A Discrete-Time Optimization-Based Sliding Mode Control Law for Linear Systems with Input and State Constraints

Matteo Rubagotti, Gian Paolo Incremona and Antonella Ferrara

***Abstract*— This paper proposes a discrete-time sliding mode control (DSMC) strategy for linear (possibly multi-input) systems with additive bounded disturbances, which guarantees the satisfaction of input and state constraints. The control law is generated by solving a finite-horizon optimal control problem at each sampling instant, aimed at obtaining a control variable that is as close as possible to a reference DSMC law, but at the same time enforces constraint satisfaction for all admissible disturbance values. Contrary to previously-proposed control approaches merging DSMC and model predictive control, our proposal guarantees the satisfaction of all standard properties of DSMC, and in particular the finite-time convergence of the state into a pseudo sliding mode band.**

I. Introduction

Sliding mode control (SMC) is a well-known approach for controlling both linear and nonlinear systems, guaranteeing complete rejection of the disturbance terms that enter the systems dynamics through the same channel as the control variables (i.e., the so-called *matched disturbances*), once the system state belongs to the so-called *sliding manifold* [1]. The discontinuous SMC action is therefore aimed at enforcing convergence of the state onto the sliding manifold in finite time, and then at guaranteeing that the sliding manifold be an invariant set for the closed-loop system. If the sliding manifold is well designed and only matched disturbances are present, the whole state vector will converge to the origin asymptotically. A sliding mode controller is usually not designed to satisfy input and state constraints, for which instead ad-hoc solutions have been proposed in [2], [3].

When implementing SMC laws in practice, an infinite-frequency switching of the control variable cannot be obtained, and the direct discretization of a discontinuous control law can lead to a non-negligible amount of *discretization chattering*, which is the high-frequency oscillation of the state around the sliding manifold due to the discretization of the control action [1, Chap. 9]. In order to solve this problem, different approaches have been proposed to define SMC strategies directly defined in discrete time, referred to as discrete-time SMC, or DSMC [1], [4]–[9]. When applying a DSMC law, the exact finite-time convergence of the state onto the sliding manifold cannot be achieved, as the disturbance terms exert an a-priori unknown action on the system dynamics during the sampling interval. Therefore, a DSMC law would only guarantee the convergence to a *pseudo-sliding mode band* (PSMB), which includes all the states within a finite, a-priori determined, distance from the sliding manifold.

Different ideas have been proposed in the literature to merge DSMC laws with model predictive control (MPC), to improve performance or to provide constraint-satisfaction properties that cannot be guaranteed by classical DSMC implementations. Earlier works consider the possibility of merging DSMC and generalized predictive control to obtain explicit control laws for linear systems [10], [11]. In [12], the reaching law defined in [6] is used to define a sliding mode trajectory reference for an MPC controller. In [13], an MPC law is formulated for linear systems, by defining the MPC cost function as the distance of the state from the sliding manifold, which leads to the guarantee of asymptotic convergence of the sliding variable to a PSMB. In [14], an MPC strategy is proposed for linear and nonlinear systems, which uses the sliding manifold as terminal constraint. Other control schemes, merging an SMC block and an MPC block within an overall feedback scheme, have been proposed in [15]–[18]. Among these, a particularly relevant solution is the use of discrete-time *integral SMC*, for which analogous results in continuous time are reported in [19], [20]. Applications of these combined schemes to nano-positioning systems, to microgrids, and to three-phase AC/DC converters, are reported in [21]–[23], respectively.

In this paper, a DSMC law for linear systems with additive disturbances is defined for (in general, multi-input) systems with linear inequality constraints on both inputs and states. Considering the fact that the perfect compensation of matched disturbances cannot be obtained for a DSMC law, the simultaneous presence of both matched and unmatched disturbances is directly considered. The control law is not based on the combination of a DSMC and an MPC block, but rather on the solution of a finite-horizon *optimal control problem* (OCP), which has the same structure of an MPC law for linear systems with quadratic cost function and polyhedral constraints, and is formulated as a quadratic programming (QP) problem. The contribution of this work consists in defining a receding-horizon control law that is indeed still a DSMC law: contrary to the above-cited works, exact finite-time convergence of the state onto the sliding manifold is guaranteed in case no disturbance terms are present, and finite-time convergence into an a-priori defined PSMB is guaranteed when disturbance terms are present, while satisfying all

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imposed constraints.

The paper is organized as follows: Section II introduces the notation used throughout the paper. Section III introduces the considered control problem, which is then solved in Section IV. An illustrative simulation example is shown in Section V, and conclusions are drawn in Section VI.

II. Notation

Let $\mathbb{N}_{\geq 0}$ denote the set of non-negative integers. Given two integers values $a_i \leq a_f$, let $\mathbb{N}_{[a_i,a_f]} \triangleq \{a_i, a_{i+1}+1, ..., a_f\}$, while $\mathbb{N}_{a_f} \triangleq \{0, 1, ..., a_f\}$. For a given set $\mathcal{A} \subseteq \mathbb{R}^n$, its interior is indicated as $\text{int}(\mathcal{A})$. Given a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes its Euclidean norm. Given a constant matrix $F \in \mathbb{R}^{m\times n}$ and a set $\mathcal{A} \subseteq \mathbb{R}^n$, we define $F\mathcal{A} \triangleq \{y \in \mathbb{R}^m \ : \ y = Fx, \ x \in \mathcal{A}\}$. The Minkowski sum of two sets $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}^n$ is defined as $\mathcal{A}_1 \oplus \mathcal{A}_2 \triangleq \{x + y : x \in \mathcal{A}_1, y \in \mathcal{A}_2\}$, while their Pontryagin difference is $\mathcal{A}_1 \ominus \mathcal{A}_2 \triangleq \{x \in \mathbb{R}^n : x + y \in \mathcal{A}_1, \ \forall y \in \mathcal{A}_2\}$. A *polyhedron* in $\mathbb{R}^n$ is defined as the set given by the intersection of a (in this paper, finite) number of half-spaces in $\mathbb{R}^n$. It is assumed that a polyhedron includes its boundaries, and is therefore always a closed set. A *polytope* is instead defined as a bounded polyhedron. For a given polytope $\mathcal{P} \in \mathbb{R}^n$, the set of its vertices $v_i \in \mathbb{R}^n$ is indicated as $\text{vert}(\mathcal{P})$. For simplicity of notation, we include the discrete-time index t in a variable of a dynamical system (e.g., the state vector x_t) when this dependence is relevant in the current expression, and we omit the time index (e.g., x) when we refer to the same variable without specific reference to time.

III. DSMC with polyhedral sets

In this section, the classical formulation of the DSMC regulation problem for linear time-invariant (LTI) systems (see, e.g., [1, Chap. 9]) is revisited by introducing explicit polytopic bounds on the control and disturbance terms, and, after that, polyhedral state constraints.

A. Unconstrained system with bounded disturbances

Consider the discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t + Ew_t \tag{1}$$

where t is the discrete-time index, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ (with $m \leq n$) is the control vector, (A, B) is a reachable pair, and $w \in \mathbb{R}^p$ is an a-priori unknown but bounded disturbance term. The disturbance term satisfies

$$w_t \in \mathcal{W}, \ t \in \mathbb{N}_{\geq 0} \tag{2}$$

where $\mathcal{W}$ is a polytope. It is assumed that $\mathcal{W}$ includes the origin, i.e., $w_t = 0$ is a possible disturbance realization. The sliding variable $s \in \mathbb{R}^m$ is defined as

$$s_t = Cx_t \tag{3}$$

with $C \in \mathbb{R}^{m\times n}$ being defined so that $s_t = 0$ for all $t \geq \bar{t}$ implies that $\lim_{t\to+\infty} x_t = 0$. The set

$$\mathcal{S} \triangleq \{x \in \mathbb{R}^n \ : \ s = Cx = 0\}, \tag{4}$$

which, assuming C has rank equal to m, is an $(n - m)$-dimensional subspace of $\mathbb{R}^n$, is referred to as *sliding manifold*. The ideal sliding motion, which takes place when s becomes and remains equal to zero, is referred to as *discrete-time sliding mode* [7]. By following one of the classical approaches to discrete-time sliding mode proposed in [1], [7], the control action u_t that solves the problem of regulation to the origin is defined so that $s_{t+1} = 0$ in case $w_t = 0$. One would therefore impose

$$s_{t+1} = Cx_{t+1} = C(Ax_t + Bu_t) = 0 \tag{5}$$

leading to the DSMC law

$$u_t^{\text{sm}} = Kx_t \tag{6}$$

with $K \triangleq -(CB)^{-1}CA$, and in turn to the closed-loop dynamics

$$x_{t+1} = Mx_t + Ew_t \tag{7}$$

where $M \triangleq A - B(CB)^{-1}CA$.

Remark 1: In order to achieve asymptotic stability of the nominal closed-loop system (i.e., when $w_t = 0$ for all $t \in \mathbb{N}_{\geq 0}$), C must be defined such that CB is invertible, and M has all eigenvalues inside the unit circle. In particular, given the fact that, for the nominal case, $s_{t+1} = CMx_t = 0$ for all $x_t \in \mathbb{R}^n$, and C has rank equal to m, then the rows of C constitute a basis for the left null space of M, which implies that M has m zero eigenvalues: the degrees of freedom of the designer will only influence the position of the other $n - m$ eigenvalues. A borderline case is obtained when $m = n$, for which $s \in \mathbb{R}^n$, leading to M being a zero matrix and $\mathcal{S} = \{0\}$. □

When the control law in (6) is applied, thus leading to the closed-loop dynamics (7), an ideal sliding motion cannot be attained, however a *pseudo-sliding motion* is typically achieved, i.e., $\|s_t\| \leq \varepsilon$ for a finite $\varepsilon \in \mathbb{R}_{\geq 0}$. The set

$$\mathcal{B} \triangleq \{x \in \mathbb{R}^n \ : \ \|Cx\| \leq \varepsilon\} \tag{8}$$

is the PSMB mentioned in Section I. A tight estimate of the value of ε is important to determine the effectiveness of the control law.

B. Input and state constraints

If constraints are introduced on the control input as

$$u_t \in \mathcal{U}, \ t \in \mathbb{N}_{\geq 0}, \tag{9}$$

where $\mathcal{U}$ is a polytope containing the origin in its interior, then (5) cannot be enforced for all $x_t \in \mathbb{R}^n$. We assume that an additional requirement is present, i.e., that of satisfying the set of state constraints

$$x_t \in \mathcal{X}, t \in \mathbb{N}_{\geq 0}, \tag{10}$$

$\mathcal{X}$ being a polyhedron (assumed to include the origin in its interior).

Remark 2: In case $\mathcal{U}$ is defined as a set of *box constraints*, i.e., a set of component-wise upper and lower bounds on each component of u_t as defined in (6), a simple solution to satisfy (9) is to impose a component-wise saturation of the control variable. Although this can guarantee, under suitable assumptions, that finite-time convergence to the PSMB is eventually achieved even if (5) is not always satisfied (see, e.g., [7]) by enforcing (9) at the same time, nothing can be said about the satisfaction of the state constraints (10). □

We might wonder if there exists a set of initial conditions $\mathcal{X}_{\rm sm}$ when it would be still possible to directly apply the control law (6) guaranteeing that both input and state constraints will never be violated during all subsequent time instants. More precisely the set,

$$\mathcal{X}_{\rm sm} \triangleq \{x_0 \in \mathbb{R}^n \ : \ x_{t+1} = Mx_t + Ew_t \in \mathcal{X} \\ \wedge \ Kx_t \in \mathcal{U}, \ \forall w_t \in \mathcal{W}, \ \forall t \in \mathbb{N}_{\geq 0}\} \quad (11)$$

can be obtained via linear programming by applying [24, Alg. 6.1]. $\mathcal{X}_{\rm sm}$ might be empty, for example if the state constraints cannot be enforced due to a relatively large disturbance term and a relatively small available control amplitude.

In order to be able to verify if an explicit expression $\mathcal{X}_{\rm sm}$ can be obtained, we begin by calculating the set of states reachable in η time steps by system (7), assuming zero initial state and any sequence of realizations of $w_t \in \mathcal{W}$ [25]. The set is defined by the following Minkowski sum:

$$\mathcal{R}_\eta = \bigoplus_{i=0}^{\eta} M^i E \mathcal{W}. \quad (12)$$

The set $\mathcal{R}_\infty \triangleq \lim_{\eta\to\infty} \mathcal{R}_\eta$ is the so-called *minimal RPI (robust positively invariant) set* for system (7). It can be shown that, for system (7), $\mathcal{R}_\infty$ is a polytope [25, Prop. 6.9]. Its exact computation is not always possible, however very efficient methods are available to obtain a very tight over-approximation [26], referred to in the following as $\hat{\mathcal{R}}_\infty$. Relying on the results of [24, Thm 6.2 and 6.3], and assuming that the origin belongs both to $\mathrm{int}(\mathcal{X} \ominus \mathcal{R}_\infty)$ and to $\mathrm{int}(\mathcal{U} \ominus K\mathcal{R}_\infty)$, then $\mathcal{X}_{\rm sm}$ can be obtained exactly with a finite number of iterations of [24, Alg. 6.1]. The over-approximation $\hat{\mathcal{R}}_\infty$ can be used in case $\mathcal{R}_\infty$ cannot be computed exactly.

Of course, when state constraints are present, one could obtain a domain of attraction larger than $\mathcal{X}_{\rm sm}$ by devising a classical strategy for computing u_t, as mentioned in Remark 2, also when $x_t \notin \mathcal{X}_{\rm sm}$. However, as already mentioned in Remark 2, the direct application of such methods cannot guarantee that the state constraints will not be violated.

The objective of this work is to modify the original SMC law (6) so as to impose the satisfaction of input and state constraints for all feasible realizations of the disturbance term, at the same time maintaining the properties of DSMC control laws, i.e.,:

- There exists a finite time index $t_1 \in \mathbb{N}_{\geq 0}$ after which the control law defined in the remainder of the paper coincides with $u_t^{\rm sm}$: this is typical of most DSMC approaches, in which however only input saturation constraints are usually taken into account.
- Pseudo-sliding is achieved, and an explicit formula will be provided to obtain the exact value of ε. The state then converges asymptotically to $\mathcal{R}_\infty$.
- In case $w_t = 0$ starting from an arbitrary time instant $t_2 \in \mathbb{N}_{\geq 0}$, then an ideal sliding motion is achieved, and the state converges asymptotically to the origin.

IV. Constrained control law

A. OCP formulation

In order to satisfy the imposed state constraints while enlarging the domain of attraction of the origin of the closed-loop system beyond $\mathcal{X}_{\rm sm}$, the idea is to predict the system trajectory within a fixed time horizon, and devise a control strategy aimed at *imitating* the original discrete-time sliding mode strategy (6), attempting at the same time to avoid the violation of the state constraints. This result can be obtained by formulating a discrete-time OCP, which is solved at each sampling instant, in a receding-horizon fashion. At each time instant t, a sequence of N future values of the term $c^* \in \mathbb{R}^m$ (which represents the difference between the actual control law and $u_t^{\rm sm}$), namely $\mathbf{c}^* \triangleq \begin{bmatrix} c_0^* & c_1^* & \dots & c_{N-1}^* \end{bmatrix}$, is obtained as solution of the above-mentioned OCP, and then only the first element is used to define the control law. The OCP is then solved again at time $t+1$, after obtaining the new state measurement x_{t+1}. In order to formulate the OCP, first notice that a generic control sequence $\mathbf{c}$ (not necessarily coinciding with the optimal one $\mathbf{c}^*$), the element of which are indicated as $c_k, \ k \in \mathbb{N}_{[0,N-1]}$, would imply a time evolution of the state along the prediction horizon given by $x_{k+1} = Mx_k + Bc_k$. A collection of "tightened" sets $\mathcal{X}_k \triangleq \mathcal{X} \ominus \mathcal{R}_k$ and $\mathcal{U}_k \triangleq \mathcal{U} \ominus K\mathcal{R}_k$ is defined, with the aim of accounting for the presence of the disturbance term w_t without explicitly considering its possible realizations along the prediction horizon. The OCP is defined as follows:

$$\mathbf{c}^* = \arg\min_{\mathbf{c}} \sum_{k=0}^{N-1} c_k^T \Psi c_k \quad (13a)$$

$$\text{subj. to } x_{k+1} = Mx_k + Bc_k, \ k \in \mathbb{N}_{[0,N-1]} \quad (13b)$$

$$Kx_k + c_k \in \mathcal{U}_k, \ k \in \mathbb{N}_{[0,N-1]} \quad (13c)$$

$$x_k \in \mathcal{X}_k, \ k \in \mathbb{N}_{[0,N-1]} \quad (13d)$$

$$x_N \in \mathcal{X}_{\rm sm} \ominus \mathcal{R}_N \quad (13e)$$

where $\Psi \in \mathbb{R}^m$ in (13a) is a symmetric positive-definite matrix. This term imposes the *imitation* of the DSMC law (6): its global (unconstrained) minimum is achieved if one can set all $c_k = 0$, which is equivalent to applying (6). The OCP (13) will lead to using (6) whenever this does not imply a constraint violation; when this is not possible, it will lead to a control law as close as possible to (6) (according to the parameters set in Ψ), which will ensure constraint satisfaction. The set of inequality constraints (13b) imposes the nominal system dynamics during the prediction. The set of inequality constraints (13c) and (13d) ensures the satisfaction of input and state constraints for any feasible realization of w_t along the prediction horizon. Finally, (13e) enforces the state at the end of the prediction horizon to be inside $\mathcal{X}_{\rm sm}$ for any feasible realization of w_t.

Once the optimal sequence $\mathbf{c}^*$ is determined, the control law applied to the system at time t is

$$u_t = u_t^{\rm sm} + c_0^*(t) = Kx_t + c_0^*(t) \quad (14)$$

where the t in brackets is used to stress the fact that, to the original sliding mode control law $u_t^{\rm sm}$, a term is added equal to the first element of the sequence $\mathbf{c}^*$ calculated at time t.

B. Theoretical results

Theorem 1: Consider system (1) with (A, B) reachable pair and the disturbance term satisfying (2), with $\mathcal{W}$ including the origin. A stabilizing DSMC law is defined as in (6), and input and state constraints are defined as in (9) and (10), respectively, both including the origin in their interior. The set $\mathcal{X}_{\rm sm}$, defined as in (11), can be calculated as it is assumed that the origin belongs both to the interior of $\mathcal{X} \ominus \mathcal{R}_\infty$ and of $\mathcal{U} \ominus K\mathcal{R}_\infty$. If the control law is defined as in (14), there exists an RPI set of initial conditions $\mathcal{X}_0$, with $\mathcal{X}_{\rm sm} \subseteq \mathcal{X}_0 \subseteq \mathcal{X}$, such that, for all feasible realizations of the disturbance term, both input and state constraints are satisfied, and the state converges asymptotically to $\mathcal{R}_\infty$.

Proof: The proof of this theorem is based on the results in [27]. To give a sketch of the proof, first notice that, if $\mathcal{X}_0 \subseteq \mathcal{X}$ is the set of states where the OCP (13) is feasible, then surely $\mathcal{X}_{\rm sm} \subseteq \mathcal{X}_0$, as $\mathcal{X}_{\rm sm}$ contains all values of x for which (13) is feasible with all $c_k^* = 0$, $k = \mathbb{N}_{[0,N-1]}$. By definition of all tightened sets, one can verify that, for any sequence of inputs and states given by a feasible solution of (13), any corresponding evolution of inputs and states of system (1) satisfies the actual constraints (9) and (10). Furthermore, the increasing of the tightening along the prediction horizon, together with the definition of $\mathcal{X}_{\rm sm}$, guarantees that, if a solution of (13) exists at time t, and we apply u_t as defined in (14), then a solution of (13) will still exist at time $t + 1$, for any feasible realization of w_t. This property is usually referred to as *recursive feasibility*, and is proven for a general case including the one here considered, in [27, Lemma 7]. Finally, one has to prove that the state converges to $\mathcal{R}_\infty$, which is obtained as a particular case of [27, Theorem 8], by verifying that all the required assumptions are satisfied in our formulation. ■

Corollary 1: If, in addition to the assumptions of Theorem 1, there exists $t_2 \in \mathbb{R}_{\geq 0}$ such that $w_t = 0$ for all $t \geq t_2$, then the state converges asymptotically to the origin.

Proof: This result is proven by noticing that $x(t_2) \in \mathcal{X}_0$, and $\mathcal{R}_\infty$ coincides (by definition) with the origin when $\mathcal{W} = \{0\}$. ■

The result in [27] refers to a generic receding-horizon control law aimed at "imitating" a given linear control law $u_t = Kx_t$, and is applicable to the formulation of our results, as it turns out that the DSMC law in this setting is actually linear. However, this does not provide anything about the properties of the control law as a DSMC law. To this aim, additional results specific for our case can be formulated, as follows.

Theorem 2: Assume that all hypoteses required in Theorem 1 hold, and, in addition, assume that $\hat{\mathcal{R}}_\infty$ is included in $\mathrm{int}(\mathcal{X}_{\rm sm})$. Then, the following holds:

i. For all initial conditions $x_0 \in \mathcal{X}_0$, there exists a finite time $t_1 \in \mathbb{R}_{\geq 0}$ such that $u_t = u_t^{\rm sm}$, for all $t \geq t_1$.
ii. A pseudo-sliding motion (as defined in Section III, i.e., $\|s_t\| \leq \varepsilon$), implying the finite-time convergence to a PSMB, is achieved for all $t > t_1$, with
$$\varepsilon = \max\left\{\|\mathrm{vert}(CE\mathcal{W})\|\right\}. \tag{15}$$
iii. An ideal sliding motion is achieved if there exists $t_2 \in \mathbb{R}_{\geq 0}$ such that $w_t = 0$ for all $t \geq t_2$.

Proof:

i. The fact that $\hat{\mathcal{R}}_\infty \subseteq \mathcal{X}_{\rm sm}$ is guaranteed by definition of $\mathcal{X}_{\rm sm}$. The strict inclusion of $\hat{\mathcal{R}}_\infty$ into $\mathcal{X}_{\rm sm}$, together with the fact that x_t asymptotically converges to $\hat{\mathcal{R}}_\infty$ for all $x_0 \in \mathcal{X}_0$ (from Theorem 1), imply that there exists a finite time $t_1 \in \mathbb{R}_{\geq 0}$ such that $u_{t_1} = u_{t_1}^{\rm sm}$. But $\mathcal{X}_{\rm sm}$ is an RPI set for the closed-loop system (7), which implies that $u_t = u_t^{\rm sm}$, for all $t \geq t_1$.
ii. Considering that $u_t^{\rm sm}$ is defined such that $Cx_{t+1} = CMx_t = 0$, the dynamics of s_t for the closed-loop system for $t \geq t_1$ is $s_{t+1} = Cx_{t+1} = CMx_t + CEw_t = CEw_t$, which does not depend on the past values of s_t, as expected. Considering all possible realizations of $w_t \in \mathcal{W}$, one can conclude that $s_t \in CE\mathcal{W}$ for all $t > t_1$. Since the pseudo-sliding condition is defined using the 2-norm, one can see immediately that the maximum value achievable by $\|s_t\|$ for $t > t_1$ is $\varepsilon = \max\left\{\|\mathrm{vert}(CE\mathcal{W})\|\right\}$.
iii. Up to time t_2, when the disturbance term is still present, Theorem 1 guarantees that $x_t \in \mathcal{X}_0$ for all $t \in \mathbb{N}_{[0,t_2]}$, and in particular $x_{t_2} \in \mathcal{X}_0$. The convergence of x_t to $\mathcal{X}_{\rm sm}$ in finite time is still guaranteed by [i], which implies that [ii] also holds, with $\mathcal{W} = \{0\}$. As a consequence, $\varepsilon = 0$, and therefore $s_t = 0$ for all $t > \max(t_1, t_2)$.

■

Theorem 2 proves that, in spite of the unconventional formulation of the DSMC law in a receding-horizon fashion, the required properties of DSMC laws, as listed at the end in Section III, are preserved.

V. Simulation Example

This section reports a numerical example related to the benchmark dynamical system in [6]. Consider a second-order system in form (1), with

$$A = \begin{bmatrix} 1.2 & 0.1 \\ 0 & 0.6 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{16}$$

The state, control and disturbance constraint sets are defined as

$$\mathcal{X} \triangleq \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times \left[-1, 1\right], \tag{17}$$
$$\mathcal{U} \triangleq \left[-1, 1\right], \tag{18}$$
$$\mathcal{W} \triangleq \left[-0.01, 0.01\right] \times \left[-0.02, 0.02\right], \tag{19}$$

while the time evolution of the disturbance term is

$$w_t = \begin{bmatrix} 0.01\cos\left(\frac{t}{4\pi}\right) \\ 0.02\sin\left(\frac{t}{4\pi}\right) \end{bmatrix}. \tag{20}$$

The sliding variable is chosen as in (3) with $C = \begin{bmatrix} 5 & 1 \end{bmatrix}$ such that the DSMC gain results being $K = \begin{bmatrix} -6 & -1.1 \end{bmatrix}$, with matrix M in (7) having all eigenvalues inside the unit circle (in particular, at 0 and 0.7). The initial conditions are $x_0 = \begin{bmatrix} -0.25 & 0 \end{bmatrix}$, while the sampling time is assumed to be equal to $0.1\,\mathrm{s}$.

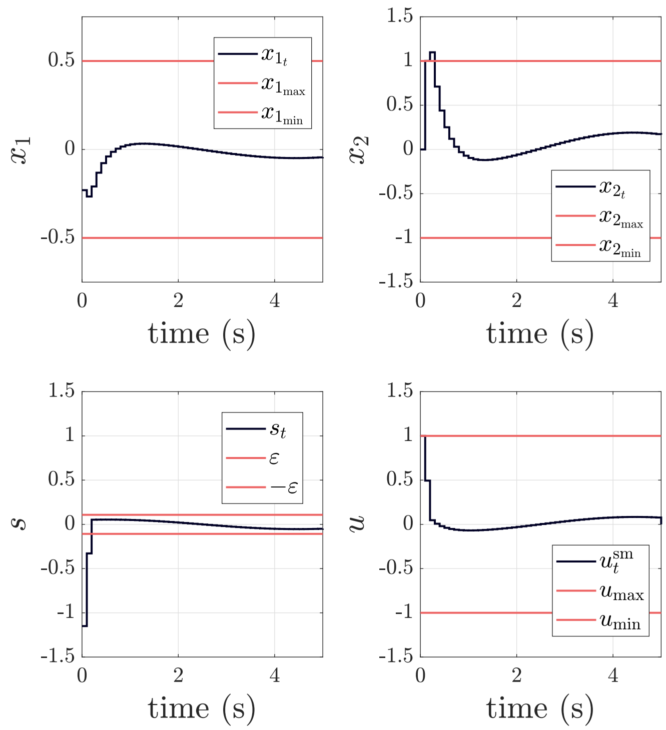

Fig. 1. DSMC with saturated input, from top left, clockwise. Time evolution of the state x_{1_t} (solid black line) and bounds $x_{1_{\min}} = -0.5$ and $x_{1_{\max}} = 0.5$ (solid red line). Time evolution of the state x_{2_t} (solid black line) and bounds $x_{2_{\min}} = -1$ and $x_{2_{\max}} = 1$ (solid red line). Time evolution of the input $u_t = u_t^{\mathrm{sm}}$ (solid black line) and bounds $u_{\min} = -1$ and $u_{\max} = 1$ (solid red line). Time evolution of the sliding variable s_t (solid black line) and bound $\varepsilon = 0.107$ (solid red line)

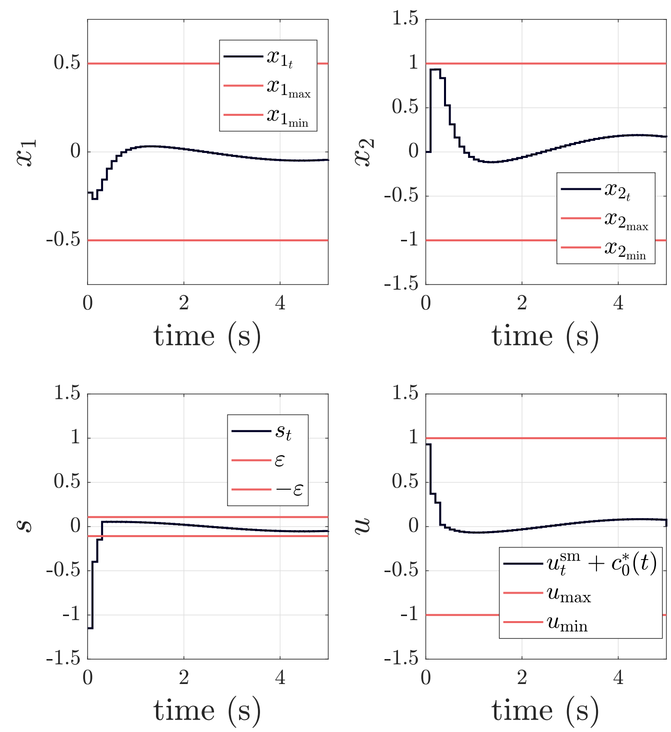

Fig. 3. Constrained DSMC, from top left, clockwise. Time evolution of the state x_{1_t} (solid black line) and bounds $x_{1_{\min}} = -0.5$ and $x_{1_{\max}} = 0.5$ (solid red line). Time evolution of the state x_{2_t} (solid black line) and bounds $x_{2_{\min}} = -1$ and $x_{2_{\max}} = 1$ (solid red line). Time evolution of the input $u_t = u_t^{\mathrm{sm}} + c_0^*(t)$ (solid black line) and bounds $u_{\min} = -1$ and $u_{\max} = 1$ (solid red line). Time evolution of the sliding variable s_t (solid black line) and bound $\varepsilon = 0.107$ (solid red line)

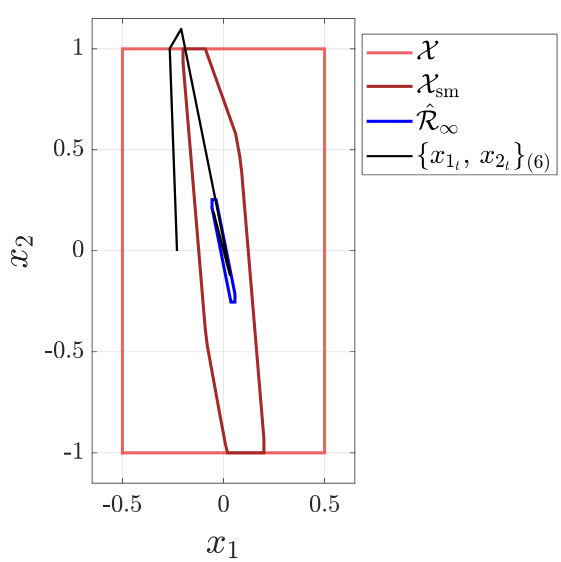

Fig. 2. DSMC with saturated input, state-space portrait. Box constraint set $\mathcal{X}$ (solid red line), set $\mathcal{X}_{\mathrm{sm}}$ (solid dark-red line), minimal RPI set $\hat{\mathcal{R}}_\infty$ (solid blue line), state trajectory when the DSMC law (6) with saturated input is used (black line)

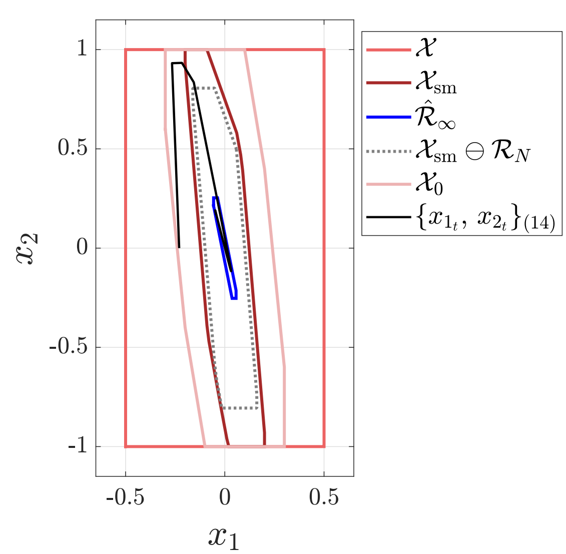

Fig. 4. Constrained DSMC, state-space portrait. Box constraint set $\mathcal{X}$ (solid red line), RPI set $\mathcal{X}_0$ (solid light-red line), set $\mathcal{X}_{\mathrm{sm}}$ (solid dark-red line), terminal set $\mathcal{X}_{\mathrm{sm}} \ominus \mathcal{R}_N$ (dotted gray line), minimal RPI set $\hat{\mathcal{R}}_\infty$ (solid blue line), state trajectory with the constrained DSMC law (14) (black line)

The sets $\mathcal{R}_t$, $t \in \mathbb{N}_{[1,N]}$, and $\hat{\mathcal{R}}_\infty$ are computed as described in Section III-B using the Multi-Parametric Toolbox 3.0 (MPT3) for MATLAB, presented in [28]. Furthermore, MPT3 also allows to generate the tightened constraints of the OCP (13) setup via YALMIP [29]. In order to solve the OCP, the prediction horizon is set equal to $N = 5$, while the input weight is $\Psi = 1$. The sets $\mathcal{X}_{\rm sm}$ and $\mathcal{X}_0$ are also numerically computed with MATLAB, according to their definitions.

In order to verify the effectiveness of the proposed approach, the DSMC law with saturated input is considered as a term of comparison. The initial conditions are the same for both strategies and such that $x_0 \in \mathcal{X}_0$, which implies that the control variable can be obtained in both cases. Figure 1 shows the behavior of the system controlled via the DSMC law (6) with saturated input. One can note that, while the first state x_{1_t} is always inside the constraints, the state x_{2_t} violates the upper bound in correspondence of the saturation of the control input u_t. As for the sliding variable, it converges to the PSMB $\mathcal{B}$ (defined as in (8) with width $\varepsilon = 0.107$ and computed according to (15)) and remains confined therein for all subsequent time instants (pseudo-sliding mode). Figure 2 shows the state-space portrait. The state trajectory violates the box constraint set $\mathcal{X}$, before reaching the invariant set $\mathcal{X}_{\rm sm}$, and then converges to the minimal RPI set $\hat{\mathcal{R}}_\infty$.

On the other hand, Figure 3 illustrates how the system evolves when it is controlled via the proposed constrained DSMC law (14). Differently from the DSMC law with saturated input, one can note that both states x_{1_t} and x_{2_t} always satisfy the constraints: this is obtained by generating a control variable u_t that is significantly different from the saturated case when the states do not belong to the set $\mathcal{X}_{\rm sm}$. As for the sliding variable, it always converges to the PSMB and then remains confined therein. Figure 4 instead shows the corresponding state-space portrait. The state trajectory in this case never violates the box constraint set $\mathcal{X}$. The initial conditions are inside the OCP feasible set $\mathcal{X}_0$, then the states enter $\mathcal{X}_{\rm sm}$, eventually converging to the minimal RPI set $\hat{\mathcal{R}}_\infty$.

VI. Conclusions

A DSMC control law guaranteeing the satisfaction of input and state constraints has been formulated, and its theoretical properties proven, also by relying on general results on robust MPC for linear systems. The simulation results show the advantages of the proposed strategy as compared to classical DSMC laws, when state constraints are present.

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