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Homogenization of the steady-state Navier-Stokes equations with prescribed flux rate or pressure drop in a perforated pipe ☆

Gianmarco Sperone

Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milan, Italy

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Abstract

The steady motion of a viscous incompressible fluid in a pipe (perforated with a large number of small holes) is modeled through the Navier-Stokes equations with mixed boundary conditions involving the Bernoulli pressure and the tangential velocity on the inlet and outlet of the tube, while either the transversal flux rate or the pressure drop is prescribed along the pipe. Applying the classical energy method in homogenization theory, we study the asymptotic behavior of the solutions to these systems, without any restriction on the magnitude of the data, as the size of the perforations goes to zero and show that the effective equations remain unmodified in the limit. The main novelty of the present work lies in the obtainment of the required uniform bounds, which are achieved (in the case of the prescribed flux problem) by a contradiction argument based on Bernoulli's law for solutions of the stationary Euler equations.

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E-mail address: gianmarcosilvio.sperone@polimi.it.

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1. Introduction and presentation of the problem

Applications in physics and engineering often lead to the analysis of steady flow of a viscous incompressible fluid through perforated pipes [7,28], where the goal is to obtain a *macroscopic* description of the system dynamics when the number of perforations goes to infinity and, simultaneously, the size of each hole vanishes. In many of such studies, of particular relevance is the possibility of calculating the velocity net fluxes from prescribed pressure drops, or alternatively, of finding the pressure drops that originate these net fluxes [27]. From a strictly mathematical point of view, starting from the pioneering works of Marchenko & Khruslov [38, Chapter IV], Sánchez-Palencia [19,45] and Tartar [48], passing through the benchmark contributions of Allaire [1–3] and Conca [12–14], homogenization methods in the context of viscous incompressible fluid flows (in stationary regime, see also [34]) have attracted the interest of several authors that adapted and expanded such techniques to models involving unsteady incompressible [22,37,41], viscous compressible [20,36,39,43] or heat-conducting [21,23,35,44] fluid flows, among many others (the list of references presented is far from being exhaustive). Concerning the boundary conditions imposed on the velocity field, while Navier boundary conditions on the surface of the perforations have been treated in [3,12], the no-slip boundary condition is typically assumed on the external boundary, that is, the velocity field is set to be zero on the boundary of the domain containing the perforations. Under a different choice for the *outer* boundary conditions, the well-posedness of the corresponding problem in the perforated domain can be guaranteed, in most cases, under a smallness assumption on the data [6,8,15,25,33,47] (we also recall that the celebrated *Leray problem* for the steady-state Navier-Stokes system with nonhomogeneous Dirichlet boundary conditions remains open in the general three-dimensional case [31]). Motivated by the works of Heywood, Rannacher & Turek [27] and Korobkov, Pileckas & Russo [32], in the present article we tackle two models associated to the steady-state Navier-Stokes equations with mixed and non-standard boundary conditions (in the sense of [15,16]) involving the Bernoulli pressure: the *prescribed net flux* (1.4) and the *prescribed pressure drop* (1.5) problems. As we will see in Section 2 (in particular, Theorem 2.1), the main difficulty encountered here lies in the obtainment of the uniform bounds (without any restriction on the magnitude of the data) that are required to describe the macroscopic behavior of the system through a compactness argument.

In the space $\mathbb{R}^3$ we use a system of cylindrical coordinates $(\rho, \theta, z) \in [0, \infty) \times [0, 2\pi] \times \mathbb{R}$, in which any spatial point will be denoted by $\xi = \rho\widehat{\rho} + z\widehat{k}$, with $\rho \geq 0$, $z \in \mathbb{R}$ and $\left\{\widehat{\rho}, \widehat{\theta}, \widehat{k}\right\} \subset \mathbb{R}^3$ the orthonormal basis in this geometry. Given $h > R > 1$, we consider an open straight cylinder Ω of radius R and length $2h$ whose axis of symmetry is directed along the z-axis:

$$\Omega = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R, \ -h < z < h\right\}.$$

For any $\xi \in \mathbb{R}^3$ and $r > 0$ we denote by $B(\xi, r) \subset \mathbb{R}^3$ the open ball of radius r with center at ξ. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of open, bounded and simply connected sets with a $\mathcal{C}^2$-boundary such that $(0, 0, 0) \in K_n$, for every $n \in \mathbb{N}$, and

$$\sup_{n\in\mathbb{N}} |K_n| < \infty .$$

Take $\varepsilon_* \in (0, 1)$ such that $\varepsilon_*|K_n| < 2\pi R^2 h$, $\forall n \in \mathbb{N}$. Following [17,21,35], given $\alpha > 3$ and $\varepsilon \in (0, \varepsilon_*]$, suppose that there exist an integer $N(\varepsilon) \geq 1$ and a collection of points $\xi_1^\varepsilon, \ldots, \xi_{N(\varepsilon)}^\varepsilon \in \mathbb{R}^3$ such that

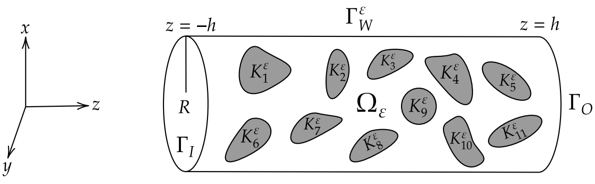

Fig. 1.1. Representation of the perforated domain Ω_ε.

$$
\begin{aligned}
&\xi_n^\varepsilon + \varepsilon^\alpha \overline{K_n} \subset B(\xi_n^\varepsilon, \delta_0 \varepsilon^\alpha) \subset B(\xi_n^\varepsilon, \delta_1 \varepsilon) \subset B\left(\xi_n^\varepsilon, \delta_2 \varepsilon\right) \subset \Omega \qquad \forall n \in \{1, ..., N(\varepsilon)\}, \\
&\partial B\left(\xi_n^\varepsilon, \delta_2 \varepsilon\right) \cap \partial B\left(\xi_m^\varepsilon, \delta_2 \varepsilon\right) = \emptyset \qquad \forall n, m \in \{1, ..., N(\varepsilon)\}, \ n \neq m, \\
&\partial B\left(\xi_n^\varepsilon, \delta_2 \varepsilon\right) \cap \partial \Omega = \emptyset \qquad \forall n \in \{1, ..., N(\varepsilon)\},
\end{aligned}
\tag{1.1}
$$

for some constants $\delta_0, \delta_1, \delta_2 > 0$ that are independent of $\varepsilon \in (0, \varepsilon_*]$ and such that $\delta_1 < \delta_2$. Setting $K_n^\varepsilon \doteq \xi_n^\varepsilon + \varepsilon^\alpha K_n$ for every $n \in \{1, ..., N(\varepsilon)\}$, we will refer to the family $\{K_n^\varepsilon\}_{n=1}^{N(\varepsilon)}$ satisfying (1.1) as the *solid obstacles*, while the set

$$
\Omega_\varepsilon \doteq \Omega \setminus \overline{K_\varepsilon} \doteq \Omega \setminus \bigcup_{n=1}^{N(\varepsilon)} \overline{K_n^\varepsilon}, \tag{1.2}
$$

represents the *perforated fluid domain* at the ε-level. We emphasize that, given $\varepsilon \in (0, \varepsilon_*]$, the family of obstacles $\{K_n^\varepsilon\}_{n=1}^{N(\varepsilon)}$ is built in such a way that the *size* of each solid is proportional to ε^α, while the mutual distance between any two consecutive holes is proportional to ε. Moreover, since we only consider those obstacles that are *strictly* contained in Ω (in the sense of $(1.1)_3$), the following bound on the number $N(\varepsilon)$ holds:

$$
N(\varepsilon) \leq \frac{3R^2 h}{2\delta_2^3 \varepsilon^3}.
$$

Notice, however, that the solids $\{K_n^\varepsilon\}_{n=1}^{N(\varepsilon)}$ may have different shapes and that they are not necessarily periodically distributed in Ω, see Fig. 1.1.

We decompose the boundary of Ω_ε as $\partial\Omega_\varepsilon = \Gamma_I \cup \Gamma_W^\varepsilon \cup \Gamma_O$, where

$$
\begin{gathered}
\Gamma_I = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R, \ z = -h\right\}, \qquad \Gamma_O = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R, \ z = h\right\}, \\
\Gamma_W^\varepsilon = \mathcal{L} \cup \partial K_\varepsilon \doteq \left\{\xi \in \mathbb{R}^3 \mid \rho = R, \ -h < z < h\right\} \cup \bigcup_{n=1}^{N(\varepsilon)} \partial K_n^\varepsilon.
\end{gathered}
\tag{1.3}
$$

The outward unit normal to $\partial\Omega_\varepsilon$ is denoted by ν (with some abuse of notation, as such vector also depends on ε). Henceforth we will refer to Γ_I and Γ_O in (1.3) as the *inlet* and *outlet* of Ω, respectively, while Γ_W^ε includes all the *physical walls* of Ω_ε. Given $\varepsilon \in (0, \varepsilon_*]$, we analyze the steady motion of a viscous incompressible fluid (with a constant kinematic viscosity $\eta > 0$) along Ω_ε, which is characterized by its velocity vector field $u_\varepsilon : \Omega_\varepsilon \longrightarrow \mathbb{R}^3$ and its scalar pressure $p_\varepsilon : \Omega_\varepsilon \longrightarrow \mathbb{R}$, under the action of an external force $f : \Omega \longrightarrow \mathbb{R}^3$. Such stationary motion will be modeled through two different boundary-value problems (with mixed boundary conditions) associated to the steady-state Navier-Stokes equations in Ω_ε. We firstly consider the **prescribed net flux problem**, which reads

$$\begin{cases} -\eta\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \nabla p_\varepsilon = f\,, \quad \nabla\cdot u_\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon\,, \\ u_\varepsilon = 0 \ \text{ on } \ \Gamma_W^\varepsilon\,, \\ u_\varepsilon\times\nu = 0\,, \quad p_\varepsilon + \dfrac{1}{2}|u_\varepsilon|^2 = p_\varepsilon^- \ \text{ on } \ \Gamma_I\,, \\ u_\varepsilon\times\nu = 0\,, \quad p_\varepsilon + \dfrac{1}{2}|u_\varepsilon|^2 = p_\varepsilon^+ \ \text{ on } \ \Gamma_O\,, \\ \displaystyle\int_{\Sigma_\varepsilon(s)} u_\varepsilon\cdot\widehat{k} = F \qquad \forall s\in[-h,h]\,. \end{cases} \tag{1.4}$$

While (1.4)$_2$ describes the usual no-slip boundary condition on the physical walls Γ_W^ε, the first equality in (1.4)$_3$-(1.4)$_4$ dictates that the fluid flow must enter and leave the domain Ω orthogonal to the inlet and outlet walls. The second identity in (1.4)$_3$-(1.4)$_4$ imposes that, respectively on the inlet Γ_I and outlet Γ_O, the *Bernoulli pressure* defined as $\Phi_\varepsilon \doteq p_\varepsilon + |u_\varepsilon|^2/2$ must equal some constants $p_\varepsilon^\mp\in\mathbb{R}$ that represent the *unknown pressure drop* $p_\varepsilon^+ - p_\varepsilon^-$ along the perforated pipe (therefore, $p_\varepsilon^\mp$ are unknown, not prescribed, constants that depend on the solution). Finally, (1.4)$_5$ dictates that the transversal flow rate of the velocity field must be constant along the pipe, given by a quantity $F\in\mathbb{R}$, where we set

$$\Sigma_\varepsilon(s) \doteq \{\xi\in\Omega_\varepsilon \mid 0<\rho<R\,,\ z=s\} \qquad \forall s\in[-h,h].$$

Secondly, we analyze the **prescribed pressure drop problem**, which reads

$$\begin{cases} -\eta\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \nabla p_\varepsilon = f\,, \quad \nabla\cdot u_\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon\,, \\ u_\varepsilon = 0 \ \text{ on } \ \Gamma_W^\varepsilon\,, \\ u_\varepsilon\times\nu = 0\,, \quad p_\varepsilon + \dfrac{1}{2}|u_\varepsilon|^2 = p^- \ \text{ on } \ \Gamma_I\,, \\ u_\varepsilon\times\nu = 0\,, \quad p_\varepsilon + \dfrac{1}{2}|u_\varepsilon|^2 = p^+ \ \text{ on } \ \Gamma_O\,, \end{cases} \tag{1.5}$$

where now the constants $p^\mp\in\mathbb{R}$ are given and prescribe the pressure drop $p^+ - p^-$ along the perforated pipe. Given a velocity field $u_\varepsilon\in\mathcal{C}^2(\Omega_\varepsilon)\cap\mathcal{C}(\overline{\Omega_\varepsilon})$ solving (1.5), observe that the transversal flux rate

$$F_\varepsilon \doteq \int_{\Sigma_\varepsilon(s)} u_\varepsilon\cdot\widehat{k} \qquad \forall s\in[-h,h]\,, \tag{1.6}$$

is constant along the pipe, but *depends* on the solution. Indeed, given $s\in(-h,h]$, we define the region

$$\Omega_\varepsilon(s) \doteq \{\xi\in\Omega_\varepsilon \mid -h<z<s\} \qquad \Longrightarrow \qquad \partial\Omega_\varepsilon(s) = \Gamma_I\cup\left\{\xi\in\Gamma_W^\varepsilon \mid -h<z<s\right\}\cup\Sigma_\varepsilon(s)\,.$$

Since u_ε vanishes on Γ_W^ε, after applying the Divergence Theorem we infer

$$0=\int_{\partial\Omega_\varepsilon(s)} u_\varepsilon\cdot\nu=-\int_{\Gamma_I} u_\varepsilon\cdot\widehat{k}+\int_{\Sigma_\varepsilon(s)} u_\varepsilon\cdot\widehat{k} \qquad\Longrightarrow\qquad \int_{\Sigma_\varepsilon(s)} u_\varepsilon\cdot\widehat{k}=\int_{\Gamma_I} u_\varepsilon\cdot\widehat{k} \qquad \forall s\in[-h,h]\,.$$

Since the scalar pressure can be determined up to an additive constant, without loss of generality we may take $p_\varepsilon^-=0$ in $(1.4)_3$. Moreover, in view of the identity

$$\nabla\left(\frac{1}{2}|u_\varepsilon|^2\right)=(\nabla u_\varepsilon)^\top u_\varepsilon \quad \text{in} \quad \Omega_\varepsilon\,,$$

it is customary (see, for example, [27,32]) to add the term $(\nabla u_\varepsilon)^\top u_\varepsilon$ to both sides of the equation of conservation of momentum $(1.4)_1$-$(1.5)_1$, thereby resulting in the problems

$$\begin{cases} -\eta\Delta u_\varepsilon+(u_\varepsilon\cdot\nabla)u_\varepsilon-(\nabla u_\varepsilon)^\top u_\varepsilon+\nabla\Phi_\varepsilon=f\,, \qquad \nabla\cdot u_\varepsilon=0 \ \text{ in } \ \Omega_\varepsilon\,,\\ u_\varepsilon=0 \ \text{ on } \ \Gamma_W^\varepsilon\,,\\ u_\varepsilon\times\nu=0\,, \quad \Phi_\varepsilon=0 \ \text{ on } \ \Gamma_I\,,\\ u_\varepsilon\times\nu=0\,, \quad \Phi_\varepsilon=p_\varepsilon^+ \ \text{ on } \ \Gamma_O\,,\\ \displaystyle\int_{\Sigma_\varepsilon(s)} u_\varepsilon\cdot\widehat{k}=F \qquad \forall s\in[-h,h]\,, \end{cases} \tag{1.7}$$

and

$$\begin{cases} -\eta\Delta u_\varepsilon+(u_\varepsilon\cdot\nabla)u_\varepsilon-(\nabla u_\varepsilon)^\top u_\varepsilon+\nabla\Phi_\varepsilon=f\,, \qquad \nabla\cdot u_\varepsilon=0 \ \text{ in } \ \Omega_\varepsilon\,,\\ u_\varepsilon=0 \ \text{ on } \ \Gamma_W^\varepsilon\,,\\ u_\varepsilon\times\nu=0\,, \quad \Phi_\varepsilon=p^- \ \text{ on } \ \Gamma_I\,,\\ u_\varepsilon\times\nu=0\,, \quad \Phi_\varepsilon=p^+ \ \text{ on } \ \Gamma_O\,. \end{cases} \tag{1.8}$$

In the case of a non-perforated pipe (that is, when $\varepsilon=0$), existence of a generalized solution to systems (1.7)-(1.8) has been recently proved by the authors of [32] without any restriction on the data, that is, for *any* prescribed flux rate or prescribed pressure drop: in the case of (1.7), the a priori estimates required by the Leray-Schauder Principle are obtained through a contradiction argument that employs Bernoulli's law [30] for solutions of the stationary Euler equations (2.43). Later, in [46], this result was extended to the case of a pipe containing a fixed obstacle with a Lipschitz boundary.

Ultimately, the main goal of the present article is to study the asymptotic behavior of the solutions of problems (1.7)-(1.8) as $\varepsilon\to 0^+$. For the prescribed flux rate problem, our homogenization result reads:

Theorem 1.1. *Let $(\Omega_\varepsilon)_{\varepsilon\in(0,\varepsilon_*]}$ be the family of perforated domains verifying (1.1) with $\alpha>3$. Given any $F\in\mathbb{R}$ and $f\in L^2(\Omega)$, let $(u_\varepsilon,\Phi_\varepsilon)\in H^1(\Omega_\varepsilon)\times L^2(\Omega_\varepsilon)$ be a weak solution of (1.7). Then, up to the extraction of a subsequence, the sequence $\{(\widetilde{u_\varepsilon},\widetilde{\Phi_\varepsilon})\}_{\varepsilon\in(0,\varepsilon_*]}\subset H^1(\Omega)\times L^2(\Omega)$*

of trivially extended functions converges strongly to a weak solution $(u, \Phi) \in H^1(\Omega) \times L^2(\Omega)$ *of problem* (1.7) *in* Ω *as* $\varepsilon \to 0^+$. *Furthermore,* $(u, \Phi) \in H^2(\Omega) \times H^1(\Omega)$ *and it satisfies in strong form the system*

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi = f\,, \qquad \nabla\cdot u = 0 \quad in \quad \Omega\,, \\ u = 0 \quad on \quad \mathcal{L}\,, \\ u\times\nu = 0\,, \qquad \Phi = 0 \quad on \quad \Gamma_I\,, \\ u\times\nu = 0\,, \qquad \Phi = p^+ \quad on \quad \Gamma_O\,, \\ \displaystyle\int_{\Sigma(s)} u\cdot\widehat{k} = F \qquad \forall s\in[-h,h]\,, \end{cases}$$

for some (unknown) constant $p^+ \in \mathbb{R}$.

Similarly, for the prescribed pressure drop problem, our homogenization result reads:

Theorem 1.2. *Let* $(\Omega_\varepsilon)_{\varepsilon\in(0,\varepsilon_*]}$ *be the family of perforated domains verifying* (1.1) *with* $\alpha > 3$. *Given any* $p^\pm \in \mathbb{R}$ *and* $f \in L^2(\Omega)$, *let* $(u_\varepsilon, \Phi_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ *be a weak solution of* (1.8). *Then, up to the extraction of a subsequence, the sequence* $\{(\widetilde{u_\varepsilon}, \widetilde{\Phi_\varepsilon})\}_{\varepsilon\in(0,\varepsilon_*]} \subset H^1(\Omega) \times L^2(\Omega)$ *of trivially extended functions converges strongly to a weak solution* $(u, \Phi) \in H^1(\Omega) \times L^2(\Omega)$ *of problem* (1.8) *in* Ω *as* $\varepsilon \to 0^+$. *Furthermore,* $(u, \Phi) \in H^2(\Omega) \times H^1(\Omega)$ *and it satisfies in strong form the system*

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi = f\,, \qquad \nabla\cdot u = 0 \quad in \quad \Omega\,, \\ u = 0 \quad on \quad \mathcal{L}\,, \\ u\times\nu = 0\,, \qquad \Phi = p^- \quad on \quad \Gamma_I\,, \\ u\times\nu = 0\,, \qquad \Phi = p^+ \quad on \quad \Gamma_O\,. \end{cases}$$

In order to prove Theorems 1.1-1.2, in Section 2 we derive uniform ε-independent bounds for the solutions of (1.7)-(1.8). Theorem 2.1, the most involved in this work, adapts the contradiction argument of [32] previously described and, additionally, employs several properties of the *relative capacity* of the perforations inside Ω (see Lemma 2.1) and a uniform *Bogovskii-type* operator over the space of square-integrable functions (not necessarily having zero mean value, see Lemma 2.2) which, in turn, relies on the construction given by Diening, Feireisl & Lu in [17]. Subsequently, applying the classical energy method in homogenization theory [45, Appendix], in Section 3 (specifically, Theorems 3.1-3.2) we show that, as $\varepsilon \to 0^+$, the effective or homogenized equations remain unchanged in the limit: up to the extraction of a subsequence, the sequences of solutions (indexed by the parameter ε) of (1.7)-(1.8) converge *strongly* (in a sense made precise in Theorems 3.1-3.2) to solutions of problems (1.7)-(1.8), respectively, in Ω as $\varepsilon \to 0^+$. This is usually interpreted by stating that *very* small perforations cannot appreciably perturb the fluid flow [2, Remark 3.3.2]. We point out that this phenomenon is not new in the literature, as it coincides with other results in homogenization theory in the regime of tiny holes

[2,20,35,36]. Therefore, as already announced, the main technical novelty of this note are precisely the uniform bounds of Section 2 without any restriction on the size of the data (external force, prescribed flux rate and prescribed pressure drop).

2. Mixed boundary-value problems at the ε-level: uniform bounds

Let $\varepsilon \in I_*$ be a fixed parameter, with $I_* \doteq (0, \varepsilon_*]$. Many of the results contained in the present article exploit the concept of *relative capacity* of K_ε with respect to Ω, defined as

$$\mathrm{Cap}_\Omega(K_\varepsilon) \doteq \min_{v \in H_0^1(\Omega)} \left\{ \int_\Omega |\nabla v|^2 \;\middle|\; v = 1 \ \text{ in } \ \overline{K_\varepsilon} \right\} . \tag{2.1}$$

The *relative capacity potential* of K_ε with respect to Ω, that is, the scalar function $\phi_\varepsilon \in H_0^1(\Omega)$ achieving the minimum in (2.1), satisfies

$$\Delta\phi_\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon\,, \quad \phi_\varepsilon = 0 \ \text{ on } \ \partial\Omega\,, \quad \phi_\varepsilon = 1 \ \text{ in } \ \overline{K_\varepsilon}\,, \quad \mathrm{Cap}_\Omega(K_\varepsilon) = \|\nabla\phi_\varepsilon\|^2_{L^2(\Omega)}\,, \tag{2.2}$$

see [40, Chapter 2] for more details. Further essential properties of the relative capacity potential are collected in the following result, in the spirit of [3, Proposition 4.3] and the examples of [11, Section 2]:

Lemma 2.1. *Let Ω_ε be as in (1.2) and $\phi_\varepsilon \in H_0^1(\Omega)$ be the function satisfying (2.2). Then, we have $\phi_\varepsilon \in H^2(\Omega_\varepsilon)$ and the following estimates hold*

$$\|1-\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_* \qquad \textit{and} \qquad \|\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla\phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_* \varepsilon^{\frac{\alpha-3}{2}}\,, \tag{2.3}$$

for some constant $C_* > 0$ that depends on Ω and $\{\delta_i\}_{i=0}^2$, but is independent of $\varepsilon \in I_*$.

Proof. In what follows, $C > 0$ will always denote a generic constant that depends on Ω and $\{\delta_i\}_{i=0}^2$ (independently of $\varepsilon \in I_*$), but that may change from line to line.
Since K_n^ε has a boundary of class $\mathcal{C}^2$ for every $n \in \{1, ..., N(\varepsilon)\}$, Ω is convex and its lateral boundary is smooth, standard elliptic regularity arguments show that $\phi_\varepsilon \in H^2(\Omega_\varepsilon)$. The first estimate in (2.3) follows directly from the Maximum Principle. Concerning the second estimate in (2.3), choose $\lambda > \delta_0$ (independent of $\varepsilon \in I_*$) such that

$$\overline{K_n^\varepsilon} \subset B(\xi_n^\varepsilon, \delta_0\varepsilon^\alpha) \subset B(\xi_n^\varepsilon, \lambda\varepsilon^\alpha) \subset B(\xi_n^\varepsilon, \delta_1\varepsilon) \qquad \forall n \in \{1, ..., N(\varepsilon)\}\,.$$

Given $n \in \{1, ..., N(\varepsilon)\}$, consider the function $\varphi_n^\varepsilon : \Omega \longrightarrow \mathbb{R}$ defined by

$$\varphi_n^\varepsilon(\xi) = \begin{cases} 1 & \text{if } \ 0 \leq |\xi - \xi_n^\varepsilon| \leq \delta_0\varepsilon^\alpha\,, \\ \dfrac{\lambda\varepsilon^\alpha - |\xi - \xi_n^\varepsilon|}{\lambda\varepsilon^\alpha - \delta_0\varepsilon^\alpha} & \text{if } \ \delta_0\varepsilon^\alpha < |\xi - \xi_n^\varepsilon| \leq \lambda\varepsilon^\alpha\,, \\ 0 & \text{if } \ |\xi - \xi_n^\varepsilon| > \lambda\varepsilon^\alpha\,, \end{cases}$$

so that, by the assumptions in (1.1)$_1$, $\varphi_n^\varepsilon \in H_0^1(\Omega)$ and $\varphi_n^\varepsilon = 1$ in $\overline{B(\xi_n^\varepsilon, \delta_0\varepsilon^\alpha)}$. Since the relative capacity is an outer measure and is increasing with respect to domain inclusion (see [40, Section 2.2]), we get

$$\mathrm{Cap}_\Omega(K_\varepsilon) \le \sum_{n=1}^{N(\varepsilon)} \mathrm{Cap}_\Omega(K_n^\varepsilon) \le \sum_{n=1}^{N(\varepsilon)} \mathrm{Cap}_\Omega(B(\xi_n^\varepsilon, \delta_0\varepsilon^\alpha)) \le \sum_{n=1}^{N(\varepsilon)} \int_\Omega |\nabla\varphi_n^\varepsilon|^2 \le C\varepsilon^{\alpha-3},$$

so that

$$\|\nabla\phi_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{\alpha-3}{2}}.$$

Since $\phi_\varepsilon \in H_0^1(\Omega)$, an application of Poincaré's inequality in Ω allows us to conclude the proof. □

Let $Q \subset \mathbb{R}^3$ be any bounded Lipschitz domain, and consider the space of square-integrable functions in Q having zero mean value:

$$L_0^2(Q) = \left\{ g \in L^2(Q) \;\middle|\; \int_Q g = 0 \right\}. \tag{2.4}$$

Another essential preliminary result concerns the construction of a uniform *Bogovskii-type* operator on the space $L^2(\Omega_\varepsilon)$, which exploits the corresponding uniform operator on $L_0^2(\Omega_\varepsilon)$ built in [17] and the particular geometry of our setting. To fix ideas, given any function (scalar or vector) $\psi \in L^2(\Omega_\varepsilon)$, hereafter we will denote by $\widetilde{\psi} \in L^2(\Omega)$ the function defined by

$$\widetilde{\psi} \doteq \begin{cases} \psi & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \overline{K_\varepsilon}. \end{cases}$$

Then, inspired by [32, Lemma 4.2], we have:

Lemma 2.2. *Let Ω_ε be as in (1.2) and $q \in L^2(\Omega_\varepsilon)$. There exists a vector field $J_\varepsilon \in H^1(\Omega_\varepsilon)$ such that*

$$\begin{cases} \nabla\cdot J_\varepsilon = q \;\; in \;\; \Omega_\varepsilon; & J_\varepsilon \times \nu = 0 \;\; on \;\; \Gamma_I; \\ J_\varepsilon = 0 \;\; on \;\; \Gamma_W^\varepsilon \cup \Gamma_O; & \|\nabla J_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C_*\|q\|_{L^2(\Omega_\varepsilon)}, \end{cases} \tag{2.5}$$

for some constant $C_* > 0$ that depends on Ω and $\{\delta_i\}_{i=0}^2$, but is independent of $\varepsilon \in I_*$.

Proof. In what follows, $C > 0$ will always denote a generic constant that depends on Ω and $\{\delta_i\}_{i=0}^2$ (independently of $\varepsilon \in I_*$), but that may change from line to line.

For every $s \in [-h, h]$ we set

$$\Sigma(s) \doteq \{\xi \in \Omega \mid 0 < \rho < R,\; z = s\}.$$

Consider a Hagen-Poiseuille flow having unit flow rate in Ω, that is,

$$U_0(\xi) \doteq \frac{2}{\pi R^4}(R^2-\rho^2)\widehat{k} \qquad \forall \xi \in \Omega \,. \tag{2.6}$$

Clearly $U_0 \in \mathcal{C}^\infty(\overline{\Omega})$ is divergence-free, it vanishes on $\mathcal{L}$, and $U_0 \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$. Moreover,

$$\int_{\Gamma_I} U_0 \cdot \widehat{k} = \int_{\Gamma_O} U_0 \cdot \widehat{k} = 1 \,. \tag{2.7}$$

Let $\phi_\varepsilon \in H^2(\Omega_\varepsilon) \cap H^1_0(\Omega)$ be the relative capacity potential of K_ε with respect to Ω, as in Lemma 2.1. Notice that $\widetilde{q} \in L^2(\Omega)$ and $\|q\|_{L^2(\Omega_\varepsilon)} = \|\widetilde{q}\|_{L^2(\Omega)}$. We then define the vector field

$$Q_\varepsilon(\xi) \doteq (1-\phi_\varepsilon(\xi)) \left(\int_z^h \int_{\Sigma(s)} \widetilde{q}(x,y,s)\, dx\, dy\, ds \right) U_0(\xi) \qquad \forall \xi \in \Omega_\varepsilon \,,$$

which is an element of $H^1(\Omega_\varepsilon)$ such that $Q_\varepsilon = 0$ on $\Gamma_O \cup \Gamma_W^\varepsilon$ and $Q_\varepsilon \times \nu = 0$ on Γ_I. Then, in view of Hölder's inequality and the Maximum Principle, the following pointwise bounds hold:

$$\begin{aligned}
&\left|\frac{\partial Q_\varepsilon}{\partial x}(\xi)\right| \leq C\|q\|_{L^2(\Omega_\varepsilon)}(1+|\nabla\phi_\varepsilon(\xi)|) \,, \qquad \left|\frac{\partial Q_\varepsilon}{\partial y}(\xi)\right| \leq C\|q\|_{L^2(\Omega_\varepsilon)}(1+|\nabla\phi_\varepsilon(\xi)|) \,, \\
&\left|\frac{\partial Q_\varepsilon}{\partial z}(\xi)\right| \leq C\|q\|_{L^2(\Omega_\varepsilon)}(1+|\nabla\phi_\varepsilon(\xi)|) + C\|\widetilde{q}\|_{L^1(\Sigma(z))} \qquad \text{for a.e. } \xi \in \Omega_\varepsilon \,.
\end{aligned} \tag{2.8}$$

The estimates in (2.3)-(2.8), combined with Jensen's inequality, give us

$$\begin{aligned}
\|\nabla Q_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &\leq C \left(\|q\|^2_{L^2(\Omega_\varepsilon)} + \int_{-h}^h \int_0^{2\pi} \int_0^R \rho \|\widetilde{q}\|^2_{L^1(\Sigma(z))}\, d\rho\, d\theta\, dz \right) \\
&\leq C \left(\|q\|^2_{L^2(\Omega_\varepsilon)} + \int_{-h}^h \int_{\Sigma(s)} |\widetilde{q}(x,y,s)|^2\, dx\, dy\, ds \right) \leq C\|q\|^2_{L^2(\Omega_\varepsilon)} \,,
\end{aligned}$$

that is,

$$\|\nabla Q_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\|q\|_{L^2(\Omega_\varepsilon)} \,. \tag{2.9}$$

On the other hand, from the Divergence Theorem and (2.7) we obtain

$$-\int_{\Omega_\varepsilon} \nabla \cdot Q_\varepsilon = -\int_{\partial\Omega_\varepsilon} Q_\varepsilon \cdot \nu = \int_{\Gamma_I} Q_\varepsilon \cdot \widehat{k} = \int_{-h}^h \int_{\Sigma(s)} \widetilde{q}(x,y,s)\, dx\, dy\, ds = \int_\Omega \widetilde{q} = \int_{\Omega_\varepsilon} q \,,$$

so that $q + \nabla \cdot Q_\varepsilon \in L^2_0(\Omega_\varepsilon)$, see (2.4). Then, there exists a vector field $X_\varepsilon \in H^1_0(\Omega_\varepsilon)$ such that

$$\nabla \cdot X_\varepsilon = q + \nabla \cdot Q_\varepsilon \quad \text{in} \quad \Omega_\varepsilon \qquad \text{and} \qquad \|\nabla X_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_B(\Omega_\varepsilon)\|q + \nabla \cdot Q_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{2.10}$$

see [9]. From [17, Theorem 2.3] we know that $C_B(\Omega_\varepsilon)$ (the so-called *Bogovskii constant* of Ω_ε, see [24, Section 2]) admits the uniform bound

$$C_B(\Omega_\varepsilon) \leq C\left(1 + \varepsilon^{\frac{\alpha-3}{2}}\right). \tag{2.11}$$

We set $J_\varepsilon \doteq X_\varepsilon - Q_\varepsilon$ which, in view of (2.9)-(2.10)-(2.11), is an element of $H^1(\Omega_\varepsilon)$ satisfying (2.5). □

As in [46], we introduce the functional spaces (of vector fields) that will be employed hereafter:

$$\mathcal{V}_*(\Omega_\varepsilon) = \left\{ v \in H^1(\Omega_\varepsilon) \;\middle|\; \begin{array}{ll} \nabla \cdot v = 0 \ \text{ in } \ \Omega_\varepsilon\,; & v \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O\,; \\ v = 0 \ \text{ on } \ \Gamma_W^\varepsilon\,; & \displaystyle\int_{\Sigma_\varepsilon(s)} v \cdot \widehat{k} = 0 \quad \forall s \in [-h, h] \end{array} \right\}$$

and

$$\mathcal{V}(\Omega_\varepsilon) = \left\{ v \in H^1(\Omega_\varepsilon) \mid \nabla \cdot v = 0 \ \text{ in } \ \Omega_\varepsilon\,; \qquad v \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O\,; \qquad v = 0 \ \text{ on } \ \Gamma_W^\varepsilon \right\},$$

which are Hilbert spaces if endowed with the Dirichlet scalar product of the gradients. Concerning the boundary-value problems (1.7)-(1.8), throughout this section we assume that $F \in \mathbb{R}$, $p^\pm \in \mathbb{R}$, $f \in L^2(\Omega)$ are a given transversal flux rate, pressure drop and external forcing term, respectively. Then, [46, Theorem 3.1] allows us to prove the existence of a vector field $\Psi_\varepsilon \in H^2(\Omega_\varepsilon)$ such that

$$\begin{cases} \nabla \cdot \Psi_\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon\,; & \Psi_\varepsilon \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O\,; \\ \Psi_\varepsilon = 0 \ \text{ on } \ \Gamma_W^\varepsilon\,; & \displaystyle\int_{\Sigma_\varepsilon(s)} \Psi_\varepsilon \cdot \widehat{k} = F \quad \forall s \in [-h, h]\,. \end{cases} \tag{2.12}$$

Moreover, as a consequence of Lemma 2.1 and [17, Theorem 2.3], there holds the bound

$$\|\nabla \Psi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_*|F|, \tag{2.13}$$

for some constant $C_* > 0$ that depends on Ω and $\{\delta_i\}_{i=0}^2$, but is independent of $\varepsilon \in I_*$. We give the following definition for the weak solutions of problems (1.7)-(1.8) (equivalently, of problems (1.4)-(1.5)):

Definition 2.1. A vector field $u \in \mathcal{V}(\Omega_\varepsilon)$ is a **weak solution** of (1.7) if $u - \Psi_\varepsilon \in \mathcal{V}_*(\Omega_\varepsilon)$ and

$$\eta \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \varphi + \int_{\Omega_\varepsilon} \left[\nabla u - (\nabla u)^\top\right] u \cdot \varphi = \int_{\Omega_\varepsilon} f \cdot \varphi \qquad \forall \varphi \in \mathcal{V}_*(\Omega_\varepsilon).$$

A vector field $u \in \mathcal{V}(\Omega_\varepsilon)$ is called a **weak solution** of (1.8) if

$$\eta \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \varphi + \int_{\Omega_\varepsilon} \Big[\nabla u - (\nabla u)^\top\Big] u \cdot \varphi + (p^+ - p^-) \int_{\Gamma_O} \varphi \cdot \widehat{k} = \int_{\Omega_\varepsilon} f \cdot \varphi \quad \forall \varphi \in \mathcal{V}(\Omega_\varepsilon) \,. \tag{2.14}$$

We refer to [32, Section 2] for an explanation of the fact that the boundary conditions involving the Bernoulli pressure in (1.7)-(1.8) are implicitly contained in the variational formulations of Definition 2.1.

The first main result of this section provides uniform bounds (with respect to $\varepsilon \in I_*$) for the solutions of the prescribed flux problem (1.7).

Theorem 2.1. *Let Ω_ε be as in (1.2), and suppose that there exists $\delta_3 \in (0, h)$ such that, for every $\varepsilon \in I_*$ and $n \in \{1, ..., N(\varepsilon)\}$, we have*

$$\partial B\left(\xi_n^\varepsilon, \delta_2\, \varepsilon\right) \cap \left\{\xi \in \overline{\Omega} \mid -h \leq z \leq -h + \delta_3 \;\; or \;\; h - \delta_3 \leq z \leq h\right\} = \emptyset \,. \tag{2.15}$$

For any $F \in \mathbb{R}$ and $f \in L^2(\Omega)$, there exists at least one weak solution $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap \mathcal{V}(\Omega_\varepsilon)$ of the prescribed net flux problem (1.7) and an associated Bernoulli pressure $\Phi_\varepsilon \in H^1(\Omega_\varepsilon)$ such that the pair $(u_\varepsilon, \Phi_\varepsilon)$ satisfies (1.7) in strong form for some constant $p_\varepsilon^+ \in \mathbb{R}$. Moreover, the uniform bound

$$\sup_{\varepsilon \in I_*} \left(\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} + |p_\varepsilon^+|\right) \leq C_* \,, \tag{2.16}$$

holds for some constant $C_* > 0$ that depends on Ω, η, f, F and $\{\delta_i\}_{i=0}^3$, but is independent of $\varepsilon \in I_*$.

Proof. In what follows, $C > 0$ will always denote a generic constant that depends on Ω, η, F and $\{\delta_i\}_{i=0}^3$ (independently of $\varepsilon \in I_*$), but that may change from line to line.

Then, given any $F \in \mathbb{R}$, $f \in L^2(\Omega)$ and $\varepsilon \in I_*$, a direct extension of [46, Theorem 3.3] ensures the existence of at least one weak solution $u_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$ of problem (1.7). Then, [46, Theorem 3.2] guarantees that $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap \mathcal{V}(\Omega_\varepsilon)$ and the existence of an associated pressure $\Phi_\varepsilon \in H^1(\Omega_\varepsilon)$ satisfying

$$\begin{cases} -\eta \Delta u_\varepsilon + \mathcal{E}(u_\varepsilon) u_\varepsilon + \nabla \Phi_\varepsilon = f \,, \qquad \nabla \cdot u_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon \,, \\ u_\varepsilon = 0 \quad \text{on} \quad \Gamma_W^\varepsilon \,, \\ u_\varepsilon \times \nu = 0 \,, \quad \Phi_\varepsilon = 0 \quad \text{on} \quad \Gamma_I \,, \\ u_\varepsilon \times \nu = 0 \,, \quad \Phi_\varepsilon = p_\varepsilon^+ \quad \text{on} \quad \Gamma_O \,, \\ \displaystyle\int_{\Sigma_\varepsilon(s)} u_\varepsilon \cdot \widehat{k} = F \qquad \forall s \in [-h, h] \,, \end{cases} \tag{2.17}$$

in strong form, for some (unknown) constant $p_\varepsilon^+ \in \mathbb{R}$. In $(2.17)_1$, $\mathcal{E}(w) \doteq \nabla w - (\nabla w)^\top$ denotes the skew-symmetric gradient of any $w \in H^1(\Omega)$. Notice that $\widetilde{\Phi_\varepsilon} \in L^2(\Omega)$, $\widetilde{u_\varepsilon} \in S_\star(\Omega)$ is divergence-free separately in Ω_ε and K_ε, where we have introduced

$$S_\star(\Omega) \doteq \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \mathcal{L}; \quad v \times \nu = 0 \text{ on } \Gamma_I \cup \Gamma_O \right\}, \tag{2.18}$$

which is a closed subspace of $H^1(\Omega)$. Moreover,

$$\|\nabla \widetilde{u_\varepsilon}\|_{L^2(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \qquad \text{and} \qquad \|\widetilde{\Phi_\varepsilon}\|_{L^2(\Omega)} = \|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{2.19}$$

We multiply the first identity in (2.17)$_1$ by u_ε and integrate by parts, each term separately, to obtain

$$-\int_{\Omega_\varepsilon} \Delta u_\varepsilon \cdot u_\varepsilon = \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \int_{\partial\Omega_\varepsilon} \frac{\partial u_\varepsilon}{\partial \nu} \cdot u_\varepsilon = \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \left(\int_{\Gamma_I} \frac{\partial u_\varepsilon}{\partial \nu} \cdot u_\varepsilon + \int_{\Gamma_O} \frac{\partial u_\varepsilon}{\partial \nu} \cdot u_\varepsilon \right).$$

Write $u_\varepsilon = (u_\varepsilon^{(1)}, u_\varepsilon^{(2)}, u_\varepsilon^{(3)})$ in Ω_ε. Notice that $\nu = \mp\widehat{k}$ on Γ_I and Γ_O, respectively, and thus $u_\varepsilon^{(1)} = u_\varepsilon^{(2)} = 0$ on $\Gamma_I \cup \Gamma_O$, in view of (1.7)$_3$-(1.7)$_4$. The regularity and incompressibility condition of u then imply

$$\frac{\partial u_\varepsilon}{\partial \nu} \cdot u_\varepsilon = u_\varepsilon^{(3)} \frac{\partial u_\varepsilon^{(3)}}{\partial z} = -u_\varepsilon^{(3)} \left(\frac{\partial u_\varepsilon^{(1)}}{\partial x} + \frac{\partial u_\varepsilon^{(2)}}{\partial y} \right) = 0 \quad \text{on} \quad \Gamma_I \cup \Gamma_O. \tag{2.20}$$

Therefore,

$$-\int_{\Omega_\varepsilon} \Delta u_\varepsilon \cdot u_\varepsilon = \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}. \tag{2.21}$$

Concerning the nonlinear term, we obviously have

$$\int_{\Omega_\varepsilon} \mathcal{E}(u_\varepsilon) u_\varepsilon \cdot u_\varepsilon = 0. \tag{2.22}$$

Regarding the pressure term, from (2.17)$_3$-(2.17)$_4$-(2.17)$_5$ we infer

$$\int_{\Omega_\varepsilon} \nabla \Phi_\varepsilon \cdot u_\varepsilon = \int_{\partial\Omega_\varepsilon} \Phi_\varepsilon (u_\varepsilon \cdot \nu) = F p_\varepsilon^+. \tag{2.23}$$

By adding the identities (2.21)-(2.22)-(2.23) we get

$$\eta \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + F p_\varepsilon^+ = \int_\Omega f \cdot \widetilde{u_\varepsilon}, \tag{2.24}$$

so that an application of Poincaré's inequality in Ω and (2.19) give us

$$|p_\varepsilon^+| \leq C \left(\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right). \tag{2.25}$$

Now, in view of Lemma 2.2, let $J_\varepsilon \in H^1(\Omega_\varepsilon)$ be a vector field such that

$$\begin{cases} \nabla \cdot J_\varepsilon = \Phi_\varepsilon \ \text{ in } \ \Omega_\varepsilon\,; & J_\varepsilon \times \nu = 0 \ \text{ on } \ \Gamma_I\,; \\ J_\varepsilon = 0 \ \text{ on } \ \Gamma_O \cup \Gamma_W^\varepsilon\,; & \|\nabla J_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,. \end{cases} \tag{2.26}$$

We multiply the first identity in $(2.17)_1$ by J_ε and integrate by parts in Ω_ε to obtain

$$\eta \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla J_\varepsilon - \eta \int_{\Gamma_I} \frac{\partial u_\varepsilon}{\partial \nu} \cdot J_\varepsilon + \int_{\Omega_\varepsilon} \mathcal{E}(u_\varepsilon) u_\varepsilon \cdot J_\varepsilon - \|\Phi_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} f \cdot J_\varepsilon\,. \tag{2.27}$$

Since $J_\varepsilon \times \nu = 0$ on Γ_I, as in (2.20) one can show that

$$\frac{\partial u_\varepsilon}{\partial \nu} \cdot J_\varepsilon = 0 \quad \text{on} \quad \Gamma_I\,. \tag{2.28}$$

Observing that $\widetilde{J_\varepsilon} \in H^1(\Omega)$ and that $\|\nabla \widetilde{J_\varepsilon}\|_{L^2(\Omega)} = \|\nabla J_\varepsilon\|_{L^2(\Omega_\varepsilon)}$, we insert (2.28) into (2.27), apply Hölder's inequality, the Sobolev and Poincaré inequalities in Ω and (2.19)-(2.26) in order to write

$$\begin{aligned} \|\Phi_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &= \eta \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla J_\varepsilon + \int_{\Omega} \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \widetilde{J_\varepsilon} - \int_{\Omega} f \cdot \widetilde{J_\varepsilon} \\ &\leq \eta \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla J_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\mathcal{E}(\widetilde{u_\varepsilon})\|_{L^2(\Omega)} \|\widetilde{u_\varepsilon}\|_{L^4(\Omega)} \|\widetilde{J_\varepsilon}\|_{L^4(\Omega)} \\ &\quad + \|f\|_{L^2(\Omega)} \|\widetilde{J_\varepsilon}\|_{L^2(\Omega)} \\ &\leq \eta \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla J_\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\|\nabla \widetilde{u_\varepsilon}\|^2_{L^2(\Omega)} \|\nabla \widetilde{J_\varepsilon}\|_{L^2(\Omega)} \\ &\quad + C\|f\|_{L^2(\Omega)} \|\nabla \widetilde{J_\varepsilon}\|_{L^2(\Omega)} \\ &\leq C\left(\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)}\right) \|\nabla J_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq C\left(\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)}\right) \|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,, \end{aligned} \tag{2.29}$$

thereby yielding

$$\|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\left(\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)}\right)\,. \tag{2.30}$$

By contradiction, suppose now that the norms $\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ are not uniformly bounded with respect to $\varepsilon \in I_*$. Then, there must exists a subsequence (not being relabeled) such that

$$\lim_{\varepsilon \to 0^+} \mathcal{Z}_\varepsilon = +\infty \quad \text{with} \ \ \mathcal{Z}_\varepsilon \doteq \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \quad \forall \varepsilon \in I_*\,. \tag{2.31}$$

The estimates in (2.25)-(2.30) (see also (2.19)) enable us to establish that, along this divergent subsequence (2.31), the following sequences are all uniformly bounded with respect to $\varepsilon \in I_*$:

$$(\widehat{u_\varepsilon})_{\varepsilon\in I_*} \doteq \left(\frac{\widetilde{u_\varepsilon}}{\mathcal{Z}_\varepsilon}\right)_{\varepsilon\in I_*} \subset S_\star(\Omega)\,; \qquad (\widehat{\Phi_\varepsilon})_{\varepsilon\in I_*} \doteq \left(\frac{\widetilde{\Phi_\varepsilon}}{\mathcal{Z}_\varepsilon^2}\right)_{\varepsilon\in I_*} \subset L^2(\Omega)\,;$$

$$(\widehat{p_\varepsilon})_{\varepsilon\in I_*} \doteq \left(\frac{p_\varepsilon^+}{\mathcal{Z}_\varepsilon^2}\right)_{\varepsilon\in I_*} \subset \mathbb{R}\,.$$

There must exist $\widehat{u} \in S_\star(\Omega)$, $\widehat{\Phi} \in L^2(\Omega)$ and $\widehat{p} \in \mathbb{R}$ such that the following convergences hold:

$$\begin{aligned} &\widehat{u_\varepsilon} \rightharpoonup \widehat{u} \quad \text{weakly in } S_\star(\Omega)\,; && \widehat{u_\varepsilon} \to \widehat{u} \quad \text{strongly in } L^4(\Omega)\,; \\ &\widehat{\Phi_\varepsilon} \rightharpoonup \widehat{\Phi} \quad \text{weakly in } L^2(\Omega)\,; && \widehat{p_\varepsilon} \to \widehat{p} \quad \text{in } \mathbb{R}\,, \end{aligned} \tag{2.32}$$

as $\varepsilon \to 0^+$, along subsequences that are not being relabeled. Notice that

$$\left| \frac{1}{\mathcal{Z}_\varepsilon^2} \int_\Omega f \cdot \widetilde{u_\varepsilon} \right| \le \frac{C}{\mathcal{Z}_\varepsilon} \|f\|_{L^2(\Omega)} \|\nabla \widehat{u_\varepsilon}\|_{L^2(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+\,.$$

If we then divide identity (2.24) by $\mathcal{Z}_\varepsilon^2$ and let $\varepsilon \to 0^+$ along the subsequences in (2.32), we obtain

$$\eta = -F\,\widehat{p}\,. \tag{2.33}$$

A contradiction will be reached in (2.33) after proving that $\widehat{p} = 0$. Firstly, given any scalar function $\phi \in \mathcal{C}_0^\infty(\Omega)$, an integration by parts and the divergence-free condition in $(2.17)_1$ imply that

$$\int_\Omega \widehat{u_\varepsilon} \cdot \nabla\phi = \frac{1}{\mathcal{Z}_\varepsilon} \int_{\Omega_\varepsilon} u_\varepsilon \cdot \nabla\phi = \frac{1}{\mathcal{Z}_\varepsilon} \int_{\partial\Omega_\varepsilon} \phi\,(u_\varepsilon \cdot \nu) = 0 \qquad \forall \varepsilon \in I_*\,,$$

since u_ε vanishes on ∂K_ε and so does ϕ on $\partial\Omega$. Then, along the subsequences (2.32), the weak convergence in $(2.32)_1$ yields

$$\int_\Omega \widehat{u} \cdot \nabla\phi = -\int_\Omega \phi\,(\nabla \cdot \widehat{u}) = 0 \qquad \forall \phi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R})\,,$$

and so $\nabla \cdot \widehat{u} = 0$ a. e. in Ω, that is, $\widehat{u} \in \mathcal{V}(\Omega)$. Secondly, given any vector field $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (not necessarily divergence-free) and the relative capacity potential $\phi_\varepsilon \in H^2(\Omega_\varepsilon)$ (see Lemma 2.1), we multiply both sides of the first identity in $(2.17)_1$ by $(1-\phi_\varepsilon)\varphi$, and integrate by parts in Ω_ε, each term separately, in the following way:

$$-\int_{\Omega_\varepsilon} \Delta u_\varepsilon \cdot (1-\phi_\varepsilon)\varphi = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla\varphi - \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot (\nabla\phi_\varepsilon \otimes \varphi + \phi_\varepsilon \nabla\varphi)\,. \tag{2.34}$$

Concerning the nonlinear term, we simply put

$$\int_{\Omega_\varepsilon} \mathcal{E}(u_\varepsilon)u_\varepsilon \cdot (1-\phi_\varepsilon)\varphi = \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \varphi - \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \phi_\varepsilon \varphi \,. \tag{2.35}$$

Regarding the pressure term, from (2.17)$_3$-(2.17)$_4$ and the properties of ϕ_ε and φ, we infer

$$\int_{\Omega_\varepsilon} \nabla \Phi_\varepsilon \cdot (1-\phi_\varepsilon)\varphi = -\int_{\Omega_\varepsilon} \Phi_\varepsilon (\nabla \cdot \varphi) + \int_{\Omega_\varepsilon} \Phi_\varepsilon \left[\phi_\varepsilon (\nabla \cdot \varphi) + \nabla \phi_\varepsilon \cdot \varphi\right] . \tag{2.36}$$

By adding the identities (2.34)-(2.35)-(2.36), and then dividing the result by $\mathcal{Z}_\varepsilon^2$, we obtain

$$\begin{aligned} &\frac{\eta}{\mathcal{Z}_\varepsilon}\int_\Omega \nabla \widehat{u_\varepsilon} \cdot \nabla \varphi + \int_\Omega \mathcal{E}(\widehat{u_\varepsilon})\widehat{u_\varepsilon}\cdot\varphi - \int_\Omega \mathcal{E}(\widehat{u_\varepsilon})\widehat{u_\varepsilon}\cdot\phi_\varepsilon\varphi - \int_\Omega \widehat{\Phi_\varepsilon}(\nabla\cdot\varphi) \\ &- \frac{\eta}{\mathcal{Z}_\varepsilon}\int_\Omega \nabla\widehat{u_\varepsilon}\cdot(\nabla\phi_\varepsilon\otimes\varphi + \phi_\varepsilon\nabla\varphi) + \int_\Omega \widehat{\Phi_\varepsilon}\left[\phi_\varepsilon(\nabla\cdot\varphi) + \nabla\phi_\varepsilon\cdot\varphi\right] = \frac{1}{\mathcal{Z}_\varepsilon^2}\int_{\Omega_\varepsilon} f\cdot(1-\phi_\varepsilon)\varphi\,, \end{aligned} \tag{2.37}$$

for every $\varepsilon \in I_*$, along the subsequences (2.32). The weak convergences in (2.32)$_1$-(2.32)$_2$ give

$$\lim_{\varepsilon\to 0^+} \frac{\eta}{\mathcal{Z}_\varepsilon}\int_\Omega \nabla\widehat{u_\varepsilon}\cdot\nabla\varphi = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+}\int_\Omega \widehat{\Phi_\varepsilon}(\nabla\cdot\varphi) = \int_\Omega \widehat{\Phi}(\nabla\cdot\varphi)\,. \tag{2.38}$$

In order to handle the nonlinear terms appearing in (2.37), we write

$$\int_\Omega \mathcal{E}(\widehat{u_\varepsilon})\widehat{u_\varepsilon}\cdot\varphi = \int_\Omega \mathcal{E}(\widehat{u_\varepsilon})\widehat{u}\cdot\varphi + \int_\Omega \mathcal{E}(\widehat{u_\varepsilon})(\widehat{u_\varepsilon}-\widehat{u})\cdot\varphi \qquad \forall \varepsilon \in I_*\,. \tag{2.39}$$

On one hand, for a fixed $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have that the application

$$\psi \in S_\star(\Omega) \longmapsto \int_\Omega \mathcal{E}(\psi)\widehat{u}\cdot\varphi$$

clearly defines a continuous functional on $S_\star(\Omega)$, so that the weak convergence in (2.32)$_1$ implies

$$\lim_{\varepsilon\to 0^+}\int_\Omega \mathcal{E}(\widehat{u_\varepsilon})\widehat{u}\cdot\varphi = \int_\Omega \mathcal{E}(\widehat{u})\widehat{u}\cdot\varphi\,. \tag{2.40}$$

On the other hand, applying Hölder and Sobolev inequalities in Ω, from (2.3) we notice that

$$
\begin{aligned}
&\left|\int_{\Omega} \mathcal{E}(\widehat{u_\varepsilon})(\widehat{u_\varepsilon}-\widehat{u})\cdot\varphi\right| \leq 2\|\nabla\widehat{u_\varepsilon}\|_{L^2(\Omega)}\|\widehat{u_\varepsilon}-\widehat{u}\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)} \qquad \forall\varepsilon\in I_*\,,\\
&\left|\int_{\Omega} \mathcal{E}(\widehat{u_\varepsilon})\widehat{u_\varepsilon}\cdot\phi_\varepsilon\varphi\right| \leq C\|\nabla\widehat{u_\varepsilon}\|_{L^2(\Omega)}\|\widehat{u_\varepsilon}\|_{L^4(\Omega)}\|\varphi\|_{L^\infty(\Omega)}\,\varepsilon^{\frac{\alpha-3}{2}} \qquad \forall\varepsilon\in I_*\,,\\
&\left|\int_{\Omega} \widehat{\Phi_\varepsilon}\,[\phi_\varepsilon(\nabla\cdot\varphi)+\nabla\phi_\varepsilon\cdot\varphi]\right| \leq C\|\widehat{\Phi_\varepsilon}\|_{L^2(\Omega)}\|\varphi\|_{W^{1,\infty}(\Omega)}\,\varepsilon^{\frac{\alpha-3}{2}} \qquad \forall\varepsilon\in I_*\,,\\
&\left|\int_{\Omega} \nabla\widehat{u_\varepsilon}\cdot(\nabla\phi_\varepsilon\otimes\varphi+\phi_\varepsilon\nabla\varphi)\right| \leq C\|\nabla\widehat{u_\varepsilon}\|_{L^2(\Omega)}\|\varphi\|_{W^{1,\infty}(\Omega)}\,\varepsilon^{\frac{\alpha-3}{2}} \qquad \forall\varepsilon\in I_*\,,\\
&\left|\int_{\Omega_\varepsilon} f\cdot(1-\phi_\varepsilon)\varphi\right| \leq C\|f\|_{L^2(\Omega)}\|\varphi\|_{L^\infty(\Omega)} \qquad \forall\varepsilon\in I_*\,,
\end{aligned}
\tag{2.41}
$$

so that the strong convergence in (2.32)$_1$ yields

$$
\lim_{\varepsilon\to 0^+}\int_{\Omega}\mathcal{E}(\widehat{u_\varepsilon})(\widehat{u_\varepsilon}-\widehat{u})\cdot\varphi = 0\,. \tag{2.42}
$$

Observing (2.38)-(2.39)-(2.40)-(2.41)-(2.42), one can take the limit as $\varepsilon\to 0^+$ in (2.37) and deduce that

$$
\int_{\Omega}\mathcal{E}(\widehat{u})\widehat{u}\cdot\varphi - \int_{\Omega}\widehat{\Phi}(\nabla\cdot\varphi) = 0 \qquad \forall\varphi\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)\,,
$$

that is, the pair $(\widehat{u},\widehat{\Phi})\in\mathcal{V}(\Omega)\times L^2(\Omega)$ satisfies in distributional form the following Euler-type equation:

$$
\left[\nabla\widehat{u}-(\nabla\widehat{u})^\top\right]\widehat{u}+\nabla\widehat{\Phi}=0\,,\quad \nabla\cdot\widehat{u}=0 \ \text{ in } \ \Omega\,. \tag{2.43}
$$

Since $\mathcal{E}(\widehat{u})\widehat{u}\in L^{3/2}(\Omega)$ (by Sobolev embedding), (2.43) proves that actually $\widehat{\Phi}\in W^{1,3/2}(\Omega)$. We set $\widehat{\Phi}_0 \doteq \widehat{\Phi}-|\widehat{u}|^2/2$, so that the pair $(\widehat{u},\widehat{\Phi}_0)\in\mathcal{V}(\Omega)\times W^{1,3/2}(\Omega)$ satisfies in strong form the Euler equation

$$
(\widehat{u}\cdot\nabla)\widehat{u}+\nabla\widehat{\Phi}_0=0\,,\quad \nabla\cdot\widehat{u}=0 \ \text{ in } \ \Omega\,.
$$

Since $\widehat{u}=0$ on $\mathcal{L}$, the Bernoulli law [29, Lemma 4] (see [5, Theorem 2.2] and [30, Theorem 1] as well) states that $\widehat{\Phi}_0$ must be constant on $\mathcal{L}$. Then, there exists $\widehat{p}_{\mathcal{L}}\in\mathbb{R}$ such that $\widehat{\Phi}=\widehat{p}_{\mathcal{L}}$ on $\mathcal{L}$. Now, in view of the embedding $H^1(\Omega_\varepsilon)\subset L^6(\Omega_\varepsilon)$, observe that $f-\mathcal{E}(u_\varepsilon)u_\varepsilon\in L^{3/2}(\Omega_\varepsilon)$, so from the Hölder and Sobolev inequalities in Ω we estimate

$$\begin{aligned}\|f-\mathcal{E}(u_\varepsilon)u_\varepsilon\|_{L^{3/2}(\Omega_\varepsilon)} &\le \|f\|_{L^{3/2}(\Omega)}+\|\mathcal{E}(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^6(\Omega_\varepsilon)}\\ &=\|f\|_{L^{3/2}(\Omega)}+\|\mathcal{E}(\widetilde{u_\varepsilon})\|_{L^2(\Omega)}\|\widetilde{u_\varepsilon}\|_{L^6(\Omega)}\\ &\le \|f\|_{L^{3/2}(\Omega)}+C\|\mathcal{E}(\widetilde{u_\varepsilon})\|_{L^2(\Omega)}\|\nabla\widetilde{u_\varepsilon}\|_{L^2(\Omega)}\\ &\le C\left(\|f\|_{L^{3/2}(\Omega)}+\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\right).\end{aligned} \tag{2.44}$$

We introduce the following subdomains of Ω:

$$\Omega_I \doteq \{\xi\in\Omega \mid -h<z<-h+\delta_3\} \qquad \text{and} \qquad \Omega_O \doteq \{\xi\in\Omega \mid h-\delta_3<z<h\}.$$

Observe that the pair $(u_\varepsilon,\Phi_\varepsilon)\in W^{2,3/2}(\Omega_\varepsilon)\times W^{1,3/2}(\Omega_\varepsilon)$ is also a strong solution to the Stokes system $(2.17)_1$ in Ω_ε, with a right-hand side given by $f-\mathcal{E}(u_\varepsilon)u_\varepsilon$. If we apply the same extension argument of [46, Theorem 3.2], we can then invoke the usual local regularity results for the Stokes equations (see [10, Teorema, page 311] or [26, Theorem IV.4.1]) and the estimates (2.30)-(2.44) to yield

$$\begin{aligned}\|u_\varepsilon\|_{W^{2,3/2}(\Omega_I)}+\|\Phi_\varepsilon\|_{W^{1,3/2}(\Omega_I)} &\le C\left(\|f-\mathcal{E}(u_\varepsilon)u_\varepsilon\|_{L^{3/2}(\Omega_\varepsilon)}+\|\nabla u_\varepsilon\|_{L^{3/2}(\Omega_\varepsilon)}+\|\Phi_\varepsilon\|_{L^{3/2}(\Omega_\varepsilon)}\right)\\ &\le C\left(1+\|f\|_{L^2(\Omega)}+\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right).\end{aligned} \tag{2.45}$$

In the same way we derive

$$\|u_\varepsilon\|_{W^{2,3/2}(\Omega_O)}+\|\Phi_\varepsilon\|_{W^{1,3/2}(\Omega_O)} \le C\left(1+\|f\|_{L^2(\Omega)}+\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}+\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right). \tag{2.46}$$

We emphasize that, in view of assumption (2.15), the constant $C>0$ entering (2.45)-(2.46) is independent of $\varepsilon\in I_*$, since the strips Ω_I and Ω_O do not contain any holes. Along the weakly convergent subsequence $(2.32)_2$, and in the light of (2.45)-(2.46), we infer that the sequences $(\widehat{\Phi_\varepsilon})_{\varepsilon\in I_*}\subset W^{1,3/2}(\Omega_I)$ and $(\widehat{\Phi_\varepsilon})_{\varepsilon\in I_*}\subset W^{1,3/2}(\Omega_O)$ are uniformly bounded. Therefore, there exist $\widehat{\Phi_I}\in W^{1,3/2}(\Omega_I)$ and $\widehat{\Phi_O}\in W^{1,3/2}(\Omega_O)$ such that the following convergences hold as $\varepsilon\to 0^+$:

$$\begin{aligned}&\widehat{\Phi_\varepsilon}\rightharpoonup\widehat{\Phi_I} \quad \text{weakly in } W^{1,3/2}(\Omega_I); && \widehat{\Phi_\varepsilon}\rightharpoonup\widehat{\Phi_O} \quad \text{weakly in } W^{1,3/2}(\Omega_O);\\ &\widehat{\Phi_\varepsilon}\to\widehat{\Phi_I} \quad \text{strongly in } L^2(\Omega_I); && \widehat{\Phi_\varepsilon}\to\widehat{\Phi_O} \quad \text{strongly in } L^2(\Omega_O);\\ &\widehat{\Phi_\varepsilon}\to\widehat{\Phi_I} \quad \text{strongly in } L^1(\partial\Omega_I); && \widehat{\Phi_\varepsilon}\to\widehat{\Phi_O} \quad \text{strongly in } L^1(\partial\Omega_O),\end{aligned} \tag{2.47}$$

along subsequences that are not being relabeled, see also [42, Theorem 6.2]. In view of $(2.17)_3$-$(2.17)_4$, the strong convergences in $(2.47)_3$ imply that $\widehat{\Phi_I}=0$ on Γ_I and $\widehat{\Phi_O}=\widehat{p}$ on Γ_O. But since we also have that $\widehat{\Phi_\varepsilon}\rightharpoonup\Phi_I$ weakly in $L^2(\Omega_I)$ and $\widehat{\Phi_\varepsilon}\rightharpoonup\Phi_O$ weakly in $L^2(\Omega_O)$ as $\varepsilon\to 0^+$, by uniqueness of the weak limit there must hold $\widehat{\Phi}=\widehat{\Phi_I}$ in Ω_I and $\widehat{\Phi}=\widehat{\Phi_O}$ in Ω_O. Therefore, since $\widehat{\Phi}\in W^{1,3/2}(\Omega)$,

$$\widehat{\Phi}=0 \quad \text{on} \quad \Gamma_I; \qquad \widehat{\Phi}=\widehat{p} \quad \text{on} \quad \Gamma_O. \tag{2.48}$$

In view of (2.48), for almost every $(\rho,\theta,z)\in(0,R)\times[0,2\pi]\times(0,h)$ we may write

$$\widehat{p} - \widehat{\Phi}(\rho,\theta,z) = \widehat{\Phi}(\rho,\theta,h) - \widehat{\Phi}(\rho,\theta,z) = \int_z^h \frac{\partial \widehat{\Phi}}{\partial z_0}(\rho,\theta,z_0)\,dz_0\,.$$

Integrating this last equality with respect to $(\theta, z) \in [0, 2\pi] \times (0, h)$ gives us

$$\begin{aligned}\rho \int_0^h \int_0^{2\pi} |\widehat{p} - \widehat{\Phi}(\rho,\theta,z)|\,d\theta\,dz &\leq \rho \int_0^h \int_0^{2\pi} \int_z^h |\nabla\widehat{\Phi}(\rho,\theta,z_0)|\,dz_0\,d\theta\,dz \\ &\leq \rho h \int_0^h \int_0^{2\pi} |\nabla\widehat{\Phi}(\rho,\theta,z_0)|\,d\theta\,dz_0\,. \end{aligned} \tag{2.49}$$

Since $\nabla\widehat{\Phi} \in L^1(\Omega)$, given any integer $j \geq 1$, the Mean Value Theorem for Lebesgue integrals can be applied to deduce that

$$\left| \left\{ \rho \in \left(R - \frac{1}{j}, R \right) \;\middle|\; \rho \int_0^h \int_0^{2\pi} |\nabla\widehat{\Phi}(\rho,\theta,z_0)|\,d\theta\,dz_0 \leq j \|\nabla\widehat{\Phi}\|_{L^1(\Omega_j)} \right\} \right| > 0\,,$$

where $\Omega_j \doteq \left\{ \xi \in \Omega \mid R - \frac{1}{j} < \rho < R \right\}$ for every integer $j \geq 1$. Therefore, we can find a sequence of numbers $(\rho_j^+)_{j\geq 1} \subset (0, R)$ such that $\rho_j^+ \to R$ as $j \to \infty$ and

$$\rho_j^+ \int_0^h \int_0^{2\pi} |\nabla\widehat{\Phi}(\rho_j^+,\theta,z_0)|\,d\theta\,dz_0 \leq j \|\nabla\widehat{\Phi}\|_{L^1(\Omega_j)} \qquad \forall j \geq 1\,. \tag{2.50}$$

In view of the Euler-type equation (2.43), Hölder's and Poincaré's inequality we have

$$\|\nabla\widehat{\Phi}\|_{L^1(\Omega_j)} = \|(\widehat{u}\cdot\nabla)\widehat{u} - (\nabla\widehat{u})^\top\widehat{u}\|_{L^1(\Omega_j)} \leq 2\|\nabla\widehat{u}\|_{L^2(\Omega_j)}\|\widehat{u}\|_{L^2(\Omega_j)} \leq \frac{C}{j}\|\nabla\widehat{u}\|^2_{L^2(\Omega_j)}\,,$$

(with some constant $C > 0$ independent of $j \geq 1$) which, once inserted into (2.50), gives

$$\rho_j^+ \int_0^h \int_0^{2\pi} |\nabla\widehat{\Phi}(\rho_j^+,\theta,z_0)|\,d\theta\,dz_0 \leq C\|\nabla\widehat{u}\|^2_{L^2(\Omega_j)} \qquad \forall j \geq 1\,.$$

Since $\nabla\widehat{u} \in L^2(\Omega)$ and $|\Omega_j| \to 0$ as $j \to \infty$, the last inequality and (2.49) imply that

$$\lim_{j\to\infty} \int_0^h \int_0^{2\pi} |\widehat{p} - \widehat{\Phi}(\rho_j^+,\theta,z)|\,d\theta\,dz = 0\,. \tag{2.51}$$

Similarly we can prove the existence of a sequence $(\rho_j^-)_{j\geq 1} \subset (0,R)$ such that $\rho_j^- \to R$ as $j \to \infty$ and

$$\lim_{j\to\infty} \int_{-h}^{0} \int_{0}^{2\pi} |\widehat{\Phi}(\rho_j^-,\theta,z)|\, d\theta\, dz = 0\,. \tag{2.52}$$

On the other hand, since $\widehat{\Phi} = \widehat{p}_{\mathcal{L}}$ on $\mathcal{L}$, for almost every $(\rho,\theta,z) \in (0,R)\times[0,2\pi]\times(-h,h)$ we have

$$\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho,\theta,z) = \widehat{\Phi}(R,\theta,z) - \widehat{\Phi}(\rho,\theta,z) = \int_{\rho}^{R} \frac{\partial \widehat{\Phi}}{\partial \rho_0}(\rho_0,\theta,z)\, d\rho_0\,,$$

so that

$$\begin{aligned}
\rho \int_{-h}^{h} \int_{0}^{2\pi} |\widehat{\Phi}(\rho,\theta,z) - \widehat{p}_{\mathcal{L}}|\, d\theta\, dz &\leq \rho \int_{-h}^{h} \int_{0}^{2\pi} \int_{\rho}^{R} |\nabla \widehat{\Phi}(\rho_0,\theta,z)|\, d\rho_0\, d\theta\, dz \\
&\leq \int_{-h}^{h} \int_{0}^{2\pi} \int_{\rho}^{R} \rho_0 |\nabla \widehat{\Phi}(\rho_0,\theta,z)|\, d\rho_0\, d\theta\, dz\,,
\end{aligned}$$

and since $\nabla\widehat{\Phi} \in L^1(\Omega)$, the last inequality implies that

$$\lim_{\rho\to R} \int_{-h}^{h} \int_{0}^{2\pi} |\widehat{\Phi}(\rho,\theta,z) - \widehat{p}_{\mathcal{L}}|\, d\theta\, dz = 0\,. \tag{2.53}$$

Given any integer $j \geq 1$ and $(\theta,z) \in [0,2\pi]\times(0,h)$ we can therefore write

$$|\widehat{p}_{\mathcal{L}} - \widehat{p}| \leq |\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho_j^+,\theta,z)| + |\widehat{\Phi}(\rho_j^+,\theta,z) - \widehat{p}|\,.$$

By integrating this last inequality for $(\theta,z) \in [0,2\pi]\times(0,h)$ we obtain

$$\begin{aligned}
|\widehat{p}_{\mathcal{L}} - \widehat{p}| &\leq \frac{1}{2\pi h}\left(\int_{0}^{h}\int_{0}^{2\pi} |\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho_j^+,\theta,z)|\, d\theta\, dz + \int_{0}^{h}\int_{0}^{2\pi} |\widehat{\Phi}(\rho_j^+,\theta,z) - \widehat{p}\,|\, d\theta\, dz \right) \\
&\leq \frac{1}{2\pi h}\left(\int_{-h}^{h}\int_{0}^{2\pi} |\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho_j^+,\theta,z)|\, d\theta\, dz + \int_{0}^{h}\int_{0}^{2\pi} |\widehat{\Phi}(\rho_j^+,\theta,z) - \widehat{p}\,|\, d\theta\, dz \right),
\end{aligned}$$

so that, by taking the limit as $j \to \infty$ in the last inequality and observing (2.51)-(2.53) we deduce that $\widehat{p}_{\mathcal{L}} = \widehat{p}$. In a similar fashion, as a consequence of (2.52)-(2.53) we obtain $\widehat{p}_{\mathcal{L}} = 0$. Therefore,

$\widehat{p} = 0$ and a contradiction is reached in (2.33), so that the norms $\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ are uniformly bounded with respect to $\varepsilon \in I_*$. Combined with (2.25)-(2.30), this concludes the proof. $\square$

Theorem 2.1 deserves some remarks and observations:

Remark 2.1. The additional assumption (2.15), which requires the existence of thin strips near Γ_I and Γ_O that are never perforated in the process as $\varepsilon \to 0^+$, is only invoked to specify the boundary values of the limit Bernoulli pressure $\widehat{\Phi} \in W^{1,3/2}(\Omega)$ on Γ_I and Γ_O. Such information cannot be directly extracted from the weak convergence $\widehat{\Phi_\varepsilon} \rightharpoonup \widehat{\Phi}$ in $L^2(\Omega)$ as $\varepsilon \to 0^+$, while it does not seem straightforward to build a $W^{1,3/2}(\Omega)$-uniform extension for the Bernoulli pressure Φ_ε inside the holes; in fact, following the approach of [39, Lemma 1.7] (based on local regularity estimates for the Stokes problem) one obtains

$$\|u_\varepsilon\|_{W^{2,3/2}(\Omega_\varepsilon)} + \|\Phi_\varepsilon\|_{W^{1,3/2}(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon^\alpha}\left(1 + \|f\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\right) \quad \forall \varepsilon \in I_* \,,$$

for some constant $C > 0$ independent of $\varepsilon \in I_*$. It is left open the possibility of recovering the result of Theorem 2.1 without hypothesis (2.15).

Remark 2.2. Looking into the proof of Theorem 2.1, we notice that

$$p_\varepsilon^+ = \frac{1}{\pi R^2} \int_{\Gamma_O} \Phi_\varepsilon \qquad \forall \varepsilon \in I_* \,,$$

in view of $(2.17)_4$. As a direct consequence of the trace inequality (applied in Ω_O) and (2.46), we can easily derive the estimate (2.25). However, we did not proceed in this way in order to highlight the precise and only point in which the additional assumption (2.15) is required, as described in Remark 2.1.

Remark 2.3. The uniform bound obtained in Theorem 2.1 can be easily achieved under a smallness assumption on the data. More precisely, it follows from (2.12)-(2.13) and [46, Theorem 3.4] the existence of a constant $\delta_* > 0$ (depending only on Ω, η, and $\{\delta_i\}_{i=0}^2$, independent of $\varepsilon \in I_*$) such that, if

$$|F| + \|f\|_{L^2(\Omega)} < \delta_* \,,$$

then problem (1.7) has a unique weak solution $u_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$ which, moreover, admits the bound

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C_* \,,$$

for some constant $C_* > 0$ that depends on Ω, η, F, f and $\{\delta_i\}_{i=0}^2$ (independent of $\varepsilon \in I_*$).

The second main result of this section provides uniform bounds (with respect to $\varepsilon \in I_*$) for the solutions of the prescribed pressure drop problem (1.8). Compared to Theorem 2.1, in this case the additional assumption (2.15) is not required and the corresponding proof is considerably simpler.

Theorem 2.2. *Let Ω_ε be as in (1.2). For any $p^\pm \in \mathbb{R}$ and $f \in L^2(\Omega)$, there exists at least one weak solution $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap \mathcal{V}(\Omega_\varepsilon)$ of the prescribed pressure drop problem (1.8) and an associated Bernoulli pressure $\Phi_\varepsilon \in H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon)$ such that the pair $(u_\varepsilon, \Phi_\varepsilon)$ satisfies (1.8) in strong form. Moreover, the uniform bound*

$$\sup_{\varepsilon \in I_*} \left(\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right) \leq C_* \,,$$

holds for some constant $C_* > 0$ that depends on Ω, η, f, $p^\pm$ and $\{\delta_i\}_{i=0}^2$, but is independent of $\varepsilon \in I_*$.

Proof. In what follows, $C > 0$ will always denote a generic constant that depends on Ω, η, $p^\pm$ and $\{\delta_i\}_{i=0}^2$ (independently of $\varepsilon \in I_*$), but that may change from line to line.

Given any $p^\pm \in \mathbb{R}$, $f \in L^2(\Omega)$ and $\varepsilon \in I_*$, [32, Theorem 3.2] ensures the existence of at least one weak solution $u_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$ of problem (1.8). Then, the same argument of [46, Theorem 3.2] can be applied to deduce $u_\varepsilon \in H^2(\Omega_\varepsilon) \cap \mathcal{V}(\Omega)$ and the existence of an associated pressure $\Phi_\varepsilon \in H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon)$ satisfying

$$\begin{cases} -\eta \Delta u_\varepsilon + \mathcal{E}(u_\varepsilon) u_\varepsilon + \nabla \Phi_\varepsilon = f \,, \qquad \nabla \cdot u_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon \,, \\ u_\varepsilon = 0 \quad \text{on} \quad \Gamma_W^\varepsilon \,, \\ u_\varepsilon \times \nu = 0 \,, \quad \Phi_\varepsilon = p^- \quad \text{on} \quad \Gamma_I \,, \\ u_\varepsilon \times \nu = 0 \,, \quad \Phi_\varepsilon = p^+ \quad \text{on} \quad \Gamma_O \,, \end{cases} \tag{2.54}$$

in strong form, see also [32, Section 2]. As in the proof of Theorem 2.1, notice that $\widetilde{\Phi_\varepsilon} \in L^2_0(\Omega)$, and $\widetilde{u_\varepsilon} \in S_\star(\Omega)$ is divergence-free separately in Ω_ε and K_ε, see (2.18). By taking $\varphi = u_\varepsilon$ in the weak formulation (2.14) and setting $p_* \doteq p^+ - p^-$ we obtain

$$\eta \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + p_* \int_{\Gamma_O} u_\varepsilon \cdot \widehat{k} = \int_\Omega f \cdot \widetilde{u_\varepsilon} \,. \tag{2.55}$$

An application of Poincaré's inequality in Ω yields

$$\left| \int_\Omega f \cdot \widetilde{u_\varepsilon} \right| \leq C \|f\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \,. \tag{2.56}$$

On the other hand, the trace inequality (applied in Ω) gives us

$$\left| \int_{\Gamma_O} u_\varepsilon \cdot \widehat{k} \right| \leq \|\widetilde{u_\varepsilon}\|_{L^1(\Gamma_O)} \leq \sqrt{\pi} R \|\widetilde{u_\varepsilon}\|_{L^2(\Gamma_O)} \leq C \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \,. \tag{2.57}$$

After plugging (2.56)-(2.57) into (2.55) we obtain

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \left(\|f\|_{L^2(\Omega)} + 1 \right) \,. \tag{2.58}$$

Concerning the Bernoulli pressure, since $\Phi_\varepsilon \in L^2_0(\Omega_\varepsilon)$, there exists $X_\varepsilon \in H^1_0(\Omega_\varepsilon)$ such that

$$\nabla \cdot X_\varepsilon = \Phi_\varepsilon \quad \text{in} \quad \Omega_\varepsilon \qquad \text{and} \qquad \|\nabla X_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \, ,$$

see [9] and [17, Theorem 2.3] again. If we multiply the first identity in $(2.54)_1$ by X_ε and integrate by parts in Ω_ε, arguing exactly as in (2.27)-(2.28)-(2.29) we derive the bound

$$\|\Phi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\left(\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega)}\right) . \tag{2.59}$$

A combination of (2.58)-(2.59) concludes the proof. □

3. Asymptotic behavior as $\varepsilon \to 0^+$: homogenized equations

By employing the renowned *energy method* of Tartar [45, Appendix] (see also [49, Chapter 15]), in this section we derive the effective (or *homogenized*) equations satisfied by the solutions of problems (1.7)-(1.8) as $\varepsilon \to 0^+$. We start with the prescribed flux problem (1.7).

Theorem 3.1. *Let $(\Omega_\varepsilon)_{\varepsilon\in I_*}$ be the family of perforated domains verifying (1.1)-(2.15). Given any $F \in \mathbb{R}$ and $f \in L^2(\Omega)$, let $(u_\varepsilon, \Phi_\varepsilon) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ be a weak solution of (1.7). Then, up to the extraction of a subsequence, the sequence $\{(\widetilde{u_\varepsilon}, \widetilde{\Phi_\varepsilon})\}_{\varepsilon\in I_*} \subset \mathcal{V}(\Omega) \times L^2(\Omega)$ converges strongly to a weak solution $(u, \Phi) \in \mathcal{V}(\Omega) \times L^2(\Omega)$ of problem (1.7) in Ω as $\varepsilon \to 0^+$. Furthermore, $(u, \Phi) \in H^2(\Omega) \times H^1(\Omega)$ and it satisfies in strong form the system*

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi = f \, , \quad \nabla\cdot u = 0 \quad in \quad \Omega \, , \\ u = 0 \quad on \quad \mathcal{L} \, , \\ u \times \nu = 0 \, , \quad \Phi = 0 \quad on \quad \Gamma_I \, , \\ u \times \nu = 0 \, , \quad \Phi = p^+ \quad on \quad \Gamma_O \, , \\ \displaystyle\int_{\Sigma(s)} u \cdot \widehat{k} = F \qquad \forall s \in [-h, h] \, , \end{cases}$$

for some (unknown) constant $p^+ \in \mathbb{R}$.

Proof. In what follows, $C > 0$ will always denote a generic constant that depends on Ω, η, F and $\{\delta_i\}_{i=0}^3$ (independently of $\varepsilon \in I_*$), but that may change from line to line.

Given any $F \in \mathbb{R}$, $f \in L^2(\Omega)$ and $\varepsilon \in I_*$, let $(u_\varepsilon, \Phi_\varepsilon) \in \mathcal{V}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ be a weak solution of (1.7). From Theorem 2.1 we know that $(u_\varepsilon, \Phi_\varepsilon) \in H^2(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ satisfies (2.17) in strong form (for some unknown constant $p^+_\varepsilon \in \mathbb{R}$) and that $\{(\widetilde{u_\varepsilon}, \widetilde{\Phi_\varepsilon})\}_{\varepsilon\in I_*} \subset S_\star(\Omega) \times L^2(\Omega)$, see (2.18). Now, given any scalar function $\phi \in C^\infty_0(\Omega)$, an integration by parts and the divergence-free condition in $(2.17)_1$ imply that

$$-\int_\Omega \phi(\nabla\cdot\widetilde{u_\varepsilon}) = \int_\Omega \widetilde{u_\varepsilon}\cdot\nabla\phi = \int_{\Omega_\varepsilon} u_\varepsilon\cdot\nabla\phi = \int_{\partial\Omega_\varepsilon} \phi(u_\varepsilon\cdot\nu) = 0 \qquad \forall \varepsilon \in I_* \, ,$$

since u_ε vanishes on ∂K_ε and so does ϕ on $\partial\Omega$. This proves $\nabla \cdot \widetilde{u_\varepsilon} = 0$ almost everywhere in Ω, that is, $\widetilde{u_\varepsilon} \in \mathcal{V}(\Omega)$ for every $\varepsilon \in I_*$. Moreover, (2.16)-(2.19) ensure that the sequences $(\widetilde{u_\varepsilon})_{\varepsilon\in I_*} \subset \mathcal{V}(\Omega)$, $(\widetilde{\Phi_\varepsilon})_{\varepsilon\in I_*} \subset L^2(\Omega)$, $(p_\varepsilon^+)_{\varepsilon\in I_*} \subset \mathbb{R}$ are all uniformly bounded, so there exist $u \in \mathcal{V}(\Omega)$, $\Phi \in L^2(\Omega)$ and $p^+ \in \mathbb{R}$ such that the following convergences hold as $\varepsilon \to 0^+$:

$$
\begin{aligned}
&\widetilde{u_\varepsilon} \rightharpoonup u \quad \text{weakly in } \mathcal{V}(\Omega)\,; \qquad \widetilde{u_\varepsilon} \to u \quad \text{strongly in } L^q(\Omega) \text{ for every } q \in [1,6)\,;\\
&\widetilde{u_\varepsilon} \to u \quad \text{strongly in } L^1(\partial\Omega)\,; \qquad \widetilde{\Phi_\varepsilon} \rightharpoonup \Phi \quad \text{weakly in } L^2(\Omega)\,; \qquad p_\varepsilon^+ \to p^+ \quad \text{in } \mathbb{R}\,,
\end{aligned}
\tag{3.1}
$$

along subsequences that are not being relabeled. From $(2.17)_5$ we also deduce

$$
\int_{\Gamma_I} \widetilde{u_\varepsilon} \cdot \widehat{k} = F \qquad \forall \varepsilon \in I_*\,,
$$

so that the strong convergence in $(3.1)_2$ gives

$$
\left| \int_{\Gamma_I} u \cdot \widehat{k} - F \right| = \left| \int_{\Gamma_I} (u - \widetilde{u_\varepsilon}) \cdot \widehat{k} \right| \leq \|\widetilde{u_\varepsilon} - u\|_{L^1(\Gamma_I)} \leq \|\widetilde{u_\varepsilon} - u\|_{L^1(\partial\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+\,.
$$

Since $u \in \mathcal{V}(\Omega)$, the previous computation combined with the Divergence Theorem allow us to conclude

$$
\int_{\Sigma(s)} u \cdot \widehat{k} = F \qquad \forall s \in [-h,h]\,,
\tag{3.2}
$$

see again (1.6). On the other hand, given any $\varphi \in \mathcal{V}_*(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^3)$ and the relative capacity potential $\phi_\varepsilon \in H^2(\Omega_\varepsilon)$ (see Lemma 2.1), we multiply both sides of the first identity in $(2.17)_1$ by $(1-\phi_\varepsilon)\varphi$ and integrate by parts in Ω_ε (following the path of (2.34)-(2.35)-(2.36)) to obtain

$$
\begin{aligned}
&\eta \int_\Omega \nabla\widetilde{u_\varepsilon} \cdot \nabla\varphi + \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \varphi - \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \phi_\varepsilon\varphi - \eta \int_\Omega \nabla\widetilde{u_\varepsilon} \cdot (\nabla\phi_\varepsilon \otimes \varphi + \phi_\varepsilon\nabla\varphi)\\
&+ \int_\Omega \widetilde{\Phi_\varepsilon}\,[\phi_\varepsilon(\nabla\cdot\varphi) + \nabla\phi_\varepsilon \cdot \varphi] = \int_{\Omega_\varepsilon} f \cdot (1-\phi_\varepsilon)\varphi\,,
\end{aligned}
\tag{3.3}
$$

for every $\varepsilon \in I_*$, along the subsequences (3.1). With the help of both convergences in $(3.1)_1$ (see again (2.38)-(2.39)-(2.40)-(2.41)-(2.42)) we can easily prove that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_\Omega \nabla\widetilde{u_\varepsilon} \cdot \nabla\varphi = \int_\Omega \nabla u \cdot \nabla\varphi\,, \qquad \lim_{\varepsilon\to 0^+} \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \varphi = \int_\Omega \mathcal{E}(u)u \cdot \varphi\,,\\
&\lim_{\varepsilon\to 0^+} \left(\left| \int_\Omega \nabla\widetilde{u_\varepsilon} \cdot \nabla(\phi_\varepsilon\varphi) \right| + \left| \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \phi_\varepsilon\varphi \right| + \left| \int_\Omega \widetilde{\Phi_\varepsilon}[\nabla\cdot(\phi_\varepsilon\varphi)] \right| \right) = 0\,.
\end{aligned}
\tag{3.4}
$$

Also, observing (2.3), notice that

$$\left| \int_{\Omega_\varepsilon} f\cdot(1-\phi_\varepsilon) - \int_\Omega f\cdot\varphi \right| \le \left| \int_{K_\varepsilon} f\cdot\varphi \right| + C\|f\|_{L^2(\Omega)}\,\varepsilon^{\frac{\alpha-3}{2}} \to 0 \quad \text{as } \varepsilon\to 0^+\,, \tag{3.5}$$

because $f\cdot\varphi \in L^1(\Omega)$ and $|K_\varepsilon| \to 0$ as $\varepsilon \to 0^+$. In view of (3.4)-(3.5), we then take the limit as $\varepsilon \to 0^+$ in (3.3) to deduce

$$\eta\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Omega \mathcal{E}(u)u\cdot\varphi = \int_\Omega f\cdot\varphi \qquad \forall\varphi\in\mathcal{V}_*(\Omega)\cap\mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^3)\,,$$

so that, by density (see [18, Theorem 1]), $u\in\mathcal{V}(\Omega)$ is a weak solution of (1.7) in Ω (recall (3.2)). Then, [46, Theorem 3.2] ensures that $u\in H^2(\Omega)$ and the existence of a pressure $\Phi\in H^1(\Omega)$ satisfying

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi = f\,, \quad \nabla\cdot u = 0 \ \text{ in } \ \Omega\,, \\ u = 0 \ \text{ on } \ \mathcal{L}\,, \\ u\times\nu = 0\,, \quad \Phi = p_*^- \ \text{ on } \ \Gamma_I\,, \\ u\times\nu = 0\,, \quad \Phi = p_*^+ \ \text{ on } \ \Gamma_O\,, \\ \displaystyle\int_{\Sigma(s)} u\cdot\widehat{k} = F \qquad \forall s\in[-h,h]\,, \end{cases} \tag{3.6}$$

in strong form, for some (unknown) constants $p_*^\pm\in\mathbb{R}$. Then, the argument of Theorem 2.1 concerning the thin strips near Γ_I and Γ_O allows us to infer that $p_*^- = 0$ and $p_*^+ = p^+$. In order to show the strong convergence in $\mathcal{V}(\Omega)\times L^2(\Omega)$, in view of the weak convergences in (3.1), it clearly suffices to show that

$$\lim_{\varepsilon\to 0^+} \|\nabla\widetilde{u_\varepsilon}\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\Omega)} \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} \|\widetilde{\Phi_\varepsilon}\|_{L^2(\Omega)} = \|\Phi\|_{L^2(\Omega)}\,. \tag{3.7}$$

Multiplying the first identity in $(3.6)_1$ by u and integrating by parts in Ω we obtain

$$\eta\|\nabla u\|^2_{L^2(\Omega)} = -Fp^+ + \int_\Omega f\cdot u\,. \tag{3.8}$$

Given $\varepsilon\in I_*$, exactly as in the proof of Theorem 2.1 we obtain identity (2.24), that is

$$\eta\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = -Fp_\varepsilon^+ + \int_\Omega f\cdot\widetilde{u_\varepsilon} \qquad \forall\varepsilon\in I_*\,. \tag{3.9}$$

In view of (3.1)-(3.8), by taking the limit in (3.9) as $\varepsilon\to 0^+$ we get the first equality in (3.7), that is, $\widetilde{u_\varepsilon}\to u$ strongly in $\mathcal{V}(\Omega)$ as $\varepsilon\to 0^+$. Now, in view of Lemma 2.2, let $J_\varepsilon\in H^1(\Omega_\varepsilon)$ be a vector field verifying (2.26). Observing then that the sequence $(\widetilde{J_\varepsilon})_{\varepsilon\in I_*} \subset H^1(\Omega)$ is uniformly bounded, we deduce the existence of $J \in H^1(\Omega)$ such that

$$\begin{gathered} J = 0 \text{ on } \Gamma_O \cup \mathcal{L}; \qquad J \times \nu = 0 \text{ on } \Gamma_I; \\ \widetilde{J_\varepsilon} \rightharpoonup J \quad \text{weakly in } H^1(\Omega); \qquad \widetilde{J_\varepsilon} \to J \quad \text{strongly in } L^q(\Omega) \text{ for every } q \in [1,6), \end{gathered} \tag{3.10}$$

as $\varepsilon \to 0^+$, along a (not relabeled) subsequence. Given any scalar function $\phi \in \mathcal{C}_0^\infty(\Omega)$, since $\nabla \cdot J_\varepsilon = \Phi_\varepsilon$ in Ω_ε and J_ε vanishes on ∂K_ε for any $\varepsilon \in I_*$, an integration by parts gives us

$$\int_\Omega \widetilde{J_\varepsilon} \cdot \nabla\phi = \int_{\Omega_\varepsilon} J_\varepsilon \cdot \nabla\phi = -\int_{\Omega_\varepsilon} \phi\, \Phi_\varepsilon = -\int_\Omega \phi\, \widetilde{\Phi_\varepsilon} \qquad \forall \varepsilon \in I_*.$$

We take the limit in this last identity as $\varepsilon \to 0^+$, observing (3.1)-(3.10), to obtain

$$\int_\Omega J \cdot \nabla\phi = -\int_\Omega \phi\, \Phi \qquad \forall \phi \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}),$$

that is, $\nabla \cdot J = \Phi$ in Ω. Multiplying the first identity in $(3.6)_1$ by J and integrating by parts in Ω we get

$$\|\Phi\|^2_{L^2(\Omega)} = \eta \int_\Omega \nabla u \cdot \nabla J + \int_\Omega \mathcal{E}(u)u \cdot J - \int_\Omega f \cdot J. \tag{3.11}$$

Given $\varepsilon \in I_*$, we multiply the first identity in $(2.17)_1$ by J_ε and, integrating by parts in Ω_ε, we reach the first identity in (2.29):

$$\|\widetilde{\Phi_\varepsilon}\|^2_{L^2(\Omega)} = \eta \int_\Omega \nabla\widetilde{u_\varepsilon} \cdot \nabla\widetilde{J_\varepsilon} + \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \widetilde{J_\varepsilon} - \int_\Omega f \cdot \widetilde{J_\varepsilon} \qquad \forall \varepsilon \in I_*. \tag{3.12}$$

Knowing that $\widetilde{u_\varepsilon} \to u$ strongly in $\mathcal{V}(\Omega)$ as $\varepsilon \to 0^+$ and using (3.10) we deduce that

$$\lim_{\varepsilon\to 0^+} \int_\Omega \nabla\widetilde{u_\varepsilon} \cdot \nabla\widetilde{J_\varepsilon} = \int_\Omega \nabla u \cdot \nabla J \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} \int_\Omega f \cdot \widetilde{J_\varepsilon} = \int_\Omega f \cdot J, \tag{3.13}$$

and also, from the Hölder and Sobolev inequalities in Ω we infer

$$\begin{aligned} &\left| \int_\Omega \mathcal{E}(\widetilde{u_\varepsilon})\widetilde{u_\varepsilon} \cdot \widetilde{J_\varepsilon} - \int_\Omega \mathcal{E}(u)u \cdot J \right| \\ &\leq \left| \int_\Omega (\mathcal{E}(\widetilde{u_\varepsilon}) - \mathcal{E}(u))\widetilde{u_\varepsilon} \cdot \widetilde{J_\varepsilon} \right| + \left| \int_\Omega \mathcal{E}(u)(\widetilde{u_\varepsilon} - u) \cdot \widetilde{J_\varepsilon} \right| + \left| \int_\Omega \mathcal{E}(u)u \cdot (\widetilde{J_\varepsilon} - J) \right| \\ &\leq C \left(\|\nabla\widetilde{u_\varepsilon} - \nabla u\|_{L^2(\Omega)} + \|\widetilde{u_\varepsilon} - u\|_{L^4(\Omega)} + \|\widetilde{J_\varepsilon} - J\|_{L^4(\Omega)} \right) \to 0 \quad \text{as } \varepsilon \to 0^+. \end{aligned} \tag{3.14}$$

In view of (3.11)-(3.13)-(3.14), by taking the limit in (3.12) as $\varepsilon \to 0^+$ we get the second equality in (3.7), that is, $\widetilde{\Phi_\varepsilon} \to \Phi$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0^+$. This concludes the proof. □

Remark 3.1. Let $(\Omega_\varepsilon)_{\varepsilon\in I_*}$ be the family of perforated domains verifying (1.1) (the additional assumption (2.15) is not required here). Given any $F \in \mathbb{R}$, [46, Theorem 3.1] ensures the existence of a vector field $\Psi_\varepsilon \in H^2(\Omega_\varepsilon)$ satisfying (2.12)-(2.13). Therefore, the sequence $(\widetilde{\Psi_\varepsilon})_{\varepsilon\in I_*} \subset S_\star(\Omega)$ is uniformly bounded, and there exists $\Psi \in S_\star(\Omega)$ for which the following convergences hold as $\varepsilon \to 0^+$:

$$\widetilde{\Psi_\varepsilon} \rightharpoonup \Psi \quad \text{weakly in } S_\star(\Omega)\,; \qquad \widetilde{\Psi_\varepsilon} \to \Psi \quad \text{strongly in } L^q(\Omega) \text{ for every } q \in [1,6)\,;$$
$$\widetilde{\Psi_\varepsilon} \to \Psi \quad \text{strongly in } L^q(\partial\Omega) \text{ for every } q \in [1,4)\,,$$

along a (not relabeled) subsequence. As in the proof of Theorem 3.1 we can show that $\Psi \in \mathcal{V}(\Omega)$ and

$$\int_{\Sigma(s)} \Psi \cdot \widehat{k} = F \qquad \forall s \in [-h,h]\,,$$

so that Ψ is a flux carrier of F in Ω, in the sense of Definition 2.1.

Concerning the prescribed pressure drop problem (1.8), we have the following result, analogous to Theorem 3.1, whose proof is omitted (for the sake of brevity, since it is very similar to the proof of Theorem 3.1 with obvious minor modifications):

Theorem 3.2. *Let $(\Omega_\varepsilon)_{\varepsilon\in I_*}$ be the family of perforated domains verifying (1.1). For any given $p^\pm \in \mathbb{R}$ and $f \in L^2(\Omega)$, let $(u_\varepsilon, \Phi_\varepsilon) \in \mathcal{V}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ be a weak solution of (1.8). Then, up to the extraction of a subsequence, the sequence $\{(\widetilde{u_\varepsilon}, \widetilde{\Phi_\varepsilon})\}_{\varepsilon\in I_*} \subset \mathcal{V}(\Omega) \times L^2_0(\Omega)$ converges strongly to a weak solution $(u,\Phi) \in \mathcal{V}(\Omega) \times L^2_0(\Omega)$ of problem (1.8) in Ω as $\varepsilon \to 0^+$. Furthermore, $(u,\Phi) \in H^2(\Omega) \times H^1(\Omega)$ and it satisfies in strong form the system*

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi = f\,, \qquad \nabla\cdot u = 0 \quad in \quad \Omega\,, \\ u = 0 \quad on \quad \mathcal{L}\,, \\ u \times \nu = 0\,, \qquad \Phi = p^- \quad on \quad \Gamma_I\,, \\ u \times \nu = 0\,, \qquad \Phi = p^+ \quad on \quad \Gamma_O\,. \end{cases}$$

Remark 3.2. All the results presented in this paper remain valid if, instead of a circular tube, we consider a container of arbitrary cross-section, that is, if we set

$$\Omega = \Big\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \Theta\,,\ -h < z < h\Big\}\,,$$

with $\Theta \subset \mathbb{R}^2$ being any open bounded domain having a smooth boundary. In this case, the corresponding Hagen-Poiseuille flow (2.6) is defined as (we use Cartesian coordinates):

$$U_0(x,y,z) = \frac{F}{\ell_0} v_0(x,y)\widehat{k} \qquad \forall (x,y,z) \in \overline{\Omega}\,,$$

where $v_0 \in H_0^1(\Theta; \mathbb{R})$ is a weak solution of the following torsion problem:

$$-\Delta v_0 = 1 \quad \text{in} \quad \Theta\,, \qquad v_0 = 0 \quad \text{on} \quad \partial\Theta\,,$$

and

$$\ell_0 \doteq \int_\Theta v_0 = \int_\Theta |\nabla v_0|^2 \neq 0\,.$$

Furthermore, adapting our techniques, we point out that similar results to the ones presented in this manuscript hold whenever the container Ω is a finite, smooth and *distorted* pipe, that is, if we set (possibly in different coordinate systems)

$$\Omega = \Big\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \Theta_1\,,\ -h < z < 0\Big\} \cup \Omega_c$$
$$\cup \Big\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \Theta_2\,,\ 0 < z < h\Big\}\,,$$

for some smooth bounded domains $\Theta_1, \Theta_2 \subset \mathbb{R}^2$, $\Omega_c = \Omega \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ being an open, bounded, smooth and simply connected set; see also [4] or [26, Chapter XIII].

Data availability

No data was used for the research described in the article.

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