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Article *in* Lecture Notes in Mathematics -Springer-verlag- · May 2009

DOI: 10.1007/978-3-642-01763-6_15

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Summary. We systematically compare the Hamiltonian and Markovian approaches of quantum open system theory, in the case of the spin-boson model. We first give a complete proof of the weak coupling limit and we compute the Lindblad generator of this model. We study properties of the associated quantum master equation such as decoherence, detailed quantum balance and return to equilibrium at inverse temperature $0 < \beta \leq \infty$. We further study the associated quantum Langevin equation, its associated interaction Hamiltonian. We finally give a quantum repeated interaction model describing the spin-boson system where the associated Markovian properties are satisfied without any assumption.

1 Introduction

In the quantum theory of irreversible evolutions two different approaches have usually been considered by physicists as well as mathematicians: the Hamiltonian and the Markovian ones.

The Hamiltonian approach consists in giving a full Hamiltonian model for the interaction of a simple quantum system with a quantum field (particle gas, heat bath...) and to study the ergodic properties of the associated quantum dynamical system. The usual tools are then typically: modular theory of von Neumann algebras, KMS states...(cf [BR96], [DJP03], [JP96a], [JP96b]).

The Markovian approach consists in giving up the idea of modeling the environment and concentrating on the effective dynamics of the small system. This dynamics is supposed to be described by a (completely positive) semigroup and the studies concentrate on its Lindblad generator, or on the associated quantum Langevin equation (cf [F06], [F99], [F93], [FR06], [FR98], [P92], [HP84], [M95]).

In this article we systematically compare the two approaches in the case of the well-known spin-boson model. The first step in relating the Hamiltonian and Markovian models is to derive the Lindblad generator from the Hamiltonian description, by means of the weak coupling limit. We indeed give a

C. Donati-Martin et al. (eds.), *Séminaire de Probabilités XLII*,
Lecture Notes in Mathematics 1979, DOI 10.1007/978-3-642-01763-6_15,

complete proof of the convergence of the Hamiltonian evolution to a Lindblad semigroup in the van Hove limit. We derive an explicit form for the generator in terms of Hamiltonian, this is treated in section 3.

In section 4, 5 and 6 we study the basic properties of the quantum master equation associated to the Lindbladian obtained in section 3. We investigate the quantum decoherence property. We show that the quantum detailed balance condition is satisfied with respect to the thermodynamical equilibrium state of the spin system and we prove the convergence to equilibrium in all cases.

In section 7 we consider the natural quantum Langevin equation associated to the Lindblad generator of the spin-boson system. We indeed introduce a natural unitary ampliation of the quantum master equation in terms of a Schrödinger equation perturbed by quantum noises. Such a quantum Langevin equation is actually a unitary evolution in the interaction picture, we compute the associated Hamiltonian which we compare to the initial Hamiltonian.

Finally, we give a quantum repeated interaction model which allows to prove that the Markovian properties of the spin-boson system are satisfied without assuming any hypothesis.

2 The Model

2.1 Spin-boson System

The model we shall consider all along this article is the spin-boson model, that is, a two level atom interacting with a reservoir modelled by a free Bose gas at thermal equilibrium for the temperature $T = \frac{1}{k\beta}$ (the case of zero temperature, i.e., $\beta = \infty$, is also treated). Let us start by defining the spin-boson system at positive temperature. We first introduce the isolated spin and the free reservoir, and we describe the coupled system.

The Hilbert space of the isolated spin is $\mathcal{K} = \mathbb{C}^2$ and its Hamiltonian is $h_S = \sigma_z$, where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The associated eigenenergies are $e_\pm = \pm 1$ and we denote the corresponding eigenstates by $\Psi_\pm$. The algebra of observables of the spin is M_2, the algebra of all complex 2×2 matrix. At inverse temperature β, the equilibrium state of the spin is the normal state defined by the Gibbs Ansatz

$$\omega_S(A) = \frac{1}{Z}\mathrm{Tr}(\exp(-\beta\sigma_z)A), \text{ for all } A \in M_2,$$

where $Z = \mathrm{Tr}(\exp(-\beta\sigma_z))$.
The dynamics of the spin is defined as

$$\tau_S^t(A) = e^{it\sigma_z} A e^{-it\sigma_z}, \quad \text{for all } \; A \in M_2, \; t \in \mathbb{R}.$$

The free reservoir is modelled by a free Bose gas which is described by the symmetric Fock space $\Gamma_s(L^2(\mathbb{R}^3))$. If we call $\omega(k) = |k|$ the energy of a single boson with momentum $k \in \mathbb{R}^3$, then the Hamiltonian of the reservoir is given by the second differential quantization $d\Gamma(\omega)$ of ω. In terms of the usual creation and annihilation operators $a^*(k)$, $a(k)$, we have

$$d\Gamma(\omega) = \int_{\mathbb{R}^3} \omega(k) a^*(k) a(k) dk.$$

The Weyl's operator associated to an element $f \in L^2(\mathbb{R}^3)$ is the operator

$$W(f) = \exp(i\varphi(f)),$$

where $\varphi(f)$ is the self-adjoint field operator defined by

$$\varphi(f) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \left(a(k)\bar{f}(k) + a^*(k) f(k)\right) dk.$$

Call $\mathcal{D}_{loc}$ the space of $f \in L^2(\mathbb{R})$ with compactly supported Fourier transform. It follows from [JP96b] that the Weyl's algebra, $\mathcal{A}_{loc} = W(\mathcal{D}_{loc})$, the algebra generated by the set $\{W(f),\ f \in \mathcal{D}_{loc}\}$ is a natural minimal set of observables associated to the reservoir. The equilibrium state of the reservoir at inverse temperature β is given by

$$\omega_R(W(f)) = \exp\Big[-\frac{\|f\|^2}{4} - \frac{1}{2} \int_{\mathbb{R}^3} |f(k)|^2 \rho(k)\, dk \Big],$$

where $\rho(k)$ is related to $\omega(k)$ by Planck's radiation law

$$\rho(k) = \frac{1}{e^{\beta\omega(k)} - 1}.$$

The dynamics of the reservoir is generated by $H_b = [d\Gamma(\omega), .]$ and it induces a Bogoliubov transformation

$$\exp(itd\Gamma(\omega)) W(f) \exp(-itd\Gamma(\omega)) = W(\exp(i\omega t) f).$$

The coupled system is described by the $\mathbb{C}^*-$algebra $M_2 \otimes \mathcal{A}_{loc}$. The free dynamics is given by

$$\tau_0^t(A) = \tau_S^t \otimes \tau_R^t(A), \quad \text{for all } A \in M_2 \otimes \mathcal{A}_{loc}.$$

2.2 Semistandard Representation

The semistandard representation of the coupled system (reservoir+spin) is the representation which is standard on its reservoir part, but not standard on the spin part (cf [DF06]). Now, let us introduce the Araki-Woods representation of the couple $(\omega_R, \mathcal{A}_{loc})$ which is the triple $(\mathcal{H}_R,\ \pi_R,\ \Omega_R)$, defined by

- $\mathcal{H}_R = l^2(\Gamma_s(L^2(\mathbb{R}^3)))$, the space of Hilbert-Schmidt on $\Gamma_s(L^2(\mathbb{R}^3))$ which is naturally identified as $\Gamma_s(L^2(\mathbb{R}^3)) \otimes \overline{\Gamma_s(L^2(\mathbb{R}^3))}$ and equipped with the scalar product $(X, Y) = \mathrm{Tr}(X^*Y)$,
- $\pi_R(W(f)) : X \longmapsto W((1+\rho)^{1/2}f)XW(\rho^{1/2}\bar{f})$ for all $X \in \mathcal{H}_R$,
- $\Omega_R = |\Omega\rangle\langle\Omega|$, where Ω is the vacuum vector of $\Gamma_s(L^2(\mathbb{R}^3))$.

Moreover a straightforward computation shows that

$$\omega_R(A) = (\Omega_R,\ \pi_R(A)\Omega_R),$$

and the relation

$$\pi_R(\exp(itd\Gamma(\omega))A\exp(-itd\Gamma(\omega))) = \exp(it[d\Gamma(\omega), .])\pi_R(A)\exp(-it[d\Gamma(\omega), .])$$

defines a dynamics on $\mathcal{M}_R = \pi_R(\mathcal{A}_{loc})''$ whose generator is the operator

$$L_R = [d\Gamma(\omega), .].$$

The free semi-Liouvillean associated to the semistandard representation of the spin-boson system is defined by

$$L_0^{semi} = \sigma_z \otimes 1 + 1 \otimes L_R.$$

The full semi-Liouvillean is the operator

$$L_\lambda^{semi} = L_0^{semi} + \lambda\sigma_x \otimes \varphi_{AW}(\alpha),$$

where $\lambda \in \mathbb{R}$, and where $\alpha \in L^2(\mathbb{R}^3)$ is called the test function (or cut-off function), $\varphi_{AW}(\alpha)$ is the field operator of the Araki-Woods representation which can be identified as follows

$$\varphi_{AW}(\alpha) \simeq \varphi((1+\rho)^{1/2}\alpha) \otimes 1 + 1 \otimes \varphi(\bar{\rho}^{1/2}\bar{\alpha})$$

(see [JP96b], [DJ03] for more details) and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following proposition follows from [JP96b].

Proposition 2.1 *If $(\omega+\omega^{-1})\alpha$ is in $L^2(\mathbb{R}^3)$, the operator L_λ^{semi} is essentially self-adjoint on $\mathbb{C}^2 \otimes D(d\Gamma(\omega)) \otimes D(d\Gamma(\omega))$ for all $\lambda \in \mathbb{R}$.*

An immediate consequence of the above proposition is that

$$\tau_\lambda^t(A) = e^{itL_\lambda^{semi}} A e^{-itL_\lambda^{semi}}$$

defines a dynamics on $\mathcal{M} = M_2 \otimes \mathcal{M}_R$.

2.3 Reservoir 1-particle Space

After taking the Araki-Woods representation of the pair $(\omega_R, \mathcal{A}_{loc})$, we distinguish that the reservoir state is a non-Fock state (i.e., it cannot be represented as a pure state on a Fock space) and this case is more complicated to treat. By using the identifications given in [DJ03] and [JP96a], we see that this state can be represented as a pure state on a Fock space. Hence we have

$$\begin{aligned} &\Gamma_s(L^2(\mathbb{R}^3)) \otimes \overline{\Gamma_s(L^2(\mathbb{R}^3))} \\ &\qquad\qquad \simeq \Gamma_s(L^2(\mathbb{R}^3)) \otimes \Gamma_s(\overline{L^2(\mathbb{R}^3)}) \simeq \Gamma_s(L^2(\mathbb{R}^3) \oplus \overline{L^2(\mathbb{R}^3)}), \\ &L_R \simeq d\Gamma(\omega \oplus -\overline{\omega}), \\ &\varphi_{AW}(\alpha) \simeq \varphi((1+\rho)^{1/2}\alpha \oplus \bar{\rho}^{1/2}\bar{\alpha}), \\ &\Omega_R \simeq \Omega \oplus \bar{\Omega}. \end{aligned}$$

Therefore, it is obvious that ω_R is a pure state which is defined on the Fock space $\Gamma_s(L^2(\mathbb{R}^3) \oplus \overline{L^2(\mathbb{R}^3)})$. Moreover we have the Bogoliubov transformation

$$e^{itd\Gamma(\omega\oplus-\bar{\omega})}\varphi_{AW}(\alpha)e^{-itd\Gamma(\omega\oplus-\bar{\omega})} = \varphi_{AW}(e^{it\omega}\alpha).$$

This simplifies our formulation.

3 Weak Coupling Limit of the Spin-Boson System

3.1 Abstract Theory of the Weak Coupling Limit

Let $\mathcal{Y}$ be a Banach space and $\mathcal{X}$ its dual, i.e., $\mathcal{X} = \mathcal{Y}^*$. Let P be a projection on $\mathcal{X}$ and $e^{it\delta_0}$ a one parameter group of isometries on $\mathcal{X}$ which commutes with P. Put $E = P\delta_0$. It is clear that E is the generator of a one parameter group of isometries on Ran$P$. Consider a perturbation Q of δ_0 such that $\mathcal{D}(Q) \supset \mathcal{D}(\delta_0)$.

We introduce the following assumptions:

(1) P is a w^*-continuous projection on $\mathcal{X}$ with norm is equal to one.
(2) $e^{it\delta_0}$ a one parameter group of w^*-continuous isometries (C_0^*-group) on $\mathcal{X}$,
(3) For $|\lambda| < \lambda_0$, $i\delta_\lambda = i\delta_0 + i\lambda Q$ is the generator of a one parameter C_0^*-semigroup of contractions.

Consider now the operator

$$K_\lambda(t) = i\int_0^{\lambda^{-2}t} e^{-is(E+\lambda PQP)}PQe^{is(1-P)\delta_\lambda(1-P)}QP\,ds.$$

For the proof of the following theorem we refer the interested reader to [DF06].

Theorem 3.1 *Suppose that assumptions* (1), (2) *and* (3) *are true. Assume that the following hypotheses are satisfied:*

(4) P is a finite range projection and $PQP = 0$,
(5) For all $t_1 > 0$, there exists a constant c such that

$$\sup_{|\lambda|<1} \sup_{0 \le t \le t_1} \|K_\lambda(t)\| \le c.$$

(6) There exists an operator K defined on Ran$P$ *such that*

$$\lim_{\lambda \to 0} K_\lambda(t) = K$$

for all $0 < t < \infty$.
Put

$$K^\sharp = \sum_{e \in spE} 1\!\!1_e(E) K 1\!\!1_e(E) = \lim_{T \to \infty} \frac{1}{T} \int_0^T e^{itE} K e^{-itE} dt.$$

Then we have
i) $e^{itK^\sharp}$ is a semigroup of contractions,
ii) For all $t_1 > 0$,

$$\lim_{\lambda \to 0} \sup_{0 \le t \le t_1} \|e^{-itE/\lambda^2} P e^{it(\delta_0 + \lambda Q)/\lambda^2} P - e^{itK^\sharp}\| = 0.$$

3.2 Application to the Spin-boson System

Recall that in the semistandard representation of the spin-boson system, the free semi-Liouvillean is the operator

$$L_0^{semi} = \sigma_z \otimes 1 + 1 \otimes L_R,$$

and the full semi-Liouvillean is given by

$$L_\lambda^{semi} = L_0^{semi} + \lambda \sigma_x \otimes \varphi_{AW}(\alpha).$$

Set $V = \sigma_x \otimes \varphi_{AW}(\alpha)$. Put

$$\delta_\lambda = [L_\lambda^{semi}, .] = \delta_0 + \lambda [V, .],$$

with $\delta_0 = [L_0^{semi}, .]$, the generator of the dynamics τ_λ^t. For $B \otimes C \in \mathcal{M}$, we define the projection P by

$$P(B \otimes C) = \omega_R(C) B \otimes 1_{\mathcal{H}_R}.$$

In particular we have

$$E = P\delta_0 = \delta_0 P = [\sigma_z, .]P \text{ and } P[V, .]P = 0.$$

Set $P_1 = 1 - P$. Then it follows that

$$K_\lambda(t) = i\int_0^{\lambda^{-2}t} e^{-isE}P[V,.]e^{isP_1[L_\lambda^{semi},.]P_1}[V,.]P\,ds.$$

Note that $P[V,.]P = 0,\ P_1$ commutes with $[L_0^{semi},.]$ and

$$e^{isP_1[L_0^{semi},.]P_1} = e^{is[L_0^{semi},.]}P_1 + P.$$

Thus, if we suppose that

$$K = i\int_0^{\infty} e^{-isE}P[V,.]e^{isP_1[L_0^{semi},.]P_1}[V,.]P\,ds$$

exists, we have

$$K = i\int_0^{\infty} e^{-isE}P[V,.]e^{is[L_0^{semi},.]}[V,.]P\,ds.$$

In the following we assume that $(\omega + \omega^{-1})\alpha \in L^2(\mathbb{R}^3)$ and we propose to show, under some conditions, that K exists and the operator K_λ converges to K when $\lambda \to 0$. Set

$$U_t^\lambda = e^{itP_1[L_\lambda^{semi},.]P_1},\ \ U_t = e^{itP_1[L_0^{semi},.]P_1}.$$

We thus have

$$U_t^\lambda = U_t + i\lambda\int_0^t U_{t-s}P_1[V,.]P_1U_s^\lambda\,ds.$$

Hence, the operator $U_{-t}U_t^\lambda$ satisfies the equation

$$U_{-t}U_t^\lambda = 1 + i\lambda\int_0^t (U_{-s}P_1[V,.]P_1U_s)(U_{-s}U_s^\lambda)\,ds.$$

Therefore, we get the following series of iterated integrals

$$U_{-t}U_t^\lambda = 1 + \sum_{n\geq 1}(i\lambda)^n \int_{0\leq t_n\leq \ldots\leq t_1\leq t}(U_{-t_1}P_1[V,.]P_1U_{t_1})\ldots$$
$$(U_{-t_n}P_1[V,.]P_1U_{t_n})\,dt_n\ldots dt_1.$$

Note that the operator U_{t_k} commutes with P_1. So, if we put

$$Q_k = U_{-t_k}[V,.]U_{t_k},$$

then

$$U_{-t}U_t^\lambda = 1 + \sum_{n\geq 1}(i\lambda)^n\int_{0\leq t_n\leq \ldots.\leq t_1\leq t}(P_1Q_1P_1)\ldots(P_1Q_nP_1)\,dt_n\ldots dt_1,$$

and

$$
\begin{aligned}
K_\lambda(t) = & \; i\int_0^{\lambda^{-2}t} e^{-isE}P[V,.]e^{isP_1[L_0^{semi},.]P_1}[V,.]P\,ds \\
& + i\sum_{n\geq 1}(i\lambda)^n \int_{0\leq t_n\leq \dots \leq t_0\leq \lambda^{-2}t} e^{-it_0E}P[V,.]U_{t_0}(P_1Q_1P_1)\dots \\
& (P_1Q_nP_1)[V,.]P\,dt_n\dots dt_0.
\end{aligned} \tag{1}
$$

Put

$$
R_n(t) = \int_{0\leq t_n\leq \dots \leq t_0\leq t} e^{-it_0E}P[V,.]U_{t_0}(P_1Q_1P_1)\dots.(P_1Q_nP_1)[V,.]P\,dt_n\dots dt_0.
$$

Recall that $PU_{-t_0} = P$. Hence, if we set $Q_{n+1} = U_{-t_{n+1}}[V,.]U_{t_{n+1}}$, with $t_{n+1} = 0$, we get

$$
R_n(t) = \int_{0\leq t_n\leq \dots \leq t_0\leq t} e^{-it_0E}PQ_0(P_1Q_1P_1)\dots(P_1Q_nP_1)Q_{n+1}P\,dt_n\dots dt_0. \tag{2}
$$

Lemma 3.2

$$
\begin{aligned}
R_n(t) = & \int_{0\leq t_n\leq \dots \leq t_0\leq t} P[\sigma_{x,0}\otimes \varphi_{AW}(e^{-it_0\omega}\alpha),.]P_1\dots \\
& P_1[\sigma_{x,n+1}\otimes \varphi_{AW}(e^{-it_{n+1}\omega}),.]P\,dt_n\dots dt_0,
\end{aligned}
$$

where $t_{n+1} = 0,\ \sigma_{x,r} = e^{-it_r\sigma_z}\sigma_x e^{it_r\sigma_z}$.

Proof. Let us start by computing $P_1Q_rP_1$ for $r \geq 1$. We have

$$
U_{t_r} = e^{it_r[\sigma_z,.]}e^{it_r[L_R,.]}P_1 + P,
$$

and

$$
U_{t_r}P_1 = e^{it_r[\sigma_z,.]}e^{it_r[L_R,.]}P_1.
$$

Therefore, it follows that

$$
P_1U_{-t_r}[V,.]U_{t_r}P_1 = P_1e^{-it_r[\sigma_z,.]}e^{-it_r[L_R,.]}[V,.]e^{it_r[\sigma_z,.]}e^{it_r[L_R,.]}P_1.
$$

Furthermore we have

$$
\begin{aligned}
& e^{-it_r[\sigma_z,.]}e^{-it_r[L_R,.]}[V,.]e^{it_r[\sigma_z,.]}e^{it_r[L_R,.]}(B\otimes C) \\
& \qquad = [\sigma_{x,r}\otimes e^{-it_rL_R}\varphi_{AW}(\alpha)e^{it_rL_R},.](B\otimes C),
\end{aligned}
$$

and

$$
e^{-it_rL_R}\varphi_{AW}(\alpha)e^{it_rL_R} = \varphi_{AW}(e^{-it_r\omega}\alpha).
$$

This gives

$$
P_1Q_rP_1 = P_1[\sigma_{x,r}\otimes \varphi_{AW}(e^{-it_r\omega}\alpha),.]P_1.
$$

Besides, $Pe^{-t_0[\sigma_z,.]} = Pe^{-it_0[\sigma_z,.]}e^{-it_0[L_R,.]}$ and

$$\begin{aligned} e^{-it_0E}PQ_0P_1 &= Pe^{-it_0[\sigma_z,.]}[V,.]e^{it_0[\sigma_z,.]}e^{it_0[L_R,.]}P \\ &= Pe^{-it_0[\sigma_z,.]}e^{-it_0[L_R,.]}[V,.]e^{it_0[\sigma_z,.]}e^{it_0[L_R,.]}P \\ &= P[\sigma_{x,0} \otimes \varphi_{AW}(e^{-it_0\omega}\alpha),.]P_1. \end{aligned}$$

Thus from relation (2), the lemma holds.

Lemma 3.3

$$R_{2n+1}(t) = 0.$$

Proof. Note that

$$\begin{aligned} &P[\sigma_{x,0} \otimes \varphi_{AW}(e^{-it_0\omega}\alpha),.]P_1....P_1[\sigma_{x,2n+2} \otimes \varphi_{AW}(e^{-it_{2n+2}\omega}\alpha),.]P \\ &= P[\sigma_{x,0} \otimes \varphi_{AW}(e^{-it_0\omega}\alpha),.](1-P)[\sigma_{x,1} \otimes \varphi_{AW}(e^{-it_1\omega}\alpha),.](1-P)... \\ &...(1-P)[\sigma_{x,2n+2} \otimes \varphi_{AW}(e^{-it_{2n+2}\omega}\alpha),.]P. \end{aligned} \tag{3}$$

Therefore, if we expand the right-hand side of equation (3), we get a sum of terms each of which is a product of elements of the form

$$P[\sigma_{x,p_k} \otimes \varphi_{AW}(e^{-it_{p_k}\omega}\alpha),.]....[\sigma_{x,p_m} \otimes \varphi_{AW}(e^{-it_{p_m}\omega}\alpha),.]P,$$

where $0 \leq p_k \leq ... \leq p_m \leq ... \leq 2n+2$. But, in each product there exists at least an element of the form

$$P[\sigma_{x,r_1} \otimes \varphi_{AW}(e^{-it_{r_1}\omega}\alpha),.]....[\sigma_{x,r_{2p+1}} \otimes \varphi_{AW}(e^{-it_{r_{2p+1}}\omega}\alpha),.]P,$$

where $0 \leq r_1 \leq ... \leq r_{2p+1} \leq ... \leq r_{2n+2}$. Furthermore, it is easy to show that

$$[\sigma_{x,r_1} \otimes \varphi_{AW}(e^{-it_{r_1}\omega}\alpha),.]....[\sigma_{x,r_{2p+1}} \otimes \varphi_{AW}(e^{-it_{r_{2p+1}}\omega}\alpha),.]P(B \otimes C)$$

is a sum of terms each of which has a second component composed by $2p+1$ number product of vector fields. But the projection P acts uniquely in the second component and the Gibbs state ω_R of the reservoir is a quasi-free state (see [BR]). Then it follows that

$$P[\sigma_{x,r_1} \otimes \varphi_{AW}(e^{-itr_1\omega}\alpha),.]....[\sigma_{x,r_{2p+1}} \otimes \varphi_{AW}(e^{-it_{r_{2p+1}}\omega}\alpha),.]P(B \otimes C) = 0,$$

and by Lemma 3.2, $R_{2n+1}(t) = 0$.

Remark 2: From the proof of Lemma 3.3 we can deduce that $R_{2n}(t)$ is a sum of 2^n terms each of which is a product containing only an even number of products of commutators of the form $[\sigma_{x,r} \otimes \varphi_{AW}(e^{-it_r\omega}\alpha),.]$ between two successive projections P.

Theorem 3.4 *Suppose that the following assumptions hold:*
(i) $\|R_{2n}(t)\| \leq c_n t^n$, *where the series* $\sum_{n\geq 1} c_n t^n$ *has infinite radius of convergence.*

(ii) There exists $0 < \varepsilon < 1$ *and a sequence* $d_n \geq 0$ *such that*

$$\|R_{2n}(t)\| \leq d_n t^{n-\epsilon}.$$

Then

$$\lim_{\lambda \to 0} \sum_{n \geq 1} (i\lambda)^n R_{2n}(\lambda^{-2} t) = 0.$$

Proof. The proof of this theorem is a straightforward application of Lebesgue's Theorem.

Now, the aim is to introduce some conditions which ensures that assumptions (i) and (ii) of the above theorem are satisfied. Set

$$h(t) = \langle e^{-itL_R} \varphi_{AW}(\alpha) e^{itL_R} \varphi_{AW}(\alpha) \Omega_R, \Omega_R \rangle.$$

Recall that

$$L_R = [d\Gamma(\omega), .] \simeq d\Gamma(\omega \oplus -\overline{\omega})$$

and

$$e^{-itL_R} \varphi_{AW}(\alpha) e^{itL_R} = \varphi_{AW}(e^{-it\omega}\alpha).$$

Therefore we get

$$h(t) = \langle \varphi_{AW}(e^{-it\omega}\alpha) \varphi_{AW}(\alpha) \Omega_R, \Omega_R \rangle.$$

Moreover, a straightforward computation shows that

$$h(t-s) = \langle \varphi_{AW}(e^{-it\omega}\alpha) \varphi_{AW}(e^{-is\omega}\alpha) \Omega_R, \Omega_R \rangle.$$

Now, for any integer n we define the set $\mathcal{P}_n$ of pairings as the set of permutations σ of $(1, ..., 2n)$ such that

$$\sigma(2r-1) < \sigma(2r) \ \text{ and } \sigma(2r-1) < \sigma(2r+1)$$

for all r. Put

$$\begin{aligned} \langle \varphi_{AW}(\alpha_1) ... \varphi_{AW}(\alpha_n) \rangle &= \omega_R(\varphi_{AW}(\alpha_1) ... \varphi_{AW}(\alpha_n)) \\ &= \langle \Omega_R, \, \varphi_{AW}(\alpha_1) ... \varphi_{AW}(\alpha_n) \Omega_R \rangle. \end{aligned}$$

If $n = 2$ then $\langle \varphi_{AW}(\alpha_1) \varphi_{AW}(\alpha_2) \rangle$ is called the two point correlations matrix. Besides, we have

$$\langle \varphi_{AW}(\alpha_1) ... \varphi_{AW}(\alpha_{2n}) \rangle = \sum_{\sigma \in \mathcal{P}_n} \prod_{r=1}^{n} \langle \varphi_{AW}(\alpha_{\sigma(2r-1)}) \varphi_{AW}(\alpha_{\sigma(2r)}) \rangle, \quad (4)$$

and

$$\langle \varphi_{AW}(\alpha_1) ... \varphi_{AW}(\alpha_{2n+1}) \rangle = 0.$$

(see [BR96] P 40 for more details).
The proof of the following lemma is similar to the one of Lemma 3.2 in [D74].

Lemma 3.5 *If* $\|h\|_1 \leq \infty$, *then for any permutation* π *of* $(0, 1, ..., 2n+1)$ *we have*

$$\Big|\sum_{\sigma\in\mathcal{P}_{(0,1,\dots,2n+1)}}\int_{0\leq t_{2n}\leq\ldots\leq t_0\leq t}\prod_{r=0}^{n} h(t_{\pi\sigma(2r)}-t_{\pi\sigma(2r+1)})dt_{2n}...dt_0\Big|$$
$$\leq \frac{1}{2^{n+1}(n+1)!}\|h\|_1^{n+1}t^n,$$

with $t_{2n+1} = 0$.

We now prove the following.

Theorem 3.6 *If* $\|h\|_1 \leq \infty$ *then*

$$\|R_{2n}(t)\| \leq 2^{2n+1}\|h\|_1^{n+1}\frac{t^n}{(n+1)!}.$$

Proof. Put

$$\begin{aligned}
&\Phi_r = \varphi_{AW}(e^{-it_r\omega}\alpha),\ \ \Phi_r^L C = \Phi_r C,\ \ \Phi_r^R C = C\Phi_r,\\
&\sigma_{x,r}^L B = \sigma_{x,r}B,\ \ \sigma_{x,r}^R B = B\sigma_{x,r},\\
&\beta:\ \text{a function from }\ \{0,1,...,2n+1\}\ \text{to}\ \{L,R\},\\
&k_\beta = \sharp\{r\in\{0,1,...,2n+1\}\ \text{such that}\ \beta(r)=R\}.
\end{aligned}$$

In the sequel, we simplify the notation $\sigma_{x,r}\otimes\Phi_r$ into $\sigma_{x,r}\Phi_r$. With this notations we have

$$[\sigma_{x,r}\Phi_r, .] = \sigma_{x,r}^L\Phi_r^L - \sigma_{x,r}^R\Phi_r^R.$$

Recall that, from remark 2 and Lemma 3.2, $R_{2n}(t)$ is a sum of 2^n terms each of which is of the form

$$\begin{aligned}
C_{2n,j}(t) =\ & (-1)^j\int_{0\leq t_{2n}\leq\ldots\leq t_0\leq t}\sum_\beta(-1)^{k_\beta}P(\sigma_{x,0}^{\beta(0)}\Phi_0^{\beta(0)})(\sigma_{x,1}^{\beta(1)}\Phi_1^{\beta(1)})...\\
&...(\sigma_{x,p_1-1}^{\beta(p_1-1)}\Phi_{p_1-1}^{\beta(p_1-1)})P(\sigma_{x,p_1}^{\beta(p_1)}\Phi_{p_1}^{\beta(p_1)})...(\sigma_{x,p_j-1}^{\beta(p_j-1)}\Phi_{p_j-1}^{\beta(p_j-1)})\times\\
&P(\sigma_{x,p_j}^{p_j}\Phi_{p_j}^{\beta(p_j)})...(\sigma_{x,2n}^{\beta(2n)}\Phi_{2n}^{\beta(2n)})(\sigma_{x,2n+1}^{\beta(2n+1)}\Phi_{2n+1}^{\beta(2n+1)})P\,dt_{2n}\ldots dt_0,
\end{aligned}$$

where $0 = p_0 < p_1 < p_2 < ... < p_j < p_{j+1} = 2n+2$, each p_k is an even number and $j = N-2$, with N is the number of projections P, which appear in the expression of $C_{2n,j}(t)$.
Hence we have

$$\begin{aligned}
&\|C_{2n,j}(t)(B\otimes C)\|\\
&\leq \|B\otimes C\|\sum_\beta\int_{0\leq t_{2n}\leq\ldots\leq t_0\leq t}\prod_{r=0}^{j}|\omega_R(\Phi_{p_r}^{\beta(p_r)}...\Phi_{p_{r+1}-1}^{\beta(p_{r+1}-1)})|\,dt_{2n}\ldots dt_0,
\end{aligned}$$

$$\leq \|B\otimes C\| \sum_{\beta} \int_{0\leq t_{2n}\leq \ldots\leq t_0\leq t} \prod_{r=0}^{j} |\langle \Phi_{p_r}^{\beta(p_r)} \ldots \Phi_{p_{r+1}-1}^{\beta(p_{r+1}-1)}\rangle|\, dt_{2n}\ldots dt_0,$$

$$\leq \|B\otimes C\| \sum_{\beta} \int_{\leq t_{2n}\leq \ldots\leq t_0\leq t} \prod_{r=0}^{j} |\langle \Phi_{\pi(p_r)} \ldots \Phi_{\pi(p_{r+1}-1)}\rangle|\, dt_{2n}\ldots dt_0,$$

where π is a permutation which depends on β.
Thus from equation (4) and Lemma 3.5 we get

$$\begin{aligned}
&\|C_{2n,j}(t)\| \\
&\leq \sum_{\beta} \sum_{\sigma\in\mathcal{P}_{(0,1,\ldots,2n+1)}} \int_{0\leq t_{2n}\leq \ldots\leq t_0\leq t} \prod_{r=0}^{n} |\langle \Phi_{\pi(\sigma(2r))}\Phi_{\pi(\sigma(2r+1))}\rangle| dt_{2n}\ldots dt_0, \\
&\leq 2^{2n+2}\|h\|_1^{n+1}\frac{t^n}{2^{n+1}(n+1)!},
\end{aligned}$$

Therefore, $C_{2n,j}$ is dominated uniformly in j. Finally, this proves that

$$\|R_{2n}(t)\| \leq 2^{2n+1}\|h\|_1^{n+1}\frac{t^n}{(n+1)!}.$$

The following theorem ensures that assumption (ii) of Theorem 3.4 holds.

Theorem 3.7 *If*

$$\int_0^\infty (1+t^\varepsilon)|h(t)|dt < \infty$$

for some $0<\varepsilon<1$, *then there exists* $d_n>0$ *such that*

$$\|R_{2n}(t)\| \leq d_n t^{n-\varepsilon}.$$

Proof. We have that $R_{2n}(t)$ is a sum of 2^n terms each of which takes the form of $C_{2n,j}$ which is defined previously. In order to prove this theorem we group those terms pairwise as follows:

$$\begin{aligned}
&(-1)^j \int_{0\leq t_{2n}\leq \ldots\leq t_0\leq t} \sum_{\beta}(-1)^{k_\beta} P(\sigma_{x,0}^{\beta(0)}\Phi_0^{\beta(0)})\ldots \\
&(\sigma_{x,p_1-1}^{\beta(p_1-1)}\Phi_{p_1-1}^{\beta(p_1-1)})P\ldots P(\sigma_{x,p_j}^{\beta(p_j)}\Phi_{p_j}^{\beta(p_j)})\ldots(\sigma_{x,2n-1}^{\beta(2n-1)}\Phi_{2n-1}^{\beta(2n-1)})(\sigma_{x,2n}^{\beta(2n)}\Phi_{2n}^{\beta(2n)}) \\
&(\sigma_{x,2n+1}^{\beta(2n+1)}\Phi_{2n+1}^{\beta(2n+1)})P\, dt_{2n}\ldots dt_0 \\
&+(-1)^{(j+1)} \int_{0\leq t_{2n}\leq \ldots\leq t_0\leq t} \sum_{\beta}(-1)^{k_\beta} P(\sigma_{x,0}^{\beta(0)}\Phi_0^{\beta(0)})\ldots(\sigma_{x,p_1-1}^{\beta(p_1-1)}\Phi_{p_1-1}^{\beta(p_1-1)})P \\
&\ldots P(\sigma_{x,p_j}^{\beta(p_j)}\Phi_{p_j}^{\beta(p_j)})\ldots(\sigma_{x,2n-1}^{\beta(2n-1)}\Phi_{2n-1}^{\beta(2n-1)})P(\sigma_{x,2n}^{\beta(2n)}\Phi_{2n}^{\beta(2n)}) \\
&(\sigma_{x,2n+1}^{\beta(2n+1)}\Phi_{2n+1}^{\beta(2n+1)})P\, dt_{2n}\ldots dt_0
\end{aligned}$$

$$= (-1)^j \int_{0\le t_{2n}\le \ldots \le t_0 \le t} \sum_{\beta} (-1)^{k_\beta} P(\sigma_{x,0}^{\beta(0)} \Phi_0^{\beta(0)})...(\sigma_{x,p_1-1}^{\beta(p_1-1)} \Phi_{p_1-1}^{\beta(p_1-1)}) P...$$

$$...\Big\{ P(\sigma_{x,p_j}^{\beta(p_j)} \Phi_{p_j}^{\beta(p_j)})...(\sigma_{x,2n-1}^{\beta(2n-1)} \Phi_{2n-1}^{\beta(2n-1)})(\sigma_{x,2n}^{\beta(2n)} \Phi_{2n}^{\beta(2n)})(\sigma_{x,2n+1}^{\beta(2n+1)} \Phi_{2n+1}^{\beta(2n+1)}) P$$

$$- P(\sigma_{x,p_j}^{\beta(p_j)} \Phi_{p_j}^{\beta(p_j)})...(\sigma_{x,2n-1}^{\beta(2n-1)} \Phi_{2n-1}^{\beta(2n-1)}) P(\sigma_{x,2n}^{\beta(2n)} \Phi_{2n}^{\beta(2n)})$$

$$(\sigma_{x,2n+1}^{\beta(2n+1)} \Phi_{2n+1}^{\beta(2n+1)}) P \Big\} \, dt_{2n} \ldots dt_0.$$

Therefore, the right-hand side of the above equation is dominated by

$$\sum_{\beta} \int_{0\le t_{2n}\le \ldots \le t_0 \le t} \prod_{k=0}^{j-1} \left| \langle \Phi_{p_k}^{\beta(p_k)} \Phi_{p_k+1}^{\beta(p_k+1)} ... \Phi_{p_{k+1}-1}^{\beta(p_{k+1}-1)} \rangle \right| \times \Big| \Big\{ \langle \Phi_{p_j}^{\beta(p_j)} ... \Phi_{2n}^{\beta(2n)} \Phi_{2n+1}^{\beta(2n+1)} \rangle - \langle \Phi_{p_j}^{\beta(p_j)} ... \Phi_{2n-1}^{\beta(2n-1)} \rangle \langle \Phi_{2n}^{\beta(2n)} \Phi_{2n+1}^{\beta(2n+1)} \rangle \Big\} \Big| \, dt_{2n} \ldots dt_0. \tag{5}$$

Note that in the between bracket terms, there is no product of two point correlation matrix where $2n$ is paired with $(2n+1)$. Moreover this term is equal to

$$\sum_{\sigma \in \mathcal{P}_{(p_j,\ldots,2n+1)}} \prod_{r=\frac{1}{2}p_j}^{n} \langle \Phi_{\sigma(\pi(2r))} \Phi_{\sigma(\pi(2r+1))} \rangle,$$

where $2n$ is not paired with $(2n+1)$ and π is a permutation which depends on β.

Thus the term in equation (5) is dominated by

$$\sum_{\sigma} \int_{0\le t_{2n}\le \ldots \le t_0 \le t} \prod_{r=0}^{n} |\langle \Phi_{\sigma(2r)} \Phi_{\sigma(2r+1)} \rangle| dt_{2n} ... dt_0,$$

where $\sum_\sigma$ indicates the sum over all pairings of $\{0, 1..., 2n+1\}$ such that $2n$ is not paired with $(2n+1)$, $(t_{2n+1} = 0)$.

But we have

$$\int_{0\le t_{2n}\le \ldots \le t_0 \le t} \prod_{r=0}^{n} \left| \langle \Phi_{\sigma(2r)} \Phi_{\sigma(2r+1)} \rangle \right| dt_{2n} ... dt_0$$

$$= \int_{0\le t_{2n}\le \ldots \le t_0 \le t} \prod_{r=0}^{n} |h(t_{\sigma(2r)} - t_{\sigma(2r+1)})| dt_{2n} ... dt_0$$

$$\le cst \, \|h\|_1^n t^k \int_0^t |h(s)| s^{n-k} ds$$

$$\le cst \, \|h\|_1^n t^{n-\varepsilon} \int_0^t |h(s)| s^{\varepsilon} ds,$$

with $0 \le k \le n-1$. This ends the proof of the above theorem.

All together applying relation (1), Lemma 3.3, Theorem 3.4 to 3.7, we have proved the following.

Theorem 3.8 *Suppose that the following assumptions are satisfied:*
(1) $(\omega+\omega^{-1})\alpha \in L^2(\mathbb{R}^3)$,
(2) $\int_0^\infty (1+t^\varepsilon)|h(t)|dt < \infty$, *for some* $0<\varepsilon<1$,
then

$$\lim_{\lambda\to 0} K_\lambda(t) = K(t),$$

for all t. *Moreover*

$$K^\sharp = i\int_0^\infty \sum_{e\in sp([\sigma_z,.])} e^{-ise} P\mathbb{1}_e([\sigma_z,.])[V,.]e^{is[L_0^{semi},.]}[V,.]\mathbb{1}_e([\sigma_z,.])P\,ds.$$

3.3 Lindbladian of the Spin-boson System

Let

$$\mathcal{L} = iK^\sharp.$$

The aim of this subsection is to give an explicit formula of $\mathcal{L}$. Moreover, we prove that this operator has the form of a Lindblad generator (or Lindbladian). Let us introduce the well known formula of distribution theory

$$\int_0^\infty e^{\pm it\omega}\,dt = \frac{\pm i}{\omega \pm i0} = \pi\delta(\omega) \pm iV_p(\frac{1}{\omega}), \tag{6}$$

where

$$\frac{1}{x+i0} = \lim_{\varepsilon\to 0}\frac{1}{x+i\varepsilon},$$

$$\int f(x)\delta(x)\,dx = f(0),$$

$$\int f(x)V_p(\frac{1}{x})\,dx = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{f(x)}{x}\,dx = PP\int\frac{f(x)}{x}\,dx,$$

$$\int f(x)\frac{1}{x+i0}\,dx = \lim_{\varepsilon\to 0}\int f(x)\frac{1}{x+i\varepsilon}\,dx,$$

for all f, such that $\mathbb{R}\ni x\mapsto f(x)$ is a continuous function and provided the integrals on the right are well defined and the limits exist.
Note that the eigenvalues of $[\sigma_z,.]$ are 2, -2 and 0 where 2, -2 are non degenerate and 0 has multiplicity two. Besides, the corresponding eigenvectors are respectively given by $|\Psi_+\rangle\langle\Psi_-|,\ |\Psi_-\rangle\langle\Psi_+|$ and $|\Psi_+\rangle\langle\Psi_+|,\ |\Psi_-\rangle\langle\Psi_-|$.

Put

$$n_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad n_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$n_+^L X = n_+ X, \ n_+^R X = X n_+, \ n_-^L X = n_- X, \ n_-^R X = X n_-,$$
$$N(\omega) = \frac{1}{e^{\beta\omega(k)} - 1}.$$

It is easy to check that

$$\begin{aligned} \mathbb{1}_2([\sigma_z, .]) &= n_+^L n_-^R, \\ \mathbb{1}_{-2}([\sigma_z, .]) &= n_-^L n_+^R, \\ \mathbb{1}_0([\sigma_z, .]) &= n_+^L n_+^R + n_-^L n_-^R. \end{aligned}$$

The explicit formula of the Lindbladian associated to the spin-boson system is given as follows.

Theorem 3.9 *If the following assumptions are met:*

i) $\int_0^\infty |h(t)|\, dt < \infty$,

ii) *α is a $\mathcal{C}^1$ function in a neighborhood of the sphere* $B(0,2) = \{k \in \mathbb{R}^3, \ |k| = 2\}$,

iii) $(1+\omega)\alpha \in L^\infty(\mathbb{R}^3)$,

then for all $X \in M_2$,

$$\begin{aligned} \mathcal{L}(X) = \ & i\Big(\mathrm{Im}(\alpha,\alpha)_+^- - \mathrm{Im}(\alpha,\alpha)_-^+\Big)[n_+, X] \\ & + i\Big(\mathrm{Im}(\alpha,\alpha)_-^- - \mathrm{Im}(\alpha,\alpha)_+^+\Big)[n_-, X] \\ & + \mathrm{Re}(\alpha,\alpha)_-^+\Big(2\sigma_+ X \sigma_- - \{n_+, X\}\Big) \\ & + \mathrm{Re}(\alpha,\alpha,)_-^-\Big(2\sigma_- X \sigma_+ - \{n_-, X\}\Big), \end{aligned}$$

where

$$\begin{aligned} \mathrm{Im}(\alpha,\alpha)_+^+ &= \int_{\mathbb{R}^3} \frac{N(\omega)+1}{\omega+2} |\alpha(k)|^2 \, dk, \\ \mathrm{Im}(\alpha,\alpha)_-^- &= PP \int \frac{N(\omega)}{\omega-2} |\alpha(k)|^2 \, dk, \\ \mathrm{Im}(\alpha,\alpha)_-^+ &= PP \int \frac{N(\omega)+1}{\omega-2} |\alpha(k)|^2 \, dk, \\ \mathrm{Im}(\alpha,\alpha)_+^- &= \int_{\mathbb{R}^3} \frac{N(\omega)}{\omega+2} |\alpha(k)|^2 \, dk, \\ \mathrm{Re}(\alpha,\alpha)_-^+ &= \pi \frac{e^{2\beta}}{e^{2\beta}-1} \int_{\mathbb{R}^3} |\alpha(k)|^2 \delta(\omega-2) \, dk, \\ \mathrm{Re}(\alpha,\alpha)_-^- &= \frac{\pi}{e^{2\beta}-1} \int_{\mathbb{R}^3} |\alpha(k)|^2 \delta(\omega-2) \, dk. \end{aligned}$$

Proof. A straightforward computation shows that for all $X \in M_2$,

$$\mathbb{1}_2([\sigma_z,.])[V,.]e^{is[L_0^{semi},.]}[V,.]\mathbb{1}_2([\sigma_z,.])PX$$
$$= \Big[\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha) + \varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\Big]\, n_+ X n_-,$$
$$\mathbb{1}_{-2}([\sigma_z,.])[V,.]e^{is[L_0^{semi},.]}[V,.]\mathbb{1}_{-2}([\sigma_z,.])PX$$
$$= \Big[\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha) + \varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\Big]\, n_- X n_+,$$
$$\mathbb{1}_0([\sigma_z,.])[V,.]e^{is[L_0^{semi},.]}[V,.]\mathbb{1}_0([\sigma_z,.])PX$$
$$= \Big[e^{-2is}\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha) + e^{2is}\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\Big]\, n_+ X n_+$$
$$+\Big[e^{2is}\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha) + e^{-2is}\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\Big]\, n_- X n_-$$
$$-\Big[e^{-2is}\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha) + e^{2is}\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\Big]\, \sigma_+ X \sigma_-$$
$$-\Big[e^{2is}\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha) + e^{-2is}\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\Big]\, \sigma_- X \sigma_+.$$

Hence, for all $X \in M_2$, we have

$$\sum_{e\in sp([\sigma_z,.])} e^{-ise} P\mathbb{1}_e([\sigma_z,.])[V,.]e^{is[L_0^{semi},.]}[V,.]\mathbb{1}_e([\sigma_z,.])(X)$$
$$= \Big[e^{-2is}\langle\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle + e^{-2is}\langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\Big]\, n_+ X n_-$$
$$+\Big[e^{2is}\langle\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle + e^{2is}\langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\Big]\, n_- X n_+$$
$$-2\mathrm{Re}\Big(e^{2is}\langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\Big)\Big[\sigma_+ X \sigma_- - n_+ X n_+\Big]$$
$$-2\mathrm{Re}\Big(e^{-2is}\langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\Big)\Big[\sigma_- X \sigma_+ - n_- X n_-\Big].$$

It follows that

$$\mathcal{L}(X) = -\Big[\int_0^\infty e^{-2is}\Big(\langle\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle + \langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\Big)\,ds\Big]$$
$$n_+ X n_-$$
$$-\Big[\int_0^\infty e^{2is}\Big(\langle\varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle + \langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\Big)\,ds\Big]$$
$$n_- X n_+$$
$$+2\mathrm{Re}\Big(\int_0^\infty e^{2is}\langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\,ds\Big)\Big[\sigma_+ X \sigma_- - n_+ X n_+\Big]$$
$$+2\mathrm{Re}\Big(\int_0^\infty e^{-2is}\langle\varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\,ds\Big)\Big[\sigma_- X \sigma_+ - n_- X n_-\Big].$$

But we have

$$\begin{aligned}
&\langle \varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle \\
&= \int_{\mathbb{R}^3} e^{is\omega}(N(\omega)+1)|\alpha(k)|^2\,dk + \int_{\mathbb{R}^3} e^{-is\omega}N(\omega)|\alpha(k)|^2\,dk \\
&= \overline{\langle \varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle}.
\end{aligned}$$

Now, by assumptions i), ii) and iii) of the above theorem, we apply formula (6) to get

$$\begin{aligned}
\int_0^\infty e^{-2is}\langle \varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle\,ds &= \mathrm{Re}(\alpha,\alpha)_-^+ + i\mathrm{Im}(\alpha,\alpha)_-^+ - i\mathrm{Im}(\alpha,\alpha)_+^-, \\
\int_0^\infty e^{-2is}\langle \varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\,ds &= \mathrm{Re}(\alpha,\alpha)_-^- + i\mathrm{Im}(\alpha,\alpha,)_-^- - i\mathrm{Im}(\alpha,\alpha)_+^+, \\
\int_0^\infty e^{2is}\langle \varphi_{AW}(e^{is\omega}\alpha)\varphi_{AW}(\alpha)\rangle\,ds &= \mathrm{Re}(\alpha,\alpha)_-^+ - i\mathrm{Im}(\alpha,\alpha)_-^+ + i\mathrm{Im}(\alpha,\alpha)_+^-, \\
\int_0^\infty e^{2is}\langle \varphi_{AW}(\alpha)\varphi_{AW}(e^{is\omega}\alpha)\rangle\,ds &= \mathrm{Re}(\alpha,\alpha)_-^- + i\mathrm{Im}(\alpha,\alpha)_+^+ - i\mathrm{Im}(\alpha,\alpha)_-^-.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\mathcal{L}(X) = \ &\Big\{ - \mathrm{Re}(\alpha,\alpha)_-^+ - \mathrm{Re}(\alpha,\alpha)_-^- + i\Big(\mathrm{Im}(\alpha)_+^- - \mathrm{Im}(\alpha,\alpha)_-^+\Big) \\
&-i\Big(\mathrm{Im}(\alpha,\alpha)_-^- - \mathrm{Im}(\alpha,\alpha_+^+)\Big)\Big\}\, n_+Xn_- + \Big\{ - \mathrm{Re}(\alpha,\alpha)_-^+ - \mathrm{Re}(\alpha,\alpha)_-^- \\
&-i\Big(\mathrm{Im}(\alpha,\alpha)_+^- - \mathrm{Im}(\alpha,\alpha)_-^+\Big) + i\Big(\mathrm{Im}(\alpha,\alpha)_-^- - \mathrm{Im}(\alpha,\alpha)_+^+\Big)\Big\}\, n_-Xn_+ \\
&+2\mathrm{Re}(\alpha,\alpha)_-^+\Big[\sigma_+X\sigma_- - n_+Xn_+\Big] + 2\mathrm{Re}(\alpha,\alpha)_-^-\Big[\sigma_-X\sigma_+ - n_-Xn_-\Big].
\end{aligned}$$

Hence, we get the following

$$\begin{aligned}
\mathcal{L}(X) = i&\Big(\mathrm{Im}(\alpha,\alpha)_+^- - \mathrm{Im}(\alpha,\alpha)_-^+\Big)\Big[n_+Xn_- - n_-Xn_+\Big] \\
&+i\Big(\mathrm{Im}(\alpha,\alpha)_-^- - \mathrm{Im}(\alpha,\alpha)_+^+\Big)\Big[n_-Xn_+ - n_+Xn_-\Big] \\
&+\mathrm{Re}(\alpha,\alpha)_-^+\Big[2\sigma_+X\sigma_- - 2n_+Xn_+ - \Big(n_+Xn_- + n_-Xn_+\Big)\Big] \\
&+\mathrm{Re}(\alpha,\alpha)_-^-\Big[2\sigma_-X\sigma_+ - 2n_-Xn_- - \Big(n_+Xn_- + n_-Xn_+\Big)\Big].
\end{aligned}$$

Note that we have

$$\begin{aligned}
n_+Xn_- + n_-Xn_+ &= \{n_+,X\} - 2n_+Xn_+ = \{n_-,X\} - 2n_-Xn_-, \\
n_+Xn_- - n_-Xn_+ &= [n_+,X], \\
n_-Xn_+ - n_+Xn_- &= [n_-,X].
\end{aligned}$$

This proves the theorem.

4 Properties of the Quantum Master Equation

In this section we state some properties of the quantum master equation associated to the spin-boson system, such as quantum decoherence and quantum detailed balance condition. Note that the log-Sobolev inequality with explicit computation of optimal constants are known in this context. We refer the interested reader to [C04].

4.1 Quantum Master Equation

Let $\rho \in M_2$ be a density matrix. Then the quantum master equation of the spin-boson system is given by

$$\begin{aligned} \frac{d\rho(t)}{dt} = & \; i\Big(\mathrm{Im}(\alpha,\alpha)_-^+ - \mathrm{Im}(\alpha,\alpha)_+^-\Big)[n_+,\rho(t)] \\ & + i\Big(\mathrm{Im}(\alpha,\alpha)_+^+ - \mathrm{Im}(\alpha,\alpha)_-^-\Big)[n_-,\rho(t)] \\ & + \mathrm{Re}(\alpha,\alpha)_-^+\Big(2\sigma_-\rho(t)\,\sigma_+ - \{n_+,\rho(t)\}\Big) \\ & + \mathrm{Re}(\alpha,\alpha)_-^-\Big(2\sigma_+\rho(t)\,\sigma_- - \{n_-,\rho(t)\}\Big). \end{aligned}$$

Put

$$\rho(t) = \rho_{11}(t)\, n_+ + \rho_{12}(t)\,\sigma_+ + \rho_{21}(t)\,\sigma_- + \rho_{22}(t)\, n_-.$$

Therefore, the above master equation is equivalent to the following system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\rho_{11}(t) &= 2\mathrm{Re}(\alpha,\alpha)_-^-\,\rho_{22}(t) - 2\mathrm{Re}(\alpha,\alpha)_-^+\,\rho_{11}(t) \\ \frac{d}{dt}\rho_{12}(t) &= \Big[-i\Big(\mathrm{Im}(\alpha,\alpha)_+^+ - \mathrm{Im}(\alpha,\alpha)_-^-\Big) + i\Big(\mathrm{Im}(\alpha,\alpha)_-^+ - \mathrm{Im}(\alpha,\alpha)_+^-\Big) \\ &\quad -\mathrm{Re}(\alpha,\alpha)_-^- - \mathrm{Re}(\alpha,\alpha)_-^+\Big]\,\rho_{12}(t) \\ \frac{d}{dt}\rho_{21}(t) &= \Big[-i\Big(\mathrm{Im}(\alpha,\alpha)_-^+ - \mathrm{Im}(\alpha,\alpha_+^-)\Big) + i\Big(\mathrm{Im}(\alpha,\alpha)_+^+ - \mathrm{Im}(\alpha,\alpha)_-^-\Big) \\ &\quad -\mathrm{Re}(\alpha,\alpha)_-^+ - \mathrm{Re}(\alpha,\alpha)_-^-\Big]\,\rho_{21}(t) \\ \frac{d}{dt}\rho_{22}(t) &= 2\mathrm{Re}(\alpha,\alpha)_-^+\,\rho_{11}(t) - 2\mathrm{Re}(\alpha,\alpha)_-^-\,\rho_{22}(t). \end{aligned}$$

Hence, it is straightforward to show that the thermodynamical equilibrium state ρ_β of the spin system is the only solution of the above equation.

4.2 Quantum Decoherence of the Spin System

Definition 1 *We say that the dynamical evolution of a quantum system describes decoherence , if there exists an orthonormal basis of $\mathcal{H}_s$ such that the off-diagonal elements of its time evolved density matrix in this basis vanish as $t \to \infty$.*

From the system of ordinary differential equations introduced in the previous subsection, we have

$$\begin{aligned}
\rho_{12}(t) &= \rho_{12}(0)\exp\Big(-i(\mathrm{Im}(\alpha,\alpha)_{+}^{-}+\mathrm{Im}(\alpha,\alpha)_{+}^{+}-\mathrm{Im}(\alpha,\alpha)_{-}^{-}-\mathrm{Im}(\alpha,\alpha)_{-}^{+})t\Big)\\
&\quad\times\exp\Big(-(\mathrm{Re}(\alpha,\alpha)_{-}^{-}+\mathrm{Re}(\alpha,\alpha)_{+}^{-})t\Big)\\
&= \rho_{12}(0)\exp\Big(-i(\mathrm{Im}(\alpha,\alpha)_{+}^{-}+\mathrm{Im}(\alpha,\alpha)_{+}^{+}-\mathrm{Im}(\alpha,\alpha)_{-}^{-}-\mathrm{Im}(\alpha,\alpha)_{-}^{+})t\Big)\\
&\quad\times\exp\Big(-\pi(\frac{e^{2\beta}+1}{e^{2\beta}-1}\int_{\mathbb{R}^3}|\alpha(k)|^2\delta(\omega-2)dk)t\Big),\\
\rho_{21}(t) &= \rho_{21}(0)\exp\Big(-i(\mathrm{Im}(\alpha,\alpha)_{-}^{+}+\mathrm{Im}(\alpha,\alpha)_{-}^{-}-\mathrm{Im}(\alpha,\alpha)_{+}^{-}-\mathrm{Im}(\alpha,\alpha)_{+}^{+})t\Big)\\
&\quad\times\exp\Big(-(\pi\frac{e^{2\beta}+1}{e^{2\beta}-1}\int_{\mathbb{R}^3}|\alpha(k)|^2\delta(\omega-2)dk)t\Big).
\end{aligned}$$

Therefore, the spin system describes quantum decoherence if and only if

$$\int_{\mathbb{R}^3}|\alpha(k)|^2\delta(\omega-2)dk\neq 0.$$

Thus, the decoherence of the spin system is controlled by the cut-off function α.

4.3 Quantum Detailed Balance Condition

The following definition is taken from [AL87].

Definition 2 *Let Θ be a generator of a quantum dynamical semigroup written as*

$$\Theta = i\,[H,.]+\Theta_0,$$

where H is a self-adjoint operator. We say that Θ satisfies a quantum detailed balance condition with respect to a stationary state ρ if

i) $[H,\rho]=0$,
ii) $\langle\Theta_0(A),B\rangle_\rho=\langle A,\Theta_0(B)\rangle_\rho$, *for all* $A, B\in D(\Theta_0)$, *with* $\langle A,B\rangle_\rho=\mathrm{Tr}(\rho A^*B)$.

Actually, we prove the following.

Theorem 4.1 *The generator $\mathcal{L}$ of the quantum dynamical semigroup $T_t = e^{itK^\sharp}$ satisfies a quantum detailed balance condition with respect to the thermodynamical equilibrium state of the spin system*

$$\rho_\beta=\frac{e^{-\beta\sigma_z}}{\mathrm{Tr}(e^{-\beta\sigma_z})}.$$

Proof. Note that

$$\mathcal{L}(A) = i\,[H,\,A] + \mathcal{L}_D(A),$$

with

$$H = \Big(\mathrm{Im}(\alpha,\alpha)_+^- - \mathrm{Im}(\alpha,\alpha)_-^+\Big)\, n_+ + \Big(\mathrm{Im}(\alpha,\alpha)_-^- - \mathrm{Im}(\alpha,\alpha)_+^+\Big)\, n_-,$$

and

$$\mathcal{L}_D(\rho) = \mathrm{Re}(\alpha,\alpha)_-^+ \Big(2\,\sigma_+\rho\,\sigma_- - \{n_+,\,\rho\}\Big) + \mathrm{Re}(\alpha,\alpha)_-^- \Big(2\,\sigma_-\rho\,\sigma_+ - \{n_-,\,\rho\}\Big).$$

Therefore, it is clear that H is a self-adjoint operator and $[H,\rho_\beta\,] = 0$. Moreover it is straightforward to show that $\mathcal{L}_D$ is self-adjoint for the $\langle\,,\,\rangle_{\rho_\beta}$ scalar product.

5 Return to Equilibrium for the Spin-boson System

5.1 Hamiltonian Case

In this subsection we recall the results of return to equilibrium for the spin-boson system proved in [JP96b].

For $f \in L^2(\mathbb{R}^3)$ we define $\tilde{f}$ on $\mathbb{R}\times S^2$ by

$$\tilde{f}(s,\hat{k}) = \begin{cases} -|s|^{1/2}\bar{f}(|s|\hat{k}), & s<0, \\ s^{1/2}f(s\hat{k}), & s\geq 0. \end{cases}$$

Put

$$\mathcal{C}(\delta) = \Big\{ z \in \mathbb{C} \text{ s.t } |Imz| < \delta \Big\},$$
$$H^2(\delta,\eta) = \Big\{ f : \mathcal{C}(\delta) \to \eta \text{ s.t } \|f\|_{H^2(\delta,\eta)} = \sup_{|a|<\delta} \int_{-\infty}^{+\infty} \|f(x+ia)\|_\eta^2 dx < \infty \Big\},$$

where η is a Hilbert space.

Definition 3 *Let $\mathcal{M}$ be a W^*-algebra, τ a dynamics on $\mathcal{M}$ and ω a faithful normal state on $\mathcal{M}$. We say that the triple $(\mathcal{M},\tau,\omega)$ has the property of return to equilibrium if for all $A \in \mathcal{M}$ and all normal state μ, we have*

$$\lim_{t\to\infty} \mu(\tau^t(A)) = \omega(A).$$

Then, in the Hamiltonian approach of the spin-boson system, the following is proved in [JP96b].

Theorem 5.1 *Assume that the following assumptions are satisfied:*
(i) $(\omega+\omega^{-1})\alpha \in L^2(\mathbb{R}^3)$,
(ii) $\int_{S^2} |\alpha(2\hat{k})|^2\, d\sigma(\hat{k}) > 0$, *where $d\sigma$ is the surface measure on S^2,*
(iii) There exists $0 < \delta < \frac{2\pi}{\beta}$ *such that* $\tilde{\alpha} \in H^2(\delta, L^2(S^2))$.

Then, for all $\beta > 0$ there exists a constant $\Lambda(\beta) > 0$ which depends only on the cut-off function α, such that the spin-boson system has the property of return to equilibrium for all $0 < |\lambda| < \Lambda(\beta)$.

Remark: In the above theorem the authors show that for any fixed temperature $\beta \in]0, +\infty[$, the spectrum of the full-Liouvillean L_λ associated to the spin-boson system is absolutely continuous uniformly on $\lambda \in]0, \Lambda(\beta)[$ and in particular for λ very small (weak coupling). Moreover they used the theory of perturbation of KMS-states for constructing the eigenvector of L_λ associated to the eigenvalue 0. Therefore, for any fixed $\beta \in]0, +\infty[$, the spin-boson system weakly coupled has the property of return to equilibrium.

5.2 Markovian Case

We shall compare the above conditions for the return to equilibrium to the one we obtain in the Markovian approach. Let $(T_t)_{t\geq 0}$ be a quantum dynamical semigroup on $\mathcal{B}(\eta)$ such that its generator has the form

$$\mathcal{L}(X) = G^* X + XG + \sum_{k\geq 1} L_k^* X L_k,$$

where $G = -\frac{1}{2}\sum_{k\geq 1} L_k^* L_k - iH$.
Put

$$\mathcal{A}(T) = \Big\{X \in \mathcal{B}(\eta) \text{ s.t } T_t(X) = X, \text{ for all } t \geq 0\Big\},$$

$$\mathcal{N}(T) = \Big\{X \in \mathcal{B}(\eta) \text{ s.t } T_t(X^*X) = T_t(X^*)T_t(X) \text{ and } T_t(XX^*) = T_t(X)T_t(X^*), \text{ for all } t \geq 0\Big\}.$$

The following result is useful for the study of approach to equilibrium in the Markovian case.

Theorem 5.2 *(Frigerio-Verri)*
If T has a faithful stationary state ρ and $\mathcal{N}(T) = \mathcal{A}(T)$, then

$$w^* - \lim_{t\to\infty} T_t(X) = T_\infty(X), \forall X \in \mathcal{B}(\eta),$$

where $X \to T_\infty(X)$ is a conditional expectation. In particular the quantum dynamical semigroup T has the property of return to equilibrium.

We state without proof the following result which is a special case of a theorem proved in [FR98].

Theorem 5.3 *Suppose that $(T_t)_t$ is a norm continuous quantum dynamical semigroup which has a faithful normal stationary state and H is a self-adjoint* *operator which has a pure point spectrum. Then* $(T_t)_t$ *has the property of return to equilibrium if and only if*

$$\Big\{L_k,\, L_k^*,\, H,\, k \geq 1\Big\}' = \Big\{L_k,\, L_k^*,\, k \geq 1\Big\}'.$$

Applying the above result, we now prove the following.

Theorem 5.4 *Suppose that the following assumptions are satisfied:*
i) $\mathrm{Im}(\alpha,\, \alpha)_{\pm}^{\pm}$ *are given by real numbers,*
ii) $\int_{S^2} |\alpha(2k)|^2\, dk > 0$.
Then the quantum dynamical semigroup of the spin-boson system at positive temperature has the property of return to equilibrium.

Proof. Set

$$\begin{aligned}
H &= \Big(\mathrm{Im}(\alpha,\alpha)_{+}^{-} - \mathrm{Im}(\alpha,\alpha)_{-}^{+}\Big)n_{+} + \Big(\mathrm{Im}(\alpha,\alpha)_{-}^{-} - \mathrm{Im}(\alpha,\alpha)_{+}^{+}\Big)n_{-},\\
L_1 &= \Big(2\mathrm{Re}(\alpha,\alpha)_{-}^{+}\Big)^{1/2}\sigma_{-},\\
L_2 &= \Big(2\mathrm{Re}(\alpha,\alpha)_{-}^{-}\Big)^{1/2}\sigma_{+},\\
G &= -\frac{1}{2}\sum_{k=1}^{2} L_k^* L_k - iH.
\end{aligned} \tag{7}$$

Then the Lindbladian of the spin-boson system takes the form

$$\mathcal{L}(X) = G^* X + XG + \sum_{k=1}^{2} L_k^* X L_k,$$

for all $X \in M_2$.

Note that the quantum dynamical semigroup T of the spin-boson system has the thermodynamical equilibrium state ρ_β of the spin system as a faithful normal stationary state. Moreover H is a self-adjoint bounded operator which has a pure point spectrum and it is clear that

$$\Big\{L_k,\, L_k^*,\, H,\, k = 1,2\Big\}' = \Big\{L_k,\, L_k^*,\, k = 1,2\Big\}' = \mathbb{C}I.$$

Thus from the previous theorem, the quantum dynamical semigroup of the spin-boson system has the property of return to equilibrium.

Note that compared to the Hamiltonian approach, we have in Theorem 5.4 a simplification of conditions for return to equilibrium of the spin-boson system. So in this theorem we need only that assumptions i) and ii) are satisfied. Hypothesis i) ensures that $\mathrm{Im}(\alpha,\, \alpha)_{\pm}^{\pm}$ exist and are finite, while if ii) holds, then $\mathrm{Re}(\alpha,\, \alpha)_{-}^{\pm}$ are not vanishing.

5.3 Spin-boson System at Zero Temperature

In the Hamiltonian case, if a quantum dynamical system which its Liouvillean L has a purely absolutely continuous spectrum, except for the simple eigenvalue 0, then this system has the property of return to equilibrium (cf [JP96b]). At inverse temperature β ($0 < \beta < \infty$), by using the perturbation theory of KMS-states (cf [DJP03]), we can give an explicit formula of the eigenstate of L associated to the eigenvalue 0. But it is not the case for zero temperature ($\beta = \infty$). On the other hand, the ground state of the spin system is not faithful and by Theorem 5.3 we cannot conclude. Let us describe the spin-boson system at zero temperature.

At zero temperature, the Hilbert space of the spin-boson system is

$$\mathcal{H} = \mathbb{C}^2 \otimes \Gamma_s(L^2(\mathbb{R}^3)).$$

The free Hamiltonian is defined as

$$h_0 = \sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega),$$

and its full Hamiltonian with interaction is the operator

$$h_\lambda = h_0 + \lambda \sigma_x \otimes \varphi(\alpha),$$

where $\alpha \in L^2(\mathbb{R}^3)$ is a test function.
The zero temperature equilibrium state of the spin system is the vector state corresponding to the ground state of σ_z and it has a density matrix

$$\rho_\infty = |\Psi_-\rangle\langle\Psi_-|.$$

The weak coupling limit of the spin-boson system at zero temperature can be proved in the same way as for positive temperature. The associated Lindbladian can be deduced from the one at positive temperature by taking $\beta = \infty$ and it has the form

$$\mathcal{L}_\infty(X) = -i\nu_1[n_+, X] - i\nu_2[n_-, X] + \nu_3\Big(2\sigma_+ X \sigma_- - \{n_+, X\}\Big),$$

where

$$\nu_1 = \int_{\mathbb{R}^3} \frac{1}{\omega + 2}|\alpha(k)|^2\,dk,$$
$$\nu_2 = PP\int \frac{1}{\omega - 2}|\alpha(k)|^2\,dk,$$
$$\nu_3 = \pi\int_{\mathbb{R}^3} |\alpha(k)|^2\delta(\omega - 2)\,dk.$$

Hence, for all density matrix $\rho \in M_2$, the associated quantum master equation is given by

$$\frac{d\rho(t)}{dt} = i\nu_1[n_+, \rho(t)] + i\nu_2[n_-, \rho(t)] + \nu_3\Big(2\sigma_-\rho(t)\,\sigma_+ - \{n_+, \rho(t)\}\Big) = \mathcal{L}_\infty^*(\rho(t)).$$

Now, in order to conclude the property of return to equilibrium for the quantum dynamical semigroup associated to the spin-boson system at zero temperature, we have to show it by direct computation.

Theorem 5.5 *Assume that:*

i) ν_2 *is given by a real number,*
ii) $\int_{S^2} |\alpha(2k)|^2 \, dk > 0$.

Then the spin-boson system at zero temperature has the property of return to equilibrium. Moreover we have

$$\lim_{t\to\infty} \mathrm{Tr}(e^{t\mathcal{L}_\infty^*}\rho A) = \mathrm{Tr}(\rho_\infty A),$$

for all $A \in M_2$ *and all* ρ *be a given density matrix.*

Proof. Consider the orthonormal basis of M_2 given by

$$\Big\{|\Psi_+\rangle\langle\Psi_+|,\ |\Psi_+\rangle\langle\Psi_-|,\ |\Psi_-\rangle\langle\Psi_+|,\ |\Psi_-\rangle\langle\Psi_-|\Big\}.$$

Then in this basis we have

$$e^{t\mathcal{L}_\infty^*} = \begin{pmatrix} e^{-2t\nu_3} & 0 & 0 & 0 \\ 0 & e^{-t\nu_3}e^{it(\nu_1-\nu_2)} & 0 & 0 \\ 0 & 0 & e^{-t\nu_3}e^{-it(\nu_1-\nu 2)} & 0 \\ -e^{-2t\nu_3}+1 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore we get

$$\lim_{t\to\infty} e^{t\mathcal{L}_\infty^*} = \Pi_\infty^*,$$

where

$$\Pi_\infty^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

A direct computation gives

$$\Pi_\infty^*(A) = \sigma_- A \sigma_+ + n_- A n_-,\ \ \forall A \in M_2.$$

Consider a density matrix ρ of the form

$$\rho = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & 1-\alpha \end{pmatrix},$$

with $\alpha \in [0,1],\ \beta \in \mathbb{C}$. We have

$$\Pi_\infty^*(\rho) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |\Psi_-\rangle\langle\Psi_-| = \rho_\infty.$$

Therefore, it follows that

$$\lim_{t\to\infty} \mathrm{Tr}(e^{t\mathcal{L}_\infty^*}\rho A) = \mathrm{Tr}(\Pi_\infty^*(\rho)A) = \mathrm{Tr}(\rho_\infty A),$$

$\forall A \in M_2$. This proves our theorem.

6 Quantum Langevin Equation and Associated Hamiltonian

It is shown in [HP84] that any quantum master equation of a simple quantum system $\mathcal{H}_S$ can be dilated into a unitary quantum Langevin equation (quantum stochastic differential equation) on a larger space $\mathcal{H}_S \otimes \Gamma$ where Γ is a Fock space in which are naturally living quantum noises. Note that in the literature it is shown that natural quantum stochastic differential equations can be obtained by the stochastic limit of the full Hamiltonian system which is developed in [ALV02].

Now, let us introduce some notations that need in the sequel.

6.1 Basic Notations

Let $\mathcal{Z}$ be a Hilbert space for which we fix an orthonormal basis $\{z_k,\ k \in J\}$. We denote by $\Gamma_s(\mathbb{R}_+)$, the symmetric Fock space constructed over the Hilbert space $\mathcal{Z} \otimes L^2(\mathbb{R}_+)$. Therefore, from the following identification

$$\mathcal{Z} \otimes L^2(\mathbb{R}_+) \simeq L^2(\mathbb{R}_+, \mathcal{Z}) \simeq L^2(\mathbb{R}_+ \times J),$$

we get

$$\Gamma_s(\mathbb{R}_+) = \Gamma_{\mathrm{sym}}(L^2(\mathbb{R}_+ \times J)).$$

The space $\mathcal{Z}$ is called the multiplicity space and $\dim \mathcal{Z}$ is called the multiplicity. The set J is equal to $\{1, ..., N\}$ in the case of finite multiplicity N and is equal to $\mathbb{N}$ in the case of infinite multiplicity.

Let us introduce another Hilbert space $\mathcal{H}$ called initial or system space and we identify the tensor product

$$\mathcal{K}(\mathbb{R}_+) = \mathcal{H} \otimes \Gamma_s(\mathbb{R}_+) = \mathcal{H} \otimes \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+ \times J)^{\otimes n} = \bigoplus_{n=0}^{\infty} \mathcal{H} \otimes L^2(\mathbb{R}_+ \times J)^{\otimes n}$$

with the direct sum

$$\bigoplus_{n=0}^{\infty} \mathcal{H} \otimes L^2_{\mathrm{sym}}((\mathbb{R}_+ \times J)^n) \simeq \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}((\mathbb{R}_+ \times J)^n, \mathcal{H}),$$

consisting of the vectors $\Psi = (\Psi_n)_{n \geq 0}$ such that $\Psi_n \in L^2_{\mathrm{sym}}((\mathbb{R}_+ \times J)^n, \mathcal{H})$ and

$$\|\Psi\|^2_{\mathcal{K}(\mathbb{R}_+)} = \sum_{n \geq 0} \frac{1}{n!} \|\Psi_n\|^2_{L^2_{\mathrm{sym}}((\mathbb{R}_+ \times J)^n, \mathcal{H})} < \infty.$$

Note that for $f \in L^2(\mathbb{R}_+ \times J)$, we define its associated exponential vector by

$$\varepsilon(f) = \sum_{n \geq 0} \frac{f^{\otimes n}}{\sqrt{n!}}\ .$$

6.2 Hudson-Parthasarathy Equation

Let $H,\ R_k$ and $S_{kl},\ k,\ l \geq 1$ be bounded operators on $\mathcal{H}$ such that

$$H = H^*,\ \sum_j S_{jk}^* S_{jl} = \sum_j S_{kj} S_{lj}^* = \delta_{kl}, \tag{8}$$

and the sum $\sum_k R_k^* R_k$ are assumed to be strongly convergent to a bounded operator. Through $H,\ R_k$ and S_{kl} we define the following operators

$$S \in \mathcal{U}(\mathcal{H} \otimes \mathcal{Z}),\ R \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{Z}),\ G \in \mathcal{B}(\mathcal{H}),$$

by

$$\begin{aligned} Ru &= \sum_k (R_k u) \otimes z_k,\ \forall u \in \mathcal{H}, \\ S &= \sum_{kl} S_{kl} \otimes |z_k\rangle\langle z_l|, \\ G &= -iH - \frac{1}{2} \sum_k R_k^* R_k = -iH - \frac{1}{2} R^* R. \end{aligned}$$

The basic quantum noises are the processes

$$\begin{aligned} A_i(t) &= A(\mathbb{1}_{(0,t)} \otimes z_i), \\ A_i^+(t) &= A^+(\mathbb{1}_{(0,t)} \otimes z_i), \\ \Lambda_{ij}(t) &= \Lambda(\pi_{(0,t)} \otimes |z_i\rangle\langle z_j|), \end{aligned}$$

where $i,\ j \in J,\ \mathbb{1}_{(0,t)}$ is the indicator function over $(0,t)$, while $\pi_{(0,t)}$ is the multiplication operator by $\mathbb{1}_{(0,t)}$ in $L^2(\mathbb{R}_+)$.
The Hudson-Parthasarathy equation is defined as follows

$$(HP) \begin{cases} dU(t) = \big\{ \sum_k R_k dA_k^+(t) + \sum_{kl} (S_{kl} - \delta_{kl}) d\Lambda_{kl}(t) \\ - \sum_{kl} R_k^* S_{kl} dA_l(t) + G dt \big\} U(t) \\ U(0) = 1. \end{cases}$$

Note that in order to have a unitary solution U of (HP), we need some conditions on the system operators. Actually the following theorem holds.

Theorem 6.1 *Suppose that the system operators H, R_k, S_{kl} satisfies (8). Then there exists a unique strongly continuous unitary adapted process $U(t)$ which satisfies equation (HP).*

Proof. For the proof of this theorem we refer the reader to [P92].

Now, in order to associate a group V to the solution U of (HP), we first introduce the one-parameter strongly continuous unitary group θ in $L^2(\mathbb{R}, \mathcal{Z})$ and its associated second quantization Θ in $\Gamma(\mathbb{R})$, defined by

$$\theta_t f(r) = f(r+t), \;\; \forall f \in L^2(\mathbb{R}, \mathcal{Z}),$$
$$\Theta_t e(f) = e(\theta_t f), \;\; \forall f \in L^2(\mathbb{R}, \mathcal{Z}). \tag{9}$$

Note that Θ and $U(t)$ can be extended to act on the space

$$\mathcal{K}(\mathbb{R}) = \mathcal{H} \otimes \Gamma_s(\mathbb{R}_+) \otimes \Gamma_s(\mathbb{R}_-) = \mathcal{K}(\mathbb{R}_+) \otimes \Gamma_s(\mathbb{R}_-) = \mathcal{H} \otimes \Gamma_s(\mathbb{R}),$$

by

$$\Theta_t = 1 \otimes \Theta_t \quad \text{in } \mathcal{H} \otimes \Gamma_s(\mathbb{R}),$$
$$U(t) = U(t) \otimes 1 \quad \text{in } \mathcal{K}(\mathbb{R}_+) \otimes \Gamma_s(\mathbb{R}_-).$$

Theorem 6.2 *Let Θ be the one-parameter strongly continuous group defined by (9) and U the solution of the EDSQ (HP) with system operators satisfying (8). Then*

$$U(t+s) = \Theta_s^* U(t) \Theta_s U(s), \;\; \forall s, \, t \geq 0,$$

and the family $V = \{V_t\}_{t \in \mathbb{R}}$ such that

$$V_t = \begin{cases} \Theta_t U(t), & t \geq 0 \\ U^*(|t|)\Theta_t, & t \leq 0, \end{cases}$$

defines a one-parameter strongly continuous unitary group. Furthermore, the family of two-parameter unitary operators

$$U(t,s) = \Theta_t^* V_{t-s} \Theta_s = \Theta_s^* U(t-s) \Theta_s, \;\; \forall s \leq t,$$

is strongly continuous in t and in s and satisfies the composition law

$$U(t,s)U(t,r) = U(t,r), \;\; \forall r \leq s \leq t.$$

Proof. See [B06] for the proof of this theorem.

The group V defined as above, describes the reversible evolution of the small system plus the reservoir which is modelled by the free Bose gas. The free evolution of the reservoir is represented by the group Θ whose generator is formally given by

$$E_0 = d\Gamma(i\frac{\partial}{\partial x}).$$

Note that $U(t) = U(t,0) = \Theta_t^* V_t$ is the evolution operator giving the dynamics state from time 0 to time t of the whole system in the interaction picture. Moreover by the Stone theorem

$$d\Theta_t = -iE_0 \Theta_t dt,$$
$$dV_t = -iKV_t dt.$$

The operators $H,\ E_0$ represent respectively the energy associated to the small system and the reservoir. The operator K represents the total energy of the combined system in the interaction picture and the system operators $R_j,\ S_{ij}$ control this interaction. Besides, if we take $R_j = 0,\ S_{ij} = \delta_{ij}$, then we get

$$U(t) = e^{itH},\ V_t = e^{-itE_0}e^{-itH},$$

and $K = E_0 + H$ which is self-adjoint operator defined on $\mathcal{H} \otimes D(E_0)$.

In [G01], Gregoratti give an essentially self-adjoint restriction of the Hamiltonian K which appears as a singular perturbation of $E_0 + H$.

6.3 Hamiltonian Associated to the Hudson-Parthasarathy Equation

Recall that the generators ϵ_0 and E_0 of the groups θ in $L^2(\mathbb{R}, \mathcal{Z})$ and Θ in $\mathcal{K}$ are self-adjoint unbounded operators. In order to explicit their domains we introduce the Sobolev space

$$\begin{aligned} &H^{\sum}((\mathbb{R} \times J)^n, \mathcal{H}) \\ &\quad = \Big\{ u \in L^2((\mathbb{R} \times J)^n, \mathcal{H}) \ \text{ such that } \sum_{k=1}^{n} \partial_k u \in L^2((\mathbb{R} \times J)^n, \mathcal{H}) \Big\}, \end{aligned}$$

where all the derivatives of u are in the sense of distributions in $(\mathbb{R} \times J)^n$ $(n \geq 1)$ and

$$H^{\sum}((\mathbb{R} \times J)^0, \mathcal{H}) = \mathcal{H}.$$

Furthermore $H^{\sum}((\mathbb{R} \times J)^n, \mathcal{H})$ is a Hilbert space with respect to the scalar product

$$\langle u, v \rangle_{H^{\sum}((\mathbb{R}\times J)^n,\mathcal{H})} = \langle u, v \rangle_{L^2((\mathbb{R}\times J)^n,\mathcal{H})} + \langle \sum_{k=1}^{n} \partial_k u, \sum_{k=1}^{n} \partial_k v \rangle_{L^2((\mathbb{R}\times J)^n,\mathcal{H})}.$$

Set

$$H^{\sum}_{\text{sym}}((\mathbb{R} \times J)^n, \mathcal{H}) = H^{\sum}((\mathbb{R} \times J)^n, \mathcal{H}) \cap L^2_{\text{sym}}((\mathbb{R} \times J)^n, \mathcal{H}).$$

We have

$$D(\epsilon_0) = H^1(\mathbb{R}, \mathcal{Z}), \ \text{ and } \ \epsilon_0 u = iu',$$

Besides, the domain of E_0 is given by

$$\begin{aligned} D(E_0) = \ &\Big\{ \Phi \in \mathcal{K} \ \ s.t \ \Phi_n \in H^{\sum}_{\text{sym}}((\mathbb{R} \times J)^n, \mathcal{H}), \ \forall n \ \text{ and} \\ &\sum_{n\geq 1} \frac{1}{n!} \| \sum_{k=1}^{n} \partial_k \Phi_n \|^2 < \infty \Big\}, \end{aligned}$$

and this operator acts on its domain by $(E_0\Phi)_n = i\sum_{k=1}^n \partial_k \Phi_n$.

Set $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$. Let us introduce the dense subspaces in $\mathcal{K}$ defined by

$$\mathcal{W} = \Big\{\Phi \in \mathcal{K} \;\; s.t\; \Phi_n \in H^{\sum}_{sym}((\mathbb{R}_* \times J)^n, \mathcal{H}),\ \forall n \;\text{ and } \sum_{n\geq 1} \frac{1}{n!}\|\sum_{k=1}^{\infty} \partial_k \Phi_n\|^2_{L^2(\mathbb{R}\times J)^n, \mathcal{H}} < \infty\Big\},$$

$$\nu_s = \Big\{\Phi \in W \;\; s.t \sum_{n\geq 0} \frac{1}{n!}\|\Phi_{n+1}|_{\{r_{n+1}=s\}}\|^2_{\mathcal{Z}\otimes L^2((\mathbb{R}\times J)^n,\mathcal{H})} < \infty\Big\},$$

$$\nu_{0^\pm} = \nu_{0^-} \cap \nu_{0^+},$$

where $\Phi_{n+1}|_{\{r_{n+1}=s\}}$ is the trace (restriction) of the function Φ_{n+1} on the hyperplane $\{r_{n+1} = s\}$, for all $s \in \mathbb{R}_* \cup \{0^-, 0^+\}$. Clearly

$$\nu_{0^\pm} \subseteq \mathcal{W}.$$

Define the trace operator $a(s) : \nu_s \to \mathcal{Z} \otimes \mathcal{K}$ such that

$$(a(s)\Phi)_n = \Phi_{n+1}|_{\{r_{n+1}=s\}}.$$

Note that $\varepsilon(H^1(\mathbb{R}^*, \mathcal{Z})) \subset \nu_s$ and

$$a(s)\Psi(u)\otimes h = u(s)\otimes \Psi(u)\otimes h,\;\; \forall u \in H^1(\mathbb{R}_*, \mathcal{Z}),\; h \in \mathcal{H},$$

where

$$\Psi(u) = (1,\, u,\, u^{\otimes 2}, ..., u^{\otimes n}, ...).$$

Moreover $\mathcal{W} \supset D(E_0)$ and E_0 can be extended to a non-symmetric unbounded operator in $\mathcal{W}$ by

$$(E\Phi)_n = i\sum_{k=1}^{n} \partial_k \Phi_n.$$

The following theorem gives an essentially self-adjoint restriction of the Hamiltonian operator associated to (HP) and it is proved in [G01].

Theorem 6.3 *Let K be the Hamiltonian operator associated to the equation (HP) such that the system operators satisfying (8). Then*
(1) $D(K)\cap \nu_{0^\pm} = \Big\{\Phi \in \nu_{0^\pm} \;\; s.t\;\; a(0^-)\Phi = Sa(0^+)\Phi + R\Phi\Big\}$,
(2) $K\Phi = \big(H + E - iR^* a(0^-) + \frac{i}{2}R^*R\big)\Phi,\;\; \forall \Phi \in D(K)\cap\nu_{0^\pm}$,
(3) $K|_{D(K)\cap\nu_{0^\pm}}$ *is a essentially self-adjoint operator.*

6.4 Hamiltonian Associated to the Stochastic Evolution of the Spin-boson System

Recall that the quantum Langevin equation of the spin-boson system is defined on $\mathbb{C}^2 \otimes \Gamma_s(L^2(\mathbb{R}_+, \mathbb{C}^2))$ by

$$\begin{cases} dU(t) = \Big\{Gdt + \sum_{k=1}^{2} L_k dA_k^+(t) - \sum_{k=1}^{2} L_k^* dA_k(t)\Big\}U(t) \\ U(0) = I, \end{cases}$$

where $G,\ L_k,\ k \in \{0,1\}$ are given by the relation (7).
Note that this equation satisfies the class of Hudson-Parthasarathy equation with $S_{ij} = \delta_{ij}$. Moreover we have

$$\begin{aligned}
&S = I, \\
&Ru = \big(2\mathrm{Re}(\alpha,\alpha)_-^+\big)^{1/2}\sigma_- u \otimes \Psi_+ + \big(2\mathrm{Re}(\alpha,\alpha)_-^-\big)^{1/2}\sigma_+ u \otimes \Psi_-,\ \forall u \in \mathbb{C}^2, \\
&R^* u \otimes \varphi = \langle \Psi_+, \varphi\rangle \big(2\mathrm{Re}(\alpha,\alpha)_-^+\big)^{1/2}\sigma_+ u + \langle \Psi_-, \varphi\rangle \big(2\mathrm{Re}(\alpha,\alpha)_-^-\big)^{1/2}\sigma_- u, \\
&\forall u,\ \varphi \in \mathbb{C}^2, \\
&R^* R = 2\mathrm{Re}(\alpha,\alpha)_-^+ n_+ + 2\mathrm{Re}(\alpha,\alpha)_-^- n_-.
\end{aligned}$$

Therefore we get

$$\nu_{0^\pm} \cap D(K) = \Big\{\Phi \in \nu_{0^\pm} \quad \text{s.t} \ \ a(0^-)\Phi = a(0^+)\Phi + R\Phi\Big\},$$

and

$$K\Phi = \Big(H + E - iR^* a(0^-) + i\big(\mathrm{Re}(\alpha,\alpha)_-^+ n_+ + \mathrm{Re}(\alpha,\alpha)_-^- n_-\big)\Big)\Phi,$$

for every $\Phi \in \nu_{0^\pm} \cap D(K)$.

Recall that the associated energy of the reservoir is given by $E = d\Gamma(i\frac{\partial}{\partial x})$. Therefore, by using the spectral theorem, $i\frac{\partial}{\partial x}$ is a multiplication operator by a variable ω in $\mathbb{R}$. Thus we get

$$E = d\Gamma(\omega),$$

and E is the same as the usual Hamiltonian. On the other hand, the operator

$$H = \Big(\mathrm{Im}(\alpha/\alpha)_+^- - \mathrm{Im}(\alpha/\alpha)_-^+\Big)n_+ + \Big(\mathrm{Im}(\alpha/\alpha)_-^- - \mathrm{Im}(\alpha/\alpha)_+^+\Big)n_-,$$

describes the energy of the spin. Note that the constants $\mathrm{Im}(\alpha/\alpha)_\pm^\pm$ have an important physical interpretation. In some sense they contain all physical information on the original Hamiltonian of the spin. The free evolution of the combined system is described by $\mathcal{H}_f = H + E$ and the Hamiltonian K appears as a singular perturbation of H_f, where the operator R defined as above controls the interaction between the spin and the reservoir.

7 Repeated Quantum Interaction Model

In this section, we start by describing the repeated quantum interaction model (cf [AP06]). We prove that the quantum Langevin equation of the spin-boson system at zero temperature can be obtained as the continuous limit of an

Hamiltonian repeated interaction model. Moreover we compare the Lindbladian of the spin-boson system at positive temperature to the one obtained by using the method introduced in [AJ07].

Consider a small system $\mathcal{H}_0$ coupled with a piece of environment $\mathcal{H}$. The interaction between the two systems is described by the Hamiltonian H which is defined on $\mathcal{H}_0 \otimes \mathcal{H}$. The associated unitary evolution during the interval $[0, h]$ of times is

$$\mathbb{L} = e^{-ihH}.$$

After the first interaction, we repeat this time coupling the same $\mathcal{H}_0$ with a new copy of $\mathcal{H}$. Therefore, the sequence of the repeated interactions is described by the space

$$\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}^*} \mathcal{H}.$$

The unitary evolution of the small system in interaction picture with the $n-th$ copy of $\mathcal{H}$, denoted by $\mathcal{H}_n$, is the operator $\mathbb{L}_n$ which acts as $\mathbb{L}$ on $\mathcal{H}_0 \otimes \mathcal{H}_n$ and acts as the identity on the copies of $\mathcal{H}$ other than $\mathcal{H}_n$. The associated evolution equation of this model is defined on $\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}^*} \mathcal{H}$ by

$$\begin{cases} u_{n+1} = \mathbb{L}_{n+1} u_n \\ u_0 = I \end{cases} \tag{10}$$

Let $\{X_i\}_{i \in \Lambda \cup \{0\}}$ be an orthonormal basis of $\mathcal{H}$ with $X_0 = \Omega$ and let us consider the coefficients $(\mathbb{L}^i_j)_{i,j \in \Lambda \cup \{0\}}$ which are operators on $\mathcal{H}_0$ of the matrix representation of $\mathbb{L}$ in the basis $\{X_i\}_{i \in \Lambda \cup \{0\}}$.

Theorem 7.1 *If*

$$\begin{aligned}
\mathbb{L}^0_0 &= I - h(iH + \frac{1}{2}\sum_k L^*_k L_k) + h\omega^0_0, \\
\mathbb{L}^0_j &= \sqrt{h} L_j + \sqrt{h}\omega^0_j, \\
\mathbb{L}^i_0 &= -\sqrt{h}\sum_k L^*_k S^k_i + \sqrt{h}\omega^i_0, \\
\mathbb{L}^i_j &= S^i_j + h\omega^i_j,
\end{aligned}$$

where H is a self-adjoint bounded operator, $(S^i_j)_{i,j}$ is a family of unitary operator, $(L_i)_i$ are operators on $\mathcal{H}_0$ and the terms ω^i_j converge to 0 when h tends to 0, then the solution $(u_n)_{n \in \mathbb{N}}$ of (10) is made of invertible operators which are locally uniformly bounded in norm. Moreover $u_{[t/h]}$ converges weakly to the solution $U(t)$ of the equation

$$\begin{cases} dU(t) = \sum_{i,j} L^i_j U(t) da^i_j(t) \\ U(0) = I \end{cases}$$

where

$$L_0^0 = -\big(iH + \frac{1}{2}\sum_k L_k^* L_k\big),$$
$$L_j^0 = L_j,$$
$$L_0^i = -\sum_k L_k^* S_i^k,$$
$$L_j^i = S_j^i - \delta_{ij} I.$$

Proof. See [AP06] for the proof of this theorem.

Now, let us put $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$ and consider the dipole interaction Hamiltonian defined on $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$H = \sigma_z \otimes I + I \otimes H_R + \frac{1}{\sqrt{h}}\big(\sigma_- \otimes a^* + \sigma_+ \otimes a\big),$$

where

$$H_R = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \text{ is the Hamiltonian of the piece of the reservoir,}$$
$$V = \sigma_-,$$
$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } a^* \text{ is the adjoint of } a.$$

Fix an orthonormal basis $\{\Omega, X\}$ of $\mathbb{C}^2$ such that

$$\Omega = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The unitary evolution during the interval $[0, h]$ of time is $\mathbb{L} = e^{-ihH}$ such that

$$\mathbb{L}_0^0 = \langle \Omega, \mathbb{L}\Omega \rangle = I - ih\,\sigma_z - \frac{1}{2}h\sigma_+\sigma_- + o(h),$$
$$\mathbb{L}_0^1 = \langle \Omega, \mathbb{L}X \rangle = -i\sqrt{h}\,\sigma_+ + o(\sqrt{h}),$$
$$\mathbb{L}_1^0 = \langle X, \mathbb{L}\Omega \rangle = -i\sqrt{h}\,\sigma_- + o(\sqrt{h}),$$
$$\mathbb{L}_1^1 = \langle X, \mathbb{L}X \rangle = I - ih\sigma_z - ihI - \frac{1}{2}h\,\sigma_-\sigma_+ + o(h).$$

Therefore we obtain

$$\frac{\mathbb{L}_0^0 - I}{h} \xrightarrow{h \to 0} G_0 = -i\sigma_z - \frac{1}{2}\sigma_+\sigma_-,$$
$$\frac{\mathbb{L}_0^1}{\sqrt{h}} \xrightarrow{h \to 0} -L^* = -i\sigma_+,$$
$$\frac{\mathbb{L}_1^0}{\sqrt{h}} \xrightarrow{h \to 0} L = -i\sigma_-.$$

Thus by Theorem 7.1, the solution $(u_n)_{n\in\mathbb{N}}$ of the equation

$$\begin{cases} u_{n+1} = \mathbb{L}_{n+1}\, u_n \\ u_0 = I \end{cases}$$

is made of invertible operators which are locally uniformly bounded in norm and in particular $u_{[t/h]}$ converges weakly to the solution $U(t)$ of the equation

$$\begin{cases} dU(t) = \Big\{ G_0\, dt + L\, dA^+(t) - L^*\, dA^-(t) \Big\}\, U(t) \\ U(0) = I. \end{cases}$$

Theorem 7.2 *The quantum dynamical semigroup of the repeated quantum interaction model associated to the spin-boson system at zero temperature converges towards to equilibrium.*

Proof. The associated Lindbladian of the above equation is of the form

$$\mathcal{L}(X) = i[\sigma_z,\, X] + 2\sigma_+ X \sigma_- - \{n_+,\, X\},$$

and the proof is similar as the one of Theorem 5.5.

Now, at inverse temperature β, we suppose that the piece of the reservoir is described by the state

$$\rho = \frac{1}{1+e^{-\beta}} e^{-\beta H_R} = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix}.$$

The GNS representation of $(\mathbb{C}^2, \rho)$ is the triple $(\pi, \widetilde{\mathcal{H}}, \Omega_R)$, such that

- $\Omega_R = I$,
- $\widetilde{\mathcal{H}} = M_2$, the algebra of all complex 2×2 matrix which equipped by the scalar product
$$\langle A, B\rangle = \mathrm{Tr}(\rho A^* B),$$
- $\pi : M_2 \to \mathcal{B}(\widetilde{\mathcal{H}})$, such that $\pi(M)A = MA,\ \forall M,\ A \in M_2$.

Set

$$X_1 = \frac{1}{\sqrt{\beta_1}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ X_2 = \frac{1}{\sqrt{\beta_0}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ X_3 = \frac{1}{\sqrt{\beta_0 \beta_1}} \begin{pmatrix} \beta_1 & 0 \\ 0 & -\beta_0 \end{pmatrix}.$$

It is easy to show that $(\Omega_R, X_1, X_2, X_3)$ is an orthonormal basis of M_2. Now, if we put $\widetilde{\mathbb{L}} = \pi(\mathbb{L})$ which is defined on $\mathbb{C}^2 \otimes M_2$, then a straightforward computation shows that the coefficients $(\widetilde{\mathbb{L}}^i_j)_{i,j}$, which are operators on $\mathbb{C}^2$, of the matrix representation of $\widetilde{\mathbb{L}}$, are given by

$$
\begin{aligned}
\widetilde{\mathbb{L}}_0^0 &= I - ih\sigma_z - ih\,\beta_1 I - \frac{1}{2}h\beta_0\sigma_+\sigma_- - \frac{1}{2}h\beta_1\sigma_-\sigma_+ + o(h^2),\\
\widetilde{\mathbb{L}}_1^0 &= -i\sqrt{\beta_1}\sqrt{h}\,\sigma_+ + o(h^{3/2}),\\
\widetilde{\mathbb{L}}_2^0 &= -i\sqrt{\beta_0}\sqrt{h}\,\sigma_- + o(h^{3/2}),\\
\widetilde{\mathbb{L}}_3^0 &= o(h),\\
\widetilde{\mathbb{L}}_0^1 &= -i\sqrt{\beta_1}\sqrt{h}\,\sigma_- + o(h^{3/2}),\\
\widetilde{\mathbb{L}}_0^2 &= -i\sqrt{\beta_0}\sqrt{h}\sigma_+ + o(h^{3/2}),\\
\widetilde{\mathbb{L}}_0^3 &= o(h),\\
\widetilde{\mathbb{L}}_1^1 &= I + o(h),\\
\widetilde{\mathbb{L}}_2^2 &= I + o(h),\\
\widetilde{\mathbb{L}}_3^3 &= I + o(h),\\
\widetilde{\mathbb{L}}_1^2 &= \widetilde{\mathbb{L}}_2^1 = \widetilde{\mathbb{L}}_1^3 = \widetilde{\mathbb{L}}_3^1 = \widetilde{\mathbb{L}}_2^3 = \widetilde{\mathbb{L}}_3^2 = 0.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
&\frac{\widetilde{\mathbb{L}}_0^0 - I}{h} \xrightarrow{h\to 0} L_0^0 = -i\sigma_z - i\beta_1 I - \frac{1}{2}\beta_0\,\sigma_+\,\sigma_- - \frac{1}{2}\beta_1\,\sigma_-\sigma_+,\\
&\frac{\widetilde{\mathbb{L}}_1^0}{\sqrt{h}} \xrightarrow{h\to 0} L_1^0 = -i\sqrt{\beta_1}\,\sigma_+,\\
&\frac{\widetilde{\mathbb{L}}_2^0}{\sqrt{h}} \xrightarrow{h\to 0} L_2^0 = -i\sqrt{\beta_0}\,\sigma_-,\\
&\frac{\widetilde{\mathbb{L}}_0^1}{\sqrt{h}} \xrightarrow{h\to 0} L_0^1 = -i\sqrt{\beta_1}\sigma_-,\\
&\frac{\widetilde{\mathbb{L}}_0^2}{\sqrt{h}} \xrightarrow{h\to 0} L_0^2 = \sqrt{\beta_0}\,\sigma_+,
\end{aligned}
$$

and the other terms converges to 0 when h tends to 0. Thus the solution $(\widetilde{u}_n)_{n\in\mathbb{N}}$ of the equation

$$
\begin{cases}
\widetilde{u}_{n+1} = \widetilde{\mathbb{L}}_{n+1}\widetilde{u}_n\\
\widetilde{u}_0 = I
\end{cases}
$$

is made of invertible operators which are locally uniformly bounded in norm and in particular $\widetilde{u}_{[t/h]}$ converges weakly to the solution $\widetilde{U}(t)$ of the equation

$$
\begin{cases}
d\widetilde{U}(t) = \Big\{ - \Big(i\sigma_z + i\beta_1 I + \frac{1}{2}\beta_0\,\sigma_+\sigma_- + \frac{1}{2}\beta_1\,\sigma_-\sigma_+\Big)\,dt\\
-i\sigma_-\Big(\sqrt{\beta_1}\,da_0^1(t) + \sqrt{\beta_0}da_2^0(t)\Big) - i\sigma_+\Big(\sqrt{\beta_1}\,da_1^0(t) + \sqrt{\beta_0}\,da_0^2(t)\Big)\Big\}\widetilde{U}(t)\\
\widetilde{U}(0) = I.
\end{cases}
$$

Theorem 7.3 *The quantum dynamical semigroup of the repeated quantum interaction model associated to the spin-boson system converges towards the equilibrium.*

Proof. It suffices to observe that the associated Lindbladian of the above equation has the form

$$\mathcal{L}(X) = i[\sigma_z, X] + \frac{1}{2}\beta_0\Big[2\sigma_- X\sigma_+ - \{n_-, X\}\Big] + \frac{1}{2}\beta_1\Big[2\sigma_+ X\sigma_- - \{n_+, X\}\Big].$$

Remark: Note that by using the repeated quantum interaction model we can prove that the Markovian properties of the spin-boson system are satisfied without using any assumption.

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