OPEN QUANTUM RANDOM WALKS, QUANTUM MARKOV CHAINS AND RECURRENCE

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Abstract. In the present paper, we construct QMCs associated with Open Quantum Random Walks such that the transition operator of the chain is defined by OQRW and the restriction of QMC to the commutative subalgebra coincides with the distribution $\mathbb{P}_\rho$ of OQRW. This sheds new light on some properties of the measure $\mathbb{P}_\rho$. As an example, we simply mention that the measure can be considered as a distribution of some functions of certain Markov process. Furthermore, we study several properties of QMC and associated measure. A new notion of φ-recurrence of QMC is studied, and it is established relations between the defined recurrence and the existing ones.

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1. Introduction

The study of asymptotic behavior of trace-preserving completely positive maps, also known as quantum channels, is a fundamental topic in quantum information theory, see for instance [12, 13, 24, 25, 26, 31, 32]. More recently, an important class of quantum channels, namely Open Quantum Random Walks (OQRWs) has been introduced by S. Attal et al. [7] and its long term behavior studied [8, 22, 23, 34]. These extensions of Markov chains, where the process retains some amount of memory which is encoded by a quantum state.

Let us recall some necessary information about OQRW. Let $\mathcal{K}$ denote a separable Hilbert space and let $\{|i\rangle\}_{i\in\Lambda}$ be its orthonormal basis indexed by the vertices of some graph Λ (here the set Λ of vertices might be finite or countable). Let $\mathcal{H}$ be another Hilbert space, which will describe the degrees of freedom given at each point of Λ. Then we will consider the space $\mathcal{H}\otimes\mathcal{K}$. For each pair i, j one associates a bounded linear operator B^i_j on $\mathcal{H}$. This operator describes the effect of passing from $|j\rangle$ to $|i\rangle$. We will assume that for each j, one has

$$\sum_i B^{i*}_j B^i_j = \mathbb{1}, \tag{1.1}$$

where, if infinite, such series is strongly convergent. This constraint means: the sum of all the effects leaving site j is $\mathbb{1}$. The operators B^i_j act on $\mathcal{H}$ only, we dilate them as operators on $\mathcal{H}\otimes\mathcal{K}$ by putting

$$M^i_j = B^i_j \otimes |i\rangle\langle j|\,.$$

The operator M^i_j encodes exactly the idea that while passing from $|j\rangle$ to $|i\rangle$ on the lattice, the effect is the operator B^i_j on $\mathcal{H}$.

According to [7] one has

$$\sum_{i,j} M^{i^*}_j M^i_j = \mathbb{1}. \tag{1.2}$$

Therefore, the operators $(M^i_j)_{i,j}$ define a completely positive mapping

$$\mathcal{M}(\rho) = \sum_i\sum_j M^i_j\,\rho\,M^{i^*}_j \tag{1.3}$$

on $\mathcal{H}\otimes\mathcal{K}$.

In what follows, we consider density matrices on $\mathcal{H}\otimes\mathcal{K}$ which take the form

$$\rho=\sum_i \rho_i\otimes|i\rangle\langle i|, \tag{1.4}$$

assuming that $\sum_i \mathrm{Tr}(\rho_i)=1$.

For a given initial state of such form, the *Open Quantum Random Walk (OQRW)* is defined by the mapping $\mathcal{M}$, which has the following form

$$\mathcal{M}(\rho)=\sum_i\Big(\sum_j B^i_j\rho_j B^{i*}_j\Big)\otimes|i\rangle\langle i|. \tag{1.5}$$

By means of the map $\mathcal{M}$ one defines a family of classical random process on $\Omega=\Lambda_+^{\mathbb{Z}}$. Namely, for any density operator ρ on $\mathcal{H}\otimes\mathcal{K}$ (see (1.4)) the probability distribution is defined by

$$\mathbb{P}_\rho(i_0,i_1,\dots,i_n)=\mathrm{Tr}(B^{i_n}_{i_{n-1}}\cdots B^{i_2}_{i_1}B^{i_1}_{i_0}\rho_{i_0}B^{i_1*}_{i_0}B^{i_2*}_{i_1}\cdots B^{i_n*}_{i_{n-1}}).$$

We point out that this distribution is not a Markov measure [9].

On the other hand, it is well-known [30] that to each classical random walk one can associate certain Markov chain and some properties of the walk can be explored by the constructed chain. Therefore, it is natural to construct Quantum Markov chain associated with OQRW and investigate its properties.

More precisely, the following arises problem: find a quantum Markov chain (QMC)[1] (or finitely correlated state (FCS)[18]) φ on the algebra $\mathcal{A}=\otimes_{i\in\mathbb{Z}_+}\mathcal{A}_i$, where $\mathcal{A}_i$ is isomorphic to $B(\mathcal{H})\otimes B(\mathcal{K})$, $i\in\mathbb{Z}_+$, such that the transition operator P (see section 3 for details) equal to the mapping $\mathcal{M}^{*2}$ and the restriction of φ to the commutative subalgebra of $\mathcal{A}$ coincides with the distribution $\mathbb{P}_\rho$, i.e.

$$\varphi\big((\mathbb{1}\otimes|i_0><i_0|)\otimes\cdots\otimes(\mathbb{1}\otimes|i_n><i_n|)\big)=\mathbb{P}_\rho(i_0,i_1,\dots,i_n). \tag{1.6}$$

We note that one can find a state with property (1.6) very easily, but the question is would that kind of state be difficult to distinguish as a QMC (or FCS)? Finding such a QMC will allow to interpret the distribution $\mathbb{P}_\rho$ as a QMC, and to study further properties of $\mathbb{P}_p$.

The main aim of this paper is to solve the initial problem and lead a further investigation in a few of the consequences of the problem. In what follows, we are going to work within QMC scheme [1, 4], and provide a concrete construction of QMC with the desired property. We stress that to construct such a state we define a notion of transpose of QMC (which is impossible to define with FCS) and using this, one defines QMC associated with $\mathcal{M}$. The solution of the problem sheds new light on some properties of the measure $\mathbb{P}_\rho$. For example, the measure can be considered as a distribution of some functions of certain Markov process [27, 19]. This together with the results of [19] will allow to compute certain physical quantities (e.g. entropy) of $\mathbb{P}_\rho$ as well as the further development of the study of repeated quantum measurements [10] via finitely correlated states [18].

R. Carbone and Y. Pautrat [14, 15] have recently studied irreducibility and periodicity aspects of the mapping $\mathcal{M}$ and as expected, the dynamical behavior of an OQRW is in general quite different from what is obtained with the usual (closed) quantum random walk [35, 36]. Simultaneously, an OQRW is quite different in general from what is obtained with the usual (closed) quantum random walk [35, 36]. In most of the existing papers, as a whole, the distribution is not well-studied therefore in the present paper, we will discover and go in depth of certain markovianity of the distribution along with establishing its ergodic properties with $\mathcal{M}$'s ergodic properties.

Furthermore, the study of the notion of recurrence motivated a large number of papers extending it in different directions: see[11] for the notion of monitored recurrence for discrete-time quantum processes; see [21] for the recurrence of discrete time unitary evolutions, see [5] for the recurrence of the quantum Markov chains; see [17] for the recurrence of the quantum Markovian semigroups. In

[1] We note that a Quantum Markov Chain is a quantum generalization of a Classical Markov Chain where the state space is a Hilbert space, and the transition probability matrix of a Markov chain is replaced by a transition amplitude matrix, which describes the mathematical formalism of the discrete time evolution of open quantum systems, see [4]-[6],[18, 20] for more details.

[2] The dual of $\mathcal{M}$ is defined by the equality $\mathrm{Tr}(\mathcal{M}(\rho)x)=\mathrm{Tr}(\rho\mathcal{M}^*(x))$ for all density operators ρ and observables x.

[5, 6] it was defined a notion of recurrence for QMC which was based on the transition expectation and initial projection. When we look at QMC it depends on an initial state and a transition expectation, therefore, we define the recurrence within QMC scheme. It turns out that the defined recurrence is connected to the recurrence of OQRW [9, 16]. However, the notions of recurrence elaborated in [9, 16] are purely classical, i.e. they depend on a classical probability distribution $\mathbb{P}_\rho$ (which is not necessary to be Markov one, therefore, it has appeared different phenomena than Markov one) and they are not connected to the noncommutative observables. In the present paper, we propose to study φ^t_ρ-recurrence which could treat more general events in the non-commutative setting. Namely, one can study φ^t_ρ-recurrence of projections rather than $\mathbb{1}\otimes|k\rangle\langle k|$. The recurrence of these kinds of projections can not treated by means of ones investigated in [9, 23]. Moreover, the present approach can be also applied to the case of finitely correlated states (see Remark 4.10).

The paper is organized as follows. In section 2, we introduce basic concepts related to the Open Quantum Random Walks and construct the corresponding measure which we stress that this measure a'priori is not a Markov one. As we move along to section 3, we recall a notion of Quantum Markov Chain (QMC) and its transpose while providing a construction of QMC. By means of the construction of section 3, we construct QMCs associated with Open Quantum Random Walks in section 4. It turns out that the transpose of the constructed QMC is naturally corresponding to the given OQRW. Additionally, in this section, we solve the posted problem and thoroughly review several properties of QMC and associated measure. Besides, it is also provided a construction of finitely correlated states with given marginal distributions on some commutative algebra. This opens new perspectives with the results of [10]. In section 5, a new notion of φ-recurrence of QMC is analyzed consequently showing that the defined recurrence is related to the recurrence investigated in [9, 16]. Section 6 is devoted to interesting examples of OQRW for which relations between φ-recurrence and the recurrence in the sense of [16] are investigate. Lastly, in sections 7 and 8 we provide the proofs of the formulated results in the sections 4 and 5, respectively.

2. Open Quantum Random Walks

In this section, we recall basic setup to define the distribution associated with Open Quantum Random Walks (OQRW).

As before, we consider the mapping $\mathcal{M}$ defined by (2.1). Hence, a measurement of the position in $\mathcal{K}$ would give that each site i is occupied with probability

$$\sum_j \mathrm{Tr}\left(B^i_j\rho_j B^{i*}_j\right).$$

If the measurement is performed after two steps, i.e.

$$\mathcal{M}^2(\rho)=\sum_i\sum_j\sum_k B^i_j B^j_k\,\rho_k\,B^{j*}_k B^{i*}_j\otimes|i\rangle\langle i|.$$

Hence measuring the position, we get the site $|i\rangle$ with probability

$$\sum_j\sum_k \mathrm{Tr}\left(B^i_j B^j_k\,\rho_k\,B^{j*}_k B^{i*}_j\right).$$

The random walk which is described in this way by the iteration of the completely positive map $\mathcal{M}$ is not a classical random walk, it is a quantum random walk.

The indicated distributions define a measure. Let us construct this measure.

Let us denote $\Omega_{\mathbb{Z}_+}=\Lambda^{\mathbb{Z}_+}$, $\Omega_{\mathbb{Z}}=\Lambda^{\mathbb{Z}}$, here $\mathbb{Z}_+$ denotes the set of all non negative integers. A subset of $\Omega_{\mathbb{Z}_+}$ (resp. $\Omega_{\mathbb{Z}}$) given by

$$A^{[l,m]}(i_l,i_{l+1},\dots,i_m)=\{\omega\in\Omega_{\mathbb{Z}_+}:\omega_l=i_l,\dots,\omega_m=i_m\}.$$

is called *thin cylindrical set*, where $i_k\in\Lambda$, $k\in\mathbb{Z}_+$. By $\mathfrak{F}$ we denote the σ-algebra generated by thin cylindrical sets.

Since the finite disjoint unions of thin cylinders form an algebra which generates $\mathfrak{F}$, therefore a measure μ on $\mathfrak{F}$ is uniquely determined by the values:

$$\mu_n(A^{[l,n]}(i_l, i_{l+1}, \dots, i_n)).$$

which should satisfy the compatibility conditions, i.e.

$$\sum_{j\in\Lambda} \mu_{n+1}(A^{[0,n+1]}(i_0, i_1, \dots, i_n, j)) = \mu_n(A^{[0,n]}(i_0, i_1, \dots, i_n)) \tag{2.1}$$

The Kolmogorov's Theorem ensures the existence of the measure μ on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$.

Now for a given $\mathcal{M}$ (see (2.1)) and a fixed ρ (see (1.4)), for every $n \in \mathbb{N}$, we define a measure $\mathbb{P}_{\rho,n}$ on $\Omega_n := \Lambda^{[0,n]}$ as the distribution of the OQRW, i.e.

$$\mathbb{P}_{\rho,n}(A^{[0,n]}(i_0, i_1, \dots, i_n)) = \mathrm{Tr}(B_{i_{n-1}}^{i_n} \cdots B_{i_1}^{i_2} B_{i_0}^{i_1} \rho_{i_0} B_{i_0}^{i_1 *} B_{i_1}^{i_2 *} \cdots B_{i_{n-1}}^{i_n *}). \tag{2.2}$$

The defined measures satisfy the compatibility condition. Indeed, due to (1.1) we have

$$\begin{aligned}
\sum_{j\in\Lambda} \mathbb{P}_{\rho,n+1}(A^{[0,n+1]}(i_0, i_1, \dots, i_n, j)) &= \sum_{j\in\Lambda} \mathrm{Tr}(B_{i_n}^{j} \cdots B_{i_1}^{i_2} B_{i_0}^{i_1} \rho_{i_0} B_{i_0}^{i_1 *} B_{i_1}^{i_2 *} \cdots B_{i_n}^{j *}) \\
&= \mathrm{Tr}\left(\left(\sum_{j\in\Lambda} \mathrm{Tr}(B_{i_n}^{j*} B_{i_n}^{j})\right) B_{i_{n-1}}^{i_n} \cdots B_{i_1}^{i_2} B_{i_0}^{i_1} \rho_{i_0} B_{i_0}^{i_1 *} B_{i_1}^{i_2 *} \cdots B_{i_{n-1}}^{i_n *}\right) \\
&= \mathbb{P}_{\rho,n}(A^{[0,n]}(i_0, i_1, \dots, i_n)).
\end{aligned}$$

Hence, we have the following result.

Proposition 2.1. *For given OQRW $\mathcal{M}$ and an initial density operator ρ there is a unique measure $\mathbb{P}_\rho$ on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$ with marginal distributions given by* (2.2).

It turns out that the measure $\mathbb{P}_\rho$ can be extended to $(\Omega_{\mathbb{Z}}, \mathfrak{F})$ under some conditions. Namely, we have the following

Proposition 2.2. *Let $\mathbb{P}_\rho$ be a measure defined on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$ associated with OQRW $\mathcal{M}$ and an initial density operator ρ. If $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$ is an invariant density operator w.r.t. $\mathcal{M}$, then the measure $\mathbb{P}_\rho$ can be extended to $(\Omega_{\mathbb{Z}}, \mathfrak{F})$.*

Proof. The invariance of $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$ w.r.t. $\mathcal{M}$ implies

$$\sum_j B_j^i \rho_j B_j^{i*} = \rho_i, \quad \forall\, i \in \Lambda. \tag{2.3}$$

Multiply the above equation by $B_i^{i_1}$ and $B_i^{i_1 *}$ on the left and right, respectively, and summing over i we get

$$\sum_{i,j} B_i^{i_1} B_j^i \rho_j B_j^{i*} B_i^{i_1 *} = \sum_i B_i^{i_1} \rho_i B_i^{i_1 *} = \rho_{i_1}, \ \forall\, i \in \Lambda. \tag{2.4}$$

By induction, we can establish that if ρ is invariant, then one has

$$\sum_{i_1, i_2, \dots, i_n} B_{i_n}^{i} B_{i_{n-1}}^{i_n} \cdots B_{i_1}^{i_2} \rho_{i_1} B_{i_1}^{i_2 *} \cdots B_{i_{n-1}}^{i_n *} B_{i_n}^{i*} = \rho_i, \ \forall\, i \in \Lambda. \tag{2.5}$$

The measure μ_ρ can be extended to $(\Omega_{\mathbb{Z}}, \mathfrak{F})$ if one has

$$\sum_{j\in\Lambda} \mathbb{P}_{\rho,n+1}(A^{[0,n+1]}(j, i_0, i_1, \dots, i_n)) = \mathbb{P}_{\rho,n}(A^{[0,n]}(i_0, i_1, \dots, i_n)). \tag{2.6}$$

Let us check the last equality. Indeed, from (2.3) we have

$$\begin{aligned}\sum_{j\in\Lambda}\mathbb{P}_{\rho,n+1}(A^{[0,n+1]}(j,i_0,i_1,\dots,i_n)) &= \sum_{j\in\Lambda}\mathrm{Tr}(B^{i_n}_{i_{n-1}}\cdots B^{i_1}_{i_0}B^{i_0}_{j}\rho_j B^{i_0*}_{j}B^{i_1*}_{i_0}\cdots B^{j*}_{i_n})\\ &= \mathrm{Tr}\left(B^{i_n}_{i_{n-1}}\cdots B^{i_2}_{i_1}B^{i_1}_{i_0}\rho_{i_0}B^{i_1*}_{i_0}B^{i_2*}_{i_1}\cdots B^{i_n*}_{i_{n-1}}\right)\\ &= \mathbb{P}_{\rho,n}(A^{[0,n]}(i_0,i_1,\dots,i_n)).\end{aligned}$$

This completes the proof. □

Remark 2.3. The existence of invariant density operators for OQRW $\mathcal{M}$ has been studied in [23]. One of the sufficient conditions is based on the irreducibility of the mapping [15].

Let us consider a random process (X_n) defined for $\omega=(i_0,i_1,\dots)\in\Omega_{\mathbb{Z}_+}$ by $X_n(\omega)=i_n$. Then the process (X_n) with distribution $\mathbb{P}_\rho$, in general, is not Markov one (see [9, Example 5.1]). Moreover, in the existing literature properties of the measure $\mathbb{P}_\rho$ are not well-studied.

In this paper, we will show that the measure $\mathbb{P}_\rho$ can be interpreted as a quantum Markov chain. This allows us to treat such quantum walks in the framework of QMC.

3. Quantum Markov Chains

In this section, we recall the definition of quantum Markov chain [1, 4, 33].

For each $i\in\mathbb{Z}_+$, (here $\mathbb{Z}_+$ denotes the set of all non negative integers) let us associate identical copies of a separable Hilbert space $\mathcal{H}$ and C^*-subalgebra M_0 of $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$:

$$\mathcal{H}_{\{i\}}=\mathcal{H},$$

$$\mathcal{A}_{\{i\}}=M_0\subset\mathcal{B}(\mathcal{H})\text{ for each } i\in\mathbb{Z}_+ \tag{3.1}$$

We assume that any minimal projection in M_0 is one dimensional.

For any bounded $\Lambda\subset\mathbb{Z}_+$, let

$$\mathcal{A}_\Lambda=\bigotimes_{i\in\Lambda}\mathcal{A}_i,\quad \mathcal{A}_{loc}=\bigcup_{\Lambda\subset\mathbb{Z}_+,|\Lambda|<\infty}\mathcal{A}_\Lambda$$

$$\mathcal{A}=\overline{\mathcal{A}_{loc}}=:\bigotimes_{i\in\mathbb{Z}_+}\mathcal{A}_i \tag{3.2}$$

where the bar denotes the norm closure.

For each $i\in\mathbb{Z}_+$, let J_i be the canonical injection of M_0 to the i-th component of $\mathcal{A}$. For each $\Lambda\subset\mathbb{Z}_+$ we identity $\mathcal{A}_\Lambda$ as a subalgebra of $\mathcal{A}$.

The basic ingredients in the construction of a stationary generalized quantum Markov chain in the sense of Accardi and Frigerio [4] consist of a *transition expectation* $\mathcal{E}: M_0\otimes M_0\to M_0$ which is completely positive unital map (i.e. $\mathcal{E}(\mathbb{1}\otimes\mathbb{1})=\mathbb{1})$), and a state ϕ_0 on M_0. In what follows, a pair $(\phi_0,\mathcal{E})$ is called a *Markov pair.*

A state φ defined on $\mathcal{A}$ associated with a Markov pair $(\phi_0,\mathcal{E})$, is called *Quantum Markov Chain (QMC)* if

$$\varphi(x_0\otimes x_1\otimes\ldots\otimes x_n)=\phi_0(\mathcal{E}(x_0\otimes\mathcal{E}(x_1\otimes\cdots\otimes\mathcal{E}(x_n\otimes\mathbb{1})\cdots)))). \tag{3.3}$$

Let $\sigma: M_0\otimes M_0\to M_0\otimes M_0$ be the flipping automorphism defined by $\sigma(x\otimes y)=y\otimes x$. For every transition expectation $\mathcal{E}$ one can associate its *transpose* by $\mathcal{E}^t=\mathcal{E}\circ\sigma$. Hence, given a Markov pair $(\phi_0,\mathcal{E})$ we naturally associate its *transpose Markov pair* $(\phi_0,\mathcal{E}^t)$. The QMC corresponding to the pair $(\phi_0,\mathcal{E}^t)$ is called *transpose QMC* of φ, and it is denoted by φ^t.

To every transition expectation one associates two kinds of Markov operators (i.e. completely positive, identity preserving map) from M_0 into itself:

$$P(a) = \mathcal{E}(\mathbb{1} \otimes a), \quad \text{(backward transition operator)} \tag{3.4}$$

$$T(a) = \mathcal{E}(a \otimes \mathbb{1}), \quad \text{(forward transition operator).} \tag{3.5}$$

Remark 3.1. It is known [4] that in the classical setting T is the identity operator, and P coincides with usual Markov transition operator.

Remark 3.2. We point out that the quantum Markov chain can be also treated as a special case of finitely correlated states (FCS) which were introduced in [18]. Let us recall the well-known construction. Let $\mathfrak{A}, \mathfrak{B}$ be two C^*-algebras with units $\mathbb{1}_{\mathfrak{A}}, \mathbb{1}_{\mathfrak{B}}$, respectively, φ_0 be a state on $\mathfrak{B}$, and $\mathcal{E} : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B}$ be a completely positive unital map such that for all $b \in \mathfrak{B}$ one has

$$\varphi_0(\mathcal{E}(\mathbb{1}_{\mathfrak{A}} \otimes b)) = \varphi_0(b).$$

For each $a \in \mathfrak{A}$ one defines a map $\mathcal{E}_a : \mathfrak{B} \to \mathfrak{B}$ by setting $\mathcal{E}_a(b) = \mathcal{E}(a \otimes b)$. The functional

$$\varphi(x_1 \otimes \cdots x_n) = \varphi_0(\mathcal{E}_{x_1} \cdots \mathcal{E}_{x_n}(\mathbb{1}_{\mathfrak{B}}))$$

uniquely defines a state on the C^*-algebra $\bigotimes_{i \in \mathbb{N}} \mathfrak{A}_i$, where $\mathfrak{A}_i$ is a copy of $\mathfrak{A}$. The state φ is the *finitely correlated state* associated to $(\mathfrak{A}, \mathfrak{B}, \mathcal{E}, \varphi_0)$. In case, $\mathfrak{A} = \mathfrak{B}$ we will recover QMC. On the other hand, we stress that, in general, we cannot define the transpose FCS on the same algebra with the initial one. Therefore, in what follows, we will work within QMC scheme.

In what follows, by $\mathcal{A}_{n]}$ we denote the subalgebra of $\mathcal{A}$, generated by the first $(n+1)$ factors, i.e.

$$a_{n]} = a_0 \otimes a_1 \otimes \cdots a_n \otimes \mathbb{1}_{[n+1} = J_0(a_0) J_1(a_1) \cdots J_n(a_n),$$

with $a_0, a_1, \dots, a_n \in M_0$. It is well known [2] that for each $n \in \mathbb{N}$ there exists a unique completely positive identity preserving mapping $E_{n]} : \mathcal{A} \to \mathcal{A}_{n]}$ such that

$$E_{n]}(a_{m]}) = a_0 \otimes \cdots \otimes a_{n-1} \otimes \mathcal{E}(a_n \otimes \mathcal{E}(a_{n+1} \otimes \cdots \otimes \mathcal{E}(a_m \otimes \mathbb{1}) \cdots)), \quad m > n \tag{3.6}$$

Remark 3.3. We notice that if the state ϕ_0 satisfies the following condition:

$$\phi_o(\mathcal{E}(\mathbb{1} \otimes x)) = \phi_0(x), \quad x \in M_0 \tag{3.7}$$

then the Markov pair $(\phi_0, \mathcal{E})$ defines local states

$$\varphi_{[i,n]}(x_i \otimes x_{i+1} \otimes \ldots \otimes x_n) = \phi_0(\mathcal{E}(x_i \otimes \mathcal{E}(x_{i+1} \otimes \cdots \otimes \mathcal{E}(x_n \otimes \mathbb{1}) \cdots)))). \tag{3.8}$$

The family of local states $\{\varphi_{[i,n]}\}$, due to (3.7), satisfies a compatibility condition, and therefore, the state φ is well defined on $A_{\mathbb{Z}} := \bigotimes_{i \in \mathbb{Z}} \mathcal{A}_i$. Moreover, φ is translation invariant, i.e. it is invariant with respect to the shift α, i.e. $\alpha(J_n(a)) = J_{n+1}(a)$.

Recall that by Tr we denote the trace on M_0 which takes the value 1 at each minimal projection, and let $\widetilde{\mathrm{Tr}}$ be the trace on $M_0 \otimes M_0$. Denote by $\widetilde{\mathrm{Tr}}^{(i)}$, $i = 1, 2$, the partial traces defined by

$$\widetilde{\mathrm{Tr}}^{(1)}(a \otimes b) = \mathrm{Tr}(a) b, \quad \widetilde{\mathrm{Tr}}^{(2)}(a \otimes b) = \mathrm{Tr}(b) a. \tag{3.9}$$

In [33] it was given a construction of a quantum Markov chain defined by a set $\{K_i\}_{i \in \mathbb{N}}$ of conditional density amplitudes [1]. Namely, let $W_0 \in M_0$ be a density matrix and $\{K_i\}_{i \in \mathbb{N}}$ be a set of the Hilbert-Schmidt operators in $M_0 \otimes M_0$ satisfying

$$\sum_i \|K_i\|^2 < \infty,$$

$$\sum_i \widetilde{\mathrm{Tr}}^{(2)}(K_i K_i^*) = \mathbb{1}. \tag{3.10}$$

Then the corresponding transition expectation [4]

$$\mathcal{E}(A) = \sum_i \widetilde{\mathrm{Tr}}^{(2)}(K_i A K_i^*), \quad A \in M_0 \otimes M_0. \tag{3.11}$$

and the density operator W_0 form a Markov pair $(W_0, \mathcal{E})$.

We point out that the transpose transition expectation associated with (3.11) has the following form:

$$\mathcal{E}^t(A) = \sum_i \widetilde{\mathrm{Tr}}^{(2)}(K_i \sigma(A) K_i^*), \quad A \in M_0 \otimes M_0. \tag{3.12}$$

Hence, $(W_0, \mathcal{E}^t)$ is a Markov pair. We stress that the QMCs associated with Markov pairs $(W_0, \mathcal{E})$ and $(W_0, \mathcal{E}^t)$, respectively, may have different properties. We will demonstrate some differences in the next sections.

Remark 3.4. We point out if additionally W_0 satisfies

$$\sum_i \widetilde{\mathrm{Tr}}^{(1)}(K_i^*(W_0 \otimes \mathbf{1})K_i) = W_0. \tag{3.13}$$

Then the associated QMC associated with the pair $(W_0, \mathcal{E})$ is well defined on the algebra $\mathcal{A}_{\mathbb{Z}}$.

4. Quantum Markov Chains associated with OQRW

In this section, we are going to construct QMCs associated with OQRW.

Let $\mathcal{M}$ be a OQRW given by (2.1). In this section we will use notations from the previous sections.

Take a density operator $\rho \in B(\mathcal{H} \otimes \mathcal{K})$, of the form

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|.$$

In what follows, we assume that $\rho_i \neq 0$ for all $i \in \mathbb{N}$.

We are going to construct a QMC associated with ρ and $\mathcal{M}$. To do so, we consider the algebra

$$\mathcal{A} = \bigotimes_{i \in \mathbb{Z}_+} \mathcal{A}_i,$$

where $\mathcal{A}_i = B(\mathcal{H} \otimes \mathcal{K})$ for all $i \in \mathbb{Z}_+$.

Define the following operators:

$$A_{ij} = \frac{1}{(\mathrm{Tr}(\rho_j))^{1/2}} \big(\rho_j^{1/2} \otimes |i\rangle\langle j|\big), \quad i, j \in \Lambda, \tag{4.1}$$

$$K_{ij} = M_j^{i*} \otimes A_{ij}. \tag{4.2}$$

Let us show that the pair $(\rho, \{K_{ij}\})$ defines a QMC on $\mathcal{A}$. Firstly note that the condition (3.10) follows from (1.2). Indeed, one has

$$\begin{aligned} \mathrm{Tr}^{(2)}\left(\sum_{i,j} K_{ij} K_{ij}^*\right) &= \sum_{i,j} M_j^{i*} M_j^i \frac{\mathrm{Tr}(\rho_j \otimes |i\rangle\langle i|)}{\mathrm{Tr}(\rho_j)} \\ &= \mathbb{1}. \end{aligned} \tag{4.3}$$

Hence, one can define a quantum Markov chain φ_ρ corresponding to the pair $(\rho, \mathcal{E})$, where the transition expectation $\mathcal{E}$ is defined by (see (3.11)):

$$\mathcal{E}(x \otimes y) = \sum_{i,j} M_j^{i*} x M_j^i \frac{\mathrm{Tr}(\rho_j \otimes |j\rangle\langle j|y)}{\mathrm{Tr}(\rho_j)}. \tag{4.4}$$

Remark 4.1. We point out that the constructed QMC is not naturally associated with the given OQRW, since the transition operator corresponding to $\mathcal{E}$ (4.4) is not equal to the dual $\mathcal{M}^*$ of $\mathcal{M}$. Indeed, we have

$$\begin{aligned} P(x) &= \sum_{i,j} M_j^{i*} M_j^i \frac{\mathrm{Tr}(\rho_j \otimes |j\rangle\langle j|x)}{\mathrm{Tr}(\rho_j)} \\ &= \sum_j \frac{\mathrm{Tr}(\rho_j \otimes |j\rangle\langle j|x)}{\mathrm{Tr}(\rho_j)} |j\rangle\langle j| \end{aligned}$$

which is clearly not equal to

$$\mathcal{M}^*(x) = \sum_{i,j} M_j^{i*} x M_j^i.$$

For the sake completeness, let us provide some properties of the QMC φ_ρ, which will allow us to distinguish the differences between the states φ and φ^t.

Proposition 4.2. *The QMC φ_ρ associated with the Markov pair $(\rho, \mathcal{E})$ can be extended to $\mathcal{A}_{\mathbb{Z}}$. Moreover, φ_ρ has the following form:*

$$\varphi_\rho(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \sum_v \mathrm{Tr}(\rho_v) \psi_v(x_1) \psi_v(x_2) \cdots \psi_v(x_n), \tag{4.5}$$

where

$$\psi_v(x) = \frac{1}{\mathrm{Tr}(\rho_v)} \sum_i \mathrm{Tr}\left(B_v^i \rho_v (B_v^i)^* \otimes |i\rangle\langle i|x\right), \quad v \in \Lambda. \tag{4.6}$$

Remark 4.3. From this theorem we infer that the constructed QMC is a convex combination of product states. On the other hand, this example is interesting to study the recurrence within QMC.

Now let us consider the transpose of $\mathcal{E}$ (see (4.4)) which is defined by

$$\mathcal{E}^t(x \otimes y) = \sum_{i,j} \frac{\mathrm{Tr}(\rho_j \otimes |j\rangle\langle j|x)}{\mathrm{Tr}(\rho_j)} (M_j^i)^* y M_j^i. \tag{4.7}$$

It is clear that

$$P(x) = \mathcal{E}^t(\mathbb{1} \otimes x) = \mathcal{M}^*(x).$$

We know that the Markov pair $(\rho, \mathcal{E}^t)$ defines the transpose QMC φ_ρ^t on $\mathcal{A}$.

Theorem 4.4. *If ρ is invariant state of $\mathcal{M}$ (i.e. $\mathcal{M}(\rho) = \rho$), then the QMC φ_ρ^t can be extended to $\mathcal{A}_{\mathbb{Z}}$. Moreover, φ_ρ^t is translation invariant.*

Remark 4.5.
- From Proposition 4.2 and Theorem 4.4 we immediately infer the difference (for example, the extendibility) between QMC φ and φ_ρ^t.
- We notice that taking into account Proposition 2.2 and the last theorem, we infer that the measure μ_ρ and the state φ_ρ^t have the same extendability property.

For any configuration $\omega \in \Omega_{\mathbb{Z}_+}$, we define a product state φ_ω on $\mathcal{A}$ as follows:

$$\varphi_\omega = \bigotimes_{k \in \omega} \varphi_k,$$

where

$$\varphi_k(x) = \frac{\mathrm{Tr}(\rho_k \otimes |k\rangle\langle k|x)}{\mathrm{Tr}(\rho_k)}. \tag{4.8}$$

It is clear that the mapping $\omega \to \varphi_\omega$ is measurable on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$.

Theorem 4.6. *The QMC φ^t_ρ on $\mathcal{A}$ has the following form:*

$$\varphi^t_\rho = \int_{\Omega_{\mathbb{Z}_+}} \varphi_\omega d\mathbb{P}_\rho(\omega), \tag{4.9}$$

where $\mathbb{P}_\rho$ is the measure given in Proposition 2.1. Moreover, one has

$$\varphi^t_\rho\big((\mathbb{1}\otimes|i_0\rangle\langle i_0|)\otimes\cdots\otimes(\mathbb{1}\otimes|i_n\rangle\langle i_n|)\big) = \mathbb{P}_\rho(A^{[0,m]}(i_0,\dots,i_m)). \tag{4.10}$$

for any $i_0,\dots,i_n\in\Lambda$, $n\in\mathbb{N}$.

Remark 4.7. We note that if one takes $\rho = p\otimes|k\rangle\langle k|$ ($k\in\mathbb{N}$), then we can define the corresponding transition expectation as follows:

$$\mathcal{E}^t(x\otimes y) \;=\; \sum_{i,j}\mathrm{Tr}(\rho\otimes|j\rangle\langle j|x)(M^i_j)^* y M^i_j. \tag{4.11}$$

The corresponding, QMC will be denoted by $\varphi^t_{p,k}$. Moreover, from the proof of Theorem 4.6 we infer that

$$\varphi^t_{p,k}\big((\mathbb{1}\otimes|k\rangle\langle k|)\otimes(\mathbb{1}\otimes|i_1\rangle\langle i_1|)\otimes\cdots\otimes(\mathbb{1}\otimes|i_n\rangle\langle i_n|)\big) = \mathbb{P}_{p,k}(k,i_1,\dots,i_n) \tag{4.12}$$

for any $i_1,\dots,i_n\in\Lambda$, $n\in\mathbb{N}$. Here

$$\mathbb{P}_{p,k}(k,i_1,\dots,i_n) = \mathrm{Tr}(B^{i_n}_{i_{n-1}}\cdots B^{i_2}_{i_1}B^{i_1}_k p B^{i_1*}_k B^{i_2*}_{i_1}\cdots B^{i_n*}_{i_{n-1}}).$$

From Theorem 4.6 we immediately infer that the QMC φ^t_ρ is a solution of the posted problem (see Introduction). Moreover, the theorem yields that the measure $\mathbb{P}_\rho$ can be considered and treated as a quantum Markov chain while the measure is not Markov one. This representation shows that the constructed QMC is a canonical one associated with OQRW, and it opens new perspectives in the investigation of $\mathbb{P}_\rho$ within QMC scheme.

The representation (4.9) allows us to investigate the ergodic properties of the state φ^t_ρ in terms of the ergodic properties of the mapping $\mathcal{M}^*$ and the measure $\mathbb{P}_\rho$, and vise-versa. We point out that in [15] it has been investigated several ergodic properties of OQRW $\mathcal{M}^*$, but there was not a connection between the erodicities of $\mathcal{M}^*$ and the measure $\mathbb{P}_\rho$. Next results shed new light on this question.

Let us first recall some necessary definitions. Consider the measure $\mathbb{P}_\rho$ on $(\Omega_{\mathbb{Z}_+},\mathfrak{F})$. By $s:\Omega_{\mathbb{Z}_+}\to\Omega_{\mathbb{Z}_+}$ we denote the *shift* transformation defined by $(s(\omega))_n=\omega_{n+1}$. Recall that the measure μ_ρ is called:

1. *ergodic* if for all $A,B\in\mathfrak{F}$ one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}_\rho(A\cap s^{-n}(B)) = \mathbb{P}_\rho(A)\mathbb{P}_\rho(B) \tag{4.13}$$

2. *weak mixing* if for all $A,B\in\mathfrak{F}$ one has

$$\lim_{n\to\infty}\mathbb{P}_\rho(A\cap s^{-n}(B)) = \mathbb{P}_\rho(A)\mathbb{P}_\rho(B) \tag{4.14}$$

Now let us recall some necessary notions about the ergodicity of C^*-dynamical systems. A C^*-dynamical system $(\mathfrak{A},T,\varphi)$ [3] is called

1. *ergodic* if for all $x,y\in\mathfrak{A}$ one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi(xT^n(y)) = \varphi(x)\varphi(y) \tag{4.15}$$

2. *weak mixing* if for all $x,y\in\mathfrak{A}$ one has

$$\lim_{n\to\infty}\varphi(xT^n(y)) = \varphi(x)\varphi(y). \tag{4.16}$$

[3] The triple $(\mathfrak{A},T,\varphi)$ is called to be a C^*-*dynamical system*, if $\mathfrak{A}$ is a C^*-algebra with unit, $T:\mathfrak{A}\to\mathfrak{A}$ is completely positive unital mapping with an invariant state φ on $\mathfrak{A}$.

Several ergodic properties of C^*-dynamical systems have been investigated in [28]. The Theorem 4.6 allows us to establish the following result.

Theorem 4.8. *Let ρ be an invariant state for $\mathcal{M}$. Cosnider the following statements:*

(i) *the C^*-dynamical system $(B(\mathcal{H})\otimes B(\mathcal{K}), \mathcal{M}^*, \rho)$ is ergodic (resp. weak mixing);*
(ii) *the C^*-dynamical system $(\mathcal{A}, \alpha, \varphi_\rho^t)$ is ergodic (resp. weak mixing);*
(iii) *the measure $\mathbb{P}_\rho$ is ergodic (resp. weak mixing).*

The following implications hold: (i)$\Rightarrow$ (ii)$\Leftrightarrow$(iii).
Moreover, if $\mathcal{E}^t(B(\mathcal{H})\otimes B(\mathcal{K})\otimes \mathbb{1}) = B(\mathcal{H})\otimes B(\mathcal{K})$, then the all statements are equivalent.

Remark 4.9. We notice that in [15, Sections 3, 4] it was given certain sufficient conditions for the ergodicity and weak mixing of $\mathcal{M}^*$. The last theorem with the results of the mentioned paper opens new insight to the properties of the measure $\mathbb{P}_\rho$.

Observation. It is known [3] that any quantum Markov state φ admits a representation

$$\varphi = \int_{\Omega_{\mathbb{Z}_+}} \psi_\omega d\lambda(\omega)$$

where ψ_ω is product states and λ is a Markov measure on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$. Comparing the last one with Theorem 4.6, we point out that in our case the measure $\mathbb{P}_\rho$ is not necessary to be a Markov one while the state φ_ρ^t is a QMC. Note that the representation (4.9) does not imply that φ_ρ^t is a product state. Since, any state on $B(\mathcal{H})$ is a limit of convex combination of vector states.

Remark 4.10. We point out that the construction of the transition expectation (4.4) will provide a more general construction of finitely correlated states associated with successive measuraments of a finite family $\mathcal{J} = \{\Phi_a\}_{a\in L}$ of completely positive maps $\Phi_a : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\Phi = \sum_{a\in L}\Phi_a$ satisfies $\Phi(\mathbb{1}) = \mathbb{1}$. Here the alphabet L describes the possible outcomes of a single measurement. Let us consider the measurable space $(\Omega, \mathfrak{F})$, where $\Omega = L^{\mathbb{Z}_+}$. A pair $(\mathcal{J}, \rho)$, where ρ is a density matrix on $\mathcal{H}$, due to $\Phi(\mathbb{1}) = \mathbb{1}$, defines a distribution on Ω by

$$\mathbb{P}_{\mathcal{J},\rho}(a_1, \dots, a_n) = \mathrm{Tr}(\rho\Phi_{a_1}\circ\cdots\circ\Phi_{a_n}(\mathbb{1})).$$

Let $\mathcal{K}$ denotes a Hilbert space with orthonormal basis $\{|a\rangle\}_{a\in L}$. By $C(\mathcal{H}\otimes\mathcal{K})$ we denote a commutative subalgebra of $B(\mathcal{H})\otimes B(\mathcal{K})$ generated by the projections $\{\mathbb{1}\otimes|a\rangle\}_{a\in L}$. Then one can see that the distribution $P_{\mathcal{J},\rho}$ can be considered as a state on the algebra $\mathfrak{C} = \bigotimes_{i\in\mathbb{Z}_+}\mathcal{C}_i$, where $\mathcal{C}_i$ is a copy of $C(\mathcal{H}\otimes\mathcal{K})$. Now using (4.4) let us define the transition expectation $\mathcal{E} : B(\mathcal{H})\otimes B(\mathcal{K}) \to B(\mathcal{H})$ by

$$\mathcal{E}(x\otimes y) = \sum_{a\in L}\mathrm{Tr}((\rho\otimes|a\rangle\langle a|)x)\Phi_a(y). \tag{4.17}$$

Assume that $\Phi^*(\rho) = \rho$, then $(B(\mathcal{H}), B(\mathcal{K}), \mathcal{E}, \rho)$ defines a FCS φ on $\mathcal{A}$ such that

$$\varphi\lceil\mathfrak{C} = \mathbb{P}_{\mathcal{J},\rho}, \quad \mathcal{E}(\mathbb{1}\otimes x) = \Phi(x).$$

Hence, one can study the state φ and $\mathbb{P}_{\mathcal{J},\rho}$ all together. This with results of [9, 10, 19] opens new insight to the entropy production and recurrence for finitely correlated states.

5. Recurrence of QMC associated with OQRW

In this section is devoted to the notion of recurrence with QMC scheme. Furthermore, some examples will be illustrated.

Following [5] we recall a definition of the stopping time associated with a projection $e \in M_0$, which is a sequence $\{\tau_k\}$ defined by

$$\begin{aligned}
&\tau_0 = e \otimes \mathbb{1}_{[1} = J_0(e),\\
&\tau_1 = e^\perp \otimes e \otimes \mathbb{1}_{[2} = J_0(e^\perp)J_1(e),\\
&\cdots\cdots,\\
&\tau_k = \underbrace{e^\perp \otimes \cdots \otimes e^\perp}_{k-1} \otimes e \otimes \mathbb{1}_{[k+1} = J_0(e^\perp)\cdots J_{k-1}(e^\perp)J_k(e),\\
&\tau_\infty^n = \underbrace{e^\perp \otimes \cdots \otimes e^\perp}_{n} \otimes \mathbb{1}_{[n+1} = J_0(e^\perp)\cdots J_n(e^\perp).
\end{aligned}$$

Since the sequence $\{\tau_\infty^n\}$ is decreasing, therefore, its strong limit exists in $\mathcal{A}''$ (bicommutant of $\mathcal{A}$), and it is denoted by

$$\tau_\infty := \lim_{n\to\infty} \tau_\infty^n$$

One can see that

$$\sum_{k\geq 0} \tau_k = \mathbb{1} - \tau_\infty, \tag{5.1}$$

where the sum is meant in the strong topology in $\mathcal{A}''$.

Definition 5.1. Let φ be a QMC on $\mathcal{A}$ associated with the pair $(\phi_0, \mathcal{E})$. A projection e is called

(i) $\mathcal{E}$-*completely accessible* if

$$E_{0]}(\tau_\infty) := \lim_{n\to\infty} E_{0]}(\tau_\infty^n) = 0;$$

(i) φ-*completely accessible* if $\varphi(\tau_\infty) = 0$;

(iii) $\mathcal{E}$-*recurrent* if $\mathrm{Tr}(\mathcal{E}(e \otimes \mathbb{1})) < \infty$ and one has

$$\frac{1}{\mathrm{Tr}(\mathcal{E}(e\otimes\mathbb{1}))}\,\mathrm{Tr}\left(E_{0]}\left(\sum_{n\geq 0} J_0(e)\otimes\tau_n\right)\right) = 1$$

(iv) φ-*recurrent* if $\varphi(J_0(e)) \neq 0$ and

$$\frac{1}{\varphi(J_0(e))}\varphi\left(\sum_{n\geq 0} J_0(e)\otimes\tau_n\right) = 1$$

Definition 5.2. Let φ be a QMC on $\mathcal{A}$ associated with the pair $(\phi_0, \mathcal{E})$ and e, f be two projections in M_0. A projection f is called

(i) $\mathcal{E}$-*accessible* from e if there is $n \in \mathbb{N}$ such that

$$E_{0]}(J_0(e)\otimes\mathbb{1}_{n-1]}\otimes J_n(f)) \neq 0;$$

(ii) φ-*accessible* from e (we denote it as $e \to^\varphi f$) if there is $n \in \mathbb{N}$ such that

$$\varphi(J_0(e)\otimes\mathbb{1}_{n-1]}\otimes J_n(f)) \neq 0.$$

If $e \to^\varphi f$ and $f \to^\varphi e$, then e and f are called φ-*communicate* and one denotes $e \leftrightarrow^\varphi f$.

Remark 5.3. We notice that the $\mathcal{E}$-accessibility and $\mathcal{E}$-recurrence have been introduced and studied in [5]. From the definitions one can infer that, due to the Markov property of φ, $\mathcal{E}$-accessibility and $\mathcal{E}$-recurrence imply φ-accessibility and φ-recurrence, respectively. The reverse is not true (see Example 6.2).

Now we are going to study several properties of φ-accessibility and φ-recurrence, respectively.

Theorem 5.4. *Let φ be a QMC on $\mathcal{A}$ associated with the pair $(\phi_0, \mathcal{E})$. The following statements hold:*

(i) $\varphi(J_n(e)) = 0$ *for all* $n \in \mathbb{N}$ *if and only if for every* $k \in \mathbb{N}$ *one has* $\varphi(\beta^k(\tau_\infty)) = 1$*, where*

$$\beta(a_0 \otimes a_1 \otimes \cdots a_n) = \mathbb{1} \otimes a_0 \otimes a_1 \otimes \cdots a_n, \quad \text{for any } n \in \mathbb{N};$$

(ii) *e is φ-recurrent if and only if* $\varphi(J_0(e) \otimes \tau_\infty) = 0$*. In particular, if e is φ-completely accessible, then e is φ-recurrent;*

(iii) *if φ is faithful, then e is φ-completely accessible if and only if e is φ-recurrent;*

(iv) *if all projections in M_0 are φ-communicating and e is φ-recurrent, then e is φ-completely accessible.*

Corollary 5.5. *Let φ be a QMC on $\mathcal{A}_{\mathbb{Z}}$ associated with the pair $(\phi_0, \mathcal{E})$. The following statements hold:*

(i) $\varphi(e) = 0$ *if and only if* $\varphi(\tau_\infty) = 1$*;*

(ii) *e is φ-recurrent if and only if e is φ-completely accessible;*

(iii) *if φ is faithful, then e is φ-completely accessible if and only if $\mathcal{E}$-completely accessible.*

Now we turn our attention of the QMC associated with OQRW. Let $\mathcal{M}$ be a OQRW given by (2.1). In what follows, we will use notations from the previous sections.

Given a density operator $\rho \in B(\mathcal{H} \otimes \mathcal{K})$, one can construct a QMC φ_ρ^t on $\mathcal{A}$. Due to (4.12) the φ_ρ^t-recurrence of a projection $\mathbb{1} \otimes |j\rangle\langle j|$ means the following one:

$$\sum_{k=1}^{\infty} \sum_{\substack{i_1,\dots,i_{k-1} \\ i_\ell \neq j, 1 \leq \ell \leq k-1}} \mathbb{P}_\rho(A^{[0,k+1]}(j, i_1, \dots, i_{k-1}, j)) = \mathbb{P}_\rho(A^{[0,0]}(j))$$

which is equivalent to

$$\mathbb{P}_{\rho,j}(t_j < \infty) = 1$$

where $t_j(\omega) = \inf\{n \in \mathbb{N} : \ \omega_n = j\}$.

Let us take $\rho = p \otimes |k\rangle\langle k|$. Then from Remark 4.7 one finds that the $\varphi_{p,k}^t$-recurrence of $\mathbb{1} \otimes |k\rangle\langle k|$ is equivalent to $\mathbb{P}_{p,k}(t_k < \infty) = 1$. If the projection $\mathbb{1} \otimes |k\rangle\langle k|$ is $\varphi_{p,k}^t$-recurrent for all p, then we obtain LS-recurrence [9, 23]. Moreover, we obtain that $\mathcal{E}$-recurrence implies LS-recurrence. Hence, the φ_ρ^t-recurrence is weaker among the recurrences investigated in [9, 23]. However, the notions of recurrence studied in [9, 16] are purely classical, i.e. they depend on a classical probability distribution (which is not necessary to be Markov one, therefore, it appeared different phenomena than Markov one) and they are not connected to the noncommutative observables. In the present paper, we propose to study φ_ρ^t-recurrence which could treat more general events in the non-commutative setting. For example, one can study φ_ρ^t-recurrence projections rather than $\mathbb{1} \otimes |k\rangle\langle k|$. The recurrence of this kind of projection can not be studied by means of ones investigated in [9, 23]. Moreover, the present approach can be also applied to the case of finitely correlated states (see Remark 4.10).

Theorem 5.6. *Let $\mathcal{M}$ be a OQRW, $\rho = p \otimes |k\rangle\langle k|$ be an initial density matrix and $Q \in B(\mathcal{H})$ be a projection. Then the following statements hold:*

(i) *if* $\mathrm{Tr}(pQ) = 1$*, then the projection $Q \otimes |k\rangle\langle k|$ is $\varphi_{p,k}^t$-recurrent if and only if* $\mathbb{P}_{p,k}(t_k < \infty) = 1$*;*

(ii) *if* $\mathrm{Tr}(pQ) < 1$*, then the projection $\mathbb{1} - Q \otimes |k\rangle\langle k|$ is $\varphi_{p,k}^t$-recurrent.*

From this theorem we infer that the essential difference between the $\varphi_{p,k}^t$-recurrence and LS-recurrence for the projection $\mathbb{1} - Q \otimes |k\rangle\langle k|$ which is a quantum phenomena that can not describe by the classical approach, i.e. using the distribution $\mathbb{P}_{p,k}$.

6. Examples

Example 6.1. We consider the example given in section 12.1 of [7]. In our notation this example is given by $\Lambda = \{1, 2\}$, $\mathcal{H} = \mathbb{C}^2$ (with canonical basis (e_1, e_2)) and transitions are given by

$$B_1^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad B_2^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B_1^2 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

where $|a|^2+|b|^2=|c|^2+|d|^2=1$, $0<|a|^2,|c|^2<1$.

1. Denote

$$\rho=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\otimes|2\rangle\langle 2| =: \rho_0\otimes|2\rangle\langle 2|.$$

A straightforward computation shows that

$$\begin{aligned}\mathcal{M}(\rho) &= \sum_{i,j=1}^{2} B_j^i\otimes|i\rangle\langle j|\big((\rho_0\otimes|2\rangle\langle 2|\big)B_j^{i*}\otimes|j\rangle\langle i|\\ &= \sum_{i=1}^{2} B_2^i\rho_0 B_2^{i*}\otimes|i\rangle\langle i|.\end{aligned}$$

Due to

$$B_2^1\rho_0B_2^{1*}=0,\quad B_2^2\rho_0B_2^{2*}=\rho_0 \tag{6.1}$$

we conclude that ρ is an invariant state for $\mathcal{M}$.

From (6.1) we immediately infer that

$$\mathbb{P}_{\rho_0,2}(2,i_1,\dots,i_n)=\begin{cases}1, & \text{if } \ n=1, i_1=2,\\ 0, & \text{if } \ i_\ell=1, 1\le\ell\le n, n\ge 1,\end{cases}$$

hence $|2\rangle\langle 2|$ is $\mathbb{P}_{\rho_0,2}$-recurrent.

Now consider the projection $e_Q=Q\otimes|2\rangle\langle 2|$, where Q is a projection in $B(\mathcal{H})$. By denoting $\lambda=\mathrm{Tr}(p_0Q)$, from (7.6),(4.8) we obtain

$$\begin{aligned}\varphi^t_{\rho_0,2}(e_Q\otimes\underbrace{e_Q^\perp\otimes\cdots\otimes e_Q^\perp}_{n}) &= \sum_{i_1,\dots,i_n}\mathbb{P}_{\rho_0,2}(2,i_1,\dots,i_n)\lambda\big(1-\lambda\delta_{i_1,2}\big)\cdots\big(1-\lambda\delta_{i_n,2}\big)\\ &= \mathbb{P}_{\rho_0,2}(2,\dots,2)\lambda\big(1-\lambda\big)^n\\ &= \lambda\big(1-\lambda\big)^n\to 0 \quad \text{as} \quad n\to\infty\end{aligned}$$

for any $\lambda\in(0,1]$. Hence, due to Theorem 5.4 (ii) from the last relation one gets that the projection e_Q is $\varphi^t_{\rho_0,2}$-recurrent, if $\mathrm{Tr}(\rho_0Q)>0$. Note that the recurrence of the projection e_Q cannot be treated by means of the classical measure $\mathbb{P}_{\rho_0,2}$.

2. Now consider

$$\tilde\rho=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\otimes|2\rangle\langle 2| =: \rho_1\otimes|2\rangle\langle 2|.$$

Then one can calculate that

$$B_2^1\rho_1B_2^{1*}=\rho_0,\quad B_2^2\rho_1B_2^{2*}=0, \tag{6.2}$$

$$B_2^1\rho_0B_2^{1*}=|c|^2\rho_0,\quad B_1^1\rho_0B_2^{1*}=|a|^2\rho_0. \tag{6.3}$$

Then from (6.2),(6.3) we obtain

$$\begin{aligned}\mathbb{P}_{\rho_1,2}(t_2<\infty) &= \sum_{n=1}^{\infty}\mathbb{P}_{\rho_1,2}(2,\underbrace{1,\dots,1}_{n},2)\\ &= \sum_{n=1}^{\infty}|c|^2|a|^{2(n-1)}\\ &= \frac{|c|^2}{1-|a|^2}\\ &= 1\end{aligned}$$

which means that $|2\rangle\langle 2|$ is $\mathbb{P}_{\rho_1,2}$-recurrent.

Let us consider the projection $e_Q = Q\otimes|2\rangle\langle 2|$. Taking into account (6.2) one gets

$$\mathbb{P}_{\rho_1,2}(2,2,i_2,\dots,i_n,2)=0$$

for any $i_2,\dots,i_n\in\{1,2\}$ $(n\geq 2)$ which with (6.3) yields

$$\begin{aligned}
\varphi^t_{\rho_1,2}(e_Q\otimes\underbrace{e_Q^\perp\otimes\cdots\otimes e_Q^\perp}_{n}\otimes e_Q) &= \sum_{i_1,\dots,i_n}\mathbb{P}_{\rho_1,2}(2,i_1,\dots,i_n,2)\lambda^2\prod_{1\leq k\leq n}\big(1-\lambda\delta_{i_k,2}\big)\\
&= \sum_{i_2,\dots,i_n}\mathbb{P}_{\rho_1,2}(2,1,i_2\dots,i_n,2)\lambda^2\prod_{2\leq k\leq n}\big(1-\lambda\delta_{i_k,2}\big)\\
&\quad+\sum_{i_2,\dots,i_n}\mathbb{P}_{\rho_1,2}(2,2,i_2,\dots,i_n,2)\lambda^2(1-\lambda)\prod_{2\leq k\leq n}\big(1-\lambda\delta_{i_k,2}\big)\\
&= \lambda^2\sum_{k=1}^n\mathbb{P}_{\rho_1,2}(2,\underbrace{1,\dots,1}_{n-k+1},\underbrace{2,\dots,2}_{k})(1-\lambda)^{k-1}\\
&= \lambda^2|c|^2\sum_{k=1}^n|a|^{2(n-k)}(1-\lambda)^{k-1}\\
&= \frac{\lambda^2|c|^2(|a|^{2n}-(1-\lambda)^n)}{|a|^2+\lambda-1}.
\end{aligned}$$

Now taking into account $\lambda=\varphi^t_{\rho_1,2}(e_Q)$, from the last equality with $|a|^2+|c|^2=1$ we find

$$\begin{aligned}
\frac{1}{\varphi^t_{\rho_1,2}(e_Q)}\sum_{n\geq 0}\varphi^t_{\rho_1,2}(e_Q\otimes\underbrace{e_Q^\perp\otimes\cdots\otimes e_Q^\perp}_{n}\otimes e_Q) &= \frac{\lambda^2|c|^2}{|a|^2+\lambda-1}\left(\frac{|a|^2}{1-|a|^2}-\frac{1-\lambda}{\lambda}\right)\\
&= \frac{\lambda^2|c|^2}{\lambda-|c|^2}\left(\frac{|a|^2}{|c|^2}-\frac{1-\lambda}{\lambda}\right)\\
&= \frac{|a|^2\lambda}{\lambda-|c|^2}-\frac{|c|^2(1-\lambda)}{\lambda-|c|^2}\\
&= 1
\end{aligned}$$

Hence, for any Q with $\mathrm{Tr}(\rho_1 Q)>0$, the projection e_Q is $\varphi^t_{\rho_1,2}$-recurrent.

3. In this case, we assume that $c=0$, and take another initial state

$$\rho=\frac{1}{2}\rho_0\otimes|1\rangle\langle 1|+\frac{1}{2}\rho_0\otimes|2\rangle\langle 2|,$$

here ρ_0 is given as above. One can see that $B_1^2\rho_0B_1^{2*}=0$, $B_1^1\rho_0B_1^{1*}=\rho_0$. Hence, we conclude that

$$\mathrm{Tr}(B^{i_n}_{i_{n-1}}\cdots B^{i_2}_{i_1}B^{i_1}_{i_0}\rho_0B^{i_1*}_{i_0}B^{i_2*}_{i_1}\cdots B^{i_n}_{i_{n-1}})=0,$$

if there is $k_0\in\{0,\dots,n\}$ such that $i_{k_0}\neq i_{k_0-1}$.

Assume that $e^\perp=Q\otimes|1\rangle\langle 1|$. Hence, we have

$$\begin{aligned}
\varphi^t_\rho(\tau^n_\infty) &= \frac{1}{2}\sum_{i_0,i_1,\dots,i_n}\mathrm{Tr}\left(B^{i_n}_{i_{n-1}}\cdots B^{i_2}_{i_1}B^{i_1}_{i_0}\rho_0B^{i_1*}_{i_0}B^{i_2*}_{i_1}\cdots B^{i_n}_{i_{n-1}}\right)\varphi_{i_0}(e^\perp)\cdots\varphi_{i_n}(e^\perp)\\
&= \frac{1}{2}(\mathrm{Tr}(\rho_0Q))^n.
\end{aligned}$$

So, if $\mathrm{Tr}(\rho_0Q)<1$ then e is φ^t-completely accessible.

Similarly, one gets

$$\varphi^t_\rho(e\otimes\tau^n_\infty)=\frac{1}{2}(1-\mathrm{Tr}(\rho_0Q))(\mathrm{Tr}(\rho_0Q))^n.$$

Hence, we infer that if $\mathrm{Tr}(\rho_0p)\neq 1$, then e is φ^t_ρ-completely accessible iff φ^t-recurrent. Otherwise (if $\mathrm{Tr}(\rho_0Q)=1$, e is φ^t_ρ-recurrent, but not φ^t_ρ-complete accessible.

Example 6.2. In this example, we are going to show that $\mathcal{E}$-recurrence is stronger than φ-recurrence. To do so, as an illustrative example, we are going work with QMC associated given by (4.5).

Let us consider a stationary OQRW on $\mathbb{Z}$ with nearest-neighbor jumps (see [7]). Let $\mathcal{H}$ be a Hilbert space and $B,C\in B(\mathcal{H})$ such that $B^*B+C^*C=\mathbb{1}$. We define the walk as follows: assume that $B_i^{i-1}=B$ and $B_i^{i+1}=C$ for all $i\in\mathbb{Z}$, all the others B_j^i being equal to 0. Then one can calculate that

$$\mathcal{M}(\rho)=\sum_j\big(B\otimes|j-1\rangle\langle j|\rho B^*\otimes|j\rangle\langle j-1|+C\otimes|j+1\rangle\langle j|\rho C^*\otimes|j+1\rangle\langle j|\big). \tag{6.4}$$

Take a density operator $\rho\in B(\mathcal{H}\otimes\mathcal{K})$, of the form

$$\rho=\sum_i\rho_i\otimes|i\rangle\langle i|,$$

with $\rho_i\neq 0$ for all i.

Take a projection $e_{Q,k}=Q\otimes|k\rangle\langle k|$ for some $k\in\mathbb{Z}$, here $Q\in B(\mathcal{H})$ is a projection. Then due to Corollary 7.2 and (6.4) we have

$$\begin{aligned}\mathcal{E}(e_{Q,k}\otimes E_{0]}(\tau^n_\infty)) &= \sum_\ell B^*\otimes|\ell\rangle\langle\ell-1|e_{Q,k}B\otimes|\ell-1\rangle\langle\ell|\big(\psi_\ell(e^\perp_{Q,k})\big)^n\\ &\quad+\sum_\ell C^*\otimes|\ell\rangle\langle\ell+1|e_{Q,k}B\otimes|\ell+1\rangle\langle\ell|\big(\psi_\ell(e^\perp_{Q,k})\big)^n\\ &= B^*QB\otimes|k+1\rangle\langle k+1|\big(\psi_{k+1}(e^\perp_{Q,k})\big)^n\\ &\quad+C^*QC\otimes|k-1\rangle\langle k-1|\big(\psi_{k-1}(e^\perp_{Q,k})\big)^n.\end{aligned}\tag{6.5}$$

Taking into account (4.6) one finds

$$\psi_\ell(e^\perp_{Q,k})=\begin{cases}1-\frac{\mathrm{Tr}(B\rho_{k+1}B^*Q)}{\mathrm{Tr}(\rho_{k+1})}, \text{if} & \ell=k+1\\ 1-\frac{\mathrm{Tr}(C\rho_{k-1}C^*Q)}{\mathrm{Tr}(\rho_{k-1})}, \text{if} & \ell=k-1\end{cases}\tag{6.6}$$

Therefore, from (6.5) with (6.6) one gets

$$\begin{aligned}\mathcal{E}(e_{Q,k}\otimes E_{0]}(\tau^n_\infty)) &= B^*QB\otimes|k+1\rangle\langle k+1|\left(1-\frac{\mathrm{Tr}(B\rho_{k+1}B^*Q)}{\mathrm{Tr}(\rho_{k+1})}\right)^n\\ &\quad+C^*QC\otimes|k-1\rangle\langle k-1|\left(1-\frac{\mathrm{Tr}(C\rho_{k-1}C^*Q)}{\mathrm{Tr}(\rho_{k-1})}\right)^n.\end{aligned}\tag{6.7}$$

Hence, if one has

$$0<\mathrm{Tr}(B\rho_{k+1}B^*Q)<\mathrm{Tr}(\rho_{k+1}),\quad 0<\mathrm{Tr}(C\rho_{k-1}C^*Q)<\mathrm{Tr}(\rho_{k-1})\tag{6.8}$$

then from (6.7) we infer that $\mathcal{E}(e_{Q,k}\otimes E_{0]}(\tau^n_\infty))\to 0$ as $n\to\infty$ which according to [5, Theorem 1, (iii)] implies that $e_{Q,k}$ is $\mathcal{E}$-recurrent.

Now let us look for the φ-recurrence. From (4.5) we have

$$\varphi(e_{Q,k}\otimes\underbrace{e^\perp_{Q,k}\otimes\cdots\otimes e^\perp_{Q,k}}_{n})=\sum_\ell\mathrm{Tr}(\rho_\ell)\psi_\ell(e_{Q,k})\big(\psi_\ell(e^\perp_{Q,k})\big)^n$$

From

$$\psi_\ell(e_{Q,k}) = \begin{cases} \frac{\mathrm{Tr}(B\rho_{k+1}B^*Q)}{\mathrm{Tr}(\rho_{k+1})}, \text{if} \quad \ell = k+1 \\ \frac{\mathrm{Tr}(C\rho_{k-1}C^*Q)}{\mathrm{Tr}(\rho_{k-1})}, \text{if} \quad \ell = k-1 \end{cases}$$

with (6.6) we obtain

$$\begin{aligned} \varphi(e_{Q,k} \otimes \tau_\infty^n) &= \mathrm{Tr}(B\rho_{k+1}B^*Q)\Big(1 - \frac{\mathrm{Tr}(B\rho_{k+1}B^*Q)}{\mathrm{Tr}(\rho_{k+1})}\Big)^n \\ &\quad + \mathrm{Tr}(C\rho_{k-1}C^*Q)\Big(1 - \frac{\mathrm{Tr}(C\rho_{k-1}C^*Q)}{\mathrm{Tr}(\rho_{k-1})}\Big)^n. \end{aligned} \tag{6.9}$$

Clearly, if (6.8) is satisfied then $e_{Q,k}$ is φ-recurrent. This means that under the condition (6.8) the $\mathcal{E}$-recurrence is equivalent to the φ-recurrence.

If one of the following conditions

(a) $supp(\rho_{k+1})supp(B) = 0$ and $0 < \mathrm{Tr}(C\rho_{k-1}C^*Q) < \mathrm{Tr}(\rho_{k-1})$,
(b) $supp(\rho_{k-1})supp(C) = 0$ and $0 < \mathrm{Tr}(B\rho_{k+1}B^*Q) < \mathrm{Tr}(\rho_{k+1})$,
(c) $supp(\rho_{k+1})supp(B) = 0$ and $supp(\rho_{k-1})supp(C) = 0$,

is satisfied, then from (6.9) we infer that $e_{Q,k}$ is still φ-recurrent, while it is not $\mathcal{E}$-recurrent.

7. Proofs for Section 4

In this section we collect all the proofs of the formulated Theorems and Propositions in Section 4. We first need the following auxiliary fact.

Lemma 7.1. *Let $\rho = \sum_\ell \rho_\ell \otimes |\ell\rangle\langle\ell|$. Then one has:*

(i) $\mathrm{Tr}(\rho M_v^{u*} x M_v^u) = \mathrm{Tr}(B_v^u \rho_v B_v^{u*} \otimes |u\rangle\langle u|x)$;

(ii) $\mathrm{Tr}(\rho_v \otimes |v\rangle\langle v| M_j^{i*} x M_j^i) = \mathrm{Tr}(B_j^i \rho_v B_j^{i*} \otimes |i\rangle\langle i|x)\delta_{vj}$.

Proof. (i). We have

$$\begin{aligned} \mathrm{Tr}(\rho M_v^{u*} x M_v^u) &= \sum_\ell \mathrm{Tr}(\rho_\ell \otimes |\ell\rangle\langle\ell| M_v^{u*} x M_v^u) \\ &= \sum_\ell \mathrm{Tr}\left(B_v^u \otimes |u\rangle\langle v|(\rho_\ell \otimes |\ell\rangle\langle\ell|)(B_v^{u*} \otimes |v\rangle\langle u|x\right) \\ &= \mathrm{Tr}(B_v^u \rho_v B_v^{u*} \otimes |u\rangle\langle u|x). \end{aligned}$$

The equality (ii) can be proven by the same way. $\square$

Proof of Proposition 4.2. The extension of φ on $\mathcal{A}_{\mathbb{Z}}$ is defined by (3.8). It is compatible, if the condition (3.13) is satisfied. Therefore, we check (3.13) for (4.2). Indeed, we have

$$\begin{aligned} \mathrm{Tr}^{(1)}\Big(\sum_{i,j} K_{ij}^*(\rho \otimes 1\!\!1)K_{ij}\Big) &= \mathrm{Tr}^{(1)}\Big(\sum_{i,j} M_j^i \otimes A_{ij}^*(\rho \otimes 1\!\!1)M_j^{i*} \otimes A_{ij}\Big) \\ &= \sum_{i,j} \mathrm{Tr}^{(1)}\left[\big(M_j^i \rho M_j^{i*}\big) \otimes A_{ij}^* A_{ij}\right] \\ &= \sum_{i,j} \mathrm{Tr}\big(M_j^i \rho M_j^{i*}\big)\frac{\rho_j \otimes |j\rangle\langle j|}{\mathrm{Tr}(\rho_j)}. \end{aligned} \tag{7.1}$$

Now using

$$\begin{aligned}
\mathrm{Tr}\big(M_j^i\rho M_j^{i*}\big) &= \mathrm{Tr}\big(B_j^i\otimes|i\rangle\langle j|\rho B_j^{i*}|j\rangle\langle i|\big)\\
&= \sum_\ell \mathrm{Tr}\big(B_j^i\otimes|i\rangle\langle j|\rho_\ell\otimes|\ell\rangle\langle\ell|B_j^{i*}|j\rangle\langle i|\big)\\
&= \mathrm{Tr}\big(B_j^i\rho_j B_j^{i*}|i\rangle\langle i|\big)\\
&= \mathrm{Tr}\big(B_j^i\rho_j B_j^{i*}\big)\\
&= \mathrm{Tr}\big(B_j^{i*}B_j^i\rho_j\big) \qquad (7.2)
\end{aligned}$$

from (7.2) one finds

$$\begin{aligned}
\mathrm{Tr}^{(1)}\Big(\sum_{i,j}K_{ij}^*(\rho\otimes\mathbb{1})K_{ij}\Big) &= \sum_{i,j}\mathrm{Tr}\big(B_j^{i*}B_j^i\rho_j\big)\frac{\rho_j\otimes|j\rangle\langle j|}{\mathrm{Tr}(\rho_j)}\\
&= \sum_j \mathrm{Tr}\Big(\sum_i B_j^{i*}B_j^i\rho_j\Big)\frac{\rho_j\otimes|j\rangle\langle j|}{\mathrm{Tr}(\rho_j)}\\
&= \sum_j \mathrm{Tr}(\rho_j)\frac{\rho_j\otimes|j\rangle\langle j|}{\mathrm{Tr}(\rho_j)}\\
&= \sum_j \rho_j\otimes|j\rangle\langle j|\\
&= \rho. \qquad (7.3)
\end{aligned}$$

Hence, the above defined QMC can be extended to $\mathcal{A}_{\mathbb{Z}}$.

To prove the equality (4.2), it is enough to prove for the case $n=2$ since the general formula can be proved by induction. From (4.4) and using Lemma 7.1 one finds

$$\begin{aligned}
\varphi(x_1\otimes x_2) &= \mathrm{Tr}(\rho\mathcal{E}(x_1\otimes\mathcal{E}(x_2\otimes\mathbb{1})))\\
&= \mathrm{Tr}\left(\rho\mathcal{E}(x_1\otimes\Big(\sum_{i,j}M_j^{i*}x_2M_j^i\Big)\right)\\
&= \sum_{i,j}\mathrm{Tr}\left(\rho\mathcal{E}(x_1\otimes M_j^{i*}x_2M_j^i)\right)\\
&= \sum_{i,j}\mathrm{Tr}\left(\rho\Big(\sum_{u,v}M_v^{u*}x_1M_v^u\Big)\right)\frac{\mathrm{Tr}(\rho_v\otimes|v\rangle\langle v|M_j^{i*}x_2M_j^i)}{\mathrm{Tr}(\rho_v)}\\
&= \sum_{u,v}\mathrm{Tr}\left(\rho M_v^{u*}x_1M_v^u\right)\sum_{i,j}\frac{\mathrm{Tr}(\rho_v\otimes|v\rangle\langle v|M_j^{i*}x_2M_j^i)}{\mathrm{Tr}(\rho_v)}\\
&= \sum_{u,v}\mathrm{Tr}\left(B_v^u\rho_vB_v^{u*}\otimes|u\rangle\langle u|x_1\right)\sum_i\frac{\mathrm{Tr}\left(B_v^i\rho_vB_v^{i*}\otimes|i\rangle\langle i|x_2\right)}{\mathrm{Tr}(\rho_v)}\\
&= \sum_v \mathrm{Tr}(\rho_v)\psi_v(x_1)\psi_v(x_2)
\end{aligned}$$

This completes the proof. ∎

Using the same idea of the proof we can get the following.

Corollary 7.2. *For any projection $e \in B(\mathcal{H} \otimes \mathcal{K}^\rho)$ one has*

$$\mathcal{E}(e \otimes E_{0]}(\tau_\infty^n)) = \sum_{u,v} M_v^{u*} e M_v^u \big(\psi_v(e^\perp)\big)^{n+1}.$$

Proof of Theorem 4.4. It is enough to check the equality (3.7) for the pair $(\rho, \mathcal{E}^t)$. From (4.7) we have

$$\varphi_0(\mathcal{E}^t(\mathbb{1} \otimes x)) = \mathrm{Tr}(\rho \mathcal{M}^*(x)) = \mathrm{Tr}(\mathcal{M}(\rho)x) = \mathrm{Tr}(\rho x) = \varphi_0(x).$$

Correspondingly, the equality

$$\varphi_\rho^t(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \mathrm{Tr}(\rho \mathcal{E}^t(x_1 \otimes \mathcal{E}^t(x_2 \otimes \cdots \otimes \mathcal{E}^t(x_n \otimes \mathbb{1})) \cdots)) \tag{7.4}$$

defines a QMC on $\mathcal{A}_\mathbb{Z}$. □

To prove Theorem 4.6 we need the following auxiliary fact.

Lemma 7.3. *One has*

$$\begin{aligned} \mathcal{E}^t(x_1 \otimes \mathcal{E}^t(x_2 \otimes \cdots \otimes \mathcal{E}^t(x_n \otimes \mathbb{1})) \cdots) \quad &= \sum_{i_1, i_2, \ldots, i_n} \big(B_{i_1}^{i_2*} B_{i_2}^{i_3*} \cdots B_{i_{n-1}}^{i_n*} B_{i_{n-1}}^{i_n} \cdots B_{i_2}^{i_3} B_{i_1}^{i_2} \otimes |i_1\rangle\langle i_1|\big) \\ &\quad \times \varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n), \end{aligned} \tag{7.5}$$

where

$$\varphi_k(x) = \frac{\mathrm{Tr}(\rho_k \otimes |k\rangle\langle k| x)}{\mathrm{Tr}(\rho_k)}.$$

Proof. Let us prove for $n = 2$. Then from (4.7) with (4.8) we obtain

$$\begin{aligned} \mathcal{E}^t(x_1 \otimes \mathcal{E}^t(x_2 \otimes \mathbb{1})) \quad &= \quad \mathcal{E}^t \left(x_1 \otimes \left(\sum_{i,j} M_j^{i*} M_j^i \varphi_j(x_2) \right) \right) \\ &= \quad \sum_{i,j} \mathcal{E}^t \left(x_1 \otimes (B_j^{i*} B_j^i \otimes |j\rangle\langle j|) \right) \varphi_j(x_2) \\ &= \quad \sum_j \mathcal{E}^t \left(x_1 \otimes (\mathbb{1} \otimes |j\rangle\langle j|) \right) \varphi_j(x_2) \\ &= \quad \sum_j \sum_{u,v} M_v^{u*} (\mathbb{1} \otimes |j\rangle\langle j|) M_v^{u*} \varphi_v(x_1) \varphi_j(x_2) \\ &= \quad \sum_{j,v} (B_v^{j*} B_v^{j*} \otimes |v\rangle\langle v|) \varphi_v(x_1) \varphi_j(x_2) \end{aligned}$$

which shows that (7.5) is true at $n = 2$. General setting can be proved by the same argument. □

Proof of Theorem 4.6. Due to the density argument, it is enough to prove the assertion for local elements of $\mathcal{A}$. Namely, one has

$$\begin{aligned} \varphi_\rho^t(x_0 \otimes x_1 \otimes \cdots \otimes x_n) \quad &= \sum_{i_0, i_1, \ldots, i_n} \mathrm{Tr} \left(B_{i_{n-1}}^{i_n} \cdots B_{i_1}^{i_2} B_{i_0}^{i_1} \rho_{i_0} B_{i_0}^{i_1*} B_{i_1}^{i_2*} \cdots B_{i_{n-1}}^{i_n} \right) \\ &\quad \times \varphi_{i_0}(x_0) \cdots \varphi_{i_n}(x_n). \end{aligned} \tag{7.6}$$

The last one immediately follows from Lemma 7.3. □

Proof of Theorem 4.8. (i)$\Rightarrow$ (ii). Assume that the map $\mathcal{M}^*$ is weak mixing, i.e. for every state κ one has $\mathcal{M}^n\kappa \to \rho$ weakly.

Due to the density argument, it is enough to prove the statement for local elements $x, y \in \mathcal{A}_{loc}$, i.e.

$$x = x_{i_0} \otimes \cdots \otimes x_{i_m}, \quad y = y_{j_0} \otimes \cdots \otimes y_{j_\ell}$$

Then due to equality (7.4) one finds

$$\begin{aligned} \varphi^t_\rho(x\alpha^n(y)) &= \mathrm{Tr}(\rho\mathcal{E}^t(x_1 \otimes \mathcal{E}^t(x_2 \otimes \cdots \mathcal{E}^t(x_m \otimes (\mathcal{M}^*)^n(\mathcal{E}^t(y_1 \otimes \mathcal{E}^t(y_2 \otimes \cdots \mathcal{E}^t(y_\ell \otimes \mathbb{1})))) \\ &\to \varphi^t_\rho(x)\varphi^t_\rho(y) \ \text{ as } \ n \to \infty \end{aligned}$$

which yields the assertion. The reverse implication (ii)$\Rightarrow$(i) follows from the last relation and $\mathcal{E}^t(B(\mathcal{H}) \otimes B(\mathcal{K}) \otimes \mathbb{1}) = B(\mathcal{H}) \otimes B(\mathcal{K})$.

The implication (ii)$\Rightarrow$ (iii) immediately follows from the equality (4.12). Therefore, we consider (iii)$\Rightarrow$ (ii). Assume that the measure μ_ρ is weak mixing. It is enough to prove the statement for local elements $x, y \in \mathcal{A}_{loc}$, i.e.

$$x = x_{i_0} \otimes \cdots \otimes x_{i_m}, \quad y = y_{j_0} \otimes \cdots \otimes y_{j_\ell}$$

Then due to (7.6) we have

$$\varphi^t_\rho(x\alpha^n(y)) = \sum_{\substack{i_0,\dots,i_n \\ j_0,\dots,j_\ell}} \mu_\rho(A^{[0,m]}(i_0,\dots,i_m)) \cap \sigma^{-n}(A^{[0,\ell]}(j_0,\dots,j_\ell))\varphi_{i_0,\dots,i_m}(x)\varphi_{j_0,\dots,j_\ell}(y)$$

Now using the weak mixing property of μ_ρ one finds

$$\begin{aligned} & \sum_{\substack{i_0,\dots,i_n \\ j_0,\dots,j_\ell}} \mu_\rho(A^{[0,m]}(i_0,\dots,i_m)) \cap \sigma^{-n}(A^{[0,\ell]}(j_0,\dots,j_\ell))\varphi_{i_0,\dots,i_m}(x)\varphi_{j_0,\dots,j_\ell}(y) \\ \to & \sum_{\substack{i_0,\dots,i_n \\ j_0,\dots,j_\ell}} \mu_\rho(A^{[0,m]}(i_0,\dots,i_m)))\mu_\rho(A^{[0,\ell]}(j_0,\dots,j_\ell))\varphi_{i_0,\dots,i_m}(x)\varphi_{j_0,\dots,j_\ell}(y) \\ = & \left(\sum_{i_0,\dots,i_n} \mu_\rho(A^{[0,m]}(i_0,\dots,i_m))\varphi_{i_0,\dots,i_m}(x)\right)\left(\sum_{j_0,\dots,j_\ell} \mu_\rho(A^{[0,\ell]}(j_0,\dots,j_\ell))\varphi_{j_0,\dots,j_\ell}(y)\right) \\ = & \ \varphi^t_\rho(x)\varphi^t_\rho(y) \ \text{ as } \ n \to \infty, \end{aligned}$$

which yields the weak mixing property of φ^t_ρ. The ergodicity can be proved by the same argument. This completes the proof. $\square$

8. Proofs for Section 5

Proof of Theorem 5.4. (i) Let $\varphi(J_n(e)) = 0$ for every $n \in \mathbb{N}$. For any $k, m \in \mathbb{N}$ we have

$$\mathbb{1}_{m]} \otimes \tau_k \le \mathbb{1}_{m+k-1]} \otimes J_{m+k}(e) \otimes \mathbb{1}_{[m+k+1},$$

therefore, one finds $\varphi(\beta^m(\tau_k)) \le \varphi(J_{m+k}(e)) = 0$. Hence, from (5.1) one gets $\varphi(\beta^m(\tau_\infty)) = 1$.

Now assume that $\varphi(\beta^m(\tau_\infty)) = 1$ for any $m \in \mathbb{N}$. Then again from (5.1) we obtain

$$\varphi\left(\beta^m\left(\sum_{k\ge 0}\tau_k\right)\right) = 0,$$

which implies $\varphi(\beta^m(\tau_k)) = 0$ for all $k \in \mathbb{N}$. This means $\varphi(\beta^m(\tau_0)) = \varphi(J_m(e)) = 0$.

(ii) Let e be φ-recurrent. Then from the definition and (5.1) one finds

$$\varphi(J_0(e)) = \varphi\left(J_0(e) \otimes \sum_{k\ge 0}\tau_k\right) = \varphi(J_0(e)) - \varphi(J_0(e) \otimes \tau_\infty),$$

which means $\varphi(J_0(e) \otimes \tau_\infty) = 0$. The reverse implication is obvious.

(iii) If φ is faithful, then the φ-completely accessibility of e is equivalent to $\tau_\infty = 0$, then from (ii) we have that e is φ-recurrent. Conversely, if e is φ-recurrent, then due to the faithfulness of φ with (ii) one gets $J_0(e)\otimes\tau_\infty = 0$, so $\tau_\infty = 0$ which means that e is φ-completely accessibility.

(iv) Assume that e is not φ-completely accessible, this means $\varphi(\tau_\infty) > 0$. Due to the φ-recurrence one has $\varphi(J_0(e)\otimes\tau_\infty) = 0$, which implies that

$$\lim_{n\to\infty}\varphi(J_0(e)\otimes\tau_\infty^n) = 0.$$

The last equality yields that

$$\lim_{n\to\infty}\varphi(J_0(e)\otimes J_1(e)\otimes\tau_\infty^n) = 0, \quad \lim_{n\to\infty}\varphi(J_0(e)\otimes J_1(e^\perp)\otimes\tau_\infty^n) = 0,$$

so

$$\lim_{n\to\infty}\varphi(J_0(e)\otimes\mathbb{1}\otimes\tau_\infty^n) = 0.$$

Hence, iterating the last equality, for every $k\in\mathbb{N}$ we have

$$\varphi(J_0(e)\otimes\mathbb{1}_{k-1]}\otimes\tau_\infty) = 0. \tag{8.1}$$

Since $\varphi(\tau_\infty) > 0$, then one can find a projection $p\in M_0$ ($p\neq 0$) such that $\tau_\infty \geq \lambda p$ for some positive number λ. Then from (8.1) we infer that

$$\varphi(J_0(e)\otimes\mathbb{1}_{k-1]}\otimes p) \leq \frac{1}{\lambda}\varphi(J_0(e)\otimes\mathbb{1}_{k-1]}\otimes\tau_\infty) = 0$$

this implies that e and p are not φ-communicate, which is a contradiction. This completes the proof. □

Proof of Corollary 5.5. Since, φ is a QMC on $\mathcal{A}_\mathbb{Z}$, then it is a translation invariant state. Therefore, the statement (i) immediately follows from (i) of Theorem 5.4. To establish (ii) again due to the translation invariance of φ we obtain

$$\varphi(\tau_\infty) = \varphi(\tau_\infty) - \varphi(e\otimes\tau_\infty)$$

which by (ii) Theorem 5.4 yields the assertion.

(iii) The faithfulness of φ implies that e is φ-completely accessible iff $\tau_\infty = 0$, which yields that e is $\mathcal{E}$-completely accessible. The reverse implication is obvious.

□

Proof of Theorem 5.6. (i) Let us consider the $\varphi^t_{p,k}$-recurrence of a projection $e_{Q,k} = Q\otimes|k\rangle\langle k|$. From (7.6),(4.8) and Remark 4.7, we obtain

$$\begin{aligned}
\varphi^t_{p,k}(e_{Q,k}\otimes\tau_n) &= \varphi^t_{p,k}(e_{Q,k}\otimes\underbrace{e^\perp_{Q,k}\otimes\cdots\otimes e^\perp_{Q,k}}_{n}\otimes e_{Q,k}) \\
&= \sum_{i_i,\dots,i_n}\mathbb{P}_{p,k}(k,i_1,\dots,i_n,k)\varphi_k(e_{Q,k})^2\varphi_{i_1}(e^\perp_{Q,k})\cdots\varphi_{i_n}(e^\perp_{Q,k}) \\
&= \sum_{i_i,\dots,i_n}\mathbb{P}_{p,k}(k,i_1,\dots,i_n,k)\,\mathrm{Tr}(pQ)^2\big(1-\mathrm{Tr}(pQ)\delta_{i_i,k}\big)\cdots\big(1-\mathrm{Tr}(pQ)\delta_{i_n,k}\big) \\
&= \sum_{\substack{i_i,\dots,i_n \\ i_\ell\neq k, 1\leq\ell\leq n}}\mathbb{P}_{p,k}(k,i_1,\dots,i_n,k)\,\mathrm{Tr}(pQ)^2 + \big(1-\mathrm{Tr}(pQ)\big)G, \qquad (8.2)
\end{aligned}$$

where G is some expression.

Due to $\mathrm{Tr}(pQ) = 1$, the last expression (8.2) implies that the projection $e_{Q,k}$ is $\varphi^t_{p,k}$-recurrent if and only if $\mathbb{P}_{p,k}(t_k<\infty) = 1$.

(ii) Now assume that $\mathrm{Tr}(pQ) < 1$, and denote $\lambda = \mathrm{Tr}(pQ)$. Clearly, $\lambda \in (0,1)$. Then again from (7.6),(4.8) we find

$$\begin{aligned}
\varphi^t_{p,k}(e^{\perp}_{Q,k}\otimes\tau^n_\infty) &= \varphi^t_{p,k}(e^{\perp}_{Q,k}\otimes\underbrace{e_{Q,k}\otimes\cdots\otimes e_{Q,k}}_{n+1}) \\
&= \sum_\ell \mathbb{P}_{p,k}(\underbrace{\ell,k,\dots,k}_{n+2})\big(1-\lambda\delta_{\ell,k})\lambda^{n+1} \\
&= \mathbb{P}_{p,k}(\underbrace{k,k,\dots,k}_{n+2})\big(1-\lambda)\lambda^{n+1} + \sum_{\ell\neq k}\mathbb{P}_{p,k}(\ell,k,\dots,k)\lambda^{n+1} \\
&\leq \big(1-\lambda)\lambda^{n+1} + \sum_{\ell\neq k}\mathbb{P}_{p,k}(\ell)\lambda^{n+1} \\
&\leq \big(1-\lambda)\lambda^{n+1} + (1-\mathbb{P}_{p,k}(k))\lambda^{n+1} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Hence, according to Theorem 5.4 (ii) from the last relation one gets that the projection $\mathbb{1} - P\otimes|k\rangle\langle k|$ is $\varphi^t_{p,k}$-recurrent. This completes the proof. $\square$

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