OPEN ACCESS
IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **34** (2021) 3813–3857 https://doi.org/10.1088/1361-6544/abf338

The stochastic Cahn–Hilliard equation with degenerate mobility and logarithmic potential

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Received 24 January 2021
Accepted for publication 23 March 2021
Published 18 June 2021

Abstract
We prove existence of martingale solutions for the stochastic Cahn–Hilliard equation with degenerate mobility and multiplicative Wiener noise. The potential is allowed to be of logarithmic or double-obstacle type. By extending to the stochastic framework a regularization procedure introduced by Elliott and Garcke in the deterministic setting, we show that a compatibility condition between the degeneracy of the mobility and the blow-up of the potential allows to confine some approximate solutions in the physically relevant domain. By using a suitable Lipschitz-continuity property of the noise, uniform energy and magnitude estimates are proved. The passage to the limit is then carried out by stochastic compactness arguments in a variational framework. Applications to stochastic phase-field modelling are also discussed.

Keywords: stochastic Cahn–Hilliard equation, degenerate mobility, logarithmic potential, stochastic compactness, monotonicity, variational approach
Mathematics Subject Classification numbers: 35K25, 35R60, 60H15, 80A22.

1. Introduction

The Cahn–Hilliard equation was firstly proposed in [14] in order to describe the spinodal decomposition occurring in binary metallic alloys. Since then it has been increasingly employed in several areas such as, among many others, physics, engineering, and biology.

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Recommended by Professor Charles R Doering.

In the recent years, the Cahn–Hilliard equation has become one of the most important models involved in phase-field theory. In such class of models, the evolution of a certain material exhibiting two different features is described by introducing a so-called *state variable* $\varphi \in [-1, 1]$, representing the difference in volume fractions. The sets $\{\varphi = 1\}$ and $\{\varphi = -1\}$ correspond to the pure regions, while the interfacial region $\{-1 < \varphi < 1\}$ where the two components coexist is supposed to have a positive thickness. For this reason, such models are usually referred to as diffuse interface models, and the time evolution of the state variable is often described by means of a Cahn–Hilliard-type equation. The field of applications of diffuse-interface modelling is enormous. In physics it is used in the context of evolution of separating materials, phase-transition phenomena, and dynamics of mixtures of fluids; in biology phase-field modelling is crucial in the description of evolution of interacting cells, tumour growths, and dynamics of interacting populations; in engineering it plays a central role in modelling of damage and deterioration in continuous media.

Given a smooth bounded domain $\mathcal{O}$ of $\mathbb{R}^d$, with $d \geqslant 2$, and a fixed final time $T > 0$, the deterministic Cahn–Hilliard equation reads

$$\partial_t\varphi - \mathrm{div}(m(\varphi)\nabla\mu) = 0 \quad \text{in } (0, T) \times \mathcal{O}, \tag{1.1}$$

$$\mu = -\Delta\varphi + F'(\varphi) \quad \text{in } (0, T) \times \mathcal{O}, \tag{1.2}$$

$$\mathbf{n} \cdot \nabla\varphi = \mathbf{n} \cdot m(\varphi)\nabla\mu = 0 \quad \text{in } (0, T) \times \partial\mathcal{O}, \tag{1.3}$$

$$\varphi(0) = \varphi_0 \quad \text{in } \mathcal{O}. \tag{1.4}$$

The variable φ is referred to as state variable, or order parameter, while μ is the chemical potential. Here, the symbol $\mathbf{n}$ denotes the outward unit vector on the boundary $\partial\mathcal{O}$, the function m is known as mobility, while $F : \mathbb{R} \to [0, +\infty]$ is a double-well potential with two global minima. Typical examples of m and F are given below.

The chemical potential μ is directly related in equation (1.2) to the subdifferential of the so-called free-energy functional

$$\varphi \mapsto \frac{1}{2}\int_{\mathcal{O}} |\nabla\varphi|^2 + \int_{\mathcal{O}} F(\varphi).$$

The double-well potential F may be thought as a singular convex function that has been perturbed by a concave quadratic function: the effect of the concave perturbation is then the creation of two global minima for F, each one representing the pure phases of the model. Minimising the F-term in free-energy above describes then the tendency of each pure phase to concentrate, whereas the gradient term penalises high oscillations of the state variable. The idea behind the minimisation of the free-energy is then a calibration between these two phenomena.

Since only the values of φ in the interval $[-1, 1]$ are relevant to the physical derivation of the model, the double-well potential F is only meaningful if defined on $[-1, 1]$. The most important example is the logarithmic one, defined as

$$F_{\log}(r) := \frac{\theta}{2}\left((1 + r)\ln(1 + r) + (1 - r)\ln(1 - r)\right) + \frac{\theta_0}{2}(1 - r^2), \quad r \in (-1, 1), \tag{1.5}$$

with $0 < \theta < \theta_0$ being given constants, which possesses two global minima in the interior of the physically relevant domain $[-1, 1]$. This choice of the nonlinearity is the most coherent with the physical derivation of the Cahn–Hilliard model itself, in relation to its thermodynamical consistency: for this reason, (1.5) is usually employed in contexts related to separation phenomena in physics. A second relevant choice for F is the so-called double-obstacle potential

$$F_{\mathrm{ob}}(r) := \begin{cases} 1 - r^2 & \text{if } r \in [-1, 1], \\ +\infty & \text{otherwise.} \end{cases} \tag{1.6}$$

Here, by contrast, the global minima corresponds exactly to the pure phases ± 1, and this choice is then often employed in the modelling contexts where the pure phases have a privileged role compared to the interface: it is the cases, for example, of tumour growth dynamics in biology. In (1.6), the derivative F'_{ob} has to be interpreted in the sense of convex analysis as the subdifferential ∂F_{ob}, and the equation takes the form of a differential inclusion. In some cases, these double-well potentials are approximated by the polynomial one

$$F_{\mathrm{pol}}(r) := \frac{1}{4}(r^2 - 1)^2, \quad r \in \mathbb{R}, \tag{1.7}$$

which nonetheless does not ensure the relevant constraint $\varphi \in [-1, 1]$. While on the one hand the polynomial potential F_{pol} is certainly much easier to handle from the mathematical point of view, on the other hand the logarithmic potential $F_{\log}$ is surely the most relevant in terms of thermodynamical consistency. Indeed, due to the physical interpretation of diffuse-interface modelling, only the values of the variable φ in $[-1, 1]$ are meaningful. For this reason, the possibility of dealing with the logarithmic potential $F_{\log}$ is crucial.

The classical choice of the mobility m is a positive constant m_{con}, independent of φ. Nevertheless, starting from the pioneering contribution [10] itself, several authors proposed the choice of a mobility depending explicitly on the order parameter. A thermodynamically relevant choice for m has been exhibited in the works [11, 12, 56] and consists of a polynomial mobility m_{pol} defined on the physically relevant domain $[-1, 1]$ with degeneracy at the extremal points:

$$m_{\mathrm{pol}}(r) := 1 - r^2, \quad r \in [-1, 1]. \tag{1.8}$$

A more general version of m_{pol} is given by

$$m_{\alpha}(r) := (1 - r^2)^{\alpha}, \quad r \in [-1, 1], \ \alpha \geqslant 1. \tag{1.9}$$

Several variants of the Cahn–Hilliard equation have been studied in the last decades. Novick–Cohen proposed in [67] the viscous regularization (see also [41, 42]), accounting also for viscous dynamics occurring in phase-transition evolution. Gurtin generalized the viscous correction in [55] possibly including nonlinear viscosity contributions in the equation. More recently, physicists have introduced the so-called dynamic boundary conditions in order to account also for possible interaction with the walls in a confined system (see for example [45, 50, 59]).

The mathematical literature on the deterministic Cahn–Hilliard equation is extremely developed: we refer to [65] and the references therein for a unifying treatment on the available literature. In particular, in the case of constant mobility existence, uniqueness, and regularity have been studied in [15, 16, 18, 19, 21, 51] both with irregular potentials and dynamic boundary conditions, and in [7, 8, 66, 75] with nonlinear viscosity terms. Significant attention has been devoted also to the asymptotic behaviour of solutions [20, 24, 52] and optimal control problems [17, 22, 23, 57]. A mathematical analysis of the framework of nonconstant and possibly degenerate mobility has been investigated in [13, 40]. In this direction we also refer to the contribution [29, 62, 84] regarding existence of solutions. Let us point out the work [54] dealing with the analysis of a Cahn–Hilliard equation with mobility depending on the chemical potential, [78] in relation to global attractors, and [61] for an approach based on gradient flows in Wasserstein spaces. A diffuse interface model with degenerate mobility has been studied also in [48]. Numerical simulations have been analysed in [4, 60].

Despite the fact that the deterministic model has been proven to be extremely effective in the description of phase-separation, there are certainly important downsides. One of the main drawbacks of the deterministic framework is the impossibility of describing the unpredictable disruptions occurring in the evolution at the microscopic scale. These may be due to several phenomena of different nature, such as uncertain movements at a microscopic level caused by configurational, electronic, or magnetic effects, which cannot be captured by the classical deterministic Cahn–Hilliard system. The most natural way to capture the randomness component which may affect phase-field evolutions is to introduce a Wiener-type noise in the Cahn–Hilliard equation itself. This was first proposed in [25] employing Wiener noises in the well-known Cahn–Hilliard–Cook stochastic version of the model, which has been then validated as the only genuine description of phase-separation in the contributions [6, 71]. Currently, this is widely studied both in the physics and applied mathematics literatures, for which we refer to [64, 72] and the references therein. The stochastic version Cahn–Hilliard equation reads

$$\mathrm{d}\varphi - \operatorname{div}(m(\varphi)\nabla\mu)\,\mathrm{d}t = G(\varphi)\,\mathrm{d}W \quad \text{in } (0,T)\times\mathcal{O}, \tag{1.10}$$

$$\mu = -\Delta\varphi + F'(\varphi) \quad \text{in } (0,T)\times\mathcal{O}, \tag{1.11}$$

$$\mathbf{n}\cdot\nabla\varphi = \mathbf{n}\cdot m(\varphi)\nabla\mu = 0 \quad \text{in } (0,T)\times\partial\mathcal{O}, \tag{1.12}$$

$$\varphi(0) = \varphi_0 \quad \text{in } \mathcal{O}. \tag{1.13}$$

Here, W is a cylindrical Wiener process defined on certain stochastic basis, while G is a suitable stochastically integrable operator with respect to W. Precise assumptions on the data are given in section 2 below.

From the mathematical point of view, the stochastic Cahn–Hilliard equation has been studied mainly in the case of polynomial potential and only with constant mobility. One the first contributions in this direction is [27], in which the authors show existence of solutions via a semigroup approach in the case of polynomial potentials. More recently, well-posedness has been investigated also in [26, 39] again in the polynomial setting. A more general framework allowing for rapidly growing potentials (e.g. more than exponentially) has been analysed in [74, 77] from a variational approach. The genuine case of logarithmic potentials has only been covered in the works [30, 31, 53] by means of so-called reflection measures.

The mathematical literature on stochastic phase-field modelling has also been increasingly developed. Let us point out in this direction the works [1] dealing with unbounded noise, [43, 44] for a study of a diffuse interface model with thermal fluctuations, and [5, 69] dealing with the stochastic Allen–Cahn equation. Beside well-posedness, optimal control problems have also been studied in [76] in the case of the stochastic Cahn–Hilliard equation, and in [68] in the context of a stochastic phase-field model for tumour growth. Besides, let us point out the mathematical literature on stochastic two-phase flows has also been expanded in the last years, in the context of coupled stochastic systems of Cahn–Hilliard–Navier–Stokes and Allen–Cahn–Navier–Stokes type. In this direction, we refer the reader to the contributions [33, 35, 80, 81] for existence of solutions, [3, 36] about asymptotic long-time behaviour, [37] on large deviation limits, and [32, 34] dealing with a nonlocal phase-field equation in the system instead.

One of the most critical problems of the stochastic model is that the presence of the random forcing term does not guarantee that the state variable φ remains in the physically relevant domain $[-1, 1]$, even if the double-well potential is singular as in (1.5) or (1.6). This is due to the presence of the additional second-order term in the energy balance, which may cause blow-up of the energy in finite time. The consequences of this fact are sever, both on the mathematical side and especially from the modelling perspective. Indeed, from the point of view of thermodynamically consistency of the equation, φ represents a local concentration and is only meaningful if belonging to the physical interval $[-1, 1]$: the impossibility of proving that the solution of the stochastic equation satisfies this constraint inevitably represents a modelling downside. Besides, as we have pointed out above, the available literature on the stochastic Cahn–Hilliard equation only handles the case of constant mobility, which unfortunately is not the most suited for describing phase-separation.

The current state-of-the-art of the stochastic Cahn–Hilliard equation inevitably calls then for a deeper investigation in the direction of degeneracy of the mobility, including the physically relevant logarithmic potential, and showing that the physically meaningful constraint $\varphi \in [-1, 1]$ is achieved. This paper is the first contribution in the mathematical literature that addresses these three points, and represents then an important step especially in terms of applicative validation of the stochastic Cahn–Hilliard–Cook model. Our results have important consequences to all fields, in particular physics and engineering, where phase-separation is usually studied under random forcing, as we provide the first mathematical validation of the stochastic model in its more relevant form, i.e. with degenerate mobility m_{pol} and logarithmic potential $F_{\log}$.

In this paper we are interested in studying the stochastic Cahn–Hilliard equation (1.10)–(1.13) from a variational approach, including the cases of degenerate mobility m_{pol}, the logarithmic double-well potential $F_{\log}$, and the double-obstacle potential F_{ob}. As we have pointed out, these choices are indeed the most relevant in terms of the thermodynamical coherency of the model. From the mathematical perspective, our approach differs from [30, 31, 53] as we do not rely on reflection measures. Our techniques extend to stochastic framework the ideas of Elliott and Garcke in [40], and consist of a compatibility condition between the degeneracy of the mobility, the coefficient G, and the possible blow-up of the potential at the extremal points ± 1.

Let us briefly explain the main difficulties arising in the case of degenerate mobility and logarithmic potential, and how we overcome these in the present work.

The first main issue appearing in the stochastic framework is the presence of a proliferation term in equation (1.10). Indeed, in the deterministic case the integration in space of equation (1.1) yields, together with the boundary conditions (1.3), the conservation of mass during the evolution. This is in turn crucial when dealing with irregular potentials such as $F_{\log}$ or F_{ob}, as it allows to control the spatial mean of the chemical potential. However, in the stochastic scenario the presence of the noise term in equation (1.10) determines a proliferation of the total mass of the system. Whereas this drawback can be overcome in the easier case of regular potentials as F_{pol}, this results in the impossibility of obtaining satisfactory estimates on the chemical potential μ in the case of logarithmic and double-obstacle potentials. The main reason is the following. On the one hand the derivatives of F_{pol} can be controlled by F_{pol} itself, i.e. $|F''_{\text{pol}}|, |F'_{\text{pol}}| \leqslant c(1 + F)$ for a certain $c > 0$, so that the usual energy estimates on F_{pol} allow to bound also F'_{pol}, hence μ as a byproduct. On the other hand, however, the derivatives of the logarithmic potential $F_{\log}$ blow up at ± 1 much more rapidly than $F_{\log}$ itself, so that the classical energy estimates on $F_{\log}$ are not enough to deduce a control on $F'_{\log}$. This problem is even more evident in the stochastic setting due to the presence in the energy estimates of the second order Itô correction, which depends of the second derivative F''. Again, while this term can be handled in the case of polynomial potentials as F_{pol}, in the case of logarithmic potential $F_{\log}$ the situation is much more critical, due to blow-ups at ± 1 pointed out above.

The second main issue is the degeneracy of the mobility. Indeed, while in the case of constant mobility m_{con}, or more generally if m is bounded from below by a positive constant, one usually deduces estimates on $\nabla\mu$ pretty directly, if m degenerates at ± 1, as in the physically relevant case m_{pol}, then there is no hope to obtain a control on the gradient of the chemical potential.

These two main problems suggest that in the case of degenerate mobility the role of the chemical potential μ must be passed by, and a different interpretation of the equation is needed. To this end, if one formally substitutes equation (1.11) into (1.10), it is possible to obtain a variational formulation on the problem only involving the variable φ. In particular, the nonlinear term resulting from such substitution (see definition 2.5 below) is in the form $m(\varphi)F''(\varphi)$. Hence, supposing that the degeneracy of the mobility compensates the blow-up of F'' at ± 1, we can obtain a coherent formulation of the problem not involving μ anymore. Such compatibility condition between m and F'' was employed in the mentioned work [40], and is very natural as it is satisfied by the physically relevant choices m_{pol} and $F_{\log}$, as we will show in remark 2.3 below.

The idea to overcome the presence of Itô correction terms in the energy estimates depending on F'' is of similar nature. If G is Lipschitz-continuous and vanishes at the extremal points ± 1, its degeneracy can compensate the blow-up of F'' and the energy estimate can be closed. Again, such Lipschitz-continuity assumption on G is very natural in applications, and has been widely employed in stochastic phase-field modelling. For example, in [5] the authors consider a diffuse-interface model based on the stochastic Allen–Cahn equation in the context of evolution of damage in certain continuum media. The state variable represents here the damage parameter, which may be though as the local ratio of active cohesive bonds at the microscopic level. Clearly, only positive values of the state variable are physically meaningful, and in order to ensure this the authors use a obstacle-type potential. As far as the noise is concerned, the idea to handle the singularity is of similar nature: the random forcing is supposed to be multiplicative and to 'switch off' whenever the φ touches the potential barriers. On the same line, the superposition-type stochastic forcing that we propose in this paper has also been widely employed in applications. In [43] for instance, this has been actively explored in the context of a model for a binary mixture of incompressible fluids.

In this work we prove two main results. The first one deals with existence of martingale solutions to the problem (1.10)–(1.13) in the case of positive mobility and regular potential. Although being a preparatory work for us, this first result is also interesting on its own, as it covers the case of non constant mobility and allows up to first-order exponential growth for the potential. The proofs rely on a double approximation involving a Faedo–Galerkin discretization in space and an Yosida regularization on the nonlinearity. The second main result that we prove is the main contribution of the paper, and states existence of martingale solutions in the case of degenerate mobility and irregular potentials, possibly including $F_{\log}$ and F_{ob}. We employ a suitable regularization on the potential, the mobility, and the diffusion coefficient so that we can solve the approximated problem thanks to the first result. We show then uniform estimates on the solutions by using energy and magnitude estimates based on the compatibility between m, F, and G. Finally we pass to the limit by a stochastic compactness argument.

As far as uniqueness of solutions is concerned, the problem is still open even in the deterministic setting. A uniqueness result for the system with degenerate mobility has been obtained in [49] in the framework of a nonlocal diffusion related to tumour growth dynamics: here, the authors exploit the regularizing properties of the nonlocal nature of the equation in order to show continuous dependence on the initial data. Nevertheless, for Cahn–Hilliard evolutions of local type with degenerate mobility, regularity of solutions is much more difficult to achieve, and uniqueness remains unknown. In the stochastic setting, this also prevents from proving existence of probabilistically strong solutions.

Let us now summarize the main contents of the work. In section 2 we introduce the main setting and we state the main results. Section 3 contains the proof of existence of martingale solutions in the case of positive mobility and regular potential. Section 4 is focused on the proof of existence of martingale solutions in the setting of degenerate mobility and irregular potential.

2. Main results

We introduce here the notation and setting of the paper, and state the main results. The first main result focuses on existence of solutions in case of nondegenerate mobility and regular potential, while the second deals with the case of degenerate mobility and logarithmic potential.

2.1. Notation and setting

For any real Banach space E, its dual will be denoted by E^*. The norm in E and the duality pairing between E^* and E will be denoted by $\|\cdot\|_E$ and $\langle\cdot,\cdot\rangle_E$, respectively. If $(A,\mathscr{A},\nu)$ is a finite measure space, we use the classical notation $L^p(A;E)$ for the space of p-Bochner integrable functions, for any $p\in[1,+\infty]$. We shall also use the classical symbol $L^0(A;E)$ for the space of $\mathscr{A}$-measurable functions with values in E. If E_1 and E_2 are separable Hilbert spaces, we use the notation $\mathscr{L}^2(E_1,E_2)$ for the space of Hilbert–Schmidt operators from E_1 to E_2.

Throughout the paper, $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ is a filtered probability space satisfying the usual conditions, with $T>0$ being a fixed final time, and W is a cylindrical Wiener process on a separable Hilbert space U. We fix once and for all a complete orthonormal system $(u_k)_k$ of U. For every separable Hilbert space E and $\ell\in[2,+\infty)$, we set

$$L^\ell_w(\Omega;L^\infty(0,T;E^*)):=\Big\{v:\Omega\to L^\infty(0,T;E^*)\text{ weak}^*\text{-measurable:}\quad \mathbb{E}\|v\|^\ell_{L^\infty(0,T;E^*)}<+\infty\Big\},$$

and recall that by [38, theorem 8.20.3] we have

$$L^\ell_w(\Omega;L^\infty(0,T;E^*))=\Big(L^{\frac{\ell}{\ell-1}}(\Omega;L^1(0,T;E))\Big)^*.$$

Moreover, we will use the symbols $C^0([0,T];E)$ and $C^0_w([0,T];E)$ for the spaces of strongly and weakly continuous functions from $[0,T]$ to E, respectively.

It is useful to recall here some general facts about the cylindrical Wiener process W and stochastic integration that will be used later on in the paper: we follow the approach of [63, sections 2.5.1–2]. Since W is cylindrical in U, we have the formal representation

$$W=\sum_{k=0}^\infty \beta_k u_k, \tag{2.1}$$

with $(\beta_k)_k$ being a family of independent real Brownian motions. Nonetheless, it is important to note that the infinite sum does *not* converge in general in U, hence W is not rigorously defined as a U-valued continuous process. In order to properly define W and the corresponding stochastic integral, it is useful to show that it is always possible to consider W as a Q_1-Wiener process on a larger space U_1, with Q_1 being of trace-class on U_1. To this end, note that there always exists a larger separable Hilbert space U_1 and a Hilbert–Schmidt operator $\iota \in \mathscr{L}^2(U, U_1)$. For example, on U one can define the norm

$$\|v\|_{U_1} := \left(\sum_{k=0}^{\infty} \frac{1}{k^2} |(v, u_k)_U|^2 \right)^{1/2}, \quad v \in U.$$

It is easy to check that $\|\cdot\|_{U_1}$ is a well-defined norm on U, weaker than the usual one $\|\cdot\|_U$. It makes sense then to define U_1 as the abstract closure of U with respect to the norm $\|\cdot\|_{U_1}$: namely, U_1 is the abstract space of infinite linear combinations v of $(u_k)_k$ for which $\|v\|_{U_1}$ is finite, i.e.

$$U_1 := \left\{ v = \sum_{k=0}^{\infty} \alpha_k u_k : \ \|v\|_{U_1}^2 := \sum_{k=0}^{\infty} \frac{\alpha_k^2}{k^2} < +\infty \right\}.$$

With this choice, $(U_1, \|\cdot\|_{U_1})$ is actually a separable Hilbert space and the inclusion $\iota : U \hookrightarrow U_1$ is Hilbert–Schmidt: indeed, we have

$$\sum_{k=0}^{\infty} \|\iota(u_k)\|_{U_1}^2 = \sum_{k=0}^{\infty} \|u_k\|_{U_1}^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{j^2} |(u_k, u_j)_U|^2$$
$$= \sum_{k=0}^{\infty} \frac{1}{k^2} \|u_k\|_U^2 = \sum_{k=0}^{\infty} \frac{1}{k^2} < +\infty.$$

Consequently, by the properties of Hilbert–Schmidt operators (see again [63]), we have that the infinite sum (2.1) actually converges in U_1: this means that we can look at W as a rigorously defined stochastic process on U_1. Moreover, it actually holds that W is a well-defined Q_1-Wiener process on U_1, with $Q_1 := \iota \circ \iota^*$ being of trace class on U_1, and such that $Q_1^{1/2}(U_1) = \iota(U)$. In the sequel we will say that W is a cylindrical Wiener process on U if and only if it is a Q_1-Wiener process on U_1. Furthermore, stochastic integration with respect to the cylindrical process W is defined in terms of the usual stochastic integration with respect to the Q_1-Wiener process. In this regard, for every Hilbert space K it holds that $B \in \mathscr{L}^2(U, K)$ if and only if $B \circ \iota^{-1} \in \mathscr{L}^2(Q_1^{1/2}(U_1), H)$. Consequently, for every progressively measurable stochastic integrand $B \in L^2(\Omega; L^2(0, T; \mathscr{L}^2(U, K)))$, it holds that $B \circ \iota^{-1} \in L^2(\Omega; L^2(0, T; \mathscr{L}^2(Q_1^{1/2}(U_1), K)))$ and

$$\int_0^{\cdot} B(s)\, \mathrm{d}W(s) := \int_0^{\cdot} B \circ \iota^{-1}(s)\, \mathrm{d}W(s),$$

where the right-hand side is the usual integral with respect to the Q_1-Wiener process, the left-hand side is the stochastic integral with respect to the cylindrical Wiener process, and the equality is intended in the sense of indistinguishable continuous K-valued processes. The definition of stochastic integral with respect to the cylindrical Wiener process W can be shown to be independent of the specific Hilbert space U_1 and the Hilbert–Schmidt embedding ι.

Let $\mathcal{O} \subset \mathbb{R}^d$ ($d \geqslant 2$) be a smooth bounded domain. We shall use the notation $Q := (0, T) \times \mathcal{O}$ and $Q_t := (0, t) \times \mathcal{O}$ for every $t \in (0, T)$. Denoting by $\mathbf{n}$ the outward normal unit vector on $\mathcal{O}$, we define the functional spaces

$$H := L^2(\mathcal{O}), \quad V_1 := H^1(\mathcal{O}), \quad V_2 := \{v \in H^2(\mathcal{O}) : \mathbf{n} \cdot \nabla v = 0 \text{ a.e. on } \partial\mathcal{O}\},$$

endowed with their natural norms $\|\cdot\|_H$, $\|\cdot\|_{V_1}$, and $\|\cdot\|_{V_2}$, respectively. For every $v \in V_1^*$, we set $v_{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\langle v, 1\rangle$ for the spatial mean of v. We also define

$$\mathcal{B}_R := \{v \in L^\infty(\mathcal{O}) : \|v\|_{L^\infty(\mathcal{O})} \leqslant R\}, \quad R > 0.$$

Moreover, we will use the symbol c to denote any arbitrary positive constant depending only on the data of the problem, whose value may be updated throughout the proofs. When we want to specify the dependence of c on specific quantities, we will indicate them through a subscript.

2.2. Nondegenerate mobility and regular potential

In case of nondegenerate mobility and regular potential, we assume the following.

ND1 $F \in C^2(\mathbb{R})$, $F \geqslant 0$, $F'(0) = 0$, and there exists a constant $C_F > 0$ such that

$$|F'(r)| \leqslant C_F\,(1 + F(r)) \quad \forall\, r \in \mathbb{R},$$
$$|F''(r)| \leqslant C_F\,(1 + F(r)) \quad \forall\, r \in \mathbb{R},$$
$$F''(r) \geqslant -C_F \quad \forall\, r \in \mathbb{R}.$$

ND2 $m \in C^0(\mathbb{R})$ and there exist two constants $m_*, m^* > 0$ such that

$$m_* \leqslant m(r) \leqslant m^* \quad \forall\, r \in \mathbb{R}.$$

ND3 $G : H \to \mathscr{L}^2(U, H)$ is measurable and there exists $(g_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(\mathbb{R})$ such that

$$G(v)u_k = g_k(v) \quad \forall\, v \in H,\ \forall\, k \in \mathbb{N},$$
$$C_G := \sum_{k=0}^{\infty} \|g_k\|^2_{W^{1,\infty}(\mathbb{R})} < +\infty.$$

ND4 The initial datum in nonrandom, and satisfies $\varphi_0 \in V_1$ and $F(\varphi_0) \in L^1(\mathcal{O})$.

Let us point out that the assumption **ND1** allows for the classical choice of the polynomial double-well potential F_{pol} defined in (1.7), but also allows to consider polynomials of any orders and even first-order exponentials. Moreover, assumption **ND2** allows of course for the constant mobility scenario, but also includes the case of positive nonconstant mobilities. Condition **ND3** on the noise is widely employed in literature (see for example [9, 43, 44]), and ensures that in particular that $G : H \to \mathscr{L}^2(U, H)$ is Lipschitz-continuous and linearly bounded, and that the restriction $G_{|V_1} : V_1 \to \mathscr{L}^2(U, V_1)$ is linearly bounded. The initial datum is assumed to be nonrandom in **ND4**: this is meaningful in relation to the physical interpretation of the model. A random initial datum could also be considered, but this would make the mathematical treatment too heavy in our opinion, as different estimates are based on different moments in Ω. Since this is not the main focus of the paper, we preferred to assume φ_0 nonrandom in order to make the treatment clearer and avoid technicalities: for an exact analysis on the moments of the initial datum we refer the reader to [77].

We precise now the definition of martingale solution in the case of nondegenerate mobility and regular potential.

Definition 2.1. Assume conditions **ND1**–**ND4**. A martingale solution to the problem (1.10)–(1.13) is a septuple $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}, \hat{\varphi}, \hat{\mu})$ such that:

- $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}})$ is a filtered probability space satisfying the usual conditions;
- $\hat{W}$ is a U-valued cylindrical Wiener process on $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}})$;
- $\hat{\varphi} \in L^0(\hat{\Omega}; C^0([0,T]; H) \cap L^2(0,T; V_2))$ is progressively measurable;
- $\hat{\mu} = -\Delta\hat{\varphi} + F'(\hat{\varphi}) \in L^0(\hat{\Omega}; L^2(0,T; V_1))$;
- for every $v \in V_1$, it holds that

$$\begin{aligned}\int_{\mathcal{O}} \hat{\varphi}(t,x)v(x)\,\mathrm{d}x + \int_{Q_t} m(\hat{\varphi}(s,x))\nabla\hat{\mu}(s,x)\cdot\nabla v(x)\,\mathrm{d}x\,\mathrm{d}s \\ = \int_{\mathcal{O}} \varphi_0(x)v(x)\,\mathrm{d}x + \int_{\mathcal{O}} \left(\int_0^t G(\hat{\varphi}(s))\,\mathrm{d}\hat{W}(s)\right)(x)v(x)\,\mathrm{d}x\end{aligned} \tag{2.2}$$

  for every $t \in [0,T]$, $\hat{\mathbb{P}}$-almost surely.

The first main result of the paper deals with existence of martingale solutions in case of positive mobility and regular potential.

Theorem 2.2. *Assume conditions* **ND1–ND4**. *Then, there exists a martingale solution*

$$\left(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}, \hat{\varphi}, \hat{\mu}\right)$$

to the problem (1.10)–(1.13) *in the sense of definition* 2.1 *such that, for every* $\ell \in [2,+\infty)$,

$$\begin{aligned}&\hat{\varphi} \in L^\ell(\hat{\Omega}; C^0([0,T]; H)) \cap L^\ell_w(\hat{\Omega}; L^\infty(0,T; V_1)) \cap L^{\ell/2}(\hat{\Omega}; L^2(0,T; V_2)),\\ &\hat{\mu} \in L^{\ell/2}(\hat{\Omega}; L^2(0,T; V_1)), \qquad \nabla\hat{\mu} \in L^\ell(\hat{\Omega}; L^2(0,T; H^d)),\\ &F'(\hat{\varphi}) \in L^{\ell/2}(\hat{\Omega}; L^2(0,T; H)),\end{aligned}$$

and the following energy inequality holds, for every $t \in [0,T]$*:*

$$\begin{aligned}\frac{1}{2}\sup_{r\in[0,t]} \hat{\mathbb{E}}\|\nabla\hat{\varphi}(r)\|_H^2 + \sup_{r\in[0,t]} \hat{\mathbb{E}}\|F(\hat{\varphi}(r))\|_{L^1(\mathcal{O})} + \hat{\mathbb{E}}\int_{Q_t} m(\hat{\varphi}(s,x))|\nabla\hat{\mu}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant \frac{1}{2}\|\nabla\varphi_0\|_H^2 + \|F(\varphi_0)\|_{L^1(\mathcal{O})} + \frac{C_G}{2}\hat{\mathbb{E}}\int_0^t \|\nabla\hat{\varphi}(s)\|_H^2\,\mathrm{d}s \\ + \frac{1}{2}\hat{\mathbb{E}}\int_0^t \sum_{k=0}^\infty \int_{\mathcal{O}} F''(\hat{\varphi}(s,x))|g_k(\hat{\varphi}(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.3}$$

If also

$$|F''(r)| \leqslant C_F(1+|r|^q) \quad \forall\, r \in \mathbb{R}, \;\; \textit{where} \quad \begin{cases} q \in [2,+\infty) & \textit{if } d = 2,\\ q := \dfrac{2}{d-2} & \textit{if } d \geqslant 3,\end{cases} \tag{2.4}$$

then it holds that

$$\hat{\varphi} \in L^{\ell/4}(\hat{\Omega}; L^2(0,T; H^3(\mathcal{O}))), \qquad F'(\hat{\varphi}) \in L^{\ell/4}(\hat{\Omega}; L^2(0,T; V_1)).$$

2.3. Degenerate mobility and irregular potential

We deal now with a degenerate mobility $m : [-1, 1] \to \mathbb{R}$ which vanishes at ± 1 and an irregular potential $F : (-1, 1) \to \mathbb{R}$ possibly of logarithmic or double-obstacle type. In this case we assume the following.

D1 $F : (-1, 1) \to [0, +\infty)$ can be decomposed as $F = F_1 + F_2$, where $F_1 \in C^2(-1, 1)$ is convex and $F_2 \in C^2([-1, 1])$.

D2 $m \in W^{1,\infty}(-1, 1)$ is such that

$$m(r) \geqslant 0 \quad \forall\, r \in [-1, 1], \qquad m(r) = 0 \quad \text{iff } r = \pm 1, \qquad mF'' \in C^0([-1, 1]).$$

In particular, it is well defined the function

$$M : (-1, 1) \to [0, +\infty), \qquad M(0) = M'(0) = 0, \qquad M''(r) = \frac{1}{m(r)}, \quad r \in (-1, 1).$$

D3 $G : \mathcal{B}_1 \to \mathscr{L}^2(U, H)$ is measurable and there exists $(g_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(-1, 1)$ such that

$$G(v)u_k = g_k(v) \quad \forall\, v \in \mathcal{B}_1,\ \forall\, k \in \mathbb{N},$$

$$g_k\sqrt{F''},\ g_k\sqrt{M''} \in L^\infty(-1, 1) \quad \forall\, k \in \mathbb{N},$$

$$L_G := \sum_{k=0}^{\infty} \left(\|g_k\|^2_{W^{1,\infty}(-1,1)} + \left\| g_k\sqrt{F''} \right\|^2_{L^\infty(-1,1)} + \left\| g_k\sqrt{M''} \right\|^2_{L^\infty(-1,1)} \right) < +\infty.$$

D4 The initial datum is nonrandom and satisfies

$$\varphi_0 \in V_1, \qquad |\varphi_0| < 1 \text{ a.e. in } \mathcal{O}, \qquad F(\varphi_0) \in L^1(\mathcal{O}), \qquad M(\varphi_0) \in L^1(\mathcal{O}).$$

Note that under assumption **D1** the irregular component of the potential F is the convex part F_1, which may explode at ± 1. In condition **D2** we assume on the other hand that the degeneracy of the mobility can only occur at ± 1, and compensates the eventual blow up of F'' at ± 1. This is expected from the point of view of application to phase-field modelling (see the remark 2.3 below). Finally, the additional summability condition in **D3** is a generalization of the classical compatibility condition between m and F'' to the stochastic framework. This can be interpreted as a compensation of the blow up of F'' and M'' in ± 1 also by the component functions $(g_k)_k$. Again, this condition is satisfied in several physically relevant scenarios (see remark 2.3 below).

Remark 2.3 (logarithmic potential). Let us show now that the assumptions **D1**–**D2** allow for the physically relevant case of degenerate mobility and logarithmic potential given by the natural choices m_{pol} and $F_{\log}$ defined in (1.8) and (1.5), respectively. Indeed, assumption **D1** holds with obvious choice of F_1 and F_2. Moreover, an elementary computation yields

$$F''_{\log}(r) = \frac{\theta}{1 - r^2} - \theta_0, \quad r \in (-1, 1),$$

so that $m_{\text{pol}}F''_{\log} \in C^0([-1, 1])$ and also condition **D2** is satisfied.

With mobility m_{pol} and potential $F_{\log}$, a sufficient condition for assumption **D3** is that

$$(g_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(-1, 1), \qquad g_k(-1) = g_k(1) = 0 \quad \forall\, k \in \mathbb{N}, \qquad \sum_{k=0}^{\infty} \|g_k'\|^2_{L^\infty(-1,1)} < +\infty, \tag{2.5}$$

meaning essentially that the components $(g_k)_k$ are Lipschitz-continuous and vanish at the extremal points. Let us show that under (2.5) also **D3** is satisfied. Indeed, for every $r \in (-1, 1)$ and $k \in \mathbb{N}$ one has

$$\begin{aligned}\left|g_k(r)\sqrt{F''_{\log}(r)}\right|^2 &= \left|\theta\frac{g_k(r)}{1-r^2} - \theta_0 g_k(r)\right|^2 \leqslant 2\theta\frac{g_k^2(r)}{1-r^2} + 2\theta_0 g_k^2(r)\\ &= 2\theta\frac{|g_k(r)-g_k(-1)||g_k(r)-g_k(1)|}{|1-r||1+r|} + 2\theta_0|g_k(r)-g_k(1)|^2\\ &\leqslant 2\theta\|g'_k\|^2_{L^\infty(-1,1)}\frac{|1+r||1-r|}{|1+r||1-r|} + 2\theta_0\|g'_k\|^2_{L^\infty(-1,1)}|r-1|^2\\ &\leqslant 2\|g'_k\|^2_{L^\infty(-1,1)}(\theta+4\theta_0),\end{aligned}$$

so that $g_k\sqrt{F''_{\log}} \in L^\infty(-1,1)$ for all $k \in \mathbb{N}$, and

$$\sum_{k=0}^{\infty}\left\|g_k\sqrt{F''_{\log}}\right\|^2_{L^\infty(-1,1)} \leqslant 2(\theta+4\theta_0)\sum_{k=0}^{\infty}\|g'_k\|^2_{L^\infty(-1\,1)} < +\infty.$$

The computations for the terms $g_k\sqrt{M''_{\mathrm{pol}}}$ are entirely analogous, and **D3** follows.

Remark 2.4 (double-obstacle potential). Note that choosing $F_1 = 0$ and $F_2(r) := 1 - r^2$, $r \in [-1, 1]$, one recovers exactly the double-obstacle potential F_{ob} defined in (1.6). Choosing also the degenerate mobility m_{pol} and G as in (1.8) and (2.5), it is not difficult to check that **D1**–**D4** are satisfied.

We give now the definition of martingale solution in the case of degenerate mobility and irregular potential. The main idea is to formally substitute equation (1.11) in the variational formulation of the problem in order to remove the dependence on the variable μ. The advantage of the degeneracy of the mobility is that the resulting variational formulation makes sense thanks to assumption **D2**.

Definition 2.5. Assume **D1**–**D4**. A martingale solution to the problem (1.10)–(1.13) is a sextuple $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}}, \hat{W}, \hat{\varphi})$ such that:

- $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}})$ is a filtered probability space satisfying the usual conditions;
- $\hat{W}$ is a U-valued cylindrical Wiener process on $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}})$;
- $\hat{\varphi} \in C^0_w([0,T]; L^2(\hat{\Omega}; H)) \cap L^0(\hat{\Omega}; L^2(0,T; V_2))$ is progressively measurable;
- $|\hat{\varphi}| \leqslant 1$ almost everywhere in $\hat{\Omega} \times Q$;
- it holds that

$$\begin{aligned}\int_{\mathcal{O}}\hat{\varphi}(t,x)v(x)\,\mathrm{d}x &+ \int_{Q_t}\Delta\hat{\varphi}(s,x)\,\mathrm{div}\,[m(\hat{\varphi}(s,x))\nabla v(x)]\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_{Q_t}m(\hat{\varphi}(s,x))F''(\hat{\varphi}(s,x))\nabla\hat{\varphi}(s,x)\cdot\nabla v(x)\,\mathrm{d}x\,\mathrm{d}s \qquad (2.6)\\ &= \int_{\mathcal{O}}\varphi_0(x)v(x)\,\mathrm{d}x + \int_{\mathcal{O}}\left(\int_0^t G(\hat{\varphi}(s))\mathrm{d}\hat{W}(s)\right)(x)v(x)\,\mathrm{d}x\end{aligned}$$

for every $v \in V_2 \cap W^{1,d}(\mathcal{O})$, $\hat{\mathbb{P}}$-almost surely, for every $t \in [0, T]$.

Remark 2.6. As we have anticipated, the variational formulation (2.6) in definition 2.5 is formally obtained substituting the definition (1.11) of chemical potential in equation (1.10) and integrating by parts. The reason why we do so is that the chemical potential μ does not inherit enough regularity in the degenerate case. The main advantage of such substitution is that all the terms in (2.6) are still well-defined. Indeed, by **D2** we have that $mF'' \in C^0([-1,1])$, so that the third term on the left-hand side makes sense. Moreover, for every $v \in V_2 \cap W^{1,d}(\mathcal{O})$ we have that $\mathrm{div}(m(\hat\varphi)\nabla v) = m'(\hat\varphi)\nabla\hat\varphi\cdot\nabla v + m(\hat\varphi)\Delta v$, where $m(\hat\varphi), m'(\hat\varphi) \in L^\infty(Q)$ by **D2**. Also, since $V_1 \hookrightarrow L^{\frac{2d}{d-2}}(\mathcal{O})$, noting that $\frac{d-2}{2d}+\frac{1}{d}=\frac{1}{2}$ by the Hölder inequality we have $\mathrm{div}(m(\varphi')\nabla v) \in H$, so that also the second term on the left-hand side of (2.6) makes sense.

We are now ready to state the second main result of the paper, ensuring existence of martingale solutions in the case of degenerate mobility and irregular potential. Both the cases of logarithmic and double-obstacle potential are covered.

Theorem 2.7. *Assume conditions* **D1**–**D4**. *Then, there exists a martingale solution*

$$\left(\hat\Omega, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}}, \hat W, \hat\varphi\right)$$

to the problem (1.10)–(1.13) *in the sense of definition* 2.5 *such that*

$$\hat\varphi \in C^0_w([0,T]; L^2(\hat\Omega; V_1)) \cap L^2(\hat\Omega; L^2(0,T; V_2)),$$
$$F(\hat\varphi),\ M(\hat\varphi) \in L^\infty(0,T; L^1(\Omega\times\mathcal{O})).$$

In particular, if

$$\lim_{|r|\to 1^-} F_1(r) = +\infty \quad \textit{or} \quad \lim_{|r|\to 1^-} M(r) = +\infty, \tag{2.7}$$

then

$$|\hat\varphi(t)| < 1 \quad \textit{a.e. in}\, \hat\Omega\times\mathcal{O} \quad \forall\, t\in[0,T].$$

Let us stress that the last assertion of theorem 2.7 ensures that under (2.7) the concentration $\hat\varphi$ is almost everywhere contained in the interior of the physically relevant domain, meaning that the contact set $\{|\hat\varphi| = 1\}$ has measure 0, or better said that

$$|\{(\hat\omega, x) \in \hat\Omega\times\mathcal{O} : |\hat\varphi(\hat\omega, t, x)| = 1\}| = 0 \quad \forall\, t\in[0,T].$$

Note that in general the degeneracy of the mobility at ± 1 may prevent M to blow up at ± 1, and (2.7) is not always satisfied. For example, an easy computation shows that for degenerate mobility m_{pol} introduced in remark 2.3 we have

$$M_{\text{pol}}(r) = \frac{1}{2}\left((1+r)\ln(1+r) + (1-r)\ln(1-r)\right), \quad r\in(-1,1).$$

Hence, in such a case M_{pol} is bounded in $(-1,1)$ and condition (2.7) is satisfied only if the potential F blows up at ± 1. On the other hand, in case of polynomial mobility m_α with $\alpha \geqslant 2$, condition (2.7) is always satisfied, irrespectively of the potential F. If (2.7) is not satisfied, then one can only infer that $|\hat\varphi| \leqslant 1$ almost everywhere, as it is natural to expect.

3. Positive mobility and regular potential

This section is devoted to the proof of theorem 2.2. The main idea is to perform two separate approximations on the problem. The first one depends on a parameter $\lambda > 0$, and is obtained replacing the nonlinearity F with its Yosida approximation. The second one depends on the parameter $n \in \mathbb{N}$ and is a Faedo–Galerkin finite-dimensional approximation. Uniform estimates are proved first uniformly in n, when λ is fixed, and a passage to the limit as $n \to \infty$ yields existence of approximated solutions for $\lambda > 0$ fixed. Secondly, further uniform estimates are proved uniformly in λ and a passage to the limit as $\lambda \to 0$ gives existence of solutions to the original problem.

3.1. The approximation

First of all, since $F'' \geqslant -C_F$, the function $\gamma : \mathbb{R} \to \mathbb{R}$ defined as $\gamma(r) := F'(r) + C_F r$, $r \in \mathbb{R}$, is nondecreasing and continuous: hence, γ can be identified with a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and satisfies $\gamma(0) = 0$. It makes sense then to introduce the Yosida approximation $\gamma_\lambda : \mathbb{R} \to \mathbb{R}$ of γ for any $\lambda > 0$ and define $\hat{\gamma}_\lambda : \mathbb{R} \to [0, +\infty)$ as $\hat{\gamma}_\lambda(r) := \int_0^r \gamma_\lambda(s)\mathrm{d}s$, $r \in \mathbb{R}$. With this notation, we introduce the approximated potential as

$$F_\lambda : \mathbb{R} \to [0, +\infty), \qquad F_\lambda(r) := F(0) + \hat{\gamma}_\lambda(r) - \frac{C_F}{2} r^2, \quad r \in \mathbb{R}.$$

Let us recall that from the general theory on monotone analysis [2, chapter 2] we know that the Yosida approximation γ_λ is $\frac{1}{\lambda}$-Lipschitz-continuous, hence also linearly bounded. Consequently, by definition of $\hat{\gamma}_\lambda$ there exists a constant $c_\lambda > 0$ such that $\hat{\gamma}(r) \leqslant c_\lambda(1 + |r|^2)$ for all $r \in \mathbb{R}$. Also, noting that $F'_\lambda(r) = \gamma_\lambda(r) - C_F r$ for all $r \in \mathbb{R}$ by definition, this readily ensures that also F'_λ is $\frac{1}{\lambda}$-Lipschitz-continuous and that, possibly renominating c_λ, it holds

$$|F_\lambda(r)| \leqslant c_\lambda(1 + |r|^2) \quad \forall\, r \in \mathbb{R}. \tag{3.1}$$

Secondly, let $(e_j)_{j \in \mathbb{N}_+} \subset V_2$ and $(\alpha_j)_{j \in \mathbb{N}_+}$ be the sequences of eigenfunctions and eigenvalues of the negative Laplace operator with homogeneous Neumann conditions on $\mathcal{O}$, respectively, i.e.

$$\begin{cases} -\Delta e_j = \alpha_j e_j & \text{in } \mathcal{O}, \\ \mathbf{n} \cdot \nabla e_j = 0 & \text{in } \partial\mathcal{O}, \end{cases} \qquad j \in \mathbb{N}_+.$$

Then, possibly using a renormalization procedure, we can suppose that $(e_j)_j$ is a complete orthonormal system of H and an orthogonal system in V_1. For every $n \in \mathbb{N}_+$, we define the finite dimensional space $H_n := \operatorname{span}\{e_1, \dots, e_n\} \subset V_2$, endowed with the $\|\cdot\|_H$-norm.

We define the approximated operator $G_n : H_n \to \mathscr{L}^2(U, H_n)$ as

$$G_n(v)u_k := \sum_{j=1}^n (G(v)u_k, e_j)_H e_j, \quad v \in H_n, \ k \in \mathbb{N}.$$

One can check that G_n is well-defined: indeed, for every $v \in H_n$ and every $n \in \mathbb{N}_+$, thanks to assumption **ND3** we have

$$\sum_{k=0}^{\infty}\|G_n(v)u_k\|_H^2=\sum_{k=0}^{\infty}\sum_{j=1}^{n}|(G(v)u_k,e_j)_H|^2\leqslant\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}|(G(v)u_k,e_j)_H|^2$$
$$=\sum_{k=0}^{\infty}\|G(v)u_k\|_H^2=\|G(v)\|^2_{\mathscr{L}^2(U,H)},$$

so that $G_n(v)\in\mathscr{L}^2(U,H_n)$ for all $v\in H_n$ and

$$\|G_n(v)\|_{\mathscr{L}^2(U,H)}\leqslant\|G(v)\|_{\mathscr{L}^2(U,H)}\quad\forall\, v\in H,\ \forall\, n\in\mathbb{N}_+. \tag{3.2}$$

A similar computations shows also that G_n is Lipschitz-continuous from H_n to $\mathscr{L}^2(U,H_n)$.

Similarly, we define the approximated initial value

$$\varphi_0^n:=\sum_{j=1}^{n}(\varphi_0,e_j)_He_j.$$

Finally, for every $n\in\mathbb{N}_+$ let $m_n:=\rho_n*m$ where (ρ_n) is a standard sequence of mollifiers. In particular, we have that

$$(m_n)_n\subset W^{1,\infty}(\mathbb{R}),\qquad m_*\leqslant m_n(r)\leqslant m^*\quad\forall\, r\in\mathbb{R},$$
$$m_n\to m\quad\text{in}\, C^0([a,b])\ \forall\, a<b.$$

We consider the approximated problem

$$\mathrm{d}\varphi_{\lambda,n}-\mathrm{div}\left(m_n(\varphi_{\lambda,n})\nabla\mu_{\lambda,n}\right)\mathrm{d}t=G_n(\varphi_{\lambda,n})\,\mathrm{d}W\quad\text{in}\,(0,T)\times\mathcal{O}, \tag{3.3}$$
$$\mu_{\lambda,n}=-\Delta\varphi_{\lambda,n}+F'_\lambda(\varphi_{\lambda,n})\quad\text{in}\,(0,T)\times\mathcal{O}, \tag{3.4}$$
$$\mathbf{n}\cdot\nabla\varphi_{\lambda,n}=\mathbf{n}\cdot m(\varphi_{\lambda,n})\nabla\mu_{\lambda,n}=0\quad\text{in}\,(0,T)\times\partial\mathcal{O}, \tag{3.5}$$
$$\varphi_{\lambda,n}(0)=\varphi_0^n\quad\text{in}\,\mathcal{O}. \tag{3.6}$$

Let us fix now $\lambda>0$ and $n\in\mathbb{N}_+$: we look for a solution $(\varphi_{\lambda,n},\ \mu_{\lambda,n})$ to (3.3)–(3.6) in the form

$$\varphi_{\lambda,n}=\sum_{j=1}^{n}a_j^{\lambda,n}e_j,\qquad\mu_{\lambda,n}=\sum_{j=1}^{n}b_j^{\lambda,n}e_j,$$

for some processes

$$a^{\lambda,n}:=(a_1^{\lambda,n},\dots,a_n^{\lambda,n}):\Omega\times[0,T]\to\mathbb{R}^n,\qquad b^{\lambda,n}:=(b_1^{\lambda,n},\dots,b_n^{\lambda,n}):\Omega\times[0,T]\to\mathbb{R}^n.$$

Plugging in the ansatz on $\varphi_{\lambda,n}$ and $\mu_{\lambda,n}$ in (3.3)–(3.6) and taking any arbitrary e_i, $i=1,\dots,n$, as test functions, we immediately see that the variational formulation of (3.3)–(3.6) is given by

$$\int_{\mathcal{O}}\varphi_{\lambda,n}(t,x)e_i(x)\,\mathrm{d}x+\int_{Q_t}m_n(\varphi_{\lambda,n}(s,x))\nabla\mu_{\lambda,n}(s,x)\cdot\nabla e_i(x)\,\mathrm{d}x\,\mathrm{d}s$$
$$=\int_{\mathcal{O}}\varphi_0^n(x)e_i(x)\,\mathrm{d}x+\int_{\mathcal{O}}\left(\int_0^tG_n(\varphi_{\lambda,n}(s))\,\mathrm{d}W(s)\right)(x)e_i(x)\,\mathrm{d}x\quad\forall\, i=1,\dots,n,$$

and

$$\int_{\mathcal{O}} \mu_{\lambda,n}(t,x) e_i(x)\,\mathrm{d}x = \int_{\mathcal{O}} \nabla\varphi_{\lambda,n}(t,x)\cdot\nabla e_i(x)\,\mathrm{d}x + \int_{\mathcal{O}} F'_\lambda(\varphi_{\lambda,n}(t,x)) e_i(x)\,\mathrm{d}x \quad \forall\, i=1,\dots,n,$$

for every $t \in [0,T]$, $\mathbb{P}$-almost surely. Using the orthogonality properties of $(e_j)_j$, we deduce then that $(\varphi_{\lambda,n},\ \mu_{\lambda,n})$ satisfy (3.3)–(3.6) if and only if the vectors $(a^{\lambda,n}, b^{\lambda,n})$ satisfy the stochastic differential equations

$$\begin{aligned}
&\mathrm{d}a_i^{\lambda,n} + \sum_{j=1}^n b_j^{\lambda,n} \int_{\mathcal{O}} m_n\left(\sum_{l=1}^n a_l^{\lambda,n} e_l(x)\right)\nabla e_j(x)\cdot\nabla e_i(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \left(G_n\left(\sum_{l=1}^n a_l^{\lambda,n} e_l\right)\mathrm{d}W, e_i\right)_H, \\
&b_i^{\lambda,n} = \alpha_i a_i^{\lambda,n} + \int_{\mathcal{O}} F'_\lambda\left(\sum_{l=1}^n a_l^{\lambda,n} e_l(x)\right) e_i(x)\,\mathrm{d}x, \\
&a_i^{\lambda,n}(0) = (\varphi_0, e_i)_H,
\end{aligned}$$

for every $i=1,\dots,n$, where the stochastic integral on the right-hand side must be interpreted for any $i=1,\dots,n$ as $G_i^{\lambda,n}\,\mathrm{d}W$ where

$$G_i^{\lambda,n} : H_n \to \mathscr{L}^2(U,\mathbb{R}), \qquad G_i^{\lambda,n} u_k := \left(G_n\left(\sum_{l=1}^n a_l^{\lambda,n} e_l\right) u_k, e_i\right)_H, \quad k\in\mathbb{N}.$$

Since the functions m_n, F'_λ, and the operator G_n are Lipschitz-continuous, the system can be seen as an abstract evolution equation on the Hilbert space H_n with Lipschitz-continuous nonlinearities: hence, the classical theory [28, section 7.1] applies and we can find a unique solution

$$a^{\lambda,n}, b^{\lambda,n} \in L^\ell\left(\Omega; C^0([0,T];\mathbb{R}^n)\right) \quad \forall\,\ell\in[2,+\infty).$$

We deduce that for every $n\in\mathbb{N}$ the approximated system (3.3)–(3.6) admits a unique solution

$$\varphi_{\lambda,n}, \mu_{\lambda,n} \in L^\ell\left(\Omega; C^0([0,T];H_n)\right) \quad \forall\,\ell\in[2,+\infty).$$

3.2. Uniform estimates in n, with λ fixed

We show now that the approximated solution $(\varphi_{\lambda,n},\ \mu_{\lambda,n})$ satisfy some energy estimates, independently of n, with $\lambda > 0$ being fixed.

First of all, integrating (3.3) on $\mathcal{O}$ and using Itô's formula yields

$$\begin{aligned}
\frac{1}{2}|(\varphi_{\lambda,n}(t))_{\mathcal{O}}|^2 &= \frac{1}{2}|(\varphi_0^n)_{\mathcal{O}}|^2 + \int_0^t (\varphi_{\lambda,n}(s))_{\mathcal{O}} (G_n(\varphi_{\lambda,n}(s))\,\mathrm{d}W(s))_{\mathcal{O}} \\
&\quad + \frac{1}{2}\int_0^t \|\mathbb{G}_n(\varphi_{\lambda,n}(s))_{\mathcal{O}}\|^2_{\mathscr{L}^2(U,\mathbb{R})}\mathrm{d}s.
\end{aligned}$$

Taking supremum in time, power $\ell/2$ and expectations, thanks to the Burkholder–Davis–Gundy inequality we have

$$\mathbb{E}\sup_{s\in[0,t]}|(\varphi_{\lambda,n}(s))_{\mathcal{O}}|^{\ell}\lesssim\mathbb{E}|(\varphi_0^n)_{\mathcal{O}}|^{\ell}+\mathbb{E}\left(\int_0^t\sum_{k=0}^{\infty}|(G_n(\varphi_{\lambda,n}(s))u_k)_{\mathcal{O}}|^2\,\mathrm{d}s\right)^{\ell/2}$$
$$+\,\mathbb{E}\left(\int_0^t|(\varphi_{\lambda,n}(s))_{\mathcal{O}}|^2\sum_{k=0}^{\infty}|(G_n(\varphi_{\lambda,n}(s))u_k)_{\mathcal{O}}|^2\,\mathrm{d}s\right)^{\ell/4}.$$

Note that $|(\varphi_0^n)_{\mathcal{O}}|=|\mathcal{O}|^{-1}|\int_{\mathcal{O}}\varphi_0^n|\leqslant|\mathcal{O}|^{-1}\|\varphi_0^n\|_{L^1(\mathcal{O})}\leqslant|\mathcal{O}|^{-1/2}\|\varphi_0^n\|_H\leqslant|\mathcal{O}|^{-1/2}\|\varphi_0\|_H$ and

$$\sum_{k=0}^{\infty}|(G_n(\varphi_{\lambda,n})u_k)_{\mathcal{O}}|^2\leqslant|\mathcal{O}|^{-1}\sum_{k=0}^{\infty}\|G_n(\varphi_{\lambda,n})u_k\|_H^2\leqslant|\mathcal{O}|^{-1}\sum_{k=0}^{\infty}\|G(\varphi_{\lambda,n})u_k\|_H^2$$
$$=|\mathcal{O}|^{-1}\sum_{k=0}^{\infty}\|g_k(\varphi_{\lambda,n})\|_H^2\leqslant\sum_{k=0}^{\infty}\|g_k\|_{L^\infty(\mathbb{R})}^2\leqslant C_G.$$

Hence, by the Young inequality we infer that there exists $c>0$, independent of λ and n, such that

$$\|(\varphi_{\lambda,n})_{\mathcal{O}}\|_{L^{\ell}(\Omega;C^0([0,T]))}\leqslant c. \tag{3.7}$$

We want now to write Itô's formula for the free energy functional

$$\mathcal{E}_\lambda(v):=\frac{1}{2}\int_{\mathcal{O}}|\nabla v|^2+\int_{\mathcal{O}}F_\lambda(v),\quad v\in H_n. \tag{3.8}$$

To this end, note that since $H_n\hookrightarrow V_2$ and F'_λ is Lipschitz-continuous, $\mathcal{E}_\lambda:H_n\to[0,+\infty)$ is Fréchet-differentiable with $\mathrm{d}\mathcal{E}_\lambda:H_n\to H_n^*$ given by

$$D\mathcal{E}_\lambda(v)h=\int_{\mathcal{O}}\nabla v(x)\cdot\nabla h(x)\,\mathrm{d}x+\int_{\mathcal{O}}F'_\lambda(v(x))h(x)\,\mathrm{d}x,\quad v,h\in H_n.$$

Let us show now that also $\mathrm{D}\mathcal{E}_\lambda$ is Fréchet-differentiable with $D^2\mathcal{E}_\lambda:H_n\to\mathscr{L}(H_n,H_n^*)$ given by

$$D^2\mathcal{E}_\lambda(v)[h,k]=\int_{\mathcal{O}}\nabla h(x)\cdot\nabla k(x)\,\mathrm{d}x+\int_{\mathcal{O}}F''_\lambda(v(x))h(x)k(x)\,\mathrm{d}x,\quad v,h,k\in H_n.$$

Indeed, for every $v,h,k\in H_n$ we have that

$$\left|D\mathcal{E}_\lambda(v+k)h-D\mathcal{E}_\lambda(v)h-\int_{\mathcal{O}}\nabla h(x)\cdot\nabla k(x)\,\mathrm{d}x-\int_{\mathcal{O}}F''_\lambda(v(x))h(x)k(x)\,\mathrm{d}x\right|$$
$$=\left|\int_{\mathcal{O}}F'_\lambda(v(x)+k(x))h(x)\,\mathrm{d}x-\int_{\mathcal{O}}F'_\lambda(v(x))h(x)\,\mathrm{d}x-\int_{\mathcal{O}}F''_\lambda(v(x))h(x)k(x)\,\mathrm{d}x\right|$$
$$=\left|\int_0^1\int_{\mathcal{O}}\left(F''_\lambda(v(x)+\tau k(x))-F''_\lambda(v(x))\right)h(x)k(x)\,\mathrm{d}x\,\mathrm{d}\tau\right|.$$

Now, since we have the continuous inclusion $H_n \hookrightarrow L^\infty(\mathcal{O})$, by Hölder inequality we infer that

$$\left|D\mathcal{E}_\lambda(v+k)h - D\mathcal{E}_\lambda(v)h - \int_\mathcal{O} \nabla h(x)\cdot\nabla k(x)\,\mathrm{d}x - \int_\mathcal{O} F''_\lambda(v(x))h(x)k(x)\,\mathrm{d}x\right|$$
$$\leqslant \|h\|_{L^\infty(\mathcal{O})}\|k\|_{L^\infty(\mathcal{O})}\int_0^1 \|F''_\lambda(v+\tau k) - F''_\lambda(v)\|_{L^1(\mathcal{O})}\,\mathrm{d}\tau$$
$$\leqslant c_n^2\|h\|_{H_n}\|k\|_{H_n}\int_0^1 \|F''_\lambda(v+\tau k) - F''_\lambda(v)\|_{L^1(\mathcal{O})}\,\mathrm{d}\tau,$$

where $c_n > 0$ is the norm of the inclusion $H_n \hookrightarrow L^\infty(\mathcal{O})$. Since F''_λ is continuous and bounded, the third factor on the right-hand side converges to 0 if $k \to 0$ in H_n by the dominated convergence theorem, hence

$$\sup_{\|h\|_{H_n}\leqslant 1}\left|D\mathcal{E}_\lambda(v+k)h - D\mathcal{E}_\lambda(v)h - \int_\mathcal{O} \nabla h(x)\cdot\nabla k(x)\,\mathrm{d}x - \int_\mathcal{O} F''_\lambda(v(x))h(x)k(x)\,\mathrm{d}x\right|$$
$$= o\left(\|k\|_{H_n}\right) \quad \text{as } k\to 0 \text{ in } H_n.$$

This shows that $D\mathcal{E}_\lambda$ is indeed Fréchet-differentiable with derivative $D^2\mathcal{E}_\lambda$ given as above. Furthermore, the derivatives $D\mathcal{E}_\lambda$ and $D^2\mathcal{E}_\lambda$ are continuous and bounded on bounded subsets of H_n, as it follows from the Lipschitz-continuity of F'_λ and the continuity and boundedness of F''_λ.

We can then apply Itô's formula to $\mathcal{E}_\lambda(\varphi_{\lambda,n})$ in the classical version of [28]. To this end, note that by (3.4) we have $\mathrm{d}\mathcal{E}_\lambda(\varphi_{\lambda,n}) = \mu_{\lambda,n}$: we obtain then

$$\begin{aligned}
&\mathcal{E}_\lambda(\varphi_{\lambda,n}(t)) + \int_{Q_t} m_n(\varphi_{\lambda,n}(s,t))|\nabla\mu_{\lambda,n}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad= \mathcal{E}_\lambda(\varphi_0^n) + \int_0^t \left(\mu_{\lambda,n}(s), G_n(\varphi_{\lambda,n}(s))\,\mathrm{d}W(s)\right)_H \\
&\qquad+ \frac{1}{2}\int_0^t\sum_{k=0}^\infty\int_\mathcal{O}\left[|\nabla G_n(\varphi_{\lambda,n}(s))u_k|^2(x)\right. \\
&\qquad\left.+ F''_\lambda(\varphi_{\lambda,n}(s,x))|G_n(\varphi_{\lambda,n}(s))u_k|^2(x)\right]\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.9}$$

Taking power $\ell/2$ at both sides, supremum in time and then expectations yields, recalling the definition (3.8) and the assumption **ND2**,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}\|\nabla\varphi_{\lambda,n}(s)\|_H^\ell + \mathbb{E}\sup_{s\in[0,t]}\|F_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(\mathcal{O})}^{\ell/2} + m_*^{\ell/2}\mathbb{E}\|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^\ell \\
&\quad\leqslant c_\ell\left[\|\nabla\varphi_0^n\|_H^\ell + \|F_\lambda(\varphi_0^n)\|_{L^1(\mathcal{O})}^{\ell/2} + \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\left(\mu_{\lambda,n}(r), G_n(\varphi_{\lambda,n}(r))\,\mathrm{d}W(r)\right)_H\right|^{\ell/2}\right. \\
&\qquad+ \mathbb{E}\|G_n(\varphi_{\lambda,n})\|_{L^2(0,t;\mathscr{L}^2(U,V_1))}^\ell \\
&\qquad\left.+ \mathbb{E}\left|\int_0^t\sum_{k=0}^\infty\int_\mathcal{O}|F''_\lambda(\varphi_n(s,x))||G_n(\varphi_{\lambda,n}(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\right|^{\ell/2}\right]
\end{aligned}$$

for every $t \in [0, T]$, $\mathbb{P}$-almost surely, for a certain constant $c_\ell > 0$ depending only on ℓ. Let us estimate the terms on the right-hand side separately. First of all, from the definition of the approximate initial value φ_0^n, since $\varphi \in V_1$ we have $\|\nabla\varphi_0^n\|_H \leqslant \|\nabla\varphi_0\|_H$. Let us focus on the stochastic integral. By the Burkholder–Davis–Gundy inequality and the estimate (3.2), we infer that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \big(\mu_{\lambda,n}(r), G_n(\varphi_n(r))\,\mathrm{d}W(r)\big)_H\right|^{\ell/2} \\
&\quad\leqslant c_\ell \mathbb{E}\left(\int_0^t \|\mu_{\lambda,n}(s)\|_H^2 \|G_n(\varphi_{\lambda,n})\|_{\mathscr{L}^2(U,H)}^2\,\mathrm{d}s\right)^{\ell/4} \\
&\quad\leqslant c_\ell \mathbb{E}\left(\int_0^t \|\mu_{\lambda,n}(s)\|_H^2 \|G(\varphi_{\lambda,n})\|_{\mathscr{L}^2(U,H)}^2\,\mathrm{d}s\right)^{\ell/4}
\end{aligned}$$

for a certain constant $c_\ell > 0$ independent of n and λ. Thanks to assumption **ND3** we have

$$\|G(\varphi_{\lambda,n})\|_{\mathscr{L}^2(U,H)}^2 = \sum_{k=0}^{\infty} \|g_k(\varphi_{\lambda,n})\|_H^2 \leqslant |\mathcal{O}| \sum_{k=0}^{\infty} \|g_k\|_{L^\infty(\mathbb{R})}^2 \leqslant |O| C_G,$$

so that, consequently, we deduce that

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \big(\mu_{\lambda,n}(r), G_n(\varphi_{\lambda,n}(r))\,\mathrm{d}W(r)\big)_H\right|^{\ell/2} \leqslant c_\ell |O|^{\ell/4} C_G^{\ell/4} \mathbb{E}\|\mu_{\lambda,n}\|_{L^2(0,t;H)}^{\ell/2}.$$

Summing and subtracting $(\mu_{\lambda,n})_{\mathcal{O}}$ on the right-hand side, using the Poincaré–Wirtinger inequality and the Young inequality we deduce that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \big(\mu_{\lambda,n}(r), G_n(\varphi_{\lambda,n}(r))\,\mathrm{d}W(r)\big)_H\right|^{\ell/2} \\
&\quad\leqslant \frac{m_*^{\ell/2}}{2}\mathbb{E}\|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^{\ell} + c\mathbb{E}\|(\mu_{\lambda,n})_{\mathcal{O}}\|_{L^2(0,t)}^{\ell/2} + c,
\end{aligned}$$

where $c = c(m_*, c_\ell, |\mathcal{O}|, \ell, C_G) > 0$ is an arbitrarily large constant independent of n and λ. Let us focus now on the trace terms in Itô's formula. Since $G(\varphi_{\lambda,n})$ takes values in $\mathscr{L}^2(U, V_1)$, we have

$$\begin{aligned}
\|G_n(\varphi_{\lambda,n})\|_{\mathscr{L}^2(U,V_1)}^2 &\leqslant \|G(\varphi_{\lambda,n})\|_{\mathscr{L}^2(U,V_1)}^2 = \sum_{k=0}^{\infty} \|g_k(\varphi_{\lambda,n})\|_{V_1}^2 \\
&\leqslant \left(|\mathcal{O}| + \|\nabla\varphi_{\lambda,n}\|_H^2\right) \sum_{k=0}^{\infty} \|g_k\|_{W^{1,\infty}(\mathbb{R})}^2,
\end{aligned}$$

so that **ND3** yields

$$\mathbb{E}\|G_n(\varphi_n)\|_{L^2(0,t;H)}^{\ell} \leqslant c\left(1 + \mathbb{E}\|\nabla\varphi_{\lambda,n}\|_{L^2(0,t;H)}^{\ell}\right)$$

for a certain constant $c > 0$ independent of n and λ. Taking the mean of (3.4) we get

$$(\mu_{\lambda,n})_{\mathcal{O}} = (F_\lambda'(\varphi_{\lambda,n}))_{\mathcal{O}} \leqslant c\|F_\lambda'(\varphi_{\lambda,n})\|_{L^1(\mathcal{O})},$$

so that from assumption **ND1** it follows that

$$|(\mu_{\lambda,n})_{\mathcal{O}}| \leqslant c\left(1 + \|F_\lambda(\varphi_{\lambda,n})\|_{L^1(\mathcal{O})}\right)$$

for a certain $c > 0$ independent of λ and n: we infer then that, possibly updating the value of c,

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\|\nabla\varphi_{\lambda,n}(s)\|_H^\ell + \mathbb{E}\sup_{s\in[0,t]}\|F_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(\mathcal{O})}^{\ell/2} \\
&\quad + \mathbb{E}\sup_{s\in[0,t]}|(\mu_{\lambda,n}(s))_{\mathcal{O}}|^{\ell/2} + \mathbb{E}\|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^\ell \\
&\leqslant c\left(1 + \mathbb{E}\|(\mu_{\lambda,n})_{\mathcal{O}}\|_{L^2(0,t)}^{\ell/2} + \mathbb{E}\|\nabla\varphi_{\lambda,n}\|_{L^2(0,t;H)}^\ell\right) + \|F_\lambda(\varphi_0^n)\|_{L^1(\mathcal{O})}^{\ell/2} \\
&\quad + \frac{1}{2}\mathbb{E}\left(\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\varphi_{\lambda,n}(s,x))||G_n(\varphi_{\lambda,n}(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\ell/2}.
\end{aligned} \tag{3.10}$$

Let us estimate the two terms on the right-hand side. Recall that here $\lambda > 0$ is fixed. First of all, since F_λ is bounded by a quadratic function by (3.1) and $(\varphi_0^n)_n$ is bounded in H thanks to the properties of the orthogonal projection on H_n, we have that

$$\|F_\lambda(\varphi_0^n)\|_{L^1(\mathcal{O})} \leqslant c_\lambda\left(1 + \|\varphi_0^n\|_H^2\right) \leqslant c_\lambda\left(1 + \|\varphi_0\|_H^2\right) \quad \forall n \in \mathbb{N}$$

for a certain $c_\lambda > 0$ independent of n. Secondly, note that since $|F_\lambda''| \leqslant c_\lambda$ for a certain $c_\lambda > 0$ independent of n, by the same computations as above we have

$$\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\varphi_{\lambda,n}(s,x))||G_n(\varphi_n(s))u_k|^2(x)\,\mathrm{d}x \leqslant c_\lambda\|G(\varphi_{\lambda,n})\|_{\mathscr{L}^2(U,H)}^2 \leqslant c_\lambda|\mathcal{O}|C_G.$$

Putting this information together we deduce from (3.10) that there exists a positive constant c_λ, independent of n, such that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\|\nabla\varphi_{\lambda,n}(s)\|_H^\ell + \mathbb{E}\sup_{s\in[0,t]}\|F_\lambda(\varphi_{\lambda,n}(s))\|_{L^1(\mathcal{O})}^{\ell/2} \\
&\quad + \mathbb{E}\sup_{s\in[0,t]}|(\mu_{\lambda,n}(s))_{\mathcal{O}}|^{\ell/2} + \mathbb{E}\|\nabla\mu_{\lambda,n}\|_{L^2(0,t;H)}^\ell \\
&\leqslant c_\lambda\left(1 + \mathbb{E}\|(\mu_{\lambda,n})_{\mathcal{O}}\|_{L^2(0,t)}^{\ell/2} + \mathbb{E}\|\nabla\varphi_{\lambda,n}\|_{L^2(0,t;H)}^\ell\right).
\end{aligned}$$

The Gronwall lemma and the estimate (3.7) yield then, after updating the constant c_λ,

$$\|\varphi_{\lambda,n}\|_{L^\ell(\Omega;C^0([0,T];V_1))} \leqslant c_\lambda, \tag{3.11}$$

$$\|\mu_{\lambda,n}\|_{L^{\ell/2}(\Omega;L^2(0,T;V_1))} + \|\nabla\mu_{\lambda,n}\|_{L^\ell(\Omega;L^2(0,T;H))} \leqslant c_\lambda. \tag{3.12}$$

Moreover, the computations performed above also imply that

$$\|G_n(\varphi_{\lambda,n})\|_{L^\infty(\Omega\times(0,T);\mathscr{L}^2(U,H))\cap L^\ell(\Omega;L^\infty(0,T;\mathscr{L}^2(U,V_1)))} \leqslant c_\lambda$$

for a positive constant c independent of λ and n, hence also by [46, lemma 2.1], for any $s \in (0, 1/2)$,

$$\left\| \int_0^{\cdot} G_n(\varphi_{\lambda,n}(s))\, \mathrm{d}W(s) \right\|_{L^\ell(\Omega;W^{s,\ell}(0,T;V_1))} \leqslant c_{\lambda,s}.$$

By comparison in (3.4) we infer then that

$$\|\varphi_{\lambda,n}\|_{L^\ell(\Omega;W^{\bar{s},\ell}(0,T;V_1^*))} \leqslant c_\lambda, \tag{3.13}$$

where c_λ is independent of n, and $\bar{s} \in (1/\ell, 1/2)$ is fixed.

3.3. Passage to the limit as $n \to \infty$, with λ fixed

We perform here the passage to the limit as $n \to \infty$, keeping $\lambda > 0$ fixed.

Let us show that the sequence of laws of $(\varphi_{\lambda,n})_n$ is tight on $C^0([0, T]; H)$. To this end, let us recall that, since $\bar{s} > 1/\ell$, by [79, corollary 5, p 86] we have the compact inclusion

$$L^\infty(0, T; V_1) \cap W^{\bar{s},\ell}(0, T; V_1^*) \stackrel{c}{\hookrightarrow} C^0([0, T]; H).$$

Hence, for every $R > 0$ the closed ball B_R in $L^\infty(0, T; V_1) \cap W^{\bar{s},\ell}(0, T; V_1^*)$ of radius R is compact in $C^0([0, T]; H)$. Moreover, thanks to the Markov inequality and the estimates (3.11) and (3.13) we have

$$\begin{aligned}\mathbb{P}\{\varphi_{\lambda,n} \in B_R^c\} &= \mathbb{P}\{\|\varphi_{\lambda,n}\|_{L^\infty(0,T;V_1)\cap W^{\bar{s},\ell}(0,T;V_1^*)} > R\} \\ &\leqslant \frac{1}{R^\ell}\mathbb{E}\|\varphi_{\lambda,n}\|^\ell_{L^\infty(0,T;V_1)\cap W^{\bar{s},\ell}(0,T;V_1^*)} \leqslant \frac{c_\lambda^\ell}{R^\ell},\end{aligned}$$

which yields

$$\lim_{R\to+\infty} \sup_{n\in\mathbb{N}_+} \mathbb{P}\{\varphi_{\lambda,n} \in B_R^c\} = 0,$$

as required. Hence, the family of laws of $(\varphi_{\lambda,n})_n$ on $C^0([0, T]; H)$ is tight. Using a similar argument, since $W^{\bar{s},\ell}(0, T; V_1)$ is compactly embedded in $C^0([0, T]; H)$, one can also show that the family of laws of

$$G_n(\varphi_{\lambda,n}) \cdot W := \int_0^{\cdot} G_n(\varphi_{\lambda,n}(s))\, \mathrm{d}W(s)$$

is tight on $C^0([0, T]; H)$. Moreover, taking into account the remarks in subsection 2.1 we identify W with a constant sequence of random variables with values in $C^0([0, T]; U_1)$. Hence, we deduce that the family of laws of $(\varphi_{\lambda,n}, G_n(\varphi_{\lambda,n}) \cdot W, W)_n$ is tight on the product space

$$C^0([0, T]; H) \times C^0([0, T]; H) \times C^0([0, T]; U_1).$$

By Prokhorov and Skorokhod theorems (see [58, theorem 2.7] and [83, theorem 1.10.4, addendum 1.10.5]) and their weaker version by Jakubowski–Skorokhod (see e.g. [9, theorem 2.7.1]), recalling the estimates (3.11)–(3.13) there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and measurable maps $\Lambda_n : (\tilde{\Omega}, \tilde{\mathscr{F}}) \to (\Omega, \mathscr{F})$ such that $\tilde{\mathbb{P}} \circ \Lambda_n^{-1} = \mathbb{P}$ for every $n \in \mathbb{N}$ and

$$\begin{aligned}
\tilde{\varphi}_{\lambda,n} &:= \varphi_{\lambda,n} \circ \Lambda_n \to \tilde{\varphi}_\lambda && \text{in } L^p(\tilde{\Omega}; C^0([0,T]; H)) \ \forall\, p < \ell, \\
\tilde{\varphi}_{\lambda,n} &\overset{*}{\rightharpoonup} \tilde{\varphi}_\lambda && \text{in } L^\ell_w(\tilde{\Omega}; L^\infty(0,T; V_1)) \cap L^\ell(\tilde{\Omega}; W^{\bar{s},\ell}(0,T; V_1^*)), \\
\tilde{\mu}_{\lambda,n} &:= \mu_{\lambda,n} \circ \Lambda_n \rightharpoonup \tilde{\mu}_\lambda && \text{in } L^{\ell/2}(\tilde{\Omega}; L^2(0,T; V_1)), \\
\nabla\tilde{\mu}_{\lambda,n} &\rightharpoonup \nabla\tilde{\mu}_\lambda && \text{in } L^\ell(\tilde{\Omega}; L^2(0,T; H)), \\
\tilde{I}_{\lambda,n} &:= (G_n(\varphi_{\lambda,n}) \cdot W) \circ \Lambda_n \to \tilde{I}_\lambda && \text{in } L^p(\tilde{\Omega}; C^0([0,T]; H)) \ \forall\, p < \ell, \\
\tilde{W}_n &:= W \circ \Lambda_n \to \tilde{W} && \text{in } L^p(\tilde{\Omega}; C^0([0,T]; U_1)) \ \forall\, p < \ell,
\end{aligned}$$

for some measurable processes

$$\begin{aligned}
&\tilde{\varphi}_\lambda \in L^\ell(\tilde{\Omega}; C^0([0,T]; H)) \cap L^\ell_w(\tilde{\Omega}; L^\infty(0,T; V_1)) \cap L^\ell(\tilde{\Omega}; W^{\bar{s},\ell}(0,T; V_1^*)), \\
&\tilde{\mu}_\lambda \in L^{\ell/2}(\tilde{\Omega}; L^2(0,T; V_1)), \qquad \nabla\tilde{\mu}_\lambda \in L^\ell(\tilde{\Omega}; L^2(0,T; H)), \\
&\tilde{I}_\lambda \in L^\ell(\tilde{\Omega}; C^0([0,T]; H)), \\
&\tilde{W} \in L^\ell(\tilde{\Omega}; C^0([0,T]; U_1)).
\end{aligned}$$

Note that possibly enlarging the new probability space, it is not restrictive to suppose that $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ is independent of λ. Now, since F'_λ is Lipschitz-continuous, we readily have

$$F'_\lambda(\tilde{\varphi}_{\lambda,n}) \to F'_\lambda(\tilde{\varphi}_\lambda) \quad \text{in } L^\ell(\tilde{\Omega}; L^2(0,T; H)),$$

and similarly, since $G : H \to \mathscr{L}^2(U,H)$ is Lipschitz-continuous,

$$\begin{aligned}
&\|G_n(\tilde{\varphi}_{\lambda,n}) - G(\tilde{\varphi}_\lambda)\|_{L^p(\tilde{\Omega}; L^2(0,T; \mathscr{L}^2(U,H)))} \\
&\quad \leqslant \|G_n(\tilde{\varphi}_{\lambda,n}) - G_n(\tilde{\varphi}_\lambda)\|_{L^p(\tilde{\Omega}; L^2(0,T; \mathscr{L}^2(U,H)))} \\
&\qquad + \|G_n(\tilde{\varphi}_\lambda) - G(\tilde{\varphi}_\lambda)\|_{L^p(\tilde{\Omega}; L^2(0,T; \mathscr{L}^2(U,H)))} \\
&\quad \leqslant \|G(\tilde{\varphi}_{\lambda,n}) - G(\tilde{\varphi}_\lambda)\|_{L^p(\tilde{\Omega}; L^2(0,T; \mathscr{L}^2(U,H)))} \\
&\qquad + \|G_n(\tilde{\varphi}_\lambda) - G(\tilde{\varphi}_\lambda)\|_{L^p(\tilde{\Omega}; L^2(0,T; \mathscr{L}^2(U,H)))} \to 0,
\end{aligned}$$

so that

$$G_n(\tilde{\varphi}_{\lambda,n}) \to G(\tilde{\varphi}_\lambda) \quad \text{in } L^p(\tilde{\Omega}; L^2(0,T; \mathscr{L}^2(U,H))) \ \forall\, p < \ell.$$

Moreover, since $\varphi_0 \in V_1$ we also have that $\varphi_0^n \to \varphi_0$ in V_1.

Let us handle now the stochastic integral and identify the limit term $\tilde{I}_\lambda$: we follow the classical arguments in [46] and [28, section 8.4]. Introducing the filtration

$$\tilde{\mathscr{F}}_{\lambda,n,t} := \sigma\{\tilde{\varphi}_{\lambda,n}(s), \tilde{I}_{\lambda,n}(s), \tilde{W}_n(s) : s \in [0,t]\}, \quad t \in [0,T],$$

clearly $\tilde{W}_n$ is adapted. Moreover, since the maps Λ_n preserve the laws, we have that $\tilde{W}_n$ is a Q_1-Wiener process on U_1 (hence a cylindrical Wiener process on U), and $\tilde{I}_{\lambda,n}$ is the H-valued martingale

$$\tilde{I}_{\lambda,n}(t) = \int_0^t G_n(\tilde{\varphi}_{\lambda,n}(s))\,\mathrm{d}\tilde{W}_n(s) \quad \forall\, t \in [0,T].$$

These statements follow directly by the fact the Λ_n preserves the laws, and by definition of Q_1-Wiener process and stochastic integral (see for example [73, section 4.5] and [82, section 4.3]).

The next step is to show that the limit process $\tilde W$ is actually a cylindrical Wiener process in U, i.e. that is a Q_1-Wiener process on U_1. To this end, we introduce the filtration

$$\tilde{\mathscr{F}}_{\lambda,t} := \sigma\{\tilde\varphi_\lambda(s), \tilde I_\lambda(s), \tilde W(s) : s \in [0,t]\}, \quad t \in [0,T],$$

so that clearly $\tilde W$ is adapted to it. Also, it holds $\tilde W(0) = 0$ since $\tilde W_n(0) = 0$ and $\tilde W_n \to \tilde W$ in $C^0([0,T];U_1)$ as $n \to \infty$. Moreover, for every $s, t \in [0,T]$ with $s \leqslant t$ and for every bounded continuous function $\psi \in C^0_b(C^0([0,s];H) \times C^0([0,s];H) \times C^0([0,s];U_1))$, since $\tilde W_n$ is a $(\tilde{\mathscr{F}}_{\lambda,n,t})_t$-martingale we have

$$\tilde{\mathbb{E}}\left[(\tilde W_n(t) - \tilde W_n(s))\psi(\tilde\varphi_{\lambda,n}, \tilde I_{\lambda,n}, \tilde W_n)\right] = 0 \quad \forall\, n \in \mathbb{N}.$$

Hence, letting $n \to \infty$ we deduce, thanks to the strong convergences above, the continuity and boundedness of ψ, and the dominated convergence theorem, that

$$\tilde{\mathbb{E}}\left[(\tilde W(t) - \tilde W(s))\psi(\tilde\varphi_\lambda, \tilde I_\lambda, \tilde W)\right] = 0,$$

yielding that $\tilde W$ is a $(\tilde{\mathscr{F}}_{\lambda,t})_t$-martingale in U_1. Similarly, since $\tilde W_n$ is a Q_1-Wiener process on U_1, by the dominated convergence theorem it holds that, for every $h, k \in U_1$,

$$\begin{aligned} 0 = \tilde{\mathbb{E}}&\Big[\big((\tilde W_n(t), h)_{U_1}(\tilde W_n(t), k)_{U_1} - (\tilde W_n(s), h)_{U_1}(\tilde W_n(s), k)_{U_1} \\ &- (t-s)(Q_1 h, k)_{U_1}\big)\psi(\tilde\varphi_{\lambda,n}, \tilde I_{\lambda,n}, \tilde W_n)\Big] \\ &\xrightarrow{n\to\infty} \tilde{\mathbb{E}}\Big[\big((\tilde W(t), h)_{U_1}(\tilde W(t), k)_{U_1} - (\tilde W(s), h)_{U_1}(\tilde W(s), k)_{U_1} \\ &- (t-s)(Q_1 h, k)_{U_1}\big)\,\psi(\tilde\varphi_\lambda, \tilde I_\lambda, \tilde W)\Big], \end{aligned}$$

from which we get that the tensor quadratic variation of $\tilde W$ on U_1 is

$$\langle\!\langle \tilde W \rangle\!\rangle(t) = tQ_1, \quad t \in [0,T].$$

Hence, by [28, thorem 4.6] we deduce that $\tilde W$ is a Q_1-Wiener process on U_1, i.e. a cylindrical Wiener process on U, with respect to the filtration $(\tilde{\mathscr{F}}_{\lambda,t})_t$.

Next, let us argue on the same same line for $\tilde I_\lambda$. First of all, it is clear that $\tilde I_\lambda$ is $(\tilde{\mathscr{F}}_{\lambda,t})_t$-adapted and $\tilde I_\lambda(0) = 0$, since $\tilde I_{\lambda,n}(0) = 0$ for all $n \in \mathbb{N}$. Secondly, for every $s, t \in [0,T]$ with $s \leqslant t$ and for every $\psi \in C^0_b(C^0([0,s];H) \times C^0([0,s];H) \times C^0([0,s];U_1))$, the martingale property of $\tilde I_{\lambda,n}$ and the dominated convergence theorem yield again

$$0 = \tilde{\mathbb{E}}\left[(\tilde I_{\lambda,n}(t) - \tilde I_{\lambda,n}(s))\psi(\tilde\varphi_{\lambda,n}, \tilde I_{\lambda,n}, \tilde W_n)\right] \xrightarrow{n\to\infty} \tilde{\mathbb{E}}\left[(\tilde I_\lambda(t) - \tilde I_\lambda(s))\psi(\tilde\varphi_\lambda, \tilde I_\lambda, \tilde W)\right].$$

We deduce that $\tilde I_\lambda$ is an H-valued $(\tilde{\mathscr{F}}_{\lambda,t})_t$-martingale. Similarly, since $\tilde I_{\lambda,n} = G_n(\tilde\varphi_{\lambda,n}) \cdot \tilde W_n$, we have the tensor quadratic variation

$$\langle\!\langle \tilde I_{\lambda,n} \rangle\!\rangle(t) = \int_0^t G_n(\tilde\varphi_{\lambda,n}(s)) \circ G_n(\tilde\varphi_{\lambda,n}(s))^*\, \mathrm{d}s, \quad t \in [0,T].$$

Consequently, given arbitrary $h, k \in H$, the dominated convergence theorem yields again that

$$\begin{aligned} 0 = \tilde{\mathbb{E}}\Bigg[\Bigg(&(\tilde{I}_{\lambda,n}(t), h)_H(\tilde{I}_{\lambda,n}(t), k)_H - (\tilde{I}_{\lambda,n}(s), h)_H(\tilde{I}_{\lambda,n}(s), k)_H \\ &- \int_s^t (G_n(\tilde{\varphi}_{\lambda,n}(r))G_n(\tilde{\varphi}_{\lambda,n}(r))^* h, k)_H \, \mathrm{d}r\Bigg) \psi(\tilde{\varphi}_{\lambda,n}, \tilde{I}_{\lambda,n}, \tilde{W}_n)\Bigg] \\ \xrightarrow{n\to\infty} \tilde{\mathbb{E}}\Bigg[\Bigg(&(\tilde{I}_{\lambda}(t), h)_H(\tilde{I}_{\lambda}(t), k)_H - (\tilde{I}_{\lambda}(s), h)_H(\tilde{I}_{\lambda}(s), k)_H \\ &- \int_s^t (G(\tilde{\varphi}_{\lambda}(r))G(\tilde{\varphi}_{\lambda}(r))^* h, k)_H \, \mathrm{d}r\Bigg) \psi(\tilde{\varphi}_{\lambda}, \tilde{I}_{\lambda}, \tilde{W})\Bigg], \end{aligned}$$

so that the tensor quadratic variation of $\tilde{I}_\lambda$ is given by

$$\langle\langle \tilde{I}_\lambda \rangle\rangle(t) = \int_0^t G(\tilde{\varphi}_\lambda(s)) \circ G(\tilde{\varphi}_\lambda(s))^* \, \mathrm{d}s, \quad t \in [0, T]. \tag{3.14}$$

Taking these remarks into account and setting $\tilde{M}_\lambda := G(\tilde{\varphi}_\lambda) \cdot \tilde{W}$, with the information collected so far we know that $\tilde{M}_\lambda$ and $\tilde{I}_\lambda$ are H-valued martingales with respect to $(\tilde{\mathscr{F}}_{\lambda,t})_t$ with tensor quadratic variations given by

$$\langle\langle \tilde{M}_\lambda \rangle\rangle(t) = \langle\langle \tilde{I}_\lambda \rangle\rangle(t) = \int_0^t G(\tilde{\varphi}_\lambda(s)) \circ G(\tilde{\varphi}_\lambda(s))^* \, \mathrm{d}s, \quad t \in [0, T]. \tag{3.15}$$

In order to conclude that actually $\tilde{M}_\lambda = \tilde{I}_\lambda$, we need to exploit their tensor quadratic covariation. To this end, recalling the definition of Q_1 in subsection 2.1, by [70, theorem 3.2, p 12] we have that

$$\begin{aligned} \langle\langle \tilde{I}_{\lambda,n}, \tilde{W}_n \rangle\rangle(t) &= \int_0^t G_n(\tilde{\varphi}_{\lambda,n}(s)) \circ \iota^{-1} \, \mathrm{d}\langle\langle \tilde{W}_n \rangle\rangle(s) \\ &= \int_0^t G_n(\tilde{\varphi}_{\lambda,n}(s)) \circ \iota^{-1} \circ Q_1 \, \mathrm{d}s \\ &= \int_0^t G_n(\tilde{\varphi}_{\lambda,n}(s)) \circ \iota^* \, \mathrm{d}s, \quad t \in [0, T], \end{aligned}$$

yielding, taking adjoints (and recalling that $\iota : U \to U_1$ is the usual inclusion),

$$\langle\langle \tilde{W}_n, \tilde{I}_{\lambda,n} \rangle\rangle(t) = \int_0^t \iota \circ G_n(\tilde{\varphi}_{\lambda,n}(s))^* \, \mathrm{d}s, \quad t \in [0, T].$$

Hence, for every $s, t \in [0, T]$ with $s \leqslant t$, for every $\psi \in C_b^0(C^0([0, s]; H) \times C^0([0, s]; H) \times C^0([0, s]; U_1))$, for every $h \in U_1$ and $k \in H$, we have again by the dominated convergence theorem

$$\begin{aligned} 0 = \tilde{\mathbb{E}}\Bigg[\Bigg(&(\tilde{W}_n(t), h)_{U_1}(\tilde{I}_{\lambda,n}(t), k)_H - (\tilde{W}_n(s), h)_{U_1}(\tilde{I}_{\lambda,n}(s), k)_H \\ &- \int_s^t (G_n(\tilde{\varphi}_{\lambda,n}(r))^* k, h)_{U_1} \, \mathrm{d}r\Bigg) \psi(\tilde{\varphi}_{\lambda,n}, \tilde{I}_{\lambda,n}, \tilde{W}_n)\Bigg] \\ \xrightarrow{n\to\infty} \tilde{\mathbb{E}}\Bigg[\Bigg(&(\tilde{W}(t), h)_{U_1}(\tilde{I}_{\lambda}(t), k)_H \\ &- (\tilde{W}(s), h)_{U_1}(\tilde{I}_{\lambda}(s), k)_H - \int_s^t (G(\tilde{\varphi}_{\lambda}(r))^* k, h)_{U_1} \, \mathrm{d}r\Bigg) \psi(\tilde{\varphi}_{\lambda}, \tilde{I}_{\lambda}, \tilde{W})\Bigg]. \end{aligned}$$

Consequently, we have that

$$\langle\!\langle \tilde{W}, \tilde{I}_\lambda \rangle\!\rangle(t) = \int_0^t \iota \circ G(\tilde{\varphi}_\lambda(s))^* \, \mathrm{d}s, \quad t \in [0,T]. \tag{3.16}$$

We are now ready to conclude: indeed, taking (3.14)–(3.16) into account, we get

$$\begin{aligned}\langle\!\langle \tilde{M}_\lambda - \tilde{I}_\lambda \rangle\!\rangle &= \langle\!\langle \tilde{M}_\lambda \rangle\!\rangle + \langle\!\langle \tilde{I}_\lambda \rangle\!\rangle - 2\langle\!\langle \tilde{M}_\lambda, \tilde{I}_\lambda \rangle\!\rangle \\ &= 2\int_0^\cdot G(\tilde{\varphi}_\lambda(s)) \circ G(\tilde{\varphi}_\lambda(s))^* \, \mathrm{d}s - 2\int_0^\cdot G(\tilde{\varphi}_\lambda(s)) \, \mathrm{d}\langle\!\langle \tilde{W}, \tilde{I}_\lambda \rangle\!\rangle(s) = 0,\end{aligned}$$

which yields that

$$\tilde{I}_\lambda = \tilde{M}_\lambda = \int_0^\cdot G(\tilde{\varphi}_\lambda(s)) \, \mathrm{d}\tilde{W}(s). \tag{3.17}$$

Eventually, testing (3.3) by arbitrary $v \in V_1$ and integrating in time yields, after letting $n \to \infty$ and taking (3.17) into account

$$\begin{aligned}&\int_{\mathcal{O}} \tilde{\varphi}_\lambda(t,x) v(x) \, \mathrm{d}x + \int_{Q_t} m(\tilde{\varphi}_\lambda(s,x)) \nabla \tilde{\mu}_\lambda(s,x) \cdot \nabla v(x) \mathrm{d}x \, \mathrm{d}s \\ &\quad = \int_{\mathcal{O}} \varphi_0(x) v(x) \mathrm{d}x + \int_{\mathcal{O}} \left(\int_0^t G(\tilde{\varphi}_\lambda(s)) \mathrm{d}\tilde{W}(s) \right)(x) v(x) \mathrm{d}x\end{aligned} \tag{3.18}$$

for every $t \in [0,T]$, $\tilde{\mathbb{P}}$-almost surely. Indeed, this follows directly form the convergences proved above, the fact that $m_n(\tilde{\varphi}_{\lambda,n}) \to m(\tilde{\varphi}_\lambda)$ almost everywhere, the fact that $|m_n| \leqslant m^*$, and the dominated convergence theorem. Moreover, testing (3.4) by $v \in V_1$ gives

$$\int_{\mathcal{O}} \tilde{\mu}_\lambda(t,x) v(x) \mathrm{d}x = \int_{\mathcal{O}} \nabla \tilde{\varphi}_\lambda(s,x) \cdot \nabla v(x) \mathrm{d}x + \int_{\mathcal{O}} F'_\lambda(\tilde{\varphi}_\lambda(t,x)) v(x) \mathrm{d}x \tag{3.19}$$

for almost every $t \in (0,T)$, $\tilde{\mathbb{P}}$-almost surely.

Finally, if we take expectations in (3.9), noting the stochastic integral is a martingale and that $\tilde{\mathbb{P}} \circ \Lambda_n^{-1} = \mathbb{P}$ for every n, we obtain

$$\begin{aligned}&\frac{1}{2}\tilde{\mathbb{E}}\|\nabla\tilde{\varphi}_{\lambda,n}(t)\|_H^2 + \tilde{\mathbb{E}}\|F_\lambda(\tilde{\varphi}_{\lambda,n}(t))\|_{L^1(\mathcal{O})} + \tilde{\mathbb{E}}\int_{Q_t} m_n(\tilde{\varphi}_{\lambda,n}(s,x))|\nabla\tilde{\mu}_{\lambda,n}(s,x)|^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\quad \leqslant \frac{1}{2}\|\nabla\varphi_0^n\|_H^2 + \|F_\lambda(\varphi_0^n)\|_{L^1(\mathcal{O})} + \frac{C_G}{2}\tilde{\mathbb{E}}\int_0^t \|\nabla\tilde{\varphi}_{\lambda,n}(s)\|_H^2 \, \mathrm{d}s \\ &\qquad + \frac{1}{2}\tilde{\mathbb{E}}\int_0^t \sum_{k=0}^\infty \int_{\mathcal{O}} F''_\lambda(\tilde{\varphi}_{\lambda,n}(s,x))|g_k(\tilde{\varphi}_{\lambda,n}(s,x))|^2 \, \mathrm{d}x \, \mathrm{d}s\end{aligned}$$

for every $t \in [0,T]$. We want to let $n \to \infty$ using the convergences proved above. To this end, the first two terms on the left-hand side and all the terms on the right-hand side pass to the limit by weak lower semicontinuity and the dominated convergence theorem (recall that F''_λ is continuous and bounded). In order to pass to the limit by lower semicontinuity in the third term on the left-hand side, it is sufficient to show that

$$\sqrt{m_n(\tilde{\varphi}_{\lambda,n})}\nabla\tilde{\mu}_{\lambda,n} \rightharpoonup \sqrt{m(\tilde{\varphi}_\lambda)}\nabla\tilde{\mu}_\lambda \quad \text{in } L^1(\tilde{\Omega} \times Q). \tag{3.20}$$

To prove this, note that since $m_n(\tilde{\varphi}_{\lambda,n}) \to m(\tilde{\varphi}_\lambda)$ a.e. in $\tilde{\Omega} \times Q$, for any arbitrary fixed $\sigma > 0$, by the Severini–Egorov theorem there is a measurable set $A_\sigma \subset \tilde{\Omega} \times Q$ such that $|A_\sigma^c| \leqslant \sigma$ and $m_n(\tilde{\varphi}_{\lambda,n}) \to m(\tilde{\varphi}_\lambda)$ uniformly in A_σ. In particular, we have that $\sqrt{m_n(\tilde{\varphi}_{\lambda,n})}1_{A_\sigma} \to \sqrt{m(\tilde{\varphi}_\lambda)}1_{A_\sigma}$ in $L^\infty(\tilde{\Omega} \times Q)$. Consequently, for any $\zeta \in L^\infty(\tilde{\Omega} \times Q)^d$ we have

$$\int_{\tilde{\Omega}\times Q} \sqrt{m_n(\tilde{\varphi}_{\lambda,n})}\nabla\tilde{\mu}_{\lambda,n} \cdot \zeta = \int_{\tilde{\Omega}\times Q} 1_{A_\sigma}\sqrt{m_n(\tilde{\varphi}_{\lambda,n})}\nabla\tilde{\mu}_{\lambda,n} \cdot \zeta + \int_{A_\sigma^c} \sqrt{m_n(\tilde{\varphi}_{\lambda,n})}\nabla\tilde{\mu}_{\lambda,n} \cdot \zeta.$$

Since $\sqrt{m_n(\tilde{\varphi}_{\lambda,n})}1_{A_\sigma} \to \sqrt{m(\tilde{\varphi}_\lambda)}1_{A_\sigma}$ in $L^\infty(\tilde{\Omega} \times Q)$ and $\zeta \in L^\infty(\tilde{\Omega} \times Q)$ we have

$$\int_{\tilde{\Omega}\times Q} 1_{A_\sigma}\sqrt{m_n(\tilde{\varphi}_{\lambda,n})}\nabla\tilde{\mu}_{\lambda,n} \cdot \zeta \to \int_{\tilde{\Omega}\times Q} 1_{A_\sigma}\sqrt{m(\tilde{\varphi}_\lambda)}\nabla\tilde{\mu}_\lambda \cdot \zeta,$$

while the Hölder inequality and the boundedness of $\nabla\tilde{\mu}_{\lambda,n}$ in $L^2(\tilde{\Omega} \times Q)$ yields

$$\begin{aligned}\left|\int_{A_\sigma^c} \sqrt{m_n(\tilde{\varphi}_{\lambda,n})}\nabla\tilde{\mu}_{\lambda,n} \cdot \zeta\right| &\leqslant (m^*)^{1/2}\|\nabla\tilde{\mu}_{\lambda,n}\|_{L^2(\tilde{\Omega}\times Q)}\|\zeta\|_{L^2(A_\sigma^c)} \\ &\leqslant c\|\zeta\|_{L^\infty(\tilde{\Omega}\times Q)}\sigma^{1/2},\end{aligned}$$

where $c > 0$ is independent of n and σ. Since σ and ζ are arbitrary, we infer that (3.20) holds. Hence, passing to the limit as $n \to \infty$ yields by weak lower semicontinuity, for every $t \in [0, T]$,

$$\begin{aligned}&\frac{1}{2}\tilde{\mathbb{E}}\|\nabla\tilde{\varphi}_\lambda(t)\|_H^2 + \tilde{\mathbb{E}}\|F_\lambda(\tilde{\varphi}_\lambda(t))\|_{L^1(\mathcal{O})} + \tilde{\mathbb{E}}\int_{Q_t} m(\tilde{\varphi}_\lambda(s,x))|\nabla\tilde{\mu}_\lambda(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\leqslant \frac{1}{2}\|\nabla\varphi_0\|_H^2 + \|F_\lambda(\varphi_0)\|_{L^1(\mathcal{O})} + \frac{C_G}{2}\tilde{\mathbb{E}}\int_0^t \|\nabla\tilde{\varphi}_\lambda(s)\|_H^2\,\mathrm{d}s \\ &\qquad + \frac{1}{2}\tilde{\mathbb{E}}\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}} F_\lambda''(\tilde{\varphi}_\lambda(s,x))|g_k(\tilde{\varphi}_\lambda(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.21}$$

3.4. *Uniform estimates in* λ

We prove here uniform estimates independently of λ.

First of all, since $\tilde{\mathbb{P}} \circ \Lambda_n^{-1} = \mathbb{P}$ for every $n \in \mathbb{N}_+$, from (3.7) and weak lower semicontinuity it follows that

$$\|(\tilde{\varphi}_\lambda)_{\mathcal{O}}\|_{L^\ell(\tilde{\Omega};C^0([0,T]))} \leqslant c$$

for a certain $c > 0$ independent of λ.

Secondly, from the estimate (3.10) we infer that

$$\begin{aligned}&\tilde{\mathbb{E}}\sup_{s\in[0,t]}\|\nabla\tilde{\varphi}_{\lambda,n}(s)\|_H^\ell + \tilde{\mathbb{E}}\sup_{s\in[0,t]}\|F_\lambda(\tilde{\varphi}_{\lambda,n}(s))\|_{L^1(\mathcal{O})}^{\ell/2} \\ &\quad + \tilde{\mathbb{E}}\sup_{s\in[0,t]}|(\tilde{\mu}_{\lambda,n}(s))_{\mathcal{O}}|^{\ell/2} + \tilde{\mathbb{E}}\|\nabla\tilde{\mu}_{\lambda,n}\|_{L^2(0,t;H)}^\ell \\ &\leqslant c\left(1 + \tilde{\mathbb{E}}\|(\tilde{\mu}_{\lambda,n})_{\mathcal{O}}\|_{L^2(0,t)}^{\ell/2} + \tilde{\mathbb{E}}\|\nabla\tilde{\varphi}_{\lambda,n}\|_{L^2(0,t;H)}^\ell\right) + \|F_\lambda(\varphi_0^n)\|_{L^1(\mathcal{O})}^{\ell/2} \\ &\quad + \frac{1}{2}\tilde{\mathbb{E}}\left(\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\tilde{\varphi}_{\lambda,n}(s,x))||G_n(\tilde{\varphi}_{\lambda,n}(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\ell/2},\end{aligned}$$

where again c is independent of λ. We want to let $n\to\infty$ using weak lower semicontinuity of the norms at both sides. To this end, since $\varphi_0^n\to\varphi_0$ in V_1 and F_λ is bounded by a quadratic function by (3.1), $F_\lambda(\varphi_0^n)\to F_\lambda(\varphi_0)$ in $L^1(\mathcal{O})$ as $n\to\infty$. Moreover, recalling also the strong convergences $G_n(\tilde\varphi_{\lambda,n})\to G(\tilde\varphi_\lambda)$ in $L^p(\tilde\Omega;L^2(0,T;\mathscr{L}^2(U,H)))$ and $\tilde\varphi_{\lambda,n}\to\tilde\varphi_\lambda$ in $L^p(\tilde\Omega;C^0([0,T];H))$ for every $p<\ell$, proved in the previous subsection, we have in particular that

$$|F_\lambda''(\tilde\varphi_{\lambda,n})||G_n(\tilde\varphi_{\lambda,n})u_k|^2\to|F_\lambda''(\tilde\varphi_\lambda)||G(\tilde\varphi_\lambda)u_k|^2\quad\text{a.e. in }\tilde\Omega\times(0,T)\times\mathcal{O},\ \forall\,k\in\mathbb{N}.$$

Since $|F_\lambda''|\leqslant c_\lambda$, by **ND3** we have

$$\int_{\mathcal{O}}|F_\lambda''(\tilde\varphi_{\lambda,n})||G_n(\tilde\varphi_{\lambda,n})u_k|^2\leqslant c_\lambda\|G_n(\tilde\varphi_{\lambda,n})u_k\|_H^2\leqslant c_\lambda\|G(\tilde\varphi_{\lambda,n})u_k\|_H^2\leqslant c_\lambda|\mathcal{O}|\|g_k\|^2_{L^\infty(\mathbb{R})},$$

and the dominated convergence theorem yields

$$\tilde{\mathbb{E}}\left(\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\tilde\varphi_{\lambda,n}(s,x))||G_n(\tilde\varphi_{\lambda,n}(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\ell/2}$$
$$\to\tilde{\mathbb{E}}\left(\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\tilde\varphi_\lambda(s,x))||G(\tilde\varphi_\lambda(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\ell/2}.$$

We infer then, letting $n\to\infty$ and using weak lower semicontinuity, that

$$\begin{aligned}&\tilde{\mathbb{E}}\sup_{s\in[0,t]}\|\nabla\tilde\varphi_\lambda(s)\|_H^\ell+\tilde{\mathbb{E}}\sup_{s\in[0,t]}\|F_\lambda(\tilde\varphi_\lambda(s))\|_{L^1(\mathcal{O})}^{\ell/2}\\&\quad+\tilde{\mathbb{E}}\sup_{s\in[0,t]}|(\tilde\mu_\lambda(s))_{\mathcal{O}}|^{\ell/2}+\tilde{\mathbb{E}}\|\nabla\mu_\lambda\|^\ell_{L^2(0,t;H)}\\&\leqslant c\left(1+\tilde{\mathbb{E}}\|(\tilde\mu_\lambda)_{\mathcal{O}}\|^{\ell/2}_{L^2(0,t)}+\tilde{\mathbb{E}}\|\nabla\tilde\varphi_\lambda\|^\ell_{L^2(0,t;H)}\right)+\|F_\lambda(\varphi_0)\|^{\ell/2}_{L^1(\mathcal{O})}\\&\quad+\frac12\tilde{\mathbb{E}}\left(\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\tilde\varphi_\lambda(s,x))||G(\tilde\varphi_\lambda(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\ell/2},\end{aligned}$$

where the constant c is independent of λ. Let us bound the last two terms on the right-hand side uniformly in λ. First of all, recalling that F is a quadratic perturbation of the convex function $\hat\gamma$, by definition of F_λ we have that $\|F_\lambda(\varphi_0)\|_{L^1(\mathcal{O})}\leqslant\|F(\varphi_0)\|_{L^1(\mathcal{O})}$. Secondly, using the Hölder inequality and assumption **ND3** we have

$$\begin{aligned}&\int_0^t\sum_{k=0}^\infty\int_{\mathcal{O}}|F_\lambda''(\tilde\varphi_\lambda(s,x))||G(\tilde\varphi_\lambda(s))u_k|^2(x)\,\mathrm{d}x\,\mathrm{d}s\\&\leqslant\int_0^t\|F_\lambda''(\tilde\varphi_\lambda(s))\|_{L^1(\mathcal{O})}\,\mathrm{d}s\sum_{k=0}^\infty\|g_k(\tilde\varphi_\lambda)\|^2_{L^\infty(Q)}\\&\leqslant|Q|C_G\int_0^t\|F_\lambda''(\tilde\varphi_\lambda(s))\|_{L^1(\mathcal{O})}\,\mathrm{d}s.\end{aligned}$$

Putting this information together, using the growth assumption on F'' in **ND1**, we have then

$$\begin{aligned}&\tilde{\mathbb{E}}\sup_{s\in[0,t]}\|\nabla\tilde{\varphi}_\lambda(s)\|_H^\ell+\tilde{\mathbb{E}}\sup_{s\in[0,t]}\|F_\lambda(\tilde{\varphi}_\lambda(s))\|_{L^1(\mathcal{O})}^{\ell/2}\\&\quad+\tilde{\mathbb{E}}\sup_{s\in[0,t]}|(\tilde{\mu}_\lambda(s))_{\mathcal{O}}|^{\ell/2}+\tilde{\mathbb{E}}\|\nabla\tilde{\mu}_\lambda\|_{L^2(0,t;H)}^\ell\\&\leqslant c\left(1+\tilde{\mathbb{E}}\|(\tilde{\mu}_\lambda)_{\mathcal{O}}\|_{L^2(0,t)}^{\ell/2}+\tilde{\mathbb{E}}\|\nabla\tilde{\varphi}_\lambda\|_{L^2(0,t;H)}^\ell+\tilde{\mathbb{E}}\|F_\lambda(\tilde{\varphi}_\lambda)\|_{L^1(0,t;L^1(\mathcal{O}))}^{\ell/2}\right)\\&\quad+\|F(\varphi_0)\|_{L^1(\mathcal{O})}^{\ell/2}\end{aligned}$$

,where the constant c (possibly updated) is independent of λ. The Gronwall lemma yields then the estimates

$$\|\tilde{\varphi}_\lambda\|_{L^\ell(\tilde{\Omega};L^\infty(0,T;V_1))}\leqslant c, \tag{3.22}$$

$$\|\tilde{\mu}_\lambda\|_{L^{\ell/2}(\tilde{\Omega};L^2(0,T;V_1))}+\|\nabla\tilde{\mu}_\lambda\|_{L^\ell(\tilde{\Omega};L^2(0,T;H))}\leqslant c. \tag{3.23}$$

Thanks to **ND3** we deduce that

$$\|G(\tilde{\varphi}_\lambda)\|_{L^\infty(\tilde{\Omega}\times(0,T);\mathscr{L}^2(U,H))\cap L^\ell(\tilde{\Omega};L^\infty(0,T;\mathscr{L}^2(U,V_1)))}\leqslant c,$$

hence also by [46, lemma 2.1], for any $s\in(0,1/2)$,

$$\left\|\int_0^\cdot G(\tilde{\varphi}_\lambda(s))\mathrm{d}\tilde{W}(s)\right\|_{L^\ell(\tilde{\Omega};W^{s,\ell}(0,T;V_1))}\leqslant c_s. \tag{3.24}$$

By comparison in (3.18) we infer that

$$\|\tilde{\varphi}_\lambda\|_{L^\ell(\tilde{\Omega};W^{\bar{s},\ell}(0,T;V_1^*))}\leqslant c, \tag{3.25}$$

where again c is independent of λ, and $\bar{s}\in(1/\ell,1/2)$ is fixed. Finally, testing the variational equation (3.19) by $\gamma_\lambda(\tilde{\varphi}_\lambda)=F'_\lambda(\tilde{\varphi}_\lambda)+C_F\tilde{\varphi}_\lambda$ and rearranging the terms we have, for almost every $t\in(0,T)$,

$$\begin{aligned}&\int_{\mathcal{O}}\gamma'_\lambda(\tilde{\varphi}_\lambda(t,x))|\nabla\tilde{\varphi}_\lambda(t,x)|^2\,\mathrm{d}x+\int_{\mathcal{O}}|F'_\lambda(\tilde{\varphi}_\lambda(t,x))|^2\,\mathrm{d}x\\&=\int_{\mathcal{O}}\tilde{\mu}_\lambda(t,x)F'_\lambda(\tilde{\varphi}_\lambda(t,x))\mathrm{d}x+C_F\int_{\mathcal{O}}\tilde{\mu}_\lambda(t,x)\tilde{\varphi}_\lambda(t,x)\,\mathrm{d}x\\&\quad-C_F\int_{\mathcal{O}}F'_\lambda(\tilde{\varphi}_\lambda(t,x))\tilde{\varphi}_\lambda(t,x)\,\mathrm{d}x.\end{aligned}$$

Since the first term on the left-hand side is nonnegative by monotonicity of γ_λ, the Young inequality yields, after integrating in time,

$$\|F'_\lambda(\tilde{\varphi}_\lambda)\|_{L^2(0,T;H)}^2\leqslant\frac{1}{2}\|F'_\lambda(\tilde{\varphi}_\lambda)\|_{L^2(0,T;H)}^2+\frac{3}{2}\|\tilde{\mu}_\lambda\|_{L^2(0,T;H)}^2+\frac{3}{2}C_F^2\|\tilde{\varphi}_\lambda\|_{L^2(0,T;H)}^2,$$

so that by (3.22)–(3.23) we have

$$\|F'_\lambda(\tilde{\varphi}_\lambda)\|_{L^{\ell/2}(\tilde{\Omega};L^2(0,T;H))}\leqslant c. \tag{3.26}$$

By comparison in (3.19) we deduce that $\Delta\tilde{\varphi}_\lambda \in L^{\ell/2}(\tilde{\Omega}; L^2(0,T;H))$ and by elliptic regularity

$$\|\tilde{\varphi}_\lambda\|_{L^{\ell/2}(\tilde{\Omega};L^2(0,T;V_2))} \leqslant c. \tag{3.27}$$

3.5. Passage to the limit as $\lambda \to 0$

We perform here the passage to the limit as $\lambda \to 0$ and recover a martingale solution to the original problem (1.10)–(1.13). Since the arguments are very similar to the ones in subsection 3.3, we shall omit the details.

Since we have the compact inclusion $L^\infty(0,T;V_1) \cap W^{\bar{s},\ell}(0,T;V_1^*) \stackrel{c}{\hookrightarrow} C^0([0,T];H)$, using the estimates (3.22) and (3.25) and arguing exactly as in subsection 3.3, one readily deduce that the family of laws of $(\tilde{\varphi}_\lambda)_\lambda$ on $C^0([0,T];H)$ is tight. Similarly, estimate (3.24) and the compact inclusion $W^{\bar{s},\ell}(0,T;V_1) \stackrel{c}{\hookrightarrow} C^0([0,T];H)$ yields that the family of laws of $(G(\tilde{\varphi}_\lambda)\cdot\tilde{W})_\lambda$ is tight on $C^0([0,T];H)$. In particular, the family of laws of $(\tilde{\varphi}_\lambda, G(\tilde{\varphi}_\lambda)\cdot\tilde{W}, \tilde{W})_\lambda$ is tight on the product space

$$C^0([0,T];H) \times C^0([0,T];H) \times C^0([0,T];U_1)\,.$$

By Prokhorov and Jakubowski–Skorokhod theorems (see again the references [58, theorem 2.7], [83, theorem 1.10.4, addendum 1.10.5], and [9, theorem 2.7.1]), there exists a further probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ and measurable maps $\Xi_\lambda : (\hat{\Omega}, \hat{\mathscr{F}}) \to (\tilde{\Omega}, \tilde{\mathscr{F}})$ such that $\hat{\mathbb{P}} \circ \Xi_\lambda^{-1} = \tilde{\mathbb{P}}$ for every $\lambda > 0$ and

$$\begin{aligned}
\hat{\varphi}_\lambda := \tilde{\varphi}_\lambda \circ \Xi_\lambda \to \hat{\varphi} \quad & \text{in } L^p(\hat{\Omega}; C^0([0,T];H))\ \forall\, p < \ell,\\
\hat{\varphi}_\lambda \stackrel{*}{\rightharpoonup} \hat{\varphi} \quad & \text{in } L^\ell_w(\hat{\Omega}; L^\infty(0,T;V_1)),\\
\hat{\varphi}_\lambda \rightharpoonup \hat{\varphi} \quad & \text{in } L^{\ell/2}(\hat{\Omega}; L^2(0,T;V_2)) \cap L^\ell(\hat{\Omega}; W^{\bar{s},\ell}(0,T;V_1^*)),\\
\hat{\mu}_\lambda := \tilde{\mu}_\lambda \circ \Xi_\lambda \rightharpoonup \hat{\mu} \quad & \text{in } L^{\ell/2}(\hat{\Omega}; L^2(0,T;V_1)),\\
\nabla\hat{\mu}_\lambda \rightharpoonup \nabla\hat{\mu} \quad & \text{in } L^\ell(\hat{\Omega}; L^2(0,T;H)),\\
F'_\lambda(\hat{\varphi}_\lambda) \rightharpoonup \hat{\xi} \quad & \text{in } L^{\ell/2}(\hat{\Omega}; L^2(0,T;H)),\\
\hat{I}_\lambda := (G(\tilde{\varphi}_\lambda)\cdot\tilde{W}) \circ \Xi_\lambda \to \hat{I} \quad & \text{in } L^p(\hat{\Omega}; C^0([0,T];H))\ \forall\, p < \ell,\\
\hat{W}_\lambda := \tilde{W} \circ \Xi_\lambda \to \hat{W} \quad & \text{in } L^p(\hat{\Omega}; C^0([0,T];U_1))\ \forall\, p < \ell,
\end{aligned}$$

for some measurable processes

$$\begin{aligned}
&\hat{\varphi} \in L^\ell(\hat{\Omega}; C^0([0,T];H)) \cap L^\ell_w(\hat{\Omega}; L^\infty(0,T;V_1)) \cap L^{\ell/2}(\hat{\Omega}; L^2(0,T;V_2)) \cap\\
&L^\ell(\hat{\Omega}; W^{\bar{s},\ell}(0,T;V_1^*)),\\
&\hat{\mu} \in L^{\ell/2}(\hat{\Omega}; L^2(0,T;V_1)), \qquad \nabla\hat{\mu} \in L^\ell(\hat{\Omega}; L^2(0,T;H)),\\
&\hat{\xi} \in L^{\ell/2}(\hat{\Omega}; L^2(0,T;H)),\\
&\hat{I} \in L^\ell(\hat{\Omega}; C^0([0,T];H)),\\
&\hat{W} \in L^\ell(\hat{\Omega}; C^0([0,T];U_1))\,.
\end{aligned}$$

Since $G : H \to \mathscr{L}^2(U, H)$ is Lipschitz-continuous, we also have

$$G(\hat{\varphi}_\lambda) \to G(\hat{\varphi}) \quad \text{in } L^p(\hat{\Omega}; L^2(0, T; \mathscr{L}^2(U, H))) \ \forall\, p < \ell.$$

Moreover, since $F' = \gamma - C_F I$ and γ is maximal monotone, by the strong-weak closure of maximal monotone operators (see [2, lemma 2.3]) and the strong convergence of $(\hat{\varphi}_\lambda)_\lambda$, we have that

$$\xi = F'(\hat{\varphi}) \quad \text{a.e. in } \hat{\Omega} \times (0, T) \times \mathcal{O}.$$

Following the exact same argument of subsection 3.3, we introduce the filtration

$$\hat{\mathscr{F}}_{\lambda,t} := \sigma\{\hat{\varphi}_\lambda(s), \hat{I}_\lambda(s), \hat{W}_\lambda(s) : s \in [0, t]\}, \quad t \in [0, T],$$

and infer that $\hat{I}_\lambda$ is the H-valued martingale given by

$$\hat{I}_\lambda(t) = \int_0^t G(\hat{\varphi}_\lambda(s))\, \mathrm{d}\hat{W}_\lambda(s) \quad \forall\, t \in [0, T].$$

Now, arguing again as in subsection 3.3, thanks to the strong convergences of $\hat{\varphi}_\lambda \to \hat{\varphi}$ and $G(\hat{\varphi}_\lambda) \to G(\hat{\varphi})$ proved above, we can suitably enlarge the probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ and find a saturated and right-continuous filtration $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$ such that $\hat{W}$ is a cylindrical Wiener process on the stochastic basis $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}})$ and

$$\hat{I}(t) = \int_0^t G(\hat{\varphi}(s))\, \mathrm{d}\hat{W}(s) \quad \forall\, t \in [0, T].$$

Now, since $\hat{\mathbb{P}} \circ \Xi_\lambda = \tilde{\mathbb{P}}$ for every $\lambda > 0$, from (3.18) and (3.19) it follows that

$$\begin{aligned}\int_{\mathcal{O}} \hat{\varphi}_\lambda(t, x) v(x)\, \mathrm{d}x &+ \int_{Q_t} m(\hat{\varphi}_\lambda(s, x)) \nabla \hat{\mu}_\lambda(s, x) \cdot \nabla v(x)\, \mathrm{d}x\, \mathrm{d}s \\ &= \int_{\mathcal{O}} \varphi_0(x) v(x)\, \mathrm{d}x + \int_{\mathcal{O}} \left(\int_0^t G(\hat{\varphi}_\lambda(s))\, \mathrm{d}\hat{W}_\lambda(s) \right)(x) v(x)\, \mathrm{d}x\end{aligned}$$

for every $t \in [0, T]$, $\hat{\mathbb{P}}$-almost surely, and

$$\int_{\mathcal{O}} \hat{\mu}_\lambda(t, x) v(x)\, \mathrm{d}x = \int_{\mathcal{O}} \nabla \hat{\varphi}_\lambda(s, x) \cdot \nabla v(x)\, \mathrm{d}x + \int_{\mathcal{O}} F'_\lambda(\hat{\varphi}_\lambda(t, x)) v(x)\, \mathrm{d}x$$

for almost every $t \in (0, T)$, $\hat{\mathbb{P}}$-almost surely. Hence, using the convergences proved above, the continuity and boundedness of m together with the dominated convergence theorem, we can let $\lambda \to 0$ in the variational formulations and obtain exactly (2.2), and $\hat{\mu} = -\Delta\hat{\varphi} + F'(\hat{\varphi})$.

In order to prove the energy inequality (2.3) we note that since $\hat{\mathbb{P}} \circ \Xi_\lambda^{-1} = \tilde{\mathbb{P}}$ for all $\lambda > 0$, from (3.21) we infer that

$$\begin{aligned}\frac{1}{2}\hat{\mathbb{E}} \|\nabla \hat{\varphi}_\lambda(t)\|_H^2 &+ \hat{\mathbb{E}} \|F_\lambda(\hat{\varphi}_\lambda(t))\|_{L^1(\mathcal{O})} + \hat{\mathbb{E}} \int_{Q_t} m(\hat{\varphi}_\lambda(s, x)) |\nabla \hat{\mu}_\lambda(s, x)|^2\, \mathrm{d}x\, \mathrm{d}s \\ &\leqslant \frac{1}{2} \|\nabla \varphi_0\|_H^2 + \|F_\lambda(\varphi_0)\|_{L^1(\mathcal{O})} + \frac{C_G}{2} \hat{\mathbb{E}} \int_0^t \|\nabla \hat{\varphi}_\lambda(s)\|_H^2\, \mathrm{d}s \\ &\quad + \frac{1}{2} \hat{\mathbb{E}} \int_0^t \sum_{k=0}^\infty \int_{\mathcal{O}} F''_\lambda(\hat{\varphi}_\lambda(s, x)) |g_k(\hat{\varphi}_\lambda(s, x))|^2\, \mathrm{d}x\, \mathrm{d}s\end{aligned} \tag{3.28}$$

for every $t \in [0, T]$. We want to let again $\lambda \to 0$ and use the convergences just proved. To this end, for the second term on the left-hand side note that F_λ is a quadratic perturbation of the convex function $\hat{\gamma}_\lambda$, where $\hat{\gamma}_\lambda(\hat{\varphi}_\lambda) \geqslant \hat{\gamma}(J_\lambda\hat{\varphi}_\lambda)$ and $J_\lambda := (I + \lambda\gamma)^{-1} : \mathbb{R} \to \mathbb{R}$ is the resolvent of γ. Noting that

$$|J_\lambda\hat{\varphi}_\lambda - \hat{\varphi}| \leqslant |J_\lambda\hat{\varphi}_\lambda - \hat{\varphi}_\lambda| + |\hat{\varphi}_\lambda - \hat{\varphi}| \leqslant \lambda|\gamma_\lambda(\hat{\varphi}_\lambda)| + |\hat{\varphi}_\lambda - \hat{\varphi}|,$$

we easily infer that $J_\lambda\hat{\varphi}_\lambda \to \hat{\varphi}$ a.e. in $\hat{\Omega} \times Q$, so that by weak lower semicontinuity and the Fatou lemma

$$\hat{\mathbb{E}}\|F(\hat{\varphi})\|_{L^1(\mathcal{O})} \leqslant \liminf_{\lambda\to 0} \hat{\mathbb{E}}\|F_\lambda(\hat{\varphi}_\lambda)\|_{L^1(\mathcal{O})}.$$

Moreover, for the third term on the left-hand side of (3.28), arguing exactly as in the proof of (3.20) we have that

$$\hat{\mathbb{E}}\int_{Q_t} m(\hat{\varphi}(s,x))|\nabla\hat{\mu}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \liminf_{\lambda\to 0}\hat{\mathbb{E}}\int_{Q_t} m(\hat{\varphi}_\lambda(s,x))|\nabla\hat{\mu}_\lambda(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Furthermore, for the second term on the right-hand side of (3.28) note that $F_\lambda(\varphi_0) \leqslant F(\varphi_0)$. Finally, the last term on the right-hand side of (3.28) can pass to the limit by the dominated convergence theorem. Indeed, since $F'_\lambda = \gamma_\lambda - C_F I = \gamma \circ J_\lambda - C_F I$, we have $F''_\lambda = (\gamma' \circ J_\lambda)J'_\lambda - C_F$, from which

$$\begin{aligned}|F''_\lambda(\hat{\varphi}_\lambda)||g_k(\hat{\varphi}_\lambda)|^2 &\leqslant \|g_k\|^2_{L^\infty(\mathbb{R})}|F''_\lambda(\hat{\varphi}_\lambda)|\\ &= \|g_k\|^2_{L^\infty(\mathbb{R})}|\gamma'(J_\lambda(\hat{\varphi}_\lambda))J'_\lambda(\hat{\varphi}_\lambda) - C_F|.\end{aligned}$$

Recalling that $J_\lambda : \mathbb{R} \to \mathbb{R}$ is one-Lipschitz continuous, it holds $|J'_\lambda| \leqslant 1$ and we deduce that

$$|F''_\lambda(\hat{\varphi}_\lambda)||g_k(\hat{\varphi}_\lambda)|^2 \leqslant \|g_k\|^2_{L^\infty(\mathbb{R})}\left(|\gamma'(J_\lambda(\hat{\varphi}_\lambda))| + C_F\right),$$

where, by definition of γ we have $\gamma'(J_\lambda(\hat{\varphi}_\lambda)) = F''(J_\lambda\hat{\varphi}_\lambda) + C_F$. From these computations and assumption **ND1** it follows that

$$\begin{aligned}|F''_\lambda(\hat{\varphi}_\lambda)||g_k(\hat{\varphi}_\lambda)|^2 &\leqslant \|g_k\|^2_{L^\infty(\mathbb{R})}\left(|F''(J_\lambda(\hat{\varphi}_\lambda))| + 2C_F\right)\\ &\leqslant c\|g_k\|^2_{L^\infty(\mathbb{R})}\left(1 + |F(J_\lambda(\hat{\varphi}_\lambda))|\right).\end{aligned}$$

The term in brackets on the right-hand side is uniformly integrable in $\hat{\Omega} \times Q$ because $F(J_\lambda(\hat{\varphi}_\lambda)) \to F(\hat{\varphi})$ in $L^1(\hat{\Omega} \times Q)$, hence so is the left-hand side by comparison. Hence, recalling that $\sum_{k=0}^{\infty}\|g_k\|^2_{L^\infty(\mathbb{R})} \leqslant C_G$, the Vitali convergence theorem yields

$$\begin{aligned}&\hat{\mathbb{E}}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}} F''_\lambda(\hat{\varphi}_\lambda(s,x))|g_k(\hat{\varphi}_\lambda(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\to \hat{\mathbb{E}}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}} F''(\hat{\varphi}(s,x))|g_k(\hat{\varphi}(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Letting then $\lambda \to 0$ in (3.28) taking into account these remarks yields exactly, by weak lower semicontinuity, the energy inequality (2.3).

In order to conclude we only need to prove the last assertion of theorem 2.2. To this end, note that if $d \geqslant 3$ and $|F''(r)| \leqslant C_F(1 + |r|^{\frac{2}{d-2}})$ for all $r \in \mathbb{R}$, recalling that $V_1 \hookrightarrow L^{\frac{2d}{d-2}}(\mathcal{O})$ and noting that $\frac{2-d}{2d} + \frac{1}{d} = \frac{1}{2}$, by Hölder inequality we have

$$\begin{aligned}\|\nabla F'_\lambda(\hat\varphi_\lambda)\|_H &= \|F''_\lambda(\hat\varphi_\lambda)\nabla\hat\varphi_\lambda\|_H \leqslant C_F\left\|\left(1+|\hat\varphi_\lambda|^{\frac{2}{d-2}}\right)\nabla\hat\varphi_\lambda\right\|_H\\ &\leqslant C_F\left\|1+|\hat\varphi_\lambda|^{\frac{2}{d-2}}\right\|_{L^d(\mathcal{O})}\|\nabla\hat\varphi_\lambda\|_{L^{\frac{2d}{d-2}}(\mathcal{O})}\\ &\leqslant C_F\|1+\hat\varphi_\lambda\|^{\frac{2}{d-2}}_{L^{\frac{2d}{d-2}}(\mathcal{O})}\|\nabla\hat\varphi_\lambda\|_{L^{\frac{2d}{d-2}}(\mathcal{O})}\\ &\leqslant c\|1+\hat\varphi_\lambda\|^{\frac{2}{d-2}}_{V_1}\|\hat\varphi_\lambda\|_{V_2},\end{aligned}$$

from which it follows that

$$\|\nabla F'_\lambda(\hat\varphi_\lambda)\|_{L^2(0,T;H)} \leqslant c\|1+\hat\varphi_\lambda\|^{\frac{2}{d-2}}_{L^\infty(0,T;V_1)}\|\hat\varphi_\lambda\|_{L^2(0,T;V_2)}.$$

Using (3.22) and (3.27), since $\frac{2}{d-2}\leqslant 2$ the right-hand side is uniformly bounded in $L^{\ell/4}(\hat\Omega)$, so that by weak lower semicontinuity we have $F'(\hat\varphi)\in L^{\ell/4}(\hat\Omega;L^2(0,T;V_1))$. Since $\hat\mu=-\Delta\hat\varphi+F'(\hat\varphi)$, we conclude by elliptic regularity. If $d=2$, the same argument works using the embedding $V_1\hookrightarrow L^q(\mathcal{O})$ for all $q\in[2,+\infty)$. This concludes the proof of theorem 2.2.

4. Degenerate mobility and irregular potential

This section is devoted to proving theorem 2.7. The main idea of the proof is the following. We approximate the irregular potential F and the mobility m using a suitable regularization, depending on a parameter $\varepsilon>0$, introduced in [40] in the deterministic setting. We show that the ε-approximated problem admits martingale solutions thanks to the already proved theorem 2.7. Finally, exploiting the compatibility assumptions between F, m, and G, we prove uniform estimates on the solutions and pass to the limit by monotonicity and stochastic compactness arguments.

Let us also mention that a similar approximation in $\varepsilon>0$ was used in [47], in the study of a nonlocal Cahn–Hilliard–Navier–Stokes deterministic model with degenerate mobility. Here the authors prove that the system with $\varepsilon>0$ fixed admits a solution by relying on a time-discretisation argument, and then they show convergence as $\varepsilon\to 0$. In our case, the idea is similar, but existence of solution at ε fixed is obtained using the λ-approximation of the **ND** case instead of the time-discretisation. This choice was meant to avoid technical time-measurability issues in the stochastic setting; nonetheless, we point out that both techniques are feasible also in the stochastic case.

4.1. The approximation

For brevity of notation, we shall denote by ε a fixed real sequence $(\varepsilon_n)_n\subset(0,1/4)$ converging to 0 as $n\to\infty$. We shall briefly say that $\varepsilon\to 0$.

Since $F_2\in C^2([-1,1])$ we can extend it to the whole $\mathbb{R}$ as

$$\tilde F_2:\mathbb{R}\to\mathbb{R},\qquad \tilde F_2(r):=\begin{cases} F_2(r) & \text{if } |r|\leqslant 1,\\ F_2(-1)+F_2'(-1)(r+1)+\dfrac{1}{2}F''(-1)(r+1)^2 & \text{if } r<-1,\\ F_2(1)+F_2'(1)(r-1)+\dfrac{1}{2}F''(1)(r-1)^2 & \text{if } r>1,\end{cases}$$

so that $\tilde{F}_2 \in C^2(\mathbb{R})$ and $\|\tilde{F}_2''\|_{C^0(\mathbb{R})} \leqslant \|F_2''\|_{C^0([-1,1])}$. Moreover, for every $\varepsilon \in (0, 1/4)$ we define $F_{1,\varepsilon} : \mathbb{R} \to \mathbb{R}$ as the unique function of class C^2 such that $F_{1,\varepsilon}(0) = F_1(0)$, $F_{1,\varepsilon}'(0) = F_1'(0)$, and

$$F_{1,\varepsilon}''(r) = \begin{cases} F_1''(r) & \text{if } |r| \leqslant 1-\varepsilon, \\ F_1''(-1+\varepsilon) & \text{if } r < -1+\varepsilon, \\ F_1''(1-\varepsilon) & \text{if } r > 1-\varepsilon. \end{cases}$$

With this notation, we introduce the regularized potential $F_\varepsilon := F_{1,\varepsilon} + \tilde{F}_2$ and note that, by **D1**, we have $F_\varepsilon \in C^2(\mathbb{R})$ with $F_\varepsilon'' \in L^\infty(\mathbb{R})$. In particular, F_ε satisfies **ND1** and (2.4). Moreover, by definition we have that $F_\varepsilon = F$ on $[-1+\varepsilon, 1-\varepsilon]$.

We define the approximated mobility

$$m_\varepsilon : \mathbb{R} \to \mathbb{R}, \qquad m_\varepsilon(r) := \begin{cases} m(r) & \text{if } |r| \leqslant 1-\varepsilon, \\ m(-1+\varepsilon) & \text{if } r < -1+\varepsilon, \\ m(1-\varepsilon) & \text{if } r > 1-\varepsilon. \end{cases}$$

Note that by assumption **D2** we have that $m_\varepsilon \in C^0(\mathbb{R})$ with

$$0 < \min\{m(-1+\varepsilon), m(1-\varepsilon)\} \leqslant m_\varepsilon(r) \leqslant \|m\|_{C^0([-1,1])} \quad \forall\, r \in \mathbb{R},$$

so that m_ε satisfies **ND2** and $m_\varepsilon = m$ on $[-1+\varepsilon, 1-\varepsilon]$. Furthermore, we define M_ε as the unique function in $C^2(\mathbb{R})$ such that

$$M_\varepsilon : \mathbb{R} \to [0, +\infty), \qquad M_\varepsilon(0) = M_\varepsilon'(0) = 0, \qquad M_\varepsilon''(r) := \frac{1}{m_\varepsilon(r)}, \quad r \in \mathbb{R}.$$

In particular, note that by definition of m_ε and **D2** we have $M_\varepsilon'' \in L^\infty(\mathbb{R})$.

We define the approximated operator $G_\varepsilon : H \to \mathscr{L}^2(U, H)$ setting

$$G_\varepsilon(v)u_k := g_{k,\varepsilon}(v) \quad v \in H,\ k \in \mathbb{N},$$

where, for every $k \in \mathbb{N}$,

$$g_{k,\varepsilon} : \mathbb{R} \to \mathbb{R}\,, \qquad g_{k,\varepsilon}(r) := \begin{cases} g_k(r) & \text{if } |r| \leqslant 1-\varepsilon, \\ g_k(-1+\varepsilon) & \text{if } r < -1+\varepsilon, \\ g_k(1-\varepsilon) & \text{if } r > 1-\varepsilon. \end{cases}$$

Note that $g_{k,\varepsilon} \in W^{1,\infty}(\mathbb{R})$ for every $k \in \mathbb{N}$ and that

$$C_{G_\varepsilon} = \sum_{k=0}^{\infty} \|g_{k,\varepsilon}\|_{W^{1,\infty}(\mathbb{R})}^2 \leqslant \sum_{k=0}^{\infty} \|g_k\|_{W^{1,\infty}(-1,1)}^2 \leqslant L_G < +\infty,$$

so that G_ε satisfies assumption **ND3** and $G_\varepsilon = G$ on $\mathcal{B}_{1-\varepsilon}$.

We deduce that the assumptions **ND1**–**ND4** are satisfied by the set of data $(F_\varepsilon, m_\varepsilon, G_\varepsilon, \varphi_0)$: hence, theorem 2.2 ensures that for every $\varepsilon \in (0, 1/4)$ there exists a martingale solution

$$\left(\Omega_\varepsilon, \mathscr{F}_\varepsilon, (\mathscr{F}_{\varepsilon,t})_{t\in[0,T]}, \mathbb{P}_\varepsilon, W_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon\right)$$

to the problem (1.10)–(1.13) in the sense of definition 2.1. Since $\varepsilon \in (0, 1/4)$ is chosen in a countable set of points (see the remark at the beginning of the subsection), by suitably enlarging the probability spaces we shall suppose that such martingale solutions are of the form

$$\left(\bar{\Omega}, \bar{\mathscr{F}}, (\bar{\mathscr{F}}_t)_{t\in[0,T]}, \bar{\mathbb{P}}, \bar{W}, \varphi_\varepsilon, \mu_\varepsilon\right),$$

for a certain stochastic basis $(\bar{\Omega}, \bar{\mathscr{F}}, (\bar{\mathscr{F}}_t)_{t\in[0,T]}, \bar{\mathbb{P}})$. Theorem 2.2 ensures also the regularities

$$\begin{aligned}
\varphi_\varepsilon &\in L^\ell(\bar{\Omega}; C^0([0,T]; H)) \cap L^\ell_w(\bar{\Omega}; L^\infty(0,T; V_1)) \\
&\qquad \cap L^{\ell/2}(\bar{\Omega}; L^2(0,T; V_2)) \cap L^{\ell/4}(\bar{\Omega}; L^2(0,T; H^3(\mathcal{O}))), \\
\mu_\varepsilon &\in L^{\ell/2}(\bar{\Omega}; L^2(0,T; V_1)), \qquad \nabla\mu_\varepsilon \in L^\ell(\bar{\Omega}; L^2(0,T; H^d)), \\
F'_\varepsilon(\varphi_\varepsilon) &\in L^{\ell/2}(\bar{\Omega}; L^2(0,T; H)) \cap L^{\ell/4}(\bar{\Omega}; L^2(0,T; V_1)).
\end{aligned}$$

for every $\ell \in [2, +\infty)$, and the variational formulation (2.2) reads

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi_\varepsilon(t,x) v(x) \mathrm{d}x + \int_{Q_t} m_\varepsilon(\varphi_\varepsilon(s,x)) \nabla\mu_\varepsilon(s,x) \cdot \nabla v(x) \, \mathrm{d}x \, \mathrm{d}s \\
&\quad = \int_{\mathcal{O}} \varphi_0(x) v(x) \, \mathrm{d}x + \int_{\mathcal{O}} \left(\int_0^t G_\varepsilon(\hat{\varphi}_\varepsilon(s)) \, \mathrm{d}W'(s) \right)(x) v(x) \, \mathrm{d}x
\end{aligned} \tag{4.1}$$

for every $v \in V_1$, for every $t \in [0,T]$, $\bar{\mathbb{P}}$-almost surely, where $\mu_\varepsilon = -\Delta\varphi_\varepsilon + F'_\varepsilon(\varphi_\varepsilon)$.

4.2. Uniform estimates in ε

We show here uniform estimates on the approximated solutions.

First estimate. First of all, taking $v = 1$ in (4.1), using Itô's formula and the Burkholder–Davis–Gundy and Young inequalities as in the proof of (3.7) yields

$$\|(\varphi_\varepsilon)_{\mathcal{O}}\|_{L^\ell(\bar{\Omega}; C^0([0,T]))} \leqslant c, \tag{4.2}$$

for a positive constant c independent of ε.

Secondly, the energy inequality (2.3) implies that

$$\begin{aligned}
&\frac{1}{2} \sup_{r\in[0,t]} \bar{\mathbb{E}} \|\nabla\varphi_\varepsilon(r)\|_H^2 + \sup_{r\in[0,t]} \bar{\mathbb{E}} \|F_\varepsilon(\varphi_\varepsilon(r))\|_{L^1(\mathcal{O})} \\
&\qquad + \bar{\mathbb{E}} \int_{Q_t} m_\varepsilon(\varphi_\varepsilon(s,x)) |\nabla\mu_\varepsilon(s,x)|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&\quad \leqslant \frac{1}{2} \|\nabla\varphi_0\|_H^2 + \|F(\varphi_0)\|_{L^1(\mathcal{O})} + \frac{C_{G_\varepsilon}}{2} \bar{\mathbb{E}} \int_0^t \|\nabla\varphi_\varepsilon(s)\|_H^2 \, \mathrm{d}s \\
&\qquad + \frac{1}{2} \bar{\mathbb{E}} \int_0^t \sum_{k=0}^\infty \int_{\mathcal{O}} F''_\varepsilon(\varphi_\varepsilon(s,x)) |g_{k_\varepsilon}(\varphi_\varepsilon(s,x))|^2 \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}$$

Now, we have already shown that $C_{G_\varepsilon} \leqslant L_G$ for every ε. Moreover, by assumption **D3** and the definitions of F_ε and m_ε, we have that for every $r \in [-1+\varepsilon, 1-\varepsilon]$

$$F''_\varepsilon(r) |g_{k,\varepsilon}(r)|^2 = F''(r) |g_k(r)|^2 \leqslant \left\| \sqrt{F''} g_k \right\|^2_{L^\infty(-1,1)},$$

while for every $r < -1 + \varepsilon$

$$
\begin{aligned}
F''_\varepsilon(r)|g_k(r)|^2 &= (F''_1(-1+\varepsilon) + \tilde{F}''_2(r))|g_k(-1+\varepsilon)|^2 \\
&= F''(-1+\varepsilon)|g_k(-1+\varepsilon)|^2 + (\tilde{F}''_2(r) - F''_2(-1+\varepsilon))|g_k(-1+\varepsilon)|^2 \\
&\leqslant \left\| \sqrt{F''} g_k \right\|^2_{L^\infty(-1,1)} + 2\|F''_2\|_{C^0([-1,1])} \|g_k\|^2_{L^\infty(-1,1)},
\end{aligned}
$$

and similarly for every $r > 1 - \varepsilon$

$$
\begin{aligned}
F''_\varepsilon(r)|g_k(r)|^2 &= (F''_1(1-\varepsilon) + \tilde{F}''_2(r))|g_k(1-\varepsilon)|^2 \\
&= F''(1-\varepsilon)|g_k(1-\varepsilon)|^2 + (\tilde{F}''_2(r) - F''_2(1-\varepsilon))|g_k(1-\varepsilon)|^2 \\
&\leqslant \left\| \sqrt{F''} g_k \right\|^2_{L^\infty(-1,1)} + 2\|F''_2\|_{C^0([-1,1])} \|g_k\|^2_{L^\infty(-1,1)}.
\end{aligned}
$$

Hence, we deduce that

$$
F''_\varepsilon(r)|g_{k,\varepsilon}(r)|^2 \leqslant \left\| \sqrt{F''} g_k \right\|^2_{L^\infty(-1,1)} + 2\|F''_2\|_{C^0([-1,1])} \|g_k\|^2_{L^\infty(-1,1)} \quad \forall\, r \in \mathbb{R}.
$$

Substituting in the energy inequality and recalling the definition of L_G yields then

$$
\begin{aligned}
&\frac{1}{2} \sup_{r\in[0,t]} \bar{\mathbb{E}} \|\nabla\varphi_\varepsilon(r)\|^2_H + \sup_{r\in[0,t]} \bar{\mathbb{E}} \|F_\varepsilon(\varphi_\varepsilon(r))\|_{L^1(\mathcal{O})} \\
&\quad + \bar{\mathbb{E}} \int_{Q_t} m_\varepsilon(\varphi_\varepsilon(s,x)) |\nabla\mu_\varepsilon(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \frac{1}{2}\|\nabla\varphi_0\|^2_H + \|F(\varphi_0)\|_{L^1(\mathcal{O})} + \frac{L_G}{2} \bar{\mathbb{E}} \int_0^t \|\nabla\varphi_\varepsilon(s)\|^2_H \,\mathrm{d}s \\
&\quad + (1 + 2\|F_2\|_{C^0([-1,1])}) L_G \frac{|Q|}{2}.
\end{aligned}
$$

The Gronwall lemma and estimate (4.2) imply then

$$
\|\varphi_\varepsilon\|_{C^0([0,T];L^2(\bar{\Omega};V_1))} \leqslant c, \tag{4.3}
$$

$$
\|F_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\bar{\Omega}\times\mathcal{O}))} \leqslant c, \tag{4.4}
$$

$$
\left\| \sqrt{m_\varepsilon(\varphi_\varepsilon)} \nabla\mu_\varepsilon \right\|_{L^2(\bar{\Omega};L^2(0,T;H))} \leqslant c. \tag{4.5}
$$

Noting that $\|g_{k,\varepsilon}\|_{W^{1,\infty}(\mathbb{R})} \leqslant \|g_k\|_{W^{1,\infty}(-1,1)}$ for every $k \in \mathbb{N}$, thanks to assumption **D3** and (4.3) we also have that

$$
\|G_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(\bar{\Omega}\times(0,T);\mathscr{L}^2(U,H)) \cap L^\infty(0,T;L^2(\bar{\Omega};\mathscr{L}^2(U,V_1)))} \leqslant c,
$$

which implies by [46, lemma 2.1] that, for every $s \in (0, 1/2)$ and for every $p \in [2, +\infty)$,

$$
\left\| \int_0^\cdot G_\varepsilon(\varphi_\varepsilon(s)) \mathrm{d}W'(s) \right\|_{L^p(\bar{\Omega};W^{s,p}(0,T;H)) \cap L^2(\bar{\Omega};W^{s,2}(0,T;V_1))} \leqslant c_{s,p}. \tag{4.6}
$$

By comparison in (4.1) we deduce from (4.5) and (4.6) that, for every $s \in (0, 1/2)$,

$$
\|\varphi_\varepsilon\|_{L^2(\bar{\Omega};W^{s,2}(0,T;V_1^*))} \leqslant c_s. \tag{4.7}
$$

Second estimate. The idea is now to write Itô's formula for $\int_{\mathcal{O}} M_\varepsilon(\varphi_\varepsilon)$. In order to do this, note that $M_\varepsilon \in C^2(\mathbb{R})$ and $M''_\varepsilon \in L^\infty(\mathbb{R})$, so that M'_ε is Lipschitz-continuous. In particular, we have that $v \mapsto M'_\varepsilon(v)$ is well defined and continuous from V_1 to V_1. Hence, we can apply Itô's formula in the variational setting [70, theorem 4.2] and obtain

$$\begin{aligned}\int_{\mathcal{O}} M_\varepsilon(\varphi_\varepsilon(t,x))\,\mathrm{d}x &+ \int_{Q_t} m_\varepsilon(\varphi_\varepsilon(s,x)) M''_\varepsilon(\varphi_\varepsilon(s,x)) \nabla\mu_\varepsilon(s,x)\cdot\nabla\varphi_\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\mathcal{O}} M_\varepsilon(\varphi_0(x))\,\mathrm{d}x + \int_0^t \left(M'_\varepsilon(\varphi_\varepsilon(s)), G_\varepsilon(\varphi_\varepsilon(s))\,\mathrm{d}W'(s)\right)_H \\ &\quad + \frac{1}{2}\int_0^t \sum_{k=0}^\infty \int_{\mathcal{O}} M''_\varepsilon(\varphi_\varepsilon(s,x)) |g_k(\varphi_\varepsilon(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Noting that $M''_\varepsilon m_\varepsilon = 1$ by definition of M_ε and recalling that $\mu_\varepsilon = -\Delta\varphi_\varepsilon + F'_\varepsilon(\varphi_\varepsilon)$, since the stochastic integral on the right-hand side is a martingale taking expectations yields

$$\begin{aligned}&\bar{\mathbb{E}}\int_{\mathcal{O}} M_\varepsilon(\varphi_\varepsilon(t,x))\,\mathrm{d}x + \bar{\mathbb{E}}\int_{Q_t} |\Delta\varphi_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \bar{\mathbb{E}}\int_{Q_t} F''_\varepsilon(\varphi_\varepsilon(s,x))|\nabla\varphi_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\mathcal{O}} M_\varepsilon(\varphi_0(x))\,\mathrm{d}x + \frac{1}{2}\bar{\mathbb{E}}\int_0^t \sum_{k=0}^\infty \int_{\mathcal{O}} M''_\varepsilon(\varphi_\varepsilon(s,x))|g_k(\varphi_\varepsilon(s,x))|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Now, by definition of M_ε and m_ε we have $M_\varepsilon(\varphi_0) \leqslant M(\varphi_0)$ almost everywhere in $\mathcal{O}$. Moreover, by assumption **D3** and the definitions of M_ε and G_ε, we have that for every $r \in [-1+\varepsilon, 1-\varepsilon]$

$$M''_\varepsilon(r)|g_{k,\varepsilon}(r)|^2 = M''(r)|g_k(r)|^2 \leqslant \left\|\sqrt{M''}g_k\right\|^2_{L^\infty(-1,1)},$$

while for every $r < -1+\varepsilon$

$$M''_\varepsilon(r)|g_k(r)|^2 = M''(-1+\varepsilon)|g_k(-1+\varepsilon)|^2 \leqslant \left\|\sqrt{M''}g_k\right\|^2_{L^\infty(-1,1)},$$

and similarly for every $r > 1-\varepsilon$

$$M''_\varepsilon(r)|g_k(r)|^2 = M''(1-\varepsilon)|g_k(1-\varepsilon)|^2 \leqslant \left\|\sqrt{M''}g_k\right\|^2_{L^\infty(-1,1)}.$$

Hence, we deduce that

$$M''_\varepsilon(r)|g_{k,\varepsilon}(r)|^2 \leqslant \left\|\sqrt{M''}g_k\right\|^2_{L^\infty(-1,1)} \quad \forall\, r\in\mathbb{R}.$$

Taking into account these remarks and recalling that $F'_\varepsilon = F'_{1,\varepsilon} + \tilde{F}'_2$, we have then

$$\begin{aligned}&\bar{\mathbb{E}}\int_{\mathcal{O}} M_\varepsilon(\varphi_\varepsilon(t,x))\,\mathrm{d}x + \bar{\mathbb{E}}\int_{Q_t} |\Delta\varphi_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \bar{\mathbb{E}}\int_{Q_t} F''_{1,\varepsilon}(\varphi_\varepsilon(s,x))|\nabla\varphi_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \int_{\mathcal{O}} M(\varphi_0(x))\,\mathrm{d}x + L_G\frac{|Q|}{2} - \bar{\mathbb{E}}\int_{Q_t} \tilde{F}''_2(\varphi_\varepsilon(s,x))|\nabla\varphi_\varepsilon(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Since $F''_{1,\varepsilon} \geqslant 0$ by definition and $\|\tilde{F}''_2\|_{L^\infty(\mathbb{R})} \leqslant \|F''_2\|_{C^0([-1,1])}$, recalling **D4** and the estimate (4.3) we deduce by elliptic regularity that

$$\|M_\varepsilon(\varphi_\varepsilon)\|_{C^0([0,T];L^1(\bar{\Omega}\times\mathcal{O}))} \leqslant c, \tag{4.8}$$

$$\|\varphi_\varepsilon\|_{L^2(\bar{\Omega};L^2(0,T;V_2))} \leqslant c. \tag{4.9}$$

Third estimate. We prove now an estimate allowing to obtain some L^∞-bounds on the limiting solution: we are inspired here by some computations performed in [40, p 414] (see also [49, section 4.1.1]). Note that by definition of M_ε and m_ε we have that, for every $r > 1$,

$$\begin{aligned} M_\varepsilon(r) &= M_\varepsilon(1-\varepsilon) + M'_\varepsilon(1-\varepsilon)(r-1+\varepsilon) + \frac{1}{2}M''_\varepsilon(1-\varepsilon)(r-1+\varepsilon)^2 \\ &\geqslant \frac{1}{2}M''_\varepsilon(1-\varepsilon)(r-1+\varepsilon)^2 = \frac{(r-1+\varepsilon)^2}{2m_\varepsilon(1-\varepsilon)} \geqslant \frac{(r-1)^2}{2m_\varepsilon(1-\varepsilon)}, \end{aligned}$$

and similarly, for every $r < -1$,

$$\begin{aligned} M_\varepsilon(r) &= M_\varepsilon(-1+\varepsilon) + M'_\varepsilon(-1+\varepsilon)(r+1-\varepsilon) + \frac{1}{2}M''_\varepsilon(-1+\varepsilon)(r+1-\varepsilon)^2 \\ &\geqslant \frac{1}{2}M''_\varepsilon(-1+\varepsilon)(r+1-\varepsilon)^2 = \frac{(r+1-\varepsilon)^2}{2m_\varepsilon(-1+\varepsilon)} \\ &\geqslant \frac{(r+1)^2}{2m_\varepsilon(-1+\varepsilon)} = \frac{(|r|-1)^2}{2m_\varepsilon(-1+\varepsilon)}. \end{aligned}$$

We infer that

$$(|r|-1)_+^2 \leqslant 2M_\varepsilon(r)\max\{m_\varepsilon(1-\varepsilon), m_\varepsilon(-1+\varepsilon)\} \quad \forall\, r \in \mathbb{R}.$$

Now, since $m_\varepsilon(1-\varepsilon) = m(1-\varepsilon)$ and $m(1) = 0$ we have

$$|m_\varepsilon(1-\varepsilon)| = |m(1-\varepsilon)| = |m(1-\varepsilon) - m(1)| \leqslant \|m'\|_{L^\infty(-1,1)}\varepsilon$$

and similarly, since $m_\varepsilon(-1+\varepsilon) = m(-1+\varepsilon)$ and $m(-1) = 0$,

$$|m_\varepsilon(-1+\varepsilon)| = |m(-1+\varepsilon)| = |m(-1+\varepsilon) - m(-1)| \leqslant \|m'\|_{L^\infty(-1,1)}\varepsilon.$$

We deduce then that

$$(|r|-1)_+^2 \leqslant 2\varepsilon\|m'\|_{L^\infty(-1,1)}M_\varepsilon(r) \quad \forall\, r \in \mathbb{R},$$

yielding, together with (4.8),

$$\|(|\varphi_\varepsilon|-1)_+\|_{C^0([0,T];L^2(\bar{\Omega};H))} \leqslant c\sqrt{\varepsilon}. \tag{4.10}$$

4.3. Passage to the limit as $\varepsilon \to 0$

The argument are similar to the ones performed in subsections 3.3 and 3.5, so we will omit the technical details for brevity.

First of all, by [79, corollary 5, p 86] we have the compact inclusion

$$L^2(0,T;V_2) \cap W^{s,2}(0,T;V_1^*) \stackrel{c}{\hookrightarrow} L^2(0,T;V_1).$$

Hence, using the estimates (4.7) and (4.9), together with the Markov inequality, arguing as in subsection 3.3 we easily infer that the laws of $(\varphi_\varepsilon)_\varepsilon$ on $L^2(0,T;V_1)$ are tight. Secondly, fixing $\bar{s} \in (0,1/2)$ and $\bar{p} \geqslant 2$ such that $\bar{s}\bar{p} > 1$, again by [79, corollary 5, p 86] we have also the compact inclusion

$$W^{\bar{s},\bar{p}}(0,T;H) \cap W^{\bar{s},2}(0,T;V_1) \overset{c}{\hookrightarrow} C^0([0,T];V_1^*) \cap L^2(0,T;H),$$

so that estimate (4.6) yields by the same argument that the sequence of laws of $(G_\varepsilon(\varphi_\varepsilon) \cdot \bar{W})_\varepsilon$ on $C^0([0,T];V_1^*) \cap L^2(0,T;H)$ are tight. In particular, the family of laws of $(\varphi_\varepsilon, G_\varepsilon(\varphi_\varepsilon) \cdot \bar{W}, \bar{W})_\varepsilon$ is tight on the product space

$$L^2(0,T;V_1) \times \left(C^0([0,T];V_1^*) \cap L^2(0,T;H)\right) \times C^0([0,T];U_1).$$

Recalling that $L^2(0,T;H)$, $L^2(0,T;V_1)$, and $L^2(0,T;V_2)$ endowed with their weak topologies are sub-Polish, by Prokhorov and Jakubowski–Skorokhod theorems (see again [58, theorem 2.7], [83, theorem 1.10.4, addendum 1.10.5], and [9, theorem 2.7.1]) and the estimates (4.3)–(4.9), there exists a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ and measurable maps $\Theta_\varepsilon : (\hat{\Omega}, \hat{\mathscr{F}}) \to (\bar{\Omega}, \bar{\mathscr{F}})$ such that $\hat{\mathbb{P}} \circ \Theta_\varepsilon^{-1} = \bar{\mathbb{P}}$ for every $\varepsilon \in (0,1/4)$ and

$$\begin{aligned}
\hat{\varphi}_\varepsilon := \varphi_\varepsilon \circ \Theta_\varepsilon \to \hat{\varphi} \quad & \text{in } L^2(0,T;V_1)\ \hat{\mathbb{P}} - \text{a.s.}, \\
\hat{\varphi}_\varepsilon \overset{*}{\rightharpoonup} \hat{\varphi} \quad & \text{in } L^\infty(0,T;L^2(\hat{\Omega};V_1)), \\
\hat{\varphi}_\varepsilon \rightharpoonup \hat{\varphi} \quad & \text{in } L^2(0,T;V_2) \cap W^{\bar{s},2}(0,T;V_1^*)\ \hat{\mathbb{P}} - \text{a.s.}, \\
\hat{\eta}_\varepsilon := (m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon) \circ \Theta_\varepsilon \rightharpoonup \hat{\eta} \quad & \text{in } L^2(0,T;H)\ \hat{\mathbb{P}} - \text{a.s.}, \\
\hat{I}_\varepsilon := (G_\varepsilon(\varphi_\varepsilon) \cdot \bar{W}) \circ \Theta_\varepsilon \to \hat{I} \quad & \text{in } L^p(\hat{\Omega};C^0([0,T];V_1^*) \cap L^2(0,T;H))\ \forall\, p \in [2,+\infty), \\
\hat{W}_\varepsilon := \bar{W} \circ \Theta_\varepsilon \to \hat{W} \quad & \text{in } L^p(\hat{\Omega};C^0([0,T];U))\ \forall\, p \in [2,+\infty),
\end{aligned}$$

for some measurable processes

$$\begin{aligned}
& \hat{\varphi} \in L^\infty(0,T;L^2(\hat{\Omega};V_1)) \cap L^2(\hat{\Omega};L^2(0,T;V_2)) \cap L^2(\hat{\Omega};W^{\bar{s},2}(0,T;V_1^*)), \\
& \hat{\eta} \in L^2(\hat{\Omega};L^2(0,T;H^d)), \\
& \hat{I} \in L^p(\hat{\Omega};C^0([0,T];V^*) \cap L^2(0,T;H)) \quad \forall\, p \in [2,+\infty), \\
& \hat{W} \in L^p(\hat{\Omega};C^0([0,T];U)) \quad \forall\, p \in [2,+\infty).
\end{aligned}$$

Furthermore, by lower semicontinuity and estimate (4.10) it follows that

$$|\hat{\varphi}(t)| \leqslant 1 \quad \text{a.e. in } \hat{\Omega} \times \mathcal{O},\ \forall\, t \in [0,T],$$

while the estimates (4.3) yields also the convergence

$$\hat{\varphi}_\varepsilon \to \hat{\varphi} \quad \text{in } L^p(\hat{\Omega};L^2(0,T;V_1))\ \forall\, p \in [1,2).$$

Now, by definition of G_ε and recalling that $\|g'_{k,\varepsilon}\|_{L^\infty(\mathbb{R})} \leqslant \|g'_k\|_{L^\infty(-1,1)}$, we have that

$$\begin{aligned}
\|G_\varepsilon(\hat{\varphi}_\varepsilon) - G(\hat{\varphi})\|^2_{\mathscr{L}^2(U,H)} &\leqslant 2\|G_\varepsilon(\hat{\varphi}_\varepsilon) - G_\varepsilon(\hat{\varphi})\|^2_{\mathscr{L}^2(U,H)} + 2\|G_\varepsilon(\hat{\varphi}) - G(\hat{\varphi})\|^2_{\mathscr{L}^2(U,H)} \\
&= 2\sum_{k=0}^{\infty} \left(\|g_{k,\varepsilon}(\hat{\varphi}_\varepsilon) - g_{k,\varepsilon}(\hat{\varphi})\|^2_H + \|g_{k,\varepsilon}(\hat{\varphi}) - g_k(\hat{\varphi})\|^2_H\right) \\
&\leqslant 2\sum_{k=0}^{\infty} \left(\|g'_{k,\varepsilon}\|^2_{L^\infty(\mathbb{R})}\|\hat{\varphi}_\varepsilon - \hat{\varphi}\|^2_H + \|g_{k,\varepsilon}(\hat{\varphi}) - g_k(\hat{\varphi})\|^2_H\right) \\
&\leqslant 2L_G\|\hat{\varphi}_\varepsilon - \hat{\varphi}\|^2_H + 2\sum_{k=0}^{\infty}\|g_{k,\varepsilon}(\hat{\varphi}) - g_k(\hat{\varphi})\|^2_H,
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{k=0}^{\infty}\|g_{k,\varepsilon}(\hat{\varphi}) - g_k(\hat{\varphi})\|^2_H \\
&= \sum_{k=0}^{\infty}\left(\left\|(g_k(-1+\varepsilon) - g_k(\hat{\varphi}))1_{\{\hat{\varphi}<-1+\varepsilon\}}\right\|^2_H + \left\|(g_k(1-\varepsilon) - g_k(\hat{\varphi}))1_{\{\hat{\varphi}>1-\varepsilon\}}\right\|^2_H\right. \\
&\leqslant \sum_{k=0}^{\infty}\|g'_k\|^2_{L^\infty(-1,1)}\left(\left\|(\hat{\varphi}+1-\varepsilon)1_{\{\hat{\varphi}<-1+\varepsilon\}}\right\|^2_H + \left\|(\hat{\varphi}-1+\varepsilon)1_{\{\hat{\varphi}>1-\varepsilon\}}\right\|^2_H\right) \\
&\leqslant L_G\left(\left\|(\hat{\varphi}+1-\varepsilon)1_{\{\hat{\varphi}<-1+\varepsilon\}}\right\|^2_H + \left\|(\hat{\varphi}-1+\varepsilon)1_{\{\hat{\varphi}>1-\varepsilon\}}\right\|^2_H\right).
\end{aligned}$$

Since $|\hat{\varphi}| \leqslant 1$ almost everywhere, we have that $1_{\{\hat{\varphi}<-1+\varepsilon\}} \to 0$ and $1_{\{\hat{\varphi}>1-\varepsilon\}} \to 0$ almost everywhere, so that the dominated convergence theorem yields

$$G_\varepsilon(\hat{\varphi}_\varepsilon) \to G(\hat{\varphi}) \quad \text{in } L^p(\hat{\Omega}; L^2(0,T;\mathscr{L}^2(U,H)))\ \forall\, p \in [1,2).$$

Following now the same argument of subsection 3.3, we define the filtration

$$\hat{\mathscr{F}}_{\varepsilon,t} := \sigma\{\hat{\varphi}_\varepsilon(s), \hat{I}_\varepsilon(s), \hat{W}_\varepsilon(s) : s \in [0,t]\}, \quad t \in [0,T],$$

and infer that

$$\hat{I}_\varepsilon(t) = \int_0^t G_\varepsilon(\hat{\varphi}_\varepsilon(s))\mathrm{d}\hat{W}_\varepsilon(s) \quad \forall\, t \in [0,T].$$

Following again subsection 3.3, thanks to the strong convergences of $\hat{\varphi}_\varepsilon \to \hat{\varphi}$ and $G_\varepsilon(\hat{\varphi}_\varepsilon) \to G(\hat{\varphi})$ proved above, we deduce that, possibly enlarging the probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}})$, there is a saturated and right-continuous filtration $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$ such that $\hat{W}$ is a cylindrical Wiener process on the stochastic basis $(\hat{\Omega}, \hat{\mathscr{F}}, (\hat{\mathscr{F}}_t)_{t\in[0,T]}, \hat{\mathbb{P}})$ and

$$\hat{I}(t) = \int_0^t G(\hat{\varphi}(s))\,\mathrm{d}\hat{W}(s) \quad \forall\, t \in [0,T].$$

Passing to the weak limit as $\varepsilon \to 0$ yields then, for every $v \in V_1$,

$$\int_{\mathcal{O}} \hat{\varphi}(t,x)v(x)\,\mathrm{d}x + \int_{Q_t} \hat{\eta}(s,x)\cdot\nabla v(x)\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathcal{O}} \varphi_0(x)v(x)\,\mathrm{d}x + \int_{\mathcal{O}} \left(\int_0^t G(\hat{\varphi}(s))\,\mathrm{d}\hat{W}(s)\right)(x)v(x)\,\mathrm{d}x \tag{4.11}$$

$\hat{\mathbb{P}}$-almost surely, for every $t \in [0,T]$. It only remains to identify the limit $\hat{\eta}$.

Let us show that, for every $\zeta \in L^\infty(0,T;V_1 \cap L^d(\mathcal{O}))$, it holds

$$\int_Q \hat{\eta}(s,x)\cdot\zeta(s,x)\,\mathrm{d}x\,\mathrm{d}s = \int_Q \Delta\hat{\varphi}(s,x)\,\mathrm{div}\,[m_\varepsilon(\hat{\varphi}(s,x))\zeta(s,x)]\,\mathrm{d}x\,\mathrm{d}s + \int_Q m(\hat{\varphi}(s,x))F''(\hat{\varphi}(s,x))\nabla\hat{\varphi}(s,x)\cdot\zeta(s,x)\,\mathrm{d}x\,\mathrm{d}s \quad \hat{\mathbb{P}}-\text{a.s.}. \tag{4.12}$$

First of all, note that all the terms in (4.12) are well defined. Indeed, by boundedness of m_ε we have $m_\varepsilon(\hat{\varphi}_\varepsilon)\zeta \in L^\infty(0,T;H^d)$, and recalling that $V_1 \hookrightarrow L^{\frac{2d}{d-2}}(\mathcal{O})$ and $\frac{d-2}{2d} + \frac{1}{d} = \frac{1}{2}$, by Hölder inequality

$$\mathrm{div}\,[m_\varepsilon(\hat{\varphi}_\varepsilon)\zeta] = m'_\varepsilon(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon\cdot\zeta + m_\varepsilon(\hat{\varphi}_\varepsilon)\,\mathrm{div}\;\zeta,$$

with

$$\begin{aligned}\|\mathrm{div}\,[m_\varepsilon(\hat{\varphi}_\varepsilon)\zeta]\|_H &\leqslant \|m'_\varepsilon\|_{L^\infty(\mathbb{R})}\|\nabla\varphi\|_{L^{\frac{2d}{d-2}}}\|\zeta\|_{L^d(\mathcal{O})} + \|m_\varepsilon\|_{L^\infty(\mathbb{R})}\|\mathrm{div}\;\zeta\|_H \\ &\leqslant c\|m\|_{W^{1,\infty}(-1,1)}\left(\|\hat{\varphi}_\varepsilon\|_{V_2}\|\zeta\|_{L^d(\mathcal{O})} + \|\eta\|_{V_1}\right).\end{aligned}$$

Hence $\mathrm{div}\,[m_\varepsilon(\hat{\varphi}_\varepsilon)\zeta] \in L^\infty(0,T;H)$. Furthermore, the second term on the right-hand side is also well defined since $m_\varepsilon F''_\varepsilon \in L^\infty(\mathbb{R})$.

By density, it is not restrictive to prove (4.12) for every $\zeta \in L^\infty(0,T;V_2 \cap H^{2\bar{m}}(\mathcal{O}))$, where $\bar{m}$ is such that $H^{2\bar{m}}(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$. Since $\hat{\mu}_\varepsilon = -\Delta\hat{\varphi}_\varepsilon + F'_\varepsilon(\hat{\varphi}_\varepsilon)$ and we have the regularity $\hat{\varphi}_\varepsilon \in L^2(0,T;V_2 \cap H^3(\mathcal{O}))$, integration by parts yields

$$\begin{aligned}&\int_Q m_\varepsilon(\hat{\varphi}_\varepsilon(s,x))\nabla\hat{\mu}_\varepsilon(s,x)\cdot\zeta(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad= -\int_Q \nabla\Delta\hat{\varphi}_\varepsilon(s,x)\cdot m_\varepsilon(\hat{\varphi}_\varepsilon(s,x))\zeta(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad+ \int_Q F''_\varepsilon(\hat{\varphi}_\varepsilon(s,x))m_\varepsilon(\hat{\varphi}_\varepsilon(s,x))\nabla\hat{\varphi}_\varepsilon(s,x)\cdot\zeta(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad= \int_Q \Delta\hat{\varphi}_\varepsilon(s,x)\,\mathrm{div}\,[m_\varepsilon(\hat{\varphi}_\varepsilon(s,x))\zeta(s,x)]\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad+ \int_Q F''_\varepsilon(\hat{\varphi}_\varepsilon(s,x))m_\varepsilon(\hat{\varphi}_\varepsilon(s,x))\nabla\hat{\varphi}_\varepsilon(s,x)\cdot\zeta(s,x)\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{4.13}$$

so that we need to show that we can pass to the limit on the right-hand side. Let us start from the first-term. This is of the form

$$\Delta\hat{\varphi}_\varepsilon m_\varepsilon'(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon\cdot\zeta+\Delta\hat{\varphi}_\varepsilon m_\varepsilon(\hat{\varphi}_\varepsilon)\,\mathrm{div}\,\zeta,$$

where

$$\Delta\hat{\varphi}_\varepsilon\rightharpoonup\Delta\hat{\varphi}\quad\text{in } L^2(0,T;H)\ \hat{\mathbb{P}}-\text{a.s.}\tag{4.14}$$

Since $m_\varepsilon(\hat{\varphi}_\varepsilon)\to m(\hat{\varphi})$ almost everywhere and $|m_\varepsilon|\leqslant|m|$, the dominated convergence theorem yields that

$$m_\varepsilon(\hat{\varphi}_\varepsilon)\to m(\hat{\varphi})\quad\text{in } L^p(Q)\ \hat{\mathbb{P}}-\text{a.s.},\ \forall\,p\in[2,+\infty).$$

Since $\zeta\in L^\infty(0,T;H^{2\bar{m}}(\mathcal{O}))$ and $H^{2\bar{m}}(\mathcal{O})\hookrightarrow L^\infty(\mathcal{O})$, this implies in particular

$$m_\varepsilon(\hat{\varphi}_\varepsilon)\,\mathrm{div}\,\zeta\to m(\hat{\varphi})\,\mathrm{div}\,\zeta\quad\text{in } L^2(0,T;H)\ \hat{\mathbb{P}}-\text{a.s.}.\tag{4.15}$$

Moreover, since $m_\varepsilon'=0$ on $(-\infty,-1+\varepsilon)\cup(1-\varepsilon,+\infty)$, we have that

$$m_\varepsilon'(\hat{\varphi}_\varepsilon)=m'(\hat{\varphi}_\varepsilon)1_{\{|\hat{\varphi}_\varepsilon|\leqslant 1-\varepsilon\}}\to m'(\hat{\varphi})\quad\text{a.e. in } \hat{\Omega}\times Q,$$

hence also, thanks to the strong convergence $\hat{\varphi}_\varepsilon\to\hat{\varphi}$ in $L^2(0,T;V_1)$, that

$$m_\varepsilon'(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon\to m'(\hat{\varphi})\nabla\hat{\varphi}\quad\text{a.e. in } Q.$$

Noting also that

$$|m_\varepsilon'(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon|^2\leqslant\|m'\|_{L^\infty(-1,1)}|\nabla\hat{\varphi}_\varepsilon|^2$$

where the right-hand side is uniformly integrable Q (because it converges strongly in $L^1(Q)$), by Vitali's generalized dominated convergence theorem we infer that

$$m_\varepsilon'(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon\to m'(\hat{\varphi})\nabla\hat{\varphi}\quad\text{in } L^2(Q)\ \hat{\mathbb{P}}-\text{a.s.}$$

Since $\zeta\in L^\infty(Q)$ by choice of $\bar{m}$, this implies that

$$m_\varepsilon'(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon\cdot\eta\to m'(\hat{\varphi})m'(\hat{\varphi})\nabla\hat{\varphi}\cdot\eta\quad\text{in } L^2(0,T;H)\ \hat{\mathbb{P}}-\text{a.s.}\tag{4.16}$$

Let us focus now on the second term on the right-hand side of (4.13). Note that since $|\hat{\varphi}|\leqslant 1$ and $\hat{\varphi}_\varepsilon\to\hat{\varphi}$ almost everywhere in Q, we have that (see [40, pp 416–7])

$$m_\varepsilon(\hat{\varphi}_\varepsilon)F_\varepsilon''(\hat{\varphi}_\varepsilon)\to m(\hat{\varphi})F''(\hat{\varphi})\quad\text{a.e. in } \hat{\Omega}\times Q.$$

Since also $\nabla\hat{\varphi}_\varepsilon\to\nabla\hat{\varphi}$ almost everywhere in Q, recalling that $|m_\varepsilon F_\varepsilon''|\leqslant\|mF''\|_{C^0([-1,1])}$ by **D2**, the Vitali dominated convergence theorem yields again

$$m_\varepsilon(\hat{\varphi}_\varepsilon)F_\varepsilon''(\hat{\varphi}_\varepsilon)\nabla\hat{\varphi}_\varepsilon\to m(\hat{\varphi})F''(\hat{\varphi})\nabla\hat{\varphi}\quad\text{in } L^2(0,T;H),\ \hat{\mathbb{P}}-\text{a.s.}\tag{4.17}$$

Consequently, using (4.14)–(4.17) in (4.13) and letting $\varepsilon\to 0$, we obtain exactly the variational formulation (4.12) for every $\zeta\in L^\infty(0,T;V_2\cap H^{2\bar{m}}(\mathcal{O}))$. A classical density argument yields then (4.12) for every $\zeta\in L^\infty(0,T;V_1\cap L^d(\mathcal{O}))$. Finally, the variational formulation (2.6) follows then from (4.11) and (4.12) with the choice $\zeta = \nabla v$. The regularity $\hat\varphi \in C^0_w([0,T]; L^2(\hat\Omega; V_1))$ is a consequence of the regularity

$$\hat\varphi \in L^\infty(0,T;L^2(\hat\Omega;V_1)) \cap C^0\left([0,T];L^2(\Omega;(V_2 \cap W^{1,d}(\mathcal{O}))^*\right),$$

obtained by comparison in (2.6).

The only thing that remains to be proved is the regularity of $F(\hat\varphi)$ and $M(\hat\varphi)$. Since we have the convergence $\hat\varphi_\varepsilon(t) \to \hat\varphi(t)$ almost everywhere in $\hat\Omega \times \mathcal{O}$, there is a measurable set $A_0 \subset \hat\Omega \times \mathcal{O}$ (possibly depending on t) of full measure such that $\hat\varphi_\varepsilon(t) \to \hat\varphi(t)$ pointwise in A_0 for every $t \in [0,T]$. Let us show that

$$F_1(\hat\varphi(t)) \leqslant \liminf_{\varepsilon\to 0} F_{1,\varepsilon}(\hat\varphi_\varepsilon(t)), \quad M(\hat\varphi(t)) \leqslant \liminf_{\varepsilon\to 0} M_\varepsilon(\hat\varphi_\varepsilon(t)) \quad \text{in } A_0 \ \forall\, t \in [0,T]. \tag{4.18}$$

Indeed, since we already know that $|\hat\varphi| \leqslant 1$, for every $(\hat\omega, x) \in A_0$ it holds either $|\hat\varphi(\hat\omega,t,x)| < 1$ or $|\hat\varphi(\hat\omega,t,x)| = 1$. In the former case, since $\hat\varphi_\varepsilon(\hat\omega,t,x) \to \hat\varphi(\hat\omega,t,x)$, we have $|\hat\varphi_\varepsilon(\hat\omega,t,x)| < 1$ for ε small enough, hence also $F_{1,\varepsilon}(\hat\varphi_\varepsilon(\hat\omega,t,x)) = F(\hat\varphi_\varepsilon(\hat\omega,t,x))$ and $M_\varepsilon(\hat\varphi_\varepsilon(\hat\omega,t,x)) = M(\hat\varphi_\varepsilon(\hat\omega,t,x))$ for ε small enough, from which (4.18) follows by continuity of F_1 and M in $(-1,1)$. In the latter case, it holds either $\hat\varphi(\hat\omega,t,x) = 1$ or $\hat\varphi(\hat\omega,t,x) = -1$. In the first possibility, for ε small we have by definition of $F_{1,\varepsilon}$

$$F_{1,\varepsilon}(\hat\varphi_\varepsilon(\hat\omega,t,x)) \geqslant \min\{F_1(\hat\varphi_\varepsilon(\hat\omega,t,x)), F_1(1-\varepsilon)\} \to L,$$

where $L := \lim_{r\to 1^-} F_1(r) \in [0,+\infty]$ is well defined by monotonicity of F_1. If $L = +\infty$, then by comparison we deduce that $F_{1,\varepsilon}(\hat\varphi_\varepsilon(\hat\omega,t,x)) \to +\infty$, so that (4.18) holds automatically. If $L < +\infty$, then F_1 can be extended by continuity in 1, and the inequality gives

$$\liminf_{\varepsilon\to 0} F_{1,\varepsilon}(\hat\varphi_\varepsilon(\hat\omega,t,x)) \geqslant L = F_1(1) = F_1(\hat\varphi(\hat\omega,t,x)),$$

so that (4.18) still holds. The argument for the point -1 and the function M is exactly the same, and (4.18) is proved. Finally, the inequalities (4.4), (4.8), and (4.18), together with the Fatou's lemma, yield the desired assertion. This completes the proof of theorem 2.7.

Acknowledgments

The author gratefully acknowledges financial support from the Austrian Science Fund (FWF) through project M 2876. The author is also very grateful to the anonymous referees for their valuable comments and constructive suggestions.

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