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Surface measures and integration by parts formula on levels sets induced by functionals of the Brownian motion in $\mathbb{R}^n$

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ABSTRACT

On the infinite dimensional space E of continuous paths from $[0,1]$ to $\mathbb{R}^n$, $n \geq 3$, endowed with the Wiener measure μ, we construct a surface measure defined on level sets of the L^2-norm of n-dimensional processes that are solutions to a general class of stochastic differential equations, and provide an integration by parts formula involving this surface measure. We follow the approach to surface measures in Gaussian spaces proposed via techniques of Malliavin calculus in [2].

1. INTRODUCTION

Let $E = C([0,1];\mathbb{R}^n)$ denote the Banach space of continuous functions from $[0,1]$ to $\mathbb{R}^n$, endowed with the sup-norm $\|f\|_\infty = \sup_{[0,1]} |f(x)|$. We denote by $\mathcal{E} = \mathcal{B}(E)$ the σ-field of Borel measurable subsets of E. Also, we introduce the Hilbert space $H = L^2(0,1;\mathbb{R}^n)$ of square integrable measurable functions.

Let us fix the notation we shall use in the sequel. The norm in $\mathbb{R}^n$ is denoted by $|x|$ and the scalar product as $\langle x, x\rangle_{\mathbb{R}^n}$. The (equivalent) L^1-norm in $\mathbb{R}^n$ is $|x|_1 = \sum |x_i|$. In the infinite dimensional spaces E and H we denote the norm respectively by $\|x\|_H$, $\|x\|_E$. Finally, the scalar product in H is $\langle x, x\rangle_H$. By E^* we we denote the dual of E.

It is known that given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a process $B = \{B(t),\, t \in [0,1]\}$ is a standard n-dimensional Brownian motion if it a centered Gaussian process with covariance function $\mathbb{E}[\langle B(t), B(s)\rangle] = (s \wedge t)I$, I being the identity matrix in $\mathbb{R}^n$. This process induces a Gaussian measure μ on the space of trajectories $(E, \mathcal{E})$. This measure is known as the *Wiener measure*; the process $B(t)(x) = x(t)$ on the probability space $(E, \mathcal{E}, \mu)$ will be denoted the standard n-dimensional Brownian motion.

On the space E we introduce the Malliavin derivative D with domain $\mathbb{D}^{1,p}$ (that is the closure in $L^p(E,\mu)$ of the class of smooth random variables) see for instance [18, 3, 9, 10]. In Section 2 below we explain its construction in more details. The adjoint operator of the Malliavin derivative operator D having domain $\mathbb{D}^{1,p}$ is the divergence operator, denoted as usual by δ, having domain $D_q(\delta)$, where $q = p'$ is the adjoint exponent of p. δ coincides with the Skorohod integral with respect to the Brownian motion B.

In addition to the Sobolev spaces $\mathbb{D}^{1,p}$, we shall consider the spaces $UC_b(E)$ of uniformly continuous and bounded functions[1] from E to $\mathbb{R}$ and $UC_b^1(E)$, of uniformly continuous and bounded functions which are Fréchet differentiable, with an uniformly continuous and bounded derivative.
In the sequel, we simplify the notation to UC_b, UC_b^1, since no confusion may arise.

Let $u \in L^p(E,\mu;H)$ be a stochastic process, indexed by $t \in [0,1]$, taking values in $\mathbb{R}^n$. The simplest example of such processes is, obviously, the Brownian motion B:

$$B(t)(x) = x(t), \qquad x \in E,\ t \in [0,1].$$

In this paper we introduce the functional $g : L^p(E,\mu;H) \to \mathbb{R}$ which associates to any such process the random variable

$$(1.1)\qquad g(u)(x) = \frac{1}{2}\|u(x)\|_H^2 = \frac{1}{2}\int_0^1 |u(x)(t)|^2\,\mathrm{d}t.$$

In case $u = B$, we shall simply write $g(x) = g(B)(x) = \frac{1}{2}\|x\|_H^2$.
The aim of this paper is to construct the surface measure induced by μ on the level sets $\{g = r\}$ and provide an integration by parts formula involving this surface measure. We shall mention here that, since the domain $\{g < r\}$ is a convex open set in E, our construction is related to that of the recent paper [1]. In particular, the integration by parts formula that we obtain in Proposition 4.8 is related to formula (1) in [1]. Notice however that our construction is quite different. For instance, they choose the measure σ on the level sets of g by appealing to the construction of [14] to fix a reference surface measure to use in the integration by parts formula. On the other hand, we construct the measure σ by following the approach initiated by Airault and Malliavin [2].
Let $X \in L^1(E,\mu)$ be a random variable (more stringent assumptions on X will be necessary,

[1] we use indifferently the terms function, functional or random variable to denote a measurable mapping $F : E \to \mathbb{R}$

compare Section 3). Then we define the function

$$(1.2)\qquad F_X(r) = \int_{\{g(u)<r\}} X(x)\,\mu(\mathrm{d}x), \qquad r \in \mathbb{R};$$

if F_X is differentiable at r, its derivative $F'_X(r)$ is candidate to be a surface integral

$$(1.3)\qquad F'_X(r) = \int_{\{g(u)=r\}} X(x)\,\sigma_r(\mathrm{d}x),$$

provided that there exists a measure σ_r, independent of X, such that (1.3) holds. Obviously, one further needs to prove that σ_r is concentrated on $\{g = r\}$. This approach was followed, among others, by [3, 10, 5, 6, 11]. The main result in this paper is given in the the following theorem, whose proof is given in Section 4.

Theorem 1.1. *Let B the standard n-dimensional Brownian motion defined on the Wiener probability space $(E, \mathcal{E}, \mu)$. Assume that the dimension n satisfies*

$$(1.4)\qquad n \geq 3.$$

Let g be the random variable defined above

$$g(x) = g(B)(x) = \frac{1}{2}\|x\|_H^2, \qquad x \in E,$$

and consider the function F_X defined in (1.2). Then, for any $r > 0$ there exists a unique Borel measure σ_r on E such that (1.3) holds for any $X \in UC_b \cup \mathbb{D}^{1,p}$ and the support of σ_r is concentrated on $\{g = r\}$.
Moreover, for fixed $r > 0$, for any $X \in \mathbb{D}^{1,p}$ and $h \in H$, the following integration by parts formula holds

$$\int_{\{g<r\}} \langle DX, h\rangle_H\,\mu(\mathrm{d}x) = -\int_{\{g=r\}} X\langle Dg, h\rangle_H\,\sigma_r(\mathrm{d}x) + \int_{\{g<r\}} XW(h)\,\mu(\mathrm{d}x),$$

with $W(h)$ the Gaussian random variable defined in (2.1).

It is necessary to emphasize that the main effort in the proof is required by proving the following proposition, which states that the random variable g satisfies, in a suitable sense, the local Malliavin condition, see [6]. Such condition was introduced by Nualart [18, Definition 2.1.2] (in a slightly different formulation) in a related context, i.e., the analysis of the density for the law of a random variable. In our construction the law of the random variable g plays a crucial role since it provides an explicit characterization of the surface measure σ_r (see (1.7)).

Proposition 1.2. *[Malliavin condition on g] There exists a process $u \in L^p(E, \mu : H)$ and a real valued random variable $\gamma \in \mathbb{D}^{1,p}$, for any $p > 1$, such that the following identity holds*

$$(1.5)\qquad \langle Dg, u\rangle_H = \gamma$$

and $\frac{u}{\gamma}$ belongs to $D_q(\delta)$ for every

$$(1.6)\qquad 1 < q < \tfrac{n}{2}.$$

The proof of this Proposition is given in Section 3. The proof of the main theorem, which is given in Section 4, follows quite naturally by the ideas provided in [6, 11].

Our construction, in particular, leads to the following identity concerning the surface measure σ_r:

$$\mathbb{E}[X \mid g = r]\, f_1(r) = \int_{\{g=r\}} X(\xi)\, \sigma_r(\mathrm{d}\xi), \tag{1.7}$$

where f_1 is the probability density function of the random variable $g = g(B)$. Corollary 4.3 below assures that f_1 is a bounded and continuous function and the identity above holds for every $r > 0$.

In the last part of the paper we extend previous results to the analysis of the random variables $g(u)$, where we assume that u is the solution of a stochastic differential equation of gradient form

$$u(t) = -\int_0^t \nabla V(u(s))\, \mathrm{d}s + B(t). \tag{1.8}$$

This is the first step in considering processes whose image law is non Gaussian. In particular, in Section 5 we prove the following result.

Theorem 1.3. *Let u be the solution of equation* (1.8)*, where $V \in C_b^3(\mathbb{R}^n;\mathbb{R}^n)$, $n \geq 3$, and $g(u)$ be the random variable defined in* (1.1)*.*
Then $g(u)$ defined on $(E, \mathcal{E}, \mu)$, has a continuous and bounded density φ_1 with respect to the Lebesgue measure on $\mathbb{R}_+$. Moreover, there exists a surface measure θ_r concentrated on $\{g(u) = r\}$ that is the restriction of μ to the level set $\{g(u) = r\}$.

We notice that the probability density function $\varphi_1(r)$ of the random variable $g(u)$ with respect to the Lebesgue measure can be computed in terms of f_1 as follows:

$$\varphi_1(r) = \mathbb{E}[\rho_1(B)^{-1} \mid g(B) = r] f_1(r),$$

where $\rho_1(B)^{-1}$ is a bounded function which is defined in terms of the coefficient V in (1.8). Moreover, it follows that $\varphi_1(r) = \theta_r(\{g(u) = r\})$. The proof is based on a Girsanov transformation of the reference Gaussian measure and it exploits the results obtained in the case $u = B$.

2. An introduction to Malliavin calculus

In literature different ways of introducing the Malliavin derivative are present. We work here in the general framework given in [18]. This approach requires to fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an isonormal Gaussian process which provides the Gaussian framework. Here, as reference probability space, we consider the Wiener space $(E, \mathcal{E}, \mu)$. The isonormal Gaussian process is given by the family of Wiener integrals.

We denote, as before, by $B(t)(x) = x(t)$, $t \in [0, 1]$, $x \in E$, the standard n-dimensional Brownian motion on the probability space $(E, \mathcal{E}, \mu)$. Given this process, we may introduce the Wiener integral

$$W(h) = \int_0^1 \langle h(s), \mathrm{d}B(s)\rangle_{\mathbb{R}^n}, \qquad h \in H = L^2(0,1;\mathbb{R}^n). \tag{2.1}$$

For any $h \in H$, $W(h)$ is a centered Gaussian random variable with variance $\|h\|_H^2$. We shall denote by $\mathcal{H}_1$ the following subspace of $L^2(E,\mu)$, called the *first Wiener chaos*, defined by

$$\mathcal{H}_1 = \{F \in L^2(E,\mu) \,:\, \exists\, h \in H,\ F = W(h)\}.$$

The map W defines a linear isometry between H and $\mathcal{H}_1$. In particular, we have $\{B(t),\, t \in [0,1]\} = \{W(\mathbf{1}_{[0,t]}),\, t \in [0,1]\}$.

Remark 2.1. In the sequel, we shall use the probabilistic notation of expectation for the integral over E

$$\mathbb{E}[F] = \int_E F(x)\,\mu(\mathrm{d}x),$$

for a measurable function (random variable) $F : (E,\mathcal{E}) \to \mathbb{R}$. In particular,

$$\mathbb{E}[W(h)] = \int_E W(h)(x)\,\mu(\mathrm{d}x) = 0, \qquad \mathbb{E}[|W(h)|^2] = \int_E |W(h)(x)|^2\,\mu(\mathrm{d}x) = \|h\|_H^2.$$

Starting from the space $\mathcal{H}_1$ we construct the class of smooth random variables

$$\mathcal{S} = \{F : (E,\mathcal{E}) \to \mathbb{R} \,:\, \exists\, f \in C_P^\infty(\mathbb{R}^d),\ h_1,\dots,h_d \in H,\ F = f(W(h_1),\dots,W(h_d)\},$$

where $C_P^\infty(\mathbb{R}^d)$ is the space of smooth functions on $\mathbb{R}^d$ with polynomial growth at infinity. We see that $\mathcal{S} \subset L^p(E,\mu)$ for any $p \geq 1$. On the class $\mathcal{S}$ of smooth random variables we consider a functional (actually, a family of functionals indexed by the order of integration p)

$$D : \mathcal{S} \subset L^p(E,\mu) \to L^p(E,\mu;H)$$

by setting

$$DF = \sum_{k=1}^d \frac{\partial f}{\partial x_k}(W(h_1),\dots,W(h_d))\,h_k.$$

Lemma 2.2. *Let $F \in \mathcal{S}$ and $h \in H$. Then it holds*

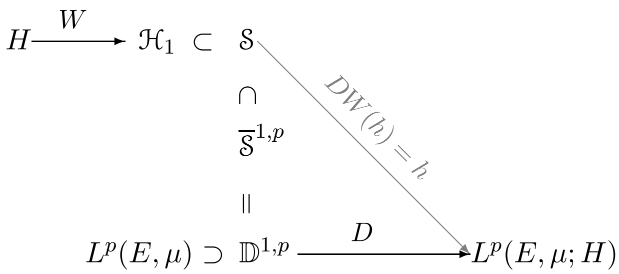
$$\mathbb{E}[\langle DF, h\rangle_H] = \mathbb{E}[FW(h)].$$

For the proof we refer to [**?**]. As a consequence, it is possible to prove that the operator D is closable from $L^p(E,\mu)$ to $L^p(E,\mu;H)$.

Definition 2.3. *We define the norm*

$$\|F\|_{1,p}^p = \mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p], \qquad F \in \mathcal{S}.$$

Then the domain *of the Malliavin derivative D, denoted by $\mathbb{D}^{1,p}$, is the closure of the class $\mathcal{S}$ in $L^p(E,\mu)$ with respect to the norm $\|\cdot\|_{1,p}$. We shall denote again by D this closure.*



Let us now introduce the divergence operator.
Fix $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. By $D_q(\delta)$ we denote the domain of the diverge operator δ. It consists of all $v \in L^q(E, \mu, ; H)$ for which there exits a $G_v \in L^q(E, \mu)$ such that

$$\mathbb{E}\left[\langle v, DF\rangle_H\right] = \mathbb{E}\left[G_v F\right], \qquad F \in \mathbb{D}^{1,p}.$$

The function G_v, if it exists, is uniquely determined. We set

$$\delta(v) := G_v, \qquad v \in D_q(\delta).$$

The divergence operator is easily seen to be closed and densely defined.

It is known that in the case $H = L^2(0, 1; \mathbb{R}^n)$, for a given $F \in \mathbb{D}^{1,p}$, its Malliavin derivative $DF \in L^p(E, \mu; H)$ can be interpreted as a stochastic process indexed by $t \in [0, 1]$. In this case we can interpret the divergence operator as a stochastic integral, the Skorohod integral and the following notation becomes significant:

$$\delta(u) = \int_0^1 \langle u(s), \delta B(s)\rangle_{\mathbb{R}^n}.$$

If the process u is adapted and Itô integrable then the Skorohod integral coincides with the Itô integral. In the special case $u = h \in H$, we have $\delta(u) = W(u)$, with $W(u)$ given by (2.1).

We conclude this Section with the following proposition, that we will exploit to prove our main result Theorem 1.1. It is an extension of [18, Proposition 1.3.3] and a proof can be found in [17, Proposition 6.9].

Proposition 2.4. *Let $1 < r, q < \infty$ be such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Assume that $F \in \mathbb{D}^{1,p}$ and $u \in D_r(\delta)$. Then $Fu \in D_q(\delta)$ and*

$$\delta(Fu) = F\delta(u) - \langle u, DF\rangle_H.$$

3. Verification of the Malliavin condition

In this section we prove that the random variable

$$g(x) = \frac{1}{2}\|x\|_H^2 = \frac{1}{2}\int_0^1 |x(t)|^2 \, \mathrm{d}t, \qquad x \in E, \tag{3.1}$$

satisfies the Malliavin condition stated in Proposition 1.2. From a probabilistic point of view, this random variable is strictly related to the Bessel process of order $\nu = \frac{n}{2} - 1$: $X(t) = |B(t)|^2$. Some results on g are given, for instance, in [7].

In the following proposition we state the construction that we aim to prove in order to achieve the verification of the Malliavin condition.

Proposition 3.1. *Assume that B is the n-dimensional standard Brownian motion with $n \geq 3$. The random variable g, defined in* (3.1)*, satisfies the Malliavin condition in Proposition 1.2 with*

$$u(s) = \frac{D_s g}{|D_s g|}, \qquad \gamma = \int_0^1 |D_s g| \, \mathrm{d}s. \tag{3.2}$$

Remark 3.2. As we shall see in the proof of Lemma 3.5, the condition on n stated in Theorem 1.1 comes from the estimate of $\mathbb{E}[\gamma^{-1/p}]$. The condition $n \geq 3$ is therefore *sufficient* for our construction to hold. It remains open the problem of whether it is also *necessary.* Notice that the existence of the probability density function f_1 for the random variable g holds (via different techniques) for every $n \geq 1$.

The proof of Proposition 3.1 is based on a chain of four lemmas that we will prove in the next Subsection.

3.1. **Preliminary lemmas.**

Lemma 3.3. *Let $\{B(t),\, t \in [0,1]\}$ be the standard n-dimensional Brownian motion. The function $g = g(B)$ as defined in* (3.1) *belongs to the space $\mathbb{D}^{1,p}$ for all $p \geq 1$ and its derivative (in the direction i) is given by*

$$D_s^i g = \int_s^1 B^i(t) \, \mathrm{d}t. \tag{3.3}$$

Proof. For every $s \in [0,1]$ we can compute the Malliavin derivative, in the direction e_i, of the function g as follows

$$D_s^i g := \langle D_s g, e_i \rangle_{\mathbb{R}^n} = \int_0^1 B^i(t) D_s(B_s^i(t)) \, \mathrm{d}t = \int_s^1 B^i(t) \, \mathrm{d}t. \tag{3.4}$$

□

Lemma 3.4. *Let $\{B(t),\, t \in [0,1]\}$ be the standard n-dimensional Brownian motion and let γ be defined as in Proposition 3.1. Then for any $0 < \eta < 1$ there exists a constant c such that*

$$\mathbb{P}\left(\gamma < \eta\right) \leq c\, \eta^n. \tag{3.5}$$

Proof. We compute, for $\eta \ll 1$,

$$\mathbb{P}(\gamma < \eta) = \mathbb{P}\left(\int_0^1 |D_s g| \, \mathrm{d}s < \eta \right).$$

Notice that every two norms in $\mathbb{R}^n$ are equivalent, so there exists c such that

$$\mathbb{P}(\gamma < \eta) = \mathbb{P}\left(\int_0^1 |D_s g|_1 \, \mathrm{d}s < c\,\eta\right).$$

Further, by Hölder's inequality,

$$\mathbb{P}(\gamma < \eta) \le \mathbb{P}\left(\left|\int_0^1 D_s g \, \mathrm{d}s\right|_1 < c\,\eta\right)$$

and, using again the equivalence of norms, we find a different c such that

$$\mathbb{P}(\gamma < \eta) \le \mathbb{P}\left(\left|\int_0^1 D_s g \, \mathrm{d}s\right| < c\,\eta\right)$$

Notice that

$$\begin{aligned}\int_0^1 D_s^i g \, \mathrm{d}s &= \int_0^1 \int_0^1 \mathbb{1}_{(0,t)}(s) B^i(t) \, \mathrm{d}t \, \mathrm{d}s = \int_0^1 \int_0^1 \mathbb{1}_{(0,t)}(s) \, \mathrm{d}s B^i(t) \, \mathrm{d}t \\ &= \int_0^1 t B^i(t) \, \mathrm{d}t = \frac{1}{2} \int_0^1 (1-t^2) \, \mathrm{d}B^i(t) =: Z_i\end{aligned}$$

is a family of independent, identically distributed Gaussian random variables with zero mean and variance $\sigma^2 = \int_0^1 \left(\frac{(1-t^2)}{2}\right)^2 \mathrm{d}t$. Therefore, for some constant c depending on σ,

$$\mathbb{P}(\gamma < \eta) \le \mathbb{P}\left(\sum_{i=1}^n \left(\tfrac{1}{\sigma} Z_i\right)^2 < c\eta^2\right).$$

Since Z_i are independent random variables, the sum in the right-hand side is a χ^2-distribution with n degrees of freedom, hence the probability above is estimated by

$$\mathbb{P}(\gamma < \eta) \le c\left(\eta^2\right)^{\frac{n}{2}} = c\eta^n.$$

□

Lemma 3.5. *Let $\{B(t),\, t \in [0,1]\}$ be the standard n-dimensional Brownian motion and let γ be defined as in Proposition 3.1. Then $\mathbb{E}\left|\ {}^1\!/_\gamma\ \right|^p < \infty$ for every $1 < p < n$.*

Proof. Fixed $p > 1$, we need to show that $\mathbb{E}\left|\ {}^1\!/_\gamma\ \right|^p < \infty$. We can write

$$\begin{aligned}\mathbb{E}\left|\ {}^1\!/_\gamma\ \right|^p &= \int_0^\infty \mathbb{P}\left(\frac{1}{\gamma^p} > \theta\right) \mathrm{d}\theta = \int_0^\infty \mathbb{P}\left(\gamma < \frac{1}{\theta^{\frac{1}{p}}}\right) \mathrm{d}\theta = p\int_0^\infty \mathbb{P}\left(\gamma < \frac{1}{\tau}\right) \tau^{p-1} \, \mathrm{d}\tau \\ &= p\int_0^1 \mathbb{P}\left(\gamma < \frac{1}{\tau}\right) \tau^{p-1} \, \mathrm{d}\tau + p\int_1^\infty \mathbb{P}\left(\gamma < \frac{1}{\tau}\right) \tau^{p-1} \, \mathrm{d}\tau.\end{aligned}$$

The first integral in the last line of the above expression is finite since $\mathbb{P}$ is a probability measure and thus $\mathbb{P}\left(\gamma < \frac{1}{\tau}\right) \le 1$. Thus it is sufficient to show the convergence of the second integral.

Thanks to Lemma 3.4 we immediately get

$$\int_1^\infty \mathbb{P}\left(\gamma < \frac{1}{\tau}\right) \tau^{p-1}\,\mathrm{d}\tau \le C \int_1^\infty \tau^{-n+p-1}\,\mathrm{d}\tau,$$

which is finite provided $n > p$. □

Lemma 3.6. *Let $\{B(t),\, t \in [0,1]\}$ be the standard n-dimensional Brownian motion and let γ be defined as in Proposition 3.1. Then $\gamma \in \mathbb{D}^{1,p}$ for every $p > 1$ and its Malliavin derivative (in the direction i) is*

$$D^i_\theta \gamma = \int_0^1 \frac{1}{|D_s g|} \int_s^1 B^i(r)\,\mathrm{d}r \int_s^1 \mathbb{1}_{(0,t)}(\theta)\,\mathrm{d}t\,\mathrm{d}s. \tag{3.6}$$

Moreover $1/_\gamma \in \mathbb{D}^{1,p}$ for any $1 < p < \frac{n}{2}$ and its Malliavin derivative is given by

$$D^i_\theta \left(1/_\gamma \right) = -\frac{D^i_\theta \gamma}{\gamma^2}. \tag{3.7}$$

Proof. By the chain rule and Lemma 3.3 we have that $\gamma \in \mathbb{D}^{1,p}$ for every $p > 1$ and

$$D^i_\theta \gamma = D^i_\theta \left(\int_0^1 \left(\sum_{i=1}^n \left| \int_s^1 B^i(t)\,\mathrm{d}t \right|^2 \right)^{\frac{1}{2}} \mathrm{d}s \right) = \int_0^1 \frac{1}{|D_s g|} \int_s^1 B^i(r)\,\mathrm{d}r \int_s^1 \mathbb{1}_{(0,t)}(\theta)\,\mathrm{d}t\,\mathrm{d}s.$$

Therefore, applying Hölder's inequality, from (3.6) we get

$$\begin{aligned} |D_\theta \gamma|^2 &= \sum_{i=1}^n \left| \int_0^1 \frac{1}{|D_s g|} \int_s^1 B^i(r)\,\mathrm{d}r \int_s^1 \mathbb{1}_{(0,t)}(\theta)\,\mathrm{d}t\,\mathrm{d}s \right|^2 \\ &\le \sum_{i=1}^n \int_0^1 \frac{1}{|D_s g|^2} \left| \int_s^1 B^i(r)\,\mathrm{d}r \int_s^1 \mathbb{1}_{(0,t)}(\theta)\,\mathrm{d}t \right|^2 \mathrm{d}s, \end{aligned}$$

taking the sum inside the integral we can simplify and we get

$$|D_\theta \gamma|^2 \le 1. \tag{3.8}$$

Equality (3.7) is straightforward to prove. It remains to prove that $\frac{1}{\gamma} \in \mathbb{D}^{1,p}$. In view of Lemma 3.5 it is sufficient to show that

$$\mathbb{E} \left\| D\left(1/_\gamma \right) \right\|_H^p < \infty. \tag{3.9}$$

By means of estimate (3.8) we get

$$\mathbb{E} \left\| D.\left(\frac{1}{\gamma}\right) \right\|_H^p = \mathbb{E} \left\| \left(\frac{D.\gamma}{\gamma^2}\right) \right\|_H^p = \mathbb{E}\left[\frac{\|D.\gamma\|_H}{|\gamma|^2} \right]^p \le \mathbb{E} \left| 1/_\gamma \right|^{2p}.$$

By Lemma 3.5, the term $\mathbb{E}\left| 1/_\gamma \right|^{2p}$ is finite provided that $n \ge 3p$ and this concludes the proof. □

We see that the condition on the dimension n appears in the statement of previous proposition, since it is necessary that $\frac{n}{2} > 1$, i.e., $n \geq 3$ in order to have a non degenerate interval for p.

We are now ready to prove Proposition 3.1.

3.2. **Proof of Proposition 3.1.** Thanks to Lemma 3.3 we know that $g \in \mathbb{D}^{1,p}$, for every $p > 1$. By definition, compare (3.2), we verify that

$$\langle Dg, u\rangle_H = \int_0^1 \langle D_s g, u(s)\rangle_{\mathbb{R}^n} \, \mathrm{d}s = \gamma. \tag{3.10}$$

It remains to prove that

$$\tfrac{u}{\gamma} \in D_q(\delta) \text{ for all } 1 < q < \tfrac{n}{2}.$$

In order to factor out a scalar random variable from a Skorohod integral we can appeal to Proposition 2.4. From Lemma 3.6 we know that $\frac{1}{\gamma} \in \mathbb{D}^{1,p}$ for every $1 < p < \frac{n}{2}$; we claim that $u \in D_r(\delta)$ for every $r > 1$. Therefore, by previous proposition, we have $\frac{u}{\gamma} \in D_q(\delta)$ for $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and, since r is arbitrary and $p < \frac{n}{2}$, we obtain that

$$\tfrac{u}{\gamma} \in D_q(\delta) \text{ for all } q < \tfrac{n}{2}.$$

It remains to verify the claim. Since u is a process adapted to the future and $|u| \leq 1$, u is backward Itô integrable. Then, we can mimic the construction given in [18, Section 1.3.3] for square integrable, adapted processes to be Skorohod integrable, and prove that u belongs to $D_r(\delta)$ for any r.

4. Proof of Theorem 1.1: existence of the surface measure for sets defined by the Brownian motion

In this section, by mimicking the construction provided in [6] we construct the surface measure induced by μ on the level sets $\{g = r\}$. Recall that g is the random variable defined by

$$g(x) = \frac{1}{2}\|x\|_H^2, \qquad x \in E;$$

by construction, $g \geq 0$. Moreover, thanks to Proposition 3.1, g satisfies the Malliavin condition of Proposition 1.2.

As stated in the Introduction, we study the family of functions $F_X(r)$ indexed by (suitably regular) random variables X, where

$$F_X(r) = \mathbb{E}[\mathbb{1}_{\{g \leq r\}} X], \qquad r \geq 0.$$

In this section, we first assume that X is a random variable in $\mathbb{D}^{1,p}$, for some $p > \frac{n}{n-2}$ (as it will be clear later on, this condition stems from the requirement $n \geq 3$ in Proposition 3.1). Let

$$\phi(u) = \int_0^u \mathbb{1}_{\{y \leq r\}} \, \mathrm{d}y.$$

ϕ is a Lipschitz continuous function, hence it is possible to compute the Malliavin derivative

$$D\phi(g) = \phi'(g)Dg; \tag{4.1}$$

scalar multiplying both sides of (4.1) with u and X implies, after a little algebra

$$X \left\langle D\phi(g), \tfrac{u}{\gamma} \right\rangle_H = \phi'(g)\, X = X\, \mathbb{1}_{\{g\le r\}};$$

thus, the duality relationship between Malliavin derivative and Skorohod integral leads to

$$F_X(r) = \mathbb{E}[X\mathbb{1}_{\{g\le r\}}] = \mathbb{E}\left[X \left\langle D\phi(g), \tfrac{u}{\gamma} \right\rangle_H\right] = \mathbb{E}\left[\delta\left(X\,\tfrac{u}{\gamma}\right)\, \phi(g)\right]. \tag{4.2}$$

Notice that in order for the last term in (4.2) to be well defined, we need to have $X\frac{u}{\gamma} \in D_\theta(\delta)$ for some θ. We postpone the verification of this fact to Subsection 4.2 and we start by considering the special case $X \equiv 1$. This case allows us to study the probability density function of g.

4.1. **Existence of the probability density function for g.** By taking $X \equiv 1$, the above reasoning leads to the existence of a density for the cumulative distribution function of the random variable g, as already proved by Nualart [18, Proposition 2.1.1]:

$$\mathbb{P}(g \le r) = F_1(r) = \mathbb{E}\left[\delta\left(\tfrac{u}{\gamma}\right) \int_0^r \mathbb{1}_{\{y<g\}}\,\mathrm{d}y\right]$$

and, by an application of Fubini's theorem, we get the following expression, which easily led to the existence of a density:

$$F_1(r) = \int_0^r \mathbb{E}\left[\delta\left(\tfrac{u}{\gamma}\right) \mathbb{1}_{\{g>y\}}\right]\,\mathrm{d}y. \tag{4.3}$$

Proposition 4.1. *Assume that there exist u and γ such that Proposition 1.2 holds. Then the mapping $s \mapsto F_1(s)$ is continuous.*

Proof. The integrand function $G : y \mapsto \mathbb{E}\left[\delta\left(\frac{u}{\gamma}\right) \mathbb{1}_{\{g>y\}}\right]$, defined in (4.3), is measurable and bounded, by assumption, hence the statement is obvious. ☐

As a consequence of previous proposition, the mapping G is also continuous, since

$$\begin{aligned}|G(y+\varepsilon) - G(y)| \le& \mathbb{E}\left[\left|\delta\left(\tfrac{u}{\gamma}\right)\right| \mathbb{1}_{\{y<g<y+\varepsilon\}}\right]\\ \le& \mathbb{E}\left[\left|\delta\left(\tfrac{u}{\gamma}\right)\right|^q\right]^{1/q} (F_1(y+\varepsilon) - F_1(y))^{1/q'},\end{aligned}$$

where q' is the conjugate exponent of q; therefore, we can apply the integral mean value theorem to get the following.

Proposition 4.2. *There exists the derivative $f_1(r) = F_1'(r)$ and it is equal to*

$$F_1'(r) = \mathbb{E}\left[\delta\left(\tfrac{u}{\gamma}\right) \mathbb{1}_{\{g>r\}}\right]$$

Proof. Since

$$\frac{1}{\varepsilon}\left(F_1(r+\varepsilon)-F_1(r)\right)=\frac{1}{\varepsilon}\int_r^{r+\varepsilon} G(y)\,\mathrm{d}y$$

and the integrand function G is continuous, the thesis follows by letting $\varepsilon \to 0$. □

Corollary 4.3. *The random variable g has a probability density function $f_1(r)$ that is continuous and bounded.*

Actually, the existence of this density is already known in the literature, as well as the explicit form of this function, see [7, Part II.4, formula (1.9.4), page 377].

4.2. **Differentiability of F_X.**

Lemma 4.4. *Let $n \ge 3$. If $X \in \mathbb{D}^{1,p}$, for some $p > \frac{n}{n-2}$ and $\frac{u}{\gamma} \in D_q(\delta)$, with $q < \frac{n}{2}$ (cfr. Proposition 1.2) then $X\frac{u}{\gamma} \in D_\theta(\delta)$ for some $\theta < \frac{np}{n+2p}$.*

Proof. By definition, this requires to show that for any smooth random variable Y

$$\left|\mathbb{E}\left\langle DY, X\tfrac{u}{\gamma}\right\rangle_H\right| < c\,\mathbb{E}[|Y|^{\theta'}]^{1/\theta'},$$

with $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Recall the integration by parts formula

$$\begin{aligned}\mathbb{E}\left\langle DY, X\tfrac{u}{\gamma}\right\rangle_H &= \mathbb{E}\left[\left\langle D(XY), \tfrac{u}{\gamma}\right\rangle_H - \left\langle DX, Y\tfrac{u}{\gamma}\right\rangle_H\right] \\ &= \mathbb{E}\left[XY\,\delta\left(\tfrac{u}{\gamma}\right) - Y\left\langle DX, \tfrac{u}{\gamma}\right\rangle_H\right];\end{aligned} \tag{4.4}$$

it follows

$$\left|\mathbb{E}\left\langle DY, X\tfrac{u}{\gamma}\right\rangle_H\right| \le \mathbb{E}\left[|Y|\,\left|X\,\delta\left(\tfrac{u}{\gamma}\right) - \left\langle DX, \tfrac{u}{\gamma}\right\rangle_H\right|\right]$$

and by Hölder's inequality (in the sequel we exploit the assumption $\theta > p'$, that is equivalent to $\theta' < p$)

$$\begin{aligned}&\le c\,\mathbb{E}[|Y|^{\theta'}]^{1/\theta'}\left(\left[\mathbb{E}\left|X\,\delta\left(\tfrac{u}{\gamma}\right)\right|^{\theta}\right]^{1/\theta} + \left[\mathbb{E}\left|\left\langle DX, \tfrac{u}{\gamma}\right\rangle_H\right|^{\theta}\right]^{1/\theta}\right) \\ &\le c\,\mathbb{E}[|Y|^{\theta'}]^{1/\theta'}\left([\mathbb{E}|X|^p]^{1/p}\left[\mathbb{E}\left|\delta\left(\tfrac{u}{\gamma}\right)\right|^{\frac{p\theta}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}} + [\mathbb{E}\|DX\|_H^p]^{1/p}\left[\mathbb{E}\left\|\tfrac{u}{\gamma}\right\|_H^{\frac{p\theta}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}\right) \\ &\le c\,\mathbb{E}[|Y|^{\theta'}]^{1/\theta'}\left(\|X\|_{\mathbb{D}^{1,p}}\left\|\tfrac{u}{\gamma}\right\|_{D_{\frac{p\theta}{p-\theta}}(\delta)}\right)\end{aligned}$$

Recall the bound in (1.6). Then we shall require

$$1 < \frac{p\theta}{p-\theta} < \frac{n}{2} \qquad \Leftrightarrow \qquad \theta < \frac{np}{n+2p}.$$

Notice that $p > \frac{n}{n-2}$ implies that $\frac{np}{n+2p} > 1$. □

We return to formula (4.2). Previous lemma guarantees the well posedness of the last term in (4.2). Proceeding now in the same way we did in previous subsection, an application of Fubini's theorem implies that

$$F_X(r) = \int_0^r \mathbb{E}\left[\delta\left(X\frac{u}{\gamma}\right)\mathbb{1}_{\{g>y\}}\right]\,\mathrm{d}y.$$

We can finally state the main result in this section.

Proposition 4.5. *Let X belongs to $\mathbb{D}^{1.p}$, $p > \frac{n}{n-2}$. Then there exists the derivative $f_X(r) = F'_X(r)$ and it is equal to*

$$F'_X(r) = \mathbb{E}\left[\delta\left(X\frac{u}{\gamma}\right)\mathbb{1}_{\{g>r\}}\right]. \tag{4.5}$$

Moreover, $f_X(r)$ is a continuous and bounded function and there exists a constant $c > 0$ such that

$$|F'_X(r)| \le c\|X\|_{\mathbb{D}^{1,p}}. \tag{4.6}$$

Proof. The proof of the first part is a straightforward extension of the computation of previous section, by taking into account the integrability of $X\frac{u}{\gamma}$ provided in Lemma 4.2.

We consider further the estimate (4.6). By the integration by parts formula for Malliavin derivative,

$$|F'_X(r)| \le \mathbb{E}\left[\left(|X\delta\left(\frac{u}{\gamma}\right)| + |\langle DX, \frac{u}{\gamma}\rangle_H|\right)\mathbb{1}_{\{g>r\}}\right]$$

and the thesis follows by Hölder's inequality and Hypothesis 1.2. □

Actually, the existence of a continuous density for the functional g implies that we can write formula (1.2) as follows (we use a probabilistic notation, since it seems more expressive)

$$F_X(r) = \mathbb{E}[X\mathbb{1}_{\{g<r\}}] = \int_{-\infty}^r \mathbb{E}[X \mid g = s] f_1(s)\,\mathrm{d}s \tag{4.7}$$

Therefore, by comparing with the results in Proposition 4.5, we obtain that the identity

$$f_X(s) = F'_X(s) = \mathbb{E}[X \mid g = s] f_1(s) \tag{4.8}$$

holds for almost every s and, since the left-hand side is continuous, we conclude that there exists a continuous version of the function $s \mapsto \mathbb{E}[X \mid g = s] f_1(s)$.

Notice that expression (4.8) for $F'_X(r)$ is more significant than (4.5). In particular it provides "a candidate" to be the surface measure. As we will formally prove in what follows this candidate is given by $f_1(r)\,\mathrm{d}\mu$, where $f_1(r)$ is the density function of g. This also highlight the dependence of the surface measure by the kind of functional g we consider.

4.3. **The surface measure.** The results in this section mimic the construction in [11, 6] and we shall skip some minor detail. Notice however that these papers only address the Hilbert setting, while we work in the Banach space E.
Let us notice that on the probability space $(E, \mathcal{E}, \mu)$, identity (4.8) formally reads

$$F'_X(r) = \int_{\{g=r\}} X(x) f_1(r)\, \mu(\mathrm{d}x).$$

We are interested in proving that there exists a surface measure σ_r on the boundary surface $\{g = r\}$ such that previous expression simplifies to

$$F'_X(r) = \int_{\{g=r\}} X(x)\, \sigma_r(\mathrm{d}x).$$

In order to achieve this results, we need to extend previous construction to the class of functionals $X \in UC_b$.

Since functions in UC_b can be uniformly approximated by elements in UC_b^1 (see [13, Section 2.2]), for every $X \in UC_b$ there exists a sequence $X_n \in UC_b^1$ such that $X_n \to X$. Moreover, since $UC_b^1 \subset \mathbb{D}^{1,p}$ for every p, results in previous section applies to the elements of the approximating sequence.

Proposition 4.6. *For every $X \in UC_b^1$, $F_X(r)$ is continuously differentiable and there exists a constant $c > 0$ such that*

$$|f_X(r)| \le c\|X\|_\infty, \qquad \text{where } \|X\|_\infty \text{ is the sup-norm in } E. \tag{4.9}$$

Proof. Only (4.9) needs to be proven. By (4.8) we get

$$|f_X(r)| \le |\mathbb{E}[X \mid g = r]|\, f_1(r) \le \mathbb{E}|X| f_1(r) \le c\, \|X\|_\infty$$

since we know, by Corollary 4.3, that f_1 is a continuous and bounded function. □

By an approximation argument we obtain that the same result holds for $X \in UC_b$.

Proposition 4.7. *For any $X \in UC_b$ the functional $F_X(r)$ is continuously differentiable and there exists a constant $c > 0$ such that*

$$|f_X(r)| \le c\|X\|_\infty. \tag{4.10}$$

We are finally in the position to conclude the proof of the main result of Theorem 1.1. For fixed r, consider a sequence $\varepsilon_n \to 0$ and define the family of measures

$$\sigma_n := \frac{1}{\varepsilon_n} \mathbb{1}_{\{r<g\le r+\varepsilon_n\}} \mu.$$

For any $X \in UC_b$ we have

$$\int_E X(x)\, \sigma_n(\mathrm{d}x) = \int_E \frac{1}{\varepsilon_n} \mathbb{1}_{\{r<g\le r+\varepsilon_n\}} X(x)\, \mu(\mathrm{d}x) = \frac{1}{\varepsilon_n} \left[F_X(r+\varepsilon_n) - F_X(r)\right];$$

thanks to Proposition 4.7 we can pass to the limit in the above formula to get

$$\lim_{n\to\infty} \int_E X(x)\, \sigma_n(\mathrm{d}x) = F'_X(r).$$

By an application of the Prokhorov's theorem (see [4, Corollary 8.6.3]) we finally obtain that the sequence σ_n converges to a measure σ_r such that

$$F'_X(r) = \int_E X(x)\,\sigma_r(\mathrm{d}x).$$

Finally, by taking suitable approximations of $X = \mathbb{1}_{\{|g-r|>\delta\}}$ we check that σ_r is concentrated on $\{g = r\}$ and the proof is complete.

4.4. **The integration by parts formula.** In this section we discuss the integration by parts formula on the level sets of the mapping g. Similar results have been obtained by [8, 1] with different techniques, see also [6, Section 4].

Proposition 4.8. *Let $r > 0$ be fixed. For any $X \in \mathbb{D}^{1,p}$ and $h \in H$ it holds*

$$\int_{\{g=r\}} X\langle Dg, h\rangle_H\,\sigma_r(\mathrm{d}x) = \int_{\{g<r\}} [XW(h) - \langle DX, h\rangle_H]\;\mu(\mathrm{d}x) \tag{4.11}$$

where $W(h)$ is the Gaussian random variable defined in (2.1)*.*

Proof. The starting point is the integration by parts formula (compare (4.4))

$$\mathbb{E}\left[X\,\langle DY, h\rangle_H\right] = \mathbb{E}\left[XY\,\delta\,(h) - Y\,\langle DX, h\rangle_H\right]; \tag{4.12}$$

which holds for random variables X and Y in the domain $\mathbb{D}^{1,p}$ of the Malliavin derivative and $h \in H$.

In a sense, we aim to apply this formula to the random variable $Y = \mathbb{1}_{\{g<r\}}$, but this cannot be obtained directly due to the lack of regularity of this mapping. We thus approximate Y by the following procedure.

Let

$$\theta_\varepsilon(a) = \frac{1}{\varepsilon}\int_a^{+\infty} \mathbb{1}_{(r-\varepsilon,r)}(s)\,\mathrm{d}s, \qquad a > 0;$$

θ_ε is a Lipschitz continuous function, hence the mapping $Y_\varepsilon = \theta_\varepsilon(g)$ is a smooth approximation of Y, in the sense that $Y_\varepsilon \to Y = \mathbb{1}_{\{g\le r\}}$ in $L^2(E,\mu)$.

The right hand side of (4.12), with Y_ε instead of Y, converges as $\varepsilon \downarrow 0$ to

$$\int_{\{g<r\}} [XW(h) - \langle DX, h\rangle_H]\;\mu(\mathrm{d}x).$$

On the other hand, we have

$$DY_\varepsilon = \theta'_\varepsilon(g)\,Dg = \frac{1}{\varepsilon}\mathbb{1}_{(r-\varepsilon,r)}(g)\,Dg$$

hence

$$\mathbb{E}\left[X\,\langle DY_\varepsilon, h\rangle_H\right] = \frac{1}{\varepsilon}\int_H X\,\mathbb{1}_{(r-\varepsilon,r)}(g)\langle Dg, h\rangle_H\,\mu(\mathrm{d}x).$$

Proceeding as in Section 4.3 we notice that $\frac{1}{\varepsilon}\mathbb{1}_{(r-\varepsilon,r)}(g)\,\mu$ converges to the measure σ_r concentrated on $\{g = r\}$. We have thus proved the thesis. □

Remark 4.9. In the special case $X = 1$, formula (4.11) reads

$$F_{W(h)}(r) = \int_{\{g<r\}} W(h)\,\mu(\mathrm{d}x) = \int_{\{g=r\}} \langle Dg, h\rangle_H \,\sigma_r(\mathrm{d}x), \qquad h \in H.$$

This is a sort of *divergence theorem* (in infinite dimensions) for the vector h; we remark that similar results are already present in the literature, compare for instance [15].

Remark 4.10. Let us further notice that Dg is explicitly known (see formula (3.4)), hence

$$\langle Dg, h\rangle_H = \sum_{i=1}^n \int_0^1 h_i(s) \int_s^1 B^i(t)\,\mathrm{d}t\,\mathrm{d}s$$

Let us define

$$\tilde h(t) = \int_0^1 (1 - t \vee r) h(r)\,\mathrm{d}r, \qquad t \in (0,1).$$

Then it holds

$$\langle Dg, h\rangle_H = W(\tilde h).$$

5. Existence of the surface measure for sets defined by the solution of gradient systems

In this section we extend previous results to cover the case of a (multidimensional) gradient system SDE (see [16]). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a potential energy function; we assume that

(5.1) $\quad V \in C_b^3(\mathbb{R}^n),$

i.e., it is continuous and bounded together with its first three derivatives.
Then we define u to be the solution of the following equation:

(5.2) $\quad \mathrm{d}u(t) = -\nabla V(u(t))\,\mathrm{d}t + \mathrm{d}B(t), \qquad u_0 = 0 \in \mathbb{R}^n$

Under our assumptions, the solution u belongs to $L^2(E, \mu; E)$ (notice that we can solve the equation in a pathwise sense).

Recall from Corollary 4.3 that $g(B)$ has a density function $f_1(r)$ that is continuous and bounded for $r > 0$. In this section we aim to study the same property for the random variable $g(u)$, where u is the solution of the equation (5.2).

Theorem 5.1. *Let $n \geq 3$. The cumulative distribution function of $g(u)$ admits a probability density function*

$$\varphi_1(r) = \int_{\{g(B)=r\}} \rho_1(B)^{-1}\,\sigma_r(\mathrm{d}x)$$

where for every process $h \in L^2(E, \mu; E)$ we let $\rho_1(h)$ be the Girsanov's density defined by

(5.3) $$\rho_1(h) = \exp\left(\int_0^1 \langle \nabla V(h(s)), \mathrm{d}B(s)\rangle - \frac{1}{2}\int_0^1 |\nabla V(h(s))|^2\,\mathrm{d}s \right),$$

and the support of σ_r is concentrated on $\{g(B) = r\}$.

5.1. **Change of measure.** First, we notice that our assumptions on V implies that

$$\sup_{t\in[0,1]} \mathbb{E}\left(\exp\left(|\nabla V(u(t))|^2\right)\right) < +\infty,$$

therefore by [12, Theorem 10.14 & Proposition 10.17] we get that the process

$$u(t) = B(t) - \int_0^t \nabla V(u(s))\,\mathrm{d}s$$

is a Brownian motion in $(E, \mathcal{E}, \nu)$, where ν is a centered Gaussian measure such that

$$\mathrm{d}\nu = \rho_1(\cdot)\,\mathrm{d}\mu.$$

Let $F : E \to \mathbb{R}$ be a bounded and Borel function; then we have that

$$\mathbb{E}_\nu[F(u)] = \mathbb{E}_\mu[F(B)].$$

Lemma 5.2. *The following representation of* $\rho_1(u)$ *holds:*

$$\rho_1(u) = \exp\left(V(u(1)) + \frac{1}{2}\int_0^1 |\nabla V(u(t))|^2\,\mathrm{d}t - \frac{1}{2}\int_0^1 \mathrm{Tr}(\nabla^2 V(u(t)))\,\mathrm{d}t\right). \tag{5.4}$$

Proof. Let us compute the Itô differential of $V(u)$:

$$\mathrm{d}V(u(t)) = \langle \nabla V(u(t)), [-\nabla V(u(t))\,\mathrm{d}t + \mathrm{d}B(t)]\rangle + \frac{1}{2}\mathrm{Tr}(\nabla^2 V(u(t)))\,\mathrm{d}t$$

Therefore, using the integral form of previous differential and recalling that $u(0) = 0$, we get

$$V(u(1)) = -\int_0^1 |\nabla V(u(t))|^2\,\mathrm{d}t + \int_0^1 \langle \nabla V(u(s)), \mathrm{d}B(s)\rangle + \frac{1}{2}\int_0^1 \mathrm{Tr}(\nabla^2 V(u(t)))\,\mathrm{d}t$$

We substitute this expression in (5.3) to get the thesis. □

Proposition 5.3. *The mapping* $x \mapsto \rho_1(B)^{-1}(x)$ *belongs to* UC_b*.*

Proof. By lemma 5.2 we can write

$$\rho_1(B)^{-1} = \exp\left[-\left(V(B(1)) + \frac{1}{2}\int_0^1 |\nabla V(B(t))|^2\,\mathrm{d}t - \frac{1}{2}\int_0^1 \mathrm{Tr}(\nabla^2 V(B(t)))\,\mathrm{d}t\right)\right].$$

Then the assumption that $V \in C_b^3(\mathbb{R}^n)$ implies that $\rho_1(B)^{-1}$ is bounded. Now, we exploit that B is the canonical Brownian motion on the Wiener space $(E, \mathcal{E}, \mu)$, hence

$$B(t)(x) - B(t)(y) = x(t) - y(t);$$

notice again that the assumption on V implies that the mappings on E defined by

$$x \mapsto V(x(1)), \qquad x \to \int_0^1 \mathrm{Tr}[\nabla^2 V(x(t))]\,\mathrm{d}t, \qquad x \to \int_0^1 |\nabla V(x(t))|^2\,\mathrm{d}t$$

are Lipschitz continuous. Therefore, if $\|x - y\|_\infty < \delta$, then $\left|\rho_1(B)^{-1}(x) - \rho_1(B)^{-1}(y)\right| \leq e^{3L\delta}$ and the proof is complete. □

5.2. **The main result.** We have now all the ingredients to prove Theorem 5.1. The proof of the existence of the density for $g(u)$ can be obtained as a corollary to the results of Section 4.2. To see this, we propose the following computation.
Using Girsanov's transform we have

$$\begin{aligned}\mu(g(u) \leq r) =&\mathbb{E}_\mu[\mathbb{1}_{\{g(u)\leq r\}}] = \mathbb{E}_\nu[\mathbb{1}_{\{g(u)\leq r\}}\, \rho_1(u)^{-1}]\\ =&\mathbb{E}_\mu[\mathbb{1}_{\{g(B)\leq r\}}\, \rho_1(B)^{-1}].\end{aligned}$$

More generally, it holds

$$\Phi_X(r) = \int_{\{g(u)\leq r\}} X(x)\, \mu(\mathrm{d}x) = F_{X\rho(B)^{-1}}(r).$$

Lemma 5.4. *The random variable $g(u)$, defined on the space $(E, \mathcal{E}, \mu)$ with values in $\mathbb{R}$, admits a probability density function with respect to the Lebesgue measure that is continuous and bounded.*

Proof. Using Proposition 5.3 we are able to apply Theorem 1.1 to obtain that the distribution function $\Phi_1(r)$ of $g(u)$ admits a derivative

$$\varphi_1(r) = f_{\rho_1(B)^{-1}}(r) = \int_E \rho_1(B)^{-1}\, \sigma_r(\mathrm{d}x)$$

where, as stated in Theorem 1.1 the support of the measure σ_r is concentrated on $\{g(B) = r\}$. Now, the thesis follows from Proposition 4.5. □

Next, we prove that there exists a surface measure on $\{g(u) = r\}$ for $r > 0$ that is the restriction of the Gaussian measure μ to the given surface. Proceeding as in Section 4 we obtain that

$$\frac{1}{\epsilon_n}\,[\Phi_X(r+\epsilon_n) - \Phi(r)] = \int_E X(x)\frac{1}{\epsilon_n}\mathbb{1}_{\{r<g(u)\leq r+\epsilon_n\}}\, \mu(\mathrm{d}x)$$

and we can pass to the limit in previous formula, since the left hand side converges to $\Phi'_X(r) = F'_{X\rho_1(B)^{-1}}(r)$ by Proposition 4.7.
Therefore, by mimicking the procedure in Section 4 we get that the sequence of measures

$$\theta_n := \frac{1}{\epsilon_n}\mathbb{1}_{\{r<g(u)\leq r+\epsilon_n\}}\, \mu(\mathrm{d}x)$$

converges to a measure θ_r and this measure is concentrated on $\{g(u) = r\}$.
In particular,

$$\varphi_1(r) = \int_{\{g(u)=r\}} \theta_r(\mathrm{d}x) = \theta_r(\{g(u) = r\}).$$

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