From the viscous Cahn-Hilliard equation to a regularized forward-backward parabolic equation*

Pierluigi Colli[(1)]
e-mail: pierluigi.colli@unipv.it

Luca Scarpa[(2)]
e-mail: luca.scarpa.15@ucl.ac.uk

[(1)] Dipartimento di Matematica "F. Casorati", Università di Pavia
via Ferrata 1, 27100 Pavia, Italy

[(2)] Department of Mathematics, University College London
Gower Street, London WC1E 6BT, United Kingdom

Abstract: A rigorous proof is given for the convergence of the solutions of a viscous Cahn–Hilliard system to the solution of the regularized version of the forward-backward parabolic equation, as the coefficient of the diffusive term goes to 0. Non-homogenous Neumann boundary condition are handled for the chemical potential and the subdifferential of a possible non-smooth double-well functional is considered in the equation. An error estimate for the difference of solutions is also proved in a suitable norm and with a specified rate of convergence.

AMS Subject Classification: 35K55, 35K50, 35B25, 35D30, 74N25.

Key words and phrases: Cahn–Hilliard system, forward-backward parabolic equation, viscosity, initial-boundary value problem, asymptotic analysis, well-posedness.

1 Introduction

In this paper we perform an asymptotic analysis on the viscous Cahn–Hilliard initial-boundary value problem

$$\partial_t y - \Delta w = 0 \quad \text{in } Q := \Omega \times (0, T), \tag{1.1}$$

$$w = \tau \partial_t y - \delta \Delta y + \beta(y) + \lambda(x, t)\pi(y) - g(x, t) \quad \text{in } Q, \tag{1.2}$$

$$\partial_{\mathbf{n}} w = h(x, t), \quad \partial_{\mathbf{n}} y = 0 \quad \text{in } \Sigma := \Gamma \times (0, T), \tag{1.3}$$

$$y(\,\cdot\,, 0) = y_0 \quad \text{in } \Omega, \tag{1.4}$$

as $\delta \searrow 0$, to obtain the regularized diffusion problem in which (1.2) and (1.3) are replaced by the respective equation and condition

$$w = \tau \partial_t y + \beta(y) + \lambda(x, t)\pi(y) - g(x, t) \quad \text{in } Q, \tag{1.5}$$
$$\partial_{\mathbf{n}} w = h(x, t) \quad \text{in } \Sigma. \tag{1.6}$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^3$ with boundary Γ and $T > 0$ stands for some final time. The variable y denotes some order parameter which may represent the concentration of a phase, and in (1.1)–(1.2) the other variable w plays as chemical potential. The data of the system are the viscosity coefficient $\tau > 0$, the given functions λ and g in Q, which depend on the space and time positions as well as the boundary datum h on Σ, and the initial value y_0 in Ω. Note that the boundary conditions in (1.3) are both of Neumann type, and the one for the chemical potential is preserved in the limit procedure, as in (1.6). Quite unusually for Cahn–Hilliard type problems, this condition is more general that the standard homegeneous one, and this makes that the mean value of y is not conserved in time, as the simple integration on (1.1) on Ω would imply.

The nonlinearities β and π have different properties, although both are equal to 0 in the critical value 0: π is simply Lipschitz continuous, while β represents a continuous increasing function that is allowed in our analysis to become a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Of course, in the case of a multivaled graph β, (1.2) and (1.5) should be properly meant as inclusions (instead of equations). The point is that, in both situations, β is the subdifferential of a convex and continuous function $\widehat{\beta} : \mathbb{R} \to [0, +\infty)$ and, letting $\widehat{\pi}(r) = \int_0^r \pi(s)ds$, $r \in \mathbb{R}$, it turns out that the contribution $\beta(y) + \lambda(x, t)\pi(y)$ results from the (sub)differentiation of the following energy functional

$$y \in L^2(\Omega) \mapsto \int_\Omega \left(\widehat{\beta}(y) + \lambda(\cdot, t)\widehat{\pi}(y) \right), \quad \text{for almost every } t \in (0, T).$$

Note that, taking for a moment λ constant, the sum

$$\psi(y) := \widehat{\beta}(y) + \lambda\widehat{\pi}(y)$$

may be seen as a generalization of the well-known double-well potential $\psi(y) = (y^2 - 1)^2/4$, actually corresponding to the choices $\beta(y) = y^3$, $\lambda = 1$, $\pi(y) = -y$. The presence of a coefficient λ varying in space and time is certainly of interest in some cases, for instance you can think to families of control problems in which some linearization has been carried out (cf., e.g., [10–12]).

Let us now comment on the two problems. If we combine (1.1) and (1.5), but without the term $\tau \partial_t y$, we obtain the nonlinear diffusion equation

$$\partial_t y - \Delta(\psi'(y) - g) = 0, \tag{1.7}$$

which can be derived starting from the mass-balance law

$$\partial_t y + \operatorname{div} \mathbf{h} = 0, \tag{1.8}$$

where $\mathbf{h}$ denotes the flux of diffusant and is related to the the gradient of the chemical potential w according to the Fick law:

$$\mathbf{h} = -m\nabla w, \quad m \text{ mobility constant taken 1 here.} \tag{1.9}$$

The equation of state

$$w = \psi'(y) - g \tag{1.10}$$

contains a nonlinear relation between w and y, via the derivative ψ' of the energy function ψ. As ψ is in general the sum of a convex function and a concave perturbation, ψ' is non-monotone and the resulting equation (1.7) may be ill-posed.

Then, suitable regularizations are in order, and the most celebrated one is the elliptic regularization

$$w = -\delta\Delta y + \psi'(y) - g, \tag{1.11}$$

with δ positive coefficient related to surface tension; (1.11) leads to the the well-known Cahn–Hilliard system [5, 15]. Other choices have been considered in the literature: in particular, Novick-Cohen and Pego [20] dealt with the viscous regularization

$$w = \tau\partial_t y + \psi'(y) - g, \tag{1.12}$$

which has been recently revisited by Tomassetti [21] (see also the list of references in [21]). In fact, the mathematical problem studied in [21] turns out to be a special case of (1.1), (1.5), (1.6), (1.4) and the existence proof in [21] is carried out by working with a Faedo–Galerkin scheme.

The reader can also examine the paper [3] in which a more general non-smooth regularization of the form

$$w = \gamma(\partial_t y) + \psi'(y) - g, \tag{1.13}$$

γ being a maximal monotone and coercive graph, is considered and the related initial–boundary value problem is investigated when taking Dirichlet boundary conditions for the chemical potential.

On the other hand, a combination of viscous and energetic regularization leads to the so-called viscous Cahn–Hilliard equation (cf. (1.2))

$$w = \tau\partial_t y - \delta\Delta y + \psi'(y) - g,$$

derived by Novick-Cohen [19] and analytically investigated in [13, 14], with numerical aspects treated in [1]. Other recent contributions [6,8,10,16,17] deal with Cahn–Hilliard or viscous Cahn–Hilliard equations with singular potentials. Moreover, and this is important for a comparison with our approach, we point out the article [9], in which a wide class of evolution equations, but with monotone nonlinearity, is obtained as asymptotic limit of special Cahn–Hilliard systems. Also, we mention the papers [7,11,12,18,22,23] regarding optimal control problems for some Cahn–Hilliard systems that possibly include dynamic boundary conditions.

In the present contribution, we first recall a precise well-posedness result for the problem (1.1)–(1.4). The statements of other results are also contained in Section 2. Then, in

Section 3 we derive a number of estimates, independent of δ, on the solutions to (1.1)–(1.4) and pass to the limit, as δ tends to 0, using properties like compactness, monotonicity and lower semicontinuinity. Thus, we prove the convergence to the solution of (1.1), (1.5), (1.6), (1.4) and this convergence proof, in our opinion, deserves some interest since at some points the result cannot be taken for granted. Of course, we have to make assumptions on the data: concerning the maximal monotone graph β in (1.2) and (1.5), we have to require that (see the later (2.16)) the growth at infinity is controlled by the one of the related convex function $\widehat{\beta}$. But this looks quite reasonable in the framework of (1.5) and less restrictive than the growth of order p assumed in [21]. Actually, by this rigorous asymptotic we give an alterative proof of the existence of solutions to the limit problem (1.1), (1.5), (1.6), (1.4). Moreover, for this problem we show the continuous dependence on the data y_0, g, h, and consequently the uniqueness of the solution, in Section 4. Last but not least, strengthening a bit the assumptions on λ and g, we are able to deduce an asymptotic error estimate for the difference of the solutions of the two problems: the proof is given in Section 5.

2   Main results

In this section, we give some precise formulation of the problems and state our results. Let us first recall the working framework:

$$\Omega \subseteq \mathbb{R}^3 \text{ smooth bounded domain}, \quad \Gamma := \partial\Omega, \quad T > 0 \text{ final time}, \tag{2.1}$$

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t) \quad \text{for all } t \in (0, T], \tag{2.2}$$

$$Q := Q_T, \quad \Sigma := \Sigma_T. \tag{2.3}$$

Then, Ω is the spatial three-dimensional domain, Γ denotes its boundary, while Q and Σ represent the related spatiotemporal sets. In addition, we adopt the notations

$$H := L^2(\Omega), \quad V := H^1(\Omega) \tag{2.4}$$

and we will denote the duality pairing between V^* and V with the symbol $\langle \cdot, \cdot \rangle$. As usual, we make the identification $H \cong H^*$, so that H is continuously embedded in V^* in the standard way: for all $u \in H$ and $v \in V$, we have $\langle u, v \rangle = (u, v)_H$, where $(\cdot, \cdot)_H$ is the inner product of H.

Secondly, let us recall some useful tools which are often used in dealing with the Cahn-Hilliard equations: the reader can refer to [10, Sec. 2, pp. 979-980]. We introduce the notation

$$v_\Omega^* := \frac{1}{|\Omega|} \langle v^*, 1 \rangle \quad \text{for all } v^* \in V^*, \tag{2.5}$$

which specifies the mean value of the elements of V^*, and we recall that

$$\exists\, C > 0: \; \|v\|_V^2 \leq C \left(\|\nabla v\|_H^2 + |v_\Omega|^2 \right) \quad \text{for all } v \in V. \tag{2.6}$$

Moreover, we define the operator

$$D(\mathcal{N}) = \{v^* \in V^* : v_\Omega^* = 0\}, \quad \mathcal{N} : D(\mathcal{N}) \to \{v \in V : v_\Omega = 0\} \tag{2.7}$$

by setting $\mathcal{N}v^*$ as the unique solution with null mean value to the generalized elliptic equation with homogeneous Neumann boundary conditions, i.e.,

$$\mathcal{N}v^* \in V, \quad (\mathcal{N}v^*)_\Omega = 0, \quad \int_\Omega \nabla \mathcal{N}v^* \cdot \nabla z = \langle v^*, z \rangle \quad \text{for all } z \in V. \tag{2.8}$$

Furthermore, if we set $\|\cdot\|_* : V^* \to [0, +\infty)$ as

$$\|v^*\|_*^2 := \|\nabla \mathcal{N}(v^* - v_\Omega^*)\|_H^2 + |v_\Omega^*|^2, \quad v^* \in V^*, \tag{2.9}$$

then $\|\cdot\|_*$ is a norm on V^*, equivalent to the usual norm $\|\cdot\|_{V^*}$, which makes V^* a Hilbert space. Finally, we recall that

$$\langle v^*, \mathcal{N}v^* \rangle = \|v^*\|_*^2 \quad \text{for all } v^* \in D(\mathcal{N}), \tag{2.10}$$

$$\exists\, C > 0 : \ \|\mathcal{N}v^*\|_V^2 \le C \|v^*\|_*^2 \quad \text{for all } v^* \in D(\mathcal{N}), \tag{2.11}$$

while for all $v^* \in H^1(0,T;V^*)$ such that $v^*(t) \in D(\mathcal{N})$ for all $t \in [0,T]$ we have that

$$\langle \partial_t v^*(t), \mathcal{N}v^*(t) \rangle = \frac{1}{2} \frac{d}{dt} \|v^*(t)\|_*^2 \quad \text{for a.e. } t \in (0,T). \tag{2.12}$$

Now, it is time make some rigorous assumptions on the data. We assume that

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty) \text{ is proper, convex and lower semicontinuous}$$
$$\text{with } \widehat{\beta}(0) = 0, \text{ and } \beta := \partial \widehat{\beta} \text{ denotes the subdifferential} \tag{2.13}$$

$$\pi : \mathbb{R} \to \mathbb{R} \text{ is a Lipschitz continuous function, with } \pi(0) = 0, \tag{2.14}$$

so that $\beta : \mathbb{R} \to 2^\mathbb{R}$ is a maximal monotone operator with domain $D(\beta)$ and satisfying the condition $\beta(0) \ni 0$. Please notice also that we have $D(\widehat{\beta}) = \mathbb{R}$.

We are now ready to focus on our problem and present the main results of the paper. As we have anticipated, the aim of the work is to take the limit as $\delta \searrow 0$ in problem (1.1)–(1.4) to show the convergence of the solutions and provide an asymptotic estimate of the error. More specifically, we present now four fundamental results. The first one ensures that problem (1.1)–(1.4) is well posed in a certain variational formulation; the second one is the effective convergence result as $\delta \searrow 0$ and provides the rigorous formulation of the limit problem; the third one is a continuous dependence result for the limit problem and the fourth one contains the asymptotic error estimate.

In the following, we assume to work in the setting (2.1)–(2.4) and (2.13)–(2.14); moreover, we let

$$\tau > 0 \text{ be fixed} \tag{2.15}$$

and additionally require that

$$\exists\, C > 0 : \ \left|\beta^0(r)\right| \le C \left(1 + \left|\widehat{\beta}(r)\right|\right) \quad \text{for all } r \in \mathbb{R}, \tag{2.16}$$

where $\beta^0(r)$ denotes the minimum-norm element of $\beta(r)$, for any $r \in \mathbb{R}$. We note that, for example, all functions with polynomial and first-order exponential growths comply

with our assumption (2.16). In addition, we point out that (2.16) implies (actually, it is equivalent to) the condition

$$|s| \leq C \left(1 + \left| \widehat{\beta}(r) \right| \right) \quad \text{for all } r \in \mathbb{R},\ s \in \beta(r) \tag{2.17}$$

for the same constant C, as checked precisely in the next remark.

Remark 2.1. Such an equivalence property holds for a more general growth condition and in the general setting of Hilbert spaces. Indeed, let X be a Hilbert space, $\widehat{\beta} : X \to [0, +\infty)$ be convex and l.s.c. (thus continuous since it is everywhere defined), and $\Psi : [0, +\infty) \to [0, +\infty)$ be a continuous function. If $\beta := \partial\widehat{\beta}$ and, for every $u \in X$, $\beta^0(u)$ is the element of $\beta(u)$ having minimal norm, from the condition

$$\left\| \beta^0(u) \right\|_X \leq \Psi \left(\widehat{\beta}(u) \right) \quad \text{for every } u \in X, \tag{2.18}$$

it follows that

$$\|\zeta\|_X \leq \Psi \left(\widehat{\beta}(u) \right) \quad \text{for all } u \in X \text{ and all } \zeta \in \beta(u). \tag{2.19}$$

Let us check that. If $u \in X$, $\zeta \in \beta(u)$ and $\varepsilon > 0$, the monotonicity of β implies

$$\left(\beta^0(u + \varepsilon\zeta) - \zeta, (u + \varepsilon\zeta) - u \right) \geq 0, \quad \text{whence} \quad \|\zeta\|_X \leq \|\beta^0(u + \varepsilon\zeta)\|_X.$$

Then, by applying (2.18) to $u + \varepsilon\zeta$, we infer

$$\|\zeta\|_X \leq \Psi \left(\widehat{\beta}(u + \varepsilon\zeta) \right)$$

and letting $\varepsilon \searrow 0$ we recover (2.19) thanks to the continuity of $\Psi \circ \widehat{\beta}$.

Now, we recall a well-posedness result for the problem with $\delta > 0$. In order to keep a convenient notation henceforth, differently from (1.1)–(1.4) now we put the subscript δ to the solution and denote by $y_{0,\delta}$ the initial value corresponding to δ.

Theorem 2.2. *Let $\delta \in (0, 1)$ and assume that*

$$\lambda \in L^\infty(Q), \tag{2.20}$$

$$g \in L^2(Q), \quad h \in L^2(\Sigma), \tag{2.21}$$

$$y_{0,\delta} \in H^1(\Omega) \quad \text{and} \quad \widehat{\beta}(y_{0,\delta}) \in L^1(\Omega). \tag{2.22}$$

Then, there exist

$$y_\delta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2\left(0, T; H^2(\Omega)\right) \tag{2.23}$$

$$w_\delta \in L^2(0, T; V), \quad \xi_\delta \in L^2(0, T; H) \tag{2.24}$$

satisfying for almost all $t \in (0, T)$ the variational equalities

$$\langle \partial_t y_\delta(t), v \rangle + \int_\Omega \nabla w_\delta(t) \cdot \nabla v = \int_\Gamma h(t)v \quad \text{for all } v \in H^1(\Omega), \tag{2.25}$$

$$\int_\Omega w_\delta(t)v = \tau \int_\Omega \partial_t y_\delta(t)v + \delta \int_\Omega \nabla y_\delta(t) \cdot \nabla v + \int_\Omega \xi_\delta(t)v$$

$$+ \int_\Omega \lambda(t)\pi(y_\delta(t))v - \int_\Omega g(t)v \qquad \text{for all } v \in H^1(\Omega) \qquad (2.26)$$

and such that

$$\xi_\delta \in \beta(y_\delta) \quad \text{a.e. in } Q, \tag{2.27}$$

$$y_\delta(0) = y_{0,\delta}. \tag{2.28}$$

Furthermore, if

$$\left(y_{0,\delta}^1, g_1, h_1\right), \ \left(y_{0,\delta}^2, g_2, h_2\right) \in H^1(\Omega) \times L^2(Q) \times L^2(\Sigma), \tag{2.29}$$

$$(y_{0,\delta}^1)_\Omega = (y_{0,\delta}^2)_\Omega, \quad \int_\Gamma h_1(t) = \int_\Gamma h_2(t) \quad \text{for a.e. } t \in (0,T) \tag{2.30}$$

and we let

$$\left(y_\delta^1, w_\delta^1, \xi_\delta^1\right), \ \left(y_\delta^2, w_\delta^2, \xi_\delta^2\right) \tag{2.31}$$

denote any respective solutions to problem (2.25)–(2.28) corresponding to the data in (2.29)–(2.30), then there exists a positive constant C, depending only on τ, the constant in (2.11), $\|\lambda\|_{L^\infty(Q)}$, a Lipschitz constant for π, Ω, and T, such that

$$\left\|y_\delta^1 - y_\delta^2\right\|_{L^\infty(0,T;V^*)} + \tau^{1/2}\left\|y_\delta^1 - y_\delta^2\right\|_{L^\infty(0,T;H)} + \delta^{1/2}\left\|\nabla(y_\delta^1 - y_\delta^2)\right\|_{L^2(0,T;H)}$$

$$\leq C\left[\left\|y_{0,\delta}^1 - y_{0,\delta}^2\right\|_* + \tau^{1/2}\left\|y_{0,\delta}^1 - y_{0,\delta}^2\right\|_H + \|g_1 - g_2\|_{L^2(Q)} + \|h_1 - h_2\|_{L^2(\Sigma)}\right]. \tag{2.32}$$

Please note that the equations (2.25)–(2.26) are the natural variational formulations of (1.1) and (1.2), obtained testing by $v \in H^1(\Omega)$ and integrating by parts, on account of the boundary conditions in (1.3).

Remark 2.3. The proof of Theorem 2.2 is omitted in this work, since the reader can refer to similar results shown in, e.g., [10, Theorems 2.2 and 2.3] and [9] for related details. For the sake of completeness, the key idea is to approximate the problem using the Yosida regularization β_ε instead of β and recover a solution of the approximating problem. Then, some uniform estimates are found for this family of solutions and a passage to the limit as $\varepsilon \searrow 0$ provides the solution to the original problem. Let us notice that here the approximation in ε should be carried out with $\delta \in (0,1)$ fixed. However, we think that the reader can reconstruct the basic steps of the proofs by examining the estimates (independent of δ) we will prove in Section 3 and compare with Section 4 for the proof of (2.32).

Theorem 2.4. *Assume (2.20)–(2.21) and*

$$y_0 \in L^2(\Omega) \quad \text{and} \quad \widehat{\beta}(y_0) \in L^1(\Omega). \tag{2.33}$$

Then, there exist a family

$$\{y_{0,\delta}\}_{\delta\in(0,1)} \subseteq H^1(\Omega), \quad y_{0,\delta} \to y_0 \text{ in } L^2(\Omega) \text{ as } \delta \searrow 0 \tag{2.34}$$

and a positive constant M such that

$$\delta^{1/2} \|\nabla y_{0,\delta}\|_H \leq M, \quad \left\|\widehat{\beta}(y_{0,\delta})\right\|_{L^1(\Omega)} \leq M \qquad \text{for all } \delta \in (0,1). \qquad (2.35)$$

Furthermore, for every $\delta \in (0,1)$ let

$$y_\delta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2\left(0,T;H^2(\Omega)\right) \qquad (2.36)$$

$$w_\delta \in L^2(0,T;V), \quad \xi_\delta \in L^2(0,T;H) \qquad (2.37)$$

be the solutions to problem (2.25)–(2.28) with data $(y_{0,\delta}, g, h)$. Then, there exist a triplet

$$y \in H^1(0,T;H), \qquad (2.38)$$

$$w \in L^2(0,T;V), \quad \xi \in L^2(0,T;H), \qquad (2.39)$$

which solves the problem

$$\int_\Omega \partial_t y(t)v + \int_\Omega \nabla w(t) \cdot \nabla v = \int_\Gamma h(t)v \quad \text{for all } v \in H^1(\Omega) \text{ and a.e. } t \in (0,T), \quad (2.40)$$

$$\int_\Omega w(t)v = \tau \int_\Omega \partial_t y(t)v + \int_\Omega \xi(t)v + \int_\Omega \lambda(t)\pi(y(t))v - \int_\Omega g(t)v$$
$$\text{for all } v \in L^2(\Omega) \text{ and a.e. } t \in (0,T), \qquad (2.41)$$

$$\xi \in \beta(y) \quad \text{a.e. in } Q, \qquad (2.42)$$

$$y(0) = y_0, \qquad (2.43)$$

and a subsequence $\{\delta_n\}_{n\in\mathbb{N}}$, with $\delta_n \searrow 0$ as $n \to \infty$, such that

$$y_{\delta_n} \rightharpoonup y \quad \text{in } H^1(0,T;H), \qquad (2.44)$$

$$y_{\delta_n} \to y \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H), \qquad (2.45)$$

$$w_{\delta_n} \rightharpoonup w \quad \text{in } L^2(0,T;V), \qquad (2.46)$$

$$\xi_{\delta_n} \rightharpoonup \xi \quad \text{in } L^2(Q). \qquad (2.47)$$

Remark 2.5. Theorem 2.4 is the effective convergence result for our problem. Note that the convergence properties (2.44)–(2.47) hold in principle for a subsequence $\{\delta_n\}$ but the next result we state will entail, in particular, the uniqueness of the solution component y. About w and ξ, they are not unique in general but their difference $w - \xi$ is uniquely determined from (2.41). Then, we can at least claim that the convergence of y_δ to y and of $w_\delta - \xi_\delta$ to $w - \xi$ is ensured for the entire family as $\delta \searrow 0$.

Theorem 2.6. *Assume (2.20) and*

$$g_1, g_2 \in L^2(Q), \quad h_1, h_2 \in L^2(\Sigma), \qquad (2.48)$$

$$y_0^1, y_0^2 \in L^2(\Omega), \quad \widehat{\beta}(y_0^1), \widehat{\beta}(y_0^2) \in L^1(\Omega), \qquad (2.49)$$

$$(y_0^1)_\Omega = (y_0^2)_\Omega, \quad \int_\Gamma h_1(t) = \int_\Gamma h_2(t) \quad \text{for a.e. } t \in (0,T). \qquad (2.50)$$

Let (y_1, w_1, ξ_1) and (y_2, w_2, ξ_2) be any respective solutions of the limit problem (2.40)–(2.43) corresponding to the data in (2.48)–(2.50). Then, there exists a positive constant C, depending only on τ, the constant in (2.11), $\|\lambda\|_{L^\infty(Q)}$, a Lipschitz constant for π, Ω, and T, such that

$$\|y_1 - y_2\|_{L^\infty(0,T;H)} \leq C \left[\|y_0^1 - y_0^2\|_H + \|g_1 - g_2\|_{L^2(Q)} + \|h_1 - h_2\|_{L^2(\Sigma)} \right] . \qquad (2.51)$$

Remark 2.7. Please note that hypothesis (2.50) is the natural generalization that takes place when dealing with a non-homogeneous Neumann boundary condition. As a matter of fact, in the case of homogeneous Neumann conditions for y, the natural requirement is that y_0^1 and y_0^2 have the same mean value (see for example [8, 10]); when a boundary datum is introduced, we need to require also that h_1 and h_2 have the same mean value on Γ, in order to recover two solutions y_1 and y_2 with same mean value on Ω at almost every time, so allowing us to prove the continuous dependence result.

Theorem 2.8. *Assume that*

$$\lambda \in L^\infty(Q) \cap L^2\big(0, T; W^{1,\infty}(\Omega)\big) , \qquad (2.52)$$
$$g \in L^2(0, T; V) , \quad h \in L^2(\Sigma) , \qquad (2.53)$$

besides (2.33). Then, there exists a positive constant C such that the following asymptotic estimate holds for all $\delta \in (0, 1)$:

$$\|y - y_\delta\|_{L^\infty(0,T;H)} \leq C \left[\delta^{1/4} + \|y_0 - y_{0,\delta}\|_H \right] . \qquad (2.54)$$

3 The convergence result

In this section, we present the proof of the convergence result contained in Theorem 2.4. In particular, we will firstly check that an approximation on the initial data satisfying (2.34)–(2.35) actually exists; then, we will find some uniform estimates on the solutions $(y_\delta, w_\delta, \xi_\delta)$ and pass to the limit as $\delta \searrow 0$.

Let us specify some useful notation that we use in the sequel. If we test equation (2.25) by $v = 1/|\Omega|$, we deduce that

$$(\partial_t y_\delta(t))_\Omega = \frac{1}{|\Omega|} \int_\Gamma h(t) \quad \text{for a.e. } t \in (0, T) , \qquad (3.1)$$

and, in view of (2.28),

$$(y_\delta(t))_\Omega = (y_{0,\delta})_\Omega + \frac{1}{|\Omega|} \int_0^t \int_\Gamma h(s) \, ds \quad \text{for all } t \in [0, T]. \qquad (3.2)$$

Hence, it is natural to introduce

$$M_\delta(t) := (y_{0,\delta})_\Omega + \frac{1}{|\Omega|} \int_0^t \int_\Gamma h(s) \, ds , \qquad (3.3)$$

so that

$$M_\delta = (y_\delta)_\Omega \,, \quad M'_\delta = (\partial_t y_\delta)_\Omega \,. \tag{3.4}$$

Owing to (2.21) and (2.33) it turns out that

$$M_\delta \text{ is bounded in } H^1(0,T) \text{ independently of } \delta \in (0,1). \tag{3.5}$$

3.1   Existence of an approximating family $\{y_{0,\delta}\}$

For every $\delta \in (0,1)$, let us define $y_{0,\delta}$ as the solution to the elliptic problem

$$\begin{cases} y_{0,\delta} - \delta \Delta y_{0,\delta} = y_0 & \text{in } \Omega \,, \\ \partial_\nu y_{0,\delta} = 0 & \text{on } \Gamma \,. \end{cases} \tag{3.6}$$

It is well known that $y_{0,\delta} \in H^2(\Omega)$ and it satisfies the variational equation

$$\int_\Omega y_{0,\delta} z + \delta \int_\Omega \nabla y_{0,\delta} \cdot \nabla z = \int_\Omega y_0 z \quad \text{for all } z \in H^1(\Omega)\,; \tag{3.7}$$

hence, testing (3.7) by $z = y_{0,\delta}$, owing to the Young inequality it is easy to see that

$$\frac{1}{2} \|y_{0,\delta}\|_H^2 + \delta \|\nabla y_{0,\delta}\|_H^2 \le \frac{1}{2} \|y_0\|_H^2 \quad \text{for all } \delta > 0\,, \tag{3.8}$$

so that the first estimate in (2.35) is satisfied. Moreover, from (3.8) it follows that there exists $\widetilde{y}_0 \in L^2(\Omega)$ and a subsequence $\{y_{0,\delta_k}\}_{k \in \mathbb{N}}$ such that

$$y_{0,\delta_k} \rightharpoonup \widetilde{y}_0 \quad \text{in } H\,, \quad \delta_k y_{0,\delta_k} \to 0 \quad \text{in } V \quad \text{as } k \to \infty\,, \tag{3.9}$$

and letting $k \to \infty$ in (3.7) we reach

$$\int_\Omega \widetilde{y}_0 z = \int_\Omega y_0 z \quad \text{for all } z \in V\,:$$

since V is dense in H, it turns out that $\widetilde{y}_0 = y_0$. Furthermore, the identification of the weak limit implies that the entire family $\{y_{0,\delta}\}$ weakly converges to y_0 in H; finally, from (3.7) we have

$$\limsup_{\delta \searrow 0} \|y_{0,\delta}\|_H^2 \le \|y_0\|_H^2 \,,$$

so that $y_{0,\delta} \to y_0$ in H and also condition (2.34) is satisfied.

It remains to check the second estimate of (2.35). For every $\varepsilon \in (0,1)$, let β_ε be the Yosida approximation of β: hence, since β_ε is Lipschitz continuous, $\beta_\varepsilon(y_{0,\delta}) \in H^1(\Omega)$ and we can test (3.7) by $z = \beta_\varepsilon(y_{0,\delta})$, obtaining

$$\int_\Omega \beta_\varepsilon(y_{0,\delta})(y_{0,\delta} - y_0) + \delta \int_\Omega \beta'_\varepsilon(y_{0,\delta}) |\nabla y_{0,\delta}|^2 = 0\,.$$

Hence, thanks to the subdifferential property and the monotonicity of β_ε we have

$$\int_\Omega \widehat{\beta}_\varepsilon(y_{0,\delta}) - \int_\Omega \widehat{\beta}_\varepsilon(y_0) \le \int_\Omega \beta_\varepsilon(y_{0,\delta})(y_{0,\delta} - y_0) \le 0\,,$$

whence (cf., e.g., [2, Thm. 2.2, p. 57])

$$\int_\Omega \widehat{\beta}_\varepsilon(y_{0,\delta}) \le \int_\Omega \widehat{\beta}_\varepsilon(y_0) \le \int_\Omega \widehat{\beta}(y_0)$$

for all $\varepsilon \in (0,1)$. Taking the limit as $\varepsilon \searrow 0$ in the above inequality, thanks to the Fatou lemma and condition (2.33) we obtain also the second part of (2.35).

3.2 The estimate on y_δ

We want to find some uniform estimates on y_δ: firstly, let us notice that equation (3.4) ensures that, for almost every $t \in (0,T)$, $y_\delta(t) - M_\delta(t) \in D(\mathcal{N})$, so that $\mathcal{N}(y_\delta(t) - M_\delta(t))$ makes sense. Hence, we can test equation (2.25) with $\mathcal{N}(y_\delta(t) - M_\delta(t))$ and (2.26) with $-(y_\delta(t) - M_\delta(t))$: summing up the two equations, the second and third integral on the left hand side cancel thanks to (2.8). Then, we obtain

$$\langle \partial_t(y_\delta(t) - M_\delta(t)), \mathcal{N}(y_\delta(t) - M_\delta(t)) \rangle + \frac{\tau}{2} \frac{d}{dt} \|y_\delta(t) - M_\delta(t)\|_H^2 + \delta \|\nabla y_\delta(t)\|_H^2$$
$$+ \int_\Omega \xi_\delta(t)(y_\delta(t) - M_\delta(t)) = -\langle M_\delta'(t), \mathcal{N}(y_\delta(t) - M_\delta(t)) \rangle + \int_\Gamma h(t)\mathcal{N}(y_\delta(t) - M_\delta(t))$$
$$- \tau \int_\Omega M_\delta'(t)(y_\delta(t) - M_\delta(t)) - \int_\Omega \lambda(t)(\pi(y_\delta(t)) - \pi(M_\delta(t)))(y_\delta(t) - M_\delta(t))$$
$$- \int_\Omega \lambda(t)\pi(M_\delta(t))(y_\delta(t) - M_\delta(t)) + \int_\Omega g(t)(y_\delta(t) - M_\delta(t)).$$

Recalling (2.7)–(2.8), we note that

$$- \langle M_\delta'(t), \mathcal{N}(y_\delta(t) - M_\delta(t)) \rangle = -M_\delta'(t)|\Omega| \left(\mathcal{N}(y_\delta(t) - M_\delta(t)) \right)_\Omega = 0\,,$$
$$- \tau \int_\Omega M_\delta'(t)(y_\delta(t) - M_\delta(t)) = -\tau M_\delta'(t)|\Omega| (y_\delta(t) - M_\delta(t))_\Omega = 0$$

while the subdifferential rule for β together with (2.27) leads to

$$\int_\Omega \xi_\delta(t)(y_\delta(t) - M_\delta(t)) \ge \int_\Omega \widehat{\beta}(y_\delta(t)) - \int_\Omega \widehat{\beta}(M_\delta(t))\,.$$

Hence, taking now into account conditions (2.11)–(2.12) and using the assumptions (2.20)–(2.21) and (2.13)–(2.14) on the data, we integrate on $(0,t)$ and, with the help of the Young inequality, we deduce that

$$\frac{1}{2} \|y_\delta(t) - M_\delta(t)\|_*^2 + \frac{\tau}{2} \|y_\delta(t) - M_\delta(t)\|_H^2 + \delta \int_0^t \|\nabla y_\delta(s)\|_H^2 \, ds + \int_0^t \int_\Omega \widehat{\beta}(y_\delta(s)) \, ds$$
$$\le \frac{1}{2} \|y_{0,\delta} - (y_{0,\delta})_\Omega\|_*^2 + \frac{\tau}{2} \|y_{0,\delta} - (y_{0,\delta})_\Omega\|_H^2 + \frac{1}{2} \|h\|_{L^2(\Sigma)}^2$$
$$+ \frac{C}{2} \int_0^t \|y_\delta(s) - M_\delta(s)\|_*^2 \, ds + \left(C_\pi \|\lambda\|_{L^\infty(Q)} + 1 \right) \int_0^t \|y_\delta(s) - M_\delta(s)\|_H^2 \, ds$$
$$+ \frac{1}{2} \left(C_\pi^2 |\Omega| \|\lambda\|_{L^\infty(Q)}^2 \|M_\delta\|_{L^2(0,T)}^2 + \|g\|_{L^2(Q)}^2 \right) + \int_0^t \int_\Omega \widehat{\beta}(M_\delta(s)) \, ds$$

for all $t \in (0, T)$ and for some Lipschitz constant C_π of π. Now, in view of (3.5) and the continuity of $\widehat{\beta}$ on $\mathbb{R}$ (ensured by (2.13) and (2.16)) we have

$$\|M_\delta\|_{L^\infty(0,T)} + \left\|\widehat{\beta}(M_\delta)\right\|_{L^\infty(0,T)} + \|M_\delta'\|_{L^2(0,T)} \leq C \tag{3.10}$$

for some constant C independent of $\delta \in (0, 1)$. Hence, taking these remarks into account and recalling that $\{y_{0,\delta}\}$ converges in $H \subseteq V^*$ by (2.34), we infer that

$$\frac{1}{2} \|y_\delta(t) - M_\delta(t)\|_*^2 + \frac{\tau}{2} \|y_\delta(t) - M_\delta(t)\|_H^2 + \delta \int_0^t \|\nabla y_\delta(s)\|_H^2 \, ds + \int_0^t \int_\Omega \widehat{\beta}(y_\delta(s)) \, ds$$
$$\leq C \left(1 + \int_0^t \|y_\delta(s) - M_\delta(s)\|_*^2 \, ds + \int_0^t \|y_\delta(s) - M_\delta(s)\|_H^2 \, ds\right).$$

Then, the Gronwall lemma ensures that

$$\|y_\delta - M_\delta\|_{L^\infty(0,T;V^*)} + \|y_\delta - M_\delta\|_{L^\infty(0,T;H)} + \delta^{1/2} \|\nabla y_\delta\|_{L^2(0,T;H)} \leq C, \tag{3.11}$$

$$\left\|\widehat{\beta}(y_\delta)\right\|_{L^1(Q)} \leq C \tag{3.12}$$

for all $\delta \in (0, 1)$; finally, since $y_\delta = (y_\delta - M_\delta) + M_\delta$, on account of (3.10) and (3.11) we find out that

$$\|y_\delta\|_{L^\infty(0,T;V^*)} + \|y_\delta\|_{L^\infty(0,T;H)} + \delta^{1/2} \|y_\delta\|_{L^2(0,T;V)} \leq C \quad \text{for all } \delta \in (0, 1). \tag{3.13}$$

3.3 The estimate on $\partial_t y_\delta$

Let us now prove some uniform estimate on $\partial_t y_\delta$. Observe that, in view of the regularity (2.23), the variational equality (2.26) yields the equation

$$w_\delta = \tau \partial_t y_\delta - \delta \Delta y_\delta + \xi_\delta + \lambda \pi(y_\delta) - g \quad \text{a.e. in } Q \tag{3.14}$$

along with the Neumann homogeneous boundary condition $\partial_\nu y_\delta = 0$ a.e. on Σ. Hence, we can take $v = \mathcal{N}(\partial_t y_\delta(t) - M_\delta'(t))$ in (2.25) and test (3.14) at time t by $-(\partial_t y_\delta - M_\delta')(t)$. Note that this makes sense since $-(\partial_t y_\delta - M_\delta')$ is in $L^2(Q)$. Summing up the two equations, the second and third integral on the left hand side cancel thanks to (2.8): hence, using condition (2.10) and integrating on $(0, t)$, for almost every $t \in (0, T)$ we have

$$\int_0^t \|\partial_t y_\delta(s) - M_\delta'(s)\|_*^2 \, ds + \tau \int_0^t \|\partial_t y_\delta(s) - M_\delta'(s)\|_H^2 \, ds + \frac{\delta}{2} \|\nabla y_\delta(t)\|_H^2$$
$$+ \int_0^t \int_\Omega \xi_\delta(s) \partial_t y_\delta(s) \, ds = \frac{\delta}{2} \|\nabla y_{0,\delta}\|_H^2 - \int_0^t \langle M_\delta'(s), \mathcal{N}(\partial_t y_\delta(s) - M_\delta'(s)) \rangle \, ds$$
$$+ \int_0^t \int_\Gamma h(s) \mathcal{N}(\partial_t y_\delta(s) - M_\delta'(s)) \, ds - \tau \int_0^t \int_\Omega M_\delta'(s)(\partial_t y_\delta(s) - M_\delta'(s)) \, ds$$
$$+ \int_0^t \int_\Omega \xi_\delta(s) M_\delta'(s) \, ds + \int_0^t \int_\Omega [g(s) - \lambda(s)\pi(y_\delta(s))] \, (\partial_t y_\delta(s) - M_\delta'(s)) \, ds.$$

As in the previous subsection, using (2.7)–(2.8) we see that

$$-\langle M'_\delta(s), \mathcal{N}(y_\delta(s) - M_\delta(s))\rangle = 0\,, \quad \int_\Omega M'_\delta(s)(\partial_t y_\delta(s) - M'_\delta(s))\,ds = 0\,,$$

while a well-known result (contained for example in [4, p. 73]) ensures that

$$\int_\Omega \xi_\delta(t)\partial_t y_\delta(t) = \frac{d}{dt}\int_\Omega \widehat{\beta}(y_\delta(t))$$

hence, using the Young inequality and taking into account conditions (2.11), (2.13)–(2.14), (2.20)–(2.21) and the growth assumption (2.17) (cf. (2.16) and Remark 2.1), we deduce that for almost every $t \in (0,T)$

$$\int_0^t \|\partial_t y_\delta(s) - M'_\delta(s)\|_*^2\,ds + \tau\int_0^t \|\partial_t y_\delta(s) - M'_\delta(s)\|_H^2\,ds + \frac{\delta}{2}\|\nabla y_\delta(t)\|_H^2 + \int_\Omega \widehat{\beta}(y_\delta(t))$$

$$\leq \frac{\delta}{2}\|\nabla y_{0,\delta}\|_H^2 + \int_\Omega \widehat{\beta}(y_{0,\delta}) + \frac{1}{2}\int_0^t \|\partial_t y_\delta(s) - M'_\delta(s)\|_*^2\,ds + \frac{C}{2}\|h\|_{L^2(\Sigma)}^2$$

$$+ \frac{\tau}{2}\int_0^t \|\partial_t y_\delta(s) - M'_\delta(s)\|_H^2\,ds + \frac{1}{2\tau}\|g - \lambda\pi(y_\delta)\|_{L^2(Q)}^2$$

$$+ C\int_0^t M'_\delta(s)\int_\Omega \widehat{\beta}(y_\delta(s))\,ds + C|\Omega|\,\|M'_\delta\|_{L^1(0,T)}\,.$$

for some constant $C > 0$. Now, thanks to conditions (2.35), (3.13), (3.10) and the Lipschitz continuity of π, we can apply the Gronwall lemma and infer that (updating the value of C, as usual)

$$\|\partial_t y_\delta - M'_\delta\|_{L^2(0,T;V^*)} + \|\partial_t y_\delta - M'_\delta\|_{L^2(0,T;H)} + \delta^{1/2}\|\nabla y_\delta\|_{L^\infty(0,T;H)} \leq C, \qquad (3.15)$$

$$\left\|\widehat{\beta}(y_\delta)\right\|_{L^\infty(0,T;L^1(\Omega))} \leq C \qquad (3.16)$$

for all $\delta \in (0,1)$. Hence, using (3.10) and (3.13) again we conclude that

$$\|y_\delta\|_{H^1(0,T;H)} + \delta^{1/2}\|y_\delta\|_{L^\infty(0,T;V)} \leq C \quad \text{for all } \delta \in (0,1)\,. \qquad (3.17)$$

3.4 The estimate on w_δ

We now take $v = w_\delta(t) - (w_\delta(t))_\Omega \in V$ in equation (2.25); by (2.6) we obtain

$$\int_\Omega |\nabla w_\delta(t)|^2 = \int_\Omega |\nabla(w_\delta(t) - (w_\delta(t))_\Omega)|^2$$

$$\leq C\|h(t)\|_{L^2(\Gamma)}\|w_\delta(t) - (w_\delta(t))_\Omega\|_V + \|\partial_t y_\delta(t)\|_{V^*}\|w_\delta(t) - (w_\delta(t))_\Omega\|_V$$

$$\leq C'\left(\|h(t)\|_{L^2(\Gamma)}^2 + \|\partial_t y_\delta(t)\|_{V^*}^2\right) + \frac{1}{2}\int_\Omega |\nabla w_\delta(t)|^2$$

for almost all $t \in (0,T)$ and for some two positive constants C and C'. Then, recalling the estimate (3.17) just proved and hypothesis (2.21) on h, we deduce that

$$\|\nabla w_\delta\|_{L^2(0,T;H)} \leq C \quad \text{for all } \delta \in (0,1)\,. \qquad (3.18)$$

Moreover, taking $v = 1$ in (2.26), thanks to (2.16) for almost all $t \in (0, T)$ we have

$$|\Omega| \, (w_\delta(t))_\Omega \leq \tau \, \|\partial_t y_\delta(t)\|_{L^1(\Omega)} + C \left(|\Omega| + \left\| \widehat{\beta}(y_\delta(t)) \right\|_{L^1(\Omega)} \right)$$
$$+ \|\lambda\|_{L^\infty(Q)} \, C_\pi \, \|y_\delta(t)\|_{L^1(\Omega)} + \|g(t)\|_{L^1(\Omega)}$$

so that recalling (3.16), (3.17), and the hypothesis (2.21) on g we infer

$$\|(w_\delta)_\Omega\|_{L^2(0,T)} \leq C \quad \text{for all } \delta \in (0, 1),$$

and this uniform bound, along with (3.18), ensures that

$$\|w_\delta\|_{L^2(0,T;V)} \leq C \quad \text{for all } \delta \in (0, 1). \tag{3.19}$$

3.5   The estimate on ξ_δ

Next, we would like to show a uniform estimate for ξ_δ in $L^2(Q)$. Let us deduce it on some approximating problem in which β is replaced by its Yosida regularization β_ε. Indeed, the estimate, independent of $\varepsilon \in (0, 1)$, will be proved for $\beta_\varepsilon(y_{\delta,\varepsilon})$, where $y_{\delta,\varepsilon}$ denotes the main component of the approximating solution. Then, passing to the limit as $\varepsilon \searrow 0$, one obtains the same estimate for ξ_δ (cf. Remark 2.3).

Let us rewrite (2.26) in tems of $w_{\delta,\varepsilon}$ and $y_{\delta,\varepsilon}$ obtaining

$$\int_\Omega w_{\delta,\varepsilon}(t)v = \tau \int_\Omega \partial_t y_{\delta,\varepsilon}(t)v + \delta \int_\Omega \nabla y_{\delta,\varepsilon}(t) \cdot \nabla v + \int_\Omega \beta_\varepsilon(y_{\delta,\varepsilon}(t)v$$
$$+ \int_\Omega \lambda(t)\pi(y_{\delta,\varepsilon}(t))v - \int_\Omega g(t)v \quad \text{for all } v \in H^1(\Omega).$$

Taking $v = \beta_\varepsilon(y_{\delta,\varepsilon}(t)) \in V$, we have

$$\delta \int_\Omega \beta_\delta'(y_{\delta,\varepsilon}(t)) \, |\nabla y_{\delta,\varepsilon}(t)|^2 + \int_\Omega |\beta_\varepsilon(y_{\delta,\varepsilon}(t))|^2$$
$$= \int_\Omega \left[g(t) - \lambda(t)\pi(y_{\delta,\varepsilon}(t)) + w_{\delta,\varepsilon}(t) - \tau \partial_t y_{\delta,\varepsilon}(t) \right] \beta_\varepsilon(y_{\delta,\varepsilon}(t))$$

for almost every $t \in (0, T)$. Hence, using the monotonicity of β_ε, the hypotheses on g, λ and π together with conditions (3.17) and (3.19), from integration with respect to time and the elementary Young inequality it follows that

$$\int_0^T \int_\Omega |\beta_\varepsilon(y_{\delta,\varepsilon})|^2 \leq \int_0^T \int_\Omega |g - \lambda\pi(y_{\delta,\varepsilon}) + w_{\delta,\varepsilon} - \tau\partial_t y_{\delta,\varepsilon}| \, |\beta_\varepsilon(y_{\delta,\varepsilon})|$$
$$\leq C + \frac{1}{2} \int_0^T \int_\Omega |\beta_\varepsilon(y_{\delta,\varepsilon})|^2$$

for a constant $C > 0$ independent of both δ and $\varepsilon \in (0, 1)$. Consequently, bearing in mind what we have anticipated before, we conclude that

$$\|\xi_\delta\|_{L^2(Q)} \leq \liminf_{\varepsilon \searrow 0} \|\beta_\varepsilon(y_{\delta,\varepsilon})\|_{L^2(Q)} \leq C \quad \text{for all } \delta \in (0, 1). \tag{3.20}$$

Let us point out a consequence of (3.20): by a comparison of the terms in (3.14) we deduce that

$$\delta \|\Delta y_\delta\|_{L^2(0,T;H)} \le C \quad \text{for all } \delta \in (0,1) . \tag{3.21}$$

Moreover, (3.13), (3.21) and well-known elliptic regularity results imply that

$$\delta \|y_\delta\|_{L^2(0,T;H^2(\Omega))} \le C \quad \text{for all } \delta \in (0,1) . \tag{3.22}$$

3.6   The passage to the limit

We are now ready to pass to the limit and conclude the proof of Theorem 2.4. Firstofall, we notice that the estimates (3.17), (3.19), (3.20), (3.21) ensure that there exist

$$y \in H^1(0,T;H) , \quad w \in L^2(0,T;V) , \quad \xi \in L^2(Q) \tag{3.23}$$

such that, at least for a subsequence,

$$y_\delta \rightharpoonup y \quad \text{in } H^1(0,T;H) , \tag{3.24}$$

$$\delta \Delta y_\delta \rightharpoonup 0 \quad \text{in } L^2(0,T;H), \tag{3.25}$$

$$w_\delta \rightharpoonup w \quad \text{in } L^2(0,T;V), \tag{3.26}$$

$$\xi_\delta \rightharpoonup \xi \quad \text{in } L^2(Q) \tag{3.27}$$

as $\delta \searrow 0$. Moreover, in view of (2.14) and (3.17) there is some $\eta \in H^1(0,T;H)$ such that

$$\pi(y_\delta) \rightharpoonup \eta \quad \text{in } H^1(0,T;H) \quad \text{as } \delta \searrow 0. \tag{3.28}$$

We also note that δy_δ tends to 0 strongly in $L^\infty(0,T;V)$ and weakly in $L^2(0,T;H^2(\Omega))$, due to (3.17) and (3.22).

Then, we can pass to the limit in (2.25) to readily recover (2.40) and in (3.14), obtaining (cf. (2.20) too)

$$w = \tau \partial_t y + \xi + \lambda\eta - g \quad \text{a.e. in } Q. \tag{3.29}$$

From (3.24), using the Ascoli theorem, it follows that

$$y_\delta \to y \quad \text{in } C^0([0,T];V^*). \tag{3.30}$$

We have to show that

$$\eta = \pi(y) \quad \text{and} \quad \xi \in \beta(y) \quad \text{a.e. in } Q. \tag{3.31}$$

To this aim, we multiply (3.14) by the test function $e^{-Rt}y_\delta(t)$ and integrate over space and time, with $R > 0$ to be chosen soon. As $e^{-Rt/2}\partial_t y_\delta(t) = \partial_t(e^{-Rt/2}y_\delta(t)) + \frac{R}{2}e^{-Rt/2}y_\delta(t)$, it is easy verify that

$$\frac{\tau}{2}\int_\Omega e^{-RT}|y_\delta(T)|^2 + \frac{R}{2}\tau \int_0^T \int_\Omega e^{-Rt}|y_\delta(t)|^2 dt + \int_0^T \int_\Omega e^{-Rt}\, \delta|\nabla y_\delta(t)|^2 dt$$

$$+ \int_0^T \int_\Omega e^{-Rt}(\xi_\delta + \lambda\pi(y_\delta))(t)\, y_\delta(t)\, dt + \int_0^T \int_\Omega e^{-Rt}g(t)\, y_\delta(t)\, dt$$

$$= \frac{\tau}{2}\|y_{0,\delta}\|_H^2 + \int_0^T e^{-Rt}\langle y_\delta(t), w_\delta(t)\rangle dt. \tag{3.32}$$

Hence, from (3.32) we infer that

$$\int_0^T \int_\Omega e^{-Rt} \left(\frac{R}{2} \tau \, y_\delta + \lambda \pi(y_\delta) + \xi_\delta \right)(t) \, y_\delta(t) \, dt$$

$$\leq \frac{\tau}{2} \|y_{0,\delta}\|_H^2 + \int_0^T e^{-Rt} \langle y_\delta(t), w_\delta(t) \rangle dt$$

$$- \int_0^T \int_\Omega e^{-Rt} g(t) \, y_\delta(t) \, dt - \frac{\tau}{2} \int_\Omega e^{-RT} |y_\delta(T)|^2. \tag{3.33}$$

We now choose R in order that the operator $L : L^2(Q) \to L^2(Q)$ defined by

$$L(v)(x,t) := \frac{R}{2} \tau \, v(x,t) + \lambda(x,t)\pi(v(x,t)), \quad (x,t) \in Q, \quad v \in L^2(Q) \tag{3.34}$$

be strongly monotone and Lipschitz continuous. In fact, we can take $R > 2\|\lambda\|_{L^\infty(Q)} C_\pi / \tau$, where C_π stands for a Lipschitz constant for π (see (2.14)). Then, it turns out that (see, e.g., [4, Lemme 2.4, p. 34]) the sum of L and of the operator induced by β on $L^2(Q)$ (still denoted by β) is maximal monotone. As

$$\int_0^T \int_\Omega e^{-Rt} v_1(t) \, v_2(t) \, dt, \quad v_1, \, v_2 \in L^2(Q),$$

is an admissible scalar product in $L^2(Q)$, if we can show that

$$\limsup_{\delta \searrow 0} \int_0^T \int_\Omega e^{-Rt} \left(\frac{R}{2} \tau \, y_\delta + \lambda \, \pi(y_\delta) + \xi_\delta \right)(t) \, y_\delta(t) \, dt$$

$$\leq \int_0^T \int_\Omega e^{-Rt} \left(\frac{R}{2} \tau \, y + \lambda \, \eta + \xi \right)(t) \, y(t) \, dt, \tag{3.35}$$

then, on account of (3.24), (3.27), (3.28) and using a standard result for maximal monotone operators (see, e.g., [4, Prop. 2.5, p. 27]), we actually prove that

$$\frac{R}{2} \tau \, y + \lambda \, \eta + \xi \in (L + \beta)(y) \quad \text{in } L^2(Q). \tag{3.36}$$

In order to check (3.35), we pass to the lim sup in the inequality (3.33) noting that

$$\lim_{\delta \searrow 0} \frac{\tau}{2} \|y_{0,\delta}\|_H^2 = \frac{\tau}{2} \|y_0\|_H^2 \quad \text{by (2.34)},$$

$$\lim_{\delta \searrow 0} \int_0^T e^{-Rt} \langle y_\delta(t), w_\delta(t) \rangle dt = \int_0^T e^{-Rt} \langle y(t), w(t) \rangle dt \quad \text{by (3.30) and (3.26)},$$

$$\lim_{\delta \searrow 0} - \int_0^T \int_\Omega e^{-Rt} g(t) \, y_\delta(t) \, dt = - \int_0^T \int_\Omega e^{-Rt} g(t) \, y(t) \, dt \quad \text{by (2.21) and (3.24)},$$

$$\limsup_{\delta \searrow 0} - \frac{\tau}{2} \int_\Omega e^{-RT} |y_\delta(T)|^2 \leq - \liminf_{\delta \searrow 0} \frac{\tau}{2} \int_\Omega e^{-RT} |y_\delta(T)|^2 \leq - \frac{\tau}{2} \int_\Omega e^{-RT} |y(T)|^2,$$

where the last inequality is a consequence of the weak convergence of $y_\delta(t)$ to $y(t)$ in H for all $t \in [0, T]$ and of the lower semicontinuity property of the norm in H. Note that

the mentioned weak convergence can also be invoked to obtain the initial condition (2.43) from (2.28). Then, using the equality (3.29) tested by $e^{-Rt}y(t)$, we easily conclude that

$$\limsup_{\delta \searrow 0} \int_0^T \int_\Omega e^{-Rt} \left(\frac{R}{2} \tau \, y_\delta + \lambda \pi(y_\delta) + \xi_\delta \right)(t) \, y_\delta(t) \, dt$$

$$\leq \frac{\tau}{2} \|y_0\|_H^2 + \int_0^T e^{-Rt} \langle y(t), w(t) \rangle dt - \int_0^T \int_\Omega e^{-Rt} g(t) \, y(t) \, dt$$

$$- \frac{\tau}{2} \int_\Omega e^{-RT} |y(T)|^2 = \int_0^T \int_\Omega e^{-Rt} \left(\frac{R}{2} \tau \, y + \lambda \, \eta + \xi \right)(t) \, y(t) \, dt,$$

whence (3.35) follows.

At this point, we derive the strong convergence

$$y_\delta(t) \to y(t) \quad \text{in } H \quad \text{for all } t \in [0, T]. \tag{3.37}$$

To this aim, we take the difference between (3.14) and (3.29), multiply by the test function $e^{-2Rs}(y_\delta - y)(s)$ and integrate over $\Omega \times (0, t)$, with $t \in (0, T]$. Taking into account the already performed computations, it is straightforward to verify that

$$\frac{\tau}{2} \int_\Omega e^{-2Rt} |(y_\delta - y)(t)|^2 + \frac{R}{2} \tau \int_0^t e^{-2Rs} \|(y_\delta - y)(s)\|_H^2 ds + \int_0^t \int_\Omega e^{-2Rs} \, \delta \, |\nabla y_\delta(s)|^2 ds$$

$$+ \int_0^t \int_\Omega e^{-2Rs} \left(\frac{R}{2} \tau \, y_\delta + \lambda \, \pi(y_\delta) + \xi_\delta - \frac{R}{2} \tau \, y - \lambda \, \eta - \xi \right)(s) \, (y_\delta - y)(s) \, ds$$

$$= \frac{\tau}{2} \int_\Omega |y_{0,\delta} - y_0|^2 - \int_0^t \int_\Omega e^{-2Rs} \, \delta \Delta y_\delta(s) \, y(s) \, ds$$

$$+ \int_0^t e^{-2Rs} \langle (y_\delta - y)(s), (w_\delta - w)(s) \rangle \, ds \quad \text{for all } t \in [0, T]. \tag{3.38}$$

Note that the integral on the second line of (3.38) is non-negative due to the monotonicity of the operator (cf. (3.34) and (2.13)) $L + \beta$ and to (2.27) and (3.36). Moreover, on account of (2.34), (3.25), (3.26) and (3.30) the right hand side of (3.38) tends to 0 for all $t \in [0, T]$. This implies (3.37) as well as

$$y_\delta \to y \quad \text{in } L^2(0, T; H), \quad \delta y_\delta \to 0 \quad \text{in } L^2(0, T; V). \tag{3.39}$$

Then, (2.45) follows from (3.30) and (3.39); moreover, the Lipschitz continuity of π enables us to conclude that $\pi(y_\delta) \to \pi(y)$ strongly in $L^2(0, T; H)$, whence $\eta = \pi(y)$ from (3.28). At this point, it is not difficult to deduce (3.31), and in particular (2.42), from (3.36), as

$$\frac{R}{2} \tau \, y + \lambda \, \eta \quad \text{is the unique element of } Ly \text{ in } L^2(Q).$$

Hence, Theorem 2.4 is completely proved. As a further remark, we note that the boundedness of $\{y_\delta\}$ in $L^\infty(0, T; H)$ and the convergence (3.37) (or the first one in (3.39)) ensure that $y_\delta \to y$ strongly in $L^p(0, T; H)$ for all $p \in [1, \infty)$.

4   Continuous dependence for the limit problem

This section is devoted to proving the continuous dependence result for the limit problem. Assume thus the hypotheses of Theorem 2.6 and let (y_i, w_i, ξ_i), $i = 1, 2$, be any respective solutions corresponding to the data in (2.48). For simplicity, let us introduce the notation

$$
\begin{aligned}
y &:= y_1 - y_2, & w &= w_1 - w_2, & \xi &= \xi_1 - \xi_2, \\
y_0 &:= y_0^1 - y_0^2, & g &:= g_1 - g_2, & h &:= h_1 - h_2.
\end{aligned}
$$

Hence, if we subtract the corresponding equations (2.40)–(2.41), for almost every $t \in (0, T)$ we have

$$
\int_\Omega \partial_t y(t) v + \int_\Omega \nabla w(t) \cdot \nabla v = \int_\Gamma h(t) v \quad \text{for all } v \in V, \tag{4.1}
$$

$$
\begin{aligned}
\int_\Omega w(t) v = \tau &\int_\Omega \partial_t y(t) v + \int_\Omega \xi(t) v \\
&+ \int_\Omega \lambda(t) \left(\pi(y_1(t)) - \pi(y_2(t)) \right) v - \int_\Omega g(t)) v \quad \text{for all } v \in V. \tag{4.2}
\end{aligned}
$$

Please note that hypothesis (2.50) ensure that $(y(t))_\Omega = 0$, so that we test (4.1) by $v = \mathcal{N}(y(t))$ and (4.2) by $v = -y(t)$: summing up, the second and third integral on the left hand side cancel thanks to (2.8). Therefore, for almost every $t \in (0, T)$ we infer that

$$
\begin{aligned}
\langle \partial_t y(t), \mathcal{N}(y(t)) \rangle &+ \frac{\tau}{2} \frac{d}{dt} \|y(t)\|_H^2 + \int_\Omega \xi(t) y(t) \\
&= \int_\Gamma h(t) \mathcal{N}(y(t)) + \int_\Omega g(t) y(t) - \int_\Omega \lambda(t) \left(\pi(y_1(t)) - \pi(y_2(t)) \right) y(t).
\end{aligned}
$$

Then, in view of (2.10)–(2.12) and the monotonicity of β, integrating on $(0, t)$ leads to

$$
\begin{aligned}
\frac{1}{2} \|y(t)\|_*^2 + \frac{\tau}{2} \|y(t)\|_H^2 \leq \frac{1}{2} \|y_0\|_*^2 + \frac{\tau}{2} \|y_0\|_H^2 + \frac{C^2}{2} \|h\|_{L^2(\Sigma)}^2 + \frac{1}{2} \|g\|_{L^2(Q)}^2 \\
+ \frac{1}{2} \int_0^t \|y(s)\|_*^2 \, ds + \left(\frac{1}{2} + \|\lambda\|_{L^\infty(Q)} C_\pi \right) \int_0^t \|y(s)\|_H^2 \, ds
\end{aligned}
$$

for a.e. $t \in (0, T)$ and a certain constant $C > 0$. Now, we can apply the Gronwall lemma that implies (updating the value of C)

$$
\|y\|_{L^\infty(0, T; H)} \leq C \left[\|y_0\|_H + \|h\|_{L^2(\Sigma)} + \|g\|_{L^2(Q)} \right],
$$

and the estimate (2.51) is proved.

5   Asymptotic error estimate

In this section, we prove the asymptotic estimate stated in Theorem 2.8. To this aim, in a first step we deduce an additional uniform estimate to improve the boundedness properties of the solution to the problem (2.25)–(2.28).

5.1   The estimate on Δy_δ

Here, we want to prove something better than (3.21). We test (3.14) by $-\delta^{1/2}\Delta y_\delta(t)$ and deduce that

$$\delta^{3/2}\int_\Omega |\Delta y_\delta(t)|^2 - \tau\delta^{1/2}\int_\Omega \partial_t y_\delta(t)\Delta y_\delta(t) = -\delta^{1/2}\int_\Omega w_\delta(t)\Delta y_\delta(t)$$

$$+ \delta^{1/2}\int_\Omega \xi_\delta(t)\Delta y_\delta(t) + \delta^{1/2}\int_\Omega \lambda(t)\pi(y_\delta(t))\Delta y_\delta(t) - \delta^{1/2}\int_\Omega g(t)\Delta y_\delta(t) \qquad (5.1)$$

for almost all $t \in (0,T)$. Please note that the second term on the left hand side can be written by integration by parts as

$$\frac{\tau}{2}\delta^{1/2}\frac{d}{dt}\int_\Omega |\nabla y_\delta(t)|^2\,;$$

let us handle the terms on the right hand side. We integrate by parts, taking into account that y_δ satisfies Neumann homogeneous boundary conditions. By the Young inequality we have

$$-\delta^{1/2}\int_\Omega w_\delta(t)\Delta y_\delta(t) = \delta^{1/2}\int_\Omega \nabla w_\delta(t)\cdot\nabla y_\delta(t) \le \frac{1}{2}\left\|\nabla w_\delta(t)\right\|_H^2 + \frac{1}{2}\left\|\delta^{1/2}\nabla y_\delta(t)\right\|_H^2\,,$$

$$-\delta^{1/2}\int_\Omega g(t)\Delta y_\delta(t) = \delta^{1/2}\int_\Omega \nabla g(t)\cdot\nabla y_\delta(t) \le \frac{1}{2}\left\|\nabla g(t)\right\|_H^2 + \frac{1}{2}\left\|\delta^{1/2}\nabla y_\delta(t)\right\|_H^2\,;$$

moreover, proceeding formally as we already did in Subsection 3.5, it is not restrictive to argue with $\beta_\varepsilon(y_\delta)$ instead of ξ_δ, so that using monotonicity we deduce that

$$\delta^{1/2}\int_\Omega \beta_\varepsilon(y_\delta(t))\Delta y_\delta(t) = -\delta^{1/2}\int_\Omega \beta_\varepsilon'(y_\delta(t))\,|\nabla y_\delta(t)|^2 \le 0\,.$$

Finally, using the hypotheses on λ and π and the Young inequality, a direct computation leads to

$$\delta^{1/2}\int_\Omega \lambda(t)\pi(y_\delta(t))\Delta y_\delta(t)$$

$$= -\delta^{1/2}\int_\Omega \pi(y_\delta(t))\nabla\lambda(t)\cdot\nabla y_\delta(t) - \delta^{1/2}\int_\Omega \lambda(t)\pi'(y_\delta(t))\,|\nabla y_\delta(t)|^2$$

$$\le \frac{1}{2}C_\pi^2\left\|y_\delta\right\|_{L^\infty(0,T;H)}^2\left\|\lambda(t)\right\|_{W^{1,\infty}(\Omega)}^2 + \frac{1}{2}\left\|\delta^{1/2}\nabla y_\delta(t)\right\|_H^2 + \left\|\lambda\right\|_{L^\infty(Q)}C_\pi\delta^{1/2}\int_\Omega |\nabla y_\delta(t)|^2\,.$$

Hence, taking all these remarks into account, integrating (5.1) with respect to time, and using (3.17), (3.19) and the hypotheses (2.20)–(2.21) we obtain

$$\delta^{3/2}\int_0^t\int_\Omega |\Delta y_\delta(s)|^2\,ds + \frac{\tau}{2}\delta^{1/2}\left\|\nabla y_\delta(t)\right\|_H^2$$

$$\le \frac{\tau}{2}\delta^{1/2}\left\|\nabla y_{0,\delta}\right\|_H^2 + C + C\delta^{1/2}\int_0^t\left\|\nabla y_\delta(s)\right\|_H^2\,ds$$

for some constant $C > 0$; then, using the Gronwall lemma and condition (2.35), we conclude that

$$\delta^{3/4}\left\|\Delta y_\delta\right\|_{L^2(0,T;H)} + \delta^{1/4}\left\|y_\delta\right\|_{L^2(0,T;V)} \le C \quad \text{for all } \delta \in (0,1)\,. \qquad (5.2)$$

5.2   Error estimate

In order to prove Theorem 2.8, we subtract (2.25) and (2.26) to (2.40) and (2.41), respectively, and for almost every $t \in (0,T)$ we have

$$\langle \partial_t(y(t) - y_\delta(t)), v \rangle + \int_\Omega \nabla(w(t) - w_\delta(t)) \cdot \nabla v = 0 \quad \text{for all } v \in H^1(\Omega), \tag{5.3}$$

$$\int_\Omega (w(t) - w_\delta(t))v = \tau \int_\Omega \partial_t(y(t) - y_\delta(t))v - \delta \int_\Omega \nabla y_\delta(t) \cdot \nabla v + \int_\Omega (\xi(t) - \xi_\delta(t))v$$
$$+ \int_\Omega \lambda(t) \left(\pi(y(t)) - \pi(y_\delta(t)) \right) v \quad \text{for all } v \in H^1(\Omega). \tag{5.4}$$

Testing the first equation by $\mathcal{N}(y(t) - y_\delta(t))$ and the second by $-(y(t) - y_\delta(t))$, we sum up with the help of (2.8) to cancel the two integrals on the left hand side, as usual. Hence, integrating by parts we obtain for almost every $t \in (0,T)$

$$\frac{1}{2} \|y(t) - y_\delta(t)\|_*^2 + \frac{\tau}{2} \|y(t) - y_\delta(t)\|_H^2 + \int_0^t \int_\Omega (\xi(s) - \xi_\delta(s))(y(s) - y_\delta(s)) \, ds$$
$$= \frac{1}{2} \|y_0 - y_{0,\delta}\|_*^2 + \frac{\tau}{2} \|y_0 - y_{0,\delta}\|_H^2 - \delta \int_0^t \int_\Omega \Delta y_\delta(s)(y(s) - y_\delta(s)) \, ds$$
$$- \int_0^t \int_\Omega \lambda(s) \left(\pi(y(s)) - \pi(y_\delta(s))(y(s) - y_\delta(s)) \right) \, ds;$$

now, the monotonicity of β, the hypotheses (2.14) and (2.20), and the Young inequality lead to

$$\frac{1}{2} \|y(t) - y_\delta(t)\|_*^2 + \frac{\tau}{2} \|y(t) - y_\delta(t)\|_H^2 \le \frac{1}{2} \|y_0 - y_{0,\delta}\|_*^2 + \frac{\tau}{2} \|y_0 - y_{0,\delta}\|_H^2$$
$$+ \frac{\delta^{1/2}}{2} \left\| \delta^{3/4} \Delta y_\delta \right\|_{L^2(Q)}^2 + \left(\frac{1}{2} + \|\lambda\|_{L^\infty(Q)} C_\pi \right) \int_0^t \|y(s) - y_\delta(s)\|_H^2 \, ds,$$

so that the Gronwall lemma and the estimate (5.2) allow us to infer that

$$\|y - y_\delta\|_{L^\infty(0,T;H)} \le C \left[\delta^{1/4} + \|y_0 - y_{0,\delta}\|_H \right]$$

for a certain constant $C > 0$. This finishes the proof. Let us point out that the error estimate (2.54) is of order $1/4$ in terms of δ provided $\|y_0 - y_{0,\delta}\|_H \le C\delta^{1/4}$: this condition is ensured for the family defined in (3.6) whenever, for instance, $y_0 \in V$ (take $z = y_{0,\delta} - y_0$ in (3.7) and perform the estimate).

References

[1] F. Bai, C. M. Elliott, A. Gardiner, A. Spence and A. M. Stuart, The viscous Cahn-Hilliard equation. I. Computations, *Nonlinearity* **8** (1995) 131-160.

[2] V. Barbu, "Nonlinear semigroups and differential equations in Banach spaces", Noordhoff, Leyden, 1976.

[3] E. Bonetti, P. Colli and G. Tomassetti, A non-smooth regularization of a forward-backward parabolic equation, preprint arXiv:1508.03225 [math.AP] (2015), pp. 1-19.

[4] H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert", *North-Holland Math. Stud.* **5**, North-Holland, Amsterdam, 1973.

[5] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, *J. Chem. Phys.* **2** (1958) 258-267.

[6] L. Cherfils, A. Miranville and S. Zelik, The Cahn–Hilliard equation with logarithmic potentials, *Milan J. Math.* **79** (2011) 561-596.

[7] P. Colli, M.H. Farshbaf-Shaker, G. Gilardi and J. Sprekels, Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials, *SIAM J. Control Optim.* **53** (2015) 2696-2721.

[8] P. Colli and T. Fukao, Cahn–Hilliard equation with dynamic boundary conditions and mass constraint on the boundary, *J. Math. Anal. Appl.*, **429** (2015) 1190-1213.

[9] P. Colli and T. Fukao, Nonlinear diffusion equations as asymptotic limits of Cahn–Hilliard systems, *J. Differential Equations* **260** (2016) 6930-6959.

[10] P. Colli, G. Gilardi and J. Sprekels, On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential, *J. Math. Anal. Appl.* **419** (2014) 972-994.

[11] P. Colli, G. Gilardi and J. Sprekels, A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions, *Adv. Nonlinear Anal.* **4** (2015) 311-325.

[12] P. Colli, G. Gilardi and J. Sprekels, A boundary control problem for the viscous Cahn-Hilliard equation with dynamic boundary conditions, *Appl. Math. Optim.* **73** (2016) 195-225.

[13] C. M. Elliott and H. Garcke, On the Cahn-Hilliard equation with degenerate mobility, *SIAM J. Math. Anal.* **27** (1996) 404-423.

[14] C. M. Elliott and A. M. Stuart, Viscous Cahn–Hilliard equation. II. Analysis, *J. Differential Equations* **128** (1996) 387-414.

[15] C. M. Elliott and S. Zheng, On the Cahn–Hilliard equation, *Arch. Rational Mech. Anal.* **96** (1986) 339-357.

[16] G. Gilardi, A. Miranville and G. Schimperna, On the Cahn–Hilliard equation with irregular potentials und dynamic boundary conditions, *Commun. Pure Appl. Anal.* **8** (2009) 881-912.

[17] G. Gilardi, A. Miranville, and G. Schimperna, Long-time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions, *Chin. Ann. Math. Ser. B* **31** (2010) 679-712.

[18] M. Hintermüller and D. Wegner, Distributed optimal control of the Cahn–Hilliard system including the case of a double-obstacle homogeneous free energy density, *SIAM J. Control Optim.* **50** (2012) 388-418.

[19] A. Novick-Cohen, On the viscous Cahn-Hilliard equation, in "Material instabilities in continuum mechanics (Edinburgh, 1985–1986)", *Oxford Sci. Publ.*, Oxford Univ. Press, New York, 1988, pp. 329–342.

[20] A. Novick-Cohen and R. L. Pego, Stable patterns in a viscous diffusion equation, *Trans. Amer. Math. Soc.* **324** (1991) 331-351.

[21] G. Tomassetti, Some remarks on a viscous regularization of the nonlinear diffusion equation, preprint arXiv:1505.07993 [math.AP] (2015), pp. 1-22.

[22] Q.-F. Wang and S.-I. Nakagiri, Optimal control of distributed parameter system given by Cahn–Hilliard equation, *Nonlinear Funct. Anal. Appl.* **19** (2014) 19-33.

[23] J. Zheng and Y. Wang, Optimal control problem for Cahn–Hilliard equations with state constraint, *J. Dyn. Control Syst.* **21** (2015) 257-272.