C_\Psi|x| = |\gamma(J_\lambda(x))| + C_\Psi|x| \\
&\le |\Psi'(J_\lambda(x))| + C_\psi|J_\lambda(x)| + C_\Psi|x| \\
&\le C_\Psi^{\frac{1}{q}}(1 + \Psi(J_\lambda(x)))^{\frac{1}{q}} + 2C_\Psi|x| \\
&= C_\Psi^{\frac{1}{q}}\left(1 + \widehat{\gamma}(J_\lambda(x)) + \frac{C_\Psi}{2}|J_\lambda(x)|^2\right)^{\frac{1}{q}} + 2C_\Psi|x| \\
&\le C_\Psi^{\frac{1}{q}}\left(1 + \Psi_\lambda(x) + C_\Psi|x|^2\right)^{\frac{1}{q}} + 2C_\Psi|x|\,,
\end{aligned}
$$

and analogously that

$$
\begin{aligned}
|\Psi_\lambda''(x)| &\le |\gamma_\lambda'(x)| + C_\Psi = |\gamma'(J_\lambda(x))J_\lambda'(x)| + C_\Psi \le |\gamma'(J_\lambda(x))| + C_\Psi \\
&\le |\Psi''(J_\lambda(x))| + 2C_\psi \\
&\le C_\Psi(1 + \Psi(J_\lambda(x))) + 2C_\Psi \\
&= C_\Psi\left(1 + \widehat{\gamma}(J_\lambda(x)) + \frac{C_\Psi}{2}|J_\lambda(x)|^2\right) + 2C_\Psi \\
&\le C_\Psi\left(1 + \Psi_\lambda(x) + C_\Psi|x|^2\right) + 2C_\Psi\,.
\end{aligned}
$$

Since $q \le 2$ by (A4) we deduce that there exists a constant $c > 0$, independent of λ, such that, for all $x \in \mathbb{R}$ and $\lambda > 0$,

$$
|\Psi_\lambda'(x)|^q + |\Psi_\lambda''(x)| \le c(1 + |\Psi_\lambda(x)| + |x|^2). \tag{3.7}
$$

Consequently, we consider the following Yosida approximation of (1.2):

$$
\begin{aligned}
&\mathrm{d}\varphi_\lambda = \mathrm{div}(m_1(\varphi_\lambda)(\nabla\mu_\lambda - \chi\nabla\sigma_\lambda))\,\mathrm{d}t + (\beta\sigma_\lambda - \alpha)f(\varphi_\lambda)\,\mathrm{d}t + G_1(\varphi_\lambda)\,\mathrm{d}W_1, \\
&\mu_\lambda = \Psi_\lambda'(\varphi_\lambda) - \varepsilon^2\Delta\varphi_\lambda, \\
&\mathrm{d}\sigma_\lambda = \mathrm{div}(m_2(\sigma_\lambda)(\nabla\sigma_\lambda - \chi\nabla\varphi_\lambda))\,\mathrm{d}t - \delta\sigma_\lambda f(\varphi_\lambda)\,\mathrm{d}t + G_2(\sigma_\lambda)\,\mathrm{d}W_2,
\end{aligned} \tag{3.8}
$$

equipped with the initial values $\varphi_\lambda(0) = \varphi_0$, $\sigma_\lambda(0) = \sigma_0$ in D and the boundary data $0 = \partial_n\varphi_\lambda = \partial_n\mu_\lambda = \partial_n\sigma_\lambda$ on $(0, T) \times \partial D$. As described before, we want to show the existence of a solution to the λ-approximated system. This is stated in the following lemma. The concept of martingale solution to the approximated problem (3.8) is analogous, *mutatis mutandis*, to the one given in Definition 3.2 for the original system.

Lemma 3.6 *There exists a martingale solution* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}_t, \widetilde{\mathbb{P}}, \widetilde{W}_1, \widetilde{W}_2, \widetilde{\varphi_\lambda}, \widetilde{\mu_\lambda}, \widetilde{\sigma_\lambda})$ *to* (3.8) *in the sense that*

$$\begin{aligned}
&\widetilde{\varphi_\lambda} \in L^2_{\mathcal{P}}(\widehat{\Omega}; C^0([0,T]; H) \cap L^2(0,T; V_2)),\\
&\widetilde{\mu_\lambda} \in L^2_{\mathcal{P}}(\widehat{\Omega}; L^2(0,T; V)),\\
&\widetilde{\sigma_\lambda} \in L^2_{\mathcal{P}}(\widehat{\Omega}; C^0([0,T]; H) \cap L^2(0,T; V)),
\end{aligned}$$

satisfy $\widetilde{\mu_\lambda} = -\varepsilon^2 \Delta \widetilde{\varphi_\lambda} + \Psi'_\lambda(\widetilde{\varphi_\lambda})$ *and*

$$\begin{aligned}
(\widetilde{\varphi_\lambda}(t), v)_D + (m_1(\widetilde{\varphi_\lambda})\nabla(\widetilde{\mu_\lambda} - \chi\widetilde{\sigma_\lambda}), \nabla v)_{D_t} &= (\varphi_0, v)_D + (\beta\widetilde{\sigma_\lambda} - \alpha, f(\widetilde{\varphi_\lambda})v)_{D_t}\\
&\quad + \left(\textstyle\int_0^t G_1(\widetilde{\varphi_\lambda}(s))\mathrm{d}\widetilde{W}_1(s), v\right)_D\\
(\widetilde{\sigma_\lambda}(t), v)_D + (m_2(\widetilde{\sigma_\lambda})\nabla(\widetilde{\sigma_\lambda} - \chi\widetilde{\varphi_\lambda}), \nabla v)_{D_t} &= (\sigma_0, v)_D - (\delta\widetilde{\sigma_\lambda}, f(\widetilde{\varphi_\lambda})v)_{D_t}\\
&\quad + \left(\textstyle\int_0^t G_2(\widetilde{\sigma_\lambda}(s))\mathrm{d}\widetilde{W}_2(s), v\right)_D
\end{aligned}$$

for every $v \in V$, *for every* $t \in [0,T]$, $\widetilde{\mathbb{P}}$*-almost surely.*

The proof of the lemma is shifted to the end of Sect. 3.3 as it requires a further approximation on the system, which we now introduce. Let $(e_j)_{j\in\mathbb{N}} \subset V_2$ and $(\alpha_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ be sequences of eigenfunctions and eigenvalues of the negative Laplace operator with homogeneous Neumann conditions, respectively, i.e. $-\Delta e_j = \alpha_j e_j$ in D with $\nabla e_j \cdot n_{\partial D} = 0$ on ∂D. We normalize, so that $(e_j)_j$ is an orthonormal system of H and an orthogonal system in V. For every $n \in \mathbb{N}$, we define the finite dimensional space $H_n := \operatorname{span}\{e_1, \dots, e_n\} \subset V_2$, endowed with the $\|\cdot\|_H$-norm. The projection onto H_n is written as $\Pi_n : H \to H_n$ and satisfies $\Pi_n h = \sum_{j=1}^n (h, e_j)_H e_j$, $\|\Pi_n h\|_H \le 1$ and $\Pi_n h \to h$ in H as $n \to \infty$.

Let now $i \in \{1,2\}$. We define the approximated operator $G_{i,n} : H_n \to \mathcal{L}^2(U_i, H_n)$ as $G_{i,n}(v)u_k^i := \sum_{j=1}^n (G_i(v)u_k^i, e_j)_H e_j$ for any $v \in H_n$ and $k \in \mathbb{N}$. One can check that $G_{i,n}$ is well-defined: indeed, for every $v \in H_n$ and every $n \in \mathbb{N}$, thanks to (A5) we have $G_{i,n}(v) \in \mathcal{L}^2(U, H_n)$ since

$$\left\|G_{i,n}(v)\right\|^2_{\mathcal{L}^2(U_i,H_n)} = \|\Pi_n \circ G_i(v)\|^2_{\mathcal{L}^2(U_i,H_n)} \le \|G_i(v)\|^2_{\mathcal{L}^2(U_i,H)} \le C_{G_i}|D|. \tag{3.9}$$

A similar computation shows that $G_{i,n}$ is C_{G_i}-Lipschitz-continuous from H_n to $\mathcal{L}^2(U_i, H_n)$. Indeed, for every for every $v_1, v_2 \in H_n$ and every $n \in \mathbb{N}$ one has thanks to (A5) that

$$\left\|G_{i,n}(v_1) - G_{i,n}(v_2)\right\|^2_{\mathcal{L}^2(U_i,H_n)} \le \|G_i(v_1) - G_i(v_2)\|^2_{\mathcal{L}^2(U_i,H)} \le C_{G_i}\|v_1 - v_2\|^2_H. \tag{3.10}$$

Note that the estimates (3.9)–(3.10) are independent of the parameter n.

Using typical approximations via mollifiers, for $i \in \{1,2\}$, the mobility function m_i is approximated by smooth functions $m_{i,n}$ satisfying $m_0 \le m_{i,n}(r) \le m_\infty$ for any

$r \in \mathbb{R}$ and $m_{i,n} \to m_i$ in $C^0([a, b])$ as $n \to \infty$ for every $[a, b] \subset \mathbb{R}$. As for the initial data, we naturally define $\varphi_0^n := \Pi_n \varphi_0$ and $\sigma_0^n := \Pi_n \sigma_0$. Summarizing, we consider the Yosida system's Galerkin approximation

$$\begin{aligned} \mathrm{d}\varphi_{\lambda,n} &= \Pi_n \operatorname{div}(m_{1,n}(\varphi_{\lambda,n})(\nabla \mu_{\lambda,n} - \chi \nabla \sigma_{\lambda,n}))\,\mathrm{d}t + \Pi_n((\beta\sigma_{\lambda,n} - \alpha) f(\varphi_{\lambda,n}))\,\mathrm{d}t \\ &\quad + G_{1,n}(\varphi_{\lambda,n})\,\mathrm{d}W_1, \\ \mu_{\lambda,n} &= \Pi_n \Psi_\lambda'(\varphi_{\lambda,n}) - \varepsilon^2 \Delta\varphi_{\lambda,n}, \\ \mathrm{d}\sigma_{\lambda,n} &= \Pi_n \operatorname{div}(m_{2,n}(\sigma_{\lambda,n})(\nabla \sigma_{\lambda,n} - \chi \nabla \varphi_{\lambda,n}))\,\mathrm{d}t - \Pi_n(\delta\sigma_{\lambda,n} f(\varphi_{\lambda,n}))\,\mathrm{d}t \\ &\quad + G_{2,n}(\sigma_{\lambda,n})\,\mathrm{d}W_2, \end{aligned} \tag{3.11}$$

equipped with the initial values $\varphi_{\lambda,n}(0) = \varphi_0^n$ and $\sigma_{\lambda,n}(0) = \sigma_0^n$ in D, as well as no-flux boundary conditions. The variational formulation of (3.11) reads

$$\begin{aligned} &(\varphi_{\lambda,n}(t), h_n)_D + (m_{1,n}(\varphi_{\lambda,n})\nabla(\mu_{\lambda,n} - \chi\sigma_{\lambda,n}), \nabla h_n)_{D_t} \\ &\quad = (\varphi_0^n, h_n)_D + (\beta\sigma_{\lambda,n} - \alpha, f(\varphi_{\lambda,n})h_n)_{D_t} \\ &\qquad + \left(\textstyle\int_0^t G_{1,n}(\varphi_{\lambda,n}(s))\,\mathrm{d}W_1(s), h_n\right)_D, \\ &(\mu_{\lambda,n}(t), h_n)_D - (\Psi_\lambda'(\varphi_{\lambda,n}), h_n)_{D_t} - \varepsilon^2(\nabla\varphi_{\lambda,n}, \nabla h_n)_{D_t} = 0, \\ &(\sigma_{\lambda,n}(t), h_n)_D + (m_{2,n}(\sigma_{\lambda,n})\nabla(\sigma_{\lambda,n} - \chi\varphi_{\lambda,n}), \nabla h_n)_{D_t} \\ &\quad = (\sigma_0^n, h_n)_D - \delta(\sigma_{\lambda,n}, f(\varphi_{\lambda,n})h_n)_{D_t} \\ &\qquad + \left(\textstyle\int_0^t G_{2,n}(\sigma_{\lambda,n}(s))\,\mathrm{d}W_2(s), h_n\right)_D, \end{aligned} \tag{3.12}$$

for any $h_n \in H_n$, for every $t \in [0, T]$, $\mathbb{P}$-almost surely.

It is a standard procedure to show the existence of a solution $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ to the Galerkin system (3.11): this is done in detail e.g. in [48, p. 3813–3814] for the stochastic Cahn–Hilliard equation, so we omit here the technical details. The idea is to represent $\varphi_{\lambda,n}$, $\mu_{\lambda,n}$ and $\sigma_{\lambda,n}$ as the sum of the basis functions in H_n and reduce the problem to an abstract evolution equation in a finite-dimensional space. Then, by standard theory, it follows that the Galerkin system (3.11) admits a unique (probabilistically strong) solution

$$\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n} \in L^\ell\big(\Omega; C^0([0, T]; H_n)\big) \quad \forall\, \ell \in [2, \infty).$$

3.2 Step 2: uniform estimates in *n*, with *λ* fixed

In this step, we show that the approximated solution $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ satisfies an energy estimate, independently of n, with $\lambda > 0$ being fixed. Ultimately, this step ensures the existence of a solution to the λ-approximated system when passing to the limit $n \to \infty$ in the next step in Sect. 3.3.

Lemma 3.7 *There holds the estimate:*

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \left\|\varphi_{\lambda,n}(s)\right\|_V^\ell + \mathbb{E} \sup_{s\in[0,t]} \left\|\Psi_\lambda(\varphi_{\lambda,n}(s))\right\|_{L^1(D)}^{\ell/2} + \mathbb{E}\left\|\nabla\mu_{\lambda,n}\right\|_{L^2(0,t;H)}^\ell \\
&\quad + \mathbb{E}\sup_{s\in[0,t]} |(\mu_{\lambda,n}(s))_D|^{\ell/2} + \mathbb{E} \sup_{s\in[0,T]} \left\|\sigma_{\lambda,n}(s)\right\|_H^\ell + \mathbb{E}\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^\ell \\
&\quad \le c_\lambda\left(1 + \|\varphi_0\|_V^\ell + \|\sigma_0\|_H^\ell\right).
\end{aligned}
\tag{3.13}
$$

Proof We split the proof into four parts. Firstly, we derive a $\varphi_{\lambda,n}$-dependent estimate for $\sigma_{\lambda,n}$ by a suitable application of Itô's formula. Similarly, we derive a $\sigma_{\lambda,n}$-dependent estimate for $\varphi_{\lambda,n}$ and its space average. Lastly, we add the three estimates and by an application of Gronwall's inequality, we obtain a n-uniform estimate.

(i) We write Itô's formula (2.3) for $\left\|\sigma_{\lambda,n}\right\|_H^2$. This procedure yields the equation

$$
\begin{aligned}
&\left\|\sigma_{\lambda,n}(t)\right\|_H^2 + 2(m_{2,n}(\sigma_{\lambda,n}), |\nabla\sigma_{\lambda,n}|^2)_{D_t} + 2\delta(f(\varphi_{\lambda,n}), |\sigma_{\lambda,n}|^2)_{D_t} \\
&\quad = \left\|\sigma_0^n\right\|_H^2 + 2\chi\left(m_{2,n}(\sigma_{\lambda,n}(s))\nabla\sigma_{\lambda,n}(s), \nabla\varphi_{\lambda,n}(s)\right)_{D_t} \\
&\qquad + \left\|G_{2,n}(\sigma_{\lambda,n}(s))\right\|_{L^2(0,t;\mathcal{L}^2(U_2,H))}^2 + 2\int_0^t \left(\sigma_{\lambda,n}(s), G_{2,n}(\sigma_{\lambda,n}(s))\mathrm{d}W_2(s)\right)_D,
\end{aligned}
\tag{3.14}
$$

for every $t \in [0, T]$, $\mathbb{P}$-almost surely. On the left-hand side of the equality, we may use $m_{2,n}(\sigma_{\lambda,n}) \ge m_0$ and $f(\varphi_\lambda) \ge 0$, see the assumptions (A3) and (A2). On the right-hand side, we exploit the definition of the approximate initial value σ_0^n to obtain $\|\sigma_0^n\|_H \le \|\sigma_0\|_H$. Concerning the second term on the right-hand side, we have by the Young inequality and the bound of the mobility function, see (A3),

$$
\begin{aligned}
&2\chi\left(m_{2,n}(\sigma_{\lambda,n}(s))\nabla\sigma_{\lambda,n}(s), \nabla\varphi_{\lambda,n}(s)\right)_{D_t} \\
&\quad \le m_0\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2 + c\chi^2\|\nabla\varphi_{\lambda,n}\|_{L^2(0,t;H)}^2.
\end{aligned}
$$

We insert the estimates back into (3.14) to obtain

$$
\begin{aligned}
&\left\|\sigma_{\lambda,n}(t)\right\|_H^2 + m_0\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2 \\
&\quad \le \|\sigma_0\|_H^2 + c\chi^2\|\nabla\varphi_{\lambda,n}\|_{L^2(0,t;H)}^2 + \left\|G_{2,n}(\sigma_{\lambda,n}(s))\right\|_{L^2(0,t;\mathcal{L}^2(U_2,H))}^2 \\
&\qquad + 2\int_0^t \left(\sigma_{\lambda,n}(s), G_{2,n}(\sigma_{\lambda,n}(s))\,\mathrm{d}W_2(s)\right)_D.
\end{aligned}
\tag{3.15}
$$

At this point, by (3.9) one has that

$$
\left\|G_{2,n}(\sigma_{\lambda,n})\right\|_{L^\infty(\Omega\times(0,T);\mathcal{L}^2(U_2,H))}^2 \le c,
$$

which implies by the Burkholder–Davis–Gundy inequality, see (2.1), that for every $\ell \geq 2$,

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} \left| \int_0^s \big(\sigma_{\lambda,n}(r), G_{2,n}(\sigma_{\lambda,n}(r))\, \mathrm{d}W_2(r)\big)_D \right|^{\ell/2} \\
&\quad \leq c_\ell \mathbb{E} \left(\int_0^t \|\sigma_{\lambda,n}(s)\|_H^2 \|G_{2,n}(\sigma_{\lambda,n}(s))\|^2_{\mathcal{L}^2(U_1,H)}\, \mathrm{d}s \right)^{\ell/4} \\
&\quad \leq c_\ell \mathbb{E} \|\sigma_{\lambda,n}\|^{\ell/2}_{L^2(0,t;H)}.
\end{aligned}$$

Consequently, we take the power $\ell/2$, the supremum in time, and then the expectation $\mathbb{E}$ in inequality (3.15): we deduce that

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,T]} \big\|\sigma_{\lambda,n}(s)\big\|_H^\ell + m_0^{\ell/2} \mathbb{E} \|\nabla \sigma_{\lambda,n}\|^\ell_{L^2(0,t;H)} \\
&\leq c_\ell \Big(1 + \|\sigma_0\|_H^\ell + \chi^\ell \mathbb{E} \|\nabla \varphi_{\lambda,n}\|^\ell_{L^2(0,t;H)} + \mathbb{E} \big\|\sigma_{\lambda,n}\big\|^{\ell/2}_{L^2(0,t;H)} \Big).
\end{aligned} \tag{3.16}$$

Especially, we note that the constant c_ℓ is independent of λ and n.

(ii) Next, we derive a bound for $\mathbb{E} \sup_{s\in[0,T]} \big\|\nabla \varphi_{\lambda,n}(s)\big\|_H^\ell$ that allows us to absorb the $\nabla \varphi_{\lambda,n}$-dependency on the right-hand side of (3.16) by a Gronwall argument. We consider the energy functional

$$\mathcal{E}_\lambda(v) := \frac{\varepsilon^2}{2} \|\nabla v\|_H^2 + \|\Psi_\lambda(v)\|_{L^1(D)}, \quad v \in H_n. \tag{3.17}$$

As proved in [48, p. 3829], $\mathcal{E}_\lambda : H_n \to [0,\infty)$ is twice Fréchet differentiable with

$$\begin{aligned}
D\mathcal{E}_\lambda(v)[h] &= \varepsilon^2 (\nabla v, \nabla h)_D + (\Psi'_\lambda(v), h)_D \qquad \forall h \in H_n, \\
D^2\mathcal{E}_\lambda(v)[h,k] &= \varepsilon^2 (\nabla h, \nabla k)_D + (\Psi''_\lambda(v) h, k)_D \quad \forall h, k \in H_n.
\end{aligned}$$

We apply Itô's formula (2.3) to $\varphi_{\lambda,n} \mapsto \mathcal{E}_\lambda(\varphi_{\lambda,n})$. To this end, note that by (3.11) we have $D\mathcal{E}_\lambda(\varphi_{\lambda,n}) = \mu_{\lambda,n}$, and thus, we obtain

$$\begin{aligned}
&\mathcal{E}_\lambda(\varphi_{\lambda,n}(t)) + (m_{1,n}(\varphi_{\lambda,n}), |\nabla \mu_{\lambda,n}|^2)_{D_t} \\
&\quad = \mathcal{E}_\lambda(\varphi_0^n) + (\mu_{\lambda,n} f(\varphi_{\lambda,n}), \beta\sigma_{\lambda,n} - \alpha)_{D_t} + \chi (m_{1,n}(\varphi_{\lambda,n}) \nabla \sigma_{\lambda,n}, \nabla \mu_{\lambda,n})_{D_t} \\
&\qquad + \int_0^t \big(\mu_{\lambda,n}(s), G_{1,n}(\varphi_{\lambda,n}(s))\, \mathrm{d}W_1(s)\big)_D + \frac{1}{2} \int_0^t \sum_{k=0}^\infty \Big[\|\nabla G_{1,n}(\varphi_{\lambda,n}(s)) u_k^1\|_H^2 \\
&\qquad + \big(\Psi''_\lambda(\varphi_{\lambda,n}(s)), |G_{1,n}(\varphi_{\lambda,n}(s)) u_k^1|^2\big)_D \Big]\, \mathrm{d}s.
\end{aligned} \tag{3.18}$$

First, we note that it holds $\mathcal{E}_\lambda(\varphi_0^n) = \frac{\varepsilon^2}{2} \|\nabla \varphi_0^n\|_H^2 + \|\Psi_\lambda(\varphi_0^n)\|_{L^1(D)}$. From the definition of the approximate initial value φ_0^n, we have $\|\nabla \varphi_0^n\|_H \leq \|\nabla \varphi_0\|_H$. Since Ψ_λ is bounded by a quadratic function due to the properties of the Yosida approximation and $(\varphi_0^n)_n$ is bounded in H thanks to the properties of the orthogonal projection on H_n, we have that

$$\left\| \Psi_\lambda(\varphi_0^n) \right\|_{L^1(D)} \le c_\lambda \big(1 + \left\| \varphi_0^n \right\|_H^2 \big) \le c_\lambda \big(1 + \|\varphi_0\|_H^2 \big).$$

Now, note that taking the space average of the equation governing $\mu_{\lambda,n}$, see $(3.11)_2$, and using the Lipschitz-continuity of Ψ_λ', we get

$$(\mu_{\lambda,n})_D = (\Psi_\lambda'(\varphi_{\lambda,n}))_D \le c_\lambda \|\varphi_{\lambda,n}\|_H.$$

Hence, we may treat the second and third terms on the right-hand side of (3.18) by the Young and Poincaré–Wirtinger inequalities as follows:

$$\begin{aligned}
(\mu_{\lambda,n} f(\varphi_{\lambda,n}), \beta\sigma_{\lambda,n} - \alpha)_{D_t} &\le \frac{m_0}{4} \|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^2 \\
&\quad + c_\lambda \Big(1 + \|\varphi_{\lambda,n}\|_{L^2(0,t;H)}^2 + \|\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2\Big), \\
\chi(m_{1,n}(\varphi_{\lambda,n})\nabla\sigma_{\lambda,n}, \nabla\mu_{\lambda,n})_{D_t} &\le \frac{m_0}{4} \|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^2 \\
&\quad + c\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2,
\end{aligned}$$

where we used the bounds for f and the mobility $m_{1,n}$ as assumed in (A2) and (A3). We insert the estimates back into (3.18) to get

$$\begin{aligned}
&\frac{\varepsilon^2}{2} \|\nabla\varphi_{\lambda,n}(t)\|_H^2 + \|\Psi_\lambda(\varphi_{\lambda,n}(t))\|_{L^1(D)} + \frac{m_0}{2} \|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^2 \\
&\quad \le c_\lambda \Big(1 + \|\varphi_0\|_V^2 + \|\varphi_{\lambda,n}\|_{L^2(0,t;H)}^2 + \|\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2\Big) \\
&\qquad + c\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2 + \int_0^t \big(\mu_{\lambda,n}(s), G_{1,n}(\varphi_{\lambda,n}(s))\,\mathrm{d}W_1(s)\big)_D \\
&\qquad + \frac{1}{2}\int_0^t \sum_{k=0}^{\infty} \Big[\|\nabla G_{1,n}(\varphi_{\lambda,n}(s))u_k^1\|_H^2 + \big(\Psi_\lambda''(\varphi_{\lambda,n}(s)), |G_{1,n}(\varphi_{\lambda,n}(s))u_k^1|^2\big)_D\Big]\,\mathrm{d}s.
\end{aligned} \tag{3.19}$$

We take the power $\ell/2$ on both sides of this inequality, the supremum in time and then the expectation. Let us focus on the stochastic integral first: we apply the Burkholder–Davis–Gundy inequality and argue as before (3.16) by exploiting (A5) to obtain

$$\mathbb{E}\sup_{s\in[0,t]} \left| \int_0^s \big(\mu_{\lambda,n}(r), G_{1,n}(\varphi_{\lambda,n}(r))\,\mathrm{d}W_1(r)\big)_D \right|^{\ell/2} \le c\mathbb{E}\left\|\mu_{\lambda,n}\right\|_{L^2(0,t;H)}^{\ell/2}.$$

We sum and subtract $(\mu_{\lambda,n})_D$ on the right-hand side, apply the Poincaré–Wirtinger and Young inequalities, which yields

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \big(\mu_{\lambda,n}(r), G_{1,n}(\varphi_{\lambda,n}(r))\,\mathrm{d}W_1(r)\big)_D\right|^{\ell/2}\\&\quad\le \widehat{\delta}\mathbb{E}\left\|\nabla\mu_{\lambda,n}\right\|^{\ell}_{L^2(0,t;H)}+c_\ell\Big(1+\mathbb{E}\left\|(\mu_{\lambda,n})_D\right\|^{\ell/2}_{L^2(0,t)}\Big),\end{aligned}$$

where $\widehat{\delta}>0$ is determined below. Taking the space average of the equation governing $\mu_{\lambda,n}$, see $(3.11)_2$, and using the assumption on the potential Ψ, see (A4) and (3.7), we get

$$(\mu_{\lambda,n})_D=(\Psi_\lambda'(\varphi_{\lambda,n}))_D\le\left\|\Psi_\lambda'(\varphi_{\lambda,n})\right\|_{L^1(D)}\le c\big(1+\left\|\Psi_\lambda(\varphi_{\lambda,n})\right\|_{L^1(D)}+\left\|\varphi_{\lambda,n}\right\|^2_H\big), \tag{3.20}$$

so that, putting everything together,

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \big(\mu_{\lambda,n}(r), G_{1,n}(\varphi_{\lambda,n}(r))\,\mathrm{d}W_1(r)\big)_H\right|^{\ell/2}\\&\quad\le \widehat{\delta}\mathbb{E}\left\|\nabla\mu_{\lambda,n}\right\|^{\ell}_{L^2(0,t;H)}+c_\ell\Big(1+\mathbb{E}\int_0^t\Big(\left\|\Psi_\lambda(\varphi_{\lambda,n}(s))\right\|^{\ell/2}_{L^1(D)}+\left\|\varphi_{\lambda,n}(s)\right\|^{\ell}_H\Big)\,\mathrm{d}s\Big).\end{aligned}$$

Let us focus now on the trace terms in Itô's formula (3.19). Since $G_1(\varphi_{\lambda,n})$ takes its values in $\mathcal{L}^2(U_1,V)$, thanks to (A5) we have

$$\begin{aligned}\int_0^t\sum_{k=0}^\infty\|\nabla G_{1,n}(\varphi_{\lambda,n}(s))u_k^1\|_H^2\,\mathrm{d}s&\le\int_0^t\sum_{k=0}^\infty\|g_{1,k}'\|^2_{L^\infty(\mathbb{R})}\|\nabla\varphi_{\lambda,n}(s)\|^2_H\,\mathrm{d}s\\&\le C_{G_1}\|\nabla\varphi_{\lambda,n}\|^2_{L^2(0,t;H)}\end{aligned}$$

Lastly, we note that it holds $|\Psi_\lambda''|\le c_\lambda$ for a certain $c_\lambda>0$ and thus, by the same computations as above, we obtain

$$\sum_{k=0}^\infty\big(\Psi_\lambda''(\varphi_{\lambda,n}),|G_{1,n}(\varphi_{\lambda,n})u_k^1|^2\big)_D\le c_\lambda\left\|G_1(\varphi_{\lambda,n})\right\|^2_{\mathcal{L}^2(U_1,H)}\le c_\lambda|D|C_{G_1}.$$

Putting everything together, we finally obtain the estimate

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left\|\nabla\varphi_{\lambda,n}(s)\right\|^{\ell}_H+\mathbb{E}\sup_{s\in[0,t]}\left\|\Psi_\lambda(\varphi_{\lambda,n}(s))\right\|^{\ell/2}_{L^1(D)}\\&\quad+\Big(\frac{m_0^{\ell/2}}{2^{\ell/2}}-\widehat{\delta}\Big)\mathbb{E}\left\|\nabla\mu_{\lambda,n}\right\|^{\ell}_{L^2(0,t;H)}+\mathbb{E}\left\|(\mu_{\lambda,n}(s))_D\right\|^{\ell/2}_{L^2(0,t)}\end{aligned}$$

$$
\begin{aligned}
&\leq c_\lambda\Big(1+\|\varphi_0\|_V^\ell+\mathbb{E}\int_0^t\Big(\big\|\Psi_\lambda(\varphi_{\lambda,n}(s))\big\|_{L^1(D)}^{\ell/2}+\big\|\varphi_{\lambda,n}(s)\big\|_H^\ell\Big)\,ds\Big)\\
&\quad+c_\lambda\Big(\mathbb{E}\|\sigma_{\lambda,n}\|_{L^2(0,t;H)}^\ell+\mathbb{E}\|\varphi_{\lambda,n}\|_{L^2(0,t;V)}^\ell\Big)+c_\ell\mathbb{E}\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^\ell, \qquad (3.21)
\end{aligned}
$$

where we obtained the last term on the left-hand side by the estimate (3.20).
(iii) We focus here on the space average of $\varphi_{\lambda,n}$. We choose $h_n = 1$ in $(3.12)_1$, integrate in time, and apply Itô's formula, see (2.3), with the function $x \mapsto |x|^2$, which yields

$$
\begin{aligned}
|(\varphi_{\lambda,n}(t))_D|^2 &= |(\varphi_0^n)_D|^2+2(\beta\sigma_{\lambda,n}-\alpha,(\varphi_{\lambda,n})_D f(\varphi_{\lambda,n}))_{D_t}\\
&\quad+\big\|(G_{1,n}(\varphi_{\lambda,n}))_D\big\|_{L^2(0,t;\mathcal{L}^2(U_1,\mathbb{R}))}^2\\
&\quad+2\int_0^t(\varphi_{\lambda,n}(s))_D\,(G_{1,n}(\varphi_{\lambda,n}(s)))_D\,\mathrm{d}W_1(s).
\end{aligned}
$$

Now, it holds by the Hölder inequality that $|(\varphi_0^n)_D| \leq c\,\|\varphi_0^n\|_H \leq c\,\|\varphi_0\|_H$ where we used the definition of the initial φ_0^n in the last step. Moreover, thanks to (A2) one has that

$$
(\beta\sigma_{\lambda,n}-\alpha,(\varphi_{\lambda,n})_D f(\varphi_{\lambda,n}))_{D_t} \leq c\big(1+\|\varphi_{\lambda,n}\|_{L^2(0,t;H)}^2+\|\sigma_{\lambda,n}\|_{L^2(0,t;H)}^2\big).
$$

Furthermore, by (3.9) we readily have

$$
\big\|(G_{1,n}(\varphi_{\lambda,n}))_D\big\|_{L^2(0,t;\mathcal{L}^2(U_1,\mathbb{R}))}^2 \leq c\,\big\|G_{1,n}(\varphi_{\lambda,n})\big\|_{L^2(0,T;\mathcal{L}^2(U_1,H))}^2 \leq c.
$$

This implies by the Burkholder–Davis–Gundy inequality, see (2.1), and the Poincaré–Wirtinger inequality, that for every $\ell \geq 2$

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\left|\int_0^s(\varphi_{\lambda,n}(r))_D\,(G_{1,n}(\varphi_{\lambda,n}(r)))_D\,\mathrm{d}W_1(r)\right|^{\ell/2}\\
&\quad\leq c\mathbb{E}\left(\int_0^t\|\varphi_{\lambda,n}(s)\|_H^2\|G_{1,n}(\varphi_{\lambda,n}(s))\|_{\mathcal{L}^2(U_1,H)}^2\,\mathrm{d}s\right)^{\ell/4}\\
&\quad\leq c\mathbb{E}\|\varphi_{\lambda,n}\|_{L^2(0,t;H)}^{\ell/2}.
\end{aligned}
$$

Putting this information together, taking supremum in time, power $\ell/2$ and expectations, yield that

$$
\mathbb{E}\sup_{s\in[0,t]}|(\varphi_{\lambda,n}(s))_D|^\ell \leq c\left(1+\mathbb{E}\|\varphi_{\lambda,n}\|_{L^2(0,t;H)}^\ell+\mathbb{E}\|\sigma_{\lambda,n}\|_{L^2(0,t;H)}^\ell+\mathbb{E}\|\varphi_{\lambda,n}\|_{L^2(0,t;H)}^{\ell/2}\right). \qquad (3.22)
$$

(iv) Next, we add the three estimates to obtain a new estimate, absorb the terms and apply Gronwall's lemma to get a n-uniform estimate. Indeed, we multiply (3.16) by a sufficiently large constant $K_\ell > 0$, possibly depending on ℓ, and add it to (3.21) and (3.22), which yields

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}|(\varphi_{\lambda,n}(s))_D|^\ell + \frac{\varepsilon^\ell}{2^{\ell/2}}\mathbb{E}\sup_{s\in[0,t]}\left\|\nabla\varphi_{\lambda,n}(s)\right\|_H^\ell + \mathbb{E}\sup_{s\in[0,t]}\left\|\Psi_\lambda(\varphi_{\lambda,n}(s))\right\|_{L^1(D)}^{\ell/2} \\
&\quad + \Big(\frac{m_0^{\ell/2}}{2^{\ell/2}} - \widehat{\delta}\Big)\mathbb{E}\left\|\nabla\mu_{\lambda,n}\right\|_{L^2(0,t;H)}^\ell + \mathbb{E}\sup_{s\in[0,t]}|(\mu_{\lambda,n}(s))_D|^{\ell/2} \\
&\quad + K_\ell\mathbb{E}\sup_{s\in[0,T]}\left\|\sigma_{\lambda,n}(s)\right\|_H^\ell + \Big(K_\ell - c_\ell\Big)\mathbb{E}\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^\ell \\
&\leq c_\lambda(1 + \|\varphi_0\|_V^\ell + \|\sigma_0\|_H^\ell) \\
&\quad + c_\lambda\mathbb{E}\int_0^t\Big(\left\|\Psi_\lambda(\varphi_{\lambda,n}(s))\right\|_{L^1(D)}^{\ell/2} + \|\sigma_{\lambda,n}(s)\|_H^\ell + \|\varphi_{\lambda,n}(s)\|_V^\ell\Big)\,\mathrm{d}s.
\end{aligned}$$

At this point, we select K_ℓ large enough and $\widehat{\delta}$ small enough to ensure that the prefactors of $\mathbb{E}\|\nabla\sigma_{\lambda,n}\|_{L^2(0,t;H)}^\ell$ and $\mathbb{E}\|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^\ell$ are positive. Hence, the Gronwall lemma and the Poincaré–Wirtinger inequality yield (3.13). □

Now, by (3.13) we infer the n-uniform bounds

$$\left\|\varphi_{\lambda,n}\right\|_{L^\ell(\Omega;C^0([0,T];V))} \leq c_\lambda, \tag{3.23}$$

$$\left\|\mu_{\lambda,n}\right\|_{L^{\ell/2}(\Omega;L^2(0,T;V))} + \left\|\nabla\mu_{\lambda,n}\right\|_{L^\ell(\Omega;L^2(0,T;H))} \leq c_\lambda, \tag{3.24}$$

$$\left\|\sigma_{\lambda,n}\right\|_{L^\ell(\Omega;C^0([0,T];H)\cap L^2(0,T;V))} \leq c_\lambda. \tag{3.25}$$

Moreover, the Lipschitz continuity of $G_{i,n}$ and (A5) yield

$$\begin{aligned}
&\left\|G_{1,n}(\varphi_{\lambda,n})\right\|_{L^\infty(\Omega\times(0,T);\mathcal{L}^2(U_1,H))\cap L^\ell(\Omega;L^\infty(0,T;\mathcal{L}^2(U_1,V)))} \leq c_\lambda, \\
&\left\|G_{2,n}(\sigma_{\lambda,n})\right\|_{L^\infty(\Omega\times(0,T);\mathcal{L}^2(U_2,H))\cap L^\ell(\Omega;L^\infty(0,T;\mathcal{L}^2(U_2,V)))} \leq c_\lambda.
\end{aligned}$$

This in turn implies, by [16, Lemma 2.1], that for every $s \in (0, 1/2)$

$$\begin{aligned}
&\left\|\int_0^\cdot G_{1,n}(\varphi_{\lambda,n}(s))\,\mathrm{d}W_1(s)\right\|_{L^\ell(\Omega;W^{s,\ell}(0,T;V))} \leq c_{\lambda,s}, \\
&\left\|\int_0^\cdot G_{2,n}(\sigma_{\lambda,n}(s))\,\mathrm{d}W_2(s)\right\|_{L^\ell(\Omega;W^{s,\ell}(0,T;V))} \leq c_{\lambda,s}.
\end{aligned}$$

Let us fix now $\bar{s} \in (1/\ell, 1/2)$: by comparison with the model equations, we directly obtain then

$$\left\|\varphi_{\lambda,n}\right\|_{L^\ell(\Omega;L^2(0,T;V_2)\cap W^{\bar{s},\ell}(0,T;V^*))} \leq c_\lambda, \tag{3.26}$$

$$\left\|\sigma_{\lambda,n}\right\|_{L^\ell(\Omega;W^{\bar{s},\ell}(0,T;V^*))} \leq c_\lambda. \tag{3.27}$$

3.3 Step 3: passage to the limit as $n \to \infty$, with λ fixed

We perform here the passage to the limit as $n \to \infty$, keeping $\lambda > 0$ fixed.

Lemma 3.8 *The sequence of laws of $(\varphi_{\lambda,n}, G_{1,n}(\varphi_{\lambda,n}) \cdot W_1, W_1)_{n\in\mathbb{N}}$ is tight on the product space*

$$\Big(C^0([0,T]; H) \cap L^2(0,T; V)\Big) \times C^0([0,T]; H) \times C^0([0,T]; \widetilde{U}_1),$$

and the sequence of laws of $(\sigma_{\lambda,n}, G_{2,n}(\sigma_{\lambda,n}) \cdot W_2, W_2)_{n\in\mathbb{N}}$ is tight on the product space

$$\Big(C^0([0,T]; V^*) \cap L^2(0,T; H)\Big) \times C^0([0,T]; H) \times C^0([0,T]; \widetilde{U}_2).$$

Proof Since it holds $\bar{s} > 1/\ell$, the Aubin–Lions compactness lemma, see [50, Cor. 5, p. 86], implies the compact inclusions

$$L^\infty(0,T; V) \cap W^{\bar{s},\ell}(0,T; V^*) \hookrightarrow\hookrightarrow C^0([0,T]; H),$$
$$L^2(0,T; V_2) \cap W^{\bar{s},\ell}(0,T; V^*) \hookrightarrow\hookrightarrow L^2(0,T; V).$$

Hence, for every $R > 0$ the closed ball B_R in $L^2(0,T; V_2) \cap L^\infty(0,T; V) \cap W^{\bar{s},\ell}(0,T; V^*)$ of radius R is compact in $C^0([0,T]; H) \cap L^2(0,T; V)$. Moreover, thanks to the Markov inequality and the estimates (3.23) and (3.26) we have

$$\begin{aligned}\mathbb{P}\{\varphi_{\lambda,n} \in B_R^c\} &= \mathbb{P}\{\|\varphi_{\lambda,n}\|_{L^2(0,T;V_2)\cap L^\infty(0,T;V)\cap W^{\bar{s},\ell}(0,T;V^*)} > R\} \\ &\le \frac{1}{R^\ell}\mathbb{E}\,\|\varphi_{\lambda,n}\|^\ell_{L^2(0,T;V_2)\cap L^\infty(0,T;V)\cap W^{\bar{s},\ell}(0,T;V^*)} \le \frac{c_\lambda^\ell}{R^\ell},\end{aligned}$$

which yields $\lim_{R\to\infty}\sup_{n\in\mathbb{N}} \mathbb{P}\{\varphi_{\lambda,n} \in B_R^c\} = 0$. Hence, the family of laws of $(\varphi_{\lambda,n})_n$ on $C^0([0,T]; H) \cap L^2(0,T; V)$ is tight. Using a similar argument, since $W^{\bar{s},\ell}(0,T; V)$ is compactly embedded in $C^0([0,T]; H)$, one can also show that the family of laws of $G_{1,n}(\varphi_{\lambda,n}) \cdot W_1 := \int_0^\cdot G_{1,n}(\varphi_{\lambda,n}(s))\, \mathrm{d}W_1(s)$ is tight on $C^0([0,T]; H)$. Eventually, we identify W_1 with a constant sequence of random variables with values in $C^0([0,T]; \widetilde{U}_1)$, which is tight. This proves the first assertion of the lemma. The second assertion follows analogously by the compact inclusions

$$L^\infty(0,T; H) \cap W^{\bar{s},\ell}(0,T; V^*) \hookrightarrow\hookrightarrow C^0([0,T]; V^*),$$
$$L^2(0,T; V) \cap W^{\bar{s},\ell}(0,T; V^*) \hookrightarrow\hookrightarrow L^2(0,T; H),$$

and this concludes the proof. □

Finally, we are in the position to prove Lemma 3.6, i.e., we prove that the Yosida approximated system admits a martingale solution.

Proof of Lemma 3.6 By the Prokhorov and Skorokhod theorems and their weaker version on sub-Polish spaces, see the discussion at the end of Sect. 2.2, recalling the estimates (3.23)–(3.26), we conclude that there exists a probability space $(\widetilde{\Omega}_\lambda, \widetilde{\mathcal{F}}_\lambda, \widetilde{\mathbb{P}}_\lambda)$ and sequences of measurable processes $(\widetilde{\varphi_{\lambda,n}}, \widetilde{\mu_{\lambda,n}}, \widetilde{\sigma_{\lambda,n}})$ with the same law as their counterpart $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ such that, passing to a further subsequence, still denoted with the index n for brevity of notation, it holds, as $n \to \infty$,

$$
\begin{aligned}
\widetilde{\varphi_{\lambda,n}} &\to \widetilde{\varphi_\lambda} && \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T];H) \cap L^2(0,T;V)) \quad \forall\, p < \ell,\\
\widetilde{\varphi_{\lambda,n}} &\to \widetilde{\varphi_\lambda} && \text{in } C^0([0,T];H) \cap L^2(0,T;V) \quad \text{a.e. in } \widetilde{\Omega}_\lambda,\\
\widetilde{\varphi_{\lambda,n}} &\overset{*}{\rightharpoonup} \widetilde{\varphi_\lambda} && \text{in } L^\ell_w(\widetilde{\Omega}_\lambda; L^\infty(0,T;V)),\\
\widetilde{\varphi_{\lambda,n}} &\rightharpoonup \widetilde{\varphi_\lambda} && \text{in } L^\ell(\widetilde{\Omega}_\lambda; L^2(0,T;V_2)) \cap L^\ell(\widetilde{\Omega}_\lambda; W^{\bar{s},\ell}(0,T;V^*)),\\
\widetilde{\mu_{\lambda,n}} &\rightharpoonup \widetilde{\mu_\lambda} && \text{in } L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;V)),\\
\nabla\widetilde{\mu_{\lambda,n}} &\rightharpoonup \nabla\widetilde{\mu_\lambda} && \text{in } L^\ell(\widetilde{\Omega}_\lambda; L^2(0,T;H)),\\
\widetilde{\sigma_{\lambda,n}} &\to \widetilde{\sigma_\lambda} && \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T];V^*) \cap L^2(0,T;H)) \quad \forall\, p < \ell,\\
\widetilde{\sigma_{\lambda,n}} &\to \widetilde{\sigma_\lambda} && \text{in } C^0([0,T];V^*) \cap L^2(0,T;H) \quad \text{a.e. in } \widetilde{\Omega}_\lambda,\\
\widetilde{\sigma_{\lambda,n}} &\overset{*}{\rightharpoonup} \widetilde{\sigma_\lambda} && \text{in } L^\ell_w(\widetilde{\Omega}_\lambda; L^\infty(0,T;H)),\\
\widetilde{\sigma_{\lambda,n}} &\rightharpoonup \widetilde{\sigma_\lambda} && \text{in } L^\ell(\widetilde{\Omega}_\lambda; L^2(0,T;V)) \cap L^\ell(\widetilde{\Omega}_\lambda; W^{\bar{s},\ell}(0,T;V^*)),
\end{aligned}
$$

for some measurable processes

$$
\begin{aligned}
&\widetilde{\varphi_\lambda} \in L^\ell(\widetilde{\Omega}_\lambda; W^{\bar{s},\ell}(0,T;V^*) \cap C^0([0,T];H) \cap L^2(0,T;V_2))\\
&\quad \cap L^\ell_w(\widetilde{\Omega}_\lambda; L^\infty(0,T;V)),\\
&\widetilde{\mu_\lambda} \in L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;V)), \quad \nabla\widetilde{\mu_\lambda} \in L^\ell(\widetilde{\Omega}_\lambda; L^2(0,T;H)),\\
&\widetilde{\sigma_\lambda} \in L^\ell(\widetilde{\Omega}_\lambda; W^{\bar{s},\ell}(0,T;V^*) \cap C^0([0,T];V^*) \cap L^2(0,T;V))\\
&\quad \cap L^\ell_w(\widetilde{\Omega}_\lambda; L^\infty(0,T;H)).
\end{aligned}
$$

By the same reasoning as above, there are sequences of measurable processes $\widetilde{W}_{1,n}$, $\widetilde{W}_{2,n}$, $\widetilde{I}^1_{\lambda,n}$ and $\widetilde{I}^2_{\lambda,n}$ with the same law as $W_{1,n}$, $W_{2,n}$, $G_{1,n}(\varphi_{\lambda,n}) \cdot W_1$, $G_{2,n}(\sigma_{\lambda,n}) \cdot W_2$, respectively, such that it holds

$$
\begin{aligned}
\widetilde{I}^1_{\lambda,n} &\to \widetilde{I}^1_\lambda && \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T];H)) \quad \forall\, p < \ell,\\
\widetilde{I}^2_{\lambda,n} &\to \widetilde{I}^2_\lambda && \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T];H)) \quad \forall\, p < \ell,\\
\widetilde{W}_{1,n} &\to \widetilde{W}_{1,\lambda} && \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T];\widetilde{U}_1)) \quad \forall\, p < \ell,\\
\widetilde{W}_{2,n} &\to \widetilde{W}_{2,\lambda} && \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T];\widetilde{U}_2)) \quad \forall\, p < \ell,
\end{aligned}
$$

with $\widetilde{I}^1_\lambda, \widetilde{I}^2_\lambda \in L^\ell(\widetilde{\Omega}_\lambda; C^0([0,T];H))$, $\widetilde{W}_{1,\lambda} \in L^\ell(\widetilde{\Omega}_\lambda; C^0([0,T];\widetilde{U}_1))$ and $\widetilde{W}_{2,\lambda} \in L^\ell(\widetilde{\Omega}_\lambda; C^0([0,T];\widetilde{U}_2))$. Now, since Ψ'_λ and $G_i : H \to \mathcal{L}^2(U_i, H)$ are Lipschitz-continuous, we readily have

$$\begin{aligned}
\Psi'_\lambda(\widetilde{\varphi_{\lambda,n}}) &\to \Psi'_\lambda(\widetilde{\varphi_\lambda}) \quad \text{in } L^p(\widetilde{\Omega}_\lambda; L^2(0,T;H)),\\
G_{1,n}(\widetilde{\varphi_{\lambda,n}}) &\to G_1(\widetilde{\varphi_\lambda}) \quad \text{in } L^p(\widetilde{\Omega}_\lambda; C^0([0,T]; \mathcal{L}^2(U_1,H))) \quad \forall\, p<\ell,\\
G_{2,n}(\widetilde{\sigma_{\lambda,n}}) &\to G_2(\widetilde{\sigma_\lambda}) \quad \text{in } L^p(\widetilde{\Omega}_\lambda; L^2(0,T; \mathcal{L}^2(U_2,H))) \quad \forall\, p<\ell,
\end{aligned}$$

Moreover, since $\varphi_0 \in V$ and $\sigma_0 \in H$, it implies $\varphi_0^n \to \varphi_0$ in V and $\sigma_0^n \to \sigma_0$ in H. By defining now the limiting filtration as

$$\widetilde{\mathcal{F}}_{\lambda,t} := \sigma\left\{(\widetilde{\varphi_\lambda}(s), \widetilde{\sigma_\lambda}(s), W_{1,\lambda}(s), W_{2,\lambda}(s), \widetilde{I}^1_\lambda(s), \widetilde{I}^1_\lambda(s)) : \ s \in [0,t]\right\}, \ t \in [0,T],$$

similarly to [48], a standard procedure allows identifying the limit terms $\widetilde{I}^1_\lambda$ and $\widetilde{I}^2_\lambda$ as the H-valued martingales given by

$$\widetilde{I}^1_\lambda = \int_0^\cdot G_1(\widetilde{\varphi_\lambda}(s))\,\mathrm{d}\widetilde{W}_{1,\lambda}(s), \quad \widetilde{I}^2_\lambda = \int_0^\cdot G_2(\widetilde{\sigma_\lambda}(s))\,\mathrm{d}\widetilde{W}_{2,\lambda}(s). \tag{3.28}$$

We test now each equation in (3.11) by an arbitrary element $v \in V$ and apply Λ_n. Taking advantage of the above convergences and the fact that, for $i = 1, 2$, $m_{i,n}(\widetilde{\varphi_{\lambda,n}}) \to m_i(\widetilde{\varphi_\lambda})$ almost everywhere thanks to the estimate $|m_{i,n}| \le m^*$ and the dominated convergence theorem, we can let $n \to \infty$ and conclude the proof of Lemma 3.6. □

3.4 Step 4: uniform estimates in λ

We prove here uniform estimates independently of λ. To this end, we start with the following lemma.

Lemma 3.9 *There holds the estimate:*

$$\begin{aligned}
&\widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]} \|\widetilde{\varphi_\lambda}(s)\|_V^\ell + \widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]} \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}^{\ell/2} + \widetilde{\mathbb{E}}_\lambda \|\nabla\mu_\lambda\|^\ell_{L^2(0,t;H)}\\
&\quad + \widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]} |(\widetilde{\mu_\lambda}(s))_D|^{\ell/2} + \widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,T]} \|\widetilde{\sigma_\lambda}(s)\|_H^\ell + \widetilde{\mathbb{E}}_\lambda \|\nabla\widetilde{\sigma_\lambda}\|^\ell_{L^2(0,t;H)}\\
&\le c\Big(1 + \|\varphi_0\|_V^\ell + \|\Psi(\varphi_0)\|_{L^1(D)}^{\ell/2} + \|\sigma_0\|_H^\ell\Big).
\end{aligned} \tag{3.29}$$

Proof Let us go back to (3.18). Since $(\mu_{\lambda,n})_D \le \|\Psi'_\lambda(\varphi_{\lambda,n})\|_{L^1(D)}$, on the right-hand side one has, by the Poincaré–Wirtinger and Hölder inequalities

$$\begin{aligned}
(\mu_{\lambda,n} f(\varphi_{\lambda,n}), \beta\sigma_{\lambda,n} - \alpha)_{D_t} &\le \widetilde{\delta}\|\nabla\mu_{\lambda,n}\|^2_{L^2(0,t;H)} + c\big(1 + \|\sigma_{\lambda,n}\|^2_{L^2(0,t;H)}\big)\\
&\quad + c\int_0^t \|\Psi'_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(D)}\big(1 + \|\sigma_{\lambda,n}(s)\|_{L^1(D)}\big)\,\mathrm{d}s,
\end{aligned}$$

as well as

$$\chi(m_{1,n}(\varphi_{\lambda,n})\nabla\sigma_{\lambda,n}, \nabla\mu_{\lambda,n})_{D_t} \leq \widetilde{\delta}\|\nabla\mu_{\lambda,n}\|^2_{L^2(0,t;H)} + c\|\nabla\sigma_{\lambda,n}\|^2_{L^2(0,t;H)},$$

for some small parameter $\widetilde{\delta} > 0$. Analogously, we have thanks to (A5), (A4), and (3.7) that

$$\begin{aligned}\int_0^t \sum_{k=0}^{\infty} \|\nabla G_{1,n}(\varphi_{\lambda,n}(s))u_k^1\|_H^2 \,\mathrm{d}s &= \int_0^t \sum_{k=0}^{\infty} \|g'_{1,k}\|^2_{L^\infty(\mathbb{R})}\|\nabla\varphi_{\lambda,n}(s)\|_H^2 \,\mathrm{d}s \\ &\leq C_{G_1}\|\nabla\varphi_{\lambda,n}\|^2_{L^2(0,t;H)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \sum_{k=0}^{\infty} \left(\Psi''_\lambda(\varphi_{\lambda,n}(s)), |G_{1,n}(\varphi_{\lambda,n}(s))u_k^1|^2\right)_D \mathrm{d}s \\ &\quad\leq c\int_0^t (1 + \|\Psi_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(D)} + \left\|\varphi_{\lambda,n}(s)\right\|_H^2)\,\mathrm{d}s.\end{aligned}$$

Taking this information into account in (3.18) we deduce that

$$\begin{aligned}&\frac{\varepsilon^2}{2}\|\nabla\varphi_{\lambda,n}(t)\|_H^2 + \|\Psi_\lambda(\varphi_{\lambda,n}(t))\|_{L^1(D)} + (m_0 - 2\widetilde{\delta})\|\nabla\mu_{\lambda,n}\|^2_{L^2(0,t;H)} \\ &\quad\leq c + \frac{\varepsilon^2}{2}\|\nabla\varphi_0\|_H^2 + \|\Psi_\lambda(\varphi_0^n)\|_{L^1(D)} + c\|\nabla\sigma_{\lambda,n}\|^2_{L^2(0,t;H)} + c\|\sigma_{\lambda,n}\|^2_{L^2(0,t;H)} \\ &\qquad + c\|\varphi_{\lambda,n}\|^2_{L^2(0,t;V)} + c\int_0^t \|\Psi'_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(D)}(1 + \|\sigma_{\lambda,n}(s)\|_H)\,\mathrm{d}s \\ &\qquad + c\int_0^t (1 + \|\Psi_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(D)})\,\mathrm{d}s + \int_0^t \left(\mu_{\lambda,n}(r), G_{1,n}(\varphi_{\lambda,n}(r))\,\mathrm{d}W_1(r)\right)_D.\end{aligned}$$

Furthermore, from (3.15) one also has

$$\begin{aligned}K\left\|\sigma_{\lambda,n}(t)\right\|_H^2 + Km_0\|\nabla\sigma_{\lambda,n}\|^2_{L^2(0,t;H)} &\leq c + K\|\sigma_0\|_H^2 + c\chi^2\|\nabla\varphi_{\lambda,n}\|^2_{L^2(0,t;H)} \\ &\quad + 2K\int_0^t \left(\sigma_{\lambda,n}(r), G_{2,n}(\sigma_{\lambda,n}(r))\,\mathrm{d}W_2(r)\right)_D,\end{aligned} \tag{3.30}$$

for some free-to-choose constant $K > 0$. Summing up the two inequalities and rearranging the terms we infer then

$$\begin{aligned}&\frac{\varepsilon^2}{2}\|\nabla\varphi_{\lambda,n}(t)\|_H^2 + \|\Psi_\lambda(\varphi_{\lambda,n}(t))\|_{L^1(D)} + (m_0 - \widetilde{\delta})\|\nabla\mu_{\lambda,n}\|^2_{L^2(0,t;H)} \\ &\qquad + K\left\|\sigma_{\lambda,n}(t)\right\|_H^2 + (Km_0 - c)\|\nabla\sigma_{\lambda,n}\|^2_{L^2(0,t;H)}\end{aligned}$$

$$
\begin{aligned}
&\leq c + \frac{\varepsilon^2}{2}\|\nabla\varphi_0\|_H^2 + \|\Psi_\lambda(\varphi_0^n)\|_{L^1(D)} + K\,\|\sigma_0\|_H^2 + c\|\sigma_{\lambda,n}\|^2_{L^2(0,t;H)} \\
&\quad + c\|\varphi_{\lambda,n}\|^2_{L^2(0,t;V)} + c\int_0^t \|\Psi'_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(D)}(1+\|\sigma_{\lambda,n}(s)\|_H)\,\mathrm{d}s \\
&\quad + c\int_0^t \|\Psi_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(D)}\,\mathrm{d}s + \int_0^t \big(\mu_{\lambda,n}(r), G_{1,n}(\varphi_{\lambda,n}(r))\,\mathrm{d}W_1(r)\big)_D \\
&\quad + 2K\int_0^t \big(\sigma_{\lambda,n}(r), G_{2,n}(\sigma_{\lambda,n}(r))\,\mathrm{d}W_2(r)\big)_D\,,
\end{aligned}
$$

where $K > 0$ and $\widetilde{\delta} > 0$ are chosen such that the prefactors on the left-hand side are positive. Now, we would like to transfer the above inequality on the novel probability space $(\widetilde{\Omega}_\lambda, \widetilde{\mathcal{F}}_\lambda, \widetilde{\mathbb{P}}_\lambda)$ obtained by the Prokhorov and Skorokhod theorems in the proof of Lemma 3.6 and to preserve the inequality as $n \to \infty$. To this end, we note that all the terms on the left-hand side can be handled by using the weak convergences obtained in the proof of Lemma 3.6 together with the weak lower semicontinuity. On the right-hand side, in the terms referring to the initial data and the deterministic integrals one can pass to the limit by using the strong convergences in the proof of Lemma 3.6. Let us show that one can pass to the limit also in the stochastic integrals. To this end, note that since $\widetilde{\mu_{\lambda,n}} \rightharpoonup \widetilde{\mu_\lambda}$ in $L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;V))$ and $G_{1,n}(\widetilde{\varphi_{\lambda,n}}) \to G_1(\widetilde{\varphi_\lambda})$ in $L^p(\widetilde{\Omega}_\lambda; C^0([0,T]; \mathcal{L}^2(U_1, H)))$ for every $\ell \geq 2$ and $p < \ell$, one has in particular that

$$
(\widetilde{\mu_{\lambda,n}}, G_{1,n}(\widetilde{\varphi_{\lambda,n}}))_D \to (\widetilde{\mu_\lambda}, G_1(\widetilde{\varphi_\lambda}))_D \quad \text{in } L^2(\widetilde{\Omega}_\lambda; L^2(0,T; \mathcal{L}^2(U_1, \mathbb{R}))),
$$

so that, arguing as in the proof of Lemma 3.6,

$$
\int_0^\cdot \big(\widetilde{\mu_{\lambda,n}}(r), G_{1,n}(\widetilde{\varphi_{\lambda,n}}(r))\mathrm{d}\widetilde{W}_{1,n}(r)\big)_D \to \int_0^\cdot \big(\widetilde{\mu_\lambda}(r), G_1(\widetilde{\varphi_\lambda}(r))\mathrm{d}\widetilde{W}_{1,\lambda}(r)\big)_D
$$

in $L^2(\widetilde{\Omega}_\lambda; C^0([0,T]))$. Analogously, since $\widetilde{\sigma_{\lambda,n}} \overset{*}{\rightharpoonup} \widetilde{\sigma_\lambda}$ in $L^p_w(\widetilde{\Omega}_\lambda; L^\infty(0,T;H))$ and $G_{2,n}(\widetilde{\sigma_{\lambda,n}}) \to G_2(\widetilde{\sigma_\lambda})$ in $L^p(\widetilde{\Omega}_\lambda; L^2(0,T; \mathcal{L}^2(U_2, H)))$ for every $\ell \geq 2$ and $p < \ell$, one has in particular that

$$
(\widetilde{\sigma_{\lambda,n}}, G_{2,n}(\widetilde{\sigma_{\lambda,n}}))_D \to (\widetilde{\sigma_\lambda}, G_2(\widetilde{\sigma_\lambda}))_D \quad \text{in } L^2(\widetilde{\Omega}_\lambda; L^2(0,T; \mathcal{L}^2(U_2, \mathbb{R}))),
$$

so that, arguing as in the proof of Lemma 3.6,

$$
\int_0^\cdot \big(\widetilde{\sigma_{\lambda,n}}(r), G_{2,n}(\widetilde{\sigma_{\lambda,n}}(r))\mathrm{d}\widetilde{W}_{2,n}(r)\big)_D \to \int_0^\cdot \big(\widetilde{\sigma_{\lambda,n}}(r), G_{2,\lambda}(\widetilde{\sigma_\lambda}(r))\mathrm{d}\widetilde{W}_{2,\lambda}(r)\big)_D
$$

in $L^2(\widetilde{\Omega}_\lambda; C^0([0,T]))$. Hence, by letting $n \to \infty$ we deduce, by possibly renominating the constants, that

$$
\begin{aligned}
&\sup_{s\in[0,t]} \|\nabla\widetilde{\varphi_\lambda}(s)\|_H^2 + \sup_{s\in[0,t]} \|\Psi_\lambda(\widetilde{\varphi_\lambda}(t))\|_{L^1(D)} + \sup_{s\in[0,t]} |(\widetilde{\mu_\lambda}(s))| + \|\nabla\widetilde{\mu_\lambda}\|^2_{L^2(0,t;H)} \\
&\quad + \sup_{s\in[0,t]} \|\widetilde{\sigma_\lambda}(s)\|_H^2 + \|\nabla\widetilde{\sigma_\lambda}\|^2_{L^2(0,t;H)} \\
&\quad \le c\left(1 + \|\nabla\varphi_0\|_H^2 + \|\Psi(\varphi_0)\|_{L^1(D)} + \|\sigma_0\|_H^2 + \|\widetilde{\sigma_\lambda}\|^2_{L^2(0,t;H)} + \|\widetilde{\varphi_\lambda}\|^2_{L^2(0,t;V)}\right) \\
&\quad + c\int_0^t \left(1 + \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)} + \|\Psi'_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}\|\widetilde{\sigma_\lambda}(s)\|_H\right) \mathrm{d}s \\
&\quad + c\sup_{s\in[0,t]}\left[\int_0^s \left(\widetilde{\mu_\lambda}(r), G_1(\widetilde{\varphi_\lambda}(r))\mathrm{d}\widetilde{W}_{1,\lambda}(r)\right)_D + \int_0^s \left(\widetilde{\sigma_\lambda}(r), G_2(\widetilde{\sigma_\lambda}(r))\mathrm{d}\widetilde{W}_{2,\lambda}(r)\right)_D\right].
\end{aligned}
$$

The Burkholder–Davis–Gundy and Poincaré–Wirtinger inequalities imply that

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]}\left|\int_0^s \left(\widetilde{\mu_\lambda}(r), G_1(\widetilde{\varphi_\lambda}(r))\,\mathrm{d}\widetilde{W}_{1,\lambda}(r)\right)_H\right|^{\ell/2} \\
&\quad \le \bar{\delta}\widetilde{\mathbb{E}}_\lambda\,\|\nabla\widetilde{\mu_\lambda}\|^\ell_{L^2(0,t;H)} + c_\ell\left(1 + \widetilde{\mathbb{E}}_\lambda\int_0^t \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|^{\ell/2}_{L^1(D)}\,\mathrm{d}s\right),
\end{aligned}
$$

where we choose $\bar{\delta} > 0$ sufficiently small. Furthermore, we have

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,T]}\left|\int_0^s \left(\widetilde{\sigma_\lambda}(r), G_2(\widetilde{\sigma_\lambda}(r))\,\mathrm{d}W_{2,\lambda}(r)\right)_D\right|^{\ell/2} \\
&\quad \le c\widetilde{\mathbb{E}}_\lambda\left(\int_0^t \|\widetilde{\sigma_\lambda}(s)\|_H^2\|G_2(\widetilde{\sigma_\lambda}(s))\|^2_{\mathcal{L}^2(U_1,H)}\,\mathrm{d}s\right)^{\ell/4} \le c\widetilde{\mathbb{E}}_\lambda\|\widetilde{\sigma_\lambda}\|^{\ell/2}_{L^2(0,t;H)}.
\end{aligned}
$$

Consequently, taking power $\ell/2$ and expectations we obtain, recalling also (3.22),

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]}\|\widetilde{\varphi_\lambda}(s)\|_V^\ell + \widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]}\|\Psi_\lambda(\widetilde{\varphi_\lambda}(t))\|^{\ell/2}_{L^1(D)} + \widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]}|(\widetilde{\mu_\lambda}(s))|^{\ell/2} \\
&\quad + \widetilde{\mathbb{E}}_\lambda\|\nabla\widetilde{\mu_\lambda}\|^\ell_{L^2(0,t;H)} + \widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]}\|\widetilde{\sigma_\lambda}(s)\|_H^\ell + \widetilde{\mathbb{E}}_\lambda\|\nabla\widetilde{\sigma_\lambda}\|^\ell_{L^2(0,t;H)} \\
&\quad \le c\left(1 + \|\nabla\varphi_0\|_H^2 + \|\Psi(\varphi_0)\|_{L^1(D)} + \|\sigma_0\|_H^2 + c\widetilde{\mathbb{E}}_\lambda\|\widetilde{\sigma_\lambda}\|^\ell_{L^2(0,t;H)} + c\widetilde{\mathbb{E}}_\lambda\|\widetilde{\varphi_\lambda}\|^\ell_{L^2(0,t;V)}\right) \\
&\quad + c\widetilde{\mathbb{E}}_\lambda\int_0^t \left(1 + \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|^{\ell/2}_{L^1(D)} + \|\Psi'_\lambda(\widetilde{\varphi_\lambda}(s))\|^{\ell/2}_{L^1(D)}\|\widetilde{\sigma_\lambda}(s)\|^{\ell/2}_H\right)\mathrm{d}s.
\end{aligned}
$$

We show how to control the last term on the right-hand side by distinguishing three cases. First, if $\chi > 0$ and m_1 is non-constant, by assumption (A4) one has that $q = 2$. Hence, one can use the Young inequality on the last term on the right-hand side and (3.7) as

$$
\widetilde{\mathbb{E}}_\lambda\int_0^t \|\Psi'_\lambda(\widetilde{\varphi_\lambda}(s))\|^{\ell/2}_{L^1(D)}\|\widetilde{\sigma_\lambda}(s)\|^{\ell/2}_H\,\mathrm{d}s
$$

$$\leq c\widetilde{\mathbb{E}}_\lambda \int_0^t \left(\|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}^{\ell/2} + \|\widetilde{\varphi_\lambda}(s)\|_H^\ell + \|\widetilde{\sigma_\lambda}(s)\|_H^\ell \right) \mathrm{d}s$$

and conclude thanks to the Gronwall lemma. Secondly, if $\chi = 0$ and m_1 is non-constant, we are in the setting $q \in (1, 2]$ by (A4): hence, one readily has from the arbitrariness of ℓ in (3.30) that

$$\widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]} \|\widetilde{\sigma_\lambda}(s)\|_H^{\frac{\ell q}{2(q-1)}} \leq c_\ell$$

so that the last term on the right-hand side can be handled as

$$\begin{aligned}
&\widetilde{\mathbb{E}}_\lambda \int_0^t \|\Psi_\lambda'(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}^{\ell/2} \|\widetilde{\sigma_\lambda}(s)\|_H^{\ell/2} \,\mathrm{d}s \\
&\quad \leq c\widetilde{\mathbb{E}}_\lambda \int_0^t \left(1 + \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}^{\frac{\ell}{2q}} + \|\varphi_\lambda(s)\|_H^{\frac{\ell}{2}} \right) \|\widetilde{\sigma_\lambda}(s)\|_H^{\frac{\ell}{2}} \mathrm{d}s \\
&\quad \leq c\widetilde{\mathbb{E}}_\lambda \int_0^t \left(1 + \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}^{\ell/2} + c\|\widetilde{\sigma_\lambda}(s)\|_H^{\frac{\ell q}{2(q-1)}} + \|\varphi_\lambda(s)\|_H^\ell + \|\sigma_\lambda(s)\|_H^\ell \right) \mathrm{d}s \\
&\quad \leq c + c\widetilde{\mathbb{E}}_\lambda \int_0^t \left(1 + \|\Psi_\lambda(\widetilde{\varphi_\lambda}(s))\|_{L^1(D)}^{\ell/2} + \|\varphi_\lambda(s)\|_H^\ell + \|\sigma_\lambda(s)\|_H^\ell \right) \mathrm{d}s
\end{aligned}$$

and one can still conclude by the Gronwall lemma. Lastly, if $m_1 = \bar{m}_1$ is constant (hence again $q > 1$), one can perform Itô formula for the H-norm of $\widetilde{\varphi_\lambda}$ as in [40, Thm. 4.2.5], which yields, together with the analogous Itô formula for the H-norm of $\widetilde{\sigma_\lambda}$, that

$$\begin{aligned}
&\sup_{s\in[0,t]} \|\widetilde{\varphi_\lambda}(s)\|_H^2 + 2\varepsilon^2\bar{m}_1\|\Delta\widetilde{\varphi_\lambda}\|_{L^2(0,t;H)}^2 + \sup_{s\in[0,t]} \|\widetilde{\sigma_\lambda}(s)\|_H^2 + 2m_0\|\nabla\widetilde{\sigma_\lambda}\|_{L^2(0,t;H)}^2 \\
&\quad \leq \|\varphi_0\|_H^2 + \|\sigma_0\|_H^2 + 2\chi(\bar{m}_1 + m_\infty) \int_0^t \|\nabla\widetilde{\varphi_\lambda}(s)\|_H \|\nabla\widetilde{\sigma_\lambda}(s)\|_H \,\mathrm{d}s \\
&\qquad - 2\int_{D_t} \Psi_\lambda''(\widetilde{\varphi_\lambda})|\nabla\widetilde{\varphi_\lambda}|^2 + 2\int_{D_t} (\beta\widetilde{\sigma_\lambda} - \alpha) f(\widetilde{\varphi_\lambda})\widetilde{\varphi_\lambda} \\
&\qquad + \|G_1(\widetilde{\varphi_\lambda})\|_{L^2(0,t;\mathcal{L}^2(U_1,H))}^2 + 2\int_0^t \left(\widetilde{\varphi_\lambda}(s), G_1(\widetilde{\varphi_\lambda}(s))\mathrm{d}\widetilde{W}_1(s)\right)_D \\
&\qquad + \|G_2(\widetilde{\sigma_\lambda})\|_{L^2(0,t;\mathcal{L}^2(U_2,H))}^2 + 2\int_0^t \left(\widetilde{\sigma_\lambda}(s), G_2(\widetilde{\sigma_\lambda}(s))\mathrm{d}\widetilde{W}_2(s)\right)_D .
\end{aligned}$$

Noting that

$$\begin{aligned}
&2\chi(\bar{m}_1 + m_\infty) \int_0^t \|\nabla\widetilde{\varphi_\lambda}(s)\|_H \|\nabla\widetilde{\sigma_\lambda}(s)\|_H \,\mathrm{d}s - 2\int_{D_t} \Psi_\lambda''(\widetilde{\varphi_\lambda})|\nabla\widetilde{\varphi_\lambda}|^2 \\
&\quad \leq m_0\|\nabla\widetilde{\sigma_\lambda}\|_{L^2(0,t;H)}^2 + \left(\frac{\chi^2(\bar{m}_1 + m_\infty)^2}{m_0} + 2C_\Psi \right) \|\nabla\widetilde{\varphi_\lambda}\|_{L^2(0,t;H)}^2
\end{aligned}$$

$$\leq m_0 \|\nabla \widetilde{\sigma_\lambda}\|^2_{L^2(0,t;H)} + \varepsilon^2 \bar{m}_1 \|\Delta \widetilde{\varphi_\lambda}\|^2_{L^2(0,t;H)} + c\|\widetilde{\varphi_\lambda}\|^2_{L^2(0,t;H)}$$

and that all the remaining terms can be handled using the arguments already performed in the previous section, we still deduce in particular that

$$\widetilde{\mathbb{E}}_\lambda \sup_{s\in[0,t]} \|\widetilde{\sigma_\lambda}(s)\|_H^{\frac{\ell q}{2(q-1)}} \leq c_\ell$$

and this allows us to conclude as in the previous case. □

We have obtained then the estimates

$$\|\widetilde{\varphi_\lambda}\|_{L^\ell(\widetilde{\Omega}_\lambda; L^\infty(0,T;V))} \leq c, \tag{3.31}$$

$$\|\widetilde{\mu_\lambda}\|_{L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;V))} + \|\nabla \widetilde{\mu_\lambda}\|_{L^\ell(\widetilde{\Omega}_\lambda; L^2(0,T;H))} \leq c, \tag{3.32}$$

$$\|\widetilde{\sigma_\lambda}\|_{L^\ell(\widetilde{\Omega}_\lambda; L^\infty(0,T;H)) \cap L^\ell(\widetilde{\Omega}_\lambda; L^2(0,T;V))} \leq c. \tag{3.33}$$

In the same way as before, we bound $G_1(\widetilde{\varphi_\lambda})$ and $G_2(\widetilde{\sigma_\lambda})$ thanks to (A5) as

$$\|G_1(\widetilde{\varphi_\lambda})\|_{L^\infty(\widetilde{\Omega}_\lambda \times (0,T); \mathcal{L}^2(U_1,H)) \cap L^\ell(\widetilde{\Omega}_\lambda; L^\infty(0,T;\mathcal{L}^2(U_1,V)))} \leq c,$$
$$\|G_2(\widetilde{\sigma_\lambda})\|_{L^\infty(\widetilde{\Omega}_\lambda \times (0,T); \mathcal{L}^2(U_2,H)) \cap L^\ell(\widetilde{\Omega}_\lambda; L^\infty(0,T;\mathcal{L}^2(U_2,V)))} \leq c.$$

As usual, these imply a bound of the stochastic integrals by by [16, Lem. 2.1] in form

$$\left\| \int_0^\cdot G_1(\widetilde{\varphi_\lambda}(s))\, \mathrm{d}\widetilde{W}_{1,\lambda}(s) \right\|_{L^\ell(\widetilde{\Omega}_\lambda; W^{s,\ell}(0,T;V))} \leq c_s,$$
$$\left\| \int_0^\cdot G_2(\widetilde{\sigma_\lambda}(s))\, \mathrm{d}\widetilde{W}_{2,\lambda}(s) \right\|_{L^\ell(\widetilde{\Omega}_\lambda; W^{s,\ell}(0,T;V))} \leq c_s.$$

for every $s \in (0, 1/2)$, and by comparison in the equations we also obtain the estimates

$$\begin{aligned} \|\widetilde{\varphi_\lambda}\|_{L^\ell(\widetilde{\Omega}_\lambda; W^{\bar{s},\ell}(0,T;V^*))} &\leq c, \\ \|\widetilde{\sigma_\lambda}\|_{L^\ell(\widetilde{\Omega}_\lambda; W^{\bar{s},\ell}(0,T;V^*))} &\leq c, \end{aligned} \tag{3.34}$$

where $\bar{s} \in (1/\ell, 1/2)$ is fixed.

Finally, since $\widetilde{\mu_\lambda} = -\varepsilon^2 \Delta \widetilde{\varphi_\lambda} + \Psi'_\lambda(\widetilde{\varphi_\lambda})$, testing by $\gamma_\lambda(\widetilde{\varphi_\lambda}) = \Psi'_\lambda(\widetilde{\varphi_\lambda}) + C_\Psi \widetilde{\varphi_\lambda}$ and rearranging the terms we have

$$\begin{aligned} (\gamma'_\lambda(\widetilde{\varphi_\lambda}), |\nabla \widetilde{\varphi_\lambda}|^2)_D + \|\gamma_\lambda(\widetilde{\varphi_\lambda})\|_H^2 &= (\widetilde{\mu_\lambda} + C_\Psi \widetilde{\varphi_\lambda}, \gamma_\lambda(\widetilde{\varphi_\lambda}))_D \\ &\leq \frac{1}{2}\|\gamma_\lambda(\widetilde{\varphi_\lambda})\|_H^2 + \|\widetilde{\mu_\lambda}\|_H^2 + C_\Psi^2 \|\widetilde{\varphi_\lambda}\|_H^2. \end{aligned}$$

Since the first term on the left-hand side is nonnegative by the monotonicity of γ_λ, the Young inequality yields, after integrating in time,

$$\left\|\Psi'_\lambda(\widetilde{\varphi_\lambda})\right\|^2_{L^2(0,T;H)} \leq 4\, \|\widetilde{\mu_\lambda}\|^2_{L^2(0,T;H)} + 6C_\Psi^2\, \|\widetilde{\varphi_\lambda}\|^2_{L^2(0,T;H)},$$

so that by (3.31)–(3.32) we have

$$\left\| \Psi'_\lambda(\widetilde{\varphi}_\lambda) \right\|_{L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;H))} \le c. \tag{3.35}$$

By comparison we deduce that $\Delta\widetilde{\varphi}_\lambda \in L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;H))$ and by elliptic regularity

$$\|\widetilde{\varphi}_\lambda\|_{L^{\ell/2}(\widetilde{\Omega}_\lambda; L^2(0,T;V_2))} \le c. \tag{3.36}$$

3.5 Step 5: passage to the limit $\lambda \to 0$

In this step, we pass to the limit as $\lambda \to 0$ in the Yosida approximation (3.8) and recover a martingale solution to the original problem (1.2). Since the arguments are very similar to the ones in Sect. 3.3 when we passed to the limit $n \to \infty$, we shall omit the exact details.

Proof of Theorem 3.4 By the Aubin–Lions compactness lemma one has the compact inclusions

$$\begin{aligned}
&L^2(0,T;V_2) \cap L^\infty(0,T;V) \cap W^{\bar{s},\ell}(0,T;V^*) \hookrightarrow\hookrightarrow C^0([0,T];H) \cap L^2(0,T;V),\\
&L^2(0,T;V) \cap L^\infty(0,T;H) \cap W^{\bar{s},\ell}(0,T;V^*) \hookrightarrow\hookrightarrow C^0([0,T];V^*) \cap L^2(0,T;H).
\end{aligned}$$

Hence, using the estimates (3.31) and (3.34) and arguing exactly as in Sect. 3.3, one readily deduces that the family of laws $(\widetilde{\varphi}_\lambda, G_1(\widetilde{\varphi}_\lambda)\cdot\widetilde{W}_1, \widetilde{W}_1)_\lambda$ is tight on the product space

$$\Big(C^0([0,T];H) \cap L^2(0,T;V)\Big) \times C^0([0,T];H) \times C^0([0,T];\widetilde{U}_1),$$

and the family of laws $(\widetilde{\sigma}_\lambda, G_2(\widetilde{\sigma}_\lambda)\cdot\widetilde{W}_2, \widetilde{W}_2)_\lambda$ is tight on the product space

$$\Big(C^0([0,T];V^*) \cap L^2(0,T;H)\Big) \times C^0([0,T];H) \times C^0([0,T];\widetilde{U}_2),$$

The Prokhorov and Jakubowski–Skorokhod theorems (see the discussion in Sect. 2.2) ensure the existence of a further probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ and sequences of measurable processes $(\widehat{\varphi}_\lambda, \widehat{\mu_\lambda}, \widehat{\sigma}_\lambda)$ with the same law as their counterpart $(\widetilde{\varphi}_\lambda, \widetilde{\mu_\lambda}, \widetilde{\sigma}_\lambda)$ such that, passing to a further subsequence, still denoted with the index λ for brevity of notation, it holds

$$\begin{aligned}
\widetilde{\varphi}_\lambda \to \widetilde{\varphi} \quad &\text{in } L^p(\widehat{\Omega}; C^0([0,T];H) \cap L^2(0,T;V)) \quad \forall\, p < \ell,\\
\widetilde{\varphi}_\lambda \to \widetilde{\varphi} \quad &\text{in } C^0([0,T];H) \cap L^2(0,T;V) \quad \text{a.e. in } \widehat{\Omega},\\
\widetilde{\varphi}_\lambda \overset{*}{\rightharpoonup} \widetilde{\varphi} \quad &\text{in } L^\ell_w(\widehat{\Omega}; L^\infty(0,T;V)),\\
\widetilde{\varphi}_\lambda \rightharpoonup \widetilde{\varphi} \quad &\text{in } L^{\ell/2}(\widehat{\Omega}; L^2(0,T;V_2)) \cap L^\ell(\widehat{\Omega}; W^{\bar{s},\ell}(0,T;V^*)),
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\mu_\lambda} &\rightharpoonup \widetilde{\mu} \quad \text{in } L^{\ell/2}(\widehat{\Omega}; L^2(0,T;V)),\\
\nabla\widetilde{\mu_\lambda} &\rightharpoonup \nabla\widetilde{\mu} \quad \text{in } L^{\ell}(\widehat{\Omega}; L^2(0,T;H)),\\
\widetilde{\sigma_\lambda} &\to \widetilde{\sigma} \quad \text{in } L^{p}(\widehat{\Omega}; C^0([0,T];V^*)\cap L^2(0,T;H)) \quad \forall\, p<\ell,\\
\widetilde{\sigma_\lambda} &\to \widetilde{\sigma} \quad \text{in } C^0([0,T];V^*)\cap L^2(0,T;H) \quad \text{a.e. in } \widehat{\Omega},\\
\widetilde{\sigma_\lambda} &\overset{*}{\rightharpoonup} \widetilde{\sigma} \quad \text{in } L^{\ell}_w(\widehat{\Omega}; L^\infty(0,T;H)),\\
\widetilde{\sigma_\lambda} &\rightharpoonup \widetilde{\sigma} \quad \text{in } L^{\ell}(\widehat{\Omega}; L^2(0,T;V))\cap L^{\ell}(\widehat{\Omega}; W^{\bar{s},\ell}(0,T;V^*)),\\
\Psi'_\lambda(\widetilde{\varphi_\lambda}) &\rightharpoonup \widetilde{\xi} \quad \text{in } L^{\ell/2}(\widehat{\Omega}; L^2(0,T;H)),
\end{aligned}
$$

for some measurable processes

$$
\begin{aligned}
&\widehat{\varphi} \in L^{\ell}(\widehat{\Omega}; W^{\bar{s},\ell}(0,T;V^*)\cap C^0([0,T];H))\cap L^{\ell}_w(\widehat{\Omega}; L^\infty(0,T;V))\\
&\qquad \cap L^{\ell/2}(\widehat{\Omega}; L^2(0,T;V_2)),\\
&\widehat{\mu} \in L^{\ell/2}(\widehat{\Omega}; L^2(0,T;V)), \quad \nabla\widetilde{\mu} \in L^{\ell}(\widehat{\Omega}; L^2(0,T;H)),\\
&\widehat{\sigma} \in L^{\ell}(\widehat{\Omega}; W^{\bar{s},\ell}(0,T;V^*)\cap C^0([0,T];V^*)\cap L^2(0,T;V))\\
&\qquad \cap L^{\ell}_w(\widehat{\Omega}; L^\infty(0,T;H)),\\
&\widehat{\xi} \in L^{\ell/2}(\widehat{\Omega}; L^2(0,T;H)).
\end{aligned}
$$

By a typical abuse of notation we write again the index λ and instead of a further subsequence. By the same reasoning as above, there are sequences of measurable processes $\widehat{W}_{1,\lambda}$, $\widehat{W}_{2,\lambda}$, $\widehat{I}_{1,\lambda}$ and $\widehat{I}_{2,\lambda}$ with the same law as $\widetilde{W}_{1,\lambda}$, $\widetilde{W}_{2,\lambda}$, $\widetilde{G}_1(\varphi_\lambda)\cdot W_{1,\lambda}$, $\widetilde{G}_2(\sigma_\lambda)\cdot W_{2,\lambda}$, respectively, such that it holds

$$
\begin{aligned}
\widehat{I}_{1,\lambda} &\to \widehat{I}_1 \quad \text{in } L^p(\widehat{\Omega}; C^0([0,T];H)) \quad \forall\, p<\ell,\\
\widehat{I}_{2,\lambda} &\to \widehat{I}_2 \quad \text{in } L^p(\widehat{\Omega}; C^0([0,T];H)) \quad \forall\, p<\ell,\\
\widehat{W}_{1,\lambda} &\to \widehat{W}_1 \quad \text{in } L^p(\widehat{\Omega}; C^0([0,T];\widetilde{U}_1)) \quad \forall\, p<\ell,\\
\widehat{W}_{2,\lambda} &\to \widehat{W}_2 \quad \text{in } L^p(\widehat{\Omega}; C^0([0,T];\widetilde{U}_2)) \quad \forall\, p<\ell,
\end{aligned}
$$

for some measurable processes $\widehat{I}_1, \widehat{I}_2 \in L^p(\widehat{\Omega}; C^0([0,T];H))$, $\widehat{W}_1 \in L^p(\widehat{\Omega}; C^0([0,T];\widetilde{U}_1))$, $\widehat{W}_2 \in L^p(\widehat{\Omega}; C^0([0,T];\widetilde{U}_2))$. Since $G_i : H \to \mathcal{L}^2(U_i, H)$ is Lipschitz-continuous for $i = 1, 2$, we also have

$$
\begin{aligned}
G_1(\widehat{\varphi_\lambda}) &\to G_1(\widehat{\varphi}) \quad \text{in } L^{\ell}(\widehat{\Omega}; L^2(0,T;\mathcal{L}^2(U_1,H))) \quad \forall\, p<\ell,\\
G_2(\widehat{\sigma_\lambda}) &\to G_2(\widehat{\sigma}) \quad \text{in } L^{\ell}(\widehat{\Omega}; L^2(0,T;\mathcal{L}^2(U_2,H))) \quad \forall\, p<\ell.
\end{aligned}
$$

Moreover, noting that, by definition of Yosida approiximation (see Sect. 4.1), $\Psi'_\lambda(\widehat{\varphi_\lambda}) = \gamma(J_\lambda(\widehat{\varphi_\lambda})) - C_\Psi\widehat{\varphi_\lambda}$ and

$$
|J_\lambda(\widehat{\varphi_\lambda}) - \widehat{\varphi}| \le |J_\lambda(\widehat{\varphi_\lambda}) - \widehat{\varphi_\lambda}| + |\widehat{\varphi_\lambda} - \widehat{\varphi}| = \lambda|\gamma_\lambda(\widehat{\varphi_\lambda})| + |\widehat{\varphi_\lambda} - \widehat{\varphi}|,
$$

by the strong convergence of $(\widehat{\varphi_\lambda})_\lambda$ obtained above, the estimate (3.35), and the continuity of γ one readily has

$$\Psi'_\lambda(\widehat{\varphi_\lambda}) \to \gamma(\widehat{\varphi}) - C_\Psi \widehat{\varphi} = \Psi'(\widehat{\varphi}) \quad \text{a.e. in } \widehat{\Omega} \times (0,T) \times D.$$

By the Severini–Egorov theorem, this implies then that

$$\widehat{\xi} = \Psi'(\widehat{\varphi}) \quad \text{a.e. in } \widehat{\Omega} \times (0,T) \times D.$$

By defining the limiting filtration as

$$\widehat{\mathcal{F}}_t := \sigma\left\{(\widehat{\varphi}(s), \widehat{\sigma}(s), \widehat{W}_1(s), \widehat{W}_2(s), \widehat{I}_1(s), \widehat{I}_2(s)) : \quad s \in [0,t]\right\}, \quad t \in [0,T],$$

following the same argument as in Sect. 3.3, thanks to the strong convergences of $\widehat{\varphi_\lambda} \to \widehat{\varphi}$ and $G_i(\widehat{\varphi_\lambda}) \to G_i(\widehat{\varphi})$ proved above, it is a classical argument to show that

$$\widehat{I}_{1,\lambda} = \int_0^\cdot G_1(\widehat{\varphi_\lambda}(s))\,\mathrm{d}\widehat{W}_{1,\lambda}(s), \quad \widehat{I}_{2,\lambda} = \int_0^\cdot G_2(\widehat{\sigma_\lambda}(s))\,\mathrm{d}\widehat{W}_{2,\lambda}(s),$$

and that $\widehat{I}_i$, $i = 1, 2$, are the H-valued martingales given by

$$\widehat{I}_1 = \int_0^\cdot G_1(\widehat{\varphi}(s))\,\mathrm{d}\widehat{W}_1(s), \quad \widehat{I}_2 = \int_0^\cdot G_2(\widehat{\sigma}(s))\,\mathrm{d}\widehat{W}_2(s).$$

Now, we note that for every $\lambda > 0$ it follows that

$$\begin{aligned}
(\widehat{\varphi_\lambda}(t), v)_D &= (\varphi_0, v)_D - (m(\widehat{\varphi_\lambda})\nabla(\widehat{\mu_\lambda} - \chi\widehat{\sigma_\lambda}), \nabla v)_{D_t} \\
&\quad + (\beta\widehat{\sigma_\lambda} - \alpha, f(\widehat{\varphi_\lambda})v)_{D_t} + \left(\int_0^t G_1(\widehat{\varphi_\lambda}(s))\,\mathrm{d}\widehat{W}_{1,\lambda}(s), v\right) D \\
(\widehat{\sigma_\lambda}(t), v)_D &= (\sigma_0, v)_D - (m_2(\widehat{\sigma_\lambda})\nabla(\widehat{\sigma_\lambda} - \chi\widehat{\varphi_\lambda}), \nabla v)_{D_t} \\
&\quad - (\delta\widehat{\sigma_\lambda}, f(\widehat{\varphi_\lambda})v)_{D_t} + \left(\int_0^t G_2(\widehat{\sigma_\lambda}(s))\,\mathrm{d}\widetilde{W}_2(s), v\right)_D
\end{aligned}$$

for every $t \in [0,T]$, $\widehat{\mathbb{P}}$-almost surely, where $\widehat{\mu_\lambda} = -\varepsilon^2\Delta\widehat{\varphi_\lambda} + \Psi'_\lambda(\widehat{\varphi_\lambda})$. Hence, using the convergences proved above, the continuity and boundedness of m_1 and m_2 together with the dominated convergence theorem, we can let $\lambda \to 0$ in the variational formulations and obtain exactly (3.1), and $\widehat{\mu} = -\varepsilon^2\Delta\widehat{\varphi} + \Psi'(\widehat{\varphi})$. Finally, the energy inequality (3.3) as stated in Theorem 3.4 follows from applying Ξ_λ to (3.29) and letting $\lambda \searrow 0$ by the weak-lower semicontinuity of the norms. □

3.6 Strong well-posedness

We focus here on the proof of the second main result: let us suppose that the two mobilities are constant, i.e. $m_1 = \bar{m}_1$ and $m_2 = \bar{m}_2$ some $\bar{m}_1, \bar{m}_2 > 0$.

Proof of Theorem 3.5 The existence of strong solutions follows by the Yamada–Watanabe theorem, see [40, Theorem E.0.8], provided to show pathwise uniqueness. To this end, let $(\varphi_1, \mu_1, \sigma_1)$, $(\varphi_2, \mu_2, \sigma_2)$ be two solutions defined on the same probability space, associated to some initial data $(\varphi_0^1, \sigma_0^1)$, (φ_0^2, σ_2) satisfying (A6). The Itô formula for the square of the H norms of $\varphi_1 - \varphi_2$ and $\sigma_1 - \sigma_2$, in the version of [40, Thm. 4.2.5], yield

$$
\begin{aligned}
&\frac{1}{2}\|(\varphi_1 - \varphi_2)(t)\|_H^2 + \frac{1}{2}\|(\sigma_1 - \sigma_2)(t)\|_H^2 + \delta(f(\varphi_1), |\sigma_1 - \sigma_2|^2)_{D_t} \\
&\quad + \bar{m}_1\varepsilon^2\|\Delta(\varphi_1 - \varphi_2)(s)\|_{L^2(0,t;H)}^2 + \bar{m}_2\|\nabla(\sigma_1 - \sigma_2)(s)\|_{L^2(0,t;H)}^2 \\
&= \frac{1}{2}\|\varphi_0^1 - \varphi_0^2\|_H^2 + \frac{1}{2}\|\sigma_0^1 - \sigma_0^2\|_H^2 + \chi(\bar{m}_1 + \bar{m}_2)(\nabla(\sigma_1 - \sigma_2), \nabla(\varphi_1 - \varphi_2))_{D_t} \\
&\quad + \beta(f(\varphi_1)(\sigma_1 - \sigma_2), \varphi_1 - \varphi_2)_{D_t} - \alpha(f(\varphi_1) - f(\varphi_2), \varphi_1 - \varphi_2)_{D_t} \\
&\quad + (f(\varphi_1) - f(\varphi_2), \sigma_2\,[\beta(\varphi_1 - \varphi_2) - \delta(\sigma_1 - \sigma_2)])_{D_t} \\
&\quad - (\Psi'(\varphi_1) - \Psi'(\varphi_2), \Delta(\varphi_1 - \varphi_2))_{D_t} + \frac{1}{2}\|G_1(\varphi_1) - G_1(\varphi_2)\|_{L^2(0,t;\mathcal{L}^2(U_1,H))}^2 \\
&\quad + \frac{1}{2}\|G_2(\sigma_1) - G_2(\sigma_2)\|_{L^2(0,t;\mathcal{L}^2(U_2,H))}^2 \\
&\quad + \int_0^t ((\varphi_1 - \varphi_2)(s), (G_1(\varphi_1(s)) - G_1(\varphi_2(s)))\mathrm{d}W_1(s))_D \\
&\quad + \int_0^t ((\sigma_1 - \sigma_2)(s), (G_2(\sigma_1(s)) - G_2(\sigma_2(s)))\mathrm{d}W_2(s))_D\,.
\end{aligned}
$$

Let us estimate the terms on the right-hand side separately. First, since V_2 is compact in V, we have

$$
\begin{aligned}
&\chi(\bar{m}_1 + \bar{m}_2)(\nabla(\sigma_1 - \sigma_2), \nabla(\varphi_1 - \varphi_2))_{D_t} \\
&\quad \le \check{\delta}\|\nabla(\sigma_1 - \sigma_2)\|_{L^2(0,t;H)}^2 + \check{\delta}\|\Delta(\varphi_1 - \varphi_2)\|_{L^2(0,t;H)}^2 + c\|\varphi_1 - \varphi_2\|_{L^2(0,t;H)}^2,
\end{aligned}
$$

where $\check{\delta}$ is a parameter that is determined below. Moreover, since f is Lipschitz and bounded, the fourth and fifth integrals on the right-hand side are controlled, thanks to the Young inequality, by

$$
\begin{aligned}
&\beta(f(\varphi_1)(\sigma_1 - \sigma_2), \varphi_1 - \varphi_2)_{D_t} - \alpha(f(\varphi_1) - f(\varphi_2), \varphi_1 - \varphi_2)_{D_t} \\
&\quad \le c\|\sigma_1 - \sigma_2\|_{L^2(0,t;H)}^2 + c\|\varphi_1 - \varphi_2\|_{L^2(0,t;H)}^2.
\end{aligned}
$$

Furthermore, for the sixth term, we use the Hölder inequality, the Lipschitz continuity of f, the inclusion $H^{\frac{7}{4}}(D) \hookrightarrow L^\infty(D)$, the compact inclusion $V_2 \hookrightarrow H^{\frac{7}{4}}(D)$, and obtain

$$
\begin{aligned}
&(f(\varphi_1) - f(\varphi_2), \sigma_2\,[\beta(\varphi_1 - \varphi_2) - \delta(\sigma_1 - \sigma_2)])_{D_t} \\
&\quad \le c\int_0^t \|\sigma_2(s)\|_H \|(\varphi_1 - \varphi_2)(s)\|_{H^{\frac{7}{4}}(D)} [\|(\varphi_1 - \varphi_2)(s)\|_H + \|(\sigma_1 - \sigma_2)(s)\|_H]\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t \|(\varphi_1-\varphi_2)(s)\|^2_{H^{\frac{7}{4}}(D)}\,\mathrm{d}s + c\int_0^t \|\sigma_2(s)\|_H^2\left[\|\varphi_1-\varphi_2\|_H^2 + \|\sigma_1-\sigma_2\|_H^2\right]\mathrm{d}s\\
&\leq \check{\delta}\|\Delta(\varphi_1-\varphi_2)\|^2_{L^2(0,t;H)}\,\mathrm{d}s + c\int_0^t (1+\|\sigma_2(s)\|_H^2)\\
&\quad\times\left[\|(\varphi_1-\varphi_2)(s)\|_H^2 + \|(\sigma_1-\sigma_2)(s)\|_H^2\right]\mathrm{d}s.
\end{aligned}
$$

As for the seventh term, we exploit assumption (3.4), the Hölder and Young inequality, and the continuous inclusions $H^{\frac{7}{4}}(D) \subset L^\infty(D)$ and $V_1 \subset L^4(D)$, to infer that

$$
\begin{aligned}
&-(\Psi'(\varphi_1)-\Psi'(\varphi_2), \Delta(\varphi_1-\varphi_2))_{D_t}\\
&\quad\leq C_\Psi \int_{D_t}\left(1+|\varphi_1|^2+|\varphi_2|^2\right)|\varphi_1-\varphi_2||\Delta(\varphi_1-\varphi_2)|\\
&\quad\leq C_\Psi \int_0^t\left(1+\|\varphi_1(s)\|^2_{L^4(D)}+\|\varphi_2(s)\|^2_{L^4(D)}\right)\\
&\qquad\times\|(\varphi_1-\varphi_2)(s)\|_{L^\infty(D)}\,\|\Delta(\varphi_1-\varphi_2)(s)\|_H\ \ \mathrm{d}s\\
&\quad\leq \check{\delta}\|\Delta(\varphi_1-\varphi_2)\|^2_{L^2(0,t;H)} + c\int_0^t\left(1+\|\varphi_1(s)\|^4_{V_1}+\|\varphi_2(s)\|^4_{V_1}\right)\\
&\qquad\times\|(\varphi_1-\varphi_2)(s)\|^2_{H^{\frac{7}{4}}(D)}\ \ \mathrm{d}s
\end{aligned}
$$

Now, recalling that by interpolation it holds that $\|v\|_{H^{\frac{7}{4}}(D)} \leq \|v\|_{V_2}^{\frac{7}{8}}\|v\|_H^{\frac{1}{8}}$ for all $v \in V_2$, by the Young inequality we deduce that

$$
\begin{aligned}
&-(\Psi'(\varphi_1)-\Psi'(\varphi_2), \Delta(\varphi_1-\varphi_2))_{D_t}\\
&\quad\leq 2\check{\delta}\|\Delta(\varphi_1-\varphi_2)\|^2_{L^2(0,t;H)} + c\int_0^t\left(1+\|\varphi_1(s)\|^{32}_{V_1}+\|\varphi_2(s)\|^{32}_{V_1}\right)\\
&\qquad\times\|(\varphi_1-\varphi_2)(s)\|^2_H\ \ \mathrm{d}s.
\end{aligned}
$$

Eventually, by the Lipschitz continuity of G_1 and G_2 one has

$$
\begin{aligned}
&\frac{1}{2}\|G_1(\varphi_1)-G_1(\varphi_2)\|^2_{L^2(0,t;\mathcal{L}^2(U_1,H))} + \frac{1}{2}\|G_2(\sigma_1)-G_2(\sigma_2)\|^2_{L^2(0,t;\mathcal{L}^2(U_2,H))}\\
&\quad\leq c\|\varphi_1-\varphi_2\|^2_{L^2(0,t;H)} + c\|\sigma_1-\sigma_2\|^2_{L^2(0,t;H)}.
\end{aligned}
$$

Similarly, for every stopping time $\tau \in [0,T]$, the Burkholder–Davis–Gundy and Young inequalities yield

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,\tau]}\int_0^s((\varphi_1-\varphi_2)(r), (G_1(\varphi_1(r))-G_1(\varphi_2(r)))\mathrm{d}W_1(r))_D\\
&\quad\leq \check{\delta}\mathbb{E}\|\varphi_1-\varphi_2\|^2_{L^\infty(0,\tau;H)} + c\mathbb{E}\|\varphi_1-\varphi_2\|^2_{L^2(0,\tau;H)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,\tau]} \int_0^s ((\sigma_1-\sigma_2)(r), (G_2(\sigma_1(r)) - G_2(\sigma_2(r)))\mathrm{d}W_2(r))_D \\
&\quad \leq \check{\delta}\mathbb{E}\|\sigma_1-\sigma_2\|^2_{L^\infty(0,\tau;H)} + c\mathbb{E}\|\sigma_1-\sigma_2\|^2_{L^2(0,\tau;H)}.
\end{aligned}
$$

We define now the sequence of stopping times

$$
\tau_n := \inf\left\{t\in[0,T] : \|\sigma_2(t)\|_H^2 + \|\varphi_1(t)\|_{V_1}^{32} + \|\varphi_2(t)\|_{V_1}^{32} \geq n\right\}, \quad n\in\mathbb{N}.
$$

Note that for every $n\in\mathbb{N}$, τ_n is well-defined and satisfies $\tau_n \nearrow T$ almost surely since $\sigma_2\in L^\infty(0,T;H)$ almost surely, as well as $\varphi_1,\varphi_2\in L^\infty(0,T;V_1)$ almost surely.

Putting everything together, by taking supremum in time $t\in[0,\tau_n]$ and then expectations, we get, after rearranging the terms,

$$
\begin{aligned}
&\left(\tfrac{1}{2}-\check{\delta}\right)\mathbb{E}\|\varphi_1-\varphi_2\|^2_{L^\infty(0,\tau_n;H)} + \left(\tfrac{1}{2}-\check{\delta}\right)\mathbb{E}\|\sigma_1-\sigma_2\|^2_{L^\infty(0,\tau_n;H)} \\
&\quad + (\bar{m}_1\varepsilon^2 - 4\check{\delta})\mathbb{E}\|\Delta(\varphi_1-\varphi_2)\|^2_{L^2(0,\tau;H)} + (\bar{m}_2-\check{\delta})\mathbb{E}\|\nabla(\sigma_1-\sigma_2)\|^2_{L^2(0,\tau_n;H)} \\
&\leq \frac{1}{2}\|\varphi_0^1-\varphi_0^2\|_H^2 + \frac{1}{2}\|\sigma_0^1-\sigma_0^2\|_H^2 \\
&\quad + c\mathbb{E}\int_0^{\tau_n}\left(1+\|\sigma_2(s)\|_H^2 + \|\varphi_1(s)\|_{V_1}^{32} + \|\varphi_2(s)\|_{V_1}^{32}\right) \\
&\quad \times\left(\|(\varphi_1-\varphi_2)(s)\|_H^2 + \|(\sigma_1-\sigma_2)(s)\|_H^2\right)\,\mathrm{d}s.
\end{aligned}
$$

By definition of τ_n, this implies that for every $n\in\mathbb{N}$ there exists a constant $c_n>0$ such that

$$
\begin{aligned}
&\left(\tfrac{1}{2}-\check{\delta}\right)\mathbb{E}\|\varphi_1-\varphi_2\|^2_{L^\infty(0,\tau_n;H)} + \left(\tfrac{1}{2}-\check{\delta}\right)\mathbb{E}\|\sigma_1-\sigma_2\|^2_{L^\infty(0,\tau_n;H)} \\
&\quad + (\bar{m}_1\varepsilon^2 - \check{\delta})\mathbb{E}\|\Delta(\varphi_1-\varphi_2)\|^2_{L^2(0,\tau;H)} + (\bar{m}_2-\check{\delta})\mathbb{E}\|\nabla(\sigma_1-\sigma_2)\|^2_{L^2(0,\tau_n;H)} \\
&\leq \frac{1}{2}\|\varphi_0^1-\varphi_0^2\|_H^2 + \frac{1}{2}\|\sigma_0^1-\sigma_0^2\|_H^2 + c_n\mathbb{E}\|\varphi_1-\varphi_2\|^2_{L^2(0,t;H)} \\
&\quad + c_n\mathbb{E}\|\sigma_1-\sigma_2\|^2_{L^2(0,t;H)}.
\end{aligned}
$$

Choosing $\check{\delta}$ small enough and using the Gronwall lemma yields the conclusion. □

4 Numerical scheme and simulations

In this section, we delve into the details of our numerical discretization scheme (Sect. 4.1) and the resulting simulations on tumor growth (Sect. 4.2).

4.1 Discretization scheme

For an overview of numerical methods for SPDEs, we refer to the work [44] and the books [34, 41, 56]. Regarding the spatial approximation of the stochastic tumor model, we introduce a finite-dimensional space $V_N \subset V$ consisting of quadrilateral elements and denote by Π_N the orthogonal projection $\Pi_N : H \to V_N$. For each variable, we select the bilinear rectangular finite element space Q_1. We seek an approximation $(\varphi, \mu, \sigma) \in V_N \times V_N \times V_N$, defined by

$$\begin{aligned}
(\mathrm{d}\varphi, v)_D &= -(m_1(\varphi)(\nabla\mu - \chi\nabla\sigma), \nabla v)_D + ((\beta\sigma - \alpha) f(\varphi), v)_D + (G_1(\varphi)\,\mathrm{d}W_1, v)_D,\\
(\mu, v)_D &= (\Psi'(\varphi), v)_D + \varepsilon^2(\nabla\varphi, \nabla v)_D\\
(\mathrm{d}\sigma, v)_D &= -(m_2(\sigma)(\nabla\sigma - \chi\nabla\varphi), \nabla v)_D - \delta(\sigma f(\varphi), v)_D + (G_2(\sigma)\,\mathrm{d}W_2, v)_D,
\end{aligned}$$

for any $v \in V_N$ and $t \in [0, T]$, equipped with the initial conditions $\varphi(0) = \Pi_N\varphi_0$ and $\sigma(0) = \Pi_N\sigma_0$. We write the solution in terms of the finite element basis $(w_j)_j$ as follows

$$\varphi(t, x) = \sum_{j=1}^N \varphi_j(t) w_j(x), \quad \mu(t, x) = \sum_{j=1}^N \mu_j(t) w_j(x), \quad \sigma(t, x) = \sum_{j=1}^N \sigma_j(t) w_j(x),$$

and we write the coefficient functions in the vector $\boldsymbol{\varphi} = [\varphi_1, \ldots, \varphi_N]^\top$ and for $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ analogously. Then inserting the ansatz into the system, it gives

$$\begin{aligned}
M\mathrm{d}\boldsymbol{\varphi} &= \left[-K_{m_1(\varphi)}\boldsymbol{\mu} + \chi K_{m_1}\boldsymbol{\sigma} + \beta M_{f(\varphi)}\boldsymbol{\sigma} - \alpha\boldsymbol{f}(\boldsymbol{\varphi})\right]\mathrm{d}t + \boldsymbol{G_1}(\boldsymbol{\varphi})\,\mathrm{d}W_1(t)\\
M\boldsymbol{\mu} &= \boldsymbol{\Psi}'(\boldsymbol{\varphi}) + \varepsilon^2 K\boldsymbol{\varphi},\\
M\mathrm{d}\boldsymbol{\sigma} &= \left[-K_{m_2(\sigma)}\boldsymbol{\sigma} + \chi K_{m_1}\boldsymbol{\varphi} - \delta M_{f(\varphi)}\boldsymbol{\sigma}\right]\mathrm{d}t + \boldsymbol{G_2}(\boldsymbol{\sigma})\,\mathrm{d}W_2(t)
\end{aligned} \tag{4.1}$$

where $M = [(w_i, w_j)_D]_{i,j=1,\ldots,N}$ is the mass matrix, $M_{f(\boldsymbol{\varphi})} = [(f(\boldsymbol{\varphi})w_i, w_j)_D]_{i,j=1,\ldots,N}$ the weighted mass matrix, $K = [(\nabla w_i, \nabla w_j)_D]_{i,j=1,\ldots,N}$ the stiffness matrix, $K_{m_1(\boldsymbol{\varphi})}$ the weighted stiffness matrix, $\boldsymbol{f}(\boldsymbol{\varphi}) = [(f(\varphi), w_j)_D]_{j=1,\ldots,N}$, and analogously for $\boldsymbol{\Psi}$. Finally, $\boldsymbol{G_1} : \mathbb{R}^N \to \mathcal{L}(U_1, \mathbb{R}^N)$ is defined by $\boldsymbol{G_1}(\boldsymbol{\varphi})u = [(G_1(\varphi)u, w_j)_D]_{j=1,\ldots,N}$ for $u \in U_1$ and in the same way for $G_2(\sigma)$.

For the discretization in time, we approximate $\boldsymbol{\varphi}(t)$ at $t = t_n = n\Delta t$ by $\boldsymbol{\varphi}_n$. Because V_N is finite-dimensional, the system consists of SODEs, and we can apply the semi-implicit Euler–Maruyama method with a time step $\Delta t > 0$ to define $\boldsymbol{\varphi}_n$. That is, given $(\boldsymbol{\varphi}_n, \boldsymbol{\mu}_n, \boldsymbol{\sigma}_n)$, we find $(\boldsymbol{\varphi}_{n+1}, \boldsymbol{\mu}_{n+1}, \boldsymbol{\sigma}_{n+1})$ by iterating:

$$\begin{aligned}
M\boldsymbol{\varphi}_{n+1} &= M\boldsymbol{\varphi}_n - K_{m_1^n}\boldsymbol{\mu}_{n+1} + \chi K_{m_1^n}\boldsymbol{\sigma}_{n+1} + \beta M_{f^n}\boldsymbol{\sigma}_n - \alpha\boldsymbol{f}(\boldsymbol{\varphi}_n) + \boldsymbol{G_1}(\boldsymbol{\varphi}_n)\Delta\boldsymbol{W}_{1,n}\\
M\boldsymbol{\mu}_{n+1} &= \boldsymbol{\Psi_c}'(\boldsymbol{\varphi}_{n+1}) + \boldsymbol{\Psi_e}'(\boldsymbol{\varphi}_n) + \varepsilon^2 K\boldsymbol{\varphi}_{n+1},\\
M\boldsymbol{\sigma}_{n+1} &= M\boldsymbol{\sigma}_n - K_{m_2^n}\boldsymbol{\sigma}_{n+1} + \chi K_{m_1^n}\boldsymbol{\varphi}_{n+1} - \delta M_{f^n}\boldsymbol{\sigma}_n + \boldsymbol{G_2}(\boldsymbol{\sigma}_n)\Delta\boldsymbol{W}_{2,n}
\end{aligned} \tag{4.2}$$

for initial data $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}(0)$ and $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}(0)$. Here, we introduced $m_1^n = m_1(\boldsymbol{\varphi}_n)$, $\boldsymbol{G_1}(\boldsymbol{\varphi}_n) \in \mathbb{R}^{N\times N}$ with $\boldsymbol{G_1}(\boldsymbol{\varphi}_n) = [(G_1(\boldsymbol{\varphi}_n)u_k, w_j)_D]_{j,k=1,\ldots,N}$ and $\Delta\boldsymbol{W}_{1,n} \in \mathbb{R}^N$

with $\Delta \boldsymbol{W}_{1,n} = [(W_1(t_{n+1}) - W_1(t_n), u_k)_D]_{k=1,\dots,N}$. and in the same way for G_2 and W_2. Having the Wiener process $W_1(t) = \sum_{j=1}^{\infty} \sqrt{q_j^1} e_j^1 \beta_j^1(t)$ where $(e_j^1)_j$ is an ONB for U_1, we compute $\boldsymbol{G_1}(\boldsymbol{\varphi}_n)\Delta \boldsymbol{W_{1,n}}$ by multiplying the matrix $\boldsymbol{G_1}(\boldsymbol{\varphi}_n) \in \mathbb{R}^{N\times N}$ by the vector

$$\left[\sqrt{q_1}(\beta_1(t_{n+1}) - \beta_1(t_n)), \dots, \sqrt{q_N}(\beta_N(t_{n+1}) - \beta_N(t_n))\right]^\top$$

In fact, we assume that $\boldsymbol{G_1}(\boldsymbol{\varphi}_n)$ is a diagonal matrix with entries $g_1(\boldsymbol{\varphi}_n^k)$ where $\boldsymbol{\varphi}_n^k$ is the k-th entry of $\boldsymbol{\varphi}_n$. The same is assumed for G_2. We use the classical convex-concave splitting of the nonlinear potential $\Psi = \Psi_e + \Psi_c$ into its expansive Ψ_e and contractive part Ψ_c. In fact, in the case of $\Psi(\varphi) = \frac{1}{4}\varphi^2(1-\varphi^2)$ we set $\Psi_e(\varphi) = \varphi^3 - \frac{3}{2}\varphi^2 - \frac{1}{4}\varphi$ and $\Psi_c(\varphi) = \frac{3}{4}\varphi$. We treat the expansive part explicitly and the contractive part implicitly. Like this, we obtain an unconditionally stable scheme in the deterministic case, see [12], which is additionally linear.

The system is implemented in the finite element library FEniCS [2].

4.2 Simulations

We set the model parameters to $\varepsilon = 0.01$, $\chi = 5$, $\alpha = 1$, $\beta = 15$, $\delta = 100$. Moreover, we choose the nonlinear potential Ψ as above and the mobility functions as $m_1(\varphi) = 10^{-16} + \varphi^2(1-\varphi)^2$ and $m_2(\sigma) = 10$. We choose the spatial domain $D = [0, 1]^2$ with uniform mesh size $\mathrm{d}x = 0.01$ and the time domain $[0, 1]$ with $\mathrm{d}t = 0.01$. The initial for the tumor volume fraction is chosen as $\varphi_0(x) = \exp(1 - 1/(1 - 16|x - \frac{1}{2}|^2))$. The nutrient's initial is set to zero, but we choose the boundary condition $\sigma = 1$ on ∂D in contrast to our mathematical analysis. However, the analysis can be carried out in the same way, e.g., see [23] in the case of the deterministic setting.

Regarding the stochasticity in the system, we consider $G_1(\varphi) = \nu\varphi_+(1 - \varphi_+)$ for different noise levels ν in the tumor equation. Here, the subscript $\cdot_+$ indicates a cut-off in the sense of $\varphi_+ = \max\{0, \min\{1, \varphi\}\}$ that ensures the boundedness of φ between 0 and 1. We motivate this choice since we only want to have noise in the interface, since it should be a fundamental part of the tumor's growth process and should not affect fully malignant or healthy cells. Moreover, we consider an additive noise $G_2(\sigma) = 1$ in the nutrients.

Test 1 (Tumor and nutrient mass over time). To comprehensively capture the influence of the noise effects, we conducted 50 simulations, each yielding a unique result. In Fig. 1, we present the mean (average) and standard deviation (a measure of variability) of these 50 samples for two different noise levels. For the low-noise-level simulation, we observe that the mean tumor volume increases over time, which aligns with our baseline expectations from the deterministic setting; for example, see [18]. However, the introduction of noise has a notable impact. In the high-noise-level simulation, the mean tumor volume exhibits a more pronounced increase. This is because the stochasticity introduced by noise acts as a proliferation term in the equation, essentially furthering tumor growth by introducing additional uncertainty into the system. The standard deviation, which quantifies the variability in tumor volume between the

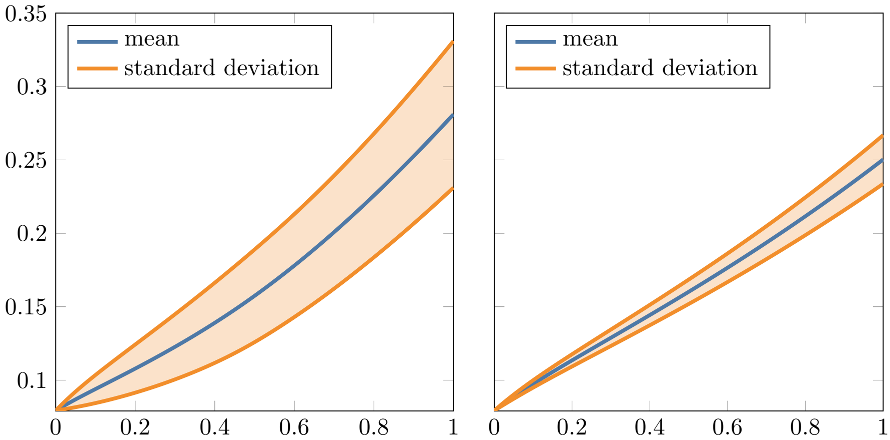

Fig. 1 Considering 50 samples, the mean and standard deviation of the tumor volume $t \mapsto \int_D \varphi(t, x)\,\mathrm{d}x$ is shown for the noise intensities $\nu = 2.5$ (left) and $\nu = 0.5$ (right)

50 samples, reflects the impact of noise. In the case of low-noise level, the standard deviation is smaller and close to the mean. This indicates a relatively consistent and predictable tumor growth pattern with less uncertainty. In contrast, the high-noise-level simulation yields a significantly larger standard deviation, indicating a wider range of possible outcomes, reflecting the increased variability and uncertainty introduced by noise. In addition, we stress that a higher noise level leads to an increase of both the volatility and the mean tumor volume: while the former is surely expectable from the stochastic forcing term, the latter is more surprising if one takes into account that the noise still has null expectation. This behaviuour could be a consequence of the combination of the nonlinear nature of the problem and the presence of $G_1(\varphi)$ as an additional source term. Indeed, while for linear SPDEs the expectation of the solution satisfies the associated deterministic equation (i.e. with no noise), this is not true in general for nonlinear problems: hence, the solutions to the systems with low and high noise intensities are significantly different in expectation. Nonetheless, since this behaviour has important consequences on the evolution of the tumor volume, it surely deserves deeper investigation.

In Fig. 2, we discuss the dynamic evolution of the nutrient volume over time. As in Fig. 1, we present both the mean and standard deviation of 50 simulations for each noise level. Notably, the standard deviation reveals a striking difference between the low- and high-noise-level cases. For the low noise level, the standard deviation is relatively smaller, indicating a more consistent and predictable behavior of the volume of nutrients. In contrast, the high-noise-level simulation yields a significantly larger standard deviation, reflecting the increased variability and uncertainty introduced by the noise. Although the means of nutrient volume in both low- and high-noise level cases initially show similar behavior, they exhibit distinctions over time. In both scenarios, the volume of nutrients initially increases due to the ongoing diffusion process, where nutrients move from the boundary and spread throughout the domain. However, the key divergence occurs as tumor growth progresses. The mean of the nutrient volume

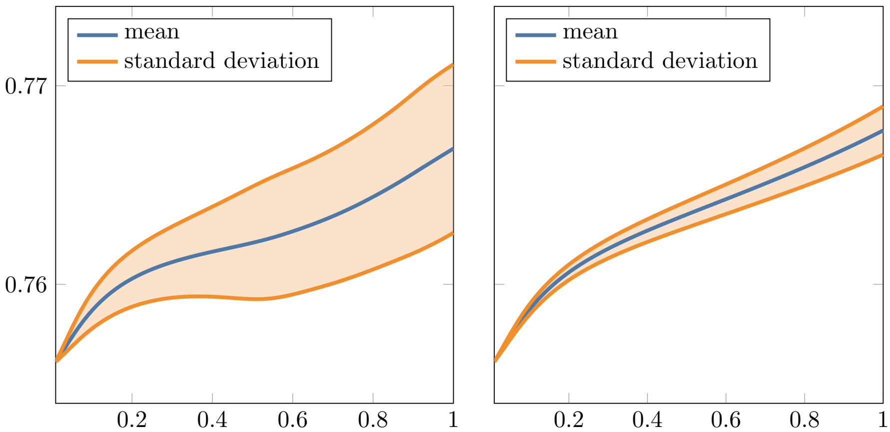

Fig. 2 Considering 50 samples, the mean and standard deviation of the nutrient volume $t \mapsto \int_D \sigma(t, x)\,\mathrm{d}x$ is shown for the noise intensities $\nu = 2.5$ (left) and $\nu = 0.5$ (right)

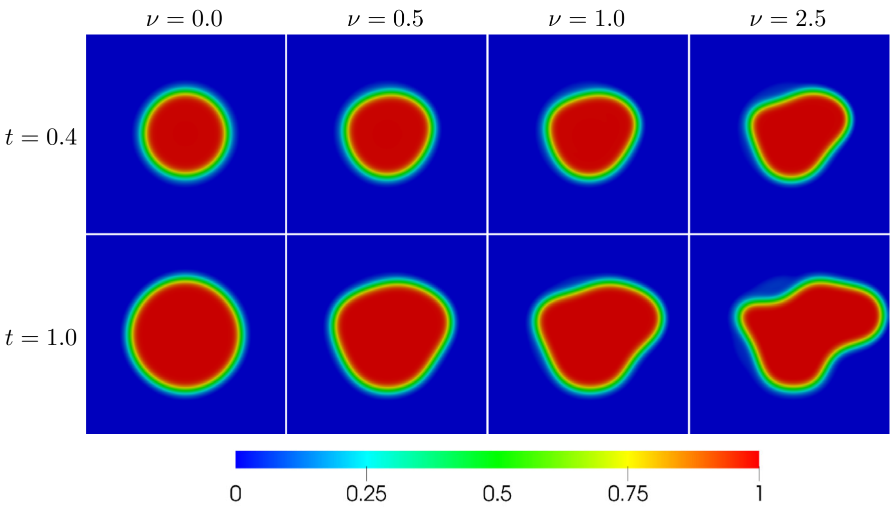

Fig. 3 Evolution of the tumor volume fraction $\varphi(t, x)$ over time in the domain D for increasing noise intensities but for a fixed seed

appears more damped after the initial increase, primarily because of the influence of tumor growth, which results in a decrease in the nutrient level. The damped effect is more pronounced in the high-noise-level simulation, where tumor growth is more substantial.

Test 2 (Tumor and nutrient evolution for different noise levels but same seed). Fig. 3 presents a series of visual representations showcasing the dynamic evolution of a tumor over several time steps. What makes this analysis distinct is the introduction of four different noise levels $\nu \in \{0, 0.5, 1, 2.5\}$. Crucially, to maintain predictability and consistency in the randomness, the same seed for generating random effects was employed in all cases. This ensures that the tumor's movement remains similar across

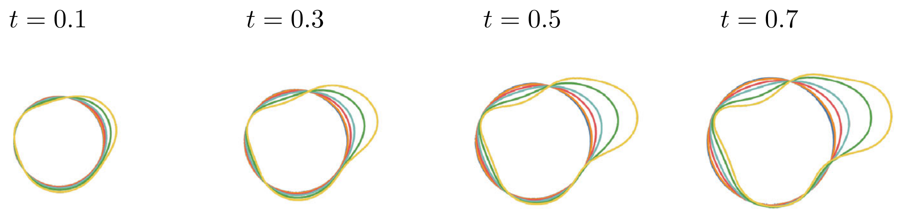

Fig. 4 Contour plots at four distinct time steps visualize the evolving shape of the tumor – contours represent the boundary where 50% of the domain contains tumor cells, showcasing the dynamic progression of the tumor's shape for different noise levels (with fixed seed)

different noise levels. In the most-left row of Fig. 3, we provide a reference point by simulating the tumor evolution in a deterministic setting, where no noise is introduced. In this case, the tumor maintains a perfectly round, circular shape throughout all time steps. This behavior aligns with existing knowledge and expectations, e.g., see [18]. The other rows of Fig. 3 represent the introduction of stochasticity through low, moderate, and high noise levels. As the noise levels increase, the tumor's shape becomes increasingly wobbly. The level of irregularity is more pronounced in the high noise level case, to the extent that it no longer resembles its initial circular shape. The primary observation here is the significant impact of stochasticity on tumor shape evolution. In the presence of noise, the tumor's behavior departs from the deterministic circular shape and takes on an irregular form. This is particularly evident at higher noise levels, where the tumor's movement becomes less predictable and more influenced by random effects.

In Fig. 4, we provide a visual representation of the different tumor shapes in four different time steps, with a particular focus on the interface of the tumor volume fraction. The contour plots in this figure delineate the contour line where the tumor volume fraction equals 50%, i.e. $\varphi(t, x) = 0.5$, effectively marking the boundary between tumor and non-tumor regions. In each of the four highlighted time steps, the contour plot vividly illustrates the evolving shape of the tumor. The most significant observation in Fig. 4 is the variability in the shape of the tumor over time. As the tumor evolves, the contour lines reveal changes in the spatial distribution of the tumor. The variability is particularly pronounced as we transition from the initial state to later time steps, reflecting the influence of noise and stochasticity on the tumor's growth patterns.

Figure 5 provides a series of visual representations showcasing the temporal evolution of nutrient distribution at the four different noise levels $\nu \in \{0, 0.5, 1, 2.5\}$. The nutrient distribution is shown at two time steps, offering insights into the spatial changes in nutrient availability within the domain. We observe that the nutrient concentration is highest at the boundary of the domain. This concentration gradient is a result of the chosen boundary condition, which provides a continuous source of nutrients at the domain's boundary. As time progresses, we witness the diffusion of nutrients from the boundary toward the center of the domain. Nutrient levels decrease as we move closer to the domain's center, reflecting the expected behavior of diffusion. Notably, the presence of the tumor has a significant impact on the nutrient distribu-

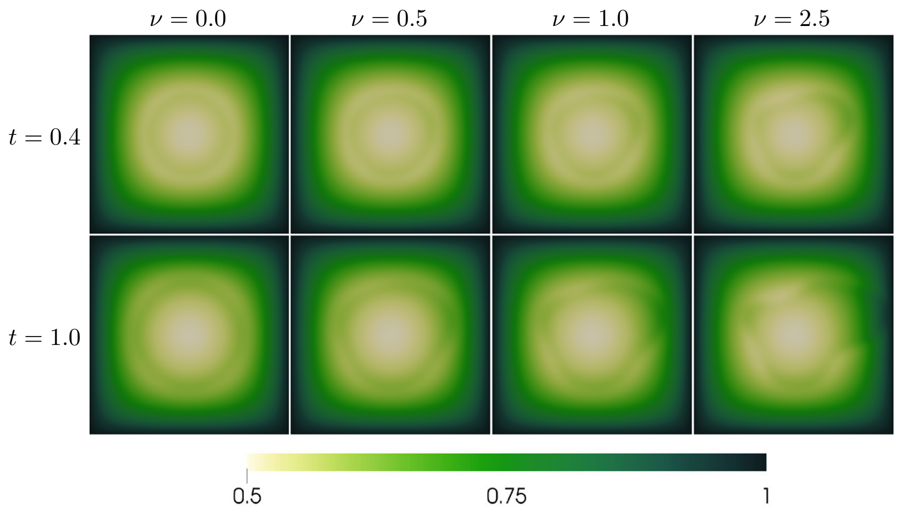

Fig. 5 Temporal evolution of nutrient distribution $\sigma(t, x)$ at the noise levels $\nu \in \{0, 0.5, 1, 2.5\}$ (using a fixed seed)

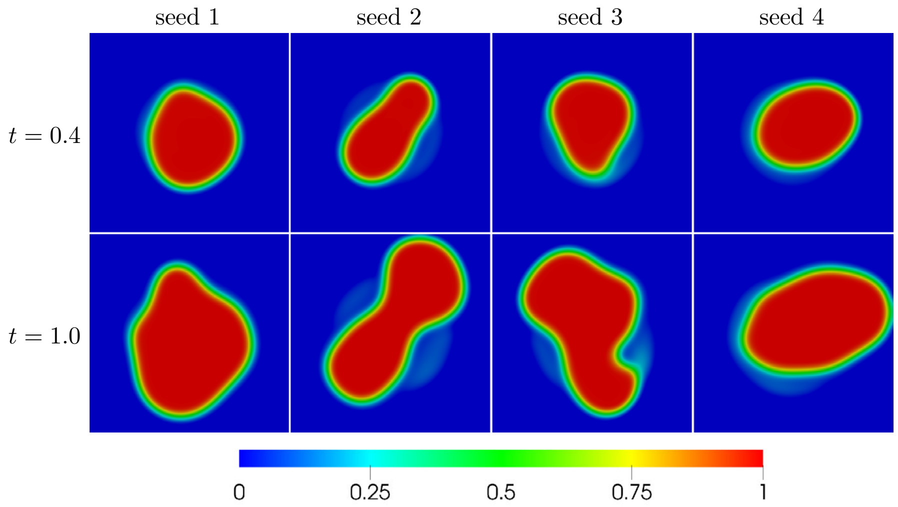

Fig. 6 Evolution of the tumor volume fraction $\varphi(t, x)$ over time in the domain D at a fixed noise intensity $\nu = 2.5$ for four random seeds

tion. As the tumor grows and absorbs nutrients, its interface becomes evident within the nutrient distribution. The nutrient levels are notably reduced in the vicinity of the tumor, indicating the tumor's influence on nutrient availability.

Test 3 (Tumor evolution for different seeds). In Fig. 6, we explore the tumor's evolution under conditions of high noise levels while considering different random seeds. Each simulation is truly random, and the figure showcases the results of four distinct samples at various time steps. At first glance, the figure highlights the significant vari-

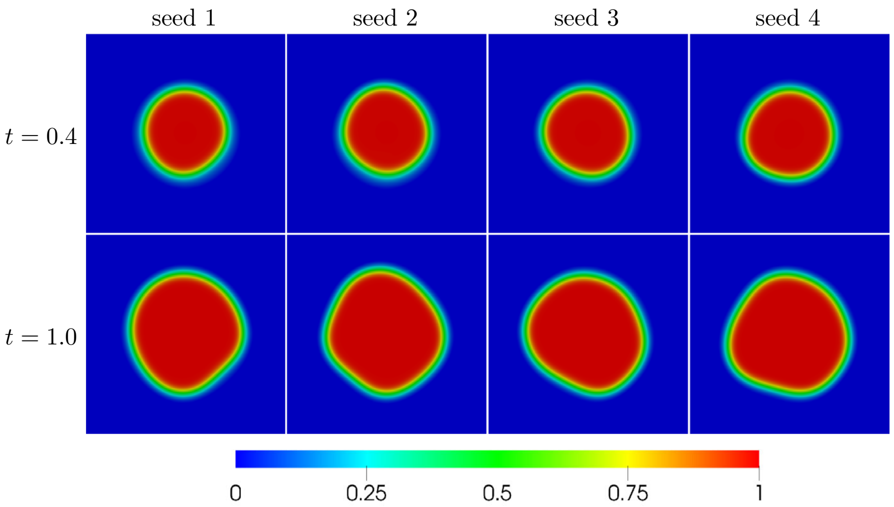

Fig. 7 Evolution of the tumor volume fraction $\varphi(t, x)$ over time in the domain D at a fixed noise intensity $\nu = 0.5$ for four random seeds

ability in tumor shape. The choice of different random seeds leads to entirely distinct tumor shapes, even at earlier time steps. This highlights the inherent randomness and unpredictability introduced by high noise levels, shaping the tumor growth patterns in unique ways for each simulation. In particular, the diversity in tumor shapes is evident even at earlier times, suggesting that the impact of randomness becomes apparent from the onset of the simulations. This underscores the role of high noise levels in influencing tumor behavior and shape diversity.

Figure 7 explores the evolution of the tumor under conditions of low noise levels while considering different random seeds. Similar to Fig. 6, the simulations are influenced by randomness, but in this case, noise levels are lower. In contrast to the high noise level case, Fig. 7 reveals that, at $t = 0.4$, the tumor shapes exhibit some resemblance among different random seeds. While the tumor shapes may slowly drift apart over time, the variability at earlier time steps is not as pronounced as in the higher noise level case. These observations underscore the effect of noise levels on tumor evolution. In low noise scenarios, the tumor shapes at early times exhibit a degree of consistency among different simulations, while high noise levels lead to a wide range of random and distinct shapes from the outset.

In Fig. 8, we present a contour plot that visualizes the impact of different random seeds on the evolution of the tumor. The contour plot shows the results of five distinct samples, with each sample representing a unique simulation outcome at different time steps. This contour plot uses the contour line of 50% tumor cells to demarcate the evolving boundary of the tumor. What is striking in Fig. 8 is the observable divergence in tumor contours over time. Each of the four distinct samples exhibits its own path of tumor growth, highlighting the variability and unpredictability introduced by high noise levels. The contour plot provides a clear visual representation of how the tumor shapes drift apart over time in this high noise level scenario. The initial similarity in

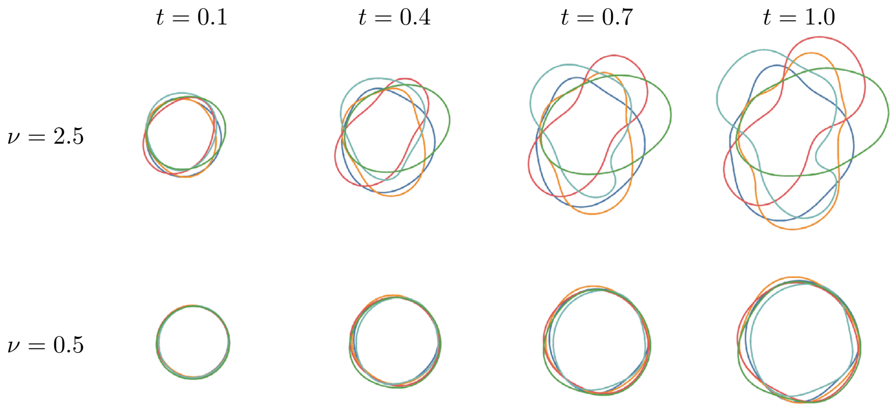

Fig. 8 Evolution of the contour lines $\varphi(t, x) = 0.5$ over time at the fixed noise intensities $\nu \in \{0.5, 2.5\}$ for random seeds

contour shapes gradually gives way to distinct and random tumor contours, underscoring the influence of randomness on tumor growth patterns. Unlike the high-noise scenario, where tumor contours drift apart, the low-noise case reveals the presence of wobbly circular shapes. The contours exhibit a certain consistency among different random seeds, emphasizing that the presence of noise, although present, does not lead to a significant divergence in tumor shapes as in the high-noise case.

Author Contributions Both authors (M.F. and L.S.) contributed equally to all aspects of the manuscript, including writing the main text, conducting the analysis, interpreting the results, contributing to the development of numerical simulations, and preparing figures. Both authors actively participated in the review and revision of the manuscript.

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Data availibility No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

1. Albano, G., Giorno, V.: A stochastic model in tumor growth. J. Theor. Biol. **242**(2), 329–336 (2006)
2. Alnæs, M., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M. E., Wells, G. N.: The FEniCS project version 1.5. *Archive of Numerical Software*, 3(100) (2015)
3. Badri, H., Leder, K.: Optimal treatment and stochastic modeling of heterogeneous tumors. Biol. Direct **11**(1), 1–17 (2016)
4. Barbu, V.: Nonlinear Differential Equations of Monotone Types in Banach Spaces. Springer, Berlin (2010)
5. Breit, D., Feireisl, E., Hofmanová, M.: Stochastically Forced Compressible Fluid Flows. Walter de Gruyter GmbH & Co KG, Berlin (2018)
6. Cherfils, L., Miranville, A., Zelik, S.: The Cahn–Hilliard equation with logarithmic potentials. Milan J. Math. **79**, 561–596 (2011)
7. Cristini, V., Lowengrub, J.: Multiscale Modeling of Cancer: An Integrated Experimental and Mathematical Modeling Approach. Cambridge University Press, Cambridge (2010)
8. Cui, J., Hong, J.: Wellposedness and regularity estimates for stochastic Cahn–Hilliard equation with unbounded noise diffusion. Anal. Comput. Stoch. Partial Differ. Equ.1–37 (2022)
9. Debussche, A., Goudenège, L.: Stochastic Cahn–Hilliard equation with double singular nonlinearities and two reflections. SIAM J. Math. Anal. **43**(3), 1473–1494 (2011)
10. Deugoue, G., Moghomye, B.J., Medjo, T.T.: Splitting-up scheme for the stochastic Cahn–Hilliard Navier–Stokes model. Stoch. Dyn. **21**(01), 2150005 (2021)
11. Edwards, R.E.: Functional Analysis: Theory and Applications. Courier Corporation, Chelmsford (2012)
12. Eyre, D.J.: Unconditionally gradient stable time marching the Cahn–Hilliard equation. MRS Proc. **529**, 39 (1998)
13. Feireisl, E., Petcu, M.: Stability of strong solutions for a model of incompressible two-phase flow under thermal fluctuations. J. Differ. Equ. **267**(3), 1836–1858 (2019)
14. Feireisl, E., Petcu, M.: A diffuse interface model of a two-phase flow with thermal fluctuations. Appl. Math. Optim. **83**(1), 531–563 (2021)
15. Ferrante, L., Bompadre, S., Possati, L., Leone, L.: Parameter estimation in a Gompertzian stochastic model for tumor growth. Biometrics **56**(4), 1076–1081 (2000)
16. Flandoli, F., Gatarek, D.: Martingale and stationary solutions for stochastic Navier–Stokes equations. Probab. Theory Relat. Fields **102**(3), 367–391 (1995)
17. Frigeri, S., Lam, K.F., Rocca, E.: On a diffuse interface model for tumour growth with non-local interactions and degenerate mobilities. In: Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs. Springer INdAM Series, vol. 22, pp. 217–254. Springer, Cham (2017)
18. Fritz, M.: Tumor evolution models of phase-field type with nonlocal effects and angiogenesis. Bull. Math. Biol. **85**(6), 44 (2023)
19. Fritz, M., Jha, P.K., Köppl, T., Oden, J.T., Wagner, A., Wohlmuth, B.: Modeling and simulation of vascular tumors embedded in evolving capillary networks. Comput. Methods Appl. Mech. Eng. **384**, 113975 (2021)
20. Fritz, M., Jha, P.K., Köppl, T., Oden, J.T., Wohlmuth, B.: Analysis of a new multispecies tumor growth model coupling 3D phase-fields with a 1D vascular network. Nonlinear Anal. Real World Appl. **61**, 103331 (2021)
21. Fritz, M., Kuttler, C., Rajendran, M.L., Wohlmuth, B., Scarabosio, L.: On a subdiffusive tumour growth model with fractional time derivative. IMA J. Appl. Math. **86**(4), 688–729 (2021)
22. Fritz, M., Lima, E.A., Nikolić, V., Oden, J.T., Wohlmuth, B.: Local and nonlocal phase-field models of tumor growth and invasion due to ECM degradation. Math. Models Methods Appl. Sci. **29**(13), 2433–2468 (2019)
23. Fritz, M., Lima, E.A., Tinsley Oden, J., Wohlmuth, B.: On the unsteady Darcy–Forchheimer–Brinkman equation in local and nonlocal tumor growth models. Math. Models Methods Appl. Sci. **29**(09), 1691–1731 (2019)
24. Garcke, H., Lam, K.F.: Analysis of a Cahn–Hilliard system with non-zero Dirichlet conditions modeling tumor growth with chemotaxis. Discrete Contin. Dynam. Syst. **37**(8), 4277–4308 (2017)
25. Garcke, H., Lam, K.F.: Well-posedness of a Cahn–Hilliard system modelling tumour growth with chemotaxis and active transport. Eur. J. Appl. Math. **28**(2), 284–316 (2017)

26. Garcke, H., Lam, K.F., Nürnberg, R., Sitka, E.: A multiphase Cahn–Hilliard–Darcy model for tumour growth with necrosis. Math. Models Methods Appl. Sci. **28**(3), 525–577 (2018)
27. Garcke, H., Lam, K.F., Rocca, E.: Optimal control of treatment time in a diffuse interface model of tumor growth. Appl. Math. Optim. **78**(3), 495–544 (2018)
28. Garcke, H., Lam, K.F., Sitka, E., Styles, V.: A Cahn–Hilliard–Darcy model for tumour growth with chemotaxis and active transport. Math. Models Methods Appl. Sci. **26**(06), 1095–1148 (2016)
29. Garcke, H., Trautwein, D.: Numerical analysis for a Cahn–Hilliard system modelling tumour growth with chemotaxis and active transport. J. Numer. Math. **30**(4), 295–324 (2022)
30. Garcke, H., Yayla, S.: Long-time dynamics for a Cahn–Hilliard tumor growth model with chemotaxis. Z. Angew. Math. Phys. **71**(4), 123 (2020)
31. Gurtin, M.E.: Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance. Physica D **92**(3–4), 178–192 (1996)
32. Hawkins-Daarud, A., van der Zee, K.G., Oden, J.T.: Numerical simulation of a thermodynamically consistent four-species tumor growth model. Int. J. Numer. Methods Biomed. Eng. **28**(1), 3–24 (2012)
33. Hawkins-Daarud, A., van der Zee, K.G., Tinsley Oden, J.: Numerical simulation of a thermodynamically consistent four-species tumor growth model. Int. J. Numer. Methods Biomed. Eng. **28**(1), 3–24 (2012)
34. Jentzen, A., Kloeden, P.E.: Taylor Approximations for Stochastic Partial Differential Equations. SIAM, Philadelphia (2011)
35. Kahle, C., Lam, K.F.: Parameter identification via optimal control for a Cahn–Hilliard-chemotaxis system with a variable mobility. Appl. Math. Optim. **82**(1), 63–104 (2020)
36. Lima, E.A., Almeida, R.C., Oden, J.T.: Analysis and numerical solution of stochastic phase-field models of tumor growth. Numer. Methods Partial Differ. Equ. **31**(2), 552–574 (2015)
37. Liotta, L.A., Saidel, G.M., Kleinerman, J.: Stochastic model of metastases formation. Biometrics **32**, 535–550 (1976)
38. Liu, W., Röckner, M.: SPDE in Hilbert space with locally monotone coefficients. J. Funct. Anal. **259**(11), 2902–2922 (2010)
39. Liu, W., Röckner, M.: Local and global well-posedness of SPDE with generalized coercivity conditions. J. Differ. Equ. **254**(2), 725–755 (2013)
40. Liu, W., Röckner, M.: Stochastic Partial Differential Equations: An introduction. Springer, Berlin (2015)
41. Lord, G.J., Powell, C.E., Shardlow, T.: An Introduction to Computational Stochastic PDEs. Cambridge University Press, Cambridge (2014)
42. Medjo, T.T.: On the existence and uniqueness of solution to a stochastic 2D Cahn–Hilliard–Navier–Stokes model. J. Differ. Equ. **263**(2), 1028–1054 (2017)
43. Niemistö, A., Dunmire, V., Yli-Harja, O., Zhang, W., Shmulevich, I.: Analysis of angiogenesis using in vitro experiments and stochastic growth models. Phys. Rev. E **72**, 062902 (2005)
44. Ondreját, M., Prohl, A., Walkington, N.J.: Numerical approximation of nonlinear SPDE's. Stoch. Partial Differ. Equ. Anal. Comput. **11**(4), 1553–1634 (2023)
45. Orrieri, C., Rocca, E., Scarpa, L.: Optimal control of stochastic phase-field models related to tumor growth. ESAIM Control Optim. Calc. Var. **26**, 104 (2020)
46. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes and Martingales: Itô Calculus, vol. 2. Cambridge University Press, Cambridge (2000)
47. Scarpa, L.: On the stochastic Cahn–Hilliard equation with a singular double-well potential. Nonlinear Anal. **171**, 102–133 (2018)
48. Scarpa, L.: The stochastic Cahn–Hilliard equation with degenerate mobility and logarithmic potential. Nonlinearity **34**(6), 3813 (2021)
49. Scarpa, L.: The stochastic viscous Cahn–Hilliard equation: well-posedness, regularity and vanishing viscosity limit. Appl. Math. Optim. **84**(1), 487–533 (2021)
50. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. **146**(1), 65–96 (1986)
51. Speer, J.F., Petrosky, V.E., Retsky, M.W., Wardwell, R.H.: A stochastic numerical model of breast cancer growth that simulates clinical data. Can. Res. **44**(9), 4124–4130 (1984)
52. Sung, H., Ferlay, J., Siegel, R.L., Laversanne, M., Soerjomataram, I., Jemal, A., Bray, F.: Global cancer statistics 2020: GLOBOCAN estimates of incidence and mortality worldwide for 36 cancers in 185 countries. CA Cancer J. Clin. **71**(3), 209–249 (2021)
53. Tan, W., Chen, C.: Stochastic modeling of carcinogenesis: some new insights. Math. Comput. Model. **28**(11), 49–71 (1998)

54. Wagner, A., Schlicke, P., Fritz, M., Kuttler, C., Oden, J.T., Schumann, C., Wohlmuth, B.: A phase-field model for non-small cell lung cancer under the effects of immunotherapy. Math. Biosci. Eng. **20**(10), 18670–18694 (2023)
55. Wise, S., Lowengrub, J., Frieboes, H., Cristini, V.: Three-dimensional multispecies nonlinear tumor growth—I: model and numerical method. J. Theor. Biol. **253**(3), 524–543 (2008)
56. Zhang, Z., Karniadakis, G.: Numerical Methods for Stochastic Partial Differential Equations with White Noise. Springer, Berlin (2017)