Sliding Mode Optimization in Robot Dynamics With LPV Controller Design

Gian Paolo Incremona, Antonella Ferrara, and Vadim I. Utkin

***Abstract*— This letter extends previous results on constrained optimization control problems of uncertain robot systems based on sliding modes generation. An equivalent linear parameter varying (LPV) state-space representation of the nonlinear robot model is considered to design a stabilizing state-feedback control law by solving linear matrix inequalities (LMI) with structural constraints. The finite-time regulation of the state trajectory to a desired reference, while minimizing a pre-specified cost function with state constraints, is then solved by a sliding mode approach relying on the considered parameter-dependent structure of the robot system. Stability conditions of the proposed approach are provided, and a realistic numerical example verifies the effectiveness of the proposed technique.**

***Index Terms*— Sliding modes, convex optimization, LPV systems, robotics.**

I. Introduction

Sliding mode control (SMC) has been proved to be an effective technique to control constrained nonlinear dynamical systems characterized by parameter uncertainties and disturbances [1]. For these reasons, SMC has been successfully applied also to robotic applications [2]–[8], among many others. In fact, these are significantly challenging systems for which the adoption of classical proportional-integral-derivative (PID) controllers is not always sufficient to guarantee desired performance and disturbance rejection.

Moreover, the presence of operational limits aimed at reducing energy consumption or minimizing wear and tear of the robots introduce other theoretical and practical challenges. Indeed, although among other methods, model predictive control seems the natural solution in case of constrained optimal control problems, its computational burden could be a bottleneck in field implementations. Furthermore, if control methods in the operative space are required, the online Jacobian matrix and the inertia matrix inversions might be needed [9, Ch. 6].

A way to introduce less costly design conditions avoiding the online matrix inversions and improve the synthesis of linear controllers to achieve some project requirement is the use of state-feedback LPV controllers [10]. If the robot model is recast into the LPV framework, then its nonlinearities are formulated as varying parameters exploiting state transformation and change of variable methods. An example of this approach is proposed in [11], where linearizing transformations are adopted to transform the LPV controller design into a feasibility problem of parameter-dependent LMI constraints.

If on the one hand the LPV approach [11] stand-alone simplifies the controller design, on the other hand there is not guarantee of compensating disturbances and fulfilling state constraints of the robot system. In this context, the use of SMC algorithms for robot manipulators has been widely investigated in the last decades [12], [13]. More specifically, a recent development of constrained SMC approaches capable of taking into account both state and input constraints is introduced in [14]. Optimization problems characterized by the generation of sliding modes were instead presented in the literature since seventies, see e.g., [15] and [16]. The specific case of sliding mode optimization in presence of constraints has been then further studied and developed for instance in [17], where sliding modes are enabled on the boundary of a feasible domain while minimizing a pre-specified cost function and relying on linear time invariant (LTI) systems.

In this letter we propose an alternative scheme which is based on the LPV approach [11] and exploits the advantages of the optimization based SMC in [17]. Specifically, the merit of this letter is to extend the SMC based optimization to the case of LPV systems. Indeed, in this letter, an LPV version of the robot model is considered to design a stabilizing state-feedback law. Consequently, the SMC component is designed by reformulating the control laws proposed in [17], so that the stabilizing control term is now a state-feedback control component obtained by solving a finite number of parameter-dependent LMIs, and the analysis of the stability property of the controlled system trajectory is different from the one reported in [17]. Therefore, the extension of SMC based optimization to LPV systems provides a new approach where not only state constraints of the robot manipulator are directly taken into account, but also a predefined cost function, related for instance to the system energy consumption, is minimized to reach an optimal equilibrium point.

The letter is organized as follows. In Section II the basic concept of constrained SMC is reviewed, while in Section III the LPV modeling of a robot manipulator is introduced. The proposed control strategy is discussed in Sections IV and V. A realistic example and some conclusions are reported in Sections VI and VII, respectively.

Notation: Given a column vector $x \in \mathbb{R}^n$, let x' denote its transpose. Analogously, given a matrix A, its transpose

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is A', while $\text{He}\{A\} := A + A'$. If A is symmetric, $A > 0$ (resp. $A \geq 0$) means that A is positive definite (resp. semi-positive definite). Given a function f, let $\text{grad}_x f$ be the gradient operator with respect to vector x.

II. Preliminary: Constrained Sliding Mode Law

Before introducing the proposed approach, we review the concept of constrained sliding mode control as in [14]. To simplify the exposition, we consider a 1-relative degree single input linear plant with matched disturbances and fully measured state.

Consider an LTI system captured by the equation

$$\dot{x}(t) = Ax(t) + Bu(t) + Bd(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the (available) state, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is the matched disturbance, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ are constant matrices with $\det(A) \neq 0$, $B \neq 0_{n\times 1}$. In [14], a general method was proposed in order to cope with the presence of state constraints, and in particular inequality constraints

$$h(x) \leq 0, \quad t \geq 0,$$

that have to be satisfied point-wise in time. The main idea of the SMC is to enable a sliding mode on a suitably selected sliding variable σ via a discontinuous control law which is a function of σ itself. If constraints are present, whenever the boundary of the admissible domain is reached, the SMC law becomes a discontinuous function of the constraint h, giving rise to the so-called constrained SMC.

The trick is then to select the control law u with

$$u(t) = \begin{cases} -\gamma\,\text{sign}(\sigma(x(t))), & h(x) < 0 \\ -\gamma\,\text{sign}(h(x(t))), & h(x) > 0, \end{cases} \tag{2}$$

where $\gamma > 0$ is the control gain. For instance, given the choice $\sigma(x(t)) = Sx(t)$, where $S \in \mathbb{R}^{1\times n}$ is a design variable such that $SB > 0$, applying the first row of (2) to (1), a regulation to zero of the sliding variable in a finite-time t_s is proved, and one has $S\dot{x}(t) = SAx(t) + SB\big(u(t) + d(t)\big) = 0, \quad t \geq t_\text{s}$. This condition leads to the definition of the *equivalent control* corresponding to u and equal to $\tilde{u}(t) = -(SB)^{-1}SAx(t) - d(t)$. This means that (2) is capable of rejecting the matched disturbance in (1) and makes the closed-loop dynamics evolve according to $\dot{x}(t) = (I_n - B(SB)^{-1}S)Ax(t)$ for all $t \geq t_\text{s}$ constrained to $Sx(t_\text{s}) = 0$. Conceptually, when instead the condition $h(x) > 0$ holds, if $\text{grad}_x h(x)B > 0$ a sliding mode is enforced on the surface $h(x) = 0$ till the point when $h(x) < 0$ again.

The idea to slide on the constraints can be then extended to solve a more sophisticated control problem. For example, the problem can consist in designing a control signal capable of steering the state trajectory to an equilibrium point while minimizing a predefined cost function without violating a feasible region defined by a number of inequality constraints. This problem, solved in [17] for LTI systems, is here extended to the LPV case, with reference to the particularly challenging application of robot manipulators. Relying on static optimization, the proposal is to select a control input split into three parts as

$$u(t) = u_0(t) + u_1(t) + u_2(t), \tag{3}$$

where u_0 is a stabilizing control law, for instance as

$$u_0(t) = Kx(t), \tag{4}$$

with $K \in \mathbb{R}^{1\times n}$ a suitable gain matrix such that $A + BK$ is Hurwitz and the nominal dynamics $\dot{x}(t) = (A + BK)x(t)$ satisfies the desired requirements (e.g., convergence, bandwidth, etc.). As for u_1, this is a control law capable of solving an unconstrained optimization problem, while u_2 is a discontinuous law chosen according to (2) to take into account constraints and disturbances.

III. LPV Modelling of the Robot Manipulator

Spurred by the motivations previously mentioned, without loss of generality and to streamline the exposition, we now present the LPV modeling of a 2-degrees-of-freedom robot manipulator with rotational joints [11]. Let $q = \left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right] \in \mathbb{R}^2$ be the vector of the joint variables, and $m_{ij} = m_{ij0} + m_{ij\text{c}}\cos(q_2)$ for all i, j, apart from $m_{22\text{c}} = 0$, be the entries of the inertia matrix. Then, the corresponding dynamic model of the robot is given by

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 - m_{11\text{c}}\sin(q_2)\big(\dot{q}_1\dot{q}_2 + \tfrac{1}{2}\dot{q}_2^2\big) \\ + f_1\dot{q}_1 + k_{\text{e}1}q_1 = \tau_1 + \tau_{d1}, \tag{5a}$$

for the first joint, and

$$m_{12}\ddot{q}_1 + m_{22}\ddot{q}_2 + \tfrac{1}{2}m_{11\text{c}}\sin(q_2)\dot{q}_1^2 \\ + f_2\dot{q}_2 + k_{\text{e}2}q_2 = \tau_2 + \tau_{d2}, \tag{5b}$$

for the second one, where f_k are the damping terms, $k_{\text{e}k}$ are the stiffness ones, while τ_k and τ_{dk} are the control torques and the load disturbances. Define the varying parameters $\rho = \left[\begin{smallmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{smallmatrix}\right] \in \mathbb{R}^3$, with $\rho_1 = \cos(q_2)$, $\rho_2 = -\dot{\rho}_1 = \dot{q}_2\sin(q_2)$ and $\rho_3 = \dot{q}_1\sin(q_2)$, which are bounded with bounded derivatives so that $\rho_k \in \big[\underline{\rho}_k, \overline{\rho}_k\big]$ and $\dot{\rho}_k \in \big[\underline{\dot{\rho}}_k, \overline{\dot{\rho}}_k\big]$. Note that, while $-1 \leq \rho_1 \leq 1$ by definition, the bounds on ρ_2 and ρ_3 are strictly related to the joint speed limits provided by the robot manufacturer or by the specific task to execute. Operating a change of variables in (5), one can write in a compact form the matrices

$$S(\rho) = \begin{bmatrix} m_{11\text{c}}\rho_2 & \frac{1}{2}m_{11\text{c}}\rho_2 \\ -\frac{1}{2}m_{11\text{c}}\rho_3 & 0 \end{bmatrix}, A_1(\rho) = \begin{bmatrix} 0_2 & I_2 \\ -K_\text{e} & (S(\rho) - F) \end{bmatrix},$$

and $B_1 = \left[\begin{smallmatrix} 0_2 \\ I_2 \end{smallmatrix}\right]$, with $K_\text{e} = \text{diag}\big(k_{\text{e}1}, k_{\text{e}2}\big)$, and $F = \text{diag}\big(f_1, f_2\big)$. Pose now the state vector equal to $x = \left[\begin{smallmatrix} q \\ \dot{q} \end{smallmatrix}\right] \in \mathbb{R}^4$, so that one has $x_1 = q_1$, $x_2 = q_2$, $x_3 = \dot{q}_1$, and $x_4 = \dot{q}_2$, the input vector $u = \left[\begin{smallmatrix} \tau_1 \\ \tau_2 \end{smallmatrix}\right] \in \mathbb{R}^2$, and the disturbance as $d = \left[\begin{smallmatrix} \tau_{d1} \\ \tau_{d2} \end{smallmatrix}\right] \in \mathbb{R}^2$, then the LPV state-space representation of the robot model is given by

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) + B(\rho(t))d(t), \tag{6}$$

with $A(\rho) = M_1^{-1}(\rho)A_1(\rho)$, $M_1(\rho) = \text{diag}\big(I_2, M(\rho)\big)$, and $B(\rho) = M_1^{-1}(\rho)B1$.

IV. LPV STABILIZING LAW

After having recast the nonlinear model of the robot in an LPV framework, we start to develop the first term of the control law in (3). To do this we exploit the LPV controller presented in [11] to find the gain matrix $K(\rho)$ in the nominal case, that is when $d = 0$. Indeed, the LPV model (6) is characterized by a rational parametric dependence, and such dependence is associated to the coupling between the inverse matrix $M^{-1}(\rho)$ and matrices $A(\rho)$ and $B(\rho)$.

The following choice will allow to compensate the rational terms and it is based on the use of a *slack variable* $V_1(\rho)$, (see also [10]). Moreover, we would like to guarantee a specific rate of convergence of the state trajectory to a desired equilibrium. Therefore, let $\lambda > 0$ be the desired exponential decay rate, and $A_{\text{cl}}(\rho) = A(\rho) + B(\rho)K(\rho)$ be the closed-loop matrix. The Lyapunov condition for the stability of the closed-loop system with decay rate λ, that is $\|x(t)\| \leq \beta \mathrm{e}^{-\lambda t}\|x(0)\|$ with $\beta > 0$, is given by the existence of a matrix $P(\rho) = P'(\rho) > 0$ such that

$$P(\rho)A_{\text{cl}}(\rho) + A'_{\text{cl}}(\rho)P(\rho) + \dot{P}(\rho) + 2\lambda P(\rho) < 0. \quad (7)$$

This condition can be transformed in an LPV control synthesis according to the following result presented in [11].

Lemma 1: Given the closed loop system $A_{\text{cl}}(\rho) = M_1^{-1}(\rho)\big(A_1(\rho) + B_1 K(\rho)\big)$ with $\rho_k \in \big[\underline{\rho}_k, \overline{\rho}_k\big]$ and $\dot{\rho}_k \in \big[\underline{\dot{\rho}}_k, \overline{\dot{\rho}}_k\big]$, this is Lyapunov stable with exponential decay rate $\lambda > 0$ if there exist matrices $\Gamma(\rho) = \Gamma'(\rho)$, $R(\rho)$ and V_1 such that the following LMI holds

$$\begin{bmatrix} -\mathrm{He}\{(V_1^{-1})'M_1(\rho)\} & A_1(\rho)V_1^{-1} + B_1R(\rho) + \Gamma(\rho) & (V_1^{-1})'M_1(\rho) \\ * & \dot{\Gamma}(\rho) + (2\lambda - 1)\Gamma(\rho) & 0 \\ * & * & -\Gamma(\rho) \end{bmatrix} < 0.$$

Moreover, the state feedback LPV controller is given by

$$u_0(t) = K(\rho)x(t), \quad (8a)$$
$$K(\rho) = V_1 R(\rho). \quad (8b)$$

Proof: In line with the proof in [11], condition (7) is equivalent to the LMI by posing $R(\rho) = V_1^{-1}K(\rho)$, $\Gamma(\rho) = -(V_1^{-1})'P(\rho)V_1^{-1}$, and by left and right multiplying by $\mathrm{diag}\big((V_1^{-1})', (V_1^{-1})', (V_1^{-1})'\big)$ and its transpose. ■

Note that, such a procedure, that is solving the previous LMI constraints, is done off-line and entirely based on the nominal model of the robot manipulator.

V. SLIDING MODE OPTIMIZATION

We are now in a position to introduce the sliding mode optimization procedure for the considered robotic case.

A. Optimal control problem

Having in mind the robotic case study, consider a suitable (in whatever appropriate sense) continuously differentiable cost function $J(x)$ subject to the inequality constraint

$$h(x) \leq 0, \quad (9)$$

so that the feasible region is defined as $\Omega = \big\{x \in \mathbb{R}^n \mid h(x) \leq 0\big\}$. We assume that $h(x)$ is a convex inequality constraint. The optimal control problem (OCP) becomes that of finding a control law $u(t)$ such that for any $x(0) \in \Omega$, then $x(t) \in \Omega$ for all $t \geq 0$ and x tends to the equilibrium point given by

$$x^o = \underset{x \in \Omega}{\operatorname{argmin}}\, J(x), \quad (10)$$

with $x^o \in \mathrm{span}\big(A^{-1}(\rho)B(\rho)\big)$.

B. Unconstrained control law

To solve the previous problem, we propose a control law as in (3), whose first term is synthesized as presented in §IV. In fact, Lemma 1 allows to satisfy all the hypotheses in [17, Prop. 3.1], guaranteeing that the previous optimization problem admits solution.

Therefore, consider now the control law (3) with $u_2 = 0$, and the nominal closed-loop system dynamics

$$\dot{x}(t) = A_{\text{cl}}(\rho(t))x(t) + B(\rho(t))u_1(t), \quad (11)$$

where the unconstrained law u_1 is chosen such that

$$\dot{u}_1(t) = -\gamma_1\big(\big(\mathrm{grad}_x J\big)'\big(-A_{\text{cl}}(\rho)^{-1}B(\rho)\big)\big)'. \quad (12)$$

Moreover, the following assumption holds.

$\boldsymbol{\mathcal{A}_1}$: Given $\gamma_1 > 0$, then $\|\dot{u}_1\|_\infty < \|u_1\|_\infty \leq \epsilon$, for some $\epsilon > 0$.

Lemma 2: Given the LPV system (11) with u_1 as in (12), let $\boldsymbol{\mathcal{A}_1}$ hold. Then, a closure of order ϵ of $x^o = \mathrm{argmin}_{x\in\Omega} J(x)$ is reached in finite-time.

Proof: Following the same proof reasoning in [17, Prop. 4.1], let $z(t) = x(t) + A_{\text{cl}}^{-1}(\rho)B(\rho)u_1$ be an auxiliary control variable, whose derivative is equal to

$$\begin{aligned}\dot{z}(t) &= A_{\text{cl}}(\rho(t))z(t) + A_{\text{cl}}^{-1}(\rho(t))B(\rho)\dot{u}_1 \\ &\quad + \big(\dot{A}_{\text{cl}}^{-1}(\rho(t))B(\rho(t)) + A_{\text{cl}}^{-1}(\rho(t))\dot{B}(\rho(t))\big)u_1(t),\end{aligned}$$

from which

$$\begin{aligned}z(t) &= 2\mathrm{e}^{A_{\text{cl}}(\rho(t))t}z(0) \\ &\quad + \int_0^t \mathrm{e}^{A_{\text{cl}}(\rho(\tau))(t-\tau)}A_{\text{cl}}^{-1}(\rho(\tau))B(\rho(\tau))\dot{u}_1(\tau)\mathrm{d}\tau \\ &\quad + \int_0^t \mathrm{e}^{A_{\text{cl}}(\rho(\tau))(t-\tau)}\big(\dot{A}_{\text{cl}}^{-1}(\rho(\tau))B(\rho(\tau)) \\ &\qquad + A_{\text{cl}}^{-1}(\rho(\tau))\dot{B}(\rho(\tau))\big)u_1(t)\mathrm{d}\tau.\end{aligned}$$

Since A_{cl} is Hurwitz, and by virtue of assumption $\boldsymbol{\mathcal{A}_1}$, then $\|z(t)\|_\infty \leq \mathcal{O}(\epsilon)$, that is $x(t) = -A_{\text{cl}}^{-1}(\rho)B(\rho)u_1 + \mathcal{O}(\epsilon)$.

Now, in line with arguments in [17], we can write the derivative of the cost function evaluated in steady-state along the solution of (11), that is

$$\begin{aligned}\dot{J}(x) &= \big(\mathrm{grad}_x J\big)'\big(-A_{\text{cl}}^{-1}(\rho)B(\rho)\big)\dot{u}_1 \\ &\quad + \big(\mathrm{grad}_x J\big)'\big(-\dot{A}_{\text{cl}}^{-1}(\rho)B(\rho) - A_{\text{cl}}^{-1}(\rho)\dot{B}(\rho)\big)u_1 + \mathcal{O}(\epsilon)\end{aligned}$$

Substituting the expression of $\dot{u}_1$ and using $\boldsymbol{\mathcal{A}_1}$, one obtains

$$\dot{J}(x) = -\gamma_1\big\|\big(\mathrm{grad}_x J\big)'\big(-A_{\text{cl}}^{-1}(\rho)B(\rho)\big)\big\|^2 + \mathcal{O}(\epsilon),$$

implying that the cost function decreases until $\|\big(\mathrm{grad}_x J\big)'\big(-A_{\text{cl}}^{-1}(\rho)B(\rho)\| \geq \mathcal{O}(\epsilon)$. As a consequence, the state $x(t)$ tends in finite-time to an ϵ-closure of the point given by $\big(\mathrm{grad}_x J\big)'\big(-A_{\text{cl}}^{-1}(\rho)B(\rho)\big) = 0$, that is x^o. ■

C. Constrained control law

Finally, we need to introduce the last component u_2 of the control law (3), so that the state remains on the constraint surface $h(x)=0$, due to enforcing a sliding mode, should the state reach this surface from the feasible domain $h(x)<0$. Consider now the perturbed system dynamics

$$\dot{x}(t)=A_{\mathrm{cl}}(\rho(t))x(t)+B(\rho(t))(u_1(t)+u_2(t))+B(\rho(t))d(t). \tag{13}$$

The term u_2 is selected such that

$$\dot{u}_2=-\gamma_2\big(-A_{\mathrm{cl}}(\rho)^{-1}B(\rho)\big)'\big(\mathrm{grad}_{\hat{x}}h(\hat{x})\big)v(\hat{x}), \tag{14a}$$

with the SMC discontinuous input

$$v(\hat{x})=\begin{cases}-1 & \tilde{h}(\hat{x})<0\\ \gamma_3 & \tilde{h}(\hat{x})>0\end{cases}, \tag{14b}$$

where, by virtue of assumption $\boldsymbol{\mathcal{A}_1}$, $\tilde{h}(\hat{x})=h(\hat{x})+\eta$, $\eta>0$, while $\hat{x}(t)=-A_{\mathrm{cl}}^{-1}(\rho)B(\rho)(u_1+u_2)$ is a static observer required to directly solve the dynamic optimization problem because of the needed discontinuities of function h in $h(x)=0$, and of the control signal given by the output of an integrator with discontinuous input (see [16] for further details). Specifically, the idea underlying (14) is conceptually the same of constrained sliding modes in §II, with an asymmetric gain due to $\gamma_3>1$, applied whenever the constraints are violated, in order to maintain the trajectory inside the set Ω even when a practical sliding mode (for instance in field implementations where ideal infinite frequency is not feasible) would be enabled on the constraint, making the state overcome the imposed limits.

Consider now the following assumption.

$\boldsymbol{\mathcal{A}_2}$: $\|d\|_\infty\le\delta$, for some $\delta>\|\mathcal{O}(\epsilon)\|_\infty$, and $\|u_2\|_\infty\le\epsilon$.

Lemma 3: Given the LPV system (13) with u_1 as in (12) and u_2 as in (14), let $\boldsymbol{\mathcal{A}_2}$ hold. If there exists $\bar{\gamma}_2\gg\delta$ such that $\gamma_2\ge\bar{\gamma}_2$ for any $x(0)\in\Omega$, then a closure of order ϵ of $x^o=\mathrm{argmin}_{x\in\Omega}J(x)$ is reached in finite-time with $x(t)\in\Omega$ for all $t\ge0$.

Proof: In line with [17, Prop. 5.1], we can select a nonnegative function V, which depends on $\hat{x}$, such that $V(\hat{x})=v'(\hat{x})\tilde{h}(\hat{x})$, where $\tilde{h}(\hat{x})=h(\hat{x})+\eta$, with $\eta>0$ being a safety margin. Computing the time derivative and exploiting $\boldsymbol{\mathcal{A}_2}$, one has

$$\begin{aligned}\dot{V}(\hat{x})&=v'(\hat{x})\big(\mathrm{grad}_{\hat{x}}\tilde{h}(\hat{x})\big)'\dot{\hat{x}}\\&=v'(\hat{x})\big(\mathrm{grad}_{\hat{x}}\tilde{h}(\hat{x})\big)'\Big(\big(-A_{\mathrm{cl}}^{-1}(\rho)B(\rho)\big)(\dot{u}_1+\dot{u}_2)\\&+\big(-\dot{A}_{\mathrm{cl}}^{-1}(\rho)B(\rho)-A_{\mathrm{cl}}^{-1}(\rho)\dot{B}(\rho)\big)(u_1+u_2)\Big)\\&=-v'(\hat{x})\big(\mathrm{grad}_{\hat{x}}\tilde{h}(\hat{x})\big)'\\&\times(-A_{\mathrm{cl}}^{-1}(\rho)B(\rho))\gamma_1\big(\big(\mathrm{grad}_xJ\big)'(-A_{\mathrm{cl}}^{-1}(\rho)B(\rho))\big)'\\&-v'(\hat{x})\big(\mathrm{grad}_{\hat{x}}\tilde{h}(\hat{x})\big)'\\&\times(-A_{\mathrm{cl}}^{-1}(\rho)B(\rho))\gamma_2(-A_{\mathrm{cl}}^{-1}(\rho)B(\rho))'(\mathrm{grad}_{\hat{x}}\tilde{h}(\hat{x}))v'(\hat{x})+\mathcal{O}(\epsilon).\end{aligned}$$

This implies that, to dominate also the disturbance, there exists a value $\bar{\gamma}_2\gg\delta$ such that $V(\hat{x})$ is a decreasing function and $\tilde{h}(\hat{x})$ will be equal to zero in finite-time, that is a sliding mode will be enforced. As a consequence, for any $x(0)\in\Omega$, the unconstrained control law will steer the state trajectory to an ϵ-closure of $x^o\in\Omega$ in finite-time (see Lemma 2), with $x(t)\in\Omega$ for all $t\ge0$. ∎

Theorem 4: Given system (6) controlled by (3), with u_0 in (8), u_1 as in (12) and u_2 as in (14), if Assumptions $\boldsymbol{\mathcal{A}_1}$ and $\boldsymbol{\mathcal{A}_2}$ hold, then the solutions to (6) are uniformly ultimately bounded with ultimate bound ϵ.

Proof: According to Lemma 1, the unperturbed version of (6) controlled via u_0 with closed-loop matrix A_{cl} is Lyapunov stable. Under $\boldsymbol{\mathcal{A}_1}$ and $\boldsymbol{\mathcal{A}_2}$, Lemmas 2 and 3 hold, which implies the uniform ultimate boundedness of the solutions to (6) when (3) is applied. ∎

VI. Case Study

In this section, the proposed control law is assessed in simulation on a realistic robotic example given by the 2-degrees-of-freedom SECAFLEX manipulator (see [18] for further details).

A. Settings

We have considered the rigid part of the SECAFLEX robot and the data are those reported in Table I. The set-point for the joint position and velocity has been selected as $x^\star=\begin{bmatrix}1.5708 & 1.7453 & 0 & 0\end{bmatrix}'$ with initial conditions of positions and velocities $x(0)=\begin{bmatrix}0.7854 & 0.7854 & 0 & 0\end{bmatrix}'$. A matched disturbance has been then added to the joint torques as $d(t)=\begin{bmatrix}\sin(10x_3(t))\\1-\cos(10x_4(t))\end{bmatrix}$, so that $\delta=2$. The time-varying parameters are finally selected as explained in §III, such that $|\rho_1|\le1$ (by construction), $|\rho_2|\le1.75\,\mathrm{rad\,s^{-1}}$ as well as $|\rho_3|\le1.75\,\mathrm{rad\,s^{-1}}$ (due to the axes speed limits). As discussed in [11], only the time derivative $\dot{\rho}_1=-\rho_2$ is used in the synthesis due to the choice of parameter dependent matrix variables, without requiring acceleration limits.

TABLE I

SECAFLEX ROBOT DATA.

m_{110}	m_{120}	m_{220}	m_{11c}	m_{12c}	k_{e1}	k_{e2}	f_1	f_2
17.5711	5.9114	3.7233	10.0462	2.8803	89.1473	456434	0.09	0.05

Before applying the sliding mode optimization, we have found the stabilizing LPV controller as in (8a) with the gain matrix $K(\rho)$ in (8b) by using the LMI approach in Lemma 1. More precisely, we have chosen the affine matrix $R(\rho)=R_0+\rho_1R_1+\rho_2R_2+\rho_3R_3$, the constant symmetric matrix $\Gamma(\rho)=\Gamma_0$, and decay rate $\lambda=0.5$. The toolbox YALMIP in MATLAB with solver SeDuMi has been adopted to solve the LMIs. Fig. 1 shows the time evolution of the joint variables and velocities in the case of the stabilized nominal (i.e., $d=0$) closed-loop system when $u_1=u_2=0_{2\times1}$, and the correspondig evolution of the parameters ρ_1, ρ_2 and ρ_3.

B. The unconstrained case

We consider now $u_2=0$, with the unconstrained law u_1 active, and the disturbance such that $d\neq0$. The cost function has been chosen as $J(x)=(x_1-x_1^\star)^2+(x_2-x_2^\star)^2+\frac{1}{2}(x_3^2+$

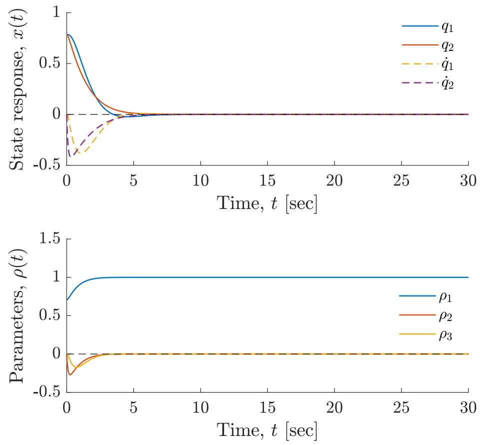

Fig. 1. Time evolution of the closed-loop dynamics when $d = 0$, only the LPV controller $u(t) = u_0(t)$ is applied, and $u_1(t) = u_2(t) = 0_{2\times 1}$ for all $t \geq 0$.

x_4^2), while the constraint is instead $h(x) = x_1 + x_2 - \pi \leq 0$. Therefore, the unconstrained law u_1 has been designed by selecting $\gamma_1 = 100$.

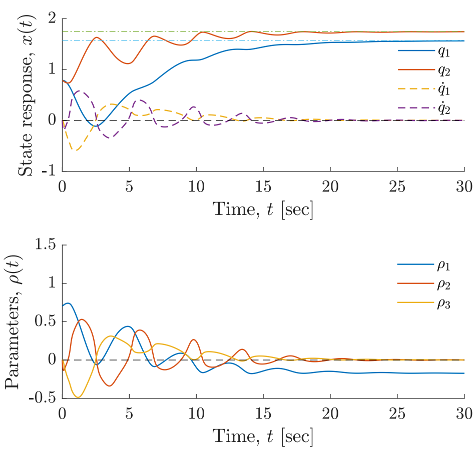

Fig. 2. Time evolution of the closed-loop dynamics when $d \neq 0$, the controller $u(t) = u_0(t) + u_1(t)$ is applied, and $u_2(t) = 0_{2\times 1}$ for all $t \geq 0$.

Fig. 2 illustrates again the time evolution of the joint states and the corresponding evolution of the parameters ρ in this second case. The system state reaches a close vicinity of the desired equilibrium point belonging to $\text{span}\big(A^{-1}(\rho)B(\rho)\big)$, which minimizes the cost function, as proved in Lemma 2, and it is apparent that the plant is significantly affected by the disturbance. The time evolution of the cost function $J(x(t))$ is shown in Fig. 3. Moreover, in Fig. 4 the state portrait in the same case is reported together with the constrained white area (delimited by the dashed line) representing the set Ω.

C. Constrained optimization

Now add the discontinuous law u_2 to the previous controller in the same condition, that is when $d \neq 0$. Specifically, the constrained law has been designed by selecting $\gamma_2 = 200$ and $\gamma_3 = 100$ to take into account the presence of the disturbance d. In the expression of $u(t)$, instead, the gain γ_1 has been maintained still equal to 100.

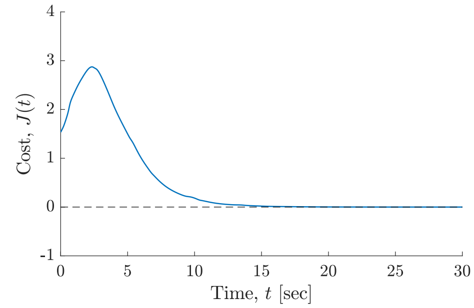

Fig. 3. Time evolution of the cost function $J(x)$ in the unconstrained case.

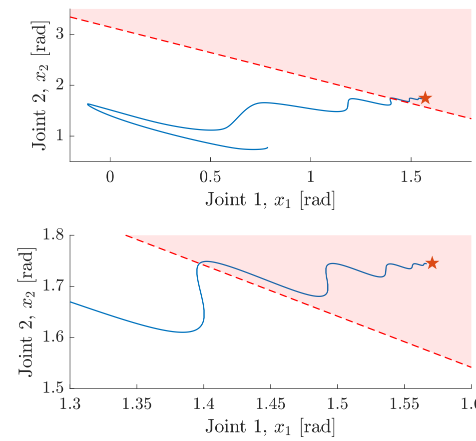

Fig. 4. State portrait (top), and zoom of the convergence area (bottom) in the unconstrained case.

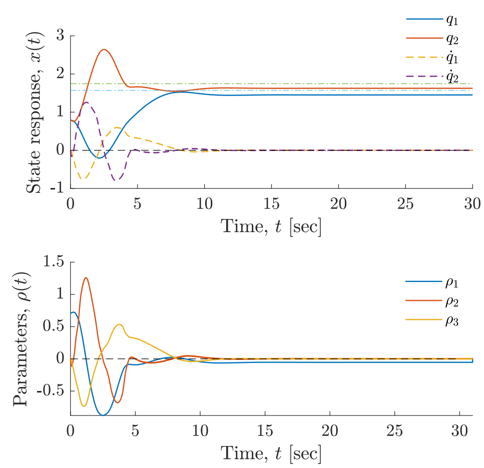

Fig. 5. Time evolution of the closed-loop dynamics when $d \neq 0$ and the controller $u(t) = u_0(t) + u_1(t) + u_2(t)$ is applied for all $t \geq 0$.

Given the constraint $h(x)$ previously introduced, it is evident that the point $J_{\min}$ is in the forbidden region. In this case, the controlled system state and parameters evolve as illustrated in Fig. 5. Fig. 6 shows that function h is steered to zero in finite-time (top), according to Lemma 3, and the sliding mode is enforced by virtue of the discontinuous input (bottom) v in (14b). The convergence occurs so that, while minimizing the cost function $J(x(t))$, the system state reaches an equilibrium point belonging to a vicinity of $\text{span}\big(A^{-1}(\rho)B(\rho)\big)$ and of the feasible region delimited

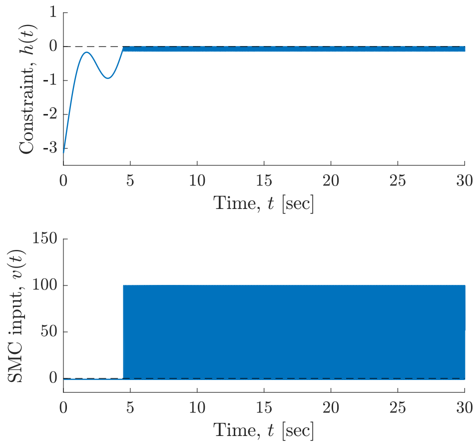

Fig. 6. Sliding mode enforced on $h = 0$ (top), and SMC input v (bottom).

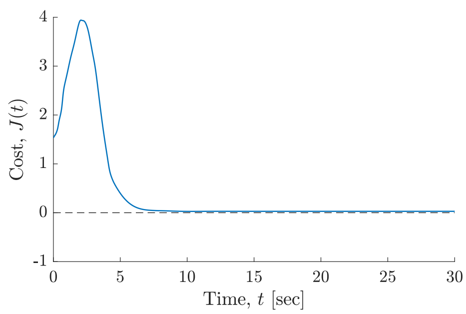

Fig. 7. Time evolution of the cost function $J(x)$ in the constrained case.

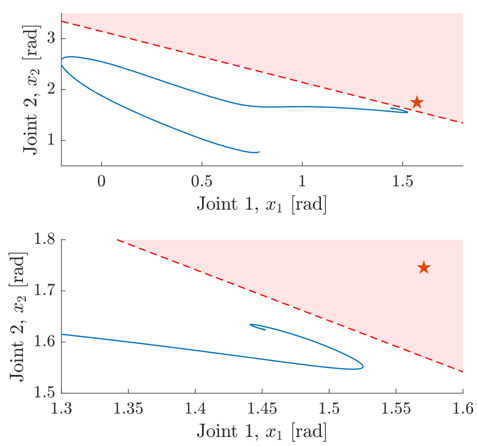

Fig. 8. State portrait (top), and zoom of the convergence area (bottom) in the constrained case.

by the constraint. The time evolution of the cost function $J(x(t))$ is shown in Fig. 7. Finally, in Fig. 8 the state portrait is reported together with the constrained area. Differently from the previous case, this time the constraint is fulfilled by virtue of the sliding mode optimization even in presence of the disturbance d.

VII. Conclusions

This letter has proposed a novel approach of constrained SMC optimization relying on an LPV formulation of a robot manipulator model. Such reformulation has the advantage to enable a relatively simple linear state-feedback control synthesis by solving a finite number of parameter-dependent LMIs. Two other control components are then added to minimize a predefined cost function, while fulfilling state constraints on the joint positions. The whole control law guarantees the uniform ultimate boundedness of the controlled systems solution, with robustness versus matched disturbances. Moreover, due to the low computational complexity of the considered optimization problem, which does not involve any model-based prediction, the present control proposal appears to be extendable to more general settings (e.g., spatial manipulators) and different applications.

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