Research Article

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Periodic perturbations of Hamiltonian systems

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Abstract: We prove existence and multiplicity results for periodic solutions of Hamiltonian systems, by the use of a higher dimensional version of the Poincaré–Birkhoff fixed point theorem. The first part of the paper deals with periodic perturbations of a completely integrable system, while in the second part we focus on some suitable global conditions, so to deal with weakly coupled systems.

Keywords: Periodic solutions, Poincaré–Birkhoff theorem, perturbation theory

MSC 2010: 34C25, 47H15

1 Introduction

This paper provides some results on the existence of periodic solutions for Hamiltonian systems which may be considered as time-periodic perturbations of an autonomous system of the type

$$\mathcal{J}\dot{z} = \nabla\mathcal{H}(z). \tag{HS}$$

Here, $\mathcal{H}\colon \mathbb{R}^{2N} \to \mathbb{R}$ is a continuously differentiable function and $\mathcal{J}$ is the standard symplectic matrix, i.e.,

$$\mathcal{J} = \begin{pmatrix} 0 & -\mathbb{I}_N \\ \mathbb{I}_N & 0 \end{pmatrix}.$$

There is a large bibliography on this problem, mainly motivated by models from classical mechanics. Remarkably, we observe that the literature on this issue can be split into two quite disjoint streams.

One of the two currents has a more topological spirit, and aims at minimal regularity assumptions. This kind of results finds a fertile ground mainly in the planar case, where, among others, a powerful tool, the *Poincaré–Birkhoff fixed point theorem*, can be used to prove the existence of periodic solutions of the perturbed system. Indeed, the case $N = 1$ is privileged by the fact that, for an autonomous planar system like (HS), any periodic orbit is always surrounded by an annulus of periodic orbits. Then, assuming that the periods of the corresponding solutions do not remain the same, the needed *twist condition* is naturally obtained, and the Poincaré–Birkhoff theorem applies (see, e.g., [29] and the references therein).

On the other hand, when $N \geq 2$, a more analytical approach has usually been followed, requiring some additional structural assumptions on the unperturbed system (HS). Usually, the system is assumed to be *completely integrable*, and more regularity is asked for the Hamiltonian function. Moreover, some rather restrictive nondegeneracy conditions are needed so to obtain the existence of periodic solutions of the perturbed system (see, e.g., [4, 9]).

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The aim of this paper is to provide a common framework for the two kinds of approach depicted above. Using a recent result by the first author and Ureña [31], where an extension of the Poincaré–Birkhoff theorem to higher dimensional Hamiltonian systems has been proposed, we will be able, on one hand, to relax the usual structural assumptions on the Hamiltonian function and, on the other hand, to extend to higher dimensions some existence results already established in the planar case.

Before entering into details, we will now spend a few words on the framework where our results are to be settled.

The general framework

A classical approach to the study of the Hamiltonian system (HS) is the search for constants of motion, since they can be used for suitably transforming the system into a simpler one. The most remarkable case occurs when (HS) has N constants of motion which are independent and in involution: In this case, the system is said to be *completely integrable*, and one has a foliation of the space in N-dimensional surfaces, which are invariant for the flow.

The Liouville–Arnold theorem then assures that, when one of these surfaces is bounded and connected, it has to be an N-dimensional torus. Moreover, for any such invariant torus Γ, there exists an open neighborhood $\mathcal{A}$ of Γ and a canonical transformation $z = (x, y) \mapsto (\varphi, I)$, mapping $\mathcal{A}$ onto $\mathbb{T}^N \times \mathscr{D}$ (where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathscr{D}$ is an open subset of $\mathbb{R}^N$), and reducing the Hamiltonian function to the simpler form $\mathcal{H}(\varphi, I) = \mathscr{K}(I)$. The coordinates $I = (I_1, \dots, I_N) \in \mathscr{D}$ are usually known as *action variables*, whereas the coordinates $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathbb{T}^N$ are called *angle variables*.

Each value $I = I^0$ is associated with an invariant torus $\Gamma^0 = \mathbb{T}^N \times \{I^0\}$, where the dynamics of the system is completely described by the frequency vector $\omega^0 = \nabla\mathscr{K}(I^0)$. When the components $\omega_1^0, \dots, \omega_N^0$ are rationally independent, the solutions are *quasiperiodic* and each orbit is a dense subset of the N-torus Γ^0. Such tori are called *nonresonant*. Otherwise, we have a foliation in M-dimensional tori, where $M < N$ is the rational rank of the components of ω^0, and the orbits will be quasiperiodic with respect to these lower dimensional tori. A special case occurs when the components of ω^0 are all pairwise commensurable. Then, all the solutions on the torus are periodic with the same period, and the N-torus Γ^0 admits a foliation in invariant 1-tori, each one defined by the orbit of a solution.

Since for every general Hamiltonian system (HS) a constant of motion is always given by the Hamiltonian function $\mathcal{H}$, we immediately deduce that every planar Hamiltonian system is completely integrable. In higher dimensions, a classical example of a completely integrable system comes from the Kepler two-body problem, or even from every central force field [38]. On the contrary, if more than two bodies are involved, the system is not completely integrable any more. However, assuming the masses of the “planets” to be small compared to the mass of the “Sun”, the system may be seen as being decomposed in n independent two-body systems, with the addition of a small perturbative term accounting for the other interactions (cf. [18, 19] and references therein). Such problems of Celestial Mechanics have probably been the main stimulus in the development of integrability and of *Hamiltonian perturbation theory*.

As a matter of fact, completely integrable Hamiltonian systems are rare, and most often the Hamiltonian function is their unique constant of motion [8, 50]. Yet, generic Hamiltonian systems may be considered as perturbations of completely integrable systems [42, 47], usually called *nearly integrable systems*. A glance of this scenario was already grasped by Henri Poincaré [46], who referred to Hamiltonian perturbation theory as the *Problème général de la Dynamique*. The efforts made by Poincaré and, among many others, by Birkhoff, led to a broad development of the theory. We suggest [5, 7] for a detailed introduction to Hamiltonian perturbation theory and [26] for a friendly overview.

As we have seen, complete integrability reveals strong properties of the dynamics. A natural question is: How much of this structure is preserved under a small perturbation? In particular, one could wonder whether, near an invariant torus of the unperturbed system, it is possible to find periodic or quasiperiodic solutions for the perturbed system with the same frequency.

A series of positive results are known for a large family of nonresonant tori, those with a Diophantine frequency. These results are usually collected under the name of KAM theory, recalling its main contributors Kolmogorov, Arnold and Moser. We remark that, beyond a nondegeneracy assumption on the torus, strong smoothness of the perturbation is always needed, cf. [2, 35, 48]. While these strongly nonresonant tori survive under small perturbations, the same is not true for the other tori [11, 41, 51] and, in particular, for those made of periodic solutions. Still, some traces of these tori can be found.

For instance, in the planar case, after the pioneering papers [39, 40], the survival of two periodic solutions was obtained as a consequence of the Poincaré–Birkhoff theorem (see, e.g., [17] and [29], where an overview on the use of the Poincaré–Birkhoff theorem for this kind of problems can be found). The required twist condition is satisfied, in this case, under some rather weak nondegeneracy assumptions. A fainter kind of traces of an invariant torus is provided by the so called Aubry–Mather theory (cf. [43] and the references therein), showing the existence of a Cantor set, called *cantorus*, that preserves, in a generalized sense, the rotational properties of the original torus.

For higher dimensional Hamiltonian systems, a local approach to the problem has been proposed by Bernstein and Katok [9], who showed the survival under small perturbations of $N + 1$ periodic solutions, requiring a convexity assumption on the Hamiltonian function (see also [4, 27, 52]). This result has been later refined by Chen [21], who replaced the convexity by a classical nondegeneracy assumption.

A rather different type of problem arises when one looks for the existence and multiplicity of periodic solutions when only the global behavior of the nonlinearity is assumed to be known. In this case, the approach is no longer perturbative, and it usually combines topological and variational methods.

In this respect, there is a large literature in the planar case, mainly motivated by some models involving scalar second order differential equations, where the Poincaré–Birkhoff theorem has been successfully applied (see, e.g., [15, 25, 30, 34, 36], or again the review in [29]). The twist condition is generated by assuming a difference between the growth of the nonlinearity near a given periodic solution and at infinity, producing a gap in the rotation numbers of the corresponding solutions in the phase plane. A sharp use of the Poincaré–Birkhoff theorem then ensures that the larger this gap, the larger the number of solutions found. Furthermore, the same strategy applies also to the search for subharmonic solutions (see, for instance, [14, 25]).

Incidentally, the twist geometry has sometimes been recovered by detecting, in the unperturbed system, an annulus of periodic orbits displaying a gap between the periods of the boundary orbits. This picture displays the same features already discussed when considering completely integrable systems. A quite common way of producing this geometry is to require the strict monotonicity of the period function associated with system (HS), a feature which has been studied by many authors (see, e.g., [22, 33, 45]) and which ensures its nondegeneracy.

The first multiplicity results extending the Poincaré–Birkhoff philosophy to higher dimensions are due to Amann and Zehnder [3], who introduced a twist condition between zero and infinity. A different perspective was followed by Conley and Zehnder [24], where the existence of $N + 1$ periodic solutions was proved for systems whose Hamiltonian function is 2π-periodic in the first N variables, and asymptotically quadratic in the other N ones. These pioneering results have been generalized in several directions, in a long series of papers (cf. [1, 20] and the references therein). See also [16, 44], where a further extension of the Poincaré–Birkhoff theorem in higher dimensions involving a monotone twist has been exploited.

The main tool and an overview of our results

Let us now recall the result in [31], which will be our main tool in the search for periodic solutions. Consider the Hamiltonian system

$$\mathcal{J}\dot{\zeta} = \nabla_\zeta \mathscr{H}(t, \zeta), \tag{1.1}$$

where the continuous function $\mathscr{H} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is also continuously differentiable in $\zeta = (\xi, \eta) \in \mathbb{R}^{2N}$. Writing $\xi = (\xi_1, \dots, \xi_N)$ and $\eta = (\eta_1, \dots, \eta_N)$, the Hamiltonian function $\mathscr{H}$ is assumed to be T-periodic in t, and 2π-periodic in each variable $\xi_1, \dots, \xi_N$. Let $D \subset \mathbb{R}^N$ be a *convex body*, i.e., a compact, convex set with nonempty interior. For every $\bar{y} \in \partial D$, we denote the normal cone by

$$\mathcal{N}_D(\bar{y}) = \{v \in \mathbb{R}^N : \langle v, y - \bar{y} \rangle \leq 0 \text{ for every } y \in D\}.$$

Moreover, let $\mathbb{B}$ be an invertible symmetric matrix.

Theorem 1.1 ([31]). *If every solution $\zeta(t) = (\xi(t), \eta(t))$ of (1.1) departing with $\eta(0) \in \partial D$ is defined for every $t \in [0, T]$ and satisfies*

$$\langle \xi(T) - \xi(0), \mathbb{B}v \rangle > 0 \quad \text{for every } v \in \mathcal{N}_D(\eta(0)) \setminus \{0\}, \tag{1.2}$$

then system (1.1) has at least $N + 1$ geometrically distinct T-periodic solutions

$$\zeta^1(t) = (\xi^1(t), \eta^1(t)), \dots, \zeta^{N+1}(t) = (\xi^{N+1}(t), \eta^{N+1}(t))$$

such that $\eta^k(0) \in D$ for every $k = 1, \dots, N + 1$.

We recall that two solutions of system (1.1) are *geometrically distinct* if one of them cannot be obtained just by adding suitable integer multiples of 2π to some components $\xi_i(t)$ of the other one.

We now briefly describe the main results of this paper, obtained by the use of Theorem 1.1.

The first part deals with small time-dependent perturbations of completely integrable systems. In Section 2, taking an invariant torus made of periodic solutions of the unperturbed system, and assuming a rather weak nondegeneracy condition, we prove the survival of $N + 1$ periodic solutions for the perturbed system. Our main theorem thus improves some previous results of Bernstein and Katok [9] and Chen [21] in two directions: First, the Hamiltonian function is assumed to be *only once* continuously differentiable and, second, our nondegeneracy assumption does not even imply the invertibility of the frequency function. Moreover, it is shown that the nondegeneracy extends also to nearby tori, so that other families of periodic solutions can coappear.

In Section 3, still dealing with completely integrable systems, we gradually abandon the local point of view and move to a large scale perspective. Assuming a twist-type condition on the product of N planar annuli, which is shown to persist for small perturbations, we thus obtain the survival of $N + 1$ periodic solutions, generalizing the planar result in [29].

In Section 4, we deal with weakly coupled systems with a T-periodic forcing term, depending on some parameters. We impose suitable conditions at zero and infinity for each of the N equations, producing a gap in the rotation numbers of the uncoupled systems. Using Theorem 1.1, we then prove the existence of $N + 1$ periodic solutions having period T, and a number of subharmonic solutions which increases with the width of the gap. As an application, we can deal with weakly coupled systems of pendulum-like equations, generalizing the main result in [32].

Notation. In all the paper, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^N$, with its associated norm $\| \cdot \|$. We write $\mathcal{B}(x_0, r)$ for the open ball centered at x_0 with radius $r > 0$, and $\mathcal{B}[x_0, r]$ for the closed ball.

2 Periodic perturbations of completely integrable systems

Let us consider a completely integrable Hamiltonian system on $\mathbb{T}^N \times \mathscr{D}$, where $\mathbb{T}^N$ is the N-dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^N$, and $\mathscr{D}$ is an open subset of $\mathbb{R}^N$. The continuously differentiable Hamiltonian function $\mathcal{H} : \mathbb{T}^N \times \mathscr{D} \to \mathbb{R}$ can be written in the form $\mathcal{H}(\varphi, I) = \mathscr{K}(I)$. We recall that $I = (I_1, \dots, I_N) \in \mathscr{D}$ are the action variables, while $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathbb{T}^N$ are the angle variables.

For every $I^* \in \mathscr{D}$, the torus $\mathscr{T}^* = \mathbb{T}^N \times \{I^*\}$ is invariant for the flow, and its evolution in time is determined by the associated frequency vector

$$\omega^* = (\omega_1^*, \dots, \omega_N^*) = \nabla \mathscr{K}(I^*).$$

We are interested in the case when the dynamics on the torus $\mathscr{T}^*$ consists of a family of periodic orbits with minimal period T^*. This happens if and only if there exist N integers $a_1, \dots, a_N$ such that

$$T^* \omega_i^* = 2\pi a_i \quad \text{for every } i = 1, \dots, N,$$

and T^* is the minimum positive real number with such a property. The integers a_i count the number of rotations made by each periodic solution around the i-th component of the torus in a period T^*; the sign of a_i describes the sense of rotation.

A standard approach to study such a system, defined on $\mathbb{T}^N \times \mathscr{D}$, is to consider its canonical lift to $\mathbb{R}^N \times \mathscr{D}$. The Hamiltonian system then becomes

$$\begin{cases} \dot{\xi} = \nabla\mathscr{K}(\eta), \\ \dot{\eta} = 0, \end{cases} \tag{CI}$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and $\eta = (\eta_1, \dots, \eta_N) \in \mathscr{D}$. To be more precise, denoting by $\mathbb{I}_N$ the identity on $\mathbb{R}^N$ and by $P_N \colon \mathbb{R}^N \to \mathbb{T}^N$ the standard projection on the torus, the map $(P_N, \mathbb{I}_N) \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{T}^N \times \mathbb{R}^N$ is a local change of variables which transforms (ξ, η) into (φ, I). Each translation of 2π in the ξ_i coordinate for system (CI) corresponds to a single rotation in the φ_i coordinate for the original system.

Let us now consider a general nearly integrable Hamiltonian system on $\mathbb{T}^N \times \mathscr{D}$, with time-dependent Hamiltonian function $\mathcal{K} \colon \mathbb{R} \times \mathbb{T}^N \times \mathscr{D} \to \mathbb{R}$, sufficiently close to $\mathscr{K}$. The canonical lift then leads to the Hamiltonian system on $\mathbb{R}^N \times \mathscr{D}$ given by

$$\begin{cases} \dot{\xi} = \nabla_\eta K(t, \xi, \eta), \\ \dot{\eta} = -\nabla_\xi K(t, \xi, \eta). \end{cases} \tag{CI$_{\mathrm{per}}$}$$

The Hamiltonian function $K \colon \mathbb{R} \times \mathbb{R}^N \times \mathscr{D} \to \mathbb{R}$ is assumed to be continuous, T-periodic in the first variable, 2π-periodic in each variable ξ_i, and continuously differentiable in $\zeta = (\xi, \eta)$.

We now fix an $I^0 \in \mathscr{D}$ and introduce some kind of nondegeneracy condition at I^0. Usually, in the literature (see, e.g., [4, 9, 21]), it is assumed that $\mathscr{K}$ is twice continuously differentiable, and that

$$\det(\mathscr{K}''(I^0)) \neq 0. \tag{2.1}$$

Here, we only ask $\mathscr{K}$ to be once continuously differentiable, and that there exists an invertible symmetric $N \times N$ matrix $\mathbb{B}$ such that

$$0 \in \operatorname{cl}\Big\{\rho \in \,]0, +\infty[\, : \min_{\|I - I^0\| = \rho} \langle \nabla\mathscr{K}(I) - \nabla\mathscr{K}(I^0), \mathbb{B}(I - I^0)\rangle > 0\Big\}, \tag{2.2}$$

where $\operatorname{cl} A$ denotes the closure of a set A. Notice that (2.1) implies (2.2), taking $\mathbb{B} = \mathscr{K}''(I^0)$. On the other hand, the function $\mathscr{K}(I) = \|I - I^0\|^\alpha$ satisfies (2.2) with $\mathbb{B} = \mathbb{I}$, but not (2.1) if $\alpha > 2$. Moreover, we observe that (2.2) does not even require the local invertibility of $\nabla\mathscr{K}$. An easy example, with $N = 1$, is provided by the function $\mathscr{K}(I) = \int_0^I f(s)\,\mathrm{d}s$ with

$$f(s) = \begin{cases} \omega^0 + |s| \sin\Big(\dfrac{1}{s}\Big) & \text{if } s \neq 0, \\ \omega^0 & \text{if } s = 0. \end{cases}$$

Clearly, this function $\mathscr{K}$ is only once continuously differentiable at $I^0 = 0$, and $\nabla\mathscr{K} = f$ is not invertible, but our nondegeneracy condition (2.2) is still satisfied, with $\mathbb{B}$ being the identity on $\mathbb{R}$.

We will show that the nondegeneracy condition (2.2) extends by continuity to a neighborhood $\mathcal{U}$ of I^0. As a consequence, we will prove that for every $I^* \in \mathcal{U}$ as above, if there exist two positive integers m^* and n^* satisfying

$$T^* = \frac{m^* T}{n^*}, \tag{2.3}$$

then the perturbed system (CI$_{\mathrm{per}}$) has at least $N + 1$ geometrically distinct m^*T-periodic solutions. These solutions stay near the corresponding solutions of the unperturbed problem, and their projections on $\mathbb{T}^N \times \mathscr{D}$ will maintain the same rotational properties of $\mathscr{T}^*$.

Here is our main result.

Theorem 2.1. *Suppose that there exist $I^0 \in \mathscr{D}$ and an invertible symmetric $N \times N$ matrix $\mathbb{B}$ such that* (2.2) *holds. Then, for every $\sigma > 0$ there exists an open neighborhood $\mathcal{U} \subseteq \mathscr{D}$ of I^0, with the following property: Given any positive integer $\overline{m}$, there exists $\varepsilon > 0$ such that if*

$$\|\nabla_\xi K(t, \xi, \eta)\| + \|\nabla_\eta K(t, \xi, \eta) - \nabla\mathscr{K}(\eta)\| < \varepsilon \quad \textit{for every } (t, \xi, \eta) \in [0, T] \times [0, 2\pi]^N \times \mathscr{D}, \tag{2.4}$$

then for every $I^* \in \mathcal{U}$ being associated with an invariant torus of periodic solutions for (CI) *with frequency vector $\omega^* = (\omega_1^*, \dots, \omega_N^*)$ and minimal period T^* satisfying* (2.3) *for suitable positive integers $m^* \leq \overline{m}$ and n^*, system* (CI$_{\text{per}}$) *has at least $N + 1$ geometrically distinct $m^* T$-periodic solutions*

$$(\xi^1(t), \eta^1(t)), \dots, (\xi^{N+1}(t), \eta^{N+1}(t))$$

with

$$\|\xi^k(t) - \xi^k(0) - t\nabla\mathscr{K}(I^*)\| + \|\eta^k(t) - I^*\| \leq \sigma \tag{2.5}$$

for every $t \in [0, m^* T]$ and $k = 1, \dots, N+1$. Moreover, for each solution $(\xi^k(t), \eta^k(t))$, its projection on $\mathbb{T}^N \times \mathscr{D}$ makes exactly $(\omega_i^*/2\pi)m^* T$ rotations around the i-th component of the torus in a period $m^* T$ for every $i = 1, \dots, N$.

Proof. We can assume, without loss of generality, the function $\mathscr{K}$ to be defined on the whole space $\mathbb{R}^N$. Indeed, after replacing the set $\mathscr{D}$ by a smaller open set, containing I^0, where $\mathscr{K}$ is bounded, we can construct a continuously differentiable extension of $\mathscr{K}$ on $\mathbb{R}^N$. The solutions we are interested in will nevertheless be contained in the smaller set, where $\mathscr{K}$ has not been modified. Similarly, for our purposes we can assume without loss of generality that the Hamiltonian system (CI$_{\text{per}}$) is defined on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

Let us fix any $\sigma > 0$ such that $\mathscr{B}[I^0, \sigma] \subseteq \mathscr{D}$. By assumption (2.2), there exist $\ell > 0$ and $\rho_1 \in]0, \sigma/4]$ such that

$$\|\eta - I^0\| = \rho_1 \;\Rightarrow\; \langle \nabla\mathscr{K}(\eta) - \nabla\mathscr{K}(I^0), \mathbb{B}(\eta - I^0)\rangle \geq 4\ell.$$

By continuity, there is an open neighborhood $\mathcal{U}$ of I^0, contained in $\mathscr{B}[I^0, \rho_1]$, such that for every $I^* \in \mathcal{U}$,

$$\|\eta - I^*\| = \rho_1 \;\Rightarrow\; \langle \nabla\mathscr{K}(\eta) - \nabla\mathscr{K}(I^*), \mathbb{B}(\eta - I^*)\rangle \geq 2\ell. \tag{2.6}$$

For any arbitrary $I^* \in \mathcal{U}$, with frequency vector $\omega^* = (\omega_1^*, \dots, \omega_N^*) = \nabla\mathscr{K}(I^*)$, let us define

$$K^*(t, \xi, \eta) = K(t, \xi + \omega^* t, \eta) - \langle \omega^*, \eta \rangle,$$

and consider the Hamiltonian system

$$\mathbb{J}\dot{\zeta} = \nabla_\zeta K^*(t, \zeta). \tag{2.7}$$

Claim. For any fixed positive real numbers $\overline{m}$ and $\bar{c}$, there exists $\varepsilon > 0$ such that if (2.4) holds, then for every $I^* \in \mathcal{U}$, every solution $\zeta(t) = (\xi(t), \eta(t))$ of (2.7) with initial point satisfying $\|\eta(0) - I^*\| \leq \rho_1$ will be such that

$$\|\xi(t) - \xi(0) - t[\nabla\mathscr{K}(\eta(0)) - \omega^*]\| + \|\eta(t) - \eta(0)\| \leq \bar{c} \quad \text{for every } t \in [0, \overline{m}T]. \tag{2.8}$$

Proof of the claim. Arguing by contradiction, assume that there is a sequence $(I_\lambda^*)_\lambda \in \mathcal{U}$ with $\omega_\lambda^* = \nabla\mathscr{K}(I_\lambda^*)$, and a sequence $(K_\lambda)_\lambda$ of Hamiltonian functions as above (in particular, they are T-periodic in t), such that, writing

$$K_\lambda^*(t, \xi, \eta) = K_\lambda(t, \xi + \omega_\lambda^* t, \eta) - \langle \omega_\lambda^*, \eta \rangle,$$

one has that

$$\|\nabla_\xi K_\lambda^*(t, \xi, \eta)\| + \|\nabla_\eta K_\lambda^*(t, \xi, \eta) - \nabla\mathscr{K}(\eta) + \omega_\lambda^*\| \leq \frac{1}{\lambda} \quad \text{for every } (t, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^N \times \mathscr{D},$$

and, accordingly, a sequence $(\zeta^\lambda)_\lambda$ with $\zeta^\lambda = (\xi^\lambda, \eta^\lambda)$, solving $\mathbb{J}\dot{\zeta}^\lambda = \nabla_\zeta K_\lambda^*(t, \zeta^\lambda)$, such that $\|\eta^\lambda(0) - I_\lambda^*\| \leq \rho_1$, while (2.8) does not hold, i.e., for every λ there exists $t_\lambda \in [0, \overline{m}T]$ for which

$$\|\xi^\lambda(t_\lambda) - \xi^\lambda(0) - t_\lambda[\nabla\mathscr{K}(\eta^\lambda(0)) - \omega_\lambda^*]\| + \|\eta^\lambda(t_\lambda) - \eta^\lambda(0)\| > \bar{c}. \tag{2.9}$$

Since the Hamiltonians K_λ^* are 2π-periodic in the variables $\xi_1, \dots, \xi_N$, we can assume that $\xi^\lambda(0) \in [0, 2\pi]^N$. Hence, passing to a subsequence, $\zeta^\lambda(0)$ converges to some point $\zeta^\sharp \in [0, 2\pi]^N \times \mathscr{B}[I^0, 2\rho_1]$. Moreover, for a subsequence, I_λ^* converges to some $I^\sharp$, and $\omega_\lambda^* = \nabla\mathscr{K}(I_\lambda^*)$ converges to $\omega^\sharp = \nabla\mathscr{K}(I^\sharp)$. Finally, for a subsequence, t_λ will converge to some $t^\sharp \in [0, \overline{m}T]$. By a lemma of Kamke (cf. [49]), for a further subsequence $(\zeta^{\lambda_l})_l$ we have uniform convergence on $[0, \overline{m}T]$ to the solution of

$$\begin{cases} \dot{\xi} = \nabla\mathscr{K}(\eta) - \omega^\sharp, \\ \dot{\eta} = 0, \end{cases}$$

given by

$$\begin{cases} \xi(t) = \xi(0) + t(\nabla\mathscr{K}(\eta(0)) - \omega^\sharp), \\ \eta(t) = \eta(0). \end{cases}$$

On the other hand, passing to the limit in (2.9) yields

$$\|\xi(t^\sharp) - \xi(0) - t^\sharp[\nabla\mathscr{K}(\eta(0)) - \omega^\sharp]\| + \|\eta(t^\sharp) - \eta(0)\| \geq \bar{c} > 0,$$

which is a contradiction, since the left-hand side is equal to zero. The claim is thus proved. □

We can now conclude the proof of Theorem 2.1. Let $\overline{m}$ be a fixed positive integer, and choose $\bar{c}$ such that

$$\bar{c} \leq \min\left\{\frac{T\ell}{\|\mathbb{B}\|\rho_1}, \frac{\sigma}{4}\right\}.$$

We now focus our attention on those $I^* \in \mathcal{U}$ whose associated invariant torus is composed of periodic solutions for (CI) with minimal period T^*, such that there exist two positive integers m^* and n^* with $m^* \leq \overline{m}$ and $T^* = m^*T/n^*$. We observe that every m^*T-periodic solution of (2.7) corresponds to an m^*T-periodic solution $(\xi(t), \eta(t))$ of (CI$_{\text{per}}$), such that every $\xi_i(t)$ makes exactly $(\omega_i^*/2\pi)m^*T$ turns around the origin in the time m^*T. We will apply Theorem 1.1 to system (2.7).

Let $D = \mathscr{B}[I^*, \rho_1]$, and let $\zeta(t) = (\xi(t), \eta(t))$ be a solution of (2.7) with $\eta(0) \in \partial D$, i.e., $\|\eta(0) - I^*\| = \rho_1$. Then, by (2.6) and (2.8), we get

$$\begin{aligned} \langle \xi(m^*T) - \xi(0), \mathbb{B}(\eta(0) - I^*)\rangle &= \langle \xi(m^*T) - \xi(0) - m^*T[\nabla\mathscr{K}(\eta(0)) - \nabla\mathscr{K}(I^*)], \mathbb{B}(\eta(0) - I^*)\rangle \\ &\quad + \langle m^*T[\nabla\mathscr{K}(\eta(0)) - \nabla\mathscr{K}(I^*)], \mathbb{B}(\eta(0) - I^*)\rangle \\ &\geq -\frac{T\ell}{\|\mathbb{B}\|\rho_1}\|\mathbb{B}\|\rho_1 + 2m^*T\ell \geq m^*T\ell > 0. \end{aligned}$$

We can therefore apply Theorem 1.1, so to get $N+1$ geometrically distinct m^*T-periodic solutions of (2.7),

$$\zeta^1(t) = (\xi^1(t), \eta^1(t)), \dots, \zeta^{N+1}(t) = (\xi^{N+1}(t), \eta^{N+1}(t)),$$

such that $\eta^k(0) \in D$ for every $k = 1, \dots, N+1$. Moreover, by (2.8), we have that $\|\eta^k(t) - I^*\| \leq \bar{c} \leq \sigma/2$ for every $t \in [0, m^*T]$. On the other hand, a continuity argument can be used, taking smaller values for $\bar{c}$ and ε, to infer that $\|\xi^k(t) - \xi^k(0) - t\nabla\mathscr{K}(I^*)\| \leq \sigma/2$ for every $t \in [0, m^*T]$. So, (2.5) holds, as well, and the proof is thus completed. □

Notice that, taking $\overline{m}$ sufficiently large, it is possible to find an arbitrarily large number of values $I^* \in \mathcal{U}$ for which the assumptions of Theorem 2.1 are satisfied, thus assuring the survival of $N+1$ subharmonic solutions from each of the corresponding invariant tori. This scenario may be compared with Birkhoff–Lewis type results [10, 13, 23], showing the existence of a family of periodic solutions with large period, accumulating towards an elliptic equilibrium. Such behavior has been observed also in the framework of Hamiltonian PDEs [6, 12].

A simple case is given by the choice $I^* = I^0$, when I^0 is associated with an invariant torus $\mathscr{T}^0$ of periodic solutions for (CI) with frequency vector ω^0 and minimal period T^0.

Corollary 2.2. *Suppose that there exists $I^0 \in \mathscr{D}$ and an invertible symmetric $N \times N$ matrix $\mathbb{B}$ such that* (2.2) *holds, and that there exist two positive integers m^0 and n^0 satisfying $T^0 = m^0T/n^0$. Then, for every $\sigma > 0$ there exists $\varepsilon > 0$ such that if*

$$\|\nabla_\xi K(t, \xi, \eta)\| + \|\nabla_\eta K(t, \xi, \eta) - \nabla\mathscr{K}(\eta)\| < \varepsilon \quad \textit{for every } (t, \xi, \eta) \in [0, T] \times [0, 2\pi]^N \times \mathscr{D},$$

then system (CI$_{\text{per}}$) *has at least $N+1$ geometrically distinct m^0T-periodic solutions*

$$(\xi^1(t), \eta^1(t)), \dots, (\xi^{N+1}(t), \eta^{N+1}(t))$$

with the same rotational properties of the torus $\mathscr{T}^0$ and such that

$$\|\xi^k(t) - \xi^k(0) - t\nabla\mathscr{K}(I^0)\| + \|\eta^k(t) - I^0\| \leq \sigma$$

for every $t \in [0, m^0T]$ and $k = 1, \dots, N+1$.

3 Twist conditions for weakly coupled period annuli

In the previous section, we have described the local phenomenon of the survival of some periodic solutions of system (CI) for the perturbed system ($\mathrm{CI_{per}}$). We now turn our attention to finding some conditions at a larger scale which guarantee the existence of multiple periodic solutions.

We still consider system ($\mathrm{CI_{per}}$) as a perturbation of system (CI), but we now look for periodic solutions $(\xi(t), \eta(t))$ starting with $\eta(0)$ in some rectangle

$$D = [\alpha_1, \beta_1] \times \cdots \times [\alpha_N, \beta_N],$$

contained in $\mathscr{D}$. We denote the faces of this rectangle by

$$\mathcal{F}_i^- = \{\eta \in D : \eta_i = \alpha_i\}, \quad \mathcal{F}_i^+ = \{\eta \in D : \eta_i = \beta_i\}.$$

Theorem 3.1. *Suppose that there exist N couples of real numbers $\omega_i^- < \omega_i^+$ such that for every $i = 1, \dots, N$, either*

$$\frac{\partial \mathscr{K}}{\partial \eta_i}(\eta) \begin{cases} \geq \omega_i^+ & \text{for every } \eta \in \mathcal{F}_i^-, \\ \leq \omega_i^- & \text{for every } \eta \in \mathcal{F}_i^+, \end{cases} \tag{3.1}$$

or

$$\frac{\partial \mathscr{K}}{\partial \eta_i}(\eta) \begin{cases} \leq \omega_i^- & \text{for every } \eta \in \mathcal{F}_i^-, \\ \geq \omega_i^+ & \text{for every } \eta \in \mathcal{F}_i^+. \end{cases} \tag{3.2}$$

Let $\omega^* = (\omega_1^*, \dots, \omega_N^*)$ be the frequency vector associated with a torus $\mathscr{T}^*$ of periodic solutions of system (CI), *with minimal period T^*. If*

$$\omega^* \in \Omega =]\omega_1^-, \omega_1^+[\times \cdots \times]\omega_N^-, \omega_N^+[,$$

and there exist two positive integers m^* and n^* such that (2.3) *holds, then there exists $\varepsilon > 0$ such that every perturbed system* ($\mathrm{CI_{per}}$) *satisfying* (2.4) *has at least $N + 1$ geometrically distinct $m^* T$-periodic solutions*

$$(\xi^1(t), \eta^1(t)), \dots, (\xi^{N+1}(t), \eta^{N+1}(t)),$$

preserving the same rotational properties of $\mathscr{T}^*$.

Proof. By the Poincaré–Miranda theorem (cf. [28, 37]), there exists an $I^* \in D$ such that $\omega^* = \nabla \mathscr{K}(I^*)$. We consider the Hamiltonian system

$$\mathcal{J}\dot{\zeta} = \nabla_\zeta K^*(t, \zeta) \tag{3.3}$$

with $K^*(t, \xi, \eta) = K(t, \xi + \omega^* t, \eta) - \langle \omega^*, \eta \rangle$.

Let us pick any $\rho > 0$ such that

$$\rho < \operatorname{dist}(D, \mathbb{R}^N \setminus \mathscr{D}) \quad \text{and} \quad \rho < m^* T \operatorname{dist}(\omega^*, \mathbb{R}^N \setminus \Omega).$$

By the same argument used in the claim within the proof of Theorem 2.1, there exists $\varepsilon_1 > 0$ such that if (2.4) holds with $\varepsilon \in]0, \varepsilon_1[$, then every solution $\zeta(t) = (\xi(t), \eta(t))$ of (3.3) with initial point $\eta(0) \in D$ remains in $\mathbb{R}^N \times \mathscr{D}$ for $t \in [0, m^* T]$, and satisfies

$$\|\xi(t) - \xi(0) - t[\nabla \mathscr{K}(\eta(0)) - \omega^*]\| + \|\eta(t) - \eta(0)\| < \rho$$

for every $t \in [0, m^* T]$. Assume that $\eta(0) \in \partial D$; we analyze four different cases.

If $\eta_i(0) = \alpha_i$ for some $i \in \{1, \dots, N\}$, and condition (3.1) holds, then

$$\xi_i(m^* T) - \xi_i(0) > m^* T[\omega_i^+ - \omega_i^*] - \rho > 0.$$

The same is true if $\eta_i(0) = \beta_i$ and (3.2) holds.

If $\eta_i(0) = \alpha_i$ and condition (3.2) holds, then

$$\xi_i(m^* T) - \xi_i(0) < m^* T[\omega_i^- - \omega_i^*] + \rho < 0,$$

and the same is true if $\eta_i(0) = \beta_i$ and (3.1) holds.

Let us define the $N \times N$ diagonal matrix $\mathbb{B}$ with, for each $i = 1, \dots, N$, $\mathbb{B}_{ii} = -1$ when (3.1) holds, and $\mathbb{B}_{ii} = +1$ when (3.2) is true. The estimates above ensure us that system (3.3) satisfies all the assumptions of Theorem 1.1, and the conclusion easily follows. □

Let us now describe a particular situation when Theorem 3.1 can be applied, generalizing the planar setting studied in [29]. We start by considering the autonomous Hamiltonian system

$$\mathcal{J}\dot{z} = \nabla\mathcal{H}(z), \tag{3.4}$$

where $\mathcal{H}\colon \mathbb{R}^{2N} \to \mathbb{R}$ is a continuously differentiable function of the special form

$$\mathcal{H}(x, y) = \mathcal{H}_1(x_1, y_1) + \cdots + \mathcal{H}_N(x_N, y_N)$$

with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$. Here we have used the notation $z = (x, y)$.

Hence, for every $i = 1, \dots, N$, the functions $\mathcal{H}_i\colon \mathbb{R}^2 \to \mathbb{R}$ are planar Hamiltonians, and we can consider the corresponding Hamiltonian systems

$$\dot{x}_i = \frac{\partial}{\partial y_i}\mathcal{H}_i(x_i, y_i), \quad \dot{y}_i = -\frac{\partial}{\partial x_i}\mathcal{H}_i(x_i, y_i), \tag{HS$_i$}$$

for each of which we assume the following:

- The planar system (HS$_i$) has a periodic solution $(\bar{x}_i(t), \bar{y}_i(t))$, which is non-constant and has minimal period $\overline{T}_i > 0$.
- Each of such solutions has a corresponding planar open tubular neighborhood $\mathcal{A}_i$ such that all the solutions of (HS$_i$) with initial point in $\mathcal{A}_i$ are periodic, and their orbits are not contractible in $\mathcal{A}_i$.
- There exist two positive real numbers T_i^-, T_i^+, with $T_i^- < \overline{T}_i < T_i^+$, such that the periods of the solutions in $\mathcal{A}_i$ cover the interval $[T_i^-, T_i^+]$.

Let us define the set

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^{2N} : (x_i, y_i) \in \mathcal{A}_i \text{ for every } i = 1, \dots, N\},$$

and consider the Hamiltonian system

$$\mathcal{J}\dot{z} = \nabla_z H(t, z), \tag{HS$_{\mathrm{per}}$}$$

where $H\colon \mathbb{R} \times \mathcal{A} \to \mathbb{R}$ is continuous, T-periodic in its first variable for some $T > 0$, and has a continuous gradient with respect to its second variable $z = (x, y)$.

For every $i = 1, \dots, N$, let us pick $T_i \in \,]T_i^-, T_i^+[$ for which there exist two positive integers m_i, n_i such that

$$T_i = \frac{m_i T}{n_i}.$$

Denoting by $a_1, \dots, a_N$ the minimal positive integers such that

$$a_1 \frac{m_1}{n_1} = \cdots = a_N \frac{m_N}{n_N},$$

we set

$$T^* = a_1 T_1 = \cdots = a_N T_N,$$

and define the frequency vector

$$\omega^* = \frac{2\pi}{T^*}(a_1, \dots, a_N).$$

Moreover, we choose the two least positive integers m^*, n^* such that

$$T^* = \frac{m^* T}{n^*}.$$

Theorem 3.2. *In the above setting, there exists* $\varepsilon > 0$ *such that every perturbed system* ($\mathrm{HS_{per}}$)*, satisfying*

$$\|\nabla_z H(t,z) - \nabla\mathcal{H}(z)\| < \varepsilon \quad \text{for every } (t,z) \in [0,T] \times \mathcal{A}, \tag{3.5}$$

has at least $N+1$ *distinct* $m^* T$*-periodic solutions*

$$z^1(t), \dots, z^{N+1}(t),$$

whose orbits lie in $\mathcal{A}$*. Moreover, for each solution* $z^k(t)$*, the number of rotations of the i-th component* $z_i^k(t)$ *along the annulus* $\mathcal{A}_i$ *in a period* $m^* T$ *is exactly equal to* $n^* a_i$ *for every* $i = 1, \dots, N$*.*

Proof. By standard arguments (cf. [29]), each of the systems (HS_i) admits a canonical transformation in action-angle coordinates (φ_i, I_i). Without loss of generality we can assume that $\dot\varphi_i(t) > 0$ for every t. The product of all such transformations is canonical, it reduces system (3.4) to the form (CI), and maps the set $\mathcal{A}$ onto $\mathbb{T}^N \times \mathscr{D}$, where $\mathscr{D} \subseteq \mathbb{R}^N$ is a product of open intervals.

For each $i = 1, \dots, N$, we define α_i and β_i as the values of the I_i-coordinate associated with two solutions of (HS_i) having periods T_i^- and T_i^+, in such a way that $\alpha_i < \beta_i$, and we set

$$\omega_i^- = \frac{2\pi}{T_i^+}, \quad \omega_i^+ = \frac{2\pi}{T_i^-}.$$

Theorem 3.1 then applies, and the proof is readily completed. □

4 Weakly coupled pendulum-like systems

In this section, we consider a weakly coupled system of the type

$$\begin{cases} J\dot z_1 = A_1 \nabla H_1(z_1) + \mathcal{R}_1(t, z_1, \dots, z_N), \\ \quad\vdots \\ J\dot z_N = A_N \nabla H_N(z_N) + \mathcal{R}_N(t, z_1, \dots, z_N), \end{cases} \tag{P}$$

where J is the 2×2 standard symplectic matrix, namely

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $A_1, \dots, A_N$ are positive real parameters. For every $i = 1, \dots, N$, we assume that $H_i : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, and $\mathcal{R}_i : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is continuous, T-periodic in t and continuously differentiable in $(z_1, \dots, z_N)$.

We assume that system (P) can be reduced to a Hamiltonian system by a linear change of variables. More precisely, there exist N invertible 2×2 matrices $\mathbb{M}_1, \dots, \mathbb{M}_N$, having positive determinant, such that the linear operator $\mathcal{L} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, defined as

$$\mathcal{L} : (z_1, \dots, z_N) \mapsto (\mathbb{M}_1 z_1, \dots, \mathbb{M}_N z_N), \tag{4.1}$$

transforms system (P) into a Hamiltonian system. With such an assumption, we will say that (P) is a *positive transformation of a Hamiltonian system*.

Let us introduce the following notation for a closed cone in $\mathbb{R}^2$ determined by two angles $\vartheta_1 < \vartheta_2$:

$$\Theta(\vartheta_1, \vartheta_2) = \{(\rho\cos\vartheta, \rho\sin\vartheta) : \rho \geq 0, \vartheta_1 \leq \vartheta \leq \vartheta_2\}.$$

We are now ready to state the main theorem of this section.

Theorem 4.1. *Let* (P) *be a positive transformation of a Hamiltonian system. For every* $i = 1, \dots, N$, *let the following assumptions hold:*
($\mathscr{A}_1$) *There exists* $C_i > 0$ *such that*

$$\|\nabla H_i(w)\| \le C_i(\|w\| + 1) \quad \textit{for every } w \in \mathbb{R}^2.$$

($\mathscr{A}_2$) *There exist* $r_i > 0$ *and* $m_i > 0$ *such that*

$$\langle \nabla H_i(w), w\rangle \ge m_i\|w\|^2 \quad \textit{for every } w \in \mathscr{B}[0, r_i].$$

($\mathscr{A}_3$) *For every* $\sigma > 0$, *there exist* $R_i > 0$ *and* $\vartheta_1^i < \vartheta_2^i$, *with* $\vartheta_2^i - \vartheta_1^i \le 2\pi$, *such that*

$$\sup\left\{\frac{\langle \nabla H_i(w), w\rangle}{\|w\|^2} : w \in \Theta(\vartheta_1^i, \vartheta_2^i) \setminus \mathscr{B}(0, R_i)\right\} \le \sigma(\vartheta_2^i - \vartheta_1^i). \tag{4.2}$$

Then, for every fixed positive integers $\nu_1, \dots, \nu_N$, *there exist* $A > 0$ *and* $\varepsilon > 0$ *such that if* $A_i \ge A$ *and*

$$\|\mathcal{R}_i(t, w_1, \dots, w_N)\| \le \varepsilon \quad \textit{for every } t \in [0, T] \textit{ and } w_1, \dots, w_N \in \mathbb{R}^2 \tag{4.3}$$

for every $i = 1, \dots, N$, *then system* (P) *has at least* $N + 1$ *distinct* T*-periodic solutions*

$$z^k(t) = (z_1^k(t), \dots, z_N^k(t))$$

such that for every $k = 1, \dots, N + 1$, *each planar component* $z_i^k(t)$, *with* $i = 1, \dots, N$, *makes exactly* ν_i *clockwise rotations around the origin in the time interval* $[0, T[$.

Some comments on the hypotheses of Theorem 4.1 are in order. Assumption ($\mathscr{A}_1$) is needed to ensure the global existence of the solutions to the Cauchy problems associated with (P). Concerning ($\mathscr{A}_2$), it will guarantee that the small amplitude planar components of the solutions do rotate around the origin, clockwise, with a least positive angular speed. Our hypothesis ($\mathscr{A}_3$), on the contrary, will ensure a small rotation number for large amplitude components. It could be compared with assumption (H'_∞) in [14, Theorem 4.1].

We now start the proof of Theorem 4.1. For a solution $z(t)$ of system (P), whose i-th component is such that $z_i(t) = (x_i(t), y_i(t)) \in \mathbb{R}^2 \setminus \{0\}$ for every $t \in [0, T]$, we denote by $\mathrm{Rot}(z_i(t); [0, T])$ the standard clockwise winding number of the path $t \mapsto z_i(t)$ around the origin, namely

$$\mathrm{Rot}(z_i(t); [0, T]) = \frac{1}{2\pi}\int_0^T \frac{\langle J\dot z_i(t), z_i(t)\rangle}{\|z_i(t)\|^2}\,\mathrm{d}t.$$

Our first lemma concerns solutions $z(t)$ whose i-th component $z_i(t)$ is small. We assume without loss of generality that $H_i(0) = 0$, and consider the level set

$$\Gamma_i^h = \{w \in \mathbb{R}^2 : H_i(w) = h\}.$$

By ($\mathscr{A}_2$), if $h > 0$ is sufficiently small, then Γ_i^h is a strictly star-shaped Jordan curve around the origin. We will denote by D_i^h the bounded, closed and connected region of $\mathbb{R}^2$ with $\partial D_i^h = \Gamma_i^h$.

Lemma 4.2. *For any* $i = 1, \dots, N$ *and every positive integer* ν_i, *if* ($\mathscr{A}_1$) *and* ($\mathscr{A}_2$) *hold, there exist three positive constants* $\bar A_i$, $\bar\varepsilon_i$ *and* $\bar h_i$ *such that, if* $A_i \ge \bar A_i$, $h \in]0, \bar h_i]$ *and*

$$\|\mathcal{R}_i(t, w_1, \dots, w_N)\| \le \bar\varepsilon_i \quad \textit{for every } t \in [0, T] \textit{ and } w_1, \dots, w_N \in \mathbb{R}^2, \tag{4.4}$$

then any solution $z(t)$ *of* (P) *with* $z_i(0) \in \Gamma_i^h$ *satisfies*

$$\mathrm{Rot}(z_i(t); [0, T]) > \nu_i.$$

Proof. Let $i \in \{1, \dots, N\}$ and ν_i be fixed. We can choose $h > 0$ and $\hat r \in]0, r_i[$, where r_i is as in assumption ($\mathscr{A}_2$), in such a way that

$$\mathscr{B}(0, \hat r) \subset D_i^h \subset D_i^{2h} \subset D_i^{3h} \subset \mathscr{B}(0, r_i). \tag{4.5}$$

We now claim that if (4.4) holds with a suitable choice of $\bar{\varepsilon}_i$, then for every solution $z(t)$ of (P) with $z_i(0) \in \Gamma_i^{2h}$ one has

$$h < H_i(z_i(t)) < 3h \quad \text{for every } t \in [0, T].$$

Indeed, set

$$C = \max\{\|\nabla H_i(w)\| : w \in \mathscr{B}[0, r_i]\}, \qquad \bar{\varepsilon}_i = \frac{h}{2CT},$$

and assume by contradiction that $z_i(0) \in \Gamma_i^{2h}$ and there exists $t_1 \in [0, T]$ such that $h < H_i(z_i(t)) < 3h$ for every $t \in [0, t_1[$, and either $H_i(z_i(t_1)) = h$ or $H_i(z_i(t_1)) = 3h$. In view of (4.5),

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t} H_i(z_i(t))\right| &= \left|\langle J\nabla H_i(z_i(t)), A_i \nabla H_i(z_i(t)) + \mathcal{R}_i(t, z_1, \dots, z_N)\rangle\right| \\ &= \left|\langle J\nabla H_i(z_i(t)), \mathcal{R}_i(t, z_1, \dots, z_N)\rangle\right| \le C\bar{\varepsilon}_i = \frac{h}{2T}\end{aligned}$$

for every $t \in [0, t_1]$, so that

$$|H_i(z_i(t_1)) - H_i(z_i(0))| \le \frac{h}{2T} t_1 < h,$$

a contradiction.

Consequently, if $z_i(0) \in \Gamma_i^{2h}$, we have that

$$\hat{r} < \|z_i(t)\| \le r_i \quad \text{for every } t \in [0, T],$$

so that the rotation number of $z_i(t)$ around the origin is well defined. Writing $z_i(t)$ in polar coordinates, namely

$$z_i(t) = (\rho_i(t) \cos \vartheta_i(t), \rho_i(t) \sin \vartheta_i(t)),$$

using ($\mathscr{A}_2$) and (4.4), we thus have

$$-\vartheta_i'(t) = \frac{\langle J\dot{z}_i(t), z_i(t)\rangle}{\|z_i(t)\|^2} = \frac{\langle A_i \nabla H_i(z_i(t)) + \mathcal{R}_i(t, z_1, \dots, z_N), z_i(t)\rangle}{\|z_i(t)\|^2} \ge A_i m_i - \frac{\bar{\varepsilon}_i}{\hat{r}}.$$

Choosing finally

$$\bar{A}_i = \frac{2\pi \hat{r} \nu_i + \bar{\varepsilon}_i T}{m_i \hat{r} T},$$

we easily conclude the proof. □

Now we need a control on the rotation number of the large planar components of the solutions.

Lemma 4.3. *For any $i = 1, \dots, N$, let $\bar{A}_i$ and $\bar{\varepsilon}_i$ be as in Lemma 4.2, and assume that $A_i \ge \bar{A}_i$ and (4.4) holds. Then, there exists $\overline{R}_i > 0$ such that any solution $z(t)$ of* (P) *with $\|z_i(0)\| \ge \overline{R}_i$ satisfies*

$$\operatorname{Rot}(z_i(t); [0, T]) < 1.$$

Proof. Fix $\sigma = 1/(2A_i T)$ and let $R_i > 0$ and $\vartheta_1^i < \vartheta_2^i$, with $\vartheta_2^i - \vartheta_1^i \le 2\pi$, be as in ($\mathscr{A}_3$). Choose $\widehat{R}_i \ge R_i$ such that

$$\widehat{R}_i > \frac{2\bar{\varepsilon}_i T}{\vartheta_2^i - \vartheta_1^i}.$$

In view of assumption ($\mathscr{A}_1$), there exists $\overline{R}_i \ge \widehat{R}_i$ such that if $\|z_i(0)\| \ge \overline{R}_i$, then $\|z_i(t)\| \ge \widehat{R}_i$ for every $t \in [0, T]$. In particular, the rotation number of $z_i(t)$ is well defined. Let us assume, by contradiction, that $\|z_i(0)\| \ge \overline{R}_i$ and $\operatorname{Rot}(z_i(t); [0, T]) \ge 1$. Then, writing

$$z_i(t) = (\rho_i(t) \cos \vartheta_i(t), \rho_i(t) \sin \vartheta_i(t)),$$

as long as $\vartheta_i(t) \in \Theta(\vartheta_1^i, \vartheta_2^i)$, since $\rho_i(t) \ge \widehat{R}_i \ge R_i$, we can use (4.2) and (4.4) to obtain

$$\begin{aligned}-\vartheta_i'(t) &= \frac{\langle A_i \nabla H_i(z_i(t)) + \mathcal{R}_i(t, z_1, \dots, z_N), z_i(t)\rangle}{\|z_i(t)\|^2} \\ &\le A_i \frac{1}{2A_i T}(\vartheta_2^i - \vartheta_1^i) + \frac{\bar{\varepsilon}_i}{\widehat{R}_i} < \frac{\vartheta_2^i - \vartheta_1^i}{T}.\end{aligned}$$

Consequently, the time needed to clockwise cross the sector $\Theta(\vartheta_1^i, \vartheta_2^i)$ is greater than T, a contradiction. □

Proof of Theorem 4.1. For any $i \in \{1, \dots, N\}$, let $\bar{A}_i > 0$ and $\bar{\varepsilon}_i > 0$ be as in Lemma 4.2, and set

$$A = \max\{\bar{A}_i : i = 1, \dots, N\}, \quad \varepsilon = \min\{\bar{\varepsilon}_i : i = 1, \dots, N\}.$$

Take $A_i \geq A$ and assume that (4.3) holds. Then, take $\overline{R}_i$ as in Lemma 4.3 for every $i = 1, \dots, N$, and consider the annulus $\mathcal{A}_i = \mathscr{B}(0, \overline{R}_i) \setminus D_i^{\bar{h}_i}$. Recall that, taking $\bar{h}_i > 0$ sufficiently small, the inner boundary of $\mathcal{A}_i$ is star-shaped. Then, by Lemmas 4.2 and 4.3, for every solution $z(t)$ of (P), if $z_i(0)$ belongs to the inner boundary of $\mathcal{A}_i$, then $z_i(t)$ makes more than ν_i clockwise rotations around the origin in the time T, while if $\|z_i(0)\| = \overline{R}_i$, it makes less than one clockwise turn in the same time.

We now use the fact that (P) is a *positive transformation of a Hamiltonian system*, and consider the linear transformation $\mathcal{L}$ defined in (4.1). Being all the matrices $\mathbb{M}_i$ invertible with positive determinant, the set

$$\mathcal{A} = \mathcal{L}(\mathcal{A}_1 \times \cdots \times \mathcal{A}_N)$$

is thus of the type $\widetilde{\mathcal{A}}_1 \times \cdots \times \widetilde{\mathcal{A}}_N$, where each $\widetilde{\mathcal{A}}_i$ is a planar annulus with star-shaped boundaries with respect to the origin. Since the change of variables preserves the above described rotational properties of the solutions, we can apply [31, Theorem 8.2] to the Hamiltonian system obtained from (P) through the change of variables given by $\mathcal{L}$. We thus obtain at least $N + 1$ distinct T-periodic solutions

$$\tilde{z}^k(t) = (\tilde{z}_1^k(t), \dots, \tilde{z}_N^k(t))$$

such that for every $k = 1, \dots, N + 1$, each component $\tilde{z}_i^k(t)$, with $i = 1, \dots, N$, makes exactly ν_i clockwise rotations around the origin in the time interval $[0, T[$. Setting

$$z^k(t) = (\mathbb{M}_1^{-1}\tilde{z}_1^k(t), \dots, \mathbb{M}_N^{-1}\tilde{z}_N^k(t)),$$

we obtain the solutions of (P) we are looking for, and the proof is thus completed. □

Remark 4.4. Theorem 4.1 exploits a gap between the rotation numbers of the solutions at zero and at infinity. With reference to the assumption at infinity, another possibility could be to replace $(\mathscr{A}_3)$ with the requirement that for some $i \in \{1, \dots, N\}$, the system $J\dot{z}_i = \nabla H_i(z_i)$ has a homoclinic orbit surrounding the origin (in the spirit of [32, Theorem 3.3]). Indeed, by continuity, small perturbations of trajectories next to the homoclinic would have small rotation number, since the homoclinic spends an infinite time to rotate around the origin. In this setting, assuming moreover $(\mathscr{A}_2)$, it would then be possible to construct the gap which allows to apply [31, Theorem 8.2], taking a level curve of H_i sufficiently near the homoclinic orbit as outer boundary of the required annulus in the i-th planar component. The same line of thought can be also adapted when the homoclinic is replaced by heteroclinics. One could also combine assumptions at infinity like $(\mathscr{A}_3)$ for some indices $i_1, \dots, i_r \in \{1, \dots, N\}$ and existence of homoclinics for the other indices $i \in \{1, \dots, N\} \setminus \{i_1, \dots, i_r\}$. We omit the details for briefness.

As a particular case, we can deal with a system of scalar second order equations like

$$\begin{cases} \ddot{x}_1 + A_1^2 f_1(x_1) = \dfrac{\partial \mathcal{W}}{\partial x_1}(t, x_1, \dots, x_N), \\ \quad\vdots \\ \ddot{x}_N + A_N^2 f_N(x_N) = \dfrac{\partial \mathcal{W}}{\partial x_N}(t, x_1, \dots, x_N), \end{cases} \tag{4.6}$$

where the continuous function $\mathcal{W} \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t, and continuously differentiable with respect to $(x_1, \dots, x_N)$. Indeed, we can write the equivalent system

$$\begin{cases} -\dot{y}_i = A_i f_i(x_i) - \dfrac{1}{A_i}\dfrac{\partial \mathcal{W}}{\partial x_i}(t, x_1, \dots, x_n), \\ \dot{x}_i = A_i y_i, \end{cases} \qquad i = 1, \dots, N,$$

which is in the form (P), with $z_i = (x_i, y_i)$, taking

$$H_i(x_i, y_i) = \frac{1}{2} y_i^2 + F_i(x_i),$$

where F_i is a primitive of f_i and

$$\mathcal{R}_i(t, x_1, y_1, \dots, x_N, y_N) = -\frac{1}{A_i}\begin{pmatrix} \frac{\partial \mathcal{W}}{\partial x_i}(t, x_1, \dots, x_n) \\ 0 \end{pmatrix}.$$

Notice that (4.6) is a *positive transformation of a Hamiltonian system*, with the linear function $\mathcal{L}$ in (4.1) given by

$$\mathbb{M}_i = \begin{pmatrix} 1 & 0 \\ 0 & A_i \end{pmatrix}, \quad i = 1, \dots, N.$$

As a consequence, we have the following statement, where, for simplicity, we only consider the case $\nu_1 = \cdots = \nu_N = 1$.

Corollary 4.5. *Assume that the continuous functions $f_i \colon \mathbb{R} \to \mathbb{R}$ satisfy*

$$\liminf_{s\to 0} \frac{f_i(s)}{s} > 0, \quad \lim_{s\to+\infty} \frac{f_i(s)}{s} = 0.$$

Moreover, for every $i = 1, \dots, N$, let $K_i > 0$ be such that

$$\left|\frac{\partial \mathcal{W}}{\partial x_i}(t, x_1, \dots, x_n)\right| \le K_i \quad \textit{for every } t \in [0, T] \textit{ and } x_1, \dots, x_N \in \mathbb{R}. \tag{4.7}$$

Then, there exists $\bar{A} > 0$ such that if $A_i \ge \bar{A}$ for every $i = 1, \dots, N$, system (4.6) *has at least $N + 1$ distinct periodic solutions*

$$x^k(t) = (x_1^k(t), \dots, x_N^k(t))$$

with minimal period T. Moreover, for every $k = 1, \dots, N + 1$, each component $x_i^k(t)$, with $i = 1, \dots, N$, has exactly two simple zeros in the interval $[0, T[$.

Proof. First, we notice that $(\mathscr{A}_1)$ is fulfilled, in view of the growth assumption on the nonlinearities. Let us now check $(\mathscr{A}_2)$. We know that there exist $\alpha_i > 0$ and $\beta_i > 0$ such that

$$0 < |s| < \beta_i \Rightarrow \frac{f_i(s)}{s} \ge \alpha_i.$$

Then, if $\|(x_i, y_i)\| \le \beta_i$,

$$\frac{\langle \nabla H_i(x_i, y_i), (x_i, y_i)\rangle}{\|(x_i, y_i)\|^2} = \frac{x_i f_i(x_i) + y_i^2}{x_i^2 + y_i^2} \ge \min\{\alpha_i, 1\} > 0,$$

as desired.

We now verify $(\mathscr{A}_3)$. Fix $\sigma \in]0, \pi[$, and take $\vartheta_1^i = 0$, $\vartheta_2^i = \sigma/2$. Writing

$$z_i = (x_i, y_i) = (\rho_i \cos\vartheta_i, \rho_i \sin\vartheta_i),$$

we have that if $z_i \in \Theta(0, \sigma/2)$, then

$$\begin{aligned}\frac{\langle \nabla H_i(z_i), z_i\rangle}{\|z_i\|^2} &= \frac{(\rho_i \cos\vartheta_i) f_i(\rho_i\cos\vartheta_i) + (\rho_i \sin\vartheta_i)^2}{\rho_i^2} \\ &\le \sin^2\vartheta_i + \left|\frac{f_i(\rho_i\cos\vartheta_i)}{\rho_i\cos\vartheta_i}\right| \le \frac{\sigma^2}{4} + \left|\frac{f_i(\rho_i\cos\vartheta_i)}{\rho_i\cos\vartheta_i}\right|.\end{aligned}$$

Taking $R_i > 0$ large enough, if $z_i \in \Theta(0, \sigma/2) \setminus \mathscr{B}(0, R_i)$, then

$$\frac{\langle \nabla H_i(z_i), z_i\rangle}{\|z_i\|^2} \le \frac{\sigma^2}{4} + \frac{\sigma^2}{4} = \sigma(\vartheta_2^i - \vartheta_1^i).$$

The proof is thus completed, noticing that it suffices to choose A_i large enough in order to make $\mathcal{R}_i(t, z_1, \dots, z_N)$ as small as desired. □

As an example, Corollary 4.5 directly applies to the following system of N coupled pendulums,

$$\begin{cases} \ddot{x}_1 + A_1^2 \sin x_1 = \dfrac{\partial \mathcal{W}}{\partial x_1}(t, x_1, \dots, x_N), \\ \qquad\qquad \vdots \\ \ddot{x}_N + A_N^2 \sin x_N = \dfrac{\partial \mathcal{W}}{\partial x_N}(t, x_1, \dots, x_N), \end{cases}$$

where $\frac{\partial \mathcal{W}}{\partial x_i}(t, x_1, \dots, x_N)$ is continuous and bounded for $i = 1, \dots, N$, and the constants $A_1, \dots, A_N$ are large enough. We are thus able to recover the results obtained in [32], by the use of the Poincaré–Birkhoff theorem, for a single equation modeling a forced pendulum having a very small length.

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