Existence and uniqueness of solutions to singular Cahn-Hilliard equations with nonlinear viscosity terms and dynamic boundary conditions*

Luca Scarpa
Department of Mathematics, University College London
Gower Street, London WC1E 6BT, United Kingdom
E-mail: luca.scarpa.15@ucl.ac.uk

Abstract

We prove global existence and uniqueness of solutions to a Cahn-Hilliard system with nonlinear viscosity terms and nonlinear dynamic boundary conditions. The problem is highly nonlinear, characterized by four nonlinearities and two separate diffusive terms, all acting in the interior of the domain or on its boundary. Through a suitable approximation of the problem based on abstract theory of doubly nonlinear evolution equations, existence and uniqueness of solutions are proved using compactness and monotonicity arguments. The asymptotic behaviour of the solutions as the the diffusion operator on the boundary vanishes is also shown.

AMS Subject Classification: 35D30, 35D35, 35K52, 35K61, 80A22

Key words and phrases: Cahn-Hilliard system, dynamic boundary conditions, nonlinear viscosity, existence of solutions, uniqueness

1 Introduction

The viscous Cahn-Hilliard equation can be written in its general form as

$$\partial_t u - \Delta\mu = 0\,, \qquad \mu = \alpha(\partial_t u) - \Delta u + \beta(u) + \pi(u) - g \qquad \text{in } (0,T)\times\Omega\,,$$

*Acknowledgments. The author is very grateful to Pierluigi Colli for his expert support and fundamental advice. The author is also thankful for the warm hospitality and excellent working conditions at the Dipartimento di Matematica "F. Casorati", Università di Pavia (Italy), where a part of this work was written.

where the unknown u and μ represent the so-called order parameter and chemical potential, respectively. Such equation is fundamental in the phase-separation of a binary alloy, for example, and describes important qualitative behaviour like the so-called spinodal decomposition: we refer to the classical works [3, 24, 25, 29] for a physical derivation of the model and some studies on the spinodal decomposition process. Here, Ω is smooth bounded domain in $\mathbb{R}^N$ $(N = 2, 3)$ with smooth boundary Γ, and $T > 0$ is the final time. As usual, the term $\beta + \pi$ represents the derivative of a double-well potential, g is a given source and α is a monotone function acting on $\partial_t u$. While in the original model α is a linear function, some generalizations have been proposed where the behaviour of α is of nonlinear type: see in this direction [21].

In the present contribution, we study the equation above coupled with the homogenous Neumann boundary condition for μ

$$\partial_{\mathbf{n}} \mu = 0 \qquad \text{in } (0, T) \times \Gamma\,,$$

which is very natural and ensures the conservation of the mass in the bulk, and a second-order doubly nonlinear dynamic boundary condition for u

$$\alpha_\Gamma(\partial_t u) + \partial_{\mathbf{n}} u - \varepsilon \Delta_\Gamma u + \beta_\Gamma(u) + \pi_\Gamma(u) = g_\Gamma \qquad \text{in } (0, T) \times \Gamma\,.$$

Here, $\varepsilon > 0$ is a fixed constant, Δ_Γ is the usual Laplace-Beltrami operator on Γ, g_Γ is a prescribed source on the boundary and the term $\beta_\Gamma + \pi_\Gamma$ represents the derivative of a double-well potential on the boundary, which may possibly differ from the one in the interior of the domain Ω. Similarly, α_Γ is a generic monotone function. Dynamic boundary conditions have been recently proposed by physicists in order to take into account also possible interactions with the walls of a confined system: for a physical motivation of this choice and some studies on parabolic-type equations with dynamic boundary conditions we mention the works [15, 23] and [16–18].

Cahn-Hilliard equations with dynamic boundary have been widely studied in the last years in the classical setting in which the viscosity terms depend linearly on the time-derivative of the order parameter. This framework corresponds in our notation to the choices $\alpha = aI$ and $\alpha_\Gamma = bI$, with $a, b > 0$ given constants and I the identity on $\mathbb{R}$. Let us mention in this direction the works [6–8, 11, 19, 20, 27] dealing with well-posedness, regularity, long-time behaviour of solutions and asymptotics, [5,9,10] for some corresponding optimal control problems, and [4,12] focused specifically on Allen-Cahn equations.

On the other hand, an important area of interest has been equally developed on the study of Cahn-Hilliard equations with possibly nonlinear viscosity terms: the reader can refer to the contributions [26] for existence-uniqueness and long-time behaviour under classical homogeneous Neumann conditions, and to [1] for a detailed thermodynamical derivation of the model and well-posedness in the case of Dirichlet conditions for the chemical potential. Let us also mention the work [28] dealing with a doubly nonlinear Cahn-Hilliard equation with a different type of nonlinearity in the viscosity, and the classical contributions [13, 30] on a variational approach to abstract doubly nonlinear equations. As the reader may notice, in this case the attention is mainly focused on the presence of a double nonlinearity in the governing equation, and, consequently, the prescription on the boundary conditions remains quite broad and classical (homogeneous Neumann or Dirichlet type).

The aim of this paper is to provide some unifying existence and uniqueness results for the more general case when both dynamic boundary conditions and nonlinear viscosity terms are present in the system. From the physical point of view, the presence of dynamic boundary conditions and nonlinear viscosity terms is more accurate, and allows for a more genuine description on the process. On the other side, from the mathematical perspective, the model is much more difficult to handle and to study. Indeed, this specific description gives rise to a system with 4 nonlinearities: α and α_Γ acting on the time-derivatives and representing the viscosities, and β and β_Γ acting on the order parameter. Besides the non-triviality of the model, the presence of several nonlinearities is strongly stimulating and challenging. In order to include also possibly non-smooth potentials in our analysis, the nonlinearities are assumed to be possibly multivalued (maximal monotone) graphs.

To summarize, we are concerned with the following system

$$\partial_t u - \Delta\mu = 0 \qquad \text{in } (0,T)\times\Omega\,, \tag{1.1}$$

$$\mu \in \alpha(\partial_t u) - \Delta u + \beta(u) + \pi(u) - g \qquad \text{in } (0,T)\times\Omega\,, \tag{1.2}$$

$$u = v\,, \quad \partial_{\mathbf{n}}\mu = 0 \qquad \text{in } (0,T)\times\Gamma\,, \tag{1.3}$$

$$\alpha_\Gamma(\partial_t v) + \partial_{\mathbf{n}} u - \varepsilon\Delta_\Gamma v + \beta_\Gamma(v) + \pi_\Gamma(v) \ni g_\Gamma \qquad \text{in } (0,T)\times\Gamma\,, \tag{1.4}$$

$$u(0) = u_0\,, \quad v(0) = v_0 \qquad \text{in } \Omega\,. \tag{1.5}$$

The paper is organized as follows. In Section 2 we state the main hypotheses of the work and the main results, commenting on the different set of assumptions that are in play. Section 3 in entirely focused on the construction of suitable approximated solutions, and is based on some abstract results on doubly nonlinear evolution equations. In Sections 4, 5 and 6 we present the proofs of the three existence results of the paper, while Section 7 contains the proof of the uniqueness result. Finally, in Section 8 we give a proof of the asymptotic limit as $\varepsilon \to 0$, recovering in this way a solution to the system corresponding to the case $\varepsilon = 0$.

2 Setting, assumptions and main results

Throughout the paper, $\Omega \subseteq \mathbb{R}^N$ $(N = 2,3)$ is a smooth bounded domain with smooth boundary Γ and $T > 0$ is a fixed final time. We use the notation $Q_t := (0,t)\times\Omega$ and $\Sigma_t := (0,t)\times\Gamma$ for every $t \in (0,T]$, with $Q := Q_T$ and $\Sigma := \Sigma_T$. The outward normal unit vector on Γ, the tangential gradient and the Laplace-Beltrami operator on Γ are denoted by $\mathbf{n}$, ∇_Γ and Δ_Γ, respectively. We shall also use the symbol $\Delta_{\mathbf{n}}$ to denote the Laplace operator with homogeneous Neumann conditions. Moreover, ε is a positive fixed number.

We introduce the spaces

$$H := L^2(\Omega)\,, \qquad H_\Gamma := L^2(\Gamma)\,, \qquad \mathcal{H} := H \times H_\Gamma\,,$$

$$V := H^1(\Omega)\,, \qquad V_\Gamma := H^1(\Gamma)\,, \qquad \mathcal{V} := \{(x,y) \in V \times V_\Gamma : x = y \text{ on } \Gamma\}\,,$$

$$W := H^2(\Omega)\,, \qquad W_\Gamma := H^2(\Gamma)\,, \qquad \mathcal{W} := \{(x,y) \in W \times W_\Gamma : x = y \text{ on } \Gamma\}\,,$$

$$W_{\mathbf{n}} := \{x \in W : \partial_{\mathbf{n}} x = 0 \text{ on } \Gamma\}\,.$$

As usual, we identify H and H_Γ with their own duals H^* and H_Γ^*, so that $H \hookrightarrow V^*$ and $H_\Gamma \hookrightarrow V_\Gamma^*$ with the inclusions given by the inner products of H and H_Γ, respectively. Moreover, we denote all norms and duality pairings by the symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, with a subscript specifying the spaces in consideration.

For any element $y \in V^*$ we define the mean

$$y_\Omega := \frac{1}{|\Omega|}\,\langle y, 1\rangle\,.$$

Moreover, recall that a norm on V, equivalent to the usual one, is given by

$$|x|_V^2 := \|\nabla x\|_H^2 + |x_\Omega|^2 \,, \quad x \in V \,, \tag{2.1}$$

and that the Laplace operator with Neumann conditions is an isomorphism between the null-mean elements in V and the null-mean elements in V^*, so that it is well defined its inverse

$$\mathcal{N} : \{y \in V^* : \ y_\Omega = 0\} \to \{y \in V : \ y_\Omega = 0\} \,,$$

where, for any $y \in V^*$ with $y_\Omega = 0$, $\mathcal{N}y$ is the unique element in V with null mean such that

$$\int_\Omega \nabla \mathcal{N}y \cdot \nabla \varphi = \langle y, \varphi \rangle \quad \forall\, \varphi \in V \,.$$

Let us specify the main hypotheses on the data: these will be in order in the whole work and will not be recalled explicitly.

We assume that

$$\widehat{\alpha}, \widehat{\alpha}_\Gamma, \widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty]$$

are proper, convex and lower semicontinuous functions such that

$$0 = \widehat{\alpha}(0) = \widehat{\alpha}_\Gamma(0) = \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) \,,$$

and we set

$$\alpha := \partial\widehat{\alpha} \,, \quad \alpha_\Gamma := \partial\widehat{\alpha}_\Gamma \,, \quad \beta := \partial\widehat{\beta} \,, \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma \,.$$

Moreover, let

$$\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R} \quad \text{Lipschitz continuous} \,, \qquad \pi(0) = \pi_\Gamma(0) = 0 \,,$$

and denote by C_π and C_{π_Γ} their respective Lipchitz constants. We shall always assume that α_Γ is coercive and that β is controlled by β_Γ, i.e.

$$\exists\, b_1, b_2 > 0 : \quad rs \geq b_1|s|^2 - b_2 \quad \forall\, s \in D(\alpha_\Gamma) \,, \quad \forall\, r \in \alpha_\Gamma(s) \,, \tag{2.2}$$

$$D(\beta_\Gamma) \subseteq D(\beta) \quad \text{and} \quad \exists\, c > 0 : \left|\beta^0(s)\right| \leq c \left(1 + \left|\beta_\Gamma^0(s)\right|\right) \quad \forall\, s \in D(\beta_\Gamma) \,. \tag{2.3}$$

These hypotheses will be always in order and will not be recalled explicitly throughout the paper. Note that (2.3) is typically not new in the literature dealing with Allen-Cahn and Cahn-Hilliard equations with dynamic boundary conditions: see for example [4,8]. Moreover, condition (2.2) appears also very natural if we recall that the evolution on the boundary is of order 2 in space, hence of Allen-Cahn type.

The first existence result that we prove requires additional assumptions on the graphs α and α_Γ: in particular, and their growth at infinity has to be no more than linear and also α has to be coercive. On the other side, no further hypothesis is made on β and β_Γ.

Theorem 2.1. *Suppose that*

$$g \in L^2(0,T;H) \,, \qquad g_\Gamma \in L^2(0,T;H_\Gamma) \,, \tag{2.4}$$

$$u_0 \in V \,, \quad u_{0|\Gamma} \in V_\Gamma \,, \qquad \widehat{\beta}(u_0) \in L^1(\Omega) \,, \qquad \widehat{\beta}_\Gamma(u_{0|\Gamma}) \in L^1(\Gamma) \,, \tag{2.5}$$

$$(u_0)_\Omega \in \operatorname{Int} D(\beta_\Gamma) \,, \tag{2.6}$$

$$D(\alpha) = D(\alpha_\Gamma) = \mathbb{R} \quad \textit{and} \quad \exists\, L > 0 : \max\{\left|\alpha^0(s)\right|, \left|\alpha_\Gamma^0(s)\right|\} \leq L\,(1 + |s|) \quad \forall\, s \in \mathbb{R} \,, \tag{2.7}$$

$$\exists\, a_1, a_2 > 0 : \quad rs \geq a_1|s|^2 - a_2 \quad \forall\, s \in D(\alpha) \,, \quad \forall\, r \in \alpha(s) \,. \tag{2.8}$$

Then, there exists a septuple $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ *such that*

$$u \in L^\infty(0,T;V) \cap H^1(0,T;H) \cap L^2(0,T;W)\,, \tag{2.9}$$
$$v \in L^\infty(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma) \cap L^2(0,T;W_\Gamma)\,, \tag{2.10}$$
$$\mu \in L^2(0,T;W_{\mathbf{n}})\,, \tag{2.11}$$
$$\eta, \xi \in L^2(0,T;H)\,, \qquad \eta_\Gamma, \xi_\Gamma \in L^2(0,T;H_\Gamma)\,, \tag{2.12}$$
$$v = u_{|\Gamma} \quad \text{a.e. in } \Sigma\,, \qquad u(0) = u_0\,, \tag{2.13}$$
$$\eta \in \alpha(\partial_t u)\,, \;\; \xi \in \beta(u) \quad \text{a.e. in } Q\,, \qquad \eta_\Gamma \in \alpha_\Gamma(\partial_t v)\,, \;\; \xi_\Gamma \in \beta_\Gamma(v) \quad \text{a.e. in } \Sigma \tag{2.14}$$

and satisfying

$$\partial_t u - \Delta\mu = 0\,, \tag{2.15}$$
$$\mu = \eta - \Delta u + \xi + \pi(u) - g\,, \tag{2.16}$$
$$\eta_\Gamma + \partial_{\mathbf{n}} u - \varepsilon\Delta_\Gamma v + \xi_\Gamma + \pi_\Gamma(v) = g_\Gamma\,. \tag{2.17}$$

Remark 2.2. Note that the setting of Theorem 2.1 includes the classical linear viscosity case, where $\alpha = aI$ and $\alpha_\Gamma = bI$, for $a, b > 0$, and allows for any choice of the potentials acting on u and v, provided that the compatibility condition (2.3) holds. In particular, we are allowed to consider in the choice of $\widehat{\beta} + \widehat{\pi}$ and $\widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma$ also logarithmic-type potentials, which are the most relevant in terms on thermodynamical consistency of the model, i.e.

$$r \mapsto ((1+r)\ln(1+r) + (1-r)\ln(1-r)) - cr^2\,, \qquad r \in (-1,1)\,, \qquad c > 0\,.$$

In the next existence result, we show how to remove the coercivity hypothesis (2.8) on α by requiring stronger assumptions on the data. Again, no further restrictions are assumed on β and β_Γ. Moreover, we stress also that if α is coercive, then the further hypotheses on the data ensure additional regularities on the solutions.

Theorem 2.3. *Assume conditions* (2.6)–(2.7) *and*

$$g \in L^2(0,T;H) \cap H^1(0,T;V^*)\,, \qquad g_\Gamma \in L^2(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma^*)\,, \tag{2.18}$$
$$g(0) \in H\,, \qquad g_\Gamma(0) \in H_\Gamma\,, \tag{2.19}$$
$$u_0 \in W\,, \qquad u_{0|\Gamma} \in W_\Gamma\,, \tag{2.20}$$
$$\exists\, y_0 \in H: \; y_0 \in \beta(u_0) \quad \text{a.e. in } \Omega\,, \qquad \exists\, y_{0\Gamma} \in H_\Gamma: \; y_{0\Gamma} \in \beta_\Gamma(u_{0|\Gamma}) \quad \text{a.e. in } \Gamma\,, \tag{2.21}$$
$$\exists\, \delta_0 > 0: \quad \{-\Delta u_0 + \beta_\delta(u_0) + (I - \delta\Delta_{\mathbf{n}})^{-1} g(0)\}_{\delta\in(0,\delta_0)} \quad \text{is bounded in } V\,. \tag{2.22}$$

Then there exists a septuple $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ *such that*

$$u \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W)\,, \tag{2.23}$$
$$v \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;W_\Gamma)\,, \tag{2.24}$$
$$\mu \in L^\infty(0,T;V) \cap L^2(0,T;W_{\mathbf{n}} \cap H^3(\Omega))\,, \tag{2.25}$$
$$\eta\,, \xi \in L^\infty(0,T;H)\,, \qquad \eta_\Gamma\,, \xi_\Gamma \in L^\infty(0,T;H_\Gamma) \tag{2.26}$$

and satisfying conditions (2.13)–(2.17)*. Moreover, if* (2.8) *holds, then the same conclusion is true without the assumption* (2.22)*, and additionally* $u \in W^{1,\infty}(0,T;H)$ *and* $\mu \in L^\infty(0,T;W_{\mathbf{n}})$.

Remark 2.4. Note that hypothesis (2.22) clearly holds if $u_0 \in H^3(\Omega)$, $g(0) \in V$ and the family $\{\beta_\delta(u_0)\}_{\delta\in(0,\delta_0)}$ is bounded in V. This condition is not new in literature: see for example the work [8, pp. 977–978] for sufficient conditions.

Remark 2.5. The setting of Theorem 2.3 allows to include in our analysis also the cases where $\alpha = \text{sign}$ for example, or $\alpha(r) = r_+$, $r \in \mathbb{R}$. Again, no further assumption are made on β or β_Γ, so that logarithmic-type potentials are included.

The third existence result that we present allows to remove the linear growth condition on α and α_Γ, but requires in turn a polynomial control of the growth of β and β_Γ. Here, the inclusions with respect to the operators α and α_Γ are satisfied in a weak sense. To this end, we shall introduce the operators $\alpha_w : V \to 2^{V^*}$ and $\alpha_{\Gamma w} : V_\Gamma \to 2^{V_\Gamma^*}$ as

$$\alpha_w(x) := \left\{ y \in V^* : \int_\Omega \widehat{\alpha}(x) + \langle y, \varphi - x\rangle_V \leq \int_\Omega \widehat{\alpha}(\varphi) \quad \forall\, \varphi \in V \right\}, \qquad x \in V\,,$$

$$\alpha_{\Gamma w}(x_\Gamma) := \left\{ y_\Gamma \in V_\Gamma^* : \int_\Gamma \widehat{\alpha}_\Gamma(x_\Gamma) + \langle y_\Gamma, \psi - x_\Gamma\rangle_{V_\Gamma} \leq \int_\Gamma \widehat{\alpha}_\Gamma(\psi) \quad \forall\, \psi \in V_\Gamma \right\}, \qquad x_\Gamma \in V_\Gamma\,,$$

which are clearly the subdifferentials of the proper, convex and l.s.c. functions induced by $\widehat{\alpha}$ and $\widehat{\alpha}_\Gamma$ on V and V_Γ, respectively. Similarly, we shall introduce the (maximal monotone) operator $\widetilde{\alpha}_w : \mathcal{V} \to 2^{\mathcal{V}^*}$ as

$$\begin{aligned}\widetilde{\alpha}_w(x, x_\Gamma) := \Big\{ y \in \mathcal{V}^* : \int_\Omega \widehat{\alpha}(x) + \int_\Gamma \widehat{\alpha}_\Gamma(x_\Gamma) + \langle y, (\varphi, \psi) - (x, x_\Gamma)\rangle_{\mathcal{V}} \\ \leq \int_\Omega \widehat{\alpha}(\varphi) + \int_\Gamma \widehat{\alpha}_\Gamma(\psi) \quad \forall\, (\varphi, \psi) \in \mathcal{V} \Big\}, \qquad (x, x_\Gamma) \in \mathcal{V}\,.\end{aligned}$$

Note that $\alpha_w(x) + \alpha_{\Gamma w}(x_\Gamma) \subseteq \widetilde{\alpha}_w(x, x_\Gamma)$ for every $(x, x_\Gamma) \in \mathcal{V}$, but equality may not hold in general as $\mathcal{V}^*$ is strictly larger than $(V \times V_\Gamma)^*$. This will result in a weaker variational formulation for both the evolution equation itself and for the inclusions with respect to the nonlinear operators acting in the viscosity terms.

Theorem 2.6. *Assume conditions* (2.6) *and* (2.18)–(2.22)*. If*

$$0 \in \operatorname{Int}\left(D(\alpha) \cap D(\alpha_\Gamma)\right) \tag{2.27}$$

and

$$\exists\, c_1, c_2 > 0 : \quad |r| \leq c_1 |s|^5 + c_2 \quad \forall\, s \in D(\beta)\,, \quad \forall\, r \in \beta(s) \tag{2.28}$$

$$\exists\, p \geq 5,\ d_1, d_2 > 0 : \quad |r| \leq d_1 |s|^p + d_2 \quad \forall\, s \in D(\beta_\Gamma)\,, \quad \forall\, r \in \beta_\Gamma(s)\,, \tag{2.29}$$

then there exists a septuple $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ *such that*

$$u \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V)\,, \qquad v \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma)\,, \tag{2.30}$$

$$\mu \in L^\infty(0,T;V) \cap L^2(0,T;W_{\mathbf{n}} \cap H^3(\Omega))\,, \tag{2.31}$$

$$\eta_w \in L^\infty(0,T;\mathcal{V}^*)\,, \qquad \eta_w \in \widetilde{\alpha}_w(\partial_t u, \partial_t v) \quad \textit{a.e. in } (0,T)\,, \tag{2.32}$$

$$\xi \in L^\infty(0,T;L^{6/5}(\Omega))\,, \qquad \xi_\Gamma \in L^\infty(0,T;L^q(\Gamma)) \quad \forall\, q \in [1,+\infty)\,, \tag{2.33}$$

$$\xi \in \beta(u) \quad \textit{a.e. in } Q\,, \qquad \xi_\Gamma \in \beta_\Gamma(v) \quad \textit{a.e. in } \Sigma\,, \tag{2.34}$$

satisfying conditions (2.13), (2.15) *and*

$$\int_\Omega \mu(t)\varphi = \langle \eta_w(t),(\varphi,\psi)\rangle_{\mathcal{V}} + \int_\Omega \nabla u(t)\cdot\nabla\varphi + \int_\Omega (\xi(t)+\pi(u(t))-g(t))\,\varphi \\ + \varepsilon\int_\Gamma \nabla_\Gamma v(t)\cdot\nabla_\Gamma\psi + \int_\Gamma (\xi_\Gamma(t)+\pi_\Gamma(v(t))-g_\Gamma(t))\psi \tag{2.35}$$

for every $(\varphi,\psi)\in\mathcal{V}$ *and a.e.* $t\in(0,T)$*. Moreover, if*

$$\exists\, c_1,c_2>0:\quad |r|\le c_1|s|^3+c_2\quad \forall\, s\in D(\beta)\,,\quad \forall\, r\in\beta(s)\,, \tag{2.36}$$

then $\xi\in L^\infty(0,T;H)$*. Furthermore, if* (2.8) *holds, then the same conclusions are true also without the assumption* (2.22)*, and additionally* $u\in W^{1,\infty}(0,T;H)$ *and* $\mu\in L^\infty(0,T;W_{\mathbf{n}})$*.*

Remark 2.7. The setting of Theorem 2.6 allows α and α_Γ to be superlinear at infinity, but in turn requires polynomial growth for β and β_Γ. In this setting, note that we can include the classical choice

$$r\mapsto \frac{1}{4}(r^2-1)^2\,,\qquad r\in\mathbb{R}\,,$$

for $\widehat\beta+\widehat\pi$, and any generic polynomial double-well potential for $\widehat\beta_\Gamma+\widehat\pi_\Gamma$. These may be seen, as usual, as suitable approximation of the more relevant logarithmic potentials.

Remark 2.8. Let us stress that the hypothesis (2.27) is the direct generalization of (2.6). Indeed, it is readily seen from (2.15) that $(u)_\Omega$ is constantly equal to $(u_0)_\Omega$, as well as $(\partial_t u)_\Omega=0$ at any time. Consequently, taking into account that α and α_Γ are acting on the time derivatives of the solutions, the hypotheses (2.6) and (2.27) clearly possess the same structure.

Remark 2.9. Let us comment on (2.35), which is the natural variational formulation in the dual space $\mathcal{V}^*$ of the couple of equations (1.2) and (1.4). Note that since $N\in\{2,3\}$, we have the continuous inclusions $V\hookrightarrow L^6(\Omega)$ and $V_\Gamma\hookrightarrow L^q(\Gamma)$ for every $q\in[1,+\infty)$. Hence, it is clear that $L^{6/5}(\Omega)\hookrightarrow V^*$ and $L^{q'}(\Gamma)\hookrightarrow V_\Gamma^*$ for every $q'\in(1,+\infty]$. For these reasons, we have in particular that $\xi\in L^\infty(0,T;V^*)$ and $\xi_\Gamma\in L^\infty(0,T;V_\Gamma^*)$, so that the dualities

$$\int_\Omega \xi\varphi \qquad\text{and}\qquad \int_\Gamma \xi_\Gamma\psi$$

in the variational formulation (2.35) make sense by the classical Hölder inequality, and must be read as $\langle\xi,\varphi\rangle_V$ and $\langle\xi_\Gamma,\psi\rangle_{V_\Gamma}$, respectively.

We turn now to uniqueness of solutions. According to different smoothness or growth assumptions on the potentials, uniqueness in proved both in the class of solutions given by Theorem 2.3 and in the largest class of Theorem 2.1.

Theorem 2.10. *Assume that* β *and* β_Γ *are single-valued, and that*

$$F:=\widehat\beta+\int_0^\cdot \pi(s)\,ds\in C^{2,1}_{loc}(\mathbb{R})\,,\qquad F_\Gamma:=\widehat\beta_\Gamma+\int_0^\cdot \pi_\Gamma(s)\,ds\in C^{2,1}_{loc}(\mathbb{R})\,, \tag{2.37}$$

$$\exists\,\widetilde{b_1}>0:\quad (s_1-s_2)(r_1-r_2)\ge \widetilde{b_1}|r_1-r_2|^2\quad \forall\,(r_i,s_i)\in\alpha_\Gamma\,,\ i=1,2\,. \tag{2.38}$$

Then, there is a unique septuple $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ *satisfying* (2.23)–(2.26) *and* (2.13)–(2.17). *Furthermore, if additionally*

$$\exists\, M > 0 : \quad F'''(r) \leq M\left(1 + |r|^3\right) \quad \textit{for a.e. } r \in \mathbb{R}\,, \tag{2.39}$$

$$\exists\, M, q > 0 : \quad F_\Gamma'''(r) \leq M\left(1 + |r|^q\right) \quad \textit{for a.e. } r \in \mathbb{R}\,, \tag{2.40}$$

there exists a unique septuple $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ *satisfying* (2.30)–(2.35), (2.13) *and* (2.15).

Finally, the last result that we present investigates the asymptotic behaviour of the solutions with respect to ε, and provides a further existence result for the problem with $\varepsilon = 0$. For sake of brevity, we only consider the case of the linearity assumption (2.7) on the growth of α and α_Γ, and provide different asymptotic convergence of the solutions depending on whether the coercivity assumption (2.8) is in order. In this direction, we need to introduce a weak formulation of the operator β_Γ induced on the space $H^{1/2}(\Gamma)$. Namely, we define $\beta_{\Gamma w} : H^{1/2}(\Gamma) \to 2^{H^{-1/2}(\Gamma)}$ as the maximal monotone operator

$$\beta_{\Gamma w}(x) := \left\{ y \in H^{-1/2}(\Gamma) : \int_\Gamma \widehat{\beta}_\Gamma(x) + \langle y, w - x\rangle_{H^{1/2}(\Gamma)} \leq \int_\Gamma \widehat{\beta}_\Gamma(w) \quad \forall\, w \in H^{1/2}(\Gamma) \right\}.$$

Theorem 2.11. *Assume conditions* (2.4), (2.6)–(2.8) *and let*

$$u_0 \in V\,, \qquad \widehat{\beta}(u_0) \in L^1(\Omega)\,, \qquad \widehat{\beta}_\Gamma(u_{0|\Gamma}) \in L^1(\Gamma)\,.$$

Let $(u_0^\varepsilon)_\varepsilon \subseteq V$ *be any family such that* u_0^ε *satisfies* (2.5) *for every* $\varepsilon > 0$,

$$u_0^\varepsilon \to u_0 \quad \textit{in } V, \qquad \varepsilon^{1/2} u_{0|\Gamma}^\varepsilon \to 0 \quad \textit{in } V_\Gamma \qquad \textit{as } \varepsilon \searrow 0\,, \tag{2.41}$$

$$\varepsilon \left\| u_{0|\Gamma}^\varepsilon \right\|_{V_\Gamma}^2 + \left\| \widehat{\beta}(u_0^\varepsilon) \right\|_{L^1(\Omega)} + \left\| \widehat{\beta}_\Gamma(u_{0|\Gamma}^\varepsilon) \right\|_{L^1(\Gamma)} \leq c \qquad \forall\, \varepsilon > 0 \tag{2.42}$$

for a positive constant c*, and let* $(u_\varepsilon, v_\varepsilon, \mu_\varepsilon, \eta_\varepsilon, \xi_\varepsilon, \eta_{\Gamma\varepsilon}, \xi_{\Gamma\varepsilon})$ *be the solutions given by Theorem 2.1 satisfying conditions* (2.9)–(2.17) *with initial datum* u_0^ε*. Then, there exists a sequence* $(\varepsilon_n)_n$ *with* $\varepsilon_n \to 0$ *as* $n \to \infty$ *and a septuple* $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ *with*

$$u \in L^\infty(0, T; V) \cap H^1(0, T; H)\,, \quad \Delta u \in L^2(0, T; H)\,, \tag{2.43}$$

$$v \in L^\infty(0, T; H^{1/2}(\Gamma)) \cap H^1(0, T; H_\Gamma)\,, \qquad \mu \in L^2(0, T; W_{\mathbf{n}})\,, \tag{2.44}$$

$$\eta, \xi \in L^2(0, T; H)\,, \quad \eta_\Gamma \in L^2(0, T; H_\Gamma)\,, \quad \xi_\Gamma \in L^2(0, T; H^{-1/2}(\Gamma))\,, \tag{2.45}$$

$$\eta \in \alpha(\partial_t u)\,, \quad \xi \in \beta(u) \quad \textit{a.e. in } Q\,, \qquad \eta_\Gamma \in \alpha_\Gamma(\partial_t v) \quad \textit{a.e. in } \Sigma\,, \tag{2.46}$$

$$\xi_\Gamma \in \beta_w(v) \quad \textit{a.e. in } (0, T)\,, \tag{2.47}$$

satisfying (2.13), (2.15)–(2.16) *and*

$$\eta_\Gamma + \partial_{\mathbf{n}} u + \xi_\Gamma + \pi_\Gamma(v) = g_\Gamma\,,$$

and such that, as $n \to \infty$,

$$u_{\varepsilon_n} \to u \quad \textit{in } C^0([0,T]; H)\,, \quad u_{\varepsilon_n} \rightharpoonup u \quad \textit{in } H^1(0,T;H)\,, \quad u_{\varepsilon_n} \overset{*}{\rightharpoonup} u \quad \textit{in } L^\infty(0,T;V)\,,$$

$$v_{\varepsilon_n} \to v \quad \textit{in } C^0([0,T]; H_\Gamma)\,, \quad v_{\varepsilon_n} \rightharpoonup v \quad \textit{in } H^1(0,T;H_\Gamma)\,, \quad v_{\varepsilon_n} \overset{*}{\rightharpoonup} v \quad \textit{in } L^\infty(0,T;H^{1/2}(\Gamma))\,,$$

$$\mu_{\varepsilon_n} \rightharpoonup \mu \quad \textit{in } L^2(0,T;W_{\mathbf{n}})\,,$$

$$\eta_{\varepsilon_n} \rightharpoonup \eta \quad \textit{in } L^2(0,T;H)\,, \qquad \xi_{\varepsilon_n} \rightharpoonup \xi \quad \textit{in } L^2(0,T;H)\,,$$

$$\eta_{\Gamma\varepsilon_n} \rightharpoonup \eta_\Gamma \quad \textit{in } L^2(0,T;H_\Gamma)\,, \qquad \xi_{\Gamma\varepsilon_n} \rightharpoonup \xi_\Gamma \quad \textit{in } L^2(0,T;H^{-1/2}(\Gamma))\,,$$

$$\varepsilon v_{\varepsilon_n} \to 0 \quad \textit{in } L^\infty(0,T;V_\Gamma)\,.$$

Furthermore, if also hypotheses (2.18)–(2.22) *hold and* $(\varepsilon u^\varepsilon_{0|\Gamma})_\varepsilon$ *is bounded in* W_Γ*, then the same conclusion is true without the coercivity assumption* (2.8)*, and we also have*

$$u \in W^{\infty}(0,T;V^*) \cap H^1(0,T;V)\,, \quad \Delta u \in L^\infty(0,T;H)\,,$$
$$v \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;H^{1/2}(\Gamma))\,, \qquad \mu \in L^\infty(0,T;V) \cap L^2(0,T;W_{\mathbf{n}} \cap H^3(\Omega))\,,$$
$$\eta, \xi \in L^\infty(0,T;H)\,, \quad \eta_\Gamma \in L^\infty(0,T;H_\Gamma)\,, \quad \xi_\Gamma \in L^\infty(0,T;H^{-1/2}(\Gamma))\,,$$

and

$$u_{\varepsilon_n} \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(0,T;V^*)\,, \qquad u_{\varepsilon_n} \rightharpoonup u \quad \text{in } H^1(0,T;V)\,,$$
$$v_{\varepsilon_n} \overset{*}{\rightharpoonup} v \quad \text{in } W^{1,\infty}(0,T;H_\Gamma)\,, \qquad v_{\varepsilon_n} \rightharpoonup v \quad \text{in } H^1(0,T;H^{1/2}(\Gamma))\,,$$
$$\mu_{\varepsilon_n} \overset{*}{\rightharpoonup} \mu \quad \text{in } L^\infty(0,T;V)\,, \qquad \mu_\varepsilon \rightharpoonup \mu \quad \text{in } L^2(0,T;W_{\mathbf{n}} \cap H^3(\Omega))\,,$$
$$\eta_{\varepsilon_n} \overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty(0,T;H)\,, \qquad \xi_{\varepsilon_n} \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(0,T;H)\,,$$
$$\eta_{\Gamma\varepsilon_n} \overset{*}{\rightharpoonup} \eta_\Gamma \quad \text{in } L^\infty(0,T;H_\Gamma)\,, \qquad \xi_{\Gamma\varepsilon_n} \overset{*}{\rightharpoonup} \xi_\Gamma \quad \text{in } L^\infty(0,T;H^{-1/2}(\Gamma))\,,$$
$$\varepsilon v_{\varepsilon_n} \to 0 \quad \text{in } H^1(0,T;V_\Gamma)\,.$$

Remark 2.12. Let us comment on the existence of an approximating family $(u^\varepsilon_0)_\varepsilon$. If the initial datum $u_0 \in V$ satisfies $a \le u_0 \le b$ almost everywhere in Ω for certain $a, b \in \mathbb{R}$ such that $[a,b] \subseteq \operatorname{Int} D(\beta_\Gamma)$, then a possible approximating sequence $(u^\varepsilon_0)_\varepsilon$ always exists. Indeed, we can set, for every $\varepsilon > 0$, u^ε_0 as the unique solution to the elliptic problem

$$\begin{cases} u^\varepsilon_0 - \varepsilon^{1/2}\Delta u^\varepsilon_0 = u_0 & \text{in } \Omega\,, \\ \partial_{\mathbf{n}} u^\varepsilon_0 = 0 & \text{in } \Gamma\,. \end{cases}$$

Such problem is well-posed by the classical theory on bilinear forms and admits a unique solution $u^\varepsilon_0 \in W_{\mathbf{n}} \cap H^3(\Omega)$. Testing by u^ε_0 and using the Young inequality one has

$$\frac{1}{2}\left\|u^\varepsilon_0\right\|^2_H + \varepsilon^{1/2}\left\|\nabla u^\varepsilon_0\right\|^2_H \le \frac{1}{2}\left\|u_0\right\|^2_H\,,$$

while testing the first equation by $-\Delta u^\varepsilon_0$ and integrating by parts yields

$$\frac{1}{2}\left\|\nabla u^\varepsilon_0\right\|^2_H + \varepsilon^{1/2}\left\|\Delta u^\varepsilon_0\right\|^2_H \le \frac{1}{2}\left\|\nabla u_0\right\|^2_H\,.$$

We infer that (along a subsequence)

$$u^\varepsilon_0 \rightharpoonup u_0 \quad \text{in } V\,, \qquad \left\|u^\varepsilon_0\right\|_V \le \left\|u_0\right\|_V \quad \forall\, \varepsilon > 0\,,$$

so that $u^\varepsilon_0 \to u_0$ in V, hence also $\varepsilon^{1/2}\Delta u^\varepsilon_0 \to 0$ in V. We deduce then that $\varepsilon^{1/2} u_\varepsilon \to 0$ in $H^3(\Omega)$, which implies in particular that $\varepsilon^{1/2} u^\varepsilon_{0|\Gamma} \to 0$ in V_Γ. Furthermore, by the maximum principle we have $a \le u^\varepsilon_0 \le b$ a.e. in Ω, hence also $a \le u^\varepsilon_{0|\Gamma} \le b$ a.e. in Σ, and we can conclude recalling that $D(\beta_\Gamma) \subseteq D(\beta)$ and the fact that every proper, convex and lower semicontinuous function is continuous in the interior of its domain.

3 The approximated problem

In this section we approximate the problem (1.1)–(1.5) and we precise the exact regularities of the approximated solutions, depending on the assumptions on the data. Note that throughout this section $\varepsilon > 0$ is fixed, so that we shall omit any specific notation for the dependence on ε.

For any $\lambda > 0$, let β_λ and $\beta_{\Gamma\lambda}$ be the Yosida approximations of the graphs β and β_Γ with approximating parameters λ and $c\lambda$, respectively, where c is the same as in (2.3): the reason why we choose this specific approximation will be clarified in Section 4.3 below. Similarly, let α_λ and $\alpha_{\Gamma\lambda}$ denote the Yosida approximations of α and α_Γ, respectively, with parameter λ. Furthermore, let $(g_\lambda)_\lambda$ and $(g_{\Gamma\lambda})_\lambda$ be two approximating sequences of g and g_Γ, respectively, such that

$$(g_\lambda)_\lambda \subseteq L^2(0,T;V) \cap H^1(0,T;V^*)\,, \qquad (g_{\Gamma\lambda})_\lambda \subseteq L^2(0,T;V_\Gamma) \cap H^1(0,T;V_\Gamma^*)\,,$$
$$g_\lambda \to g \quad \text{in } L^2(0,T;H)\,, \qquad g_{\Gamma\lambda} \to g_\Gamma \quad \text{in } L^2(0,T;H_\Gamma)\,.$$

It will be implicitly intended that the convergences hold also in the spaces $H^1(0,T;V^*)$ and $H^1(0,T;V_\Gamma^*)$ whenever (2.18) is in order. For example, we can define $g := (I - \lambda\Delta_{\mathbf{n}})^{-1}g$ and $g_\Gamma := (I - \lambda\Delta_\Gamma)^{-1}g_\Gamma$, i.e. as the solutions to the following elliptic problems:

$$\begin{cases} g_\lambda - \lambda\Delta g_\lambda = g & \text{in } \Omega\,, \\ \partial_{\mathbf{n}} g_\lambda = 0 & \text{in } \Gamma\,, \end{cases} \qquad g_{\Gamma\lambda} - \lambda\Delta_\Gamma g_{\Gamma\lambda} = g_\Gamma \quad \text{in } \Gamma\,.$$

The idea is to consider the regularized system given by

$$\begin{aligned} \partial_t u_\lambda + \lambda\mu_\lambda - \Delta\mu_\lambda &= 0 && \text{in } Q\,, \\ \mu_\lambda = \lambda\partial_t u_\lambda + \alpha_\lambda(\partial_t u_\lambda) + \lambda u_\lambda - \Delta u_\lambda + \beta_\lambda(u_\lambda) + T_\lambda\pi(u_\lambda) - g_\lambda && & \text{in } Q\,, \\ u_\lambda = v_\lambda\,, \quad \partial_{\mathbf{n}}\mu_\lambda &= 0 && \text{in } \Sigma\,, \\ \lambda\partial_t v_\lambda + \alpha_{\Gamma\lambda}(\partial_t v_\lambda) + \partial_{\mathbf{n}} u_\lambda - \varepsilon\Delta_\Gamma v_\lambda + \beta_{\Gamma\lambda}(v_\lambda) + T_\lambda\pi_\Gamma(v_\lambda) &= g_{\Gamma\lambda} && \text{in } \Sigma\,, \\ u_\lambda(0) &= u_0 && \text{in } \Omega\,, \end{aligned}$$

where $T_\lambda : \mathbb{R} \to \mathbb{R}$ is the usual truncation operator at level $\frac{1}{\lambda}$ defined by

$$T_\lambda(r) := \max\left\{-\frac{1}{\lambda}, \min\left\{\frac{1}{\lambda}, r\right\}\right\}\,, \qquad r \in \mathbb{R}\,.$$

In order to show that such regularized problem is well-posed, we use an abstract result on doubly nonlinear evolution equations on the product space $\mathcal{H}$. To this end, we introduce the operator

$$G_\lambda : H \to H\,, \qquad G_\lambda x := \lambda x - \Delta x\,, \quad x \in D(G_\lambda) := W_{\mathbf{n}}\,,$$

which is maximal monotone and invertible on H with $G_\lambda^{-1} : H \to W_{\mathbf{n}}$; in particular, the first equation together with the boundary condition for μ_λ can be written as $\mu_\lambda = -G_\lambda^{-1}(\partial_t u_\lambda)$. Hence, it is natural to define

$$\begin{aligned} A_\lambda : \mathcal{H} \to \mathcal{H}\,, &\qquad A_\lambda(x,y) := (\lambda x + \alpha_\lambda(x) + G_\lambda^{-1}(x), \lambda y + \alpha_{\Gamma\lambda}(y))\,, \\ B_\lambda : \mathcal{H} \to \mathcal{H}\,, &\qquad B_\lambda(x,y) := (\lambda x - \Delta x + \beta_\lambda(x), \partial_{\mathbf{n}} x - \varepsilon\Delta_\Gamma y + \beta_{\Gamma\lambda}(y))\,, \end{aligned}$$

where

$$D(A_\lambda) := \mathcal{H}\,, \qquad D(B_\lambda) := \mathcal{W}\,.$$

Taking into account the definition of A_λ and B_λ, the entire approximated system can be formulated as a doubly nonlinear evolution equation in the variable (u_λ, v_λ) on the product space $\mathcal{H}$ in the following compact form:

$$A_\lambda \partial_t(u_\lambda, v_\lambda) + B_\lambda(u_\lambda, v_\lambda) = (g_\lambda, g_{\Gamma\lambda}) - (T_\lambda \pi(u_\lambda), T_\lambda \pi_\Gamma(v_\lambda))\,, \qquad (u_\lambda, v_\lambda)(0) = (u_0, u_{0|\Gamma})\,.$$

We collect some useful properties of the operators A_λ and B_λ in the following lemma.

Lemma 3.1. *The operators A_λ and B_λ are maximal monotone on $\mathcal{H}$ and $D(B_\lambda) \subseteq \mathcal{V}$. Moreover, the following conditions hold:*

$$\begin{aligned}
&(i) && \forall\,(x,y) \in \mathcal{H} \quad (A_\lambda(x,y),(x,y))_{\mathcal{H}} \geq \lambda \left\|(x,y)\right\|^2_{\mathcal{H}}\,,\\
&(ii) && \exists\, k_\lambda > 0: \quad \forall\,(x,y) \in \mathcal{H} \quad \left\|A_\lambda(x,y)\right\|_{\mathcal{H}} \leq k_\lambda \left\|(x,y)\right\|_{\mathcal{H}}\,,\\
&(iii) && B_\lambda = \partial\psi_\lambda\,, \quad \psi_\lambda : \mathcal{H} \to (-\infty,+\infty] \text{ proper, convex and l.s.c.}\,, \quad D(\psi_\lambda) \subseteq \mathcal{V}\\
&(iv) && \exists\, \ell_1, \ell_2 > 0: \quad \psi_\lambda(x,y) \geq \ell_1 \left\|(x,y)\right\|^2_{\mathcal{V}} - \ell_2 \left\|(x,y)\right\|^2_{\mathcal{H}} \quad \forall\,(x,y) \in D(\psi_\lambda)\,,\\
&(v) && A_\lambda = \partial\phi_\lambda\,, \quad \phi_\lambda : \mathcal{H} \to (-\infty,+\infty] \text{ proper, convex and l.s.c.}\,, \quad D(\phi_\lambda) = \mathcal{H}\,,\\
&(vi) && A_\lambda \quad \text{is bounded in } \mathcal{H}\,,\\
&(vii) && B_\lambda : \mathcal{V} \to \mathcal{V}^* \quad \text{is Lipschitz continuous and strongly monotone}\,.
\end{aligned}$$

Proof. It is clear that A_λ and B_λ are maximal monotone. By monotonicity of α_λ and $\alpha_{\Gamma\lambda}$ and the definition of G_λ, we have that

$$\begin{aligned}
(A_\lambda(x,y),(x,y))_{\mathcal{H}} &= \lambda \int_\Omega |x|^2 + \lambda \int_\Gamma |y|^2 + \int_\Omega \alpha_\lambda(x)x + \int_\Gamma \alpha_{\Gamma\lambda}(y)y + \int_\Omega G_\lambda^{-1}(x)x\\
&\geq \lambda \left\|(x,y)\right\|^2_{\mathcal{H}} + \lambda \int_\Omega |G_\lambda^{-1}(x)|^2 + \int_\Omega |\nabla G_\lambda^{-1}(x)|^2 \geq \lambda \left\|(x,y)\right\|^2_{\mathcal{H}}
\end{aligned}$$

for every $(x,y) \in \mathcal{H}$, from which the first condition. Secondly, for every $(x,y) \in \mathcal{H}$, the Lipschitz continuity of α_λ, $\alpha_{\Gamma\lambda}$ and the continuity of $G_\lambda^{-1} : \mathcal{H} \to W_{\mathbf{n}}$, we have

$$\left\|A_\lambda(x,y)\right\|_{\mathcal{H}} \leq \left(\lambda + \frac{1}{\lambda}\right) \left\|x,y\right\|_{\mathcal{H}} + \left\|G_\lambda^{-1}(x)\right\|_H \leq \left(\lambda + \frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \left\|x,y\right\|_{\mathcal{H}}\,,$$

from which the second condition. Furthermore, it is a standard matter to check that (iii) holds with the choice $\psi_\lambda : \mathcal{H} \to [0,+\infty]$

$$\psi_\lambda(x,y) := \begin{cases} \frac{\lambda}{2}\int_\Omega |x|^2 + \frac{1}{2}\int_\Omega |\nabla x|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma y|^2 + \int_\Omega \widehat{\beta}_\lambda(x) + \int_\Gamma \widehat{\beta}_{\Gamma\lambda}(y) & \text{if } (x,y) \in \mathcal{V}\,,\\ +\infty & \text{otherwise}\,. \end{cases}$$

It is clear that $D(\psi_\lambda) \subseteq \mathcal{V}$ and that, for every $(x,y) \in \mathcal{V}$,

$$\psi_\lambda(x,y) \geq \frac{\lambda}{2}\int_\Omega |x|^2 + \frac{1}{2}\int_\Omega |\nabla x|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma y|^2 \geq \frac{1}{2}\min\{1,\lambda,\varepsilon\} \left\|(x,y)\right\|^2_{\mathcal{V}} - \frac{\varepsilon}{2}\left\|(x,y)\right\|^2_{\mathcal{H}}$$

and also condition (iv) is proved. Moreover, it is readily seen that (v) holds with

$$\phi_\lambda(x,y) := \frac{\lambda}{2}\int_\Omega |x|^2 + \int_\Omega \widehat{\alpha}_\lambda(x) + F_\lambda^*(x) + \frac{\lambda}{2}\int_\Gamma |y|^2 + \int_\Gamma \widehat{\alpha}_{\Gamma\lambda}(y)\,, \qquad (x,y) \in \mathcal{H}\,,$$

where F_λ^* is the convex conjugate of the proper, convex, l.s.c. function

$$F_\lambda(x) := \begin{cases} \frac{\lambda}{2}\int_\Omega |x|^2 + \int_\Omega |\nabla x|^2 & \text{if } x \in V\,, \\ +\infty & \text{if } x \in H \setminus V\,. \end{cases}$$

Since $\partial\phi_\lambda = G_\lambda^{-1}$ is Lipschitz continuous on H, it is also clear that $D(\phi_\lambda) = H$, and (v) is proved. Moreover, (vi) is an easy consequence of the Lipschitz continuity of α_λ, $\alpha_{\Gamma\lambda}$ and G_λ^{-1} on H. Finally, let us focus on (vii). In this case, we are looking at B_λ as its weak formulation $B_\lambda : \mathcal{V} \to \mathcal{V}^*$ given by

$$\langle B_\lambda(x, y), (z, w)\rangle_{\mathcal{V}} = \lambda\int_\Omega xz + \int_\Omega \nabla x\cdot\nabla z + \int_\Omega \beta_\lambda(x)z + \varepsilon\int_\Gamma \nabla_\Gamma y\cdot\nabla_\Gamma w + \int_\Gamma \beta_{\Gamma\lambda}(y)w\,.$$

Hence, it follows by the Lipschitz continuity of β_λ and $\beta_{\Gamma\lambda}$ that, for every $(x_1, y_1), (x_2, y_2) \in \mathcal{V}$

$$\begin{aligned} &\|B_\lambda(x_1, y_1) - B_\lambda(x_2, y_2)\|_{\mathcal{V}^*} \\ &\quad\le \left(\lambda + \frac{1}{\lambda}\right)\|x_1 - x_2\|_H + \|\nabla(x_1 - x_2)\|_H + \varepsilon\,\|\nabla_\Gamma(y_1 - y_2)\|_{H_\Gamma} + \frac{1}{c\lambda}\|y_1 - y_2\|_{H_\Gamma} \end{aligned}$$

from which the Lipschitz continuity of B_λ. Similarly, by the monotonicity of β_λ and $\beta_{\Gamma\lambda}$,

$$\begin{aligned} &\langle B_\lambda(x_1, y_1) - B_\lambda(x_2, y_2), (x_1, y_1) - (x_2, y_2)\rangle_{\mathcal{V}} \\ &\quad\ge \lambda\,\|x_1 - x_2\|_H^2 + \|\nabla(x_1 - x_2)\|_H^2 + \varepsilon\,\|\nabla_\Gamma(y_1 - y_2)\|_H^2 \ge C_{\lambda\varepsilon}\,\|(x_1, y_1) - (x_2, y_2)\|_{\mathcal{V}}^2 \end{aligned}$$

for a certain positive constant $C_{\lambda\varepsilon}$, from which the strong monotonicity of B_λ.

Now, we fix $\lambda > 0$ and we show that the approximated problem is well-posed. Given $(f, f_\Gamma) \in L^2(0, T; \mathcal{H})$, Lemma 3.1 and the hypotheses (2.4)–(2.5) ensure that we can apply the existence result contained in [13, Thm. 2.1] and infer that there exists

$$(u_\lambda, v_\lambda) \in H^1(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{V})\,, \qquad A_\lambda(\partial_t u_\lambda, \partial_t v_\lambda)\,,\ B_\lambda(u_\lambda, v_\lambda) \in L^2(0, T; \mathcal{H})$$

such that

$$A_\lambda\partial_t(u_\lambda, v_\lambda) + B_\lambda(u_\lambda, v_\lambda) = (g_\lambda, g_{\Gamma\lambda}) - (T_\lambda\pi(f), T_\lambda\pi_\Gamma(f_\Gamma)) \quad \text{a.e. in } (0, T)\,, \tag{3.1}$$

$$(u_\lambda, v_\lambda)(0) = (u_0, u_{0|\Gamma})\,. \tag{3.2}$$

Let us show that such solution (u_λ, v_λ) is indeed unique and satisfies useful estimates.

Lemma 3.2. *For every $\lambda > 0$, there exists $c_\lambda > 0$ such that*

$$\|\partial_t u_\lambda\|_{L^2(0,T;H)} + \|\partial_t v_\lambda\|_{L^2(0,T;H_\Gamma)} + \|u_\lambda\|_{L^\infty(0,T;V)} + \|v_\lambda\|_{L^\infty(0,T;V_\Gamma)} \le c_\lambda\,.$$

Moreover, there is $c'_\lambda > 0$ such that, for every $(f^i, f_\Gamma^i) \in L^2(0, T; \mathcal{H})$, if $(u_\lambda^i, v_\lambda^i)$ are any respective solution to (3.1)–(3.2)*, $i = 1, 2$, we have*

$$\begin{aligned} &\left\|\partial_t(u_\lambda^1 - u_\lambda^2)\right\|_{L^2(0,T;H)} + \left\|\partial_t(v_\lambda^1 - v_\lambda^2)\right\|_{L^2(0,T;H_\Gamma)} + \left\|u_\lambda^1 - u_\lambda^2\right\|_{L^\infty(0,T;V)} + \left\|v_\lambda^1 - v_\lambda^2\right\|_{L^\infty(0,T;V_\Gamma)} \\ &\qquad\qquad \le c'_\lambda\left(\left\|f^1 - f^2\right\|_{L^2(0,T;H)} + \left\|f_\Gamma^1 - f_\Gamma^2\right\|_{L^2(0,T;H_\Gamma)}\right)\,. \end{aligned}$$

Proof. Testing (3.1) by $\partial_t(u_\lambda, v_\lambda)$ and integrating on $(0,t)$, thanks to the monotonicity of the operators α_λ, $\alpha_{\Gamma\lambda}$ and G_λ^{-1}, using the Young inequality and the fact that $|T_\lambda| \leq \frac{1}{\lambda}$ we have

$$\begin{aligned}
&\lambda \int_0^t \|\partial_t u_\lambda(s)\|_H^2 \, ds + \frac{\lambda}{2}\int_\Omega |u_\lambda(t)|^2 + \frac{1}{2}\int_\Omega |\nabla u_\lambda(t)|^2 + \int_\Omega \widehat{\beta}_\lambda(u_\lambda(t)) \\
&\qquad + \lambda \int_0^t \|\partial_t v_\lambda(s)\|_{H_\Gamma}^2 \, ds + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma v_\lambda(t)|^2 + \int_\Gamma \widehat{\beta}_{\Gamma\lambda}(v_\lambda(t)) \\
&\leq \frac{\lambda}{2}\|u_0\|_H^2 + \frac{1}{2}\|\nabla u_0\|_H^2 + \frac{\varepsilon}{2}\left\|\nabla_\Gamma u_{0|\Gamma}\right\|_{H_\Gamma}^2 + \int_\Omega \widehat{\beta}_\lambda(u_0) + \int_\Gamma \widehat{\beta}_{\Gamma\lambda}(u_{0|\Gamma}) \\
&\qquad + \int_0^t\int_\Omega (g_\lambda(s) - T_\lambda \pi(f(s)))\,\partial_t u_\lambda(s)\, ds + \int_0^t\int_\Gamma (g_{\Gamma\lambda}(s) - T_\lambda \pi_\Gamma(f_\Gamma(s)))\,\partial_t v_\lambda(s)\, ds \\
&\leq c_\lambda \left\|(u_0, u_{0|\Gamma})\right\|_{\mathcal{V}}^2 + \frac{\lambda}{2}\int_0^t \|\partial_t u_\lambda(s)\|_H^2\, ds + \frac{\lambda}{2}\int_0^t \|\partial_t v_\lambda(s)\|_{H_\Gamma}^2\, ds \\
&\qquad + \frac{1}{\lambda}\|(g, g_\Gamma)\|_{L^2(0,T;\mathcal{H})}^2 + \frac{1}{\lambda^2}(|Q| + |\Sigma|)
\end{aligned}$$

for a certain $c_\lambda > 0$, so that rearranging the terms we obtain the first estimate. Similarly, given (f^i, f_Γ^i) and any respective solutions $(u_\lambda^i, v_\lambda^i)$ to (3.1)–(3.2), for $i = 1, 2$, taking the difference of (3.1) and testing by $\partial_t(u_\lambda^1 - u_\lambda^2, v_\lambda^1 - v_\lambda^2)$, using the monotonicity of α_λ, $\alpha_{\Gamma\lambda}$ and G_λ^{-1}, the Lipschitz continuity of β_λ, $\beta_{\Gamma\lambda}$, π, π_Γ and T_λ, an easy computation shows that

$$\begin{aligned}
&\lambda \int_0^t \left\|\partial_t(u_\lambda^1 - u_\lambda^2)(s)\right\|_H^2 ds + \lambda \int_0^t \left\|\partial_t(v_\lambda^1 - v_\lambda^2)(s)\right\|_{H_\Gamma}^2 ds \\
&\quad + \frac{\lambda}{2}\int_\Omega |(u_\lambda^1 - u_\lambda^2)(t)|^2 + \frac{1}{2}\int_\Omega |\nabla(u_\lambda^1 - u_\lambda^2)(t)|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla(v_\lambda^1 - v_\lambda^2)(t)|^2 \\
&\leq \int_0^t\int_\Omega \big(|\beta_\lambda(u_\lambda^1(s)) - \beta_\lambda(u_\lambda^2(s))| + |T_\lambda\pi(f^1(s)) - T_\lambda\pi(f^2(s))|\big)\,|\partial_t(u_\lambda^1 - u_\lambda^2)(s)|\, ds \\
&\quad + \int_0^t\int_\Gamma \big(|\beta_{\Gamma\lambda}(v_\lambda^1(s)) - \beta_{\Gamma\lambda}(v_\lambda^2(s))| + |T_\lambda\pi_\Gamma(f_\Gamma^1(s)) - T_\lambda\pi_\Gamma(f_\Gamma^2(s))|\big)\,|\partial_t(v_\lambda^1 - v_\lambda^2)(s)|\, ds \\
&\leq \frac{1}{\lambda}\int_0^t\int_\Omega |u_\lambda^1(s) - u_\lambda^2(s)||\partial_t(u_\lambda^1 - u_\lambda^2)(s)|\, ds + \frac{1}{\lambda}\int_0^t\int_\Gamma |v_\lambda^1(s) - v_\lambda^2(s)||\partial_t(v_\lambda^1 - v_\lambda^2)(s)|\, ds \\
&\quad + C_\pi \int_0^t\int_\Omega |f^1(s) - f^2(s)||\partial_t(u_\lambda^1 - u_\lambda^2)(s)|\, ds + C_{\pi_\Gamma}\int_0^t\int_\Gamma |f_\Gamma^1(s) - f_\Gamma^2(s)||\partial_t(v_\lambda^1 - v_\lambda^2)(s)|\, ds \\
&\leq \frac{\lambda}{2}\int_0^t \left\|\partial_t(u_\lambda^1 - u_\lambda^2)(s)\right\|_H^2 ds + \frac{\lambda}{2}\int_0^t \left\|\partial_t(v_\lambda^1 - v_\lambda^2)(s)\right\|_{H_\Gamma}^2 ds + \frac{1}{\lambda^2}\int_0^t \left\|u_\lambda^1(s) - u_\lambda^2(s)\right\|_H^2 ds \\
&\quad + \frac{1}{\lambda^2}\int_0^t \left\|v_\lambda^1(s) - v_\lambda^2(s)\right\|_{H_\Gamma}^2 ds + \frac{C_\pi^2}{\lambda}\left\|f^1 - f^2\right\|_{L^2(0,T;H)}^2 + \frac{C_{\pi_\Gamma}^2}{\lambda}\left\|f_\Gamma^1 - f_\Gamma^2\right\|_{L^2(0,T;H)}^2,
\end{aligned}$$

and the second inequality follows from the Gronwall lemma.

Lemma 3.2 ensures that, for any $\lambda > 0$, it is well-defined the map

$$\Theta_\lambda : E_\lambda \to E_\lambda\,, \qquad (f, f_\Gamma) \mapsto (u_\lambda, v_\lambda)\,,$$

where

$$\begin{aligned}
E_\lambda := \Big\{&(x,y) \in H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}) : \\
&\|\partial_t x\|_{L^2(0,T;H)} + \|\partial_t y\|_{L^2(0,T;H_\Gamma)} + \|x\|_{L^\infty(0,T;V)} + \|y\|_{L^\infty(0,T;V_\Gamma)} \leq c_\lambda\Big\}\,.
\end{aligned}$$

Since E_λ is compact and convex in $L^2(0,T;\mathcal{H})$ and Θ_λ is continuous on $L^2(0,T;\mathcal{H})$ by Lemma 3.2, Shauder's fixed point theorem ensures that there is a fixed point $(u_\lambda, v_\lambda) \in E_\lambda$ for Θ_λ. It is also clear by the second inequality in the previous lemma and the Gronwall lemma that (u_λ, v_λ) is also unique. As it is natural, we set $\mu_\lambda := -G_\lambda^{-1}\partial_t u_\lambda$.

Let us collect the properties of $(u_\lambda, v_\lambda, \mu_\lambda)$ in the following lemmata. The first result states precisely the regularities of the approximated solutions under the weakest assumptions of Theorem 2.1 on the data, while the second specifies some additional regularity provided by the strongest hypotheses of Theorems 2.3–2.6.

Lemma 3.3. *Under the assumptions* (2.4)–(2.5) *we have*

$$u_\lambda \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,,$$
$$v_\lambda \in H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;W_\Gamma)\,,$$
$$\mu_\lambda \in L^2(0,T;W_{\mathbf{n}})$$

and

$$\partial_t u_\lambda + \lambda\mu_\lambda - \Delta\mu_\lambda = 0 \qquad \text{in } Q\,, \tag{3.3}$$
$$\mu_\lambda = \lambda\partial_t u_\lambda + \alpha_\lambda(\partial_t u_\lambda) + \lambda u_\lambda - \Delta u_\lambda + \beta_\lambda(u_\lambda) + T_\lambda\pi(u_\lambda) - g_\lambda \qquad \text{in } Q\,, \tag{3.4}$$
$$u_\lambda = v_\lambda\,, \quad \partial_{\mathbf{n}}\mu_\lambda = 0 \qquad \text{in } \Sigma\,, \tag{3.5}$$
$$\lambda\partial_t v_\lambda + \alpha_{\Gamma\lambda}(\partial_t v_\lambda) + \partial_{\mathbf{n}} u_\lambda - \varepsilon\Delta_\Gamma v_\lambda + \beta_{\Gamma\lambda}(v_\lambda) + T_\lambda\pi_\Gamma(v_\lambda) = g_{\Gamma\lambda} \qquad \text{in } \Sigma\,, \tag{3.6}$$
$$u_\lambda(0) = u_0 \qquad \text{in } \Omega\,, \tag{3.7}$$

Proof. Thanks to classical elliptic regularity results (see [22, Thm. 3.2]), the regularities of the approximated solutions u_λ and v_λ easily follow from the fact that $(u_\lambda, v_\lambda) \in E_\lambda$ and $B_\lambda(u_\lambda, v_\lambda) \in L^2(0,T;\mathcal{H})$. Indeed, from this last condition it follows that $\Delta u_\lambda \in L^2(0,T;H)$ and $\partial_{\mathbf{n}} u_\lambda - \varepsilon\Delta v_\lambda \in L^2(0,T;H_\Gamma)$. The conditions $u_\lambda \in L^\infty(0,T;V)$, $\Delta u_\lambda \in L^2(0,T;H)$ and $v_\lambda \in L^\infty(0,T;V_\Gamma)$ imply that $u_\lambda \in L^2(0,T;H^{3/2}(\Omega))$, hence also $\partial_{\mathbf{n}} u_\lambda \in L^2(0,T;H_\Gamma)$. It follows then by comparison that $\Delta_\Gamma v_\lambda \in L^2(0,T;H_\Gamma)$, from which $v_\lambda \in L^2(0,T;W_\Gamma)$ and also $u_\lambda \in L^2(0,T;W)$. Finally, the regularity of μ is straightforward from the definition of G_λ, and (3.3)–(3.7) follow from the definition of Θ_λ itself. □

Lemma 3.4. *Under the further assumptions* (2.18)–(2.21) *we also have*

$$u_\lambda \in H^1(0,T;V) \cap L^2(0,T;H^3(\Omega)) \cap C^0([0,T];W) \cap C^1([0,T];H)\,,$$
$$v_\lambda \in H^1(0,T;V_{\Gamma\varepsilon}) \cap L^2(0,T;H^3(\Gamma)) \cap C^0([0,T];W_\Gamma) \cap C^1([0,T];H_\Gamma)\,,$$
$$\mu_\lambda \in L^2(0,T;H^3(\Omega)) \cap C^0([0,T];W_{\mathbf{n}})\,.$$

Proof. Thanks to conditions (v)–(vii) in Lemma 3.1 and the hypotheses (2.18)–(2.21), the result [13, Thm 2.2] ensures that the range of the function Θ_λ is contained in $H^1(0,T;\mathcal{V})$, hence $u_\lambda \in H^1(0,T;V)$ and $v_\lambda \in H^1(0,T;V_\Gamma)$. Consequently, by comparison in (3.3), we have $\mu_\lambda \in L^2(0,T;V)$, so that $\mu_\lambda \in L^2(0,T;H^3(\Omega))$ by elliptic regularity. Moreover, by comparison in (3.4)–(3.6), thanks to (2.18) and the fact that $\partial_t u_\lambda \in L^2(0,T;V)$ and $\partial_t v_\lambda \in L^2(0,T;V_\Gamma)$, we deduce that $-\Delta u_\lambda \in L^2(0,T;V)$ and $\partial_{\mathbf{n}} u_\lambda - \varepsilon\Delta v_\lambda \in L^2(0,T;V_\Gamma)$. Since we have $\Delta u_\lambda \in L^2(0,T;V)$ and (by Lemma 3.3) $v \in L^2(0,T;W_\Gamma)$, then $u_\lambda \in L^2(0,T;H^{5/2}(\Omega))$ and $\partial_{\mathbf{n}} u_\lambda \in L^2(0,T;V_\Gamma)$. By difference then we deduce that $\Delta_\Gamma v_\lambda \in L^2(0,T;V_\Gamma)$, so that $v_\lambda \in L^2(0,T;H^3(\Gamma))$ by elliptic regularity on the boundary, and consequently also $u_\lambda \in$

$L^2(0,T;H^3(\Omega))$. Furthermore, we have $u_\lambda \in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;V) \hookrightarrow C^0([0,T];W)$ and $v_\lambda \in L^2(0,T;H^3(\Gamma)) \cap H^1(0,T;V_\Gamma) \hookrightarrow C^0([0,T];W_\Gamma)$; in particular, we deduce that $\partial_{\mathbf{n}} u_\lambda \in C^0([0,T];H^{1/2}(\Gamma))$. Hence, setting $z_\lambda := g_\lambda - \lambda u_\lambda + \Delta u_\lambda - \beta_\lambda(u_\lambda) - T_\lambda \pi(u_\lambda)$ and $w_\lambda := g_{\Gamma\lambda} - \partial_{\mathbf{n}} u_\lambda - \beta_{\Gamma\lambda}(v_\lambda) - T_\lambda \pi_\Gamma(v_\lambda)$, from (3.3)–(3.6) we have that $A_\lambda(\partial_t u_\lambda, \partial_t v_\lambda) = (z_\lambda, w_\lambda) \in C^0([0,T];\mathcal{H})$: since $A_\lambda^{-1} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous, we infer that $u_\lambda \in C^1([0,T];H)$ and $v_\lambda \in C^1([0,T];H_\Gamma)$, hence also $\mu_\lambda \in C^0([0,T];W_{\mathbf{n}})$ from (3.3). $\square$

4 The first existence result

We present here the proof of the first main result. Recall that here we are working under the assumptions (2.4)–(2.8), so that the regularity of the approximated solutions is the one specified in Lemma 3.3. Since the passage to the limit will consist in letting $\lambda \searrow 0$, it is not restrictive to consider $\lambda \in (0,1)$ for example.

4.1 The first estimate

Testing (3.3) by μ_λ, (3.4) by $\partial_t u_\lambda$ and taking the difference, by integration by parts we have that, for every $t \in (0,T)$,

$$\begin{aligned}
&\lambda \int_{Q_t} |\mu_\lambda|^2 + \int_{Q_t} |\nabla \mu_\lambda|^2 + \lambda \int_{Q_t} |\partial_t u_\lambda|^2 + \int_{Q_t} \alpha_\lambda(\partial_t u_\lambda)\partial_t u_\lambda + \frac{\lambda}{2}\int_\Omega |u_\lambda(t)|^2 + \frac{1}{2}\int_\Omega |\nabla u_\lambda(t)|^2 \\
&\qquad + \lambda \int_{\Sigma_t} |\partial_t v_\lambda|^2 + \int_{\Sigma_t} \alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda + \frac{\varepsilon}{2}\int_\Sigma |\nabla_\Gamma v_\lambda(t)|^2 + \int_\Omega \widehat{\beta}_\lambda(u_\lambda(t)) + \int_\Sigma \widehat{\beta}_{\Gamma\lambda}(v_\lambda(t)) \\
&= \frac{\lambda}{2}\int_\Omega |u_0|^2 + \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \frac{\varepsilon}{2}\int_\Sigma |\nabla_\Gamma u_{0|\Gamma}|^2 + \int_\Omega \widehat{\beta}_\lambda(u_0) + \int_\Sigma \widehat{\beta}_{\Gamma\lambda}(u_{0|\Gamma}) \\
&\qquad + \int_{Q_t} (g_\lambda - T_\lambda \pi(u_\lambda))\,\partial_t u_\lambda + \int_{\Sigma_t} (g_{\Gamma\lambda} - T_\lambda \pi_\Gamma(v_\lambda))\,\partial_t v_\lambda\,.
\end{aligned}$$

Now, let $J_\lambda := (I + \lambda\alpha)^{-1} : \mathbb{R} \to \mathbb{R}$ and $J_{\Gamma_\lambda} := (I + \lambda\alpha_\Gamma)^{-1} : \mathbb{R} \to \mathbb{R}$ denote the resolvents of α and α_Γ, respectively. By elementary properties of maximal monotone graphs it is well known that J_λ and $J_{\Gamma\lambda}$ are contractions on $\mathbb{R}$, and that $\alpha_\lambda(\cdot) \in \alpha(J_\lambda(\cdot))$ and $\alpha_{\Gamma\lambda}(\cdot) \in \alpha_\Gamma(J_{\Gamma\lambda}\cdot)$: consequently, by the coercivity assumptions (2.8) and (2.2) we deduce that

$$\begin{aligned}
\alpha_\lambda(\partial_t u_\lambda)\partial_t u_\lambda &= \alpha_\lambda(\partial_t u_\lambda) J_\lambda \partial_t u_\lambda + \lambda|\alpha_\lambda(\partial_t u_\lambda)|^2 \geq a_1 |J_{J_\lambda}\partial_t u_\lambda|^2 - a_2 + \lambda|\alpha_\lambda(\partial_t u_\lambda)|^2\,, \\
\alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda &= \alpha_{\Gamma\lambda}(\partial_t v_\lambda) J_{\Gamma\lambda}\partial_t v_\lambda + \lambda|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2 \geq b_1 |J_{\Gamma\lambda}\partial_t v_\lambda|^2 - b_2 + \lambda|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2\,.
\end{aligned}$$

Taking into account these relations, the left-hand side of the last inequality is bounded from below by

$$\begin{aligned}
&\lambda \int_{Q_t} |\mu_\lambda|^2 + \int_{Q_t} |\nabla \mu_\lambda|^2 + \lambda \int_{Q_t} |\partial_t u_\lambda|^2 + a_1 \int_{Q_t} |J_\lambda \partial_t u_\lambda|^2 + \lambda \int_{Q_t} |\alpha_\lambda(\partial_t u_\lambda)|^2 \\
&\qquad + \frac{\lambda}{2}\int_\Omega |u_\lambda(t)|^2 + \frac{1}{2}\int_\Omega |\nabla u_\lambda(t)|^2 + \lambda \int_{\Sigma_t} |\partial_t v_\lambda|^2 + b_1 \int_{\Sigma_t} |J_{\Gamma\lambda}\partial_t v_\lambda|^2 + \lambda \int_{\Sigma_t} |\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2 \\
&\qquad + \frac{\varepsilon}{2}\int_\Sigma |\nabla_\Gamma v_\lambda(t)|^2 + \int_\Omega \widehat{\beta}_\lambda(u_\lambda(t)) + \int_\Sigma \widehat{\beta}_{\Gamma\lambda}(v_\lambda(t))
\end{aligned}$$

while the right-hand side can be handled using the Young inequality by

$$
\begin{aligned}
&a_2|Q| + b_2|\Sigma| + \frac{1}{2}\left\|u_0\right\|_V^2 + \frac{\varepsilon}{2}\left\|u_{0|\Gamma}\right\|_{V_\Gamma}^2 + \left\|\widehat{\beta}(u_0)\right\|_{L^1(\Omega)} + \left\|\widehat{\beta}_\Gamma(u_{0|\Gamma})\right\|_{L^1(\Gamma)} \\
&\quad + \int_{Q_t} (g_\lambda - T_\lambda\pi(u_\lambda))\,\partial_t u_\lambda + \int_{\Sigma_t} (g_{\Gamma\lambda} - T_\lambda\pi(v_\lambda))\,\partial_t v_\lambda \\
&\leq a_2|Q| + b_2|\Sigma| + \frac{1}{2}\left\|u_0\right\|_V^2 + \frac{\varepsilon}{2}\left\|u_{0|\Gamma}\right\|_{V_\Gamma}^2 + \left\|\widehat{\beta}(u_0)\right\|_{L^1(\Omega)} + \left\|\widehat{\beta}_\Gamma(u_{0|\Gamma})\right\|_{L^1(\Gamma)} + \frac{\delta}{2}\int_{Q_t}|\partial_t u_\lambda|^2 \\
&\quad + \frac{\delta}{2}\int_{\Sigma_t}|\partial_t v_\lambda|^2 + \frac{1}{\delta}\left\|g\right\|_{L^2(0,T;H)}^2 + \frac{1}{\delta}\left\|g_\Gamma\right\|_{L^2(0,T;H_\Gamma)}^2 + \frac{C_\pi^2}{\delta}\int_{Q_t}|u_\lambda|^2 + \frac{C_{\pi_\Gamma}^2}{\delta}\int_{\Sigma_t}|v_\lambda|^2
\end{aligned}
$$

for every $\delta > 0$. Now, by definition of α_λ and $\alpha_{\Gamma\lambda}$,

$$
\frac{\delta}{2}\int_{Q_t}|\partial_t u_\lambda|^2 \leq \delta\int_{Q_t}|J_\lambda\partial_t u_\lambda|^2 + \delta\lambda^2\int_{Q_t}|\alpha_\lambda(\partial_t u_\lambda)|^2
$$

and

$$
\frac{\delta}{2}\int_{\Sigma_t}|\partial_t v_\lambda|^2 \leq \delta\int_{\Sigma_t}|J_{\Gamma\lambda}\partial_t v_\lambda|^2 + \delta\lambda^2\int_{\Sigma_t}|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2 .
$$

Let us handle the last two terms on the right hand side. Testing (3.3) by $\frac{1}{|\Omega|}$ we easily have

$$
\partial_t(u_\lambda)_\Omega + \lambda(\mu_\lambda)_\Omega = 0 ,
$$

which yields

$$
(u_\lambda(t))_\Omega = (u_0)_\Omega - \lambda\int_0^t (\mu_\lambda(s))_\Omega\,ds \quad \forall\, t \in [0,T] , \quad \forall\,\lambda > 0 . \tag{4.1}
$$

As a consequence, by the Poincaré inequality, an easy computation yields

$$
\begin{aligned}
\left\|u_\lambda(t)\right\|_H &\leq \left\|u_\lambda(t) - (u_\lambda(t))_\Omega\right\|_H + \left\|(u_0)_\Omega\right\|_H + \lambda\int_0^t \left\|(\mu_\lambda(s))_\Omega\right\|_H\,ds \\
&\leq C\left(\left\|\nabla u_\lambda(t)\right\|_H + \left\|u_0\right\|_H + \lambda\int_0^t \left\|\mu_\lambda(s)\right\|_H\,ds\right)
\end{aligned} \tag{4.2}
$$

for a positive constant C independent of λ, from which (updating C)

$$
\int_{Q_t}|u_\lambda|^2 \leq C\left(\int_{Q_t}|\nabla u_\lambda|^2 + \left\|u_0\right\|_H^2 + \lambda^2\int_{Q_t}|\mu_\lambda|^2\right) .
$$

Moreover, by the Poincaré inequality on the boundary we also have

$$
\int_{\Sigma_t}|v_\lambda|^2 \leq C\int_{\Sigma_t}|\nabla_\Gamma v_\lambda|^2 .
$$

Taking these considerations into account on the right hand side of the estimate we obtain

$$\begin{aligned}
&\lambda\int_{Q_t}|\mu_\lambda|^2+\int_{Q_t}|\nabla\mu_\lambda|^2+\lambda\int_{Q_t}|\partial_t u_\lambda|^2+a_1\int_{Q_t}|J_\lambda\partial_t u_\lambda|^2+\lambda\int_{Q_t}|\alpha_\lambda(\partial_t u_\lambda)|^2\\
&\quad+\frac{\lambda}{2}\int_\Omega|u_\lambda(t)|^2+\frac12\int_\Omega|\nabla u_\lambda(t)|^2+\lambda\int_{\Sigma_t}|\partial_t v_\lambda|^2+b_1\int_{\Sigma_t}|J_{\Gamma\lambda}\partial_t v_\lambda|^2+\lambda\int_{\Sigma_t}|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2\\
&\quad+\frac{\varepsilon}{2}\int_\Sigma|\nabla_\Gamma v_\lambda(t)|^2+\int_\Omega\widehat{\beta}_\lambda(u_\lambda(t))+\int_\Sigma\widehat{\beta}_{\Gamma\lambda}(v_\lambda(t))\\
&\le a_2|Q|+b_2|\Sigma|+C\left\|u_0\right\|_V^2+\frac{\varepsilon}{2}\left\|u_{0|\Gamma}\right\|_{V_\Gamma}^2+\left\|\widehat{\beta}(u_0)\right\|_{L^1(\Omega)}+\left\|\widehat{\beta}_\Gamma(u_{0|\Gamma})\right\|_{L^1(\Gamma)}+\delta\int_{Q_t}|J_\lambda\partial_t u_\lambda|^2\\
&\quad+\delta\lambda^2\int_{Q_t}|\alpha_\lambda(\partial_t u_\lambda)|^2+\delta\int_{\Sigma_t}|J_{\Gamma\lambda}\partial_t v_\lambda|^2+\delta\lambda^2\int_{\Sigma_t}|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2\\
&\quad+\frac1\delta\left\|g\right\|_{L^2(0,T;H)}^2+\frac1\delta\left\|g_\Gamma\right\|_{L^2(0,T;H_\Gamma)}^2+C_\delta\int_{Q_t}|\nabla u_\lambda|^2+C_\delta\lambda^2\int_{Q_t}|\mu_\lambda|^2
\end{aligned}$$

where we have updated step by step the constant C independent of λ and $C_\delta>0$ depends only on δ. Fix now $\delta:=\min\{\frac{a_1}{2},\frac{b_1}{2},\frac12\}$: since it is not restrictive to consider $\lambda\in(0,\frac{1}{2C_\delta}]$, rearranging the terms and using the Gronwall lemma yields

$$\left\|\nabla u_\lambda\right\|_{L^\infty(0,T;H)}+\lambda^{1/2}\left\|u_\lambda\right\|_{H^1(0,T;H)\cap L^\infty(0,T;H)}\le C\,, \tag{4.3}$$

$$\varepsilon^{1/2}\left\|v_\lambda\right\|_{L^\infty(0,T;V_\Gamma)}+\lambda^{1/2}\left\|v_\lambda\right\|_{H^1(0,T;H_\Gamma)}\le C\,, \tag{4.4}$$

$$\left\|J_\lambda\partial_t u_\lambda\right\|_{L^2(0,T;H)}+\lambda^{1/2}\left\|\alpha_\lambda(\partial_t u_\lambda)\right\|_{L^2(0,T;H)}\le C\,, \tag{4.5}$$

$$\left\|J_{\Gamma\lambda}\partial_t v_\lambda\right\|_{L^2(0,T;H_\Gamma)}+\lambda^{1/2}\left\|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)\right\|_{L^2(0,T;H_\Gamma)}\le C\,, \tag{4.6}$$

$$\lambda^{1/2}\left\|\mu_\lambda\right\|_{L^2(0,T;H)}+\left\|\nabla\mu_\lambda\right\|_{L^2(0,T;H)}\le C\,, \tag{4.7}$$

$$\left\|\widehat{\beta}_\lambda(u_\lambda)\right\|_{L^\infty(0,T;L^1(\Omega))}+\left\|\widehat{\beta}_{\Gamma\lambda}(v_\lambda)\right\|_{L^\infty(0,T;L^1(\Gamma))}\le C\,. \tag{4.8}$$

From estimates (4.3), (4.7), condition (4.2) and equation (3.3), it follows that

$$\left\|u_\lambda\right\|_{L^\infty(0,T;V)}+\left\|u_\lambda\right\|_{H^1(0,T;V^*)}\le C\,. \tag{4.9}$$

Moreover, from (4.5), (4.7) and the fact that $\partial_t u_\lambda=\lambda\alpha_\lambda(\partial_t u_\lambda)+J_\lambda\partial_t u_\lambda$ (by definition of Yosida approximation), by comparison in (3.3) we have

$$\left\|\Delta\mu_\lambda\right\|_{L^2(0,T;H)}\le C\,. \tag{4.10}$$

Finally, (2.7) and (4.5)–(4.6) ensure that

$$\left\|\alpha_\lambda(\partial_t u_\lambda)\right\|_{L^2(0,T;H)}+\left\|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)\right\|_{L^2(0,T;H_\Gamma)}\le C\,. \tag{4.11}$$

4.2 The second estimate

We show here an additional estimate for μ_λ in the space $L^2(0,T;W_{\mathbf{n}})$. By (4.7), (4.10) and (2.1), it is enough to show that $(\mu_\lambda)_\Omega$ is bounded in $L^2(0,T)$ uniformly in λ. To this end, we are inspired by the computations in [8].

We test (3.3) by $G_\lambda^{-1}(u_\lambda - (u_\lambda(t))_\Omega)$, (3.4) by $u_\lambda - (u_\lambda(t))_\Omega$, take the difference, but not integrate in time: we deduce that, for almost every $t \in (0,T)$,

$$\begin{aligned}\int_\Omega |\nabla u_\lambda(t)|^2 + \int_\Omega \beta_\lambda(u_\lambda(t))(u_\lambda(t) - (u_\lambda(t))_\Omega) + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma v_\lambda(t)|^2 \\ + \int_\Gamma \beta_{\Gamma\lambda}(v_\lambda(t))(v_\lambda(t) - (u_\lambda(t))_\Omega) = -\int_\Omega \partial_t u_\lambda(t) G_\lambda^{-1}(u_\lambda(t) - (u_\lambda(t))_\Omega) \\ + \int_\Omega \left(g_\lambda(t) - T_\lambda\pi(u_\lambda(t)) - \lambda\partial_t u_\lambda(t) - \alpha_\lambda(u_\lambda(t))\right)(u_\lambda(t) - (u_\lambda(t))_\Omega) \\ + \int_\Gamma \left(g_{\Gamma\lambda}(t) - T_\lambda\pi_\Gamma(v_\lambda(t)) - \lambda\partial_t v_\lambda(t) - \alpha_{\Gamma\lambda}(v_\lambda(t))\right)(v_\lambda(t) - (u_\lambda(t))_\Omega).\end{aligned}$$

Let us show that the right hand side is bounded in $L^2(0,T)$ uniformly in λ. It is clear that the last two terms are bounded in $L^2(0,T)$ by the Hölder inequality and the estimates (4.4), (4.9) and (4.11). Moreover, by definition of G_λ^{-1} it is immediate to check that $(G_\lambda^{-1}(y))_\Omega = \frac{1}{\lambda}y_\Omega$ for every $y \in H$: hence, we deduce that $(G_\lambda^{-1}(u_\lambda(t) - (u_\lambda(t))_\Omega))_\Omega = 0$ and by the Poincaré inequality we have

$$\begin{aligned}-\int_\Omega \partial_t u_\lambda(t) G_\lambda^{-1}(u_\lambda(t) - (u_0)_\Omega) &\le \|\partial_t u_\lambda(t)\|_{V^*} \left\|G_\lambda^{-1}(u_\lambda(t) - (u_\lambda(t))_\Omega)\right\|_V \\ &\le C\,\|\partial_t u_\lambda(t)\|_{V^*} \left\|\nabla G_\lambda^{-1}(u_\lambda(t) - (u_\lambda(t))_\Omega)\right\|_H\end{aligned}$$

for a positive constant C. Now, for any $y \in H$ with $y_\Omega = 0$, setting $y_\lambda := G_\lambda^{-1}(y) \in W_{\mathbf{n}}$, we have $\lambda y_\lambda - \Delta y_\lambda = y$, so that testing by y_λ we infer that

$$\lambda\int_\Omega |y_\lambda|^2 + \int_\Omega |\nabla y_\lambda|^2 = \int_\Omega y y_\lambda \le \frac{1}{4\delta}\|y\|_{V^*}^2 + \delta\|y_\lambda\|_V^2\,,$$

for every $\delta > 0$, where $\|y_\lambda\|_V^2 \le C\|\nabla y_\lambda\|_H^2$ for a positive constant C. Choosing $\delta = \frac{1}{2C}$ yields

$$\lambda\left\|G_\lambda^{-1}(y)\right\|_H^2 + \left\|\nabla G_\lambda^{-1}(y)\right\|_H^2 \le C\|y\|_{V^*}^2 \qquad \forall\, y \in H:\ y_\Omega = 0\,,$$

so that going back to the last inequality we have

$$-\int_\Omega \partial_t u_\lambda(t) G_\lambda^{-1}(u_\lambda(t) - (u_0)_\Omega) \le C\,\|\partial_t u_\lambda(t)\|_{V^*}\|u_\lambda(t)\|_{V^*}\,.$$

By (4.9) we deduce that also this last term is bounded in $L^2(0,T)$.

Now, by assumption (2.6) we know that $(u_0)_\Omega$ belongs to the interior of $D(\beta_\Gamma)$ (hence, also of $D(\beta)$ by (2.3)). This implies that there are two constants $k_0', k_0'' > 0$ (depending only on $(u_0)_\Omega$) such that

$$\beta_\lambda(r)(r - (u_0)_\Omega) \ge k_0'|\beta_\lambda(r)| - k_0''\,, \quad \beta_{\Gamma\lambda}(r)(r - (u_0)_\Omega) \ge k_0'|\beta_\lambda(r)| - k_0' \qquad \forall\, r \in \mathbb{R}$$

(see for example [8, p. 984], [19, p. 908] and [27, Prop. A.1]). Moreover, note that by (4.1) and (4.7) we have

$$|(u_\lambda(t))_\Omega - (u_0)_\Omega| \le \lambda\int_0^t |(\mu_\lambda(s))_\Omega|\,ds \le C\lambda^{1/2} \qquad \forall\, t \in [0,T]\,.$$

Consequently, we have

$$\begin{aligned}\int_\Omega \beta_\lambda(u_\lambda(t))(u_\lambda(t)-(u_\lambda(t))_\Omega) &= \int_\Omega \beta_\lambda(u_\lambda(t))(u_\lambda(t)-(u_0)_\Omega) + \int_\Omega \beta_\lambda(u_\lambda(t))((u_0)_\Omega-(u_\lambda(t))_\Omega)\\ &\geq k_0'\int_\Omega|\beta_\lambda(u_\lambda(t))| - k_0''|\Omega| - C\lambda^{1/2}\int_\Omega|\beta_\lambda(u_\lambda(t))|\end{aligned}$$

and similarly

$$\int_\Gamma \beta_{\Gamma\lambda}(v_\lambda(t))(v_\lambda(t)-(u_\lambda(t))_\Omega) \geq k_0'\int_\Gamma|\beta_{\Gamma\lambda}(v_\lambda(t))| - k_0''|\Gamma| - C\lambda^{1/2}\int_\Gamma|\beta_{\Gamma\lambda}(v_\lambda(t))|$$

Putting this information together, we deduce that

$$\|\beta_\lambda(u_\lambda)\|_{L^2(0,T;L^1(\Omega))} + \|\beta_{\Gamma\lambda}(v_\lambda)\|_{L^2(0,T;L^1(\Gamma))} \leq C\,.$$

Hence, testing (3.4) by ±1 we have

$$\begin{aligned}\pm|\Omega|(\mu_\lambda)_\Omega \leq \int_\Omega|\beta_\lambda(u_\lambda)| + \int_\Gamma|\beta_{\Gamma\lambda}(v_\lambda)| + \int_\Omega|\lambda\partial_t u_\lambda + \alpha_\lambda(\partial_t u_\lambda) + \lambda u_\lambda + T_\lambda\pi(u_\lambda) - g_\lambda|\\ + \int_\Gamma|\lambda\partial_t v_\lambda + \alpha_{\Gamma\lambda}(\partial_t v_\lambda) + \lambda v_\lambda + T_\lambda\pi_\Gamma(v_\lambda) - g_{\Gamma\lambda}|\,,\end{aligned}$$

where the right hand side is bounded in $L^2(0,T)$ by the estimates already computed and by (4.3)–(4.4) and (4.9)–(4.11). Hence, we have that

$$\|\mu_\lambda\|_{L^2(0,T;W_{\mathbf{n}})} \leq C\,. \tag{4.12}$$

4.3 The third estimate

We test (3.4) by $\beta_\lambda(u_\lambda)$: integrating by parts yields

$$\begin{aligned}&\lambda\int_\Omega\widehat{\beta}_\lambda(u_\lambda(t)) + \lambda\int_{Q_t}\beta_\lambda(u_\lambda)u_\lambda + \int_{Q_t}\beta_\lambda'(u_\lambda)|\nabla u_\lambda|^2 + \int_{Q_t}|\beta_\lambda(u_\lambda)|^2\\ &\quad+\lambda\int_\Gamma\widehat{\beta}_\lambda(v_\lambda(t)) + \int_{\Sigma_t}\beta_\lambda'(v_\lambda)|\nabla_\Gamma v_\lambda|^2 + \int_{\Sigma_t}\beta_{\Gamma\lambda}(v_\lambda)\beta_\lambda(v_\lambda) = \lambda\int_\Omega\widehat{\beta}_\lambda(u_0) + \lambda\int_\Gamma\widehat{\beta}_\lambda(u_{0|\Gamma})\\ &\quad+\int_{Q_t}(g_\lambda - T_\lambda\pi(u_\lambda) - \alpha_\lambda(\partial_t u_\lambda))\,\beta_\lambda(u_\lambda) + \int_{\Sigma_t}(g_{\Gamma\lambda} - T_\lambda\pi_\Gamma(v_\lambda) - \alpha_{\Gamma\lambda}(\partial_t v_\lambda))\,\beta_\lambda(v_\lambda)\end{aligned}$$

By the Young inequality, the estimates (4.3)–(4.4) and (4.9)–(4.11), the hypotheses (2.4)–(2.5) and the monotonicity of β and β_Γ, we infer that for every $\delta>0$, we have

$$\frac12\int_{Q_t}|\beta_\lambda(u_\lambda)|^2 + \int_{\Sigma_t}\beta_{\Gamma\lambda}(v_\lambda)\beta_\lambda(v_\lambda) \leq C_\delta + \left\|\widehat{\beta}(u_0)\right\|_{L^1(\Omega)} + \left\|\widehat{\beta}_\Gamma(u_{0|\Gamma})\right\|_{L^1(\Gamma)} + \delta\int_{\Sigma_t}|\beta_\lambda(v_\lambda)|^2$$

for a positive constant C_δ, independent of λ. Now, by the assumption (2.3) and [4, Lemma 4.4], recalling the definition of β_λ and $\beta_{\Gamma\lambda}$, it follows that

$$|\beta_\lambda(r)| \leq c\,(|\beta_{\Gamma\lambda}(r)|+1)\quad \forall\, r\in\mathbb{R}\,.$$

Hence, substituting in the last inequality and using the Young inequality we get (updating the constant C_δ at each step)

$$\frac{1}{2}\int_Q |\beta_\lambda(u_\lambda)|^2 + \frac{1}{c}\int_\Sigma |\beta_\lambda(v_\lambda)|^2 \le C_\delta + \delta\int_\Sigma |\beta_\lambda(v_\lambda)|^2 + \int_\Sigma |\beta_\lambda(v_\lambda)| \le C_\delta + 2\delta\int_\Sigma |\beta_\lambda(v_\lambda)|^2 .$$

Choosing $\delta := \frac{1}{4c}$, we infer that

$$\|\beta_\lambda(u_\lambda)\|_{L^2(0,T;H)} + \|\beta_\lambda(v_\lambda)\|_{L^2(0,T;H_\Gamma)} \le C . \tag{4.13}$$

By comparison in (3.4), recalling also (4.3), (4.11) and (4.12), we deduce that

$$\|\Delta u_\lambda\|_{L^2(0,T;H)} \le C . \tag{4.14}$$

Hence, thanks to the classical results on elliptic regularity (see [22, Thm. 3.2]), (4.4), (4.9) and (4.14) yield

$$\varepsilon^{1/2}\,\|u_\lambda\|_{L^2(0,T;H^{3/2}(\Omega))} + \varepsilon^{1/2}\,\|\partial_{\mathbf{n}} u_\lambda\|_{L^2(0,T;H_\Gamma)} \le C , \tag{4.15}$$

and by comparison in (3.6) also

$$\left\|-\varepsilon^{3/2}\Delta_\Gamma v_\lambda + \varepsilon^{1/2}\beta_{\Gamma\lambda}(v_\lambda)\right\|_{L^2(0,T;H_\Gamma)} \le C .$$

Now, since the operators $-\Delta_\Gamma$ and $\beta_{\Gamma\lambda}$ are monotone on H_Γ, testing $-\varepsilon^{3/2}\Delta_\Gamma v_\lambda + \varepsilon^{1/2}\beta_{\Gamma\lambda}(v_\lambda)$ by either $-\varepsilon^{3/2}\Delta_\Gamma v_\lambda$ or $\varepsilon^{1/2}\beta_{\Gamma\lambda}(v_\lambda)$, integrating by parts on Γ, using monotonicity, the last estimate and the Young inequality implies by a classical argument that

$$\varepsilon^{3/2}\,\|\Delta_\Gamma v_\lambda\|_{L^2(0,T;H_\Gamma)} + \varepsilon^{1/2}\,\|\beta_{\Gamma\lambda}(v_\lambda)\| \le C . \tag{4.16}$$

4.4 The passage to the limit

In this section, we pass to the limit in the approximated problem (3.3)–(3.7) and we prove the existence of a solution for the original problem.

First of all, thanks to the estimates (4.3)–(4.16), there are

$$u \in L^\infty(0,T;V) \cap L^2(0,T;W) , \qquad v \in L^\infty(0,T;V_\Gamma) \cap L^2(0,T;W_\Gamma) , \qquad \mu \in L^2(0,T;W_{\mathbf{n}}) ,$$
$$\eta, \xi \in L^2(0,T;H) , \qquad \eta_\Gamma, \xi_\Gamma \in L^2(0,T;H_\Gamma) ,$$

such that, along a subsequence that we still denote by λ for simplicity,

$$u_\lambda \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V) , \qquad u_\lambda \rightharpoonup u \quad \text{in } L^2(0,T;W) , \tag{4.17}$$
$$v_\lambda \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;V_\Gamma) , \qquad v_\lambda \rightharpoonup v \quad \text{in } L^2(0,T;W_\Gamma) , \tag{4.18}$$
$$\mu_\lambda \to \mu \quad \text{in } L^2(0,T;W_{\mathbf{n}}) , \tag{4.19}$$
$$\alpha_\lambda(\partial_t u_\lambda) \rightharpoonup \eta \quad \text{in } L^2(0,T;H) , \qquad \alpha_{\Gamma\lambda}(\partial_t v_\lambda) \rightharpoonup \eta_\Gamma \quad \text{in } L^2(0,T;H_\Gamma) , \tag{4.20}$$
$$\beta_\lambda(u_\lambda) \rightharpoonup \xi \quad \text{in } L^2(0,T;H) , \qquad \beta_{\Gamma\lambda}(v_\lambda) \rightharpoonup \xi_\Gamma \quad \text{in } L^2(0,T;H_\Gamma) \tag{4.21}$$

and

$$\lambda u_\lambda \to 0 \quad \text{in } H^1(0,T;H) , \quad \lambda v_\lambda \to 0 \quad \text{in } H^1(0,T;H_\Gamma) , \quad \lambda\mu_\lambda \to 0 \quad \text{in } L^2(0,T;H) . \tag{4.22}$$

Moreover, noting that, by definition of Yosida approximation,

$$|\partial_t u_\lambda - J_\lambda \partial_t u_\lambda| = \lambda |\alpha_\lambda(\partial_t u_\lambda)|\,, \qquad |\partial_t v_\lambda - J_{\Gamma\lambda} \partial_t v_\lambda| = \lambda |\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|\,,$$

it is readily seen that (4.5)–(4.6) imply that $u \in H^1(0,T;H)$, $v \in H^1(0,T;H_\Gamma)$ and

$$J_\lambda \partial_t u_\lambda \rightharpoonup \partial_t u \quad \text{in } L^2(0,T;H)\,, \qquad J_{\Gamma\lambda} \partial_t v_\lambda \rightharpoonup \partial_t v \quad \text{in } L^2(0,T;H_\Gamma)\,. \tag{4.23}$$

It is clear that $u_{|\Gamma} = v$. Moreover, since the inclusion $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact, by the classical compactness results for functions with values in Banach spaces (see [31, Cor. 4, p. 85]), we have

$$u_\lambda \to u \quad \text{in } C^0([0,T];H)\,, \qquad v_\lambda \to v \quad \text{in } C^0([0,T];H_\Gamma)\,, \tag{4.24}$$

which together with (4.21) and the strong-weak closure of the maximal monotone operators β and β_Γ ensure that

$$\xi \in \beta(u) \quad \text{a.e. in } Q\,, \qquad \xi_\Gamma \in \beta_\Gamma(v) \quad \text{a.e. in } \Sigma\,.$$

Furthermore, by the Lipschitz continuity of T_λ, π and π_Γ, using the strong convergences of u_λ and v_λ it is a standard matter to check that

$$T_\lambda \pi(u_\lambda) \to \pi(u) \quad \text{in } L^2(0,T;H)\,, \qquad T_\lambda \pi_\Gamma(v_\lambda) \to \pi_\Gamma(v) \quad \text{in } L^2(0,T;H_\Gamma)\,.$$

Taking this information into account and letting $\lambda \searrow 0$ in (3.3)–(3.7), we get

$$\partial_t u - \Delta \mu = 0\,, \tag{4.25}$$

$$\mu = \eta - \Delta u + \xi + \pi(u) - g\,, \qquad \eta_\Gamma + \partial_{\mathbf{n}} u - \varepsilon \Delta_\Gamma v + \xi_\Gamma + \pi_\Gamma(v) = g_\Gamma\,. \tag{4.26}$$

The last thing that we have to prove is that $\eta \in \alpha(\partial_t u)$ a.e. in Q and $\eta_\Gamma \in \alpha_\Gamma(\partial_t v)$ a.e. in Σ. To this end, performing the same test as in Section 4.1, one can easily infer that

$$\begin{aligned}
&\int_Q |\nabla \mu_\lambda|^2 + \int_Q \alpha_\lambda(\partial_t u_\lambda)\partial_t u_\lambda + \frac{1}{2}\int_\Omega |\nabla u_\lambda(T)|^2 + \int_\Omega \widehat{\beta}_\lambda(u_\lambda(t)) \\
&\quad + \int_\Sigma \alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma v_\lambda(t)|^2 + \int_\Gamma \widehat{\beta}_{\Gamma\lambda}(v_\lambda(t)) \\
&\le \frac{\lambda}{2}\int_\Omega |u_0|^2 + \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \int_\Omega \widehat{\beta}(u_0) + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma u_{0|\Gamma}|^2 + \int_\Gamma \widehat{\beta}_\Gamma(u_{0|\Gamma}) \\
&\quad + \int_Q (g_\lambda - T_\lambda \pi(u_\lambda))\, \partial_t u_\lambda + \int_\Sigma (g_{\Gamma\lambda} - T_\lambda \pi(v_\lambda))\, \partial_t v_\lambda\,.
\end{aligned}$$

Now, since $u \in H^1(0,T;H)$, $v \in H^1(0,T;H_\Gamma)$, $\xi \in \beta(u)$ a.e. in Q and $\xi_\Gamma \in \beta_\Gamma(v)$ a.e. in Σ, by [2, Lemma 3.3] the functions

$$t \mapsto \int_\Omega \widehat{\beta}(u(t))\,, \qquad t \mapsto \int_\Gamma \widehat{\beta}_\Gamma(v(t))\,,$$

are absolutely continuous on $[0,T]$ with derivatives given by $(\xi, \partial_t u)_H$ and $(\xi_\Gamma, \partial_t v)_{H_\Gamma}$, respectively. Moreover, the strong convergence of u_λ and v_λ together with [2, Prop. 2.11] ensure that

$$\int_\Omega \widehat{\beta}_\lambda(u_\lambda(T)) \to \int_\Omega \widehat{\beta}(u(T))\,, \qquad \int_\Omega \widehat{\beta}_{\Gamma\lambda}(v_\lambda(T)) \to \int_\Omega \widehat{\beta}_\Gamma(v(T))\,.$$

Hence, by (4.17)–(4.22) and the weak lower semicontinuity of the convex integrands, we infer

$$
\begin{aligned}
&\limsup_{\lambda\searrow 0}\left[\int_Q \alpha_\lambda(\partial_t u_\lambda)\partial_t u_\lambda + \int_\Sigma \alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda\right]\\
&\le \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma u_{0|\Gamma}|^2 + \int_\Omega \widehat{\beta}(u_0) + \int_\Gamma \widehat{\beta}_\Gamma(u_{0|\Gamma}) + \int_Q (g-\pi(u))\,\partial_t u\\
&\quad + \int_\Sigma (g_\Gamma - \pi(v))\,\partial_t v - \liminf_{\lambda\searrow 0}\left[\int_Q |\nabla\mu_\lambda|^2 + \frac{1}{2}\int_\Omega |\nabla u_\lambda(T)|^2 + \int_\Omega \widehat{\beta}(u_\lambda(T)) + \int_\Gamma \widehat{\beta}_\Gamma(v_\lambda(T))\right]\\
&\le \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma u_{0|\Gamma}|^2 + \int_\Omega \widehat{\beta}(u_0) + \int_\Gamma \widehat{\beta}_\Gamma(u_{0|\Gamma}) + \int_Q (g-\pi(u))\,\partial_t u\\
&\quad + \int_\Sigma (g_\Gamma - \pi(v))\,\partial_t v - \int_Q |\nabla\mu|^2 - \frac{1}{2}\int_\Omega |\nabla u(T)|^2 - \int_\Omega \widehat{\beta}(u(T)) - \int_\Gamma \widehat{\beta}_\Gamma(v(T))
\end{aligned}
$$

Now, testing equation (4.25) by μ, the first equation in (4.26) by $\partial_t u$ and taking the difference, it is a standard matter to check that the right hand side of the last inequality coincides with

$$\int_Q \eta\partial_t u + \int_\Sigma \eta_\Gamma \partial_t v\,,$$

so that

$$\limsup_{\lambda\searrow 0}\left[\int_Q \alpha_\lambda(\partial_t u_\lambda)\partial_t u_\lambda + \int_\Sigma \alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda\right] \le \int_Q \eta\partial_t u + \int_\Sigma \eta_\Gamma\partial_t v\,.$$

This implies by a classical argument on maximal monotone operators that $\eta \in \alpha(\partial_t u)$ a.e. in Q and $\eta_\Gamma \in \alpha_\Gamma(\partial_t v)$ a.e. in Σ. This concludes the proof of Theorem 2.1.

5 The second existence result

We present here the proof of the second main result of the paper. Recall that we are working now under the stronger conditions (2.18)–(2.21), so that the regularity of the approximated solutions is the one given by Lemma 3.4.

5.1 The first estimate

We proceed as in Section 4.1, using the monotonicity of α_λ on the left hand side. For every $t\in[0,T]$ we obtain

$$
\begin{aligned}
&\lambda\int_{Q_t}|\mu_\lambda|^2 + \int_{Q_t}|\nabla\mu_\lambda|^2 + \lambda\int_{Q_t}|\partial_t u_\lambda|^2 + \frac{\lambda}{2}\int_\Omega |u_\lambda(t)|^2 + \frac{1}{2}\int_\Omega|\nabla u_\lambda(t)|^2 + \int_\Omega \widehat{\beta}_\lambda(u_\lambda(t))\\
&\qquad + \lambda\int_{\Sigma_t}|\partial_t v_\lambda|^2 + \int_{\Sigma_t}\alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda + \frac{\varepsilon}{2}\int_\Sigma |\nabla_\Gamma v_\lambda(t)|^2 + \int_\Sigma \widehat{\beta}_{\Gamma\lambda}(v_\lambda(t))\\
&\quad \le \frac{\lambda}{2}\int_\Omega |u_0|^2 + \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \frac{\varepsilon}{2}\int_\Sigma |\nabla_\Gamma u_{0|\Gamma}|^2 + \int_\Omega \widehat{\beta}_\lambda(u_0) + \int_\Sigma \widehat{\beta}_{\Gamma\lambda}(u_{0|\Gamma})\\
&\qquad + \int_{Q_t}(g_\lambda - T_\lambda\pi(u_\lambda))\,\partial_t u_\lambda + \int_{\Sigma_t}(g_{\Gamma\lambda} - T_\lambda\pi_\Gamma(v_\lambda))\,\partial_t v_\lambda\,.
\end{aligned}
$$

Now, in order to handle the terms on the boundary, we proceed exactly as in Section 4.1 using the coercivity of α_Γ on the left hand side combined with the weighted Young inequality on the last term in right-hand side. Furthermore, thanks to hypothesis (2.18) and (3.3), integrating by parts and taking into account that $\lambda \in (0,1)$ and the Lipschitz continuity of T_λ and π, we have

$$\begin{aligned}
&\int_{Q_t} (g_\lambda - T_\lambda\pi(u_\lambda))\,\partial_t u_\lambda = \int_{Q_t} g_\lambda \partial_t u_\lambda + \int_{Q_t} T_\lambda\pi(u_\lambda)\,(\lambda\mu_\lambda - \Delta\mu_\lambda)\\
&= -\int_0^t \langle \partial_t g(s), u_\lambda(s)\rangle_V \,ds + \int_\Omega g_\lambda(t)u_\lambda(t) - \int_\Omega g_\lambda(0)u_0\\
&\qquad + \lambda\int_{Q_t} T_\lambda\pi(u_\lambda)\mu_\lambda + \int_{Q_t} \nabla T_\lambda\pi(u_\lambda)\cdot\nabla\mu_\lambda\\
&\le \frac{1}{2}\|g\|^2_{H^1(0,T;V^*)} + \frac{1}{2}\|u_\lambda\|_{L^2(0,t;V)} + \frac{1}{4\delta}\|g\|^2_{L^\infty(0,T;V^*)} + \delta\|u_\lambda(t)\|^2_V + \|g\|_{L^\infty(0,T;V^*)}\|u_0\|_V\\
&\qquad + \frac{\lambda}{2}\int_{Q_t}|\mu_\lambda|^2 + \frac{1}{2}\int_{Q_t}|\nabla\mu_\lambda|^2 + \frac{C_\pi^2+1}{2}\|u_\lambda\|_{L^2(0,t;V)}
\end{aligned}$$

for every $\delta > 0$. Now, we write

$$\|u_\lambda\|^2_V = \|\nabla u_\lambda\|^2_H + \|u_\lambda\|^2_H\,,$$

where the first term can be handled using Gronwall's lemma and the second by (4.2). Hence, choosing δ small enough and rearranging the terms, thanks to the Gronwall lemma we still obtain the estimates (4.3)–(4.4) and (4.6)–(4.9).

5.2 The second estimate

First of all, in order to perform this estimate, we need to identify the initial values at $t = 0$ of $\partial_t u_\lambda$, $\partial_t v_\lambda$ and μ_λ: to this end, it is natural to require that these satisfy the system (3.3)–(3.6) at $t = 0$. We have the following result.

Lemma 5.1. *There is a unique triplet $(u'_{0\lambda}, v'_{0\lambda}, \mu_{0\lambda}) \in H \times H_\Gamma \times W_{\mathbf{n}}$ such that*

$$\begin{cases}
u'_{0\lambda} + \lambda\mu_{0\lambda} - \Delta\mu_{0\lambda} = 0 & \text{in } \Omega\,,\\
\mu_{0\lambda} = \lambda u'_{0\lambda} + \alpha_\lambda(u'_{0\lambda}) + \lambda u_0 - \Delta u_0 + \beta_\lambda(u_0) + T_\lambda\pi(u_0) - g_\lambda(0) & \text{in } \Omega\,,\\
\lambda v'_{0\lambda} + \alpha_{\Gamma\lambda}(v'_{0\lambda}) + \partial_{\mathbf{n}}u_0 - \varepsilon\Delta_\Gamma u_{0|\Gamma} + \beta_{\Gamma\lambda}(u_{0|\Gamma}) + T_\lambda\pi_\Gamma(u_{0|\Gamma}) = g_{\Gamma\lambda}(0) & \text{in } \Gamma\,.
\end{cases}$$

Furthermore, there exists $C > 0$, independent of λ, such that

$$\lambda\|\mu_{0\lambda}\|^2_H + \|\nabla\mu_{0\lambda}\|^2_H + \lambda\|u'_{0\lambda}\|^2_H + \left\|\widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(u'_{0\lambda}))\right\|_{L^1(\Omega)} + \lambda\|v'_{0\lambda}\|^2_H + \left\|\widehat{\alpha_{\Gamma\lambda}^{-1}}(\alpha_{\Gamma\lambda}(v'_{0\lambda}))\right\|_{L^1(\Gamma)} \le C\,.$$

Proof. Setting $z_{0\lambda} := g_\lambda(0) - T_\lambda\pi(u_0) - \beta_\lambda(u_0) + \Delta u_0 - \lambda u_0$ and $w_{0\lambda} := g_{\Gamma\lambda}(0) - T_\lambda\pi_\Gamma(u_{0|\Gamma}) - \beta_{\Gamma\lambda}(u_{0|\Gamma}) + \varepsilon\Delta_\Gamma u_{0|\Gamma} - \partial_{\mathbf{n}}u_0$, by the hypothesis (2.20) we have $(z_{0\lambda}, w_{0\lambda}) \in \mathcal{H}$. Moreover, the system which we are interested in reduces to $A_\lambda(u'_{0\lambda}, v'_{0\lambda}) = (z_{0\lambda}, w_{0\lambda})$, with $\mu_{0\lambda} = -G_\lambda^{-1}(u'_{0\lambda})$. Since A_λ is bi-Lipschitz continuous on $\mathcal{H}$, there is a unique pair $(u'_{0\lambda}, v'_{0\lambda}) \in \mathcal{H}$ solving the system with $\mu_{0\lambda} = -G_\lambda^{-1}(u'_{0\lambda}) \in W_{\mathbf{n}}$ by definition of G_λ. Furthermore, testing the first equation by $\mu_{0\lambda}$, the second by $u'_{0\lambda}$, taking the difference and recalling the hypotheses (2.20)–(2.21), we have

$$\begin{aligned}
\lambda\int_\Omega|\mu_{0\lambda}|^2 + \int_\Omega|\nabla\mu_{0\lambda}|^2 + \lambda\int_\Omega|u'_{0\lambda}|^2 + \int_\Omega \alpha_\lambda(u'_{0\lambda})u'_{0\lambda} + \lambda\int_\Gamma|v'_{0\lambda}| + \int_\Gamma \alpha_{\Gamma\lambda}(v'_{0\lambda})v'_{0\lambda}\\
\le \int_\Omega z_{0\lambda}u'_{0\lambda} + \int_\Gamma w_{0\lambda}v'_{0\lambda}\,.
\end{aligned}$$

On the left hand side, we use (2.2) and the fact that $\alpha_{\Gamma\lambda} \in \alpha_\Gamma(J_{\Gamma\lambda})$ to infer that

$$\int_\Gamma \alpha_{\Gamma\lambda}(v'_{0\lambda})v'_{0\lambda} \geq b_1 \int_\Gamma |J_{\Gamma\lambda}v'_{0\lambda}|^2 + \lambda \int_\Gamma |\alpha_{\Gamma\lambda}(v'_{0\lambda})|^2 - b_2|\Gamma|\,,$$

while on the right hand side, since $v'_{0\lambda} - J_{\Gamma\lambda}v'_{0\lambda} = \lambda\alpha_{\Gamma\lambda}(v'_{0\lambda})$, for every $\delta > 0$ we have

$$\int_\Gamma w_{0\lambda}v'_{0\lambda} \leq \frac{1}{2\delta}\int_\Gamma |w_{0\lambda}|^2 + \frac{\delta}{2}\int_\Gamma |v'_{0\lambda}|^2 \leq \frac{1}{2\delta}\int_\Gamma |w_{0\lambda}|^2 + \delta\lambda^2 \int_\Gamma |\alpha_{\Gamma\lambda}(v'_{0\lambda})|^2 + \delta\int_\Gamma |J_{\Gamma\lambda}v'_{0\lambda}|^2\,,$$

where $w_{0\lambda}$ is bounded in H_Γ uniformly in λ by (2.19)–(2.20). Now, recall that either (2.8) or (2.22) is in order: we distinguish the two cases. Under hypothesis (2.8), we have, on the left hand side,

$$\int_\Omega \alpha_\lambda(u'_{0\lambda})u'_{0\lambda} \geq a_1 \int_\Omega |J_\lambda u'_{0\lambda}|^2 + \lambda \int_\Omega |\alpha_\lambda(u'_{0\lambda})|^2 - a_2|\Omega|$$

while on the right hand side, since $u'_{0\lambda} - J_\lambda u'_{0\lambda} = \lambda\alpha_\lambda(u'_{0\lambda})$,

$$\int_\Omega z_{0\lambda}u'_{0\lambda} \leq \frac{1}{2\delta}\int_\Omega |z_{0\lambda}|^2 + \frac{\delta}{2}\int_\Omega |u'_{0\lambda}|^2 \leq \frac{1}{2\delta}\int_\Omega |z_{0\lambda}|^2 + \delta\lambda^2 \int_\Omega |\alpha_\lambda(u'_{0\lambda})|^2 + \delta\int_\Omega |J_\lambda u'_{0\lambda}|^2\,.$$

Since $z_{0\lambda}$ is uniformly bounded in H by (2.19)–(2.21), choosing $\delta > 0$ sufficiently small and rearranging the terms yields the desired estimate. Otherwise, if (2.22) is in order, then $z_{0\lambda}$ is uniformly bounded also in V by (2.22) and we can estimate the term on the right hand side in the duality V–V^*:

$$\int_\Omega z_{0\lambda}u'_{0\lambda} = -\lambda\int_\Omega z_{0\lambda}\mu_{0\lambda} - \int_\Omega \nabla z_{0\lambda}\cdot\nabla\mu_{0\lambda} \leq \frac{\lambda}{2}\int_\Omega |\mu_{0\lambda}|^2 + \frac{1}{2}\int_\Omega |\nabla\mu_{0\lambda}|^2 + \|z_{0\lambda}\|_V^2\,,$$

from which the desired estimate follows rearranging the terms. Note that we have used the fact that

$$\widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(u'_{0\lambda})) \leq \widehat{\alpha}_\lambda(u'_{0\lambda}) + \widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(u'_{0\lambda})) = \alpha_\lambda(u'_{0\lambda})u'_{0\lambda}$$

and the equivalent statement for α_Γ. □

We are ready now to perform the estimate. The intuitive idea is to test equation (3.3) by $\partial_t\mu_\lambda$, the time-derivative of equation (3.4) by $\partial_t u_\lambda$, take the difference and integrate. However, the regularity of the approximated solutions does not allow us to do so. Consequently, we prove by hand that the resulting estimate holds anyway. To this end, we proceed in a technical way through a discrete-time argument as in [1, Section 5.2], to which we refer for further detail; however, we avoid any detailed computation for sake of conciseness.

Fix $t \in [0,T]$ and set, for every $n \in \mathbb{N}$, $\tau_n := \frac{t}{n}$ and $t := i\tau_n$ for $i \in \{0,\ldots,n\}$. Now, by the regularities given by Lemma 3.4, we note that (3.3)–(3.4) and (3.6) hold for every $s \in [0,T]$. Hence, it makes sense to test (3.3) at time t_n^i by $\mu_\lambda(t_n^i) - \mu_\lambda(t_n^{i-1})$, the difference between (3.4) at t_n^i and at t_n^{i-1} by $\partial_t u_\lambda(t_n^i)$, and take the difference. Moreover, since $\partial\widehat{\alpha_\lambda^{-1}} = \alpha_\lambda^{-1}$, for every $i = 1,\ldots,n$, we have that

$$\left(\alpha_\lambda(\partial_t u_\lambda(t_n^i)) - \alpha_\lambda(\partial_t u_\lambda(t_n^{i-1}))\right)\partial_t u_\lambda(t_n^i) \geq \widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(\partial_t u_\lambda(t_n^i))) - \widehat{\alpha_\lambda^{-1}}(\alpha_{\Gamma\lambda}(\partial_t u_\lambda(t_n^{i-1})))$$

and similarly for the terms in $\alpha_{\Gamma\lambda}$. Hence, integrating by parts and summing over i yields (after some technical computations analogue to the ones in [1, Section 5.2]),

$$
\begin{aligned}
&\frac{\lambda}{2}\int_\Omega |\mu_\lambda(t)| + \frac{1}{2}\int_\Omega |\nabla\mu_\lambda(t)|^2 + \frac{\lambda}{2}\int_\Omega |\partial_t u_\lambda(t)|^2 + \int_\Omega \widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(\partial_t u_\lambda(t)))\\
&\quad + \lambda\int_{Q_t} |\partial_t u_\lambda|^2 + \int_{Q_t} |\nabla\partial_t u_\lambda|^2 + \int_{Q_t} \beta_\lambda'(u_\lambda)|\partial_t u_\lambda|^2 + \frac{\lambda}{2}\int_\Gamma |\partial_t v_\lambda(t)|^2\\
&\quad + \int_\Gamma \widehat{\alpha_{\Gamma\lambda}^{-1}}(\alpha_{\Gamma\lambda}(\partial_t v_\lambda(t))) + \varepsilon\int_{\Sigma_t} |\nabla_\Gamma\partial_t v_\lambda|^2 + \int_{\Sigma_t} \beta_{\Gamma\lambda}'(v_\lambda)|\partial_t v_\lambda|^2\\
&\leq \frac{\lambda}{2}\int_\Omega |\mu_{0\lambda}|^2 + \frac{1}{2}\int_\Omega |\nabla\mu_{0\lambda}|^2 + \frac{\lambda}{2}\int_\Omega |u_{0\lambda}'|^2 + \int_\Omega \widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(u_{0\lambda}')) + \frac{\lambda}{2}\int_\Gamma |v_{0\lambda}'|^2 + \int_\Gamma \widehat{\alpha_{\Gamma\lambda}^{-1}}(\alpha_{\Gamma\lambda}(v_{0\lambda}'))\\
&\quad + \int_{Q_t} \partial_t g_\lambda \partial_t u_\lambda - \int_{Q_t} T_\lambda'(\pi(u_\lambda))\pi'(u_\lambda)|\partial_t u_\lambda|^2 + \int_{\Sigma_t} \partial_t g_{\Gamma\lambda}\partial_t v_\lambda - \int_{\Sigma_t} T_\lambda'(\pi_\Gamma(v_\lambda))\pi_\Gamma'(v_\lambda)|\partial_t v_\lambda|^2\,.
\end{aligned}
$$

Now, the first six terms and the last term on the right-hand side are bounded uniformly in λ thanks to Lemma 5.1 and the estimate (4.6), respectively (recall that $|T_\lambda'| \leq 1$ and $|\pi_\Gamma'| \leq C_{\pi_\Gamma}$). Moreover, the three remaining terms can be estimated using the duality V–V^*, the assumption (2.18), the Young inequality and (4.2) by

$$
C_\delta + \delta\left\|\partial_t u_\lambda\right\|_{L^2(0,t;V)}^2 \leq C_\delta + \delta\left\|\nabla\partial_t u_\lambda\right\|_{L^2(0,t;H)}^2 + \delta\lambda^2\left\|\mu_\lambda\right\|_{L^2(0,t;H)}^2\,,
$$

for every $\delta > 0$. Hence, choosing δ sufficiently small, we deduce that there is a positive constant C such that

$$
\left\|\nabla\partial_t u_\lambda\right\|_{L^2(0,T;H)} + \lambda^{1/2}\left\|\partial_t u_\lambda\right\|_{L^\infty(0,T;H)} \leq C\,, \tag{5.1}
$$

$$
\left\|\nabla_\Gamma\partial_t v_\lambda\right\|_{L^2(0,T;H_\Gamma)} + \lambda^{1/2}\left\|\partial_t v_\lambda\right\|_{L^\infty(0,T;H_\Gamma)} \leq C\,, \tag{5.2}
$$

$$
\left\|\nabla\mu_\lambda\right\|_{L^\infty(0,T;H)} + \lambda^{1/2}\left\|\mu_\lambda\right\|_{L^\infty(0,T;H)} \leq C\,, \tag{5.3}
$$

$$
\left\|\widehat{\alpha_\lambda^{-1}}(\alpha_\lambda(\partial_t u_\lambda))\right\|_{L^\infty(0,T;L^1(\Omega))} + \left\|\widehat{\alpha_{\Gamma\lambda}^{-1}}(\alpha_{\Gamma\lambda}(\partial_t v_\lambda))\right\|_{L^\infty(0,T;L^1(\Gamma))} \leq C \tag{5.4}
$$

Thanks to (5.1)–(5.2), condition (4.2) and (4.3) and (4.7), it follows that $\partial_t u_\lambda$ and $\partial_t v_\lambda$ are uniformly bounded in $L^2(0,T;V)$ and $L^2(0,T;V_\Gamma)$, respectively. Moreover, integrating (2.7) it easily follows that $\widehat{\alpha}_\lambda$ and $\widehat{\alpha}_{\Gamma\lambda}$ are uniformly bounded in λ from above by a quadratic function: hence, $\widehat{\alpha_\lambda^{-1}} = (\widehat{\alpha}_\lambda)^*$ and $\widehat{\alpha_{\Gamma\lambda}^{-1}} = (\widehat{\alpha}_{\Gamma\lambda})^*$ are uniformly bounded from below by a quadratic function. Consequently, from the estimate (5.4) we infer also that

$$
\left\|\alpha_\lambda(\partial_t u_\lambda)\right\|_{L^\infty(0,T;H)} + \left\|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)\right\|_{L^\infty(0,T;H_\Gamma)} \leq C\,. \tag{5.5}
$$

Moreover, from the coercivity of α_Γ and the Young inequality, we have

$$
b_1|J_{\Gamma\lambda}\partial_t v_\lambda|^2 - b_2 \leq \alpha_{\Gamma\lambda}(\partial_t v_\lambda)J_{\Gamma\lambda}\partial_t v_\lambda \leq \frac{b_1}{2}|J_{\Gamma\lambda}\partial_t v_\lambda|^2 + \frac{1}{2b_1}|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)|^2,
$$

so that by (5.5) we deduce that

$$
\left\|J_{\Gamma\lambda}\partial_t v_\lambda\right\|_{L^\infty(0,T;H)} \leq C\,. \tag{5.6}
$$

Finally, arguing exactly as in Section 4.2 but using the stronger estimates (5.1)–(5.5), it is readily seen that $(\mu_\lambda)_\Omega$ is uniformly bounded in $L^\infty(0,T)$, so that by (5.3) we have

$$\|\mu_\lambda\|_{L^\infty(0,T;V)} \leq C\,.$$

Moreover, thanks to (5.1) and (5.3), by comparison in (3.3) and elliptic regularity we have

$$\|\mu_\lambda\|_{L^\infty(0,T;V)\cap L^2(0,T;W_{\mathbf{n}}\cap H^3(\Omega))} + \|\partial_t u_\lambda\|_{L^\infty(0,T;V^*)} \leq C\,. \tag{5.7}$$

It is clear that under the assumption (2.8), the same argument ensures that $J_\lambda \partial_t u_\lambda$ is uniformly bounded in $L^\infty(0,T;H)$ as well, hence also μ_λ in $L^\infty(0,T;W_{\mathbf{n}})$ form (3.3), from which the last sentence of Theorem 2.3 follows.

5.3 The third estimate

For every $t\in[0,T]$, we test equation (3.4) by $-\Delta u_\lambda(t)$ and integrate by parts:

$$\begin{aligned}\int_\Omega |\Delta u_\lambda(t)|^2 + \int_\Omega \beta_\lambda'(u_\lambda(t))|\nabla u_\lambda(t)|^2 + \varepsilon\int_\Gamma \beta_\lambda'(v_\lambda(t))|\nabla_\Gamma v_\lambda(t)|^2 + \int_\Gamma \beta_\lambda(v_\lambda(t))\beta_{\Gamma\lambda}(v_\lambda(t))&\\ = -\int_\Omega \left(g_\lambda(t) - T_\lambda\pi(u_\lambda(t)) - \lambda u_\lambda(t) + \mu_\lambda(t) - \lambda\partial_t u_\lambda(t) - \alpha_\lambda(\partial_t u_\lambda(t))\right)\Delta u_\lambda(t)&\\ - \int_\Gamma \left(g_{\Gamma\lambda}(t) - T_\lambda\pi_\Gamma(v_\lambda(t)) - \lambda\partial_t v_\lambda(t) - \alpha_{\Gamma\lambda}(\partial_t v_\lambda(t))\right)\beta_\lambda(v_\lambda(t))\,.&\end{aligned}$$

Thanks to (2.4) the estimates (5.1)–(5.5), the terms in brackets on the right hand side are bounded uniformly in λ. Hence, using the weighted Young inequality and the hypothesis (2.3) as in Section 4.3, we infer that

$$\|\Delta u_\lambda\|_{L^\infty(0,T;H)} + \|\beta_\lambda(u_\lambda)\|_{L^\infty(0,T;H_\Gamma)} \leq C\,. \tag{5.8}$$

By comparison in (3.4) we deduce that

$$\|\beta_\lambda(u_\lambda)\|_{L^\infty(0,T;H)} \leq C\,. \tag{5.9}$$

Moreover, by the classical results on elliptic regularity (see [22, Thm. 3.2]), estimate (5.8) implies, together with (4.4) and (4.9), that

$$\varepsilon^{1/2}\,\|u_\lambda\|_{L^\infty(0,T;H^{3/2}(\Omega))} + \varepsilon^{1/2}\,\|\partial_{\mathbf{n}} u_\lambda\|_{L^\infty(0,T;H_\Gamma)} \leq C \tag{5.10}$$

and, by comparison in (3.6),

$$\left\|-\varepsilon^{3/2}\Delta_\Gamma v_\lambda + \varepsilon^{1/2}\beta_{\Gamma\lambda}(v_\lambda)\right\|_{L^\infty(0,T;H_\Gamma)} \leq C\,.$$

We deduce, as usual, that

$$\varepsilon^{3/2}\,\|\Delta_\Gamma v_\lambda\|_{L^\infty(0,T;H_\Gamma)} + \varepsilon^{1/2}\,\left\|\beta_{\Gamma\lambda(v_\lambda)}\right\|_{L^\infty(0,T;H_\Gamma)} \leq C\,. \tag{5.11}$$

5.4 The passage to the limit

Taking into account (4.3)–(4.4), (4.6)–(4.9) and (5.1)–(5.11), recalling that $\varepsilon > 0$ is fixed, we infer that there are

$$
\begin{gathered}
u \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W)\,,\\
v \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;W_\Gamma)\,,\\
\mu \in L^\infty(0,T;W_{\mathbf{n}}) \cap L^2(0,T;H^3(\Omega))\,,\\
\eta\,,\xi \in L^\infty(0,T;H)\,, \qquad \eta_\Gamma\,,\xi_\Gamma \in L^\infty(0,T;H_\Gamma)\,,
\end{gathered}
$$

such that, along a subsequence that we still denote by λ for simplicity,

$$
\begin{aligned}
u_\lambda \overset{*}{\rightharpoonup} u \quad &\text{in } W^{1,\infty}(0,T;V^*) \cap L^\infty(0,T;W)\,, & u_\lambda \rightharpoonup v \quad &\text{in } H^1(0,T;V)\,,\\
v_\lambda \overset{*}{\rightharpoonup} u \quad &\text{in } W^{1,\infty}(0,T;H_\Gamma) \cap L^\infty(0,T;W_\Gamma)\,, & v_\lambda \rightharpoonup v \quad &\text{in } H^1(0,T;V_\Gamma)\,,\\
\mu_\lambda \overset{*}{\rightharpoonup} \mu \quad &\text{in } L^\infty(0,T;V)\,, & \mu_\lambda \rightharpoonup \mu \quad &\text{in } L^2(0,T;W_{\mathbf{n}} \cap H^3(\Omega))\,,\\
\alpha_\lambda(\partial_t u_\lambda) \overset{*}{\rightharpoonup} \eta \quad &\text{in } L^\infty(0,T;H)\,, & \alpha_{\Gamma\lambda}(\partial_t v_\lambda) \overset{*}{\rightharpoonup} \eta_\Gamma \quad &\text{in } L^\infty(0,T;H_\Gamma)\,,\\
\beta_\lambda(u_\lambda) \overset{*}{\rightharpoonup} \xi \quad &\text{in } L^\infty(0,T;H)\,, & \beta_{\Gamma\lambda}(v_\lambda) \overset{*}{\rightharpoonup} \xi_\Gamma \quad &\text{in } L^\infty(0,T;H_\Gamma)
\end{aligned}
$$

and

$$
\lambda u_\lambda \to 0 \quad \text{in } W^{1,\infty}(0,T;H)\,, \quad \lambda v_\lambda \to 0 \quad \text{in } W^{1,\infty}(0,T;H_\Gamma)\,, \quad \lambda \mu_\lambda \to 0 \quad \text{in } L^\infty(0,T;H)\,.
$$

At this point, it is straightforward to conclude as in Section 4.4 and Theorem 2.3 is proved.

6 The third existence result

First of all, note that all the estimates which do not involve the assumption (2.7) continue to hold also in this setting. Namely, going back to Sections 4.1 and 5.1, it is readily seen that (4.3)–(4.4), (4.6)–(4.9), (5.1)–(5.4) are satisfied.

Secondly, by (2.27), there is $\delta > 0$ such that $\pm\delta \in D(\alpha) \cap D(\alpha_\Gamma)$. Hence, by the Young inequality we have

$$
\begin{gathered}
\pm\delta\alpha_\lambda(\partial_t u_\lambda) \le \widehat{\alpha}_\lambda(\pm\delta) + \widehat{\alpha_\lambda^{-1}}(\partial_t u_\lambda) \le \widehat{\alpha}(\pm\delta) + \widehat{\alpha_\lambda^{-1}}(\partial_t u_\lambda)\,,\\
\pm\delta\alpha_{\Gamma\lambda}(\partial_t v_\lambda) \le \widehat{\alpha}_{\Gamma\lambda}(\pm\delta) + \widehat{\alpha_{\Gamma\lambda}^{-1}}(\partial_t v_\lambda) \le \widehat{\alpha}_\Gamma(\pm\delta) + \widehat{\alpha_{\Gamma\lambda}^{-1}}(\partial_t v_\lambda)\,,
\end{gathered}
$$

so that by (5.4) we deduce that

$$
\|\alpha_\lambda(\partial_t u_\lambda)\|_{L^\infty(0,T;L^1(\Omega))} + \|\alpha_{\Gamma\lambda}(\partial_t v_\lambda)\|_{L^\infty(0,T;L^1(\Gamma))} \le C\,.
$$

Furthermore, thanks to the assumptions (2.28)–(2.29), the estimates (4.4) and (4.9), as well as the continuous inclusions $V \hookrightarrow L^6(\Omega)$ and $V_\Gamma \hookrightarrow L^q(\Gamma)$ (for every $q \ge 1$), we have that for every $q \in [1,+\infty)$

$$
\|\beta_\lambda(u_\lambda)\|_{L^\infty(0,T;L^{6/5}(\Omega))} + \|\beta_{\Gamma\lambda}(v_\lambda)\|_{L^\infty(0,T;L^q(\Gamma))} \le C \tag{6.12}
$$

for every $q \geq 1$. Consequently, testing (3.4) by the constants ± 1 we get

$$\pm|\Omega|(\mu_\lambda(t))_\Omega \leq \int_\Omega |\lambda\partial_t u_\lambda + \alpha_\lambda(\partial_t u_\lambda) + \lambda u_\lambda + \beta_\lambda(u_\lambda) + T_\lambda \pi(u_\lambda)|(t) + |\Omega||(g_\lambda(t))_\Omega|$$
$$+ \int_\Gamma |\lambda\partial_t v_\lambda + \alpha_{\Gamma\lambda}(\partial_t v_\lambda) + \beta_{\Gamma\lambda}(v_\lambda) + T_\lambda \pi_\Gamma(v_\lambda)|(t) + |\Gamma||(g_{\Gamma\lambda}(t))_\Gamma|,$$

where the right-hand side is bounded in $L^\infty(0,T)$ thanks to the estimates already shown, (5.1)–(5.2) and by assumption (2.18). We infer together with (5.3) that

$$\|\mu_\lambda\|_{L^\infty(0,T;V)} \leq C.$$

Furthermore, by comparison in (3.3) and the estimates (5.1)–(5.3) that

$$\|\mu_\lambda\|_{L^\infty(0,T;V)\cap L^2(0,T;W_{\mathbf{n}}\cap H^3(\Omega))} + \|\partial_t u_\lambda\|_{L^\infty(0,T;V^*)} \leq C. \tag{6.13}$$

Again, if also (2.8) holds, the same argument ensures that $J_\lambda \partial_t u_\lambda$ is uniformly bounded in $L^\infty(0,T;H)$, hence also μ_λ in $L^\infty(0,T;W_{\mathbf{n}})$ form (3.3), from which the last sentence of Theorem 2.6 follows.

Let us focus now on the main estimate. We know that the approximated problem can be written as

$$A_\lambda(\partial_t u_\lambda, \partial_t v_\lambda) + B_\lambda(u_\lambda, v_\lambda) = (g_\lambda, g_{\Gamma\lambda}) - (T_\lambda \pi(u_\lambda), T_\lambda \pi_\Gamma(v_\lambda)),$$

where the operators A_λ and B_λ have been introduced in Section 3. Note that by (4.4) and (4.9), we have that $(u_\lambda, v_\lambda)_\lambda$ is bounded in $L^\infty(0,T;\mathcal{V})$: hence, by linearity and boundedness of the operator

$$(-\Delta, \partial_{\mathbf{n}} - \varepsilon\Delta_\Gamma) : \mathcal{V} \to \mathcal{V}^*,$$

we deduce that $(-\Delta u_\lambda, \partial_{\mathbf{n}} u_\lambda - \varepsilon \Delta v_\lambda)_\lambda$ is bounded uniformly in $L^\infty(0,T;\mathcal{V}^*)$. Moreover, since $L^{6/5}(\Omega) \hookrightarrow V^*$ and $L^{q'}(\Gamma) \hookrightarrow V_\Gamma^*$ for every $q' \in (1,+\infty]$, by (6.12) we deduce that $(\beta_\lambda(u_\lambda), \beta_{\Gamma\lambda}(v_\lambda))_\lambda$ is bounded in $L^\infty(0,T;\mathcal{V}^*)$ as well. Hence, we infer that

$$\|B_\lambda(u_\lambda, v_\lambda)\|_{L^\infty(0,T;\mathcal{V}^*)} \leq C.$$

By comparison in the equation written above we have then

$$\|(\alpha_\lambda(\partial_t u_\lambda), \alpha_{\Gamma\lambda}(\partial_t v_\lambda))_\lambda\|_{L^\infty(0,T;\mathcal{V}^*)} \leq C.$$

Let us pass to the limit. The estimates that we have collected ensure that there are

$$u \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;W),$$
$$v \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;W_\Gamma),$$
$$\mu \in L^\infty(0,T;W_{\mathbf{n}}) \cap L^2(0,T;H^3(\Omega)),$$
$$\xi \in L^\infty(0,T;L^{6/5}(\Omega)), \qquad \xi_\Gamma \in L^\infty(0,T;L^q(\Gamma)) \quad \forall\, q \in [1,+\infty),$$
$$\eta_w \in L^\infty(0,T;\mathcal{V}^*),$$

such that, along a subsequence that we still denote by λ for simplicity,

$$u_\lambda \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(0,T;V^*) \cap L^\infty(0,T;W), \qquad u_\lambda \rightharpoonup v \quad \text{in } H^1(0,T;V),$$
$$v_\lambda \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(0,T;H_\Gamma) \cap L^\infty(0,T;W_\Gamma), \qquad v_\lambda \rightharpoonup v \quad \text{in } H^1(0,T;V_\Gamma),$$
$$\mu_\lambda \overset{*}{\rightharpoonup} \mu \quad \text{in } L^\infty(0,T;V), \qquad \mu_\lambda \rightharpoonup \mu \quad \text{in } L^2(0,T;W_{\mathbf{n}} \cap H^3(\Omega)),$$
$$\beta_\lambda(u_\lambda) \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(0,T;L^{6/5}(\Omega)), \qquad \beta_{\Gamma\lambda}(v_\lambda) \overset{*}{\rightharpoonup} \xi_\Gamma \quad \text{in } L^\infty(0,T;H_\Gamma),$$
$$(\alpha_\lambda(\partial_t u_\lambda), \alpha_{\Gamma\lambda}(\partial_t v_\lambda)) \overset{*}{\rightharpoonup} \eta_w \qquad \text{in } L^\infty(0,T;\mathcal{V}^*)$$

and

$$\lambda u_\lambda \to 0 \quad \text{in } W^{1,\infty}(0,T;H)\,, \quad \lambda v_\lambda \to 0 \quad \text{in } W^{1,\infty}(0,T;H_\Gamma)\,, \quad \lambda \mu_\lambda \to 0 \quad \text{in } L^\infty(0,T;H)\,.$$

If the stronger condition (2.36) is in order, then the continuous embedding $V \hookrightarrow L^6(\Omega)$ and (4.9) imply that $(\beta_\lambda(u_\lambda))_\lambda$ is bounded in $L^\infty(0,T;H)$, from which $\xi \in L^\infty(0,T;H)$ as well. Testing the approximated equations (3.4)–(3.6) by a generic element $(\varphi,\psi) \in \mathcal{V}$, integrating by parts and letting $\lambda \to 0^+$, it is a standard matter to check that

$$\int_\Omega \mu(t)\varphi = \langle \eta_w(t), (\varphi,\psi)\rangle_{\mathcal{V}} + \int_\Omega \nabla u(t)\cdot\nabla\varphi + \int_\Omega (\xi(t) + \pi(u(t)) - g(t))\,\varphi \\ + \varepsilon\int_\Gamma \nabla_\Gamma v(t)\cdot\nabla_\Gamma\psi + \int_\Gamma (\xi_\Gamma(t) + \pi_\Gamma(v(t)) - g_\Gamma(t))\psi\,.$$

Moreover, proceeding as in the previous sections, we also have $\xi \in \beta(u)$ a.e. in Q and $\xi_\Gamma \in \beta_\Gamma(v)$ a.e. in Σ. Finally, as in Section 4.4, comparing the approximated equations (3.4)–(3.6) and the corresponding limit ones, we can infer that

$$\limsup_{\lambda\searrow 0}\left[\int_Q \alpha_\lambda(\partial_t u_\lambda)\partial_t u_\lambda + \int_\Sigma \alpha_{\Gamma\lambda}(\partial_t v_\lambda)\partial_t v_\lambda\right] \leq \int_0^T \langle \eta_w(t), (\partial_t u(t), \partial_t v(t))\rangle_{\mathcal{V}}\,dt\,,$$

which implies by a well-known criterion on maximal monotonicity that $\eta_w \in \widetilde{\alpha}_w(\partial_t u, \partial_t v)$.

7 The uniqueness result

In the hypotheses (2.37)–(2.38) of Theorem 2.10, we clearly have $\xi_i = F'(u_i) - \pi(u_i)$ and $\xi_{\Gamma i} = F'_\Gamma(v_i) - \pi_\Gamma(v_i)$ for $i = 1,2$. Now, we write the difference of the equations (2.15)–(2.17) at $i=1$ and $i=2$, test (2.15) by $\mu_1 - \mu_2$, (2.16) by $-\partial_t(u_1-u_2)$ and sum: by standard computations, the monotonicity of α and (2.38) we obtain

$$\int_{Q_t} |\nabla(\mu_1-\mu_2)|^2 + \frac{1}{2}\int_\Omega |\nabla(u_1-u_2)(t)|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma(v_1-v_2)(t)|^2 + \widetilde{b_1}\int_{\Sigma_t} |\partial_t(v_1-v_2)|^2 \\ + \int_{Q_t} (F'(u_1) - F'(u_2))\,\partial_t(u_1-u_2) + \int_{\Sigma_t} (F'_\Gamma(v_1) - F'_\Gamma(v_2))\partial_t(v_1-v_2) \leq 0$$

for every $t \in [0,T]$. We are now inspired by the argument contained in the works [14, Thm. 2.2] and [26, p. 689]: note that

$$(F'(u_1) - F'(u_2))\,\partial_t(u_1-u_2) \\ = \partial_t\left[F(u_1) - F(u_2) - F'(u_2)(u_1-u_2)\right] - \left[F'(u_1) - F'(u_2) - F''(u_2)(u_1-u_2)\right]\partial_t u_2$$

and similarly

$$(F'_\Gamma(v_1) - F'_\Gamma(v_2))\,\partial_t(v_1-v_2) \\ = \partial_t\left[F_\Gamma(v_1) - F_\Gamma(v_2) - F'_\Gamma(v_2)(v_1-v_2)\right] - \left[F'_\Gamma(v_1) - F'_\Gamma(v_2) - F''_\Gamma(v_2)(v_1-v_2)\right]\partial_t v_2\,,$$

so that

$$\int_{Q_t} |\nabla(\mu_1-\mu_2)|^2 + \frac{1}{2}\int_\Omega |\nabla(u_1-u_2)(t)|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma(v_1-v_2)(t)|^2 + \widetilde{b_1}\int_{\Sigma_t} |\partial_t(v_1-v_2)|^2 \\ + \int_\Omega \left[F(u_1) - F(u_2) - F'(u_2)(u_1-u_2)\right](t) + \int_\Gamma \left[F_\Gamma(v_1) - F_\Gamma(v_2) - F'_\Gamma(v_2)(v_1-v_2)\right](t) \\ \leq \int_{Q_t} \left[F'(u_1) - F'(u_2) - F''(u_2)(u_1-u_2)\right]\partial_t u_2 + \int_{\Sigma_t} \left[F'_\Gamma(v_1) - F'_\Gamma(v_2) - F''_\Gamma(v_2)(v_1-v_2)\right]\partial_t v_2$$

for every $t \in [0,T]$. Now, by the mean value theorem it is readily seen that

$$
\begin{aligned}
F(u_1) - F(u_2) - F'(u_2)(u_1-u_2) &\geq -C_\pi |u_1-u_2|^2\,,\\
F_\Gamma(v_1) - F_\Gamma(v_2) - F_\Gamma'(v_2)(v_1-v_2) &\geq -C_{\pi_\Gamma} |v_1-v_2|^2\,,
\end{aligned}
$$

while the usual Taylor expansion for F' yields

$$
\left[F'(u_1) - F'(u_2) - F''(u_2)(u_1-u_2)\right]\partial_t u_2 = \frac12 F'''(\tilde u_{12})|u_1-u_2|^2\partial_t u_2
$$

for a certain $\tilde u_{12}$ between u_1 and u_2. Now, recall that $\partial_t u_2 \in L^2(0,T;V) \hookrightarrow L^2(0,T;L^6(\Omega))$ and $u_i \in L^\infty(0,T;W) \hookrightarrow L^\infty(Q)$ for $i=1,2$: this implies in particular that $F'''(\tilde u_{12}) \in L^\infty(Q)$, because $F''' \in L^\infty_{loc}(\mathbb{R})$ by (2.37). Hence, recalling also that u_1-u_2 has null mean, we have that

$$
\begin{aligned}
\int_{Q_t} F'''(\tilde u_{12})|u_1-u_2|^2\partial_t u_2 &\leq \|F'''(\tilde u_{12})\|_{L^\infty(Q)} \int_0^t \|\partial_t u_2(s)\|_{L^6(\Omega)} \left\||u_1-u_2|^2(s)\right\|_{L^{6/5}(\Omega)}\,ds\\
&\leq C\int_0^t \|\partial_t u_2(s)\|_V \|\nabla(u_1-u_2)(s)\|_H^2\,ds
\end{aligned}
$$

for a certain constant $C>0$. Similarly, we obtain

$$
\int_{\Sigma_t}\left[F_\Gamma'(v_1) - F_\Gamma'(v_2) - F_\Gamma''(v_2)(v_1-v_2)\right]\partial_t v_2 \leq C\int_0^t \|\partial_t v_2(s)\|_{V_\Gamma} \|\nabla_\Gamma(v_1-v_2)(s)\|_{H_\Gamma}^2\,ds\,.
$$

Furthermore, by the Young inequality we can write (updating the constant C at each step)

$$
\begin{aligned}
C_\pi \int_\Omega |u_1-u_2|^2(t) &= 2C_\pi \int_{Q_t} \partial_t(u_1-u_2)(u_1-u_2)\\
&\leq \frac12 \|\partial_t(u_1-u_2)\|_{L^2(0,t;V^*)}^2 + C\|u_1-u_2\|_{L^2(0,t;V)}^2\\
&\leq \frac12 \int_{Q_t} |\nabla(\mu_1-\mu_2)|^2 + C\|\nabla(u_1-u_2)\|_{L^2(0,t;H)}^2
\end{aligned}
$$

and similarly

$$
C_{\pi_\Gamma}\int_\Gamma |v_1-v_2|^2(t) \leq \frac{\widetilde{b_1}}{2}\int_{Q_t} |\partial_t(v_1-v_2)|^2 + C\|\nabla_\Gamma(v_1-v_2)\|_{L^2(0,t;H_\Gamma)}^2\,.
$$

Taking into account this information and rearranging the terms yields

$$
\begin{aligned}
&\frac12\int_{Q_t}|\nabla(\mu_1-\mu_2)|^2 + \frac12\int_\Omega |\nabla(u_1-u_2)(t)|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma(v_1-v_2)(t)|^2 + \frac{\widetilde{b_1}}{2}\int_{\Sigma_t}|\partial_t(v_1-v_2)|^2\\
&\quad\leq C\int_0^t \left(1+\|\partial_t u_2(s)\|_V\right)\|\nabla(u_1-u_2)(s)\|_H^2\,ds\\
&\qquad + C\int_0^t \left(1+\|\partial_t v_2(s)\|_{V_\Gamma}\right)\|\nabla_\Gamma(v_1-v_2)(s)\|_{H_\Gamma}^2\,ds \qquad \forall\, t\in[0,T]\,,
\end{aligned}
$$

and the thesis follows from the Gronwall lemma.

In order to prove the second part of the theorem, we proceed in exactly the same way: we test (2.15) by $\mu_1-\mu_2$, (2.35) by -$(\partial_t(u_1-u_2), \partial_t(v_1-v_2)) \in \mathcal{V}$ and sum. The only difference here is that the estimate on the term involving F''' has to performed using the weaker regularity of the solutions and the hypothesis (2.39), together with the fact that $V \hookrightarrow L^6(\Omega)$, as follows:

$$\begin{aligned}
&\int_{Q_t} F'''(\tilde{u}_{12})(u_1-u_2)\partial_t u_2 \\
&\qquad \leq M \left\| |Q| + |u_1|^3 + |u_2|^3 \right\|_{L^\infty(0,T;H)} \int_0^t \|\partial_t u_2(s)\|_{L^6(\Omega)} \left\| |u_1-u_2|^2(s) \right\|_{L^3(\Omega)} ds \\
&\qquad \leq C \left(1 + \|u_1\|^2_{L^\infty(0,T;V)} + \|u_2\|^2_{L^\infty(0,T;V)}\right) \int_0^t \|\partial_t u_2(s)\|_V \|\nabla(u_1-u_2)(s)\|^2_H \, ds
\end{aligned}$$

Similarly, the term involving F_Γ''' is handled using (2.40) and the inclusion $V_\Gamma \hookrightarrow L^q(\Gamma)$ for every $q \in [1, +\infty)$.

8 The asymptotic as $\varepsilon \searrow 0$

For every $\varepsilon > 0$, the septuple $(u_\varepsilon, v_\varepsilon, \mu_\varepsilon, \eta_\varepsilon, \xi_\varepsilon, \eta_{\Gamma\varepsilon}, \xi_{\Gamma\varepsilon})$ is the solution satisfying (2.9)–(2.17) given by Theorem 2.1. Hence, recalling how such solutions were built from the approximated ones, all the estimates that we performed in Section 4 (and that are ε-independent) are preserved. In particular, going back to Section 4 and taking (2.42) into account, it is readily seen that

$$\begin{gathered}
\|u_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} + \|v_\varepsilon\|_{L^\infty(0,T;H^{1/2}(\Gamma))\cap H^1(0,T;H_\Gamma)} + \varepsilon^{1/2}\|v_\varepsilon\|_{L^\infty(0,T;V_\Gamma)} \leq C\,, \\
\|\mu_\varepsilon\|_{L^2(0,T;W_{\mathbf{n}})} \leq C\,, \\
\|\eta_\varepsilon\|_{L^2(0,T;H)} + \|\eta_{\Gamma\varepsilon}\|_{L^2(0,T;H_\Gamma)} + \|\xi_\varepsilon\|_{L^2(0,T;H)} \leq C\,, \\
\|\Delta u_\varepsilon\|_{L^2(0,T;H)} + \|\partial_{\mathbf{n}} u_\varepsilon - \varepsilon\Delta_\Gamma v_\varepsilon + \xi_{\Gamma\varepsilon}\|_{L^2(0,T;H_\Gamma)} \leq C\,.
\end{gathered}$$

By the classical results on elliptic regularity, we can only infer that

$$\|\partial_{\mathbf{n}} u_\varepsilon\|_{L^2(0,T;H^{-1/2}(\Gamma))} + \varepsilon^{1/2}\|\partial_{\mathbf{n}} u_\varepsilon\|_{L^2(0,T;H_\Gamma)} \leq c\,.$$

Taking into account that $-\Delta_\Gamma : V_\Gamma \to V_\Gamma^*$ is continuous and monotone, we also have that

$$\varepsilon^{1/2}\|\Delta_\Gamma v_\varepsilon\|_{L^\infty(0,T;V_\Gamma^*)} + \varepsilon^{3/2}\|\Delta_\Gamma v_\varepsilon\|_{L^2(0,T;H_\Gamma)} \leq c\,,$$

which yields by interpolation

$$\varepsilon\|\Delta v_\varepsilon\|_{L^2(0,T;H^{-1/2}(\Gamma))} \leq c\,,$$

hence also, by comparison,

$$\|\xi_{\Gamma\varepsilon}\|_{L^2(0,T;H^{-1/2}(\Gamma))} \leq c\,.$$

It readily seen that, along a subsequence $(\varepsilon_n)_n$, the weak convergences of Theorem 2.11 hold. Furthermore, by the classical compactness results [31, Cor. 4, p. 85] we also have

$$u_{\varepsilon_n} \to u \quad \text{in } C^0([0,T];H)\,, \qquad v_{\varepsilon_n} \to v \quad \text{in } C^0([0,T];H_\Gamma)\,,$$

which yields $\xi \in \beta(u)$ a.e. in Q by the strong-weak closure of β. Passing to the weak limit as $n \to \infty$ in (2.15)–(2.17) we deduce that $(u, v, \mu, \eta, \xi, \eta_\Gamma, \xi_\Gamma)$ satisfies the limit equations stated in Theorem 2.11. Moreover, testing (2.15) by μ_ε, (2.16) by $-\partial_t u_\varepsilon$ and summing we get

$$\int_Q |\nabla\mu_\varepsilon|^2 + \int_Q \eta_\varepsilon \partial_t u_\varepsilon + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon(T)|^2 + \int_\Omega \widehat{\beta}(u_\varepsilon(T))$$
$$+ \int_\Sigma \eta_{\Gamma\varepsilon}\partial_t v_\varepsilon + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma v_\varepsilon(T)|^2 + \int_\Gamma \widehat{\beta}_\Gamma(v_\varepsilon(T)) = \frac{1}{2}\int_\Omega |\nabla u_0^\varepsilon|^2 + \frac{\varepsilon}{2}\int_\Gamma |\nabla_\Gamma u_0^\varepsilon|^2$$
$$+ \int_\Omega \widehat{\beta}(u_0^\varepsilon) + \int_\Gamma \widehat{\beta}_\Gamma(u_0^\varepsilon) + \int_Q (g - \pi(u_\varepsilon))\partial_t u_\varepsilon + \int_\Sigma (g_\Gamma - \pi_\Gamma(v_\varepsilon))\partial_t v_\varepsilon\,,$$

from which, by standard weak lower semicontinuity results, the convergence $u_0^\varepsilon \to u_0$ in V and the estimate (2.42),

$$\limsup_{n\to\infty}\left(\int_Q \eta_\varepsilon\partial_t u_\varepsilon + \int_\Sigma \eta_{\Gamma\varepsilon}\partial_t v_\varepsilon\right) \le \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \int_\Omega \widehat{\beta}(u_0) + \int_\Gamma \widehat{\beta}_\Gamma(u_0)$$
$$- \int_Q |\nabla\mu|^2 - \frac{1}{2}\int_\Omega |\nabla u(T)|^2 - \int_\Omega \widehat{\beta}(u(T)) - \int_\Gamma \widehat{\beta}_\Gamma(v(T))$$
$$+ \int_Q (g - \pi(u))\partial_t u + \int_\Sigma (g_\Gamma - \pi_\Gamma(v))\partial_t v\,.$$

Now, performing the analogue estimate on the limiting equations, we easily deduce that the right-hand side coincides with

$$\int_Q \eta\partial_t u + \int_\Sigma \eta_\Gamma \partial_t v\,.$$

Hence, we also have that $\eta \in \alpha(\partial_t u)$ a.e. in Q and $\eta_\Gamma \in \alpha_\Gamma(\partial_t v)$ a.e. in Σ. It remains to prove that $\xi_\Gamma \in \beta_{\Gamma w}(v)$ a.e. in $(0,T)$. To this end, we test (2.15) by $\mathcal{N}(u_\varepsilon - (u_0^\varepsilon)_\Omega)$, (2.16) by $-(u_\varepsilon - (u_0^\varepsilon)_\Omega)$, and sum:

$$\|\nabla\mathcal{N}(u_\varepsilon(T) - (u_0^\varepsilon)_\Omega)\|_H^2 + \int_Q \eta_\varepsilon(u_\varepsilon - (u_0^\varepsilon)_\Omega) + \int_Q |\nabla u_\varepsilon|^2 + \int_Q \xi_\varepsilon(u_\varepsilon - (u_0^\varepsilon)_\Omega)$$
$$+ \int_\Sigma \eta_{\Gamma\varepsilon}(v_\varepsilon - (u_0^\varepsilon)_\Omega) + \varepsilon\int_\Sigma |\nabla_\Gamma v_\varepsilon|^2 + \int_\Sigma \xi_{\Gamma\varepsilon}(v_\varepsilon - (u_0^\varepsilon)_\Omega)$$
$$= \int_Q (g - \pi(u_\varepsilon))(u_\varepsilon - (u_0^\varepsilon)_\Omega) + \int_\Sigma (g_\Gamma - \pi_\Gamma(v_\varepsilon))(v_\varepsilon - (u_0^\varepsilon)_\Omega)\,.$$

Now, recalling that $u_\varepsilon - (u_0^\varepsilon)_\Omega$ has null mean and that $u_\varepsilon \to u$ in $C^0([0,T];H)$, we have in particular that $u_\varepsilon(T) - (u_0^\varepsilon)_\Omega \to u(T) - (u_0)_\Omega$ in V^*, hence also, by the properties of $\mathcal{N}$, $\mathcal{N}(u(T)_\varepsilon - (u_0)_\Omega) \to \mathcal{N}(u(T) - (u_0)_\Omega)$ in V. Furthermore, using the convergences already proved and the weak lower semicontinuity of the norms, we infer that

$$\limsup_{\varepsilon\searrow 0}\int_\Sigma \xi_{\Gamma\varepsilon}v_\varepsilon \le \int_Q (g - \pi(u))(u - (u_0)_\Omega) + \int_\Sigma (g_\Gamma - \pi_\Gamma(v))(v - (u_0)_\Omega) - \int_Q \eta(u - (u_0)_\Omega)$$
$$+ \int_\Sigma \xi_\Gamma(u_0)_\Omega - \|\nabla\mathcal{N}(u(T) - (u_0)_\Omega)\|_H^2 - \int_Q |\nabla u|^2 - \int_Q \xi(u - (u_0)_\Omega) - \int_\Sigma \eta_\Gamma(v - (u_0)_\Omega)\,.$$

As before, performing the same estimate on the limiting equations, we see that the right-hand side coincides with

$$\int_0^T \langle \xi_\Gamma(t), v(t)\rangle_{H^{1/2}(\Gamma)}\, dt\,,$$

and we can conclude by the maximal monotonicity of $\beta_{\Gamma w}$.

Finally, if the further assumptions (2.18)–(2.22) hold and $(\varepsilon u^\varepsilon_{0|\Gamma})_\varepsilon$ is bounded in W_Γ, we can proceed similarly performing the estimates in Section 5 instead. In particular, note that with these hypotheses the constant C appearing in Lemma 5.1 is independent of ε. Hence, we infer

$$\|u_\varepsilon\|_{W^{1,\infty}(0,T;V^*)\cap H^1(0,T;V)} + \|v_\varepsilon\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;H^{1/2}(\Gamma))} + \varepsilon^{1/2}\,\|v_\varepsilon\|_{H^1(0,T;V_\Gamma)} \leq c\,,$$
$$\|\mu_\varepsilon\|_{L^\infty(0,T;V)\cap L^2(0,T;W_{\mathbf{n}}\cap H^3(\Omega))} \leq c\,,$$
$$\|\eta_\varepsilon\|_{L^\infty(0,T;H)} + \|\eta_{\Gamma\varepsilon}\|_{L^\infty(0,T;H_\Gamma)} + \|\xi_\varepsilon\|_{L^\infty(0,T;H)} \leq c\,,$$
$$\|\Delta u_\varepsilon\|_{L^\infty(0,T;H)} + \|\partial_{\mathbf{n}} u_\varepsilon - \varepsilon\Delta_\Gamma v_\varepsilon + \xi_{\Gamma\varepsilon}\|_{L^\infty(0,T;H_\Gamma)} \leq c\,.$$

Now, arguing as before by elliptic regularity and interpolation arguments, we deduce that

$$\|\partial_{\mathbf{n}} u_\varepsilon\|_{L^\infty(0,T;H^{-1/2}(\Gamma))} + \varepsilon\,\|\Delta_\Gamma v_\varepsilon\|_{L^\infty(0,T;H^{-1/2}(\Gamma))} + \|\xi_{\Gamma\varepsilon}\|_{L^\infty(0,T;H^{-1/2}(\Gamma))} \leq c\,.$$

Hence, the conclusion of the proof follows easily by a completely similar argument.

References

[1] E. Bonetti, P. Colli, L. Scarpa, and G. Tomassetti. A doubly nonlinear Cahn-Hilliard system with nonlinear viscosity. *Commun. Pure Appl. Anal.*, 17(3):1001–1022, 2018.

[2] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[3] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. i. interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958.

[4] L. Calatroni and P. Colli. Global solution to the Allen-Cahn equation with singular potentials and dynamic boundary conditions. *Nonlinear Anal.*, 79:12–27, 2013.

[5] P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, and J. Sprekels. Optimal boundary control of a viscous Cahn-Hilliard system with dynamic boundary condition and double obstacle potentials. *SIAM J. Control Optim.*, 53(4):2696–2721, 2015.

[6] P. Colli and T. Fukao. Cahn-Hilliard equation with dynamic boundary conditions and mass constraint on the boundary. *J. Math. Anal. Appl.*, 429(2):1190–1213, 2015.

[7] P. Colli and T. Fukao. Equation and dynamic boundary condition of Cahn-Hilliard type with singular potentials. *Nonlinear Anal.*, 127:413–433, 2015.

[8] P. Colli, G. Gilardi, and J. Sprekels. On the Cahn-Hilliard equation with dynamic boundary conditions and a dominating boundary potential. *J. Math. Anal. Appl.*, 419(2):972–994, 2014.

[9] P. Colli, G. Gilardi, and J. Sprekels. A boundary control problem for the pure Cahn-Hilliard equation with dynamic boundary conditions. *Adv. Nonlinear Anal.*, 4(4):311–325, 2015.

[10] P. Colli, G. Gilardi, and J. Sprekels. A boundary control problem for the viscous Cahn-Hilliard equation with dynamic boundary conditions. *Appl. Math. Optim.*, 73(2):195–225, 2016.

[11] P. Colli and L. Scarpa. From the viscous Cahn-Hilliard equation to a regularized forward-backward parabolic equation. *Asymptot. Anal.*, 99(3-4):183–205, 2016.

[12] P. Colli and J. Sprekels. Optimal control of an Allen-Cahn equation with singular potentials and dynamic boundary condition. *SIAM J. Control Optim.*, 53(1):213–234, 2015.

[13] P. Colli and A. Visintin. On a class of doubly nonlinear evolution equations. *Comm. Partial Differential Equations*, 15(5):737–756, 1990.

[14] M. Efendiev and S. Zelik. Finite-dimensional attractors and exponential attractors for degenerate doubly nonlinear equations. *Math. Methods Appl. Sci.*, 32(13):1638–1668, 2009.

[15] H. P. Fischer, P. Maass, and W. Dieterich. Novel surface modes in spinodal decomposition. *Phys. Rev. Lett.*, 79:893–896, Aug 1997.

[16] C. G. Gal. On a class of degenerate parabolic equations with dynamic boundary conditions. *J. Differential Equations*, 253(1):126–166, 2012.

[17] C. G. Gal. The role of surface diffusion in dynamic boundary conditions: Where do we stand? *Milan J. Math.*, 83(2):237–278, 2015.

[18] C. G. Gal and M. Grasselli. The non-isothermal Allen-Cahn equation with dynamic boundary conditions. *Discrete Contin. Dyn. Syst.*, 22(4):1009–1040, 2008.

[19] G. Gilardi, A. Miranville, and G. Schimperna. On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions. *Commun. Pure Appl. Anal.*, 8(3):881–912, 2009.

[20] G. Gilardi, A. Miranville, and G. Schimperna. Long time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions. *Chin. Ann. Math. Ser. B*, 31(5):679–712, 2010.

[21] M. E. Gurtin. Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance. *Phys. D*, 92(3-4):178–192, 1996.

[22] H. Kardestuncer and D. H. Norrie, editors. Chapters 1–3 in *Finite element handbook*. McGraw-Hill Book Co., New York, 1987.

[23] R. Kenzler, F. Eurich, P. Maass, B. Rinn, J. Schropp, E. Bohl, and W. Dieterich. Phase separation in confined geometries: Solving the Cahn-Hilliard equation with generic boundary conditions. *Computer Physics Communications*, 133(2):139 – 157, 2001.

[24] S. Maier-Paape and T. Wanner. Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions. I. Probability and wavelength estimate. *Comm. Math. Phys.*, 195(2):435–464, 1998.

[25] S. Maier-Paape and T. Wanner. Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions: nonlinear dynamics. *Arch. Ration. Mech. Anal.*, 151(3):187–219, 2000.

[26] A. Miranville and G. Schimperna. On a doubly nonlinear Cahn-Hilliard-Gurtin system. *Discrete Contin. Dyn. Syst. Ser. B*, 14(2):675–697, 2010.

[27] A. Miranville and S. Zelik. Robust exponential attractors for Cahn-Hilliard type equations with singular potentials. *Math. Methods Appl. Sci.*, 27(5):545–582, 2004.

[28] A. Miranville and S. Zelik. Doubly nonlinear Cahn-Hilliard-Gurtin equations. *Hokkaido Math. J.*, 38(2):315–360, 2009.

[29] A. Novick-Cohen. On the viscous Cahn-Hilliard equation. In *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford Sci. Publ., pages 329–342. Oxford Univ. Press, New York, 1988.

[30] G. Schimperna, A. Segatti, and U. Stefanelli. Well-posedness and long-time behavior for a class of doubly nonlinear equations. *Discrete Contin. Dyn. Syst.*, 18(1):15–38, 2007.

[31] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.