Predefined-Time Output Stabilization with Second Order Sliding Mode Generation

Antonella Ferrara, *Fellow, IEEE*, and Gian Paolo Incremona, *Member, IEEE*

***Abstract*—In this paper a novel second order sliding mode control strategy is proposed for relative-degree-2 nonlinear uncertain SISO systems. To design the strategy, it is studied the phase portrait of the second order system in normal form associated with the formulated sliding mode control problem. A sliding surface switching between arcs of parabolas is conceived to ensure the convergence in a predefined time to the desired second order sliding mode. Lyapunov-based analysis and the reformulation of LaSalle-Yoshizawa results for nonsmooth systems are used to prove the uniform finite-time stability of the equilibrium consisting in the second order sliding mode enforcement. This in turns allows to prove the asymptotic convergence of the original plant state to the origin in spite of the uncertainties.**

***Index Terms*—Sliding mode control, nonlinear control systems, uncertain systems, output stabilization, predefined-time.**

I. Introduction

In this paper we focus on a class of relative-degree-2 nonlinear systems with uncertain model, which may be reduced to perturbed double integrators while designing controllers capable of generating second order sliding modes in a "predefined time".

Stabilization of double integrators is a classical fundamental problem in control literature [1]–[3]. In fact, the dynamics of several mechanical, electro-mechanical, thermal and chemical plants are captured by this class of systems. Several approaches have been adopted in the literature to solve control problems related to second order systems, ranging from PID control, to nonlinear and adaptive strategies [4]. When the double integrator model is affected by uncertainties, the use of sliding mode control is particularly appropriate [5]. Yet, classical sliding mode control can deal with relative-degree-1 uncertain systems, and cannot be a viable approach in the considered case.

When a sliding mode control approach is adopted, the controlled system state regulation is obtained through the finite time stabilization of a particular system output called "sliding variable". In case of uncertain nonlinear systems of relative-degree-2, a possible solution to the finite time regulation problem is offered by higher-order sliding mode control [6]. This kind of control has the capability of steering to zero in finite time not only the sliding variable, but also its time derivatives up to a certain order, so that the control signal actually fed into the plant is continuous [7]–[9]. This fact makes the approach adequate to be applied to mechanical and electro-mechanical systems, as testified by [10]–[14], and to other kind of processes which require a smooth control action.

In order to control uncertain nonlinear systems of relative-degree-2 via higher-order sliding mode control, a second order sliding mode control law must be designed. Many important results have already been established for second order sliding mode controllers [15]–[17], also including the presence of constraints on both control and state variables [18]. In contrast, the problem of the generation of second order sliding modes in fixed, prescribed or predefined time (note that predefined time convergence cannot be reached with first order sliding modes) has been faced only recently (see, for instance [19]–[24], and the references therein contained for definitions and details), so that room for novel solutions still exists.

As a matter of fact, in the classical formulation of second order sliding mode control, global finite-time output stabilization of uncertain systems of relative-degree-2 has been proved, the output being the selected sliding variable. Also solutions guaranteeing the minimum convergence time with respect to the worst realization of the uncertain terms have been provided [25], [26]. However, some applications (e.g., electromechanical systems that employ power converters, electric motors, robots or positioning of machine tools in manufacturing) require the sliding variable to converge in a finite time which is predefined, even in presence of varying disturbance and uncertainty terms, which instead, in general, make the finite time duration of the convergence phase totally unpredictable.

In the present paper, as previously mentioned, we are focused on "predefined-time" output stabilization. To this end, we present a novel second order sliding mode control solution to tackle uncertain systems of relative-degree-2, regulating the selected sliding variable to zero in a finite time which is given, compatibly with the feasibility limits. According to the definitions introduced in [21], the control solution proposed in this paper can be classified as a "predefined-time based sliding mode control", since the true fixed stabilization time is directly tuned by a particular selection of the control parameters.

The second order sliding mode strategy proposed in the paper exploits the feature of perturbed double integrators of switching over branches of parabola arcs in the state plane, when a discontinuous feedback control law is applied. By solving the predefined-time stabilization problem for the state of the perturbed double integrator system, associated with the plant to control, the asymptotic state regulation of the original nonlinear plant is attained. The merit of this paper, apart from proposing a new predefined-time sliding mode control strategy, is also to provide an explicit expression of the predefined convergence time and of the equivalent control gain needed

This is the final version of the accepted paper submitted to IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2020.2995667. Antonella Ferrara is with the Dipartimento di Ingegneria Industriale e dell'Informazione, University of Pavia, Via Ferrata 5, 27100 Pavia, Italy (e-mail: antonella.ferrara@unipv.it).

Gian Paolo Incremona is with the Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy (e-mail: gianpaolo.incremona@polimi.it).

to assign a specific transient evolution to the sliding variable [8], [12], [27]. Moreover, conditions are derived under which such a controller provides the desired predefined-time output regulation with feasibility guarantees. Finally, an illustrative example shows the efficacy of the proposal and its comparison with another valid higher-order sliding mode strategy [8].

The paper is organized as follows. In Section II the main notation is defined. The considered control problem is formulated in Section III, while in Section IV the proposed control strategy is presented. In Section V the formal stability analysis is reported. An illustrative example is discussed in Section VI, and some conclusions are gathered in Section VII.

II. Notation

Let $\mathbb{R}^+$ be the set of non-negative real numbers and $|\cdot|$ be the Euclidean norm. Given a column vector $w \in \mathbb{R}^n$, let w' denote its transpose. A function $g(w) : \mathcal{G} \to \mathbb{R}^p$ belongs to $\mathcal{L}^\infty$, i.e., $g(w) \in \mathcal{L}^\infty(\mathcal{G}, \mathbb{R}^p)$, if it is essentially bounded. Let $F[g](w)$ be an upper semi-continuous, non-empty, compact and convex valued map on $\mathcal{G}$, and $\bigcap_{|\mathcal{G}|=0}$ denotes the intersections over sets $\mathcal{G}$ of measure zero. Let $\underline{K}(g)$, $\overline{K}(g)$, $\underline{\mathrm{sgn}}(g)$ and $\overline{\mathrm{sgn}}(g)$ be the following functions,

$$\underline{K}(g) := \begin{cases} 0, & g \le 0 \\ 1, & g > 0 \end{cases} \quad \overline{K}(g) := \begin{cases} 0, & g < 0 \\ 1, & g \ge 0 \end{cases}$$
$$\underline{\mathrm{sgn}}(g) := \begin{cases} -1, & g \le 0 \\ 1, & g > 0 \end{cases} \quad \overline{\mathrm{sgn}}(g) := \begin{cases} -1, & g < 0 \\ 1, & g \ge 0 \;. \end{cases}$$

Let $F[\mathrm{sgn}(g)] = \mathrm{SGN}(g)$ be a function such that $\mathrm{SGN}(g) = 1$ if $g > 0$, $[-1, 1]$ if $g = 0$ and -1 if $g < 0$.

III. Problem Formulation

Consider a single-input single-output system affine in the control variable

$$\begin{cases} \dot{x}(t) = a(x(t)) + b(x(t))u(t) \\ y(t) = c(x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $x \in \mathcal{D}$ ($\mathcal{D} \subset \mathbb{R}^n$ an open connected set) is an absolutely continuous state trajectory satisfying the differential equation almost everywhere (a.e.), $u \in \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$ is a bounded scalar input, while $a, b \in \mathrm{C}^1(\mathcal{D}, \mathbb{R}^n)$ are uncertain functions. The output function y is given by $c \in \mathrm{C}^2(\mathcal{D}, \mathbb{R})$. In the following, the dependence on time t is omitted when obvious. Moreover, the following assumption holds.

Assumption 1 *System* (1) *is complete in $\mathcal{D}$ meaning that $x(t) \in \mathcal{D}$ and, $\forall\, x_0$ and $\forall\, u$, $x(t)$ is defined for almost all $t \in \mathbb{R}^+$. Moreover, system* (1) *has uniform and time-invariant relative degree equal to 2 and admits a global normal form in a region $\mathcal{D}_0 \subset \mathcal{D}$.*

By virtue of Assumption 1, there exists a global diffeomorphism of the form $\Phi(x) : \mathcal{D}_0 \to \Phi(\mathcal{D}_0)$, with $\Phi(\mathcal{D}_0) \subset \mathbb{R}^n$,

$$\Phi(x) = \begin{pmatrix} \Psi(x) \\ c(x) \\ a(x) \cdot \nabla c(x) \end{pmatrix} = \begin{pmatrix} z \\ \xi \end{pmatrix}$$
$$\Psi : \mathcal{D}_0 \to \mathbb{R}^{n-2}, \quad z \in \mathbb{R}^{n-2}, \quad \xi = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \in \mathbb{R}^2 \;,$$

such that $(\partial\Psi(x)/\partial x) b(x) = 0$, $\forall\, x \in \mathcal{D}_0$, and

$$\begin{cases} \dot{z} = f_0(z, \xi) & \text{(2a)} \\ \dot{\xi}_1 = \xi_2 & \text{(2b)} \\ \dot{\xi}_2 = f_1(z, \xi) + f_2(z, \xi)u & \text{(2c)} \\ y = \xi_1 & \text{(2d)} \\ \xi(t_0) = \xi_0 & \text{(2e)} \end{cases}$$

with

$$f_0(z, \xi) = \frac{\partial \Psi}{\partial x}(\Phi^{-1}(z, \xi)) a(\Phi^{-1}(z, \xi))$$
$$f_1(z, \xi) = a(\Phi^{-1}(z, \xi)) \cdot \nabla(a(\Phi^{-1}(z, \xi)) \cdot \nabla c(\Phi^{-1}(z, \xi)))$$
$$f_2(z, \xi) = b(\Phi^{-1}(z, \xi)) \cdot \nabla(a(\Phi^{-1}(z, \xi)) \cdot \nabla c(\Phi^{-1}(z, \xi))).$$

The normal form (2b)–(2e) is also called "auxiliary system" in the literature of second-order sliding mode control (see, e.g., [17]). As a consequence of the uniform relative degree assumption, it yields

$$f_2(z, \xi) \neq 0, \quad \forall\, (z, \xi) \in \Phi(\mathcal{D}_0) \;. \tag{3}$$

Since u is bounded, $f_0(\cdot)$, $f_1(\cdot)$, $f_2(\cdot)$ are continuous functions and $\Phi(\mathcal{D}_0)$ is a bounded set, one has

$$\exists\, \mu > 0 : \ |u(t)| \le \mu \quad \text{a.e.} \tag{4}$$
$$\exists\, \beta > 0 : \ |f_1(z, \xi)| \le \beta, \ \forall\, (z, \xi) \in \Phi(\mathcal{D}_0) \tag{5}$$
$$\exists\, \delta > 0 : \ f_2(z, \xi) \ \le \delta, \ \forall\, (z, \xi) \in \Phi(\mathcal{D}_0) \tag{6}$$
$$\exists\, \gamma > 0 : \ f_2(z, \xi) \ \ge \gamma, \ \forall\, (z, \xi) \in \Phi(\mathcal{D}_0) \;. \tag{7}$$

Note that characterizing the bounds of the uncertainties requires a careful analysis of the considered system, relying on e.g., data analysis or physical insights. In the following we assume that β, δ and γ are known. Moreover, the following assumptions hold.

Assumption 2 *The control amplitude upperbound μ is such that*

$$\mu > \frac{\beta}{\gamma} \;. \tag{8}$$

Assumption 3 *Given the normal form* (2)*, the zero dynamics $f_0(z, 0)$ is globally asymptotically stable.*

Relying on (2)-(7) and Assumptions 1, 2 and 3, the following control objective is introduced.

Control Objective *Given a predefined reaching time $T \ge 0$, design a feedback control law*

$$u(t) = \kappa(\xi_1(t), \xi_2(t)) \tag{9}$$

such that $\forall\, x_0 \in \mathcal{D}_0$, $y(t) = 0$, $\forall\, t \ge T$ in spite of the uncertainties.

Hence, the first problem (Problem 1) to solve is to guarantee the global finite-time stability of the origin as an equilibrium point of the normal form (2) and impose that, given any admissible initial condition, this point is reached in a predefined time. Then, the resolution of Problem 1 is instrumental to solve the major control problem (Problem 2), i.e., that of making the origin of the controlled system (1) be an asymptotically stable equilibrium point in spite of the uncertainties. The solution to Problem 2 requires to suitably select the output $y(t)$, while complying with the indicated assumptions. In classical sliding mode control $y(t)$ is called sliding variable and $c(x)$ is often selected as linear combination of the original system state [6]. In order to solve the considered control problem, it is convenient to re-interpret equations (2b)-(2e) as the following differential inclusion,

$$\begin{cases} \dot{\xi}_1 = \xi_2 & \text{(10a)} \\ \dot{\xi}_2 \in [-\beta,\, \beta] + [\gamma,\, \delta] u & \text{(10b)} \\ y = \xi_1 & \text{(10c)} \\ \xi(t_0) = \xi_0 \; . & \text{(10d)} \end{cases}$$

Solutions of the differential inclusion $\dot{\xi} \in F[f](\xi)$, with $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, are the solutions of all the differential equations

$$\dot{\xi} = f(\xi) \; . \tag{11}$$

Reading (10) in Fillipov's sense, one can claim that the solutions of (11) are absolutely continuous functions such that

$$\dot{\xi}(t) \in \mathrm{co} \left\{ \bigcap_{\epsilon > 0,\, |\mathcal{N}| = 0} f(B_\epsilon(\xi(t)) \setminus \mathcal{N}) \right\}, \text{ a.e.} \tag{12}$$

where co is the convex hull, $|\mathcal{N}|$ is the measure of sets $\mathcal{N}$, and $B_\epsilon(\xi_0) := \{\xi : \|\xi - \xi_0\| \le \epsilon\}$. Note that $\xi_2(t) = \dot{y}(t)$ can be either measurable or retrievable via a suitable estimation technique. In the context of sliding mode control the most widely used technique is that based on the so-called Levant's differentiator of the suitable order [28]. In this case, the predefined reaching time T has to be suitably selected in order to provide a sufficient time for the differentiator to converge.

IV. Predefined-Time Sliding Mode

The previous control problem is hereafter solved via a second order sliding mode controller with predefined convergence time. Let $\alpha \in [\underline{\alpha},\, \overline{\alpha}]$ where

$$\underline{\alpha} := \mu\gamma - \beta \tag{13}$$

$$\overline{\alpha} := \mu\delta + \beta \; , \tag{14}$$

such that the following assumption holds.

Assumption 4 *The reduced control amplitude $\underline{\alpha}$ is strictly positive.*

Consider now the following sliding mode controller

$$u = -\mu \,\mathrm{sgn}(s(\xi)) \tag{15}$$

$$s(\xi) := \xi_1 + \zeta(\xi, t) \tag{16}$$

where the sliding function consists of two parts: the first one ξ_1 is the output (i.e., the sliding variable), while the second part ζ is the so-called transient function, which will be defined in the following.

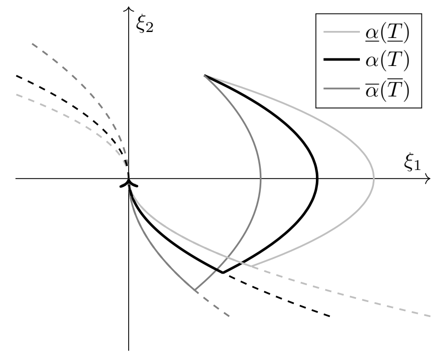

Figure 1. State portrait for a perturbed double integrator controlled via the proposed second order sliding mode with $\underline{\alpha}$ (light gray line), $\overline{\alpha}$ (dark gray line) and α (black line)

Our philosophy is to require that the output variable is steered to zero in a predefined time, i.e.,

$$\xi_1(t) = 0, \; \forall\, t \ge T(\xi_0, \alpha) \tag{17}$$

and the feedback control is able to reject all the uncertainties from the initial time instant, so that the equivalent control is defined as

$$v := \frac{\alpha}{\mu} u \; . \tag{18}$$

In order to satisfy conditions (17) and (18), we design a second order sliding mode controller. Consider the following family of switching curves

$$\sigma = \xi_1 + \frac{\xi_2 |\xi_2|}{2\alpha} \tag{19}$$

$$\underline{\sigma} = \xi_1 + \frac{\xi_2 |\xi_2|}{2\underline{\alpha}} \tag{20}$$

$$\overline{\sigma} = \xi_1 + \frac{\xi_2 |\xi_2|}{2\overline{\alpha}} \; . \tag{21}$$

Let $T(\xi_0, \alpha) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ be the settling time function such that (17) holds. The dependence of time T on parameters ξ_0 and α will be omitted when obvious. Choosing a priori a predefined-time T, and associating it to the related control gain α, the transient function is defined as follows

$$\zeta := \begin{cases} -\xi_{1_0} - \xi_{2_0} t + \alpha \,\mathrm{sgn}(\sigma_0) \frac{t^2}{2}, \; t_0 \le t \le \tau & \text{(22a)} \\ \dfrac{\xi_2 |\xi_2|}{2\alpha}, \; \tau < t < T & \text{(22b)} \\ 0, \quad t \ge T \; , & \text{(22c)} \end{cases}$$

with $t_0 = 0$, and τ being the time when the equivalent control switches its sign. Note that the choice of α implies the reaching time T which is, by construction, included in the allowable range $[\overline{T},\, \underline{T}]$ associated with $\overline{\alpha}$ and $\underline{\alpha}$, respectively (see Figure 1 for an illustrative example where $\underline{\alpha} = 1$, $\overline{\alpha} = 3$ and $\alpha = 1.5$ imply convergence times equal to $\underline{T}$ =8.09 s, $\overline{T}$ =3.16 s and T =3.83 s, respectively).

V. Analysis of Predefined-Time Stability

Consider now to apply the control law (15) to the differential inclusion (10). One obtains

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 \in [-\beta,\, \beta] - \mu[\gamma,\, \delta]\, \mathrm{sgn}(s(\xi)) \ , \end{cases} \tag{23}$$

from which, depending on $\mathrm{sgn}(s(\xi))$, one has

$$\begin{cases} \dot{\xi}_2 \in [-(\beta+\mu\delta),\, (\beta-\mu\gamma)] = -[\underline{\alpha},\, \overline{\alpha}] & s(\xi) > 0 \\ \dot{\xi}_2 \in [(\mu\gamma-\beta),\, (\mu\delta+\beta)] = [\overline{\alpha},\, \underline{\alpha}] & s(\xi) < 0 \ . \end{cases} \tag{24}$$

As for the case $s(\xi) = 0$, since $\xi(t)$ is absolutely continuous, we say that $\xi_2\dot{\xi}_2 > 0$ and restrict $\dot{\xi}_2$ to be less than $\overline{\alpha}$. Define now the following switching function

$$\psi(\xi) := \overline{K}(\xi_2)\overline{\mathrm{sgn}}(s(\xi)) + \underline{K}(-\xi_2)\underline{\mathrm{sgn}}(s(\xi)) \ , \tag{25}$$

so that (23) can be written as

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 \in -[\underline{\alpha},\, \overline{\alpha}]\psi(\xi) \ . \end{cases} \tag{26}$$

Let $\alpha \in [\underline{\alpha},\, \overline{\alpha}]$, and consider now the following equivalent system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 \in -\alpha\psi(\xi) \ . \end{cases} \tag{27}$$

Now, the following results on the predefined convergence time can be proved.

Lemma 1 *If $s(\xi(t)) = 0$, $\forall\, t \geq 0$, $\forall\, \xi_0 \in \mathbb{R}^2$, and $\xi(T) = 0$ is a global uniformly finite-time stable equilibrium point of the differential inclusion* (27)*, the finite-time T is*

$$T = \frac{\xi_{2_0}}{\alpha}\psi(\xi_0) + 2\sqrt{\frac{\xi_{2_0}^2}{2\alpha^2} + \frac{\xi_{1_0}}{\alpha}\psi(\xi_0)} \ . \tag{28}$$

Proof: Consider the differential inclusion (27) and the solution of the differential equation obtained with $\psi(\xi) = 1$ and $\forall\, \xi_0 \in \mathbb{R}^2$ such that $\xi_{1_0} > -\xi_{2_0}|\xi_{2_0}|/2\alpha$. Since it obeys the Newton's laws and the input assumes constant value, it is possible to get

$$\xi_2(t) = \xi_{2_0} - \alpha t \tag{29}$$

$$\xi_1(t) = \xi_{1_0} + \xi_{2_0}t - \frac{\alpha}{2}t^2 \tag{30}$$

which represent the velocity and the corresponding position associated with the sliding variable, respectively. The state is driven along the parabola passing through ξ_0 to the switching line at which time τ the control is switched to α. Note that for $\xi_2 < 0$ the switching line has the form $\xi_1 = \xi_2^2/2\alpha$. Squaring and dividing (29) by 2α, one obtains

$$\frac{\xi_2^2(t)}{2\alpha} = \frac{\xi_{2_0}^2}{2\alpha} + \frac{1}{2}\alpha t^2 - \xi_{2_0}t \ . \tag{31}$$

Subtracting (31) to (30), on the switching curve, it holds

$$\xi_1(t) - \frac{\xi_2^2(t)}{2\alpha} = \xi_{1_0} - \frac{\xi_{2_0}^2}{2\alpha} + 2\xi_{2_0}t - \alpha t^2 = 0 \tag{32}$$

and the following equation can be written

$$\alpha t^2 - 2\xi_{2_0}t - \xi_{1_0} + \frac{\xi_{2_0}^2}{2\alpha} = 0 \ . \tag{33}$$

Hence, solving (33) with $t = \tau$, considering only the positive root, the time instant when the control switches its sign is

$$\tau = \frac{\xi_{2_0}}{\alpha} + \sqrt{\frac{\xi_{2_0}^2}{2\alpha^2} + \frac{\xi_{1_0}}{\alpha}} \ . \tag{34}$$

From here on, the state is on the switching curve. Computing (29) in τ and recasting it for the time interval $(T - \tau)$ with $u = \alpha$, one has

$$\xi_2(\tau) = \xi_{2_0} - \alpha\tau \tag{35}$$

$$\xi_2(T) = \xi_2(\tau) + \alpha(T - \tau) = 0 \ . \tag{36}$$

Finally, the combination of (35) with (36) gives

$$T = \frac{\xi_{2_0}}{\alpha} + 2\sqrt{\frac{\xi_{2_0}^2}{2\alpha^2} + \frac{\xi_{1_0}}{\alpha}} \ . \tag{37}$$

Analogously, for the specular case with $\psi(\xi) = -1$ and $\forall\, \xi_0 \in \mathbb{R}^2$ such that $\xi_{1_0} < -\xi_{2_0}|\xi_{2_0}|/2\alpha$ one obtains

$$T = -\frac{\xi_{2_0}}{\alpha} + 2\sqrt{\frac{\xi_{2_0}^2}{2\alpha^2} - \frac{\xi_{1_0}}{\alpha}} \ . \tag{38}$$

Considering the switching function (25), equations (37) and (38) can be written in the compact way (28). ■

Note that from Lemma 1, it is immediate to verify that $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$ implies that $\overline{T} \leq T \leq \underline{T}$. Define now the following switching function

$$\varphi(\xi) := \overline{K}(-\xi_2)\overline{\mathrm{sgn}}(\overline{\sigma}) + \underline{K}(\xi_2)\underline{\mathrm{sgn}}(\overline{\sigma})\underline{K}(\underline{\sigma}\overline{\sigma}) \ . \tag{39}$$

Since in our case T is instead a design parameter, the following lemma holds.

Lemma 2 *If $T \in [\overline{T},\, \underline{T}]$ is the convergence predefined time such that $\xi(T) = 0$ is a global uniformly finite-time stable equilibrium point of the differential inclusion* (27)*, then the corresponding control amplitude $\alpha \in [\underline{\alpha},\, \overline{\alpha}]$ is computed as*

$$\alpha = \left(2\frac{\xi_{1_0}}{T^2} + \frac{\xi_{2_0}}{T}\right)\varphi(\xi_0) + 2\underline{K}(\underline{\sigma}_0\overline{\sigma}_0)\sqrt{\frac{\xi_{1_0}^2}{T^4} + \frac{\xi_{1_0}\xi_{2_0}}{T^3} + \frac{\xi_{2_0}^2}{2T^2}} . \tag{40}$$

Proof: The proof directly follows from Lemma 1. By solving (37) and (38) as functions of α and compacting the solutions by using (39), equation (40) is achieved. ■

Consider now the switching surface (16) and the associated system

$$\dot{s} \in \mathcal{F}(s, t) \tag{41}$$

where $s \in \mathcal{S}$ ($\mathcal{S} \subset \mathbb{R}$ is an open connected set) and $\mathcal{F} : \mathcal{S} \times [0,\, \infty) \to \mathbb{R}$ contains a discontinuity at any point in $\mathcal{S}$. Before introducing the main result of this work, the following lemmas and corollaries to the LaSalle-Yoshizawa Theorem (see [2, Theorem 8.4] and [29, Theorem A.8], and results in [30]) are recalled.

Lemma 3 *[30] Let $s(t)$ be a Filippov's solution of* (41) *and*

$V : \mathcal{S} \times [0, \infty) \to \mathbb{R}$ be a locally Lipschitz, regular function, then $V(s(t), t)$ is absolutely continuous, $\dot{V}(s(t), t)$ exists for almost all $t \in [0, \infty)$ and $\dot{V}(s(t), t) \overset{a.e.}{\in} \dot{\tilde{V}}(s(t), t)$ where

$$\dot{\tilde{V}}(s(t), t) := \bigcap_{\varsigma \in \partial V(s,t)} \varsigma' \begin{bmatrix} F[\mathcal{F}] \\ 1 \end{bmatrix} (s, t) \ , \tag{42}$$

or equivalently, almost everywhere,

$$\dot{V} = \varsigma' \begin{bmatrix} \dot{s} \\ 1 \end{bmatrix} , \ \forall \varsigma \in \partial V(s, t) \ , \tag{43}$$

and some $\dot{s} \in F[\mathcal{F}](s, t)$.

Lemma 4 *[30] Let $s(t)$ be any Filippov's solution of (41) and $V : \mathcal{S} \times [0, \infty) \to \mathbb{R}$ be a locally Lipschitz, regular function, if $\dot{V}(s(t), t) \overset{a.e.}{\leq} 0$ then $V(s(t), t) \leq V(s_0, 0), \forall t > 0$.*

Lemma 5 *[30] Let $\phi(t) : \mathbb{R}^+ \to \mathbb{R}$ be an uniformly continuous function, if $\lim_{t\to\infty} \int_0^t \phi(\theta) d\theta$ exists and is finite, then $\lim_{t\to\infty} \phi(t) = 0$.*

Lemma 6 *[26] If $\xi = 0$ is a globally uniformly asymptotically stable equilibrium point of (26), then it is also a globally uniformly finite-time stable equilibrium point.*

Corollary 1 *[30] Given system (41), let $\mathcal{S} \subset \mathbb{R}$ be an open and connected set containing $s = 0$, assume that $\mathcal{F}$ is a regular function, essentially locally bounded, uniformly in t. Let $V : \mathcal{S} \times [0, \infty) \to \mathbb{R}$ be a locally Lipschitz, regular function such that*

$$W_1(s) \leq V(s, t) \leq W_2(s), \quad \forall t \geq 0, \quad \forall s \in \mathcal{S} \tag{44}$$

$$\dot{V}(s(t), t) \overset{a.e.}{\leq} -W(s(t)) \tag{45}$$

where W_1 and W_2 are continuous positive definite functions, and W is a continuous positive semi-definite function on $\mathcal{S}$. Let $r > 0$ and $k > 0$ such that $B_r \subset \mathcal{S}$ and $k < \min_{|s|=r} W_1(s)$ and $s(t)$ is a Filippov's solution of (41) with $s_0 \in \{s \in B_r : W_2(s) \leq k\}$, then $s(t)$ is bounded and $\lim_{t\to\infty} W(s(t)) = 0$.

Corollary 2 *[30] Given system (41), let $\mathcal{S} \subset \mathbb{R}$ be an open and connected set containing $s = 0$, assume that $\mathcal{F}$ is a regular function, essentially locally bounded, uniformly in t. Let $V : \mathcal{S} \times [0, \infty) \to \mathbb{R}$ be a locally Lipschitz, regular function such that*

$$W_1(s) \leq V(s, t) \leq W_2(s), \quad \forall t \geq 0, \quad \forall s \in \mathcal{S} \tag{46}$$

$$\dot{\tilde{V}}(s, t) \leq -W(s) \tag{47}$$

where W_1 and W_2 are continuous positive definite functions, and W is a continuous positive semi-definite function on $\mathcal{S}$. Let $r > 0$ and $k > 0$ such that $B_r \subset \mathcal{S}$ and $k < \min_{|s|=r} W_1(s)$, then all Filippov's solutions of (41) with $s_0 \in \{s \in B_r : W_2(s) \leq k\}$ are bounded and $\lim_{t\to\infty} W(s(t)) = 0$.

Now, we can to introduce the main convergence results, first proving that the sliding surface $s = 0$ is a region of attraction.

Theorem 1 *Given the sliding function (16) with (22), and the closed-loop system (41) with (15), then $\xi = 0$ is an unformly finite-time stable equilibrium point of (10). Moreover, the function*

$$V(s(\xi), \xi) = \begin{cases} \frac{1}{2} s^2(\xi) & 0 \leq t \leq \tau \\ \xi_2 \psi(\xi) + K_3 \sqrt{K_1 \xi_2^2 + (K_2 |\xi_2| \xi_2 + \xi_1 \underline{\alpha}) \psi(\xi)} & t > \tau \end{cases} \tag{48}$$

is a Lyapunov function for system (10) where

$$K_1 = \tfrac{1}{4} \left(1 + \tfrac{\underline{\underline{\alpha}}}{\overline{\alpha}}\right) , K_2 = \tfrac{1}{4} \left(1 - \tfrac{\underline{\underline{\alpha}}}{\overline{\alpha}}\right) , K_3 = \tfrac{2\sqrt{2K_1}}{1 - 2K_2} \ . \tag{49}$$

Proof: In order to prove that $s = 0$ is a region of attraction, separately consider the cases when $\zeta(t)$ is equal to (22a) and (22b).

Step 1 ($s(t) = \xi_1(t) - \xi_{1_0} - \xi_{2_0} t + \alpha \operatorname{sgn}(\sigma_0) \frac{t^2}{2}, \ 0 \leq t \leq \tau$): In this case, the LaSalle-Yoshizawa Theorem and Corollaries are used to provide boundedness and convergence of solutions. Consider temporarily that $s(0) = s_0 \neq 0$. Since the points of the curve obey the Newton's laws, $s(t) = \xi_2(t) - \xi_{2_0} + \alpha \operatorname{sgn}(\sigma_0) t$ holds. Moreover, relying on (27), with $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, one has $\dot{\xi}_2 \in -\alpha \psi(\xi)$. Compute now the time derivative of $s(t)$ in order to make the control explicitly appear. Then, the first-order nonlinear differential inclusion (41) may be written as follows

$$\dot{s} \in \alpha \operatorname{sgn}(\sigma_0) - \alpha \psi(\xi) \ . \tag{50}$$

Consider now the Lyapunov function candidate $V(s, t) = s^2/2$ which certainly satisfies (44) where $W_1, W_2 : \mathbb{R} \to \mathbb{R}^+$ are defined as $W_1(s) := \lambda_1 |s|^2$ and $W_2(s) := \lambda_2 |s|^2$, with $\lambda_1, \lambda_2 \in \mathbb{R}^+$ known positive constants. By virtue of Lemma 3, one has $\dot{V}(s(t), t) \overset{\text{a.e.}}{\in} \dot{\tilde{V}}(s(t), t)$ where

$$\dot{\tilde{V}}(s(t), t) := \bigcap_{\varsigma \in \partial V(s,t)} \varsigma' F \begin{bmatrix} \dot{s} \\ 1 \end{bmatrix} (s, t) \ . \tag{51}$$

Since it holds that

$$\dot{\tilde{V}}(s(t), t) \subset s F \left[\dot{s}\right] (s, t) \ , \tag{52}$$

one can equivalently write

$$\dot{\tilde{V}}(s(t), t) \subset s(\alpha \operatorname{sgn}(\sigma_0) - \alpha \mathrm{SGN}(s)) \ . \tag{53}$$

Bounding the previous expression, it becomes

$$\dot{\tilde{V}}(s(t), t) \leq -(\overline{\alpha} - \alpha)|s| \ . \tag{54}$$

Hence, (47) holds with $W(s) = (\overline{\alpha} - \alpha)|s|$ where $W : \mathcal{S} \to \mathbb{R}^+$. From (44) one has that $V(s, t) \in \mathcal{L}^\infty(\mathcal{S}, \mathbb{R})$, hence $s(t) \in \mathcal{L}^\infty(\mathcal{S}, \mathbb{R})$. Since the control (15) is bounded and $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\dot{s}(t) \in \mathcal{L}^\infty(\mathcal{S}, \mathbb{R})$, and $s(t)$ is uniformly continuous in $\mathcal{S}$. Let $r > 0$ such that there exists $B_r \subset \mathcal{S}$ and $k < \min_{|s|=r} \lambda_1 |s|^2$, invoking Corollary 2, one has that $W(s)$ tends to zero as t tends to infinity. In our case, this necessarily implies that

$$\lim_{t\to\infty} s(t) = 0 \ , \tag{55}$$

$\forall s_0 \in \{s \in B_r : W_2(s) \leq k\}$. Yet, by construction, $s_0 = 0$ so that $s(t) = 0$, $\forall 0 \leq t \leq \tau$.

Step 2 ($s(t) = \sigma(t), \ \tau < t < T$): Consider the case with $\zeta(t)$ equal to (22b) for $t > \tau$. Making reference to [26, Theorem 2], $V(\xi) = \xi_2 \psi(\xi) + K_3 \sqrt{K_1 \xi_2^2 + (K_2 |\xi_2| \xi_2 + \xi_1 \underline{\alpha}) \psi(\xi)}$

is such that $V(\xi) > 0$, $\forall\, \xi \neq 0$. Moreover, with $\alpha \in [\underline{\alpha},\, \overline{\alpha}]$, one has that

$$\frac{\partial V}{\partial t} \in \xi_2 \frac{\partial V}{\partial \xi_1} - \alpha\psi(\xi)\frac{\partial V}{\partial \xi_2} \tag{56}$$

$$= -\,\alpha\left(1 + \frac{K_2 K_3 |\xi_2|}{\sqrt{K_1\xi_2^2 + (K_2|\xi_2|\xi_2 + \xi_1\underline{\alpha})\psi(\xi)}}\right) + \\ + \frac{K_3\xi_2\psi(\xi)}{\sqrt{K_1\xi_2^2 + (K_2|\xi_2|\xi_2 + \xi_1\underline{\alpha})\psi(\xi)}}\left(\frac{\underline{\alpha}}{2} - \alpha K_1\right) \tag{57}$$

$$< \frac{K_3|\xi_2|\left(|\frac{\underline{\alpha}}{2} - \alpha K_1| - \alpha K_2\right)}{\sqrt{K_1\xi_2^2 + (K_2|\xi_2|\xi_2 + \xi_1\underline{\alpha})\psi(\xi)}} \tag{58}$$

which is less or equal to zero if $|\underline{\alpha}/2 - \alpha K_1| - \alpha K_2 \leq 0$ or equivalently

$$\frac{\underline{\alpha}}{2(K_1 + K_2)} \leq \alpha \leq \frac{\underline{\alpha}}{2(K_1 - K_2)} \tag{59}$$

that is $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$. This implies that $\xi = 0$ is an asymptotically stable equilibrium point of (10). Then, the thesis directly follows by virtue of Lemma 6. ■

As a consequence of Theorem 1, one can conclude that $s(t)$ is continuously kept at zero. Indeed, while from the first part of the proof we get $s(t) = 0$, $\forall\, 0 \leq t \leq \tau$, from τ onward, using the Filippov's solution, given $s(\tau) = \sigma(\tau) = 0$, the vector field $\dot{\xi} = [\xi_2,\, \alpha]'$, $\xi_2 < 0$ (or $\dot{\xi} = [\xi_2,\, -\alpha]'$, $\xi_2 > 0$), results always tangent to the switching curve (see e.g., [18]), i.e., $s(t) = \sigma(t) = 0$, $\forall\, \tau < t < T$. Hence, it follows that $s(t) = 0, \forall\, 0 \leq t < T$.

Theorem 2 *Given the plant* (1)*, with Assumptions 1–4 and the control law* (15) *with* (16)*, then $y(t) = 0$, $\forall\, t \geq T$, $\forall\, x_0 \in \mathcal{D}_0$, and $x = 0$ is an asymptotically stable equilibrium point of* (1) *in spite of the uncertainties.*

Proof: By virtue of Theorem 1, $s(t) = 0$ is a region of attraction for all $0 \leq t < T$, which directly implies that the equivalent dynamics is exactly that described in (27). Furthermore, $\xi = 0$ is a global uniformly finite-time stable equilibrium point of (27) with convergence time T given by (28). This result implies that $\xi_1(t) = y(t) = 0$, $\forall\, t \geq T$. On the basis of what previously said about the choice of the sliding function (16) with (22), $s(t)$ is in turn kept at zero $\forall\, t$ such that $t \geq T$. Since there exists the global diffeomorphism $\Phi(x)$ such that $x = \Phi^{-1}(z,\, \xi)$ and, by Assumption 3, $f_0(z, 0)$ is globally asymptotically stable, then, following the arguments of [2, Chapter 13], one can conclude that $\forall\, x_0 \in \mathcal{D}_0$ the origin $x = 0$ is an asymptotically stable equilibrium point of (1) in spite of the uncertainties. ■

VI. Illustrative Example

In this section, the proposed control strategy is assessed in simulation relying on the benchmark of a field-controlled dc motor with negligible shaft damping [2].

The plant (see Figure 2) is described by the following equations,

$$\begin{cases} \dot{x}_1(t) = -\frac{R_f}{L_f}x_1(t) + u(t) + d(t) \\ \dot{x}_2(t) = -\frac{R_a}{L_a}x_2(t) - \frac{c_1}{L_a}x_1(t)x_3(t) + \frac{v_a}{L_a} \\ \dot{x}_3(t) = \frac{c_2}{J}x_1(t)x_2(t) \end{cases} \tag{60}$$

where x_1, x_2 and x_3 are the field current, the armature current and the angular velocity, respectively. In order solve a speed control problem, letting $x_3^\star$ be a constant reference, the output is selected as the speed error $y = x_3 - x_3^\star$. The field circuit is described by the first equation with $u = v_f/L_f$, i_f, R_f and L_f being its input voltage, current, resistance and inductance. A random bounded disturbance d is also added to the input voltage. The armature circuit is instead captured by the second equation with v_a, i_a, R_a and L_a being the corresponding variables. Finally, the third one is the torque equation for the motor shaft with J being the rotor inertia. The term $c_1x_1x_3$ represents the back e.m.f. induced in the armature circuit, while $c_2x_1x_2$ is the torque generated by the interaction of the armature current and the field circuit flux. The used numerical data are $R_a = R_f = 1\,\Omega$, $L_f = 0.2\,\mathrm{H}$, $L_a = 2\times10^{-2}\,\mathrm{H}$, $v_a = 1\,\mathrm{V}$, while e.m.f. and torque parameters have nominal values $c_1 = \sqrt{2}\times10^{-2}\,\mathrm{V\,s\,A^{-1}\,rad^{-1}}$, $c_2 = \sqrt{2}\times10^{-2}\,\mathrm{N\,m\,A^{-2}}$, and $J = 1\times10^{-6}\,\mathrm{N\,m\,s^2\,rad^{-1}}$. The initial conditions are $x_0 = [0.5,\, 0.8,\, 1]'$, while $x_3^\star = 30\,\mathrm{rad\,s^{-1}}$.

Computing the time derivatives of the output, it is possible to verify that the system has relative degree 2 in the region $\mathcal{D}_0 = \{x \in \mathbb{R}^3 \,:\, x_2 \neq 0\}$. Moreover, given the output y and its derivatives, to define the zero dynamics, we restrict x to the set $\mathcal{Z} = \{x \in \mathcal{D}_0 \,:\, x_3 = x_3^\star, x_1 = 0\}$, and one has that $\dot{x}_2 = -(R_a/L_a)x_2 + v_a/L_a$, which has an asymptotically equilibrium point at v_a/R_a. Hence, in order to transform the system into the normal form, we need to find a function $\Psi(x)$ such that $\partial\Psi/\partial x_1 = 0$ and $\Phi(x) = [\Psi(x),\, x_3 - x_3^\star,\, (c_2/J)x_1x_2]'$ is the diffeomorphism on domain $\mathcal{D}_0$. Selecting $\Psi(x) = x_2 - v_a/R_a$ and $z = \Psi(x)$ fulfills the previous conditions and make $\Phi(x)$ transform the equilibrium point of the zero dynamics to the origin. Specifically, system (2) becomes

$$\begin{cases} \dot{z} = -\frac{R_a}{L_a}\left(z + J\frac{c_1}{c_2}\frac{(\xi_1 + x_3^\star)\xi_2}{R_az + v_a}\right) \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -\xi_2\left(\frac{R_f}{L_f} + \frac{R_a^2}{L_a(R_az + v_a)}\dot{z}\right) + \frac{c_2}{JR_a}(R_az + v_a)(d + u) \end{cases}$$

such that $\beta = 4\times10^4$, $\gamma = 1.1\times10^4$ and $\delta = 1.4\times10^4$. According to Assumption 2, we choose $\mu = 35.4$ so that $\underline{\alpha} = 36\times10^4$, $\overline{\alpha} = 54\times10^4$ with $\underline{T} = 0.0288\,\mathrm{s}$, $\overline{T} = 0.0126\,\mathrm{s}$, respectively. Now, we require to steer the output y to zero in $T = 0.02\,\mathrm{s}$. By using (40), the equivalent control gain to design the sliding surface (16) with (22) is $\alpha = 45.3\times10^4$.

Figure 3 shows the state trajectory in the state-plane with respect to time. The state is steered to zero following parabola arcs which are bounded by the minimum time trajectory with respect to the worst realization of the uncertainties and the trajectory obtained when the uncertainties favor the convergence, that is when $\underline{\alpha}$ and $\overline{\alpha}$ are used, respectively. Figure 4 further assesses the proposed control strategy, which allows one to steer the state of the perturbed double integrator to zero exactly in the desired predefined time $T = 0.02\,\mathrm{s}$. As a term of comparison, we have considered a second order sliding mode control designed according to [8] (dotted lines). In this case the transient term is not designed as an optimal reaching curve, but is selected as a smooth homogeneous function of the initial conditions. Finally, Figure 4 also shows the time evolution of the state x and of the zero dynamics $f_0(z,0) = -(R_a/L_a)z$, which, according to Assumption 3, has an asymptotically equilibrium point in zero associated with the equilibrium state $x_2 = v_a/R_a$, with $v_a/R_a = 1$.

VII. Conclusions

In this paper, predefined-time output stabilization via second order sliding modes is discussed. Starting from a general class of systems, the control problem of stabilizing a nonlinear uncertain plant is reformulated into the simpler problem of stabilizing the associated system in normal form through a discontinuous bounded input. A novel sliding surface is introduced, and predefined-time stability of the state trajectory of the normal form is proved. The expressions of the convergence time and the corresponding control gain are also provided.

References

[1] M. Athans and P. L. Falb, *Optimal Control*. New York: McGraw Hill, 1966.

[2] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 1996.

[3] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*. Prentice Hall, 2001.

[4] W. S. Levine, *The Control Systems Handbook*, 2nd ed. College Park, MD,USA: CRC Press, Inc., 2010.

[5] V. I. Utkin, *Sliding Modes in Control and Optimization*, ser. Communication and control engineering. Springer-Verlag, 1992.

[6] A. Ferrara, G. P. Incremona, and M. Cucuzzella, *Advanced and Optimization Based Sliding Mode Control: Theory and Applications*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2019.

[7] A. Levant, "Sliding order and sliding accuracy in sliding mode control," *International Journal of Control*, vol. 58, no. 6, pp. 1247–1263, 1993.

[8] A. Levant and L. Alelishvili, "Integral high-order sliding modes," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1278–1282, 2007.

[9] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant, "Higher-order sliding mode controllers and differentiators," in *Sliding Mode Control and Observation*, ser. Control Engineering. Springer New York, 2014, pp. 213–249.

[10] V. I. Utkin, J. Guldner, and J. Shi, *Sliding Model Control in Electromechanical Systems*. Taylor and Francis, 1999.

[11] G. Bartolini, W. Caputo, M. Cecchi, A. Ferrara, and L. Fridman, "Vibration damping in elastic robotic structures via sliding modes," *Journal of Robotic Systems*, vol. 14, no. 9, pp. 675–696, 1998.

[12] A. Ferrara and G. P. Incremona, "Design of an integral suboptimal second order sliding mode controller for the robust motion control of robot manipulators," *IEEE Transactions on Control Systems Technology*, vol. 23, no. 6, pp. 2316–2325, 2015.

[13] M. Cucuzzella, G. P. Incremona, and A. Ferrara, "Decentralized sliding mode control of islanded AC microgrids with arbitrary topology," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 8, pp. 6706–6713, 2017.

[14] A. Ferrara, G. P. Incremona, and E. Regolin, "Optimization-based adaptive sliding mode control with application to vehicle dynamics control," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 550–564, 2019.

[15] G. Bartolini, A. Ferrara, and E. Usai, "Output tracking control of uncertain nonlinear second-order systems," *Automatica*, vol. 33, no. 12, pp. 2203–2212, 1997.

[16] ——, "Chattering avoidance by second-order sliding mode control," *IEEE Transactions on Automatic Control*, vol. 43, no. 2, pp. 241–246, 1998.

[17] G. Bartolini, A. Ferrara, A. Levant, and E. Usai, "On second order sliding mode controllers," in *Variable Structure Systems, Sliding Mode and Nonlinear Control*, ser. Lecture Notes in Control and Information, K. D. Young and Ü. Özgüner, Eds. Springer-Verlag, 1999, pp. 329–350.

[18] G. P. Incremona, M. Rubagotti, and A. Ferrara, "Sliding mode control of constrained nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 2965–2972, 2017.

[19] E. Cruz-Zavala, J. A. Moreno, and L. M. Fridman, "Uniform robust exact differentiator," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2727–2733, 2011.

[20] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[21] J. D. Sánchez-Torres, E. N. Sanchez, and A. G. Loukianov, "Predefined-time stability of dynamical systems with sliding modes," in *Proc. American Control Conference*, Chicago, IL, USA, Jul. 2015, pp. 5842–5846.

[22] M. L. Corradini and A. Cristofaro, "Nonsingular terminal sliding-mode control of nonlinear planar systems with global fixed-time stability guarantees," *Automatica*, vol. 95, pp. 561–65, 2018.

[23] Y. Song, Y. Wang, and M. Krstic, "Time-varying feedback for stabilization in prescribed finite time," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 618–633, 2019.

[24] A. K. Pal, S. Kamal, S. K. Nagar, B. Bandyopadhyay, and L. Fridman, "Design of controllers with arbitrary convergence time," *Automatica*, vol. 112, p. 108710, 2020.

[25] F. Dinuzzo and A. Ferrara, "Higher order sliding mode controllers with optimal reaching," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2126–2136, 2009.

[26] ——, "Finite-time output stabilization with second order sliding modes," *Automatica*, vol. 45, no. 9, pp. 2169–2171, 2009.

[27] V. I. Utkin and J. Shi, "Integral sliding mode in systems operating under uncertainty conditions," in *Proc. 35th IEEE Conference on Decision and Control*, vol. 4, Kobe, Japan, Dec. 1996, pp. 4591–4596.

[28] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *International Journal of Control*, vol. 76, no. 9-10, pp. 924–941, 2003.

[29] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*, 1st ed. New York, NY, USA: John Wiley & Sons, Inc., 1995.

[30] N. Fischer, R. Kamalapurkar, and W. E. Dixon, "Lasalle-Yoshizawa corollaries for nonsmooth systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2333–2338, 2013.

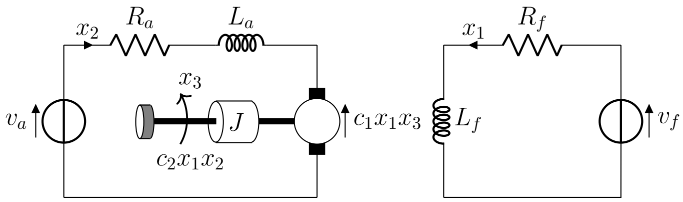

Figure 2. Field-controlled dc motor with negligible shaft damping

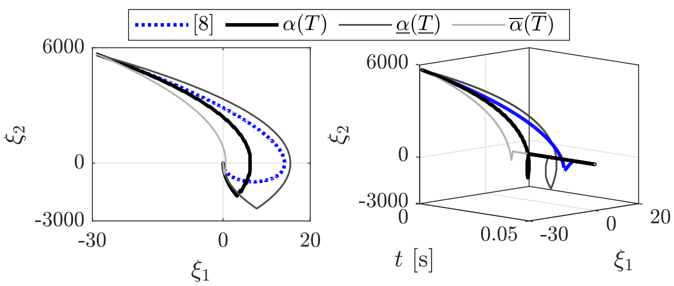

Figure 3. State portrait of the system in normal form

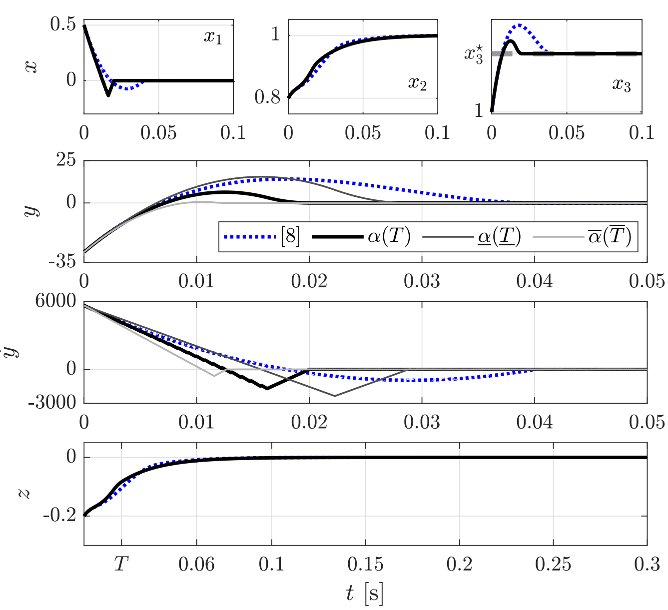

Figure 4. Time evolution of states x, output y, $\dot{y}$, and zero dynamics z