# PARABOLIC KAZHDAN-LUSZTIG $R$-POLYNOMIALS FOR QUASI-MINUSCULE QUOTIENTS

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**Abstract** We give explicit combinatorial formulas for the parabolic Kazhdan-Lusztig $R$-polynomials of the quasi-minuscule quotients of the classical Weyl groups. As an application of our results we obtain explicit combinatorial formulas for certain sums and alternating sums of ordinary Kazhdan-Lusztig $R$-polynomials.

## 1. Introduction

In their fundamental paper [18] Kazhdan and Lusztig defined, for any Coxeter group $W$, a family of polynomials, indexed by pairs of elements of $W$, which have become known as the Kazhdan-Lusztig polynomials of $W$ (see, e.g., [16, Chap.7] or [2, Chap.5]). These polynomials play an important role in several areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [2, Chap.5], and the references cited there). In order to prove the existence of these polynomials Kazhdan and Lusztig introduced another family of polynomials, usually called the $R$-polynomials, whose knowledge is equivalent to that of the Kazhdan-Lusztig polynomials.

In 1987 Deodhar ([9]) introduced parabolic analogues of all these polynomials. These parabolic Kazhdan-Lusztig and $R$-polynomials reduce to the ordinary ones for the trivial parabolic subgroup of $W$ and are also related to them in other ways (see, e.g., Proposition 1 below). Besides these connections the parabolic polynomials also play a direct role in several areas including the geometry of partial flag manifolds [17], the theory of Macdonald polynomials [14], [15], tilting modules [22], [23], generalized Verma modules [7], [1], canonical bases [12], [26], the representation theory of the Lie algebra $\mathfrak{gl}_n$ [20], quantized Schur algebras [27], quantum groups [11], and physics (see, e.g., [13], and the references cited there).

In this paper we study the parabolic Kazhdan–Lusztig $R$-polynomials for the quasi-minuscule quotients of Weyl groups. These quotients possess noteworthy combinatorial and geometric properties (see, e.g., [19] and [25]). The parabolic Kazhdan–Lusztig $R$-polynomials for the minuscule quotients have been computed in [3], and [4]. In this work we turn our attention to the quasi-minuscule quotients that are not minuscule (also known as (co)-adjoint quotients). More precisely, we obtain explicit combinatorial formulas for the parabolic Kazhdan-Lusztig $R$-polynomials of these quotients for classical Weyl groups, and derive some consequences of these results for the ordinary Kazhdan-Lusztig $R$-polynomials.

*Key words and phrases.* Kazhdan-Lusztig $R$-polynomial, quasi-minuscule quotient, Weyl group, combinatorics, Kazhdan-Lusztig polynomial.

The organization of the paper is as follows. In the next section we recall definitions, notation and results that are used in the rest of this work. In §3 we prove our main results, and derive some consequences of them.

## 2. Preliminaries

In this section we collect some definitions, notation and results that are used in the rest of this work. We let $\mathbb{P} \stackrel{\mathrm{def}}{=} \{1,2,3,\dots\}$ and $\mathbb{N} \stackrel{\mathrm{def}}{=} \mathbb{P} \cup \{0\}$. The cardinality of a set $A$ will be denoted by $|A|$. For $n \in \mathbb{P}$ we let $[n] \stackrel{\mathrm{def}}{=} \{1,2,\dots,n\}$ and $[\pm n] \stackrel{\mathrm{def}}{=} \{-n,\dots,-2,-1,1,2,\dots,n\}$. Given a sequence $(a_1,\dots,a_n) \in \mathbb{Z}^n$ we let $N_1(a_1,\dots,a_n) \stackrel{\mathrm{def}}{=} |\{i \in [n] \,|\, a_i < 0\}|$. Given $a,b \in \mathbb{Z}$ we let $\chi(a<b) \stackrel{\mathrm{def}}{=} 1$ if $a<b$ and $\chi(a<b) \stackrel{\mathrm{def}}{=} 0$ otherwise.

We follow [2] and [16] for general Coxeter groups notation and terminology. Given a Coxeter system $(W,S)$ and $u,v \in W$ we denote by $\ell(u)$ the length of $u$ in $W$, with respect to $S$, and we define $\ell(u,v) \stackrel{\mathrm{def}}{=} \ell(v)-\ell(u)$. If $s_1,\dots,s_r \in S$ are such that $u = s_1 \cdots s_r$ and $r = \ell(u)$ then we call $s_1 \cdots s_r$ a reduced word for $u$. We let $D(u) \stackrel{\mathrm{def}}{=} \{s \in S |\, \ell(us) < \ell(u)\}$ be the set of (right) descents of $u$ and we denote by $e$ the identity of $W$. Given $J \subseteq S$ we let $W_J$ be the parabolic subgroup generated by $J$ and

$$W^J \stackrel{\mathrm{def}}{=} \{u \in W |\, \ell(su) > \ell(u) \text{ for all } s \in J\}.$$

Note that $W^{\emptyset} = W$. We always assume that $W^J$ is partially ordered by *Bruhat order*. Recall (see e.g. [2, §2.2]) that this means that $x \le y$ if and only if for one reduced word of $y$ (equivalently for all) there exists a subword that is a reduced word for $x$. Given $u,v \in W^J$, $u \le v$ we let

$$[u,v]^J \stackrel{\mathrm{def}}{=} \{w \in W^J | u \le w \le v\},$$

and $[u,v] \stackrel{\mathrm{def}}{=} [u,v]^{\emptyset}$.

The following two results are due to Deodhar, and we refer the reader to [9, §§2-3] for their proofs.

**Theorem 1.** *Let $(W,S)$ be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1,q\}$, there is a unique family of polynomials $\{R^{J,x}_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbb{Z}[q]$ such that, for all $u,v \in W^J$:*

- **i):** *$R^{J,x}_{u,v}(q) = 0$ if $u \not\le v$;*
- **ii):** *$R^{J,x}_{u,u}(q) = 1$;*
- **iii):** *if $u < v$ and $s \in D(v)$ then*

$$R^{J,x}_{u,v}(q) = \begin{cases} R^{J,x}_{us,vs}(q), & \text{if } us < u, \\ (q-1)R^{J,x}_{u,vs}(q) + qR^{J,x}_{us,vs}(q), & \text{if } u < us \in W^J, \\ (q-1-x)R^{J,x}_{u,vs}(q), & \text{if } u < us \notin W^J. \end{cases}$$

**Theorem 2.** *Let $(W,S)$ be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1,q\}$, there is a unique family of polynomials $\{P^{J,x}_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbb{Z}[q]$, such that, for all $u,v \in W^J$:*

- **i):** *$P^{J,x}_{u,v}(q) = 0$ if $u \not\le v$;*
- **ii):** *$P^{J,x}_{u,u}(q) = 1$;*
- **iii):** *$deg(P^{J,x}_{u,v}(q)) < \frac{1}{2}\ell(u,v)$ if $u < v$;*

**iv):**

$$q^{\ell(u,v)}\, P^{J,x}_{u,v}\left(\frac{1}{q}\right) = \sum_{z\in[u,v]^J} R^{J,x}_{u,z}(q)\, P^{J,x}_{z,v}(q)$$

*if* $u \leq v$.

The polynomials $R^{J,x}_{u,v}(q)$ and $P^{J,x}_{u,v}(q)$, whose existence is guaranteed by the two previous theorems, are called the *parabolic $R$-polynomials* and *parabolic Kazhdan-Lusztig polynomials* (respectively) of $W^J$ of type $x$. It follows immediately from Theorems 1 and 2 and from well known facts (see, e.g., [16, §7.5] and [16, §§7.9-11]) that $R^{\emptyset,-1}_{u,v}(q)$ $(= R^{\emptyset,q}_{u,v}(q))$ and $P^{\emptyset,-1}_{u,v}(q)$ $(= P^{\emptyset,q}_{u,v}(q))$ are the (ordinary) *$R$-polynomials* and *Kazhdan-Lusztig polynomials* of $W$ which we will denote simply by $R_{u,v}(q)$ and $P_{u,v}(q)$, as customary.

The parabolic $R$-polynomials are related to their ordinary counterparts in several ways, including the following one.

**Proposition 1.** *Let $(W,S)$ be a Coxeter system, $J \subseteq S$, and $u,v \in W^J$. Then we have that*

$$R^{J,x}_{u,v}(q) = \sum_{w\in W_J} (-x)^{\ell(w)} R_{wu,v}(q).$$

A proof of this result can be found in [9] (see Proposition 2.12).

There is one more property of the parabolic $R$-polynomials that we will use and that we recall for the reader's convenience. A proof of it can be found in [10, Corollary 2.2].

**Proposition 2.** *Let $(W,S)$ be a Coxeter system, and $J \subseteq S$. Then*

$$q^{\ell(u,v)} R^{J,q}_{u,v}\left(\frac{1}{q}\right) = (-1)^{\ell(u,v)} R^{J,-1}_{u,v}(q)$$

*for all $u,v \in W^J$.*

The purpose of this work is to study the parabolic $R$-polynomials for the quasi-minuscule quotients of Weyl groups. The parabolic $R$-polynomials for the minuscule quotients have been computed in [3] and [5]. In this work we consider the quasi-minuscule quotients that are not minuscule. These quotients (also known as (co)-adjoint quotients) have been classified (see, e.g., [8]) and there are three infinite families and four exceptional ones. Using the standard notation for the classification of the finite Coxeter systems, the non-trivial (co)-adjoint quotients are: $(A_n, S\setminus\{s_1,s_n\})$, $(B_n, S\setminus\{s_{n-2}\})$, $(D_n, S\setminus\{s_{n-2}\})$, $(E_6, S\setminus\{s_0\})$, $(E_7, S\setminus\{s_1\})$, $(E_8, S\setminus\{s_7\})$, and $(F_4, S\setminus\{s_4\})$, where we number the generators as in [2] (see Appendix A1 and Exercises 20,21,22,23 in Chapter 8, and also below). The following result follows from the above classification and standard facts. Given a Weyl group $W$ we denote by $\Phi(W)$ its root system and by $\Phi_\ell(W)$ its set of long roots (see, e.g., [16, §2.10]) where, if $W$ is of type $B_n$, we mean the root system of type $B_n$.

**Proposition 3.** *Let $(W,S)$ be a Weyl group and $J \subseteq S$ be such that $(W,J)$ is a (co)-adjoint quotient. Then $|W^J| = |\Phi_\ell(W)|$.* $\square$

It is well known (see, e.g., [2, Chap. 1]) that the symmetric group $S_n$ is a Coxeter group with respect to the generating set $S = \{s^A_1,\ldots,s^A_{n-1}\}$ where $s^A_i = (i,i+1)$ for all $i \in [n-1]$. The following result is also well known (see, e.g., [2, §1.5]).

**Proposition 4.** *Let* $v \in S_n$. *Then* $\ell(v) = |\{(i,j) \in [n]^2 : i < j, v(i) > v(j)\}|$ *and* $D(v) = \{(i,i+1) \in S : v(i) > v(i+1)\}$.

We follow [2, Chap. 8] for combinatorial descriptions of the Coxeter systems of type $B_n$ and $D_n$ as permutation groups. In particular, we let $S_n^B$ be the group of all bijections $w$ of $[\pm n]$ to itself such that $w(-i) = -w(i)$ for all $i \in [n]$, $s_j \stackrel{\text{def}}{=} (j, j+1)(-j, -j-1)$ for $j = 1, ..., n-1$, $s_0 \stackrel{\text{def}}{=} (1, -1)$, and $S_B \stackrel{\text{def}}{=} \{s_0, ..., s_{n-1}\}$. If $v \in S_n^B$ then we write $v = [a_1, \dots, a_n]$ to mean that $v(i) = a_i$, for $i = 1, \dots, n$. It is well known that $(S_n^B, S_B)$ is a Coxeter system of type $B_n$ and that the following holds (see, e.g., [2, §8.1]). Given $v \in S_n^B$ we let

$$inv(v) \stackrel{\text{def}}{=} |\{(i,j) \in [n]^2 :\ i < j,\ v(i) > v(j)\}|,$$

$N_1(v) \stackrel{\text{def}}{=} N_1(v(1), \dots, v(n))$ and

$$N_2(v) \stackrel{\text{def}}{=} |\{(i,j) \in [n]^2 :\ i < j,\ v(i) + v(j) < 0\}|.$$

**Proposition 5.** *Let* $v \in S_n^B$. *Then* $\ell(v) = inv(v) - \sum_{\{j \in [n]:\ v(j) < 0\}} v(j)$, *and* $D(v) = \{s_i \in S_B :\ v(i) > v(i+1)\}$, *where* $v(0) \stackrel{\text{def}}{=} 0$.

We let $S_n^D$ be the subgroup of $S_n^B$ defined by

$$S_n^D \stackrel{\text{def}}{=} \{w \in S_n^B :\ N_1(w) \equiv 0 \pmod 2\}, \tag{1}$$

$\tilde{s}_0 \stackrel{\text{def}}{=} (1, -2)(2, -1)$, and $S_D \stackrel{\text{def}}{=} \{\tilde{s}_0, s_1, ..., s_{n-1}\}$. It is then well known that $(S_n^D, S_D)$ is a Coxeter system of type $D_n$, and that the following holds (see, e.g., [2, §8.2]).

**Proposition 6.** *Let* $v \in S_n^D$. *Then* $\ell(v) = inv(v) + N_2(v)$, *and* $D(v) = \{s_i \in S_D : v(i) > v(i+1)\}$ , *where* $v(0) \stackrel{\text{def}}{=} -v(2)$.

Let, for simplicity, $S_n^{[2,n-2]} \stackrel{\text{def}}{=} (S_n)^{S \setminus \{s_1^A, s_{n-1}^A\}}$. Then by Proposition 4 we have that

$$S_n^{[2,n-2]} = \{v \in S_n :\ v^{-1}(2) < \cdots < v^{-1}(n-1)\}.$$

Hence the map $v \mapsto (v^{-1}(1), v^{-1}(n))$ is a bijection between $S_n^{[2,n-2]}$ and $\{(i,j) \in [n]^2 :\ \ i \neq j\}$. For this reason we will freely identify these two sets and write $v = (i,j)$ if $v \in S_n^{[2,n-2]}$ and $i = v^{-1}(1)$, $j = v^{-1}(n)$. The following result is proved in [6, Prop. 10].

**Proposition 7.** *Let* $(a,b), (i,j) \in S_n^{[2,n-2]}$. *Then* $(a,b) \leq (i,j)$ *if and only if* $a \leq i$ *and* $b \geq j$. *Furthermore* $\ell((a,b)) = a - b + n - 1 - N_1(b-a)$.

Let, for simplicity, $B_n^{(n-2)} \stackrel{\text{def}}{=} (S_n^B)^{S_B \setminus \{s_{n-2}\}}$. Then, by Proposition 5, we have that

$$B_n^{(n-2)} = \{v \in B_n :\ 0 < v^{-1}(1) < \cdots < v^{-1}(n-2), v^{-1}(n-1) < v^{-1}(n)\}.$$

Hence the map $v \mapsto (v^{-1}(n-1), v^{-1}(n))$ is a bijection between $B_n^{(n-2)}$ and $\{(i,j) \in [\pm n]^2 :\ i < j,\ i \neq -j\}$. For this reason we freely identify these two sets and write $v = (i,j)$ if $v \in B_n^{(n-2)}$ and $i = v^{-1}(n-1)$, $j = v^{-1}(n)$. The following result is proved in [6, Prop. 11].

**Proposition 8.** *Let $u, v \in B_n^{(n-2)}$, $u = (a, b)$, $v = (i, j)$. Then $u \le v$ if and only if $a \ge i$ and $b \ge j$. Furthermore, $\ell(u) = 2n - 1 - a - b - N_1(a, b, a + b)$.*

Let, for brevity, $(D_n)^{(n-2)} \stackrel{\text{def}}{=} (S_n^D)^{S_D \setminus \{s_{n-2}\}}$. Then, by Proposition 6, we have that

$$(D_n)^{(n-2)} = \{v \in D_n :\ v^{-1}(-2) < v^{-1}(1) < \cdots < v^{-1}(n-2),\ v^{-1}(n-1) < v^{-1}(n)\}.$$

Hence, if $v \in D_n^{(n-2)}$, then $v^{-1}(-1) < v^{-1}(2)$ and $v^{-1}(-2) < v^{-1}(2)$ so $0 < v^{-1}(2) < v^{-1}(3) < \ldots < v^{-1}(n-2)$ and $v^{-1}(-2) < v^{-1}(1)$, $v^{-1}(-1) < v^{-1}(2)$. Since $N_1(v) \equiv 0 \pmod 2$ for all $v \in S_n^D$ we conclude that the map $v \mapsto (v^{-1}(n-1), v^{-1}(n))$ is a bijection between $(D_n)^{(n-2)}$ and $\{(i,j) \in [\pm n]^2 :\ \ i < j,\ \ i \neq -j\}$. The following result is proved in [6, Prop. 12].

**Proposition 9.** *Let $u, v \in D_n^{(n-2)}$, $u = (a, b)$, $v = (i, j)$. Then $u \le v$ if and only if $a \ge i$, $b \ge j$, $(1, -1) \notin \{(a, i), (b, j)\}$, $(a, b, i, j) \notin \{(1, 2, -2, 1), (-1, 2, -2, -1)\}$. Furthermore, $\ell(u) = 2n - 1 - a - b - 2N_1(a, b) - N_1(a + b)$.*

For $w = (i, j) \in S_n^{[2,n-2]}$ let $\tilde{w} \stackrel{\text{def}}{=} (-i, j) \in B_n^{(n-2)}$. The following result is a special case of Proposition 13 in [6].

**Proposition 10.** *Let $u, v \in S_n^{[2,n-2]}$. Then*

$$R_{u,v}^{[2,n-2],x} = R_{\tilde{u},\tilde{v}}^{(n-2),x}$$

*for all $x \in \{-1, q\}$.*

## 3. Main results

In this section we prove our main results, namely we give explicit combinatorial formulas for the parabolic $R$-polynomials of the (co)-adjoint quotients of classical Weyl groups.

Note that, by Proposition 2, it is enough to compute the parabolic $R$-polynomials of type $q$.

Let $u, v \in B_n^{(n-2)}$, $u \le v$, $u = (a, b)$, $v = (i, j)$. We let $D(u, v) \stackrel{\text{def}}{=} \{a, b\} \setminus \{i, j\}$ and

$$d_a(u, v) \stackrel{\text{def}}{=} \begin{cases} 2, & \text{if } j < a, \\ 1, & \text{otherwise.} \end{cases}$$

We also find it convenient to let $d_b(u, v) \stackrel{\text{def}}{=} 1$. We say that $(u, v)$ is *generic* if $a \neq i$, $b \neq j$, and $(a, b) \neq (-j, -i)$. Then, by Proposition 8, $i < a$, $j < b$. We say that $(u, v)$ are in *relative position* 1 (respectively, 0, $-1$) if $a < j$ (respectively, $= j$, $> j$). Note that the multiset $\{d_r(u, v) :\ r \in D(u, v)\}$ depends only on the relative position of $u$ and $v$.

**Theorem 3.** *Let $u, v \in B_n^{(n-2)}$, $u < v$, $(a, b) = u$, $(i, j) = v$. Then*

$$R_{u,v}^{(n-2),q} = \begin{cases} \varepsilon_u \varepsilon_v \prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}) + (q-1) q^{\frac{\ell(u,v)+1}{2}}, & \textit{if } (a, b) = (-j, -i), \\ \varepsilon_u \varepsilon_v \prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}), & \textit{otherwise.} \end{cases}$$

*Proof.* Let, for simplicity, $R_{x,y} \stackrel{\mathrm{def}}{=} R^{(n-2),q}_{x,y}$ for all $x, y \in B^{(n-2)}_n$. We will prove the equivalent statement that

$$R_{u,v} = \begin{cases} (q-1)(q^{a+b-1} - q^2 + 1), & \text{if } a = -j > 0 \text{ and } b = -i, \\ (q-1)(q^{a+b} - q + 1), & \text{if } a = -j < 0 \text{ and } b = -i, \\ \varepsilon_u \varepsilon_v \prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}), & \text{otherwise.} \end{cases}$$

Note that

$$\prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}) = (1-q)^{|D(u,v)|} (1+q)^{\chi(j<a)}.$$

We proceed by induction on $\ell(v) \geq 1$, the result being easy to check if $\ell(v) = 1$. So assume that $\ell(v) \geq 2$.

Assume first that $a = i$. We may clearly assume that $\ell(u, v) > 1$.

If $j = -1$ let $s \stackrel{\mathrm{def}}{=} s_0$. Then $s \in D(v)$ and $u(1) = 1$ so $u < us \notin B^{(n-2)}_n$ and we have from Theorem 1 and our induction hypotheses that $R_{u,v} = -R_{u,vs} = -\varepsilon_u \, \varepsilon_{vs} (1 - q)$, as desired. We may therefore assume that $j \neq -1$. Let $s \stackrel{\mathrm{def}}{=} (j, j+1)(-j, -j-1)$. Then $s \in D(v)$. We have three cases to consider.

**i):** $s \in D(u)$.

Then, since $u \in B^{(n-2)}_n$, $-(n-1) < u(j) < n-1$ and $u(j+1) \leq -(n-1)$ so we have from Theorem 1 and our induction hypotheses that

$$R_{u,v} = R_{us,vs} = \varepsilon_{us} \, \varepsilon_{vs} (1 - q),$$

as desired.

**ii):** $u < us \in B^{(n-2)}_n$.

Then, since $u \in B^{(n-2)}_n$, $u(j) \leq -(n-1)$ and $n-1 > u(j+1) > -(n-1)$, so $v(j+1) > -(n-1)$ and from Theorem 1 and our induction hypotheses we conclude that

$$\begin{aligned} R_{u,v} &= q \, R_{us,vs} + (q-1) R_{u,vs} \\ &= q \, \varepsilon_{us} \, \varepsilon_{vs} (1-q) + (q-1) \, \varepsilon_u \, \varepsilon_{vs} (1-q) \\ &= \varepsilon_u \varepsilon_v [q(1-q) + (1-q)^2], \end{aligned}$$

as desired.

**iii):** $u < us \notin B^{(n-2)}_n$.

Then $-(n-1) < u(j) < u(j+1) < n-1$ so from Theorem 1 and our induction hypothesis we conclude that $R_{u,v} = -R_{u,vs} = -\varepsilon_u \varepsilon_{vs} (1-q)$, and the result again follows.

Assume now that $b = j$. We may again assume that $\ell(u, v) > 1$. It is easy to see that we may also assume that $b = j = n$.

If $i = -1$ let $s \stackrel{\mathrm{def}}{=} s_0$. Then $s \in D(v)$ and $u(1) = 1$ and the result follows esactly as in the case $a = i$, $j = -1$ above. We may therefore assume that $i \neq -1$. Let $s \stackrel{\mathrm{def}}{=} (i, i+1)(-i, -i-1)$. Then $s \in D(v)$. We again have three cases to consider.

**i):** $s \in D(u)$.

Then, since $u \in B^{(n-2)}_n$, $-(n-1) = u(i+1) < u(i) < n-1$ and we conclude exactly as in case **i)** above.

**ii):** $u < us \in B^{(n-2)}_n$.

Then $-(n-1) = u(i) < u(i+1) < n-1$ so $i < 0 < a = -i$ and, since $\ell(u,v) > 1$, $a > 1$, and we conclude exactly as in case **ii)** above.

**iii):** $u < us \notin B_n^{(n-2)}$.

Then $-(n-1) < u(i) < u(i+1) < n-1$ and, since $\ell(u,v) > 1$, we conclude as in case **iii)** above.

If $a = -j$ and $b = -i$ then $v = u(a,-b)(b,-a)$ so $\ell(u,v)$ is odd and $a + b > 0$ hence $b > 1$.

Assume first that $a + 1 < b$. Let $s \stackrel{\text{def}}{=} s_{b-1}$. Then $s \in D(v)$. If $b - 1 = -a$ then $u = vs$ and the result follows. Else we have from Theorem 1 and our definitions that

$$R_{(a,b),(i,j)} = q\, R_{(a,b-1),(i+1,j)} + (q-1)\, R_{(a,b),(i+1,j)}.$$

But, by our induction hypotheses,

$$R_{(a,b-1),(i+1,j)} = -\prod_{r\in D(u,v)} (1 - q^{d_r(u,v)}) + (q-1)\, q^{\frac{\ell(us,vs)+1}{2}}$$

while

$$R_{(a,b),(i+1,j)} = \prod_{r\in D(u,v)} (1 - q^{d_r(u,v)})$$

(note that $\ell((a,b),(i+1,j))$ is even) so the result follows.

If $a+1 = b$ then $s_{a-1} \in D(v)$. If $a = 1$ then $s_0 \in D(v)$ so we have from Theorem 1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= q\, R_{us_0,vs_0} + (q-1)\, R_{u,vs_0} \\ &= q\left(-(1-q)^2 + (q-1)q^{\frac{\ell(us_0,vs_0)+1}{2}}\right) + (q-1)(1-q), \end{aligned}$$

and the result follows. If $a \neq 1$ then let $s \stackrel{\text{def}}{=} (a, a-1)(-a, -a+1)$. Then $s \in D(v)$, $u < us \in B_n^{(n-2)}$, $D(us,vs) = \{a-1, b\}$, $D(u,vs) = \{a,b\} = D(u,v)$ and $d_{a-1}(us,vs) = d_a(u,vs) = d_a(u,v)$, $d_b(us,vs) = d_b(u,vs) = d_b(u,v)$, so we have from Theorem 1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= q\left(-\prod_{r\in D(u,v)} (1 - q^{d_r(u,v)}) + (q-1)^{\frac{\ell(u,v)-1}{2}}\right) + \\ &\quad (q-1)\prod_{r\in D(u,v)} (1 - q^{d_r(u,v)}) \end{aligned}$$

and the result again follows.

We may therefore assume that $(u,v)$ is generic. Then $(us,vs)$ is generic for all $s \in S_B$ such that $us, vs \in B_n^{(n-2)}$ and $us \leq vs$.

Suppose first that $D(v) \cap \{s_1, \ldots, s_{n-1}\} \neq \emptyset$. Let $k \in [n-1]$ be such that $s_k \in D(v)$. Then $v(k) > v(k+1)$. Hence, since $v \in B_n^{(n-2)}$, $v(k+1) < n-1$.

Suppose first that $s_k \in D(u)$. Then, since $u \in B_n^{(n-2)}$, $u(k+1) < n-1$. Hence $(u,v)$ and $(us_k, vs_k)$ are in the same relative position so the result follows from Theorem 1 and our induction hypothesis.

Suppose now that $u < us_k \notin B_n^{(n-2)}$. Then, since $u \in B_n^{(n-2)}$, either $u(k), u(k+1) \leq -(n-1)$, or $-(n-1) < u(k)$, $u(k+1) < n-1$, or $n-1 \leq u(k), u(k+1)$. In the second case $(u, vs_k)$ is generic and $(u, vs_k)$ and $(u,v)$ are in the same relative position so the result follows from Theorem 1 and our induction hypotheses. In the first case, $(u, vs_k)$ is generic and $(u, vs_k)$ and $(u, v)$ are in the same relative position except if $v(-k-1) > n-1$ in which case we have that $R_{u,v} = -R_{u,vs_k} = -\varepsilon_u \varepsilon_{v\, s_k}(1-q)$ and the result follows. In the third case, since $v(k+1) < n-1$, the situation is the same except if $v(k) \geq n-1$ in which case we have that $R_{u,v} = -R_{u,vs_k} = -\varepsilon_u \varepsilon_{vs_k}(1-q)$ and the result again follows.

We may therefore assume that $u < us_k \in B_n^{(n-2)}$. We have two main cases to consider. Let, for brevity, $s \stackrel{\text{def}}{=} s_k$.

**1):** $v(k) \geq n-1$.

We then have three cases to consider.

**a):** $u(k) \leq -(n-1) < u(k+1) < n-1$.

If $v(k+1) > -(n-1)$ then $(u, vs)$ is generic, $us \leq vs$ and $(u, v)$, $(u, vs)$, $(us, vs)$ are all in the same relative position and the result follows from Theorem 1 and our induction hypotheses. If $v(k+1) \leq -(n-1)$ then $us \not\leq vs$ and $R_{u,vs} = \varepsilon_u \varepsilon_{vs}(1-q)$ so $R_{u,v} = (q-1)R_{u,vs} = \varepsilon_v \varepsilon_u (1-q)^2$ and the result again follows.

**b):** $u(k) \leq -(n-1) < n-1 \leq u(k+1)$.

Then, since $(u, v)$ is generic, $v(k+1) > -(n-1)$ so $R_{u,vs} = \varepsilon_u \varepsilon_{vs}(1-q)$, $us \not\leq vs$ so $R_{u,v} = (q-1)\, R_{u,vs} = (q-1)\varepsilon_u \varepsilon_{vs}\, (1-q)$, and the result follows.

**c):** $-(n-1) < u(k) < n-1 \leq u(k+1)$.

Then $R_{u,vs} = \varepsilon_u \varepsilon_{vs}(1-q)$. If $us \leq vs$ then $R_{us,vs} = \varepsilon_{us} \varepsilon_{vs}(1-q)^2$, so

$$R_{u,v} = q\, R_{us,vs} + (q-1)\, R_{u,vs} = \varepsilon_u \varepsilon_v\, (1+q)(1-q)^2,$$

and the result follows. If $us \not\leq vs$ then $R_{u,v} = (q-1)\, R_{u,vs} = \varepsilon_u \varepsilon_v\, (1-q)^2$, and the result again follows.

**2):** $n-1 > v(k) > -(n-1)$.

Then, since $v \in B_n^{(n-2)}$, $v(k+1) \leq -(n-1)$ and we have two cases to consider.

**a):** $u(k) \leq -(n-1) < u(k+1)$.

Then we conclude exactly as in case 1)c) above.

**b):** $-(n-1) < u(k) < n-1 \leq u(k+1)$.

Then, $us \leq vs$, $(u, vs)$ is generic, and the relative positions of $(u, v)$, $(u, vs)$, and $(us, vs)$ are the same so the result follows.

Suppose now that $D(v) \cap \{s_1, \ldots, s_{n-1}\} = \emptyset$. Then, since $(u, v)$ is generic, $v(1) < v(2) \leq -(n-1) < v(3) < \cdots < v(n) < n-1$. There are then three cases to consider. Let, for brevity, $s \stackrel{\text{def}}{=} s_0$.

**a):** $u(1) \geq n-1$.

Then $u < us \in B_n^{(n-2)}$, $R_{u,vs} = \varepsilon_u \varepsilon_{vs}(1-q)$, $us \leq vs$, so

$$R_{u,v} = q\, R_{us,vs} + (q-1)\, R_{u,vs} = q\, \varepsilon_{us} \varepsilon_{vs}(1-q)^2 + \varepsilon_u \varepsilon_v (1-q)^2,$$

as desired.

**b):** $n-1 > u(1) > -(n-1)$.

Then $u < us \notin B_n^{(n-2)}$, $(u, vs)$ is generic and $(u, v)$ and $(u, vs)$ are in the same relative position so the result follows.

**c):** $u(1) \leq -(n-1)$.

Then $u > us$, and $(u,v)$ and $(us,vs)$ are in the same relative position so the result again follows from Theorem 1 and our induction hypotheses.

This concludes the induction step and hence the proof. □

As a consequence of the previous theorem we obtain the following explicit combinatorial formula for certain sums and alternating sums of ordinary $R$-polynomials of the Weyl groups of type $B$.

**Corollary 1.** *Let $u,v \in B_n^{(n-2)}$, $u < v$, $x \in \{-1,q\}$, and $J \stackrel{\text{def}}{=} S_B \setminus \{s_{n-2}\}$. Then*

$$\sum_{w \in W_J} (-x)^{\ell(w)} R_{wu,v}(q)$$

*equals*

$$(q-x-1)^{\ell(u,v)} \Big[ \prod_{r \in D(u,v)} \Big(1 - \left(\frac{x^2}{q}\right)^{d_r(u,v)}\Big) + \Big(1 - \frac{x^2}{q}\Big) \left(\frac{x^2}{q}\right)^{\frac{\ell(u,v)+1}{2}} \Big]$$

*if $(u^{-1}(n-1), u^{-1}(n)) = (-v^{-1}(n), -v^{-1}(n-1))$, and*

$$(q-x-1)^{\ell(u,v)} \prod_{r \in D(u,v)} \Big(1 - \left(\frac{x^2}{q}\right)^{d_r(u,v)}\Big)$$

*otherwise.*

*Proof.* This follows immediately from Theorem 3, and Propositions 1 and 2. □

As a further consequence of Theorem 3 and of Proposition 10 we obtain the following result, which computes the parabolic $R$-polynomials for the (co)-adjoint quotients of type $A$, and also follows from Theorem 4.2 of [21].

**Corollary 2.** *Let $u,v \in S_n^{[2,n-2]}$, $u < v$, $u = (a,b)$, $v = (i,j)$. Then*

$$R_{u,v}^{[2,n-2],q} = \left\{ \begin{array}{ll} \varepsilon_u \varepsilon_v (1-q), & \text{if } a = i \text{ or } b = j, \\ (q-1)(q^{b-a} - q + 1), & \text{if } (a,b) = (j,i), \\ \varepsilon_u \varepsilon_v (1-q)^2, & \text{otherwise.} \end{array} \right]$$

*Proof.* This follows easily from Theorem 3, Proposition 10, and the fact that, by Propositions 7 and 8, $\ell(\tilde{z}) = \ell(z) + n - 1$ for all $z \in S_n^{[2,n-2]}$ where the first length is taken in $B_n^{(n-2)}$ and the second one in $S_n^{[2,n-2]}$. □

We now compute the parabolic $R$-polynomials of the (co)-adjoint quotients of classical Weyl groups of type $D$.

Given $u \in D_n^{(n-2)}$, $u = (a,b)$, we let $u^* \stackrel{\text{def}}{=} (-b,-a)$. For $u,v \in D_n^{(n-2)}$, $u = (a,b)$, $v = (i,j)$, we let

$$\chi(u,v) \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} 1, & \text{if } a = -j \text{ or } b = -i, \\ 0, & \text{otherwise,} \end{array} \right.$$

$$\tilde{\chi}(u,v) \stackrel{\text{def}}{=} 1 - \chi(a<j)\chi(u,v)$$

and

$$D^*(u,v) \stackrel{\text{def}}{=} D(u,v) \setminus D(u,v^*).$$

Note that $D^*(u,v) = \{a,b\} \cap \{-i,-j\}$. In particular, $|D^*(u,v)| = 2$ if and only if $(a,b) = (-j,-i)$, while $|D^*(u,v)| = 1$ if and only if $u \neq v^*$ and either $a = -i$ or $a = -j$ or $b = -i$ or $b = -j$.

**Theorem 4.** *Let $u, v \in D_n^{(n-2)}$, $u < v$, $u = (a, b)$, $v = (i, j)$. Then $\varepsilon_u \varepsilon_v R_{u,v}^{(n-2),q}$ equals*

$$\prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}) + (1-q) q^{\frac{\ell(u,v)+1-2N_1(-a)}{2}} - (1 - q^{N_1(-a)})^2 (q^{b-2} + q^a),$$

*if $D^*(u, v) = \{a, b\}$,*

$$\prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}) - (1 - q^{\tilde{\chi}(u,v)})^{|D(u,v)|} q^{k - d_{[\pm k]}(u,v)},$$

*if $D^*(u, v) = \{k\}$, where $d_{[\pm k]}(u, v) \stackrel{\text{def}}{=} |\{a, i, j\} \cap [\pm(k-1)]| + \chi(a < j)$, and*

$$\prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}),$$

*otherwise.*

*Proof.* Let, for simplicity, $R_{x,y} \stackrel{\text{def}}{=} R_{x,y}^{(n-2),q}$ for all $x, y \in D_n^{(n-2)}$, and $s_0 \stackrel{\text{def}}{=} \tilde{s}_0$.

Note first that

$$\prod_{r \in D(u,v)} (1 - q^{d_r(u,v)}) = (1-q)^{|D(u,v)|} (1+q)^{\chi(j<a)}.$$

We proceed by induction on $\ell(v) \geq 1$, the result being easy to check if $\ell(v) = 1$. So assume that $\ell(v) \geq 2$.

Let $D^*(u, v) = \{a, b\}$; in this case $a = -j$, $b = -i$ and $|D(u, v)| = 2$. We have two main cases to consider.

**1):** $a > 0$.

Then $i < j < 0 < a < b$ and $\ell(u, v) = 2(a + b) - 5$. In this case we have to prove that $R_{u,v} = \varepsilon_u \varepsilon_v (1-q)(q^{a+b-3} + (q-1)(q^{b-2} + q^a) + 1 - q^2)$. We have three cases to consider.

If $a > 1$ then $s_{a-1} \in D(v) \setminus D(u)$ and $us_{a-1} \in D_n^{(n-2)}$. Therefore

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(-b,-(a-1))} + (q-1) R_{(a,b),(-b,-(a-1))} \\ &= \varepsilon_u \varepsilon_v q((1-q)(1-q^2) + (1-q) q^{a+b-4} \\ &\quad -(1-q)^2 (q^{b-2} + q^{a-1})) + \varepsilon_u \varepsilon_v (1-q)^3 (1 + q - q^{b-2}), \end{aligned}$$

and the result follows.

If $a = 1$ and $b > 2$ then $s_{b-1} \in D(v) \setminus D(u)$ and $us_{b-1} \in D_n^{(n-2)}$ and the result follows as in the previous case. If $a = 1$ and $b = 2$ then $\ell(u, v) = 1$ so $R_{u,v} = \varepsilon_u \varepsilon_v (1 - q)$, as claimed.

**2):** $a < 0$;

Then $i < a < 0 < j < b$ and $\ell(u, v) = 2(a + b) - 1$; moreover $s_j \in D(v) \setminus D(u)$ and $us_j \in D_n^{(n-2)}$. In this case we have to show that $R_{u,v} = \varepsilon_u \varepsilon_v (1-q)(q^{a+b} - q + 1)$. We have two cases to consider.

If $a + b > 1$ then

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(-b,-(a-1))} + (q-1) R_{(a,b),(-b,-(a-1))} \\ &= \varepsilon_u \varepsilon_v q(1-q)(q^{a+b-1} - q + 1) + \varepsilon_u \varepsilon_v (1-q)^3, \end{aligned}$$

as desired.

If $a + b = 1$ then $\ell(u, v) = 1$ and the result again follows.

Suppose now that $D^*(u,v)=\{k\}$ and $a>j$. In this case $|D(u,v)|=2$ and we have to show that $R_{u,v}=\varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{k-d_{[\pm k]}(u,v)})$. There are four main cases to consider, depending on whether $a=-i$, $a=-j$, $b=-i$ or $b=-j$.

**1):** $a=-i$.

Then $k=a$, $b\neq -j$ and $i<0<a<b$. In this case $d_{[\pm k]}(u,v)=1$. We then have five cases to consider. Define

$$s \stackrel{\text{def}}{=} \begin{cases} s_j, & \text{if } j>0,\\ s_{-j-1}, & \text{if } j<0.\end{cases}$$

**a):** If $j>0$ and $a-j>1$ then $s\in D(v)\setminus D(u)$ and $us\notin D_n^{(n-2)}$. In this case

$$R_{u,v}=-R_{(a,b),(-a,j+1)}=\varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a-1}).$$

**b):** If $j>0$ and $a=j+1$ then $s\in D(v)\setminus D(u)$ and $us\in D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(-(a-1),a)}+(q-1)R_{(a,b),(-(a-1),a)}\\ &= \varepsilon_u\varepsilon_v q(1-q)^2(1-q^{a-2})+\varepsilon_u\varepsilon_v(1-q)^2\\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a-1}).\end{aligned}$$

**c):** If $j<-1$ then $s\in D(v)\setminus D(u)$ and $us\notin D_n^{(n-2)}$. So

$$R_{u,v}=-R_{(a,b),(-a,j+1)}=\varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a-1}).$$

**d):** If $j=-1$ and $a>2$ then $s\in D(v)\setminus D(u)$ and $us\notin D_n^{(n-2)}$. Then

$$R_{u,v}=-R_{(a,b),(-a,2)}=\varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a-1}).$$

**e):** If $j=-1$ and $a=2$ then $s\in D(v)\setminus D(u)$ and $us\in D_n^{(n-2)}$. So $us\not\leq vs$ and

$$R_{u,v}=(q-1)R_{(2,b),(1,2)}=\varepsilon_u\varepsilon_v(1-q)^2,$$

since $\ell(u,vs)=1$.

**2):** $a=-j$.

Then $k=a$ and $i<j<0<a<b$; in this case $d_{[\pm k]}(u,v)=0$. Let $s\stackrel{\text{def}}{=}s_a$; then $s\in D(v)\setminus D(u)$ and $us\in D_n^{(n-2)}$. If $a>1$ we have that

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(i,-(a-1))}+(q-1)R_{(a,b),(i,-(a-1))}\\ &= \varepsilon_u\varepsilon_v q(1-q)^2(1+q-q^{a-1})+\varepsilon_u\varepsilon_v(1+q)(1-q)^3\\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a}).\end{aligned}$$

If $a=1$ we find the result by similar calculations.

**3):** $b=-i$.

Then $k=b$, $a\neq -i$ and $i<0<b$; in this case $d_{[\pm k]}(u,v)=2$ and there are two main cases to consider. Define

$$s \stackrel{\text{def}}{=} \begin{cases} s_{b-1}, & \text{if } a+j>0,\\ s_{-j-1}, & \text{if } a+j<0.\end{cases}$$

**a):** Let $a+j>0$. Then $s\in D(v)\setminus D(u)$. If $b-a>1$ then $us\in D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= qR_{(a,b-1),(-(b-1),j)}+(q-1)R_{(a,b),(-(b-1),j)}\\ &= \varepsilon_u\varepsilon_v q(1-q)^2(1+q-q^{b-3})+\varepsilon_u\varepsilon_v(1+q)(1-q)^3\\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{b-2}).\end{aligned}$$

If $b-a=1$ then $us \notin D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= -R_{(a,a+1),(-a,j)} \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a-1}) \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{b-2}). \end{aligned}$$

**b):** Let $a+j<0$. Then $j<0$ and $s \in D(v) \setminus D(u)$. If $j+|a|<-1$ we have that $us \notin D_n^{(n-2)}$ and

$$R_{u,v} = -R_{(a,b),(-b,j+1)} = \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{b-2}).$$

Otherwise, if $j+|a|=-1$ and $a<0$ then $us \in D_n^{(n-2)}$ and

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(-b,a)} + (q-1)R_{(a,b),(-b,a)} \\ &= \varepsilon_u\varepsilon_v q(1-q)^2 + \varepsilon_u\varepsilon_v(1-q)^2(1-q^{b-2}) \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{b-2}). \end{aligned}$$

If $j+|a|=-1$ and $a>0$ then $s \in D(u)$ and

$$R_{u,v} = R_{(a+1,b),(-b,-a)} = \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{b-2}).$$

**4):** $b=-j$.

Then $k=b$, $a \neq -i$ and $i<j<0<1<b$. In this case $d_{[\pm k]}(u,v)=1$ and there are two cases to consider. Let $s \overset{\text{def}}{=} s_{b-1}$.

**a):** If $b-a>1$ then $s \in D(v) \setminus D(u)$ and $us \in D_n^{(n-2)}$. Therefore

$$\begin{aligned} R_{u,v} &= qR_{(a,b-1),(i,-(b-1))} + (q-1)R_{(a,b),(i,-(b-1))} \\ &= \varepsilon_u\varepsilon_v q(1-q)^2(1+q-q^{b-2}) + \varepsilon_u\varepsilon_v(1+q)(1-q)^3 \\ &= (1-q)^2(1+q-q^{b-1}). \end{aligned}$$

**b):** If $b-a=1$ then $s \in D(v) \setminus D(u)$ and $us \notin D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= -R_{(a,a+1),(i,-a)} = \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{a}) \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q-q^{b-1}). \end{aligned}$$

Suppose now that $D^*(u,v)=\{k\}$ and $a=j$; in this case $b=-i$, $d_{[\pm k]}(u,v)=2$, and $|D(u,v)|=1$. We have two main cases to consider. Define

$$s \overset{\text{def}}{=} \begin{cases} s_a, & \text{if } a>0, \\ s_{-a-1}, & \text{if } a<0. \end{cases}$$

**1):** $a>0$.

If $b-a>1$ then $s \in D(v) \cap D(u)$. So

$$R_{u,v} = R_{(a+1,b),(-b,a+1)} = \varepsilon_u\varepsilon_v(1-q)(1-q^{b-2}).$$

If $b-a=1$, then $s \in D(v) \setminus D(u)$ and $us \notin D_n^{(n-2)}$. Hence

$$R_{u,v} = -R_{(a,a+1),(-a,a+1)} = \varepsilon_u\varepsilon_v(1-q)(1-q^{a-1}) = \varepsilon_u\varepsilon_v(1-q)(1-q^{b-2}).$$

**2):** $a<0$.

In this case $s \in D(v) \cap D(u)$. So, if $a < -1$, we have that

$$R_{u,v} = R_{(a+1,b),(-b,a+1)} = \varepsilon_u \varepsilon_v (1-q)(1-q^{b-2}).$$

If $a = -1$ then, by Proposition 9, $b > 2$ and $us \in D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= qR_{(2,b),(-b,2)} + (q-1)R_{(-1,b),(-b,2)} \\ &= \varepsilon_u \varepsilon_v q(1-q)(1-q^{b-2}) + \varepsilon_u \varepsilon_v (1-q)^3 = \varepsilon_u \varepsilon_v (1-q)(1-q^{b-2}). \end{aligned}$$

Suppose now that $D^*(u,v) = \{k\}$ and $a < j$; in this case $d_{[\pm k]}(u,v) = 1$. We have to prove that $R_{u,v} = \varepsilon_u \varepsilon_v (1-q)^{|D(u,v)|}(1-(1-\chi(u,v))q^{k-1})$. There are again four main cases to consider depending on whether $a = -i$, $a = -j$, $b = -i$ or $b = -j$.

**1):** $a = -i$.

Then, by Proposition 9, $i < 0 < 1 < a < j < b$ so $|D(u,v)| = 2$; moreover $s_{a-1} \in D(v) \setminus D(u)$ and $us_{a-1} \in D_n^{(n-2)}$. Then

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(-(a-1),j)} + (q-1)R_{(a,b),(-(a-1),j)} \\ &= \varepsilon_u \varepsilon_v q(1-q)^{2-\delta_{b,j}}(1-q^{a-2}) + \varepsilon_u \varepsilon_v (1-q)^{3-\delta_{b,j}} \\ &= \varepsilon_u \varepsilon_v (1-q)^{|D(u,v)|}(1-q^{a-1}). \end{aligned}$$

**2):** $a = -j$.

Then $b \neq -i$ and $i < a < 0 < j < b$; in this case $s_j \in D(v) \setminus D(u)$ and $us_j \in D_n^{(n-2)}$. Then, if $b + a > 1$,

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(i,-(a-1))} + (q-1)R_{(a,b),(i,-(a-1))} \\ &= \varepsilon_u \varepsilon_v q(1-q)^2 + \varepsilon_u \varepsilon_v (1-q)^3 = \varepsilon_u \varepsilon_v (1-q)^2. \end{aligned}$$

If $b + a = 1$ then

$$R_{u,v} = qR_{(a-1,-a),(i,-(a-1))} + (q-1)R_{(a,-(a-1)),(i,-(a-1))} = \varepsilon_u \varepsilon_v (1-q)^2.$$

**3):** $b = -i$.

Then $a \neq -j$ and $i < 0 < b$; if $b - j > 1$ then $s_{b-1} \in D(v) \setminus D(u)$ and $us_{b-1} \in D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= qR_{(a,b-1),(-(b-1),j)} + (q-1)R_{(a,b),(-(b-1),j)} \\ &= \varepsilon_u \varepsilon_v q(1-q)^2 + \varepsilon_u \varepsilon_v (1-q)^3 = \varepsilon_u \varepsilon_v (1-q)^2. \end{aligned}$$

If $b - j = 1$ then $s_{b-1} \in D(v) \setminus D(u)$ and $us_{b-1} \in D_n^{(n-2)}$. Then

$$R_{u,v} = qR_{(a,b-1),(-(b-1),b)} + (q-1)R_{(a,b),(-(b-1),b)} = \varepsilon_u \varepsilon_v (1-q)^2.$$

**4):** $b = -j$.

Then, again by Proposition 9, $i < a < j < 0 < 1 < b$. In this case $s_{b-1} \in D(v) \setminus D(u)$ and $us_{b-1} \in D_n^{(n-2)}$. Then

$$\begin{aligned} R_{u,v} &= qR_{(a,b-1),(i,-(b-1))} + (q-1)R_{(a,b),(i,-(b-1))} \\ &= \varepsilon_u \varepsilon_v q(1-q)^{2-\delta_{a,i}}(1-q^{b-2}) + \varepsilon_u \varepsilon_v (1-q)^{3-\delta_{a,i}} \\ &= \varepsilon_u \varepsilon_v (1-q)^{|D(u,v)|}(1-q^{b-1}). \end{aligned}$$

Finally, suppose that $D^*(u,v) = \emptyset$. We have three main cases to consider, depending on whether $a > j$, $a = j$, or $a < j$.

**1):** $a > j$.

In this case $\prod_{r\in D(u,v)}(1-q^{d_r(u,v)}) = (1-q)^2(1+q)$. Define

$$s \stackrel{\text{def}}{=} \begin{cases} s_j, & \text{if } j>0, \\ s_{-j-1}, & \text{if } j<0. \end{cases}$$

Therefore $s \in D(v)$.

**a):** If $j>0$ then $s \notin D(u)$.

**i):** If $j+i=-1$ or $j+i\neq -1$ and $a-j>1$ then $us \notin D_n^{(n-2)}$ and

$$\begin{aligned} R_{u,v} &= \begin{cases} -R_{(a,b),(i+1,-i)}, & \text{if } j+i=-1, \\ -R_{(a,b),(i,j+1)}, & \text{otherwise,} \end{cases} \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q). \end{aligned}$$

**ii):** If $a-j=1$ then $us \in D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(i,a)} + (q-1)R_{(a,b),(i,a)} \\ &= \varepsilon_u\varepsilon_v q(1-q)^2 + \varepsilon_u\varepsilon_v(1-q)^2 = (1-q)^2(1+q). \end{aligned}$$

**b):** Let $j=-1$. Then $a>1$.

**i):** If $i=-2$ or $i<-2$ and $a>2$ we have that $us \notin D_n^{(n-2)}$. So

$$\begin{aligned} R_{u,v} &= \begin{cases} -R_{(a,b),(1,2)}, & \text{if } i=-2, \\ -R_{(a,b),(i,2)}, & \text{otherwise,} \end{cases} \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q). \end{aligned}$$

**ii):** If $a=2$ then $s \notin D(u)$ and $us \in D_n^{(n-2)}$. Therefore

$$\begin{aligned} R_{u,v} &= qR_{(-1,b),(i,2)} + (q-1)R_{(2,b),(i,2)} \\ &= \varepsilon_u\varepsilon_v q(1-q)^2 + \varepsilon_u\varepsilon_v(1-q)^2 = (1-q)^2(1+q). \end{aligned}$$

**c):** Let $j<-1$.

**i):** If $b+j\neq -1$ and $|a|+j\neq -1$ then $s\notin D(u)$ and $us \notin D_n^{(n-2)}$. So

$$R_{u,v} = -R_{(a,b),(i,j+1)} = \varepsilon_u\varepsilon_v(1-q)^2(1+q).$$

**ii):** If $b+j=-1$ or $b+j\neq -1$ and $a+j=-1$ then $s\in D(u)$. Hence

$$\begin{aligned} R_{u,v} &= \begin{cases} R_{(a,b+1),(i,-b)}, & \text{if } b+j=-1, \\ R_{(a+1,b),(i,-a)}, & \text{otherwise,} \end{cases} \\ &= \varepsilon_u\varepsilon_v(1-q)^2(1+q). \end{aligned}$$

**iii):** If $b+j\neq -1$ and $-a+j=-1$ then $s\notin D(u)$ and $us \in D_n^{(n-2)}$. Therefore

$$\begin{aligned} R_{u,v} &= qR_{(a-1,b),(i,a)} + (q-1)R_{(a,b),(i,a)} \\ &= \varepsilon_u\varepsilon_v q(1-q)^2 + \varepsilon_u\varepsilon_v(1-q)^2 = (1-q)^2(1+q). \end{aligned}$$

**2):** $a=j$. Then $j<n$ and $s\in D(v)$, where

$$s \stackrel{\text{def}}{=} \begin{cases} s_j, & \text{if } j>0, \\ s_{-j-1}, & \text{if } j<0. \end{cases}$$

We have to prove that $R_{u,v} = \varepsilon_u\varepsilon_v(1-q)$.

**i):** Let $j>0$. Then, if $b-a\neq 1$ we have that $us<u$ and, if $j+i\neq -1$

$$\begin{aligned} R_{u,v} &= \begin{cases} R_{(a+1,b),(-a,a+1)}, & \text{if } j+i=-1, \\ R_{(a+1,b),(i,a+1)}, & \text{otherwise,} \end{cases} \\ &= \varepsilon_u\varepsilon_v(1-q). \end{aligned}$$

If $b-a=1$; then $us\notin D_n^{(n-2)}$ and $j+i\neq -1$, so

$$R_{u,v}=-R_{(a,b),(i,b)}=\varepsilon_u\varepsilon_v(1-q).$$

**ii):** The cases $j=-1$ and $j<-1$ are analogous to the previous one and are therefore omitted.

**3):** $a<j$.

In this case $\prod_{r\in D(u,v)}(1-q^{d_r(u,v)})=(1-q)^{|D(u,v)|}$. We have six main cases to distinguish.

**a):** $j=n$. Then $b=n$. In this case, if $u\neq v$, then $(1-q)^{|D(u,v)|}=1-q$. Define

$$s\stackrel{\text{def}}{=}\begin{cases} s_i, & \text{if } i>0, \\ s_{-i-1}, & \text{if } i<0. \end{cases}$$

So $s\in D(v)$ and, if $u<us\neq v$ then $D^*(u,vs)=\varnothing$ and $|D(u,vs)|=1$.

We assume $i>0$, the cases $i=-1$ and $i<-1$ being analogous, and simpler. Then, if $a-i\neq 1$ we have that $us\notin D_n^{(n-2)}$ and $R_{u,v}=-R_{(a,n),(i+1,n)}=\varepsilon_u\varepsilon_v(1-q)$. If $a-i=1$ then $u<us=v$ so $R_{u,v}=\varepsilon_u\varepsilon_v(1-q)$, and the result follows.

**b):** $0<j<n$.

Let $s\stackrel{\text{def}}{=}s_j$. Then $s\in D(v)$. If $b=j$ then $s\in D(u)$ so $R_{u,v}=R_{us,vs}=\varepsilon_{us}\varepsilon_{vs}(1-q)$, and the result follows. We may therefore assume that $j<b$. If $b=j+1$ then $s\notin D(u)$ and $us\in D_n^{(n-2)}$ so

$$R_{u,v}=qR_{(a,b-1),(i,b)}+(q-1)R_{(a,b),(i,b)}=\varepsilon_u\varepsilon_v(1-q)^{2-\delta_{a,i}}$$

and the result follows. We may therefore assume that $b>j+1$. If $j+1=-a$ then $s\in D(u)$ and $R_{u,v}=R_{(a+1,b),(i,-a)}=\varepsilon_u\varepsilon_v(1-q)^2$, as desired. If $j+1\neq -a$ and $j+1=-i$ then $s\notin D(u)$ and $us\notin D_n^{(n-2)}$ so $R_{u,v}=-R_{(a,b),(-j,j+1)}=\varepsilon_u\varepsilon_v(1-q)^2$. Finally, if $j+1\notin\{-a,b,-i\}$ then $s\notin D(u)$ and $us\notin D_n^{(n-2)}$. So $R_{u,v}=-R_{(a,b),(i,j+1)}=\varepsilon_u\varepsilon_v(1-q)^{2-\delta_{a,i}}$, and the result again follows.

**c):** $j=-1$.

Then $s_{-i-1}\in D(v)$. If $b=-i-1$ then $u<us_{-i-1}\in D_n^{(n-2)}$ and

$$\begin{aligned} R_{u,v} &= qR_{(a,b+1),(-b,j)}+(q-1)R_{(a,b),(-b,j)} \\ &= \varepsilon_u\varepsilon_v q(1-q)^2+\varepsilon_u\varepsilon_v(1-q)^3=\varepsilon_u\varepsilon_v(1-q)^2 \end{aligned}$$

as desired. We may therefore assume that $b\neq -i-1$. If $-a=-i-1$ then $u<us_{-i-1}\in D_n^{(n-2)}$ and

$$R_{u,v}=qR_{(a-1,b),(a,j)}+(q-1)R_{(a,b),(a,j)}=\varepsilon_u\varepsilon_v(1-q)^2$$

so the result again follows. Finally, if $-i-1\notin\{-a,b\}$ then $u<us_{-i-1}\notin D_n^{(n-2)}$ so $R_{u,v}=-R_{(a,b),(i+1,j)}=\varepsilon_u\varepsilon_v(1-q)^2$ and our claim again follows.

**d):** $j < -1 < 0 < b$.

Then $s_{-j-1} \in D(v)$. If $b = -j-1$ then $s_{-j-1} \in D(u)$ and $R_{u,v} = R_{(a,b+1),(i,-b)} = \varepsilon_u \varepsilon_v (1-q)^{2-\delta_{a,i}}$ as claimed. If $b \neq -j-1$ then $u < us_{-j-1} \notin D_n^{(n-2)}$ so $R_{u,v} = -R_{(a,b),(i,j+1)} = \varepsilon_u \varepsilon_v (1-q)^{2-\delta_{a,i}}$ and the result again follows

**e):** $j < -1$, $b < 0$.

Let $s \stackrel{\text{def}}{=} s_{-i-1}$. Then $s \in D(v)$. If $a = i$ and $b = j+1$ then $\ell(u,v) = 1$ so $R_{u,v} = \varepsilon_u \varepsilon_v (1-q)$ as desired. If $a = i$ and $b > j+1$ then $s_{-j-1} \in D(v) \setminus D(u)$ and $us_{-j-1} \notin D_n^{(n-2)}$ so $R_{u,v} = -R_{(a,b),(a,j+1)} = \varepsilon_u \varepsilon_v (1-q)$ and the result again follows. We may therefore assume that $a > i$. If $a = i+1$ then $u < us \in D_n^{(n-2)}$ and, by Proposition 9, $us \not\leq vs$, so

$$R_{u,v} = (q-1) R_{(a,b),(a,j)} = \varepsilon_u \varepsilon_v (1-q)^{|D(u,vs)|+1}$$

and the result follows since $|D(u,vs)| + 1 = |D(u,v)|$. Finally, if $a > i+1$ then $us \notin D_n^{(n-2)}$ so

$$R_{u,v} = -R_{(a,b),(i+1,j)} = \varepsilon_u \varepsilon_v (1-q)^{|D(u,vs)|}$$

and the result again follows since $|D(u,vs)| = |D(u,v)|$.

This concludes the induction step and hence the proof. □

We illustrate the preceding theorem with an example. Let $u \stackrel{\text{def}}{=} [1,2,8,3,4,5,6,9,7]$, $v \stackrel{\text{def}}{=} [-1,2,-8,3,4,9,5,6,7]$. Then $u,v \in D_9^{(7)}$, $u \leq v$, $u = (3,8)$ and $v = (-3,6)$. Hence $D(u,v) = \{3,8\}$, $D^*(u,v) = \{3\}$ so $k = 3$, $\chi(u,v) = 0$, $\tilde{\chi}(u,v) = 1$ and $d_{[\pm 3]}(u,v) = |\{3,-3,6\} \cap [\pm 2]| + 1 = 1$. Therefore by Theorem 4 we have that $R_{u,v}^{(7),q}(q) = (1-q)^2 - (1-q)^2 q^2 = 1 - 2q + 2q^3 - q^4$.

In the same way as Corollary 1 follows from Theorem 3 we obtain from Theorem 4 the following explicit formulas for certain sums and alternating sums of ordinary $R$-polynomials of the Weyl groups of type $D$.

**Corollary 3.** *Let $u,v \in D_n^{(n-2)}$, $u < v$, $x \in \{-1,q\}$, and $J \stackrel{\text{def}}{=} S_D \setminus \{s_{n-2}\}$. Then*

$$\sum_{w \in W_J} (-x)^{\ell(w)} R_{wu,v}(q)$$

*equals*

$$(q-1-x)^{\ell(u,v)} \left[ \prod_{r \in D(u,v)} \left( 1 - \left(\frac{x^2}{q}\right)^{d_r(u,v)} \right) + \left( 1 - \left(\frac{x^2}{q}\right) \right) \left(\frac{x^2}{q}\right)^{\frac{\ell(u,v)+1-2N_1(-u^{-1}(n-1))}{2}} \right.$$
$$\left. - \left( 1 - \left(\frac{x^2}{q}\right)^{N_1(-u^{-1}(n-1))} \right)^2 \left( \left(\frac{x^2}{q}\right)^{u^{-1}(n)-2} + \left(\frac{x^2}{q}\right)^{u^{-1}(n-1)} \right) \right],$$

*if* $(u^{-1}(n-1), u^{-1}(n)) = (-v^{-1}(n), -v^{-1}(n-1))$,

$$(q-1-x)^{\ell(u,v)} \left[ \prod_{r \in D(u,v)} \left( 1 - \left(\frac{x^2}{q}\right)^{d_r(u,v)} \right) - \left( 1 - \left(\frac{x^2}{q}\right)^{\tilde{\chi}(u,v)} \right)^{|D(u,v)|} \left(\frac{x^2}{q}\right)^{k - d_{[\pm k]}(u,v)} \right],$$

*if* $\{u^{-1}(n-1), u^{-1}(n)\} \cap \{-v^{-1}(n-1), -v^{-1}(n)\} = \{k\}$, *where* $d_{[\pm k]}(u,v) \stackrel{\text{def}}{=} |\{u^{-1}(n-1), v^{-1}(n-1), v^{-1}(n)\} \cap [\pm(k-1)]| + \chi(u^{-1}(n-1) < v^{-1}(n))$, *and*

$$(q-1-x)^{\ell(u,v)} \prod_{r \in D(u,v)} \left(1 - \left(\frac{x^2}{q}\right)^{d_r(u,v)}\right),$$

*otherwise.*

We conclude by noting the following consequence of Theorems 3 and 4. Its proof is a routine verification, which we omit.

**Corollary 4.** *Let* $u, v \in W^J$, $u \leq v$, $W^J \in \{B_n^{(n-2)}, D_n^{(n-2)}\}$. *Then*

$$R_{u,v}^{J,q} = R_{v^*,u^*}^{J,q}. \square$$

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