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Quasi-classical dynamics

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Abstract. We study quantum particles in interaction with a force-carrying field, in the quasi-classical limit. This limit is characterized by the field having a very large number of excitations (it is therefore macroscopic), while the particles retain their quantum nature. We prove that the interacting microscopic dynamics converges, in the quasi-classical limit, to an effective dynamics where the field acts as a classical environment that drives the quantum particles.

Keywords. Quasi-classical limit, interaction of matter and light, open quantum systems, semiclassical analysis

Contents

1. Introduction and main results

This paper is devoted to the study of the quasi-classical dynamics of a coupled quantum system composed of finitely many non-relativistic particles interacting with a bosonic

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field. The quasi-classical regime is concretely realized by taking a suitable partial semiclassical limit, introduced by the authors in [16, 17] to derive external potentials as effective interactions emerging from the particle-field coupling. The physical meaning of such limit is discussed in §1.1.

Our analysis clarifies, both mathematically and physically, the role played by external macroscopic classical force fields acting on quantum systems, and in which regime such macroscopic fields provide an accurate description of the interaction between an open quantum system and its environment (bosonic field).

In order to study the dynamical quasi-classical limit, we develop a mathematical framework of infinite-dimensional quasi-classical analysis, in analogy with the semiclassical scheme initially introduced in [6–9], and further discussed in [22, 23]. Such a framework allows us to characterize the quasi-classical behavior of quantum states which are not factorized, i.e., in which the degrees of freedom of the quantum particles and the bosonic field are entangled. Although our mathematical scheme is more general, we are going to focus our attention on three concrete models of interaction between particles and force-carrying fields: the Nelson, Pauli–Fierz, and Fröhlich polaron models (see §1.4). Note that partial semiclassical limits have already been studied, with somewhat different purposes, in [10–12, 32], as well as in the context of adiabatic theories (see, e.g., [48, 53–55]).

The paper is organized as follows. In the rest of §1 we introduce the paper's mathematical framework, and we formulate and motivate our results. In §2 we develop the main technical tools for the subsequent analysis, which we call quasi-classical analysis, in analogy with the more familiar semiclassical analysis. In fact, quasi-classical analysis is semiclassical analysis on a bipartite system, where only one part is semiclassical, and the other is quantum. In §3 we describe the relevant features of the microscopic Nelson model, which we use as a reference to explain the strategy of the proof of Theorem 1.6 below. We then take the limit as $\varepsilon \to 0$ of the microscopic integral equation of motion in §4, while in §5 we discuss the uniqueness of solutions to the quasi-classical equation obtained by performing the aforementioned limit. In §6 we put together the results obtained in §§2 to 5, and prove Theorem 1.6 for the Nelson model, and thus consequently also Corollary 1.14 and Theorem 1.16. In §7, we provide the technical modifications needed to prove the aforementioned theorems for the Pauli–Fierz and polaron models.

1.1. Physical motivation

The quasi-classical description, combining a quantum system with a classical force field, is often used in physics to model external macroscopic forces acting on a quantum particle system. The best known examples are atoms and electrons in a classical electromagnetic field (see, e.g., [15]), and particles subjected to external potentials, such as systems of trapped atoms and of particles in optical lattices. Since these external force fields are macroscopic, they are heuristically taken as classical, and inserted in the particles' Hamiltonian in the same way their microscopic counterparts would appear. Note that in the literature the terminology "quasi-classical" is often used as synonymous with semiclassical, while here we use it to stress that the classical limit we consider is not complete, but applies only to a part (radiation field or environment) of the microscopic system.

In this paper we provide a detailed analysis of the quasi-classical dynamical scheme, and discuss its validity as an approximation of a more fundamental microscopic model, thus justifying and completing the above heuristic picture. The basic idea is the following: in experiments, the external force fields are considered macroscopic because they live on an energy scale much larger than the ones of the quantum particles under study: the number of field's excitations is much larger than the number of quantum particles in the system. Let us denote by N the number of particles in the system. The force field is itself a quantum object, and its excitations are created and annihilated by the interaction with the particles. Let us denote the field's number operator by

$$\mathrm{d}\mathscr{G}(1) = \int \mathrm{d}k \, a^{\dagger}(k) a(k), \tag{1.1}$$

where $\mathscr{G}$ stands for the second quantization functor. Therefore, the field is macroscopic if the state Ψ of the coupled system particles+field is such that $\langle \Psi | \mathrm{d}\mathscr{G}(1) | \Psi \rangle \gg N$. The number of particles N is fixed, and therefore of order 1. In other words, the quasi-classical configurations are the ones for which

$$\langle \Psi | \mathrm{d}\mathscr{G}(1) | \Psi \rangle \gg 1. \tag{1.2}$$

We thence introduce a quasi-classical parameter ε, playing the role of a semiclassical parameter but only for the field's degrees of freedom: when $\varepsilon \to 0$, the system becomes quasi-classical. We quantify ε as follows: a quasi-classical state Ψ_ε is a state such that

$$\langle \Psi_\varepsilon | \mathrm{d}\mathscr{G}(1) | \Psi_\varepsilon \rangle \sim 1/\varepsilon.$$

In other words, ε is proportional to the inverse of the average number of excitations of the force-carrying field. It follows that on quasi-classical states,

$$\begin{aligned} \langle \Psi_\varepsilon | \varepsilon \mathrm{d}\mathscr{G}(1) | \Psi_\varepsilon \rangle &= \int \mathrm{d}\mathbf{k} \, \langle \Psi_\varepsilon | \varepsilon a^{\dagger}(\mathbf{k}) a(\mathbf{k}) | \Psi_\varepsilon \rangle \\ &= \int \mathrm{d}\mathbf{k} \, \langle \Psi_\varepsilon | a_\varepsilon^{\dagger}(\mathbf{k}) a_\varepsilon(\mathbf{k}) | \Psi_\varepsilon \rangle \leq C, \end{aligned} \tag{1.3}$$

where $a_\varepsilon^{\#}(\cdot) := \sqrt{\varepsilon}\, a^{\#}(\cdot)$. The creation and annihilation operators $a_\varepsilon^{\#}$ satisfy ε-dependent semiclassical canonical commutation relations:

$$[a_\varepsilon(\mathbf{k}), a_\varepsilon^{\dagger}(\mathbf{k}')] = \varepsilon \delta(\mathbf{k} - \mathbf{k}'). \tag{1.4}$$

It is therefore clear that a quasi-classical state is a state that behaves semiclassically only with respect to the field's degrees of freedom.

It remains to understand which microscopic dynamics would yield, in the quasi-classical limit, an external potential acting on the particles and generated by the macroscopic field. In concrete applications, the macroscopic field is not affected by the quantum system and acts as an environment. Therefore, the coupling should be such that the particles do not back-react on the environment, at least to leading order in ε and for times of order 1. In addition, we may think that the environment itself either evolves freely, or remains constant in time. The absence of back-reaction is determined by the ε-scaling of the microscopic interaction, while the dynamical behavior of the environment is determined by the ε-scaling of the field's free part. It turns out that it is indeed possible to tune the scaling of the interaction in such a way that, in the limit as $\varepsilon \to 0$, the latter is precisely *weak enough* to make the system decouple only partially: the classical field obeys a linear, unperturbed, evolution, while the quantum system's dynamics is driven by the classical field itself. The scaling yielding such a behavior is introduced in §1.4, and discussed in §1.5. Let us stress that, in contrast to a complete semiclassical limit, in the quasi-classical regime the aforementioned partial decoupling prevents any nonlinearity from appearing in the effective dynamics of both the classical field and the quantum system.

In §1.5 we prove that the quasi-classical description can be rigorously obtained from microscopic models of particle-field interaction in the limit $\varepsilon \to 0$ of a very large number of average field's excitations. Since such limit is a semiclassical limit on the field only, the resulting structure of quasi-classical systems is that of a hybrid quantum/classical probability theory. The quantum system is driven by the classical environment, whose configuration is a classical probability with values in the quantum states for the particles. This mathematical structure is described in detail in §1.3.

1.2. Notation

Since we are going to consider a tensor product Hilbert space of the form $\mathcal{H} \otimes \mathcal{K}_\varepsilon$, we will distinguish between the full trace $\mathrm{Tr}(\,\cdot\,)$ of operators on $\mathcal{H} \otimes \mathcal{K}_\varepsilon$ and the partial traces $\mathrm{tr}_{\mathcal{H}}(\,\cdot\,)$ and $\mathrm{tr}_{\mathcal{K}_\varepsilon}(\,\cdot\,)$ with respect to $\mathcal{H}$ and $\mathcal{K}_\varepsilon$, respectively.

We adopt the following convenient notation: an operator acting *only* on the particle space $\mathcal{H}$ is denoted by a *calligraphic capital letter* (e.g., $\mathcal{T}$ or $\mathcal{T}_\varepsilon$), whereas an operator on the full space $\mathcal{H} \otimes \mathcal{K}_\varepsilon$ is identified by an *italic capital letter* (e.g., H_ε). Given an operator $\mathcal{T}$ on $\mathcal{H}$, we also conveniently denote its extension to $\mathcal{H} \otimes \mathcal{K}_\varepsilon$, i.e. $T = \mathcal{T} \otimes 1$, by the italic counterpart T.

Given a Hilbert space $\mathcal{X}$, we denote by $\mathcal{L}^p(\mathcal{X})$, $p \in [1, \infty]$, the p-th Schatten ideal of $\mathcal{B}(\mathcal{X})$, the space of bounded operators on $\mathcal{X}$. More generally, the set $\mathscr{L}(\mathcal{X})$ consists of all linear operators on $\mathcal{X}$. We also denote by $\mathcal{L}^p_+(\mathcal{X})$ and $\mathcal{B}_+(\mathcal{X})$ the cones of positive elements, and by $\mathcal{L}^p_{+,1}(\mathcal{X})$ the set of positive elements of norm 1. The corresponding norms are denoted by keeping track of the space, except for the case of the operator norm, for which we use the short notation $\|\cdot\| := \|\cdot\|_{\mathcal{B}(\mathcal{X})}$.

The space of finite measures on a measure space (X, Σ) is denoted by $\mathcal{M}(X, \Sigma)$, while the subset of probability measures is $\mathcal{P}(X, \Sigma)$. If X is a Hausdorff topological space and Σ is the Borel σ-algebra, we denote by $\mathcal{M}(X)$ the finite Radon Borel measures on X, and by $\mathcal{P}(X)$ the subset of probability measures.

Throughout the paper, given a set S we denote by $\mathbb{1}_S$ its indicator function. The symbol C also stands for a finite positive constant, whose value may vary from line to line.

1.3. Quasi-classical system

We consider a microscopic system consisting of two parts in interaction. The first one contains objects whose microscopic nature remains relevant, while the second is a semiclassical environment. For the sake of clarity, we focus on a specific class of systems: non-relativistic quantum particles in interaction with a semiclassical bosonic force-carrying field (electromagnetic, vibrational, etc.). It is not difficult to adapt the techniques to other coupled systems as well, consisting of a quantum and a semiclassical part. We denote by $\mathcal{H}$ the Hilbert space of the quantum part, and by $\mathcal{K}_\varepsilon$ the Hilbert space of the semiclassical part, which carries an ε-dependent, semiclassical, representation of the canonical commutation relations as in (1.4). Therefore, the microscopic theory is set in the Hilbert space $\mathcal{H} \otimes \mathcal{K}_\varepsilon$.

We restrict our attention to Fock representations of the canonical commutation relations. Therefore, we assume that

$$\mathcal{K}_\varepsilon = \mathcal{G}_\varepsilon(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h}^{\otimes_s n},$$

the symmetric Fock space constructed over a separable Hilbert space $\mathfrak{h}$. The space $\mathfrak{h}$ is the space of classical fields.[1] The canonical commutation relation (1.4) in $\mathcal{K}_\varepsilon$ reads, for any $z, w \in \mathfrak{h}$,

$$[a_\varepsilon(z), a_\varepsilon^\dagger(w)] = \varepsilon \langle z | w \rangle_{\mathfrak{h}},$$

and the quasi-classical limit corresponds to the limit as $\varepsilon \to 0$.

According to the notation above, a microscopic Fock-normal state is thus described by a density matrix

$$\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon). \tag{1.5}$$

The dynamics is generated by a self-adjoint and bounded-from-below Hamiltonian on $\mathcal{H} \otimes \mathcal{K}_\varepsilon$, which we denote by H_ε. Given the unitary dynamics e^{-itH_ε}, the evolved state is

$$\Gamma_\varepsilon(t) := e^{-itH_\varepsilon} \Gamma_\varepsilon e^{itH_\varepsilon}. \tag{1.6}$$

Let us now turn our attention to the effective quasi-classical system in the limit as $\varepsilon \to 0$. This is a hybrid quantum-classical system, in which the classical part acts as an environment for the quantum part. In fact, as we will see, the classical field affects the quantum particles, but the converse is not true: the interaction is not strong enough to cause a back-reaction of the particles on the classical field.

The basic observables for the classical fields are the elements $z \in \mathfrak{h}$, or, more precisely, the real vectors of the form $z + \bar{z}$. Scalar observables in a generalized sense are functions $z \mapsto f(z) \in \mathbb{C}$ semiclassically called *symbols*. In addition to scalar or field observables,

[1] Strictly speaking, the space $\mathfrak{h}$ should be the Hilbert completion of the set of test functions for the classical fields, but in the following we are going to restrict our attention to classical fields belonging to such space.

there are more general observables involving both subsystems, which are thus represented by operator-valued functions $z \mapsto \mathscr{F}(z)$, where $\mathscr{F}(z)$ is a linear operator on the particle Hilbert space $\mathcal{H}$. Note that one can easily associate an operator-valued function to a scalar symbol as well, by simply setting $\mathscr{F}(z) = f(z) \cdot \mathbb{1}$, where $\mathbb{1} \in \mathcal{B}(\mathcal{H})$ stands for the identity operator.

A state of the classical field (environment) is a Borel probability measure $\mu \in \mathcal{P}(\mathfrak{h})$, while a state of the quantum particles is a density matrix $\gamma \in \mathcal{L}^1_{+,1}(\mathcal{H})$. Since in the quasi-classical regime the environment affects the behavior of the quantum particle system, a quasi-classical state is a *state-valued probability measure*

$$\mathfrak{m} \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_+(\mathcal{H})). \tag{1.7}$$

A state-valued measure thus takes values in $\mathcal{L}^1_+(\mathcal{H})$, but it can also be conveniently described by its norm Radon–Nikodým decomposition (see Proposition 2.2): a pair $(\mu_{\mathfrak{m}}, \gamma_{\mathfrak{m}}(z))$ consisting of a scalar Borel (probability) measure $\mu_{\mathfrak{m}}$, and a $\mu_{\mathfrak{m}}$-integrable, almost everywhere defined function $\gamma_{\mathfrak{m}}(z) \in \mathcal{L}^1_{+,1}(\mathcal{H})$ taking values in normalized density matrices, i.e.,

$$\mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu_{\mathfrak{m}}(z). \tag{1.8}$$

In other words, a generic normalized quasi-classical state consists of a measure $\mu_{\mathfrak{m}}$ describing the environment, and a function $\gamma_{\mathfrak{m}}(z)$ describing how (almost) each configuration of the field affects the quantum particles' state. The quasi-classical equivalent of taking the partial trace with respect to the field's degrees of freedom is integrating with respect to the quasi-classical state-valued measure, i.e., for any operator-valued function $\mathscr{F}(z) \in \mathcal{B}(\mathcal{H})$,

$$\int_{\mathfrak{h}} \mathrm{d}\mathfrak{m}(z)\, \mathscr{F}(z) = \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z)\mathscr{F}(z). \tag{1.9}$$

Note that when integrating against the state-valued measure, it is a priori relevant to keep the order as in the above expression, since $\mathscr{F}(z)$ might not commute with $\gamma_{\mathfrak{m}}(z)$.

The quasi-classical evolution also consists of two parts: an evolution of the environment's probability measure $\mu_{\mathfrak{m}}$, and one of the quantum system for (almost) every configuration of the classical field. The evolution of the environment depends on the choice of a scaling parameter for the field's part in H_ε, and we consider two cases: either the environment is stationary, e.g., it is at equilibrium, or it evolves freely. Concretely, the environment is evolved by a unitary, linear, flow $e^{-it\nu\omega} : \mathfrak{h} \to \mathfrak{h}$, $t \in \mathbb{R}$, of classical fields, where ω is a positive self-adjoint operator on $\mathfrak{h}$ (typically, a multiplication operator by the dispersion relation of the field), and $\nu \in \{0, 1\}$, depending on the chosen scaling. This flow pushes forward the measure $\mu_{\mathfrak{m}}$, yielding

$$\mu_{\mathfrak{m},t} := (e^{-it\nu\omega})_\star \mu_{\mathfrak{m}}. \tag{1.10}$$

The explicit action of the pushforward, as is well-known, is as follows: for all measurable Borel sets $B \subset \mathfrak{h}$,

$$[(e^{-it\nu\omega})_\star \mu](B) := \mu(e^{it\nu\omega} B), \tag{1.11}$$

where $e^{it\nu\omega}B$ stands for the preimage of B with respect to the map $e^{-it\nu\omega}$. The quantum part of the evolution is generated by a map from field configurations to two-parameter groups of unitary operators $z \mapsto (\mathcal{U}_{t,s}(z))_{t,s\in\mathbb{R}}$, and it acts as

$$\gamma_{\mathfrak{m},t,s}(z) := \mathcal{U}_{t,s}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^{\dagger}_{t,s}(z). \tag{1.12}$$

Let us remark that the pushforward of the measure *does not affect* the Radon–Nikodým derivative $\gamma_{\mathfrak{m},t,s}(z)$, but only the integrated functions.

The quantum evolution is unitary for (almost) all configurations of the field. However, a measurement on the classical system modifies the quantum state in a non-unitary, but explicit, way. Let $f(z)$ be a scalar field's observable and suppose it is $\mu_{\mathfrak{m}}$-measurable. For $\lambda \in \mathbb{C}$, let us define the level set of f as

$$B_\lambda = \{z \in \mathfrak{h} \mid f(z) = \lambda\}.$$

Then the conditional quantum state $\gamma_{\mathfrak{m},t,s}|_{f=\lambda} \in \mathcal{L}^1_{+,1}(\mathcal{H})$ at time $t \in \mathbb{R}$, describing the state of the quantum system conditioned on an observed value λ of the classical observable f, is given by

$$\begin{aligned}\gamma_{\mathfrak{m},t,s}|_{f=\lambda} &= \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m},\lambda,t-s}(z)\, \gamma_{\mathfrak{m},t}(z)\, \mathbb{1}_{B_\lambda}(z)\\ &= \int_{e^{i(t-s)\nu\omega}B_\lambda} \mathrm{d}\mu_{\mathfrak{m},\lambda}(z)\, \mathcal{U}_{t,s}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^{\dagger}_{t,s}(z),\end{aligned}$$

where $(\mu_{\mathfrak{m},\lambda})_{\lambda\in\mathbb{C}}$ is the disintegration of $\mu_{\mathfrak{m}}$ with respect to the function f. The conditional evolution $(t,s) \mapsto \gamma_{\mathfrak{m},t,s}|_{f=\lambda}$ is clearly non-unitary but it preserves positivity: the dynamics is in general non-Markovian, unless either $B_\lambda = \{z_\lambda\}$ or $\mu_{\mathfrak{m}} = \delta_{z_0}$, i.e., the group property might not be satisfied. One should not indeed expect that, for any $t, s, \tau > 0$, there exists some two-parameter unitary group $\mathcal{W}_{t,s} \in \mathcal{B}(\mathcal{H})$ such that

$$\gamma_{\mathfrak{m},t,s}|_{f=\lambda} = \mathcal{W}_{t,\tau}\gamma_{\mathfrak{m},\tau,s}|_{f=\lambda}\mathcal{W}^{\dagger}_{t,\tau}.$$

The quantum state at time $t \in \mathbb{R}$, conditioned on the fact that f (or any other observable) is observed, irrespective of its value, is denoted by $\gamma_{\mathfrak{m},t,s}$, is independent of f, and is given by

$$\begin{aligned}\gamma_{\mathfrak{m},t,s} &= \int_{\mathbb{C}} \mathrm{d}\lambda \int_{e^{i(t-s)\nu\omega}B_\lambda} \mathrm{d}\mu_{\mathfrak{m},\lambda}(z)\, \mathcal{U}_{t,s}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^{\dagger}_{t,s}(z)\\ &= \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \mathcal{U}_{t,s}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^{\dagger}_{t,s}(z).\end{aligned}$$

Again the conditional evolution $(t,s) \mapsto \gamma_{\mathfrak{m},t,s}$ preserves both positivity and the trace, but it is still non-Markovian in general. It would be interesting to study the states of the environment, if any, not concentrated in a single field configuration, that make the conditional evolution Markovian, and possibly non-unitary. Such measures would yield a quasi-classical evolution on the open quantum system of Lindblad type (see e.g. [40,44]).

1.4. The concrete models: Nelson, Pauli–Fierz, and polaron

Let us define more concretely the three models of interaction between non-relativistic particles and bosonic force carrier fields that we consider throughout the paper: the Nelson, Pauli–Fierz, and polaron models.

1.4.1. Nelson model. The Nelson model describes quantum particles (e.g., nucleons), interacting with a force-carrying scalar field (e.g., a meson field), and was the first to be rigorously studied [46]. In this paper, we restrict our attention to the regularized Nelson model, where the interaction is smeared by an ultraviolet cutoff. We consider N, d-dimensional, non-relativistic, spinless particles, and therefore $\mathcal{H} = L^2(\mathbb{R}^{dN})$. The classical fields are usually taken to be in $\mathfrak{h} = L^2(\mathbb{R}^d)$, but other choices may be possible, e.g., a cavity field, whose classical space would then be $\ell^2(\mathbb{Z}^d)$. The Hamiltonian H_ε has the form

$$H_\varepsilon = K_0 + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega) + \sum_{j=1}^{N} [a_\varepsilon^\dagger(\lambda(\mathbf{x}_j)) + a_\varepsilon(\lambda(\mathbf{x}_j))],$$

where $K_0 = \mathcal{K}_0 \otimes 1$, with $\mathcal{K}_0$ self-adjoint and bounded from below on $\mathcal{H}$; ω is a positive operator on $\mathfrak{h}$ and $\mathrm{d}\mathcal{G}_\varepsilon(\omega)$ its second quantization, i.e., the Wick quantization of the symbol

$$\kappa(z) := \langle z|\omega|z\rangle_{\mathfrak{h}}; \tag{1.13}$$

and $\lambda \in L^\infty(\mathbb{R}^d; \mathfrak{h})$ is the coupling factor.

If one naively replaces the quantum canonical variables $a^\#$ with their classical counterparts, i.e., $z^\#$, one can easily deduce that the quasi-classical effective potential for the model above is given by the symbol $z \mapsto \mathcal{V}(z)$, where (see also [16, §2.2])

$$\mathcal{V}(z) = \sum_{j=1}^{N} 2\,\mathrm{Re}\,\langle z|\lambda(\mathbf{x}_j)\rangle_{\mathfrak{h}} \in \mathcal{B}(\mathcal{H}). \tag{1.14}$$

In most practical applications $\lambda(\mathbf{x};\cdot) \in \mathfrak{h} = L^2(\mathbb{R}^d)$ has the following explicit form:

$$\lambda(\mathbf{x};\mathbf{k}) = \lambda_0(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \lambda_0 \in L^2(\mathbb{R}^d).$$

This leads to the effective potential $\mathcal{V}(z)$ being the Fourier transform of an integrable function, and thus continuous and vanishing at infinity. In order to obtain more singular potentials, it is necessary to consider microscopic states whose measures are not concentrated as Radon measures in $\mathfrak{h}$ [16, §2.5]. This would, however, make the analysis more involved. We thus restrict our attention to states whose measures are indeed concentrated in $\mathfrak{h}$ (see Remark 1.11 for additional details).

1.4.2. Pauli–Fierz model. We consider the class of Pauli–Fierz models describing N non-relativistic, spinless, extended d-dimensional charges moving in $\mathbb{R}^d$, $d \geq 2$, interacting with electromagnetic radiation in the Coulomb gauge. Adding spin, adopting a different gauge, or constraining particles to an open subset of $\mathbb{R}^d$ would not affect the results, but make the analysis more involved. The particles' Hilbert space is thus $\mathcal{H} = L^2(\mathbb{R}^{dN})$, while the classical fields are in $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C}^{d-1})$. The Hamiltonian H_ε is customarily written as

$$H_\varepsilon = \sum_{j=1}^{N} (-i\nabla_j + \mathbf{A}_\varepsilon(\mathbf{x}_j))^2 + W(\mathbf{x}_1, \dots, \mathbf{x}_N) + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega),$$

with

$$\mathbf{A}_\varepsilon(\mathbf{x}) = a_\varepsilon^\dagger(\boldsymbol{\lambda}(\mathbf{x})) + a_\varepsilon(\boldsymbol{\lambda}(\mathbf{x})).$$

In the above, $W = W_1 + W_2$ is a multiplicative potential describing the interaction among charges, with $\mathcal{W}_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^{dN}; \mathbb{R}^+)$ and $\mathcal{W}_2(-\Delta + 1)^{-1/2} \in \mathcal{B}(\mathcal{H})$; ω is the field's dispersion relation, a positive multiplication operator on $\mathfrak{h}$ such that ω^{-1} is also a positive self-adjoint operator on $\mathfrak{h}$, e.g., $\omega(\mathbf{k}) = |\mathbf{k}|$; and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)$ with $\lambda_\ell \in L^\infty(\mathbb{R}^d; \mathcal{D}(\omega^{-1/2} + \omega^{1/2}))$ for all $\ell \in \{1, \dots, d\}$ and $\nabla \cdot \boldsymbol{\lambda}(x) = 0$, is the particles' charge distribution. We denote by $\mathcal{D}(\omega^{-1/2} + \omega^{1/2}) \subset \mathfrak{h}$ the intersection of the self-adjointness domains of $\omega^{-1/2}$ and $\omega^{1/2}$.

In this case, we have $\mathcal{K}_0 = -\Delta + \mathcal{W}$ and the effective potential can be easily seen to be [17, §1.2]

$$\mathcal{V}(z) = 4\sum_{j=1}^{N} \left[-i \,\mathrm{Re}\, \langle z | \boldsymbol{\lambda}(\mathbf{x}_j)\rangle_{\mathfrak{h}} \cdot \nabla_j + (\mathrm{Re}\, \langle z | \boldsymbol{\lambda}(\mathbf{x}_j)\rangle_{\mathfrak{h}})^2\right]. \tag{1.15}$$

Notice that the interaction term in H_ε is not the Wick quantization of the above symbol $\mathcal{V}(z)$, because H_ε is not normal ordered and an additional term is missing, i.e.,

$$\varepsilon \sum_{j=1}^{N} \sum_{\ell=1}^{d} \|\lambda_\ell(\mathbf{x}_j)\|^2_{\mathfrak{h}} = \mathcal{O}(\varepsilon),$$

but such a contribution vanishes as $\varepsilon \to 0$. Similarly to the Nelson model, the effective interaction $\mathcal{V}(z)$ describes the minimal coupling of the particles with a magnetic potential that is continuous and vanishing at infinity.

1.4.3. Polaron. Fröhlich's polaron [29] describes electrons moving in a quantum lattice crystal. The N d-dimensional electrons are modeled as non-relativistic spinless particles, and thus again $\mathcal{H} = L^2(\mathbb{R}^{dN})$. For the phonon vibrational field, $\mathfrak{h} = L^2(\mathbb{R}^d)$. The Hamiltonian H_ε is formally written as

$$H_\varepsilon = -\Delta + a_\varepsilon^\dagger(\phi(\mathbf{x}_j)) + a_\varepsilon(\phi(\mathbf{x}_j)) + W(\mathbf{x}_1, \dots, \mathbf{x}_N) + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(1),$$

with the particles' potential W satisfying the same assumptions as in §1.4.2 for the Pauli–Fierz model. In addition,

$$\phi(\mathbf{x}; \mathbf{k}) := \alpha \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^{(d-1)/2}},$$

with $\alpha \in \mathbb{R}$, is the polaron's form factor and, for all $\mathbf{x} \in \mathbb{R}^d$, it does not belong to $\mathfrak{h}$. Hence, H_ε as written above is only a formal expression. However, it makes sense as a closed and bounded-from-below quadratic form: one can find a parameter $r \in \mathbb{R}^+$, a splitting $\phi = \phi_r + \chi_r$ with

$$\phi_r(\mathbf{x}; \mathbf{k}) := \mathbb{1}_{\{|\mathbf{k}| \leq r\}}(\mathbf{k}) \phi(\mathbf{x}; \mathbf{k}),$$

and some $\boldsymbol{\lambda}_r \in L^\infty(\mathbb{R}^d; \mathfrak{h}^d)$ such that, as a quadratic form,

$$\begin{aligned} H_\varepsilon = -\Delta + a_\varepsilon^\dagger(\phi_r(\mathbf{x}_j)) + a_\varepsilon(\phi_r(\mathbf{x}_j)) + [-i\nabla_j, a_\varepsilon(\boldsymbol{\lambda}_r(\mathbf{x}_j)) - a_\varepsilon^\dagger(\boldsymbol{\lambda}_r(\mathbf{x}_j))] \\ + W(\mathbf{x}_1, \ldots, \mathbf{x}_N) + \nu(\varepsilon) \mathrm{d}\mathcal{G}_\varepsilon(1), \end{aligned}$$

where the commutator between two vectors of operators involves a scalar product.

In the polaron model $\mathcal{K}_0 = -\Delta + \mathcal{W}$, and the effective potential is given by [16, §2.3]

$$\mathcal{V}(z) = 2 \sum_{j=1}^{N} \mathrm{Re}\, \langle z | \phi_r(\mathbf{x}_j) \rangle_{\mathfrak{h}} + [-i\nabla_j, \mathrm{Im} \langle \boldsymbol{\lambda}_r(\mathbf{x}_j) | z \rangle_{\mathfrak{h}}].$$

Notice that one could formally resum the two terms above, obtaining the same expression (1.14) as in the Nelson model. In the case of the polaron, the potential $\mathcal{V}(z)$ is not necessarily bounded, but still relatively form bounded with respect to $-\Delta$. In fact, $\mathcal{V}(z)$ can be any function in $\dot{H}^{(d-1)/2}(\mathbb{R}^d) \cap L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

Let us also remark that in the polaron case, the quasi-classical limit is mathematically analogous to the strong coupling limit. Strongly coupled polarons have been widely studied in the mathematical literature both from a dynamical and a variational point of view (see, e.g., [25–28, 33, 34, 36, 41–43]). Compared to the available dynamical results [25, 27, 33, 41], our quasi-classical approach has the advantage of being applicable to a very general class of microscopic initial states. However, we have no control on the errors and we are not able to derive the higher order corrections to the effective dynamics, i.e., the ones given by the Landau–Pekar equations.

1.5. Main results

Before stating our main results, we provide more technical details about the general structure of the models we are considering in this paper, by specifying some assumptions that are sufficient to prove our main results, and that are satisfied in the above concrete models. We do not strive for the optimal assumptions nor for the most general setting.

First of all, we remark that all the Hamiltonians introduced in §1.4 can be cast in the form

$$H_\varepsilon = \mathcal{K}_0 \otimes 1 + \nu(\varepsilon)\, 1 \otimes \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\kappa) + \mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\mathcal{V}) + \mathcal{O}(\varepsilon), \tag{1.16}$$

where $\mathcal{K}_0$ is self-adjoint and bounded from below on $\mathcal{H}$, and describes the particle's system when it is isolated; $\nu(\varepsilon)$ is a quasi-classical scaling factor such that

$$\nu = \lim_{\varepsilon \to 0} \varepsilon \nu(\varepsilon) \in \{0, 1\}, \tag{1.17}$$

and the two relevant scalings are $\nu(\varepsilon) = 1$, yielding an environment that remains constant in time, and $\nu(\varepsilon) = 1/\varepsilon$, yielding an environment that evolves freely; κ is the symbol given by (1.13) for a densely defined, positive operator ω on $\mathfrak{h}$. Given a symbol $z \mapsto \mathcal{F}(z)$, we denote by $\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\mathcal{F})$ its Wick quantization, so that in particular $\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\kappa) = \mathrm{d}\mathcal{G}_\varepsilon(\omega)$. The symbol $z \mapsto \mathcal{V}(z)$ is operator-valued and polynomial, and it describes the interaction between the particles and the environment. The possible concrete choices of $\mathcal{V}$ have been presented in §1.4. Finally, $\mathcal{O}(\varepsilon)$ is a bounded particle operator of order ε.

To study the limit as $\varepsilon \to 0$ of the evolved states $\Gamma_\varepsilon(t)$, we make the following very general assumption on[2] $\Gamma_\varepsilon(0) = \Gamma_\varepsilon$:

$$\exists \delta > 0, \exists C_\delta < +\infty : \ \mathrm{Tr}\Big(\Gamma_\varepsilon (\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta\Big) \leq C_\delta, \tag{A1}$$

which is for instance satisfied if the state scales with ε as in (1.3), or if it is formed by a coherent superposition of vectors with a finite number of force carriers. This assumption is sufficient to prove the existence of a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \to 0$ such that Γ_{ε_n} converges to a quasi-classical state $\mathfrak{m}$ in the sense of Definition 1.1 below. For the polaron and Pauli–Fierz models, an additional assumption is necessary to study the limit as $\varepsilon \to 0$ of $\Gamma_\varepsilon(t)$, due to the fact that such models are "more singular" than the Nelson model:

$$\exists C < +\infty : \ \mathrm{Tr}\Big(\Gamma_\varepsilon (K_0 + \mathrm{d}\mathcal{G}_\varepsilon(\omega))\Big) \leq C. \tag{A1$'$}$$

Finally, in order to ensure that no loss of mass occurs along the weak limit, or equivalently, that the quasi-classical limit point $\mathfrak{m}$ is still normalized and $\|\mathfrak{m}(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})} = 1$, we also need control of the particle component of the state Γ_ε. We thus define the *reduced density matrix* for the particles as

$$\gamma_\varepsilon := \mathrm{tr}_{\mathcal{K}_\varepsilon} \Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H}), \tag{1.18}$$

and impose the following alternative conditions on γ_ε:

$$\exists \mathcal{A} > 0, \mathcal{A}^{-1} \in \mathcal{L}^\infty(\mathcal{H}) : \ \mathrm{tr}_{\mathcal{H}}(\mathcal{A}\gamma_\varepsilon) \leq C < +\infty, \tag{A2}$$

or

$$\exists \gamma \in \mathcal{L}^1_+(\mathcal{H}) : \ \gamma_\varepsilon \leq \gamma. \tag{A2$'$}$$

We are going to comment further on the above conditions in Remarks 1.9 and 1.10, but we point out here that the second is stronger than the first, in the sense that (A2′) implies (A2). A simple but relevant case in which (A2′) is trivially satisfied is given by product states of the form $\gamma \otimes \varsigma_\varepsilon$ with $\gamma \in \mathcal{L}^1_{+,1}(\mathcal{H})$ independent of ε. By contrast, the more general assumption (A2) seems at first glance to be also more arbitrary, but it could be put in relation to the physics of the model (see Remark 1.10).

Let us denote by $\widehat{\Gamma}_\varepsilon$ the non-commutative Fourier transform or generating map of a state $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$, i.e.,

$$\widehat{\Gamma}_\varepsilon(\eta) := \mathrm{tr}_{\mathcal{K}_\varepsilon}(\Gamma_\varepsilon W_\varepsilon(\eta)) \in \mathcal{L}^1(\mathcal{H}) \tag{1.19}$$

[2]For simplicity, we set the initial time s equal to 0.

for any $\eta \in \mathfrak{h}$, where $W_\varepsilon(\eta)$ is the Weyl operator on $\mathcal{K}_\varepsilon$:

$$W_\varepsilon(\eta) := e^{i(a_\varepsilon^\dagger(\eta)+a_\varepsilon(\eta))}. \tag{1.20}$$

Analogously, to any state-valued measure $\mathfrak{m} \in \mathcal{M}(\mathfrak{h}; \mathcal{L}^1_+(\mathcal{H}))$ there corresponds the Fourier transform

$$\widehat{\mathfrak{m}}(\eta) := \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z) e^{2i\,\mathrm{Re}\,\langle \eta | z\rangle_{\mathfrak{h}}} \in \mathcal{L}^1(\mathcal{H}) \tag{1.21}$$

for $\eta \in \mathfrak{h}$.

Definition 1.1 (Quasi-classical convergence). Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ and $\mathfrak{m} \in \mathcal{M}(\mathfrak{h}; \mathcal{L}^1_+(\mathcal{H}))$. We say that

$$\Gamma_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathrm{qc}} \mathfrak{m} \tag{1.22}$$

if $\widehat{\Gamma}_\varepsilon(\eta) \xrightarrow[\varepsilon\to 0]{\mathrm{w}*} \widehat{\mathfrak{m}}(\eta)$ pointwise for all $\eta \in \mathfrak{h}$ in the weak-$*$ topology in $\mathcal{L}^1(\mathcal{H})$, i.e., when testing against compact operators in $\mathcal{L}^\infty(\mathcal{H})$.

This definition is given in terms of the Fourier transforms in order to completely characterize the limit quasi-classical measure $\mathfrak{m}$. On the other hand, from the physical point of view, it is relevant to study the convergence of expectation values of quantum observables, which is discussed in §2 and specifically in Theorem 1.16. Note that in light of Proposition 2.3, assumption (A1) guarantees that any such Γ_ε admits at least one limit point in the sense of Definition 1.1.

In the following, we may omit the superscript $^{\mathrm{qc}}$ in $\Gamma_\varepsilon \xrightarrow{\mathrm{qc}} \mathfrak{m}$ if it is clear from the context that we are considering the quasi-classical convergence of Definition 1.1 (and not its stronger counterpart of Definition 1.4 below).

Remark 1.2 (Reduced density matrix). We point out that the reduced density matrix γ_ε for the particle system given in (1.18) can be obtained by evaluating the non-commutative Fourier transform (1.19) at $\eta = 0$, i.e.,

$$\gamma_\varepsilon := \mathrm{tr}_{\mathcal{K}_\varepsilon} \Gamma_\varepsilon = \widehat{\Gamma}_\varepsilon(0).$$

Hence, the convergence $\Gamma_{\varepsilon_k} \xrightarrow[k\to+\infty]{} \mathfrak{m}$ can be easily seen to imply that

$$\gamma_{\varepsilon_k} \xrightarrow[k\to+\infty]{\mathrm{w}*} \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z), \tag{1.23}$$

where we have denoted by w$*$ the *weak-$*$ operator* topology.

Remark 1.3 (Product states). As a special case, we observe that if Γ_ε is a *physical product state*,[3] i.e., there exist $\gamma \in \mathcal{L}^1_{+,1}(\mathcal{H})$ and $\varsigma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{K}_\varepsilon)$ such that $\Gamma_\varepsilon = \gamma \otimes \varsigma_\varepsilon$, then

$$\Gamma_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathrm{qc}} \mathfrak{m} \iff \varsigma_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mu_{\mathfrak{m}} \text{ and } \gamma_{\mathfrak{m}}(z) = \gamma. \tag{1.24}$$

[3]Product states are the mathematical formulation of the fact that the two parts of the system are independent. Since ε characterizes only the behavior of the field, it is not physically relevant to put an ε-dependence on the particle part.

The proper definition of convergence of scalar measures was given in [16], but it coincides with Definition 1.1 when $\mathcal{H} = \mathbb{C}$.

A stronger notion of quasi-classical convergence can be given, lifting the weak-$*$ convergence of the Fourier transform in Definition 1.1 to weak convergence, i.e., by testing with bounded operators $\mathscr{B} \in \mathcal{B}(\mathcal{H})$. This leads to the following definition.

Definition 1.4 (Bounded quasi-classical convergence). Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ and $\mathfrak{m} \in \mathcal{M}(\mathfrak{h}; \mathcal{L}^1_+(\mathcal{H}))$. Then we write

$$\Gamma_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathrm{bqc}} \mathfrak{m} \tag{1.25}$$

if $\widehat{\Gamma}_\varepsilon(\eta) \xrightarrow[\varepsilon \to 0]{\mathrm{w}} \widehat{\mathfrak{m}}(\eta)$ pointwise for all $\eta \in \mathfrak{h}$ in the weak topology in $\mathcal{L}^1(\mathcal{H})$, i.e., when testing against bounded operators in $\mathcal{B}(\mathcal{H})$.

Remark 1.5 (Mass conservation). Bounded convergence ensures that no mass is lost in the quasi-classical limit: in fact, if $\mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon) \ni \Gamma_\varepsilon \xrightarrow{\mathrm{bqc}} \mathfrak{m}$, then

$$\mathrm{tr}_{\mathcal{H}} \int_{\mathfrak{h}} \mathrm{d}\mathfrak{m}(z) = \mathrm{tr}_{\mathcal{H}}(\hat{\mathfrak{m}}(0)) = \lim_{\varepsilon \to 0} \mathrm{tr}_{\mathcal{H}}(\hat{\Gamma}_\varepsilon(0)) = 1, \tag{1.26}$$

and therefore $\mathfrak{m} \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_+(\mathcal{H}))$. By contrast, there are normalized states $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ that converge to a quasi-classical measure with mass less than 1, and possibly zero, thus occurring in a loss of mass phenomenon. For example, let us consider a state similar to the one defined in Remark 1.3, where however $\gamma = \gamma_\varepsilon$ also depends on ε,

$$\gamma_\varepsilon = |e_{n(\varepsilon)}\rangle\langle e_{n(\varepsilon)}|,$$

$\{e_n\}_{n \in \mathbb{N}}$ being an orthonormal basis in $\mathcal{H}$, and $n(\varepsilon) \xrightarrow[\varepsilon \to 0]{} +\infty$. Then $\gamma_\varepsilon \otimes \varsigma_\varepsilon \to 0$, the zero state-valued measure, and thus all the mass is lost in the limit.

Our main result (see Theorem 1.6 and Corollary 1.14 below) is that initial convergence is propagated in time: for all $t \in \mathbb{R}$, $\Gamma_{\varepsilon_n}(t)$ converges to the quasi-classical state $\mathfrak{m}_t$ defined by the norm Radon–Nikodým decomposition

$$\boxed{\mathrm{d}\mathfrak{m}_t = \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^\dagger_{t,0}(z)\mathrm{d}((e^{-it\nu\omega})_\star \mu_{\mathfrak{m}}),} \tag{1.27}$$

where $\mathcal{U}_{t,s}(z)$ is the above mentioned quasi-classical two-parameter unitary group of evolution, which turns out to be weakly generated by the time-dependent Schrödinger operator

$$\mathcal{K}_t := \mathcal{K}_0 + \mathcal{V}_t(z) \tag{1.28}$$

with

$$\mathcal{V}_t(z) := \mathcal{V}(e^{-it\nu\omega} z). \tag{1.29}$$

Notice again that the pushforward in (1.27) does not affect the Radon–Nikodým derivative $\mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^\dagger_{t,0}(z)$. The interplay between the quasi-classical limit and the time evolution can be summed up in the following commutative diagram involving the Radon–Nikodým derivatives:

$$
\begin{tikzcd}
\Gamma_\varepsilon \arrow[r, mapsto, "e^{-iH_\varepsilon t}"] \arrow[d, mapsto, "\varepsilon\to 0"] & \Gamma_\varepsilon(t) \arrow[d, mapsto, "\varepsilon\to 0"] \\
\gamma_{\mathfrak{m}}(z)\mathrm{d}\mu_{\mathfrak{m}}(z) \arrow[r, mapsto, "(1.27)"] & \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^\dagger_{t,0}(z)\mathrm{d}((e^{-it\nu\omega})_\star\mu_{\mathfrak{m}})
\end{tikzcd}
\tag{1.30}
$$

where we have decomposed the initial state-valued measure as $\mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu_{\mathfrak{m}}(z)$ with $\gamma_{\mathfrak{m}} \in \mathcal{L}^1_{+,1}(\mathcal{H})$ and $\mu_{\mathfrak{m}} \in \mathcal{M}(\mathfrak{h})$, and the convergence is always along a given subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$.

We now state the first result in detail. Recall that we say that $\mathfrak{m} \in \mathcal{M}(\mathfrak{h}; \mathcal{L}^1_{+,1})$ is a probability measure, and thus $\mathfrak{m} \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_{+,1})$ whenever $\|\mathfrak{m}(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})} = 1$.

Theorem 1.6 (Quasi-classical evolution in the Schrödinger picture). *Let $\nu(\varepsilon)$ be such that $\varepsilon\nu(\varepsilon) \to \nu \in \{0, 1\}$ as $\varepsilon \to 0$, and let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be a state satisfying assumption (A1). Let also (A1′) be satisfied for the polaron and Pauli–Fierz models. Then there exist a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and a measure $\mathfrak{m} \in \mathcal{M}(\mathfrak{h}; \mathcal{L}^1_{+,1}(\mathcal{H}))$ such that*

$$\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathrm{qc}} \mathfrak{m}, \tag{1.31}$$

and if (1.31) holds, then for all $t \in \mathbb{R}$,

$$\Gamma_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{\mathrm{qc}} \mathfrak{m}_t, \tag{1.32}$$

where $\mathfrak{m}_t$ is given by (1.27).

Corollary 1.7 (Mass conservation). *If in addition Γ_ε satisfies assumption (A2) or (A2′), then $\mathfrak{m} \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_{+,1}(\mathcal{H}))$ and thus $\mathfrak{m}_t \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_{+,1}(\mathcal{H}))$ for all $t \in \mathbb{R}$. Furthermore, if (A2) also holds for $\gamma_\varepsilon(t)$ for any $t \in [0, T)$, $T \in \mathbb{R}^+$, then for all $t \in [0, T)$,*

$$\Gamma_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{\mathrm{bqc}} \mathfrak{m}_t.$$

Remark 1.8 (Extraction of a subsequence). Let us point out that, as anticipated above, the limit measure $\mathfrak{m}$ at initial time, according to Definition 1.1, might depend on the choice of the subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \to 0$. However, we stress that the convergence at time t stated in (1.32) occurs along the *same* subsequence.

Remark 1.9 (Loss of mass). Theorem 1.6 holds irrespective of any possible loss of mass for the initial-time convergence. The quasi-classical evolution preserves the mass, thus proving that *the same amount of mass is lost at any time*. Conditions (A2) and (A2′) ensure that no mass is lost at initial time, and thus at any further time. Another sufficient condition to ensure no loss of mass is the so-called (PI) condition, which will be discussed in detail in §1.6. However, as suggested by the fact that physical factorized states $\gamma \otimes \varsigma_\varepsilon$ do not lose mass, this peculiar loss of mass phenomenon is due either to a "bad" correlation between the field and particle subsystems, or to a somewhat artificial dependence of the particle subsystem on the quasi-classical parameter in an uncorrelated state (see §1.6 for a more detailed discussion).

Remark 1.10 (Assumptions (A2) and (A2′)). The implications and the meaning of assumptions (A2) and (A2′) are a priori quite different. For instance, (A2′) provides uniform control on the reduced density matrix γ_ε but has little physical motivation, unless the two subsystems in the state are uncorrelated (i.e., the state has a tensor product structure). Assumption (A2) on the other hand implies the stronger convergence of Γ_ε to $\mathfrak{m}$ in the bounded quasi-classical sense of Definition 1.4. Such a stronger convergence holds true however only at initial time, and its propagation along the time evolution is typically very difficult to prove. A notable exception is given by trapped particle systems, i.e., when $\mathcal{K}_0$ has compact resolvent and thus one can take $\mathcal{A} = (\mathcal{K}_0 + 1)^\delta$, for some $\delta > 0$, in (A2). Thus, in this case the assumption $\mathrm{Tr}(\Gamma_\varepsilon(K_0 + \mathrm{d}\mathcal{G}_\varepsilon(\omega))^\delta) \leq C$ on the initial state is sufficient to strengthen the convergence at any time (see, e.g., [3, Lemma 3.4] for the Nelson model and §7.2 for the polaron and Pauli–Fierz models). As already remarked, the two assumptions are in fact related, because (A2′) *implies* (A2).[4] However, due to the different physical implications (particles' trapping on the one hand, isolation of the subsystems on the other), we preferred to keep the two separated.

Remark 1.11 (Rougher potentials). As already remarked in §1.4, states satisfying (A1) yield effective potentials $\mathcal{V}_t$ that are "regular". For example, no confining potential can be obtained with such quasi-classical states. It is possible to obtain more general effective potentials relaxing assumption (A1) to accommodate states whose limit are *cylindrical measures* [23], but the analysis becomes more complicated. In the polaron model, for coherent states, whose cylindrical measure is concentrated in a single "singular" point (a suitable tempered distribution), the analysis has been carried out in [14] to obtain an effective (time-dependent) point interaction.

Before proceeding further we discuss in some detail the scaling factor $\nu(\varepsilon)$ that appears in front of the free energy of the field in the Hamiltonian H_ε. Physically, one should distinguish between two relevant situations: $\nu(\varepsilon) = 1$ and $\nu(\varepsilon) = 1/\varepsilon$; all the other

[4] The inference (A2′)⇒(A2) can be proved as follows. Since γ is a positive trace class operator, it can be decomposed as $\gamma = \sum_{j\in\mathbb{N}} |\varphi_j\rangle\langle\varphi_j|$, where $(\varphi_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $\lambda_j \geq 0$ are the singular values satisfying $\sum_{j\in\mathbb{N}} \lambda_j < +\infty$. Therefore, there exists a nonnegative sequence $(\mu_j)_{j\in\mathbb{N}}$ such that $\mu_j \to +\infty$ and $\sum_{j\in\mathbb{N}} \mu_j\lambda_j < +\infty$. In fact, if there are only a finite number of nonzero λ_js, then the existence is trivial, while if the number of nonzero λ_js is infinite, one can set, for all $k \in \mathbb{N}$, $J_k = \min\{J \in \mathbb{N} \mid \sum_{J\leq j} \lambda_j < 2^{-k}\}$, and $\mu_j = 1$ for $j < J_0$, and $\mu_j = 2^{k/2}$ for $J_k \leq j < J_{k+1}$. Then, by construction, the inverse of the operator $\mathcal{A}_\gamma := \sum_{j\in\mathbb{N}} \mu_j |\varphi_j\rangle\langle\varphi_j|$ is compact and $\mathrm{tr}_{\mathcal{H}}(\mathcal{A}_\gamma\gamma_\varepsilon) = \mathrm{tr}_{\mathcal{H}}(\mathcal{A}_\gamma^{1/2}\gamma_\varepsilon\mathcal{A}_\gamma^{1/2}) \leq \mathrm{tr}_{\mathcal{H}}(\mathcal{A}_\gamma^{1/2}\gamma\mathcal{A}_\gamma^{1/2}) = \sum_{j\in\mathbb{N}} \mu_j\lambda_j < +\infty$, so that γ_ε satisfies assumption (A2).

possibilities are physically less relevant, and yield the same qualitative results, up to rescaling of the parameters. Let us remark first, however, that despite the fact that the two cases yield different evolutions for the classical field, *the interaction is always too weak to cause a back-reaction of the particles on the field* when $\varepsilon \to 0$. Thus, the quasi-classical field can indeed be seen as an environment.

Remark 1.12 ($\nu = 0$). When $\nu(\varepsilon) = \mathcal{O}(\varepsilon^{-\delta})$, $\delta < 1$, the quasi-classical field remains constant in time. In fact, in that case $\nu = \lim_{\varepsilon\to 0} \varepsilon\nu(\varepsilon) = 0$, and therefore $\mathcal{U}_{t,s}(z) = \mathcal{U}_{t-s}(z)$ is the strongly continuous group generated by the self-adjoint operator $\mathcal{K}_0 + \mathcal{V}(z)$, with, e.g., $\mathcal{V}(z) = \sum_{j=1}^{N} 2\,\mathrm{Re}\,\langle\lambda(\mathbf{x}_j)|z\rangle_{\mathfrak{h}}$ for the Nelson model. Also, the measure μ_t is constant: $\mu_t = \mu$ for all $t \in \mathbb{R}$. Therefore, in the scaling determined by $\nu(\varepsilon) = \mathcal{O}(\varepsilon^{-\delta})$ the radiation field does not evolve. Let us remark that, in the case of the polaron, this is the scaling equivalent, up to suitable rescalings, to the well-known strong coupling regime.

Remark 1.13 ($\nu = 1$). When $\nu = 1$, e.g., if $\nu(\varepsilon) = 1/\varepsilon$, the quasi-classical radiation field evolves in time in a non-trivial way, obeying a free field equation, and therefore the effective evolution operator for the particles $\mathcal{U}_{t,s}(z)$ has a time-dependent generator. For the regularized Nelson model, such a free evolution is given by the Klein–Gordon-like equation

$$(\partial_t^2 + \omega^2(D))A = 0, \tag{1.33}$$

where $\omega(D)$ is the pseudodifferential operator defined by the Fourier transform of the function ω; for the Pauli–Fierz model it is given by the free Maxwell equations in the Coulomb gauge, and for the polaron by the equation $(\partial_t^2 + 1)A = 0$. Here, for clarity, we have written such equations in the usual form, which involves the real field A and its time derivatives. Throughout the paper, however, we use the complex counterpart of that real field, denoted by z, and which is given in terms of A, e.g., in the regularized Nelson model, by

$$z = \tfrac{1}{2}\big(\omega^{1/2}(D)A + i\omega^{-1/2}(D)\partial_t A\big),$$

Hence, the evolution equation for z becomes $i\partial_t z = \omega z$.

A consequence of Theorem 1.6 is that, for any compact operator $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$, its Heisenberg evolution satisfies

$$\mathrm{Tr}(\Gamma_{\varepsilon_n} e^{itH_{\varepsilon_n}} B e^{-itH_{\varepsilon_n}}) \xrightarrow[n\to+\infty]{} \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\ \mathrm{tr}_{\mathcal{H}}[\gamma_{\mathfrak{m}}(z)\mathcal{U}_{t,0}^\dagger(z)\mathcal{B}\mathcal{U}_{t,0}(z)]. \tag{1.34}$$

There is also a counterpart of the above statement for the particle degrees of freedom alone: for any Γ_ε as in Theorem 1.6, the following weak-$*$ convergence holds in $\mathcal{L}^1(\mathcal{H})$:

$$\mathrm{tr}_{\mathcal{K}_\varepsilon}(e^{-itH_{\varepsilon_n}}\Gamma_{\varepsilon_n} e^{itH_{\varepsilon_n}}) \xrightarrow[n\to+\infty]{\mathrm{w}*} \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\ \mathcal{U}_{t,0}(z)\,\gamma_{\mathfrak{m}}(z)\mathcal{U}_{t,0}^\dagger(z), \tag{1.35}$$

i.e., the particle state obtained by tracing out the field degrees of freedom evolves as $\varepsilon \to 0$ into the right hand side of the above expression. When the state is a product state, the above result can be made more explicit (see also Remark 1.3):

Corollary 1.14 (Quasi-classical evolution of product states). *Let $\varsigma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{K}_\varepsilon)$ be a field's state such that for all $\gamma \in \mathcal{L}^1_{+,1}(\mathcal{H})$, $\gamma \otimes \varsigma_\varepsilon$ satisfies assumption* (A1)*, and* (A1′) *for the polaron and Pauli–Fierz models, so that there exists $\mu \in \mathcal{M}(\mathfrak{h})$ such that*

$$\varsigma_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mu. \tag{1.36}$$

Then, for all $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$ and all $t \in \mathbb{R}$,

$$\mathrm{tr}_{\mathcal{K}_{\varepsilon_n}}(\varsigma_{\varepsilon_n} e^{itH_{\varepsilon_n}} B e^{-itH_{\varepsilon_n}}) \xrightarrow[n\to+\infty]{\mathrm{w}} \int_{\mathfrak{h}} \mathcal{U}_{0,t}(z)\mathcal{B}\mathcal{U}_{t,0}(z)\,\mathrm{d}\mu(z). \tag{1.37}$$

Remark 1.15 (Bounded operators). It would obviously be more satisfactory to extend the above result to bounded operators $\mathcal{B} \in \mathcal{B}(\mathcal{H})$. However, this cannot be done in full generality because the convergence in Definition 1.1 holds in the weak-$*$ topology. As explained in Remark 1.10 and §1.6, one can lift the convergence to the weak topology, and thus extend the statement above to bounded observables, if an additional regularity on the initial state is assumed and such a regularity can be propagated by the dynamics, which can be done for example whenever the particle system is trapped.

The analogue of Corollary 1.14 for non-product states and more complicated observables, i.e., self-adjoint operators acting on the full Hilbert space, is more involved to state and holds true only for a subclass of such operators. We indeed introduce a class of operators on $\mathcal{H} \otimes \mathcal{K}_\varepsilon$, consisting of polynomials with m creation and n annihilation normal ordered operators, with arguments possibly depending on the particle's positions: explicitly, we consider operators $\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F})$ obtained as the *Wick quantization* of symbols $\mathcal{F} \in \mathcal{S}_{n,m}$, i.e., of the form

$$\mathcal{F}(z) = \sum_{j=1}^{N} \langle z|\lambda_1(\mathbf{x}_j)\rangle_{\mathfrak{h}} \cdots \langle z|\lambda_\ell(\mathbf{x}_j)\rangle_{\mathfrak{h}} \langle \lambda_{\ell+1}(\mathbf{x}_j)|z\rangle_{\mathfrak{h}} \cdots \langle \lambda_{\ell+m}(\mathbf{x}_j)|z\rangle_{\mathfrak{h}}, \tag{$\mathcal{S}_{\ell,m}$}$$

where $\lambda_j \in L^\infty(\mathbb{R}^d;\mathfrak{h})$, $j = 1,\ldots,m+\ell$.

To state the result, we also need to make more restrictive assumptions on the initial state Γ_ε:

$$\begin{cases} \exists\delta > \frac{1}{4}, \exists C_\delta < +\infty \ : \ \mathrm{Tr}(\Gamma_\varepsilon(\mathrm{d}\Gamma_\varepsilon(1)+1)^{2\delta}) \le C_\delta, & \text{Nelson model,} \\ \delta = 1, \exists C < +\infty \ : \ \mathrm{Tr}(\Gamma_\varepsilon H_\varepsilon^2) \le C\nu(\varepsilon)^2, & \text{Pauli–Fierz model,} \\ \exists\delta \in \mathbb{N}_*, \exists C_\delta < +\infty \ : \ \mathrm{Tr}(\Gamma_\varepsilon H_\varepsilon^{2\delta}) \le C_\delta \nu(\varepsilon)^{2\delta}, & \text{polaron.} \end{cases} \tag{A_δ}$$

Theorem 1.16 (Quasi-classical evolution in the Heisenberg picture). *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be a state satisfying assumption* (A_δ)*, so that there exists $\mathfrak{m} \in \mathcal{M}(\mathfrak{h};\mathcal{L}^1_+(\mathcal{H}))$ such that*

$$\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathrm{qc}} \mathfrak{m}. \tag{1.38}$$

Then, for all $\mathscr{F} \in \mathcal{S}_{\ell,m}$ with $(\ell+m)/2 < 2\delta$, all $t \in \mathbb{R}$ and all $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$ such that either $\mathcal{S}$ or $\mathcal{T}$ is in $\mathcal{L}^\infty(\mathcal{H})$,

$$\begin{aligned}\mathrm{Tr}(\Gamma_{\varepsilon_n}(t) T \mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathscr{F}) S) &\xrightarrow[n\to+\infty]{} \mathrm{tr}_{\mathcal{H}}\left(\int_{\mathfrak{h}} \mathrm{d}\mathfrak{m}_t(z)\, \mathcal{T}\mathscr{F}(z)\mathcal{S}\right) \\ &= \mathrm{tr}_{\mathcal{H}}\left(\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}_{0,t}(z)\, \mathcal{T}\mathscr{F}(e^{-it\nu\omega}z)\mathcal{S}\right). \qquad (1.39)\end{aligned}$$

Remark 1.17 (Regularity assumptions for the Pauli–Fierz model). The constraint $\delta = 1$ for the Pauli–Fierz model is due to some technical difficulties in propagating in time higher order regularity of the number operator, due to the fact that the number operator and the field's kinetic term are not comparable in that case, since the field carriers may be massless.

1.6. Semiclassical analysis and sketch of the proof

In this section we present a short sketch of the proof and discuss some of the key features of semiclassical analysis for infinite-dimensional systems, which is the core tool of our analysis. This discussion is meant to clarify the role of our assumptions and propose alternative approaches.

One of the main points in our investigation is the convergence of a family of quantum states as $\varepsilon \to 0$ to a quasi-classical Wigner measure in the sense of either Definition 1.1 or Definition 1.4. The latter is clearly preferable but there are known obstructions. Indeed, in infinite-dimensional semiclassical analysis with no additional degrees of freedom (which we refer to as the *scalar case*), i.e., when the limit Wigner measure is a conventional scalar measure, there can be two types of defects of convergence for a given family $\{\Gamma_\varepsilon\}_{\varepsilon\in(0,1)}$ of normalized states:

- a *loss of mass*, as in the finite-dimensional case, i.e., the limit measure may not be a probability measure and have a total mass strictly less than 1;
- a dimensional *loss of compactness* that is characteristic of the infinite-dimensional setting (see [6, §7.4]), where the mass is preserved but the expectation values of operators obtained as Wick quantizations of non-compact symbols do not converge to their limit expressions.

These defects are prevented by formulating conditions that are both sufficient and reasonable to verify in a relevant class of concrete examples: the loss of mass is prevented by imposing an ε-uniformity condition on the expectation of some power of the number operator, analogous to assumption (A1) given above (see also [6, §6.1]); loss of compactness is prevented by the so-called (PI) sufficient condition [2, 8], which reads as follows: for all $k \in \mathbb{N}$,

$$\lim_{\varepsilon\to 0} \mathrm{Tr}(\mathrm{d}\mathcal{G}_\varepsilon(1)^k \Gamma_\varepsilon) = \int \mathrm{d}\mu(z)\, \|z\|^{2k}.$$

If the above condition holds, then the expectations of all Wick polynomial bounded symbols converge.

A key difference of our quasi-classical setting compared to the scalar case is that assumption (A1) is *not sufficient* to ensure that no mass is lost in the limit: the correlation with the $\mathcal{H}$-degrees of freedom may cause a new type of mass defect, as exemplified by the product state $\gamma_\varepsilon \otimes \varsigma_\varepsilon$ introduced above in Remark 1.5, which satisfies assumption (A1) and converges to the state-valued measure 0 with no mass. Assumptions (A2) and (A2′) are both conditions on the $\mathcal{H}$-degrees of freedom that are sufficient to prevent this quasi-classical defect and whose usefulness has been discussed in Remark 1.10. Note, however, that the defect of compactness mentioned above may also occur: it is indeed not difficult to produce examples of states with no loss of mass but a defect of compactness, simply tensoring any scalar example of such defect with an $\mathcal{H}$-state that is independent of the quasi-classical parameter ε. In order to overcome the defect of compactness in the quasi-classical limit, a straightforward analogue of the scalar (PI) condition can be formulated:

$$\lim_{\varepsilon\to 0} \mathrm{Tr}(\mathrm{d}\mathcal{G}_\varepsilon(1)^k \Gamma_\varepsilon) = \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \|z\|_{\mathfrak{h}}^{2k}, \quad \forall 0 \le k \le K, \tag{PI$_K$}$$

so that the scalar condition (PI) of [2, 8] corresponds here to condition (PI$_\infty$). We allow for a possibly finite index K for a motivation to be explained in detail below, related to the propagation in time of the condition.

Restricting to states that satisfy condition (PI$_K$), for suitable $K \in \mathbb{N}_*$, allows one to use some powerful tools of infinite-dimensional semiclassical analysis (see, e.g., [2, Appendix B]) that make the study of quasi-classical dynamics less involved, and allow one to obtain stronger results. In particular, the convergence in Theorem 1.6 can be lifted to bounded quasi-classical convergence of Definition 1.4, and Theorem 1.16 holds for a more general class of symbols. There are, however, also some drawbacks. The most relevant one is that there are states of physical relevance that do not satisfy condition (PI$_K$), or that are defined by abstract and a priori considerations, in a way that does not provide enough information to test the validity of such a condition. The primary examples of this kind are states satisfying suitable variational problems (e.g., ground states of physical problems related to the ones under study, perhaps with an additional external potential that is removed at the initial time, or states belonging to some minimizing sequence of the model). In addition, there is a technical difficulty: condition (PI$_K$) is in general difficult to propagate in time. As will be explained later, to prove its propagation one has to rely on a propagation estimate for the number operator up to power K. This is possible for the Nelson model for all $K \in \mathbb{N}_*$ [20], and for the Pauli–Fierz model, at least for $K \le 2$ [5]. However, it does not seem feasible for the polaron model.

In view of the above considerations, in this paper we mainly focus on the more general class of states satisfying only assumption (A1), which can thus be defective both in mass and compactness, as in the main results presented in §1.5. Let us remark that, in order to consider more general states, a finer technical analysis on our part is required; this, however, makes the proofs also slightly more involved. We believe that it is interesting to present the results in such generality, both from a physical and a mathematical standpoint. Nonetheless, we also believe that it makes sense to informally present the proof of our results that can be obtained using condition (PI$_K$), for arbitrary $K \in \mathbb{N}_*$ in the Nelson model, and for $K \leq 2$ in the Pauli–Fierz model. The purpose of the outline is twofold: on the one hand it serves as a summary of the semiclassical strategies used throughout the paper, on the other hand it allows us to emphasize the simplifications obtained by using condition (PI_K), and thus also the subtleties we had to face otherwise.

Let us thus assume, only in this section, that (PI_K) holds true, so that the statement of Theorem 1.6 takes the following stronger form: there exist a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and a probability measure $\mathfrak{m} \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_{+,1}(\mathcal{H}))$ such that

$$\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathrm{bqc}} \mathfrak{m}, \tag{1.40}$$

and if (1.40) holds, then for all $t \in \mathbb{R}$,

$$\Gamma_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{\mathrm{bqc}} \mathfrak{m}_t, \tag{1.41}$$

where $\mathfrak{m}_t$ is given by (1.27). In addition,

$$\begin{aligned}\mathrm{Tr}[\Gamma_{\varepsilon_n}(t) T \mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\langle z^{\otimes \ell} | \tilde{b} z^{\otimes m}\rangle_{\mathfrak{h}^{\otimes_s \ell}}) S] &\xrightarrow[n\to+\infty]{} \mathrm{tr}_{\mathcal{H}}\left[\int_{\mathfrak{h}} \mathrm{d}\mathfrak{m}_t(z)\, \mathcal{T} \langle z^{\otimes \ell} | \tilde{b} z^{\otimes m}\rangle_{\mathfrak{h}^{\otimes_s \ell}}\, \mathcal{S}\right] \\ &= \mathrm{tr}_{\mathcal{H}}\left[\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}_{0,t}(z)\mathcal{T}\langle (e^{-it\nu\omega} z)^{\otimes \ell} | \tilde{b} (e^{-it\nu\omega} z)^{\otimes m}\rangle_{\mathfrak{h}^{\otimes_s \ell}} \mathcal{S}\right]\end{aligned}$$

for all $\mathcal{T}, \mathcal{S} \in \mathcal{B}(\mathcal{H})$ and all $\tilde{b} \in \mathcal{B}(\mathcal{H} \otimes \mathfrak{h}^{\otimes_s m}, \mathcal{H} \otimes \mathfrak{h}^{\otimes_s \ell})$ with $(m + \ell)/2 < K$.

The main steps of the proof of the above results are the following:

(i) First of all, we pass to the interaction representation, setting

$$\Upsilon_\varepsilon(t) := e^{it(K_0 + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))} \Gamma_\varepsilon(t) e^{-it(K_0+\nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))},$$

and write Duhamel's formula for the time evolution of its Fourier transform:

$$\begin{aligned}[\widehat{\Upsilon}_\varepsilon(t)](\eta) &= [\widehat{\Upsilon}_\varepsilon(s)](\eta) \\ &\quad - i \sum_{j=1}^N \int_s^t \mathrm{d}\tau\, e^{i\tau\mathcal{K}_0}\, \mathrm{tr}_{\mathcal{K}_\varepsilon}\big([\varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j)), \Upsilon_\varepsilon(\tau)] W_\varepsilon(\eta)\big) e^{-i\tau\mathcal{K}_0},\end{aligned} \tag{1.42}$$

where $\varphi_\varepsilon(z) := a^\dagger_\varepsilon(z) + a_\varepsilon(z)$ is the field operator.

(ii) Now, the goal is to extract a common subsequence of $\Upsilon_{\varepsilon_n}(t)$ that converges for all times $t \in \mathbb{R}$. Hence, one first needs to show that, at any time $t \in \mathbb{R}$, $\Upsilon_{\varepsilon_n}(t)$ converges along a suitable subsequence. In order to do that, we need to verify that condition (PI_K) (resp. (A1), in the case of Theorem 1.6) is satisfied for any $t \in \mathbb{R}$, which guarantees convergence in the sense of Definition 1.4 (resp. Definition 1.1) at all times. Let us sketch how it is possible to propagate (PI_K) in the Nelson model: the Duhamel formula

$$\begin{aligned}\mathrm{Tr}(\mathrm{d}\mathcal{G}_\varepsilon(1)^k \Upsilon_\varepsilon(t)) &= \mathrm{Tr}(\mathrm{d}\mathcal{G}_\varepsilon(1)^k \Upsilon_\varepsilon(0)) \\ &\quad - i \int_0^t \mathrm{Tr}\big([\mathrm{d}\mathcal{G}_\varepsilon(1)^k, \varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j))]\Upsilon_\varepsilon(\tau)\big)\, \mathrm{d}\tau\end{aligned}$$

yields

$$\begin{aligned}\|[\mathrm{d}\mathcal{G}_\varepsilon(1)^k, \varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j))]\Upsilon_\varepsilon(\tau)\|_{\mathcal{L}^1(\mathcal{H}\otimes\mathcal{K}_\varepsilon)} \\ \leq \varepsilon\, C_k\|(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^k\Upsilon_\varepsilon(\tau)\|_{\mathcal{L}^1(\mathcal{H}\otimes\mathcal{K}_\varepsilon)}.\end{aligned}$$

The trace norm on the right hand side is uniformly bounded with respect to ε (see, e.g., Proposition 3.2 below), yielding

$$\mathrm{Tr}(\mathrm{d}\mathcal{G}_\varepsilon(1)^k\Upsilon_\varepsilon(t)) = \mathrm{Tr}(\mathrm{d}\mathcal{G}_\varepsilon(1)^k\Upsilon_\varepsilon(0)) + \mathcal{O}_{k,t}(\varepsilon).$$

This implies that condition (PI_K) holds for all times provided it holds at the initial time. Let us remark again that such propagation estimate (more precisely, the ε-uniform number estimate) is not available for the polaron model, nor for the Pauli–Fierz model whenever $k \geq 3$.

(iii) Once convergence of $\Upsilon_{\varepsilon_n}(t)$ is obtained, one has to prove that $\{\hat{\Upsilon}_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly equicontinuous as a family of functions of time. This can be done by exploiting (PI_K) once more:

$$\begin{aligned}&\mathrm{Tr}\big([\varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j)), \Upsilon_\varepsilon(t)]W_\varepsilon(\eta)\big) \\ &\quad= \mathrm{Tr}\big(\Upsilon_\varepsilon(t)(\mathrm{d}\mathcal{G}_\varepsilon(1))^\delta(\mathrm{d}\mathcal{G}_\varepsilon(1))^{-\delta}[W_\varepsilon(\eta), \varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j))]\big) = \mathcal{O}(\varepsilon^\delta).\end{aligned}$$

(iv) After the extraction of a subsequence $\Upsilon_{\varepsilon_{n_k}}(t)$ converging at all times, we take the limit as $\varepsilon_{n_k} \to 0$ of (1.42); by (PI_K), the convergence follows by a direct generalization to operator-valued symbols of the analysis done in the scalar case, e.g. in [2, Appendix B]. On the other hand, to study the limit under assumption (A1) alone, we have to develop a specific quasi-classical calculus for the symbols appearing in the energy functionals of the three models. We take advantage of an approximation by simple functions that allows one to separate the two types of degrees of freedom, at the same time making the symbol compact and thus convergent without additional assumptions (see §2).

(v) The equation obtained in the limit from the Duhamel equation is a transport equation for the Fourier transform of the quasi-classical measures in interaction picture $\mathfrak{n}_t$:

$$\hat{\mathfrak{n}}_t(\eta) = \hat{\mathfrak{n}}_s(\eta) - i\int_s^t \mathrm{d}\tau \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{n}_\tau}(z)\,[\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)]e^{2i\,\mathrm{Re}\langle\eta|z\rangle_{\mathfrak{h}}}.$$

We prove that this equation has a unique solution, given by

$$\mathfrak{n}_t = \big(\mu_{\mathfrak{m}}, \widetilde{\mathcal{U}}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\widetilde{\mathcal{U}}^\dagger_{t,0}(z)\big);$$

once the interaction representation is removed, that yields

$$\mathrm{d}\mathfrak{m}_t = \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^\dagger_{t,0}(z)\,\mathrm{d}((e^{-it\nu\omega})_\star\mu_{\mathfrak{m}}),$$

as expected (see §5).

(vi) The aforementioned uniqueness allows one finally to extend the convergence to the original sequence $\Upsilon_{\varepsilon_n}(t)$.

2. Quasi-classical analysis

In this section we introduce the quasi-classical asymptotic analysis, needed to study the dynamical limit of quasi-classical systems. In particular, we have to develop a semiclassical theory for operator-valued symbols, since the latter are crucial to characterize the interaction part of the dynamics. The key tools presented here are

- the convergence of regular states to state-valued measures in the quasi-classical limit (Proposition 2.3) in the sense of Definition 1.1;
- the convergence of the expectation values of suitable classes of operators to their classical counterparts (Proposition 2.6).

Note that, in the context of finite-dimensional semiclassical analysis, operator-valued symbols corresponding to additional degrees of freedom have already been studied [13, 24, 30, 31] (see also [56, Appendix B] and references therein), although with different applications in mind.

We start by clarifying the notion of state-valued measure.

Definition 2.1 (State-valued measure). An additive measure $\mathfrak{m}$ on a measurable space (X, Σ) is $\mathcal{H}$-*state-valued* if

- $\mathfrak{m}(S) \in \mathcal{L}^1_+(\mathcal{H})$ for any $S \in \Sigma$;
- $\mathfrak{m}(\emptyset) = 0$;
- $\mathfrak{m}$ is unconditionally σ-additive in the trace norm.

An $\mathcal{H}$-state-valued measure is a *probability measure* if $\|\mathfrak{m}(X)\|_{\mathcal{L}^1} = 1$. We denote by $\mathcal{M}(X, \Sigma; \mathcal{L}^1_+(\mathcal{H}))$ and $\mathcal{P}(X, \Sigma; \mathcal{L}^1_+(\mathcal{H}))$ and by $\mathcal{M}(X; \mathcal{L}^1_+(\mathcal{H}))$ and $\mathcal{P}(X; \mathcal{L}^1_+(\mathcal{H}))$ the spaces of $\mathcal{H}$-state-valued measures and probability measures, respectively, with respect to either a generic or the Borel σ-algebra, in case X is a topological space.

Using the Radon–Nikodým property and positivity, there is a simple characterization of state-valued measures:

Proposition 2.2 (Radon–Nikodým decomposition). *For any $\mathfrak{m} \in \mathcal{M}(X, \Sigma; \mathcal{L}^1_+(\mathcal{H}))$, there exists a scalar measure $\mu_{\mathfrak{m}} \in \mathcal{M}(X, \Sigma)$ with*

$$\mu_{\mathfrak{m}}(X) = \|\mathfrak{m}(X)\|_{\mathcal{L}^1(\mathcal{H})}$$

and a $\mu_{\mathfrak{m}}$-a.e. defined measurable function $\gamma_{\mathfrak{m}} : X \to \mathcal{L}^1_{+,1}(\mathcal{H})$ such that for any $S \in \Sigma$,

$$\mathfrak{m}(S) = \int_S \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z), \tag{2.1}$$

with the right hand side meant as a Bochner integral. In addition, $\mathfrak{m} \in \mathcal{M}(X, \Sigma; \mathcal{L}^1_+(\mathcal{H}))$ is an $\mathcal{H}$-state-valued probability measure iff $\mu_{\mathfrak{m}}$ is a probability measure. We call $(\mu_{\mathfrak{m}}; \gamma_{\mathfrak{m}}(z))$ the Radon–Nikodým decomposition *of* $\mathfrak{m}$.

Proof. First of all we point out that the separable Schatten space $\mathcal{L}^1(\mathcal{H})$ of trace class operators has the Schatten space $\mathcal{L}^\infty(\mathcal{H})$ of compact operators as predual, and therefore it has the Radon–Nikodým property (see, e.g., [18, 19]). In addition, since $\mathfrak{m}$ takes values in positive operators, we can define its "norm" measure as

$$m(\cdot) := \|\mathfrak{m}(\cdot)\|_{\mathcal{L}^1(\mathcal{H})}. \tag{2.2}$$

In fact, m is a scalar measure such that $\mathfrak{m} \ll m \ll \mathfrak{m}$, i.e., $\mathfrak{m}$ and m are absolutely continuous with respect to each other. The latter property can indeed be easily seen as follows: $\mathfrak{m}(S) = 0$, as an element of the vector space $\mathcal{L}^1_+(\mathcal{H})$, if and only if $m(S) = \|\mathfrak{m}(S)\|_{\mathcal{L}^1(\mathcal{H})} = 0$.

Moreover, the Radon–Nikodým property guarantees the existence of the Radon–Nikodým derivative $\frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mu} \in L^1(X, \mathrm{d}\mu; \mathcal{L}^1_+(\mathcal{H}))$ such that

$$\mathfrak{m}(S) = \int_S \mathrm{d}\mu(z)\, \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mu}(z) \tag{2.3}$$

for any measurable $S \in \Sigma$ and any scalar measure μ such that $\mathfrak{m}$ is absolutely continuous with respect to μ.

In our setting, compared to the more general case of Banach-space-valued vector measures, there is an additional notion of positivity, as discussed above. That notion naturally singles out a given scalar measure, with respect to which $\mathfrak{m}$ is absolutely continuous. The measure is the "norm" measure m defined in (2.2). Indeed, combining the mutual absolute continuity of $\mathfrak{m}$ and m with the existence of the Radon–Nikodým derivative, we deduce that, for any measurable $S \subset \mathfrak{h}$,

$$\mathfrak{m}(S) = \int_S \mathrm{d}m(z)\, \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z), \tag{2.4}$$

and that, m-a.e.,

$$\frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m} \neq 0.$$

Therefore, we can rewrite

$$\mathfrak{m}(S) = \int_S \mathrm{d}m(z) \left\| \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z) \right\|_{\mathcal{L}^1(\mathcal{H})} \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z) \left\| \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z) \right\|^{-1}_{\mathcal{L}^1(\mathcal{H})}, \tag{2.5}$$

and setting

$$\mathrm{d}\mu_{\mathfrak{m}}(z) := \left\| \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z) \right\|_{\mathcal{L}^1(\mathcal{H})} \mathrm{d}m(z), \tag{2.6}$$

$$\gamma_{\mathfrak{m}}(z) := \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z) \left\| \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}m}(z) \right\|^{-1}_{\mathcal{L}^1(\mathcal{H})}, \tag{2.7}$$

we obtain the sought Radon–Nikodým decomposition. ∎

Let now $F : X \to \mathcal{B}(\mathcal{H})$ be a measurable function with respect to the weak-$*$ topology on $\mathcal{B}(\mathcal{H})$. It is then natural to define the ($\mathcal{L}^1$-Bochner) integrals of f with respect to $\mathfrak{m}$ as follows: for any $S \in \Sigma$,

$$\int_S \mathrm{d}\mathfrak{m}(z)\, F(z) := \int_S \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z) F(z), \tag{2.8}$$

$$\int_S F(z)\, \mathrm{d}\mathfrak{m}(z) := \int_S \mathrm{d}\mu_{\mathfrak{m}}(z)\, F(z)\gamma_{\mathfrak{m}}(z). \tag{2.9}$$

Notice that one has to keep track of the order inside the integral, i.e., putting the measure on the right or on the left of the integrand is not the same, because $\gamma_{\mathfrak{m}}$ might not commute with $F(z)$, since both are operators on $\mathcal{H}$.

State-valued measures are important since they are the quasi-classical counterparts of quantum states (see [23] for a detailed discussion). Operator-valued symbols, such as the aforementioned $\mathcal{F}$, are correspondingly the quasi-classical counterparts of quantum observables. From a general point of view, we can summarize the main objective of quasi-classical analysis as follows.

Let $\mathrm{Op}_\varepsilon(\mathcal{F})$ be a "quantization" of $\mathcal{F}$ acting on $\mathcal{H} \otimes \mathcal{K}_\varepsilon$, where the space $\mathcal{K}_\varepsilon$ carries a semiclassical representation of the canonical commutation relations corresponding to a symplectic space (V, σ) of test functions, and let Γ_ε be a quantum state converging to the Borel state-valued measure $\mathfrak{m}$ on the space V' of suitably regular classical fields. Then we would like to prove that

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathrm{tr}_{\mathcal{K}_\varepsilon}(\Gamma_\varepsilon \mathrm{Op}_\varepsilon(\mathcal{F})) &= \int_{V'} \mathrm{d}\mathfrak{m}(z)\, \mathcal{F}(z), \\ \lim_{\varepsilon\to 0} \mathrm{tr}_{\mathcal{K}_\varepsilon}(\mathrm{Op}_\varepsilon(\mathcal{F})\Gamma_\varepsilon) &= \int_{V'} \mathcal{F}(z)\, \mathrm{d}\mathfrak{m}(z), \end{aligned} \tag{2.10}$$

where the convergence holds in a suitable topology of $\mathcal{L}^1(\mathcal{H})$.

It is however difficult to obtain such results for general symbols and quantum states. The most important obstruction is indeed the difficulty of defining a proper quantization procedure for symbols acting on infinite-dimensional spaces. However, for the theories of particle-field interaction under consideration (Nelson, polaron, Pauli–Fierz), the interaction terms in the quasi-classical Hamiltonians contain only symbols of a specific form. We can therefore restrict our analysis to such type of symbols.

Let us recall that we are considering the following concrete setting: $\mathcal{H} = L^2(\mathbb{R}^{dN})$, where d is the spatial dimension on which the particles move and N is the number of quantum particles; $\mathcal{K}_\varepsilon = \mathcal{G}_\varepsilon(\mathfrak{h})$, the symmetric Fock space over the complex separable Hilbert space $\mathfrak{h}$, carrying the standard ε-dependent Fock representation of the canonical commutation relations

$$[a_\varepsilon(z), a_\varepsilon^\dagger(\eta)] = \varepsilon\langle z|\eta\rangle_{\mathfrak{h}}.$$

Finally, we are interested in the case $V' = \mathfrak{h}$, i.e., the space of test functions coincides with the space of classical fields. The type of symbols $\mathcal{F}$ is given by the class defined in ($\mathcal{S}_{\ell,m}$), i.e.,

$$\mathcal{F}(z) = \sum_{j=1}^{N} \langle z|\lambda_1(\mathbf{x}_j)\rangle_{\mathfrak{h}} \cdots \langle z|\lambda_\ell(\mathbf{x}_j)\rangle_{\mathfrak{h}} \langle \lambda_{\ell+1}(\mathbf{x}_j)|z\rangle_{\mathfrak{h}} \cdots \langle \lambda_{\ell+m}(\mathbf{x}_j)|z\rangle_{\mathfrak{h}},$$

where the functions $\lambda_j \in L^\infty(\mathbb{R}^d;\mathfrak{h})$ for any $j \in \{1,\ldots,\ell+m\}$ should be considered as fixed "parameters", and $\mathcal{F}(z)$ acts as a multiplication operator on $L^2(\mathbb{R}^{dN})$.

Since $\mathcal{F}$ is a polynomial symbol with respect to z and $\bar{z}$, it is natural to quantize it by the Wick quantization rule. For such simple symbols the Wick rule has a very simple form: substitute each z with a_ε and each $\bar{z}$ with $a_\varepsilon^\dagger$, and then put the expression so obtained in normal order, by moving all the creation operators to the left of the annihilation operators. Then we obtain

$$\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\mathcal{F}) = \sum_{j=1}^{N} a_\varepsilon^\dagger(\lambda_1(\mathbf{x}_j)) \cdots a_\varepsilon^\dagger(\lambda_\ell(\mathbf{x}_j)) a_\varepsilon(\lambda_{\ell+1}(\mathbf{x}_j)) \cdots a_\varepsilon(\lambda_{\ell+m}(\mathbf{x}_j)), \tag{2.11}$$

as a densely defined operator on $L^2(\mathbb{R}^{dN}) \otimes \mathcal{G}_\varepsilon(\mathfrak{h})$.

In order to prove weak convergence as in (2.10) for $T\mathrm{Op}_\varepsilon^{\mathrm{Wick}}(\mathcal{F})S$ with $\mathcal{S},\mathcal{T} \in \mathcal{B}(\mathcal{H})$, we need suitable hypotheses on the quantum state Γ_ε, and some preparatory results. The following condition ensures that all the quasi-classical Wigner measures corresponding to a state $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ are concentrated as Radon L^2-state-valued probability measures on $\mathfrak{h}$. Recall the definition (1.20) of the Weyl operator $W_\varepsilon(\eta)$, $\eta \in \mathfrak{h}$, and the Fourier transform (1.21) of a measure $\mathfrak{m} \in \mathcal{M}(\mathfrak{h};\mathcal{L}^1_+(\mathcal{H}))$. Recall that in this section and the rest of the paper we consider only the convergence defined in Definition 1.1 and so we simply write $\Gamma_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{m}$ instead of $\Gamma_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathrm{qc}} \mathfrak{m}$.

Proposition 2.3 (Convergence of quantum to classical states). *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be such that there exists $\delta > 0$ such that*

$$\mathrm{Tr}(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta) \le C. \tag{2.12}$$

Then there exists at least one subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and an $\mathcal{H}$-state-valued cylindrical measure $\mathfrak{m}$ (which may depend on the sequence) such that

$$\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m} \tag{2.13}$$

in the sense of Definition 1.1. Furthermore, all cluster points $\mathfrak{m}$ of Γ_ε are state-valued Radon measures on $\mathfrak{h}$, and, for any $0 \le \delta' \le \delta$, there exists $C_{\delta'} \le C_\delta$, with $C_0 = 1$, such that

$$\left\| \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z)(\|z\|_{\mathfrak{h}}^2 + 1)^{\delta'} \right\|_{\mathcal{L}^1(\mathcal{H})} = \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, (\|z\|_{\mathfrak{h}}^2 + 1)^{\delta'} \le C_{\delta'}. \tag{2.14}$$

If in addition Γ_ε satisfies either assumption (A2) or (A2′), then $\mathfrak{m}$ is a probability measure, i.e.,

$$\left\| \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z) \right\|_{\mathcal{L}^1(\mathcal{H})} = \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z) = 1. \tag{2.15}$$

In order to prove the last part of the above proposition, we need a couple of preparatory results that will be useful in §7.1 as well.

Lemma 2.4. *Let $\mathcal{T}$ be a densely defined self-adjoint operator on $\mathcal{H}$, and $\mathbb{1}_m(\mathcal{T})$ its spectral projection on the interval $[-m, m]$, $m \in \mathbb{N}$. Then the set of operators*

$$\mathfrak{K} := \left\{ \mathcal{B}_m := \mathbb{1}_m(\mathcal{T}) \mathcal{B} \, \mathbb{1}_m(\mathcal{T}) \mid \mathcal{B} \in \mathcal{L}^{\infty}_{+}(\mathcal{H}),\, m \in \mathbb{N} \right\} \tag{2.16}$$

separates the points in $\mathcal{L}^1_+(\mathcal{H})$ with respect to the weak-$*$ topology.

Proof. Let $\gamma \in \mathcal{L}^1_+(\mathcal{H})$ be such that, for all $\mathcal{B}_m \in \mathfrak{K}$,

$$\mathrm{tr}_{\mathcal{H}}(\gamma \mathcal{B}_m) = 0.$$

Let $\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ be the decomposition of γ. Then

$$\sum_{j \in \mathbb{N}} \lambda_j \langle \psi_j | \mathcal{B}_m | \psi_j \rangle_{\mathcal{H}} = 0 \implies \langle \psi_j | \mathcal{B}_m | \psi_j \rangle_{\mathcal{H}} = 0, \; \forall j \in \mathbb{N},$$

by positivity of $\mathcal{B}$. Taking the limit $m \to +\infty$ of the last equation, one finds that for any $\mathcal{B} \in \mathcal{L}^{\infty}_{+}(\mathcal{H})$ and $j \in \mathbb{N}$,

$$\langle \psi_j | \mathcal{B} | \psi_j \rangle_{\mathcal{H}} = 0, \tag{2.17}$$

but, taking in particular $\mathcal{B} = |\psi_j\rangle\langle\psi_j|$, we get $\psi_j = 0$ for any $j \in \mathbb{N}$, and therefore $\gamma = 0$. ∎

Proposition 2.5 (Convergence of general state sequences). *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be such that*

- *assumption (A1) is satisfied,*
- $\Gamma_{\varepsilon_n} \xrightarrow[n \to +\infty]{} \mathfrak{m}$,
- $|\mathrm{tr}_{\mathcal{H}}(\mathcal{T} \gamma_\varepsilon \mathcal{T})| \leq C$ *for some self-adjoint $\mathcal{T} \in \mathcal{L}(\mathcal{H})$, where γ_ε is given by (1.18).*

Then

$$T \Gamma_{\varepsilon_n} T \xrightarrow[n \to +\infty]{} \mathcal{T} \mathfrak{m} \mathcal{T}, \tag{2.18}$$

where the latter is defined by the Radon–Nikodým decomposition $(\mu_{\mathfrak{m}}, \mathcal{T} \gamma_{\mathfrak{m}}(z) \mathcal{T})$.

Proof. Since $|\mathrm{tr}_{\mathcal{H}}(\mathcal{T} \gamma_\varepsilon \mathcal{T})| \leq C$, $T \, \Gamma_\varepsilon \, T$ is a quasi-classical family of states and thus there exists a generalized subsequence $(\Gamma_{\varepsilon_{n_\alpha}})_{\alpha \in A}$ of Γ_{ε_n} and a cylindrical state-valued measure $\mathfrak{n}$ such that (see [23] for additional details)

- $T \Gamma_{\varepsilon_{n_\alpha}} T$ converges to $\mathfrak{n}$ when tested on the Weyl quantization of smooth cylindrical symbols,
- $\mathrm{tr}_{\mathcal{H}}(\mathcal{T} \widehat{\Gamma}_{\varepsilon_{n_\alpha}}(\eta) \mathcal{T} \mathcal{B})$ converges to $\mathrm{tr}_{\mathcal{H}}(\gamma(\eta) \mathcal{B})$ for all $\eta \in \mathfrak{h}$ and $\mathcal{B} \in \mathcal{L}^{\infty}(\mathcal{H})$, where $\gamma(\eta) \in \mathcal{L}^1(\mathcal{H})$ has yet to be determined.

Now, let $\mathcal{K} \in \mathfrak{K}$. Then $\mathcal{T}\mathcal{K}\mathcal{T} \in \mathcal{L}^\infty(\mathcal{H})$ and therefore

$$\lim_{\alpha\in A} \operatorname{tr}_{\mathcal{H}}(\mathcal{T}\widehat{\Gamma}_{\varepsilon_{n_\alpha}}(\eta)\mathcal{T}\mathcal{K}) = \lim_{\alpha\in A} \operatorname{tr}_{\mathcal{H}}(\widehat{\Gamma}_{\varepsilon_{n_\alpha}}(\eta)\,\mathcal{T}\mathcal{K}\mathcal{T}) = \operatorname{tr}_{\mathcal{H}}(\widehat{\mathfrak{m}}(\eta)\,\mathcal{T}\mathcal{K}\mathcal{T})$$
$$= \operatorname{tr}_{\mathcal{H}}(\mathcal{T}\,\widehat{\mathfrak{m}}(\eta)\,\mathcal{T}\mathcal{K}).$$

However, the set $\mathfrak{K}$ separates points by Lemma 2.4, and therefore we can conclude that

$$\gamma(\eta) = \mathcal{T}\,\widehat{\mathfrak{m}}(\eta)\mathcal{T}.$$

On the other hand, an analogous reasoning when testing with the Weyl quantization of smooth cylindrical symbols yields

$$\mathfrak{n} = \mathcal{T}\,\mathfrak{m}\mathcal{T}.$$

Therefore, we conclude that $T\Gamma_{\varepsilon_{n_\alpha}}T \xrightarrow[\alpha\in A]{} \mathcal{T}\,\mathfrak{m}\mathcal{T}$. Finally, let $\Gamma_{\varepsilon_{n_{\alpha'}}}$ be any generalized subsequence such that for any $\eta \in \mathfrak{h}$ and $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$,

$$\lim_{\alpha'\in A'} \operatorname{tr}_{\mathcal{H}}(\mathcal{T}\widehat{\Gamma}_{\varepsilon_{n_{\alpha'}}}(\eta)\mathcal{T}\mathcal{B}) = \operatorname{tr}_{\mathcal{H}}(\gamma'(\eta)\mathcal{B}).$$

Then, repeating the above reasoning it follows that $\gamma'(\eta) = \mathcal{T}\,\widehat{\mathfrak{m}}(\eta)\mathcal{T}$. In other words, the cluster point is unique, and therefore $T\Gamma_{\varepsilon_n}T \xrightarrow[n\to+\infty]{} \mathcal{T}\,\mathfrak{m}\mathcal{T}$. ■

Proof of Proposition 2.3. The key result about the weak-$*$ convergence in the semiclassical case is proven in [6, Theorem 6.2]. The generalization to the quasi-classical setting is trivial: for all compact operators $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$ and all $\eta \in \mathfrak{h}$, one immediately gets

$$\lim_{n\to\infty} \operatorname{Tr}(\Gamma_{\varepsilon_n} W_{\varepsilon_n}(\eta) B) = \operatorname{tr}_{\mathcal{H}}\Big(\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\,\gamma_{\mathfrak{m}}(z) e^{2i\operatorname{Re}\langle\eta|z\rangle_{\mathfrak{h}}}\,\mathcal{B}\Big)$$
$$= \operatorname{tr}_{\mathcal{H}}(\widehat{\mathfrak{m}}(\eta)\mathcal{B}), \tag{2.19}$$

where $B := \mathcal{B} \otimes 1$. Moreover, the Fourier transform $\widehat{\mathfrak{m}} : \mathfrak{h} \to \mathcal{L}^1(L^2)$ identifies uniquely the measure $\mathfrak{m}$ by Bochner's theorem [23]. The bound (2.14) is also an immediate extension of [6, Theorem 6.2] to the quasi-classical case.

It remains to prove that under either assumption (A2) or (A2′), $\mathfrak{m} \in \mathcal{P}(\mathfrak{h}; \mathcal{L}^1_+(\mathcal{H}))$. Let us start by assuming (A2). Then, by Proposition 2.5, for any *bounded* $\mathcal{B} \in \mathcal{B}(\mathcal{H})$, and $\eta \in \mathfrak{h}$,

$$\lim_{n\to+\infty} \operatorname{tr}_{\mathcal{H}}(\widehat{\Gamma}_{\varepsilon_n}(\eta)B) = \lim_{n\to+\infty} \operatorname{tr}_{\mathcal{H}}(A^{1/2}\widehat{\Gamma}_{\varepsilon_n}(\eta)A^{1/2}A^{-1/2}BA^{-1/2})$$
$$= \operatorname{tr}_{\mathcal{H}}(\mathcal{A}^{1/2}\widehat{\mathfrak{m}}(\eta)\mathcal{A}^{1/2}\mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2}) = \operatorname{tr}_{\mathcal{H}}(\widehat{\mathfrak{m}}(\eta)\mathcal{B}).$$

In particular, for $\eta = 0$ and $\mathcal{B} = \mathbb{1}$,

$$1 = \lim_{n\to+\infty} \operatorname{tr}_{\mathcal{H}}(\gamma_{\varepsilon_n}) = \lim_{n\to+\infty} \operatorname{tr}_{\mathcal{H}}(\widehat{\Gamma}_{\varepsilon_n}(0)) = \operatorname{tr}_{\mathcal{H}}(\widehat{\mathfrak{m}}(0)) = \operatorname{tr}_{\mathcal{H}}(\mathfrak{m}(\mathfrak{h})).$$

If we instead assume (A2′), the proof goes as follows. Since $\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m}$, it follows that γ_{ε_n} converges in the weak operator topology to $\widehat{\mathfrak{m}}(0)$, by compactness of rank-one operators. Let $\{m_j\}_{j\in\mathbb{N}}$ be the eigenvalues of $\widehat{\mathfrak{m}}(0)$, and $\{\psi_j\}_{j\in\mathbb{N}}$ the corresponding eigenvectors. By the aforementioned weak operator convergence, it follows that for any $j \in \mathbb{N}$,

$$\lim_{n\to+\infty} \langle \psi_j | \gamma_{\varepsilon_n} | \psi_j \rangle_{\mathcal{H}} = \langle \psi_j | \widehat{\mathfrak{m}}(0) | \psi_j \rangle_{\mathcal{H}} = m_j. \tag{2.20}$$

On the other hand, by (A2′), we can apply Lebesgue's dominated convergence theorem to the series

$$\sum_{j=0}^{+\infty} \langle \psi_j | \gamma_{\varepsilon_n} | \psi_j \rangle_{\mathcal{H}},$$

since $\langle \psi_j | \gamma_{\varepsilon_n} | \psi_j \rangle_{\mathcal{H}} \le \langle \psi_j | \gamma | \psi_j \rangle_{\mathcal{H}}$, and

$$\mathrm{tr}_{\mathcal{H}}(\gamma) = \sum_{j=0}^{+\infty} \langle \psi_j | \gamma | \psi_j \rangle_{\mathcal{H}} < +\infty.$$

Therefore,

$$1 = \lim_{n\to+\infty} \sum_{j=0}^{+\infty} \langle \psi_j | \gamma_{\varepsilon_n} | \psi_j \rangle_{\mathcal{H}} = \sum_{j=0}^{+\infty} \lim_{n\to+\infty} \langle \psi_j | \gamma_{\varepsilon_n} | \psi_j \rangle_{\mathcal{H}} = \sum_{j=0}^{+\infty} m_j = \mathrm{tr}_{\mathcal{H}}(\widehat{\mathfrak{m}}(0)). \quad \blacksquare$$

It is clear that together with Proposition 2.3, all the other results that hold in semiclassical analysis for infinite dimensions can be adapted to quasi-classical analysis, considering the semiclassical symbols and corresponding quantizations in tensor product with the identity acting on $\mathcal{H}$, replacing Wigner scalar measures with state-valued Wigner measures, and replacing convergence of the trace with $\mathcal{L}^1(\mathcal{H})$-weak-$*$ convergence of the partial trace, i.e., one should test the partial traces and integrals with compact operators.

Proposition 2.6 (Convergence of expectation values). *Let $\mathcal{F} \in \mathcal{S}_{\ell,m}$, and let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$. Assume that there exist $\delta > (n+m)/2$ such that*

$$\mathrm{Tr}\big(\Gamma_\varepsilon (\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta\big) \le C. \tag{2.21}$$

If $\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m}$, then for any $\mathcal{S}, \mathcal{T} \in \mathcal{B}(\mathcal{H})$, $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$ and $\eta \in \mathfrak{h}$,

$$\lim_{n\to+\infty} \mathrm{Tr}\big(\Gamma_{\varepsilon_n} \mathcal{T} \mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F}) \mathcal{S} (\mathcal{B} \otimes W_{\varepsilon_n}(\eta))\big)$$
$$= \mathrm{tr}_{\mathcal{H}}\left[\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z) \mathcal{T} \mathcal{F}(z) \mathcal{S} e^{2i \,\mathrm{Re}\langle \eta | z\rangle_{\mathfrak{h}}} \mathcal{B}\right], \tag{2.22}$$

with an analogous statement with Γ_{ε_n} and $\mathcal{T}\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F})\mathcal{S}$ interchanged.

To prove Proposition 2.6, we need the following preparatory lemma, which introduces approximation of $\mathcal{F}$ by simple functions.

Lemma 2.7. *Let $\mathcal{F} \in \mathcal{S}_{\ell,m}$. Then there exists a sequence $\{\mathcal{F}_M\}_{M\in\mathbb{N}}$ of operator-valued functions $\mathcal{F}_M : \mathfrak{h} \to \mathcal{B}(\mathcal{H})$ such that*

- *for all $z \in \mathfrak{h}$,*

$$\lim_{M\to\infty} \|\mathcal{F}(z) - \mathcal{F}_M(z)\|_{\mathcal{H}} = 0; \tag{2.23}$$

- *$\mathcal{F}_M(z)$ acts as the multiplication operator by*

$$\mathcal{F}_M(z) = \sum_{j=1}^{N}\sum_{k=1}^{J(M)} \langle z|\varphi_{k,1}\rangle_{\mathfrak{h}} \cdots \langle z|\varphi_{k,\ell}\rangle_{\mathfrak{h}} \langle \varphi_{k,\ell+1}|z\rangle_{\mathfrak{h}} \cdots \langle \varphi_{k,\ell+m}|z\rangle_{\mathfrak{h}} \mathbb{1}_{B_k}(\mathbf{x}_j), \tag{2.24}$$

where $J : \mathbb{N} \to \mathbb{N}$, $\varphi_{j,l} \in \mathfrak{h}$, $l \in \{1, \ldots, \ell + m\}$, and $\mathbb{1}_{B_j}$ is the characteristic function of the Borel set $B_j \subseteq \mathbb{R}^d$ and the B_j are pairwise disjoint.

Proof. It is sufficient to prove the convergence in the case $N = 1, n = 1, m = 0$, since the case $N = 1$, $n = 0$, $m = 1$ is perfectly analogous, and the general one, $N \in \mathbb{N}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, can be obtained by combining the approximation for each term of the product within each term of the sum and possibly reordering the sum.

So let us restrict to the case $\mathcal{F}(z) = \langle z|\lambda(\mathbf{x})\rangle_{\mathfrak{h}}$, $\mathbf{x} \in \mathbb{R}^d$, acting as a multiplication operator on $\mathcal{H} = L^2(\mathbb{R}^d)$. Since both $\mathcal{F}(z)$ and

$$\mathcal{F}_M(z) = \sum_{k=1}^{J(M)} \langle z|\varphi_k\rangle_{\mathfrak{h}} \mathbb{1}_{B_k}(\mathbf{x}) =: \langle z|\lambda_M(\mathbf{x})\rangle_{\mathfrak{h}} \tag{2.25}$$

are multiplication operators, we have

$$\|\mathcal{F}(z) - \mathcal{F}_M(z)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d} |\langle z|\lambda(\mathbf{x}) - \lambda_M(\mathbf{x})\rangle_{\mathfrak{h}}|.$$

Now, let us fix $z \in \mathfrak{h}$ and consider $\mathcal{F}(z) = \mathcal{F}_z(\mathbf{x})$ only as a function of $\mathbf{x} \in \mathbb{R}^d$. We can decompose $\mathcal{F}_z(\mathbf{x}) = \mathcal{F}_{\mathbb{R}}(\mathbf{x}) + i\mathcal{F}_{\mathbb{C}}(\mathbf{x})$, and split both the real and imaginary parts as $\mathcal{F}_{\mathbb{R}/\mathbb{C}}(\mathbf{x}) = \mathcal{F}_{\mathbb{R}/\mathbb{C},+}(\mathbf{x}) - \mathcal{F}_{\mathbb{R}/\mathbb{C},-}(\mathbf{x})$. Setting $K := \|\lambda\|_{L^\infty(\mathbb{R}^d,\mathfrak{h})}$, we can partition the positive real half-line as

$$\mathbb{R}_+ = A \cup \bigcup_{m=1}^{M} A_m,$$

where

$$A := [K\|z\|_{\mathfrak{h}}, \infty), \quad A_m := K\big[\tfrac{m-1}{M}\|z\|_{\mathfrak{h}}, \tfrac{m}{M}\|z\|_{\mathfrak{h}}\big). \tag{2.26}$$

Let us now focus on the positive real part $\mathcal{F}_{\mathbb{R},+}(\mathbf{x})$. We can introduce the measurable sets

$$D^+ := \mathcal{F}_{\mathbb{R},+}^{-1}(A), \quad D_m^+ := \mathcal{F}_{\mathbb{R},+}^{-1}(A_m).$$

By construction, $D^+ = \emptyset$, while, for all $m \in \{1, \ldots, M\}$, there exists $\eta_m^+ \in \mathfrak{h}$ such that

$$\langle \eta_m^+|z\rangle_{\mathfrak{h}} \in A_m.$$

For any given $\mathbf{x} \in \mathbb{R}^d$, there is a single $\tilde{m} \in \{1, \ldots, 2^M\}$ such that $\mathcal{F}_{\mathbb{R},+}(\mathbf{x}) \in A_{\tilde{m}}^+$. Therefore, uniformly with respect to $\mathbf{x} \in D_{\tilde{m}}^+$,

$$|\mathcal{F}_{\mathbb{R},+}(\mathbf{x}) - \langle \eta_m^+|z\rangle_{\mathfrak{h}}| < K\|z\|_{\mathfrak{h}}/M. \tag{2.27}$$

Repeating the same procedure for the real negative and complex positive and negative parts, we obtain collections of sets and elements, respectively,

$$\{D_m^-\}_{m=1}^M, \ \{\eta_m^-\}_{m=1}^M; \quad \{E_m^\pm\}_{m=1}^M, \ \{\xi_m^\pm\}_{m=1}^M,$$

approximating $\mathcal{F}_{\mathbb{R},-}$ and $\mathcal{F}_{\mathbb{C},\pm}$.

Let us now define a collection $\{B_k\}_{k=1}^{M^4}$ of disjoint Borel sets of $\mathbb{R}^d$ for simple approximation of $\mathcal{F}(z)$. We first identify $k \in \{1, \dots, M^4\}$ with the image $\jmath(m_1, m_2, m_3, m_4)$ with respect to some fixed set bijection $\jmath : \{1, \dots, M\}^4 \to \{1, \dots, M^4\}$, and set

$$B_k := D_{m_1}^+ \cap D_{m_2}^- \cap E_{m_3}^+ \cap E_{m_4}^-. \tag{2.28}$$

Then we define $\varphi_k := \eta_{m_1}^+ - \eta_{m_2}^- + i(\xi_{m_3}^+ - \xi_{m_4}^-)$ and

$$\begin{aligned} \mathcal{F}_M(z) &= \sum_{k=1}^{M^4} \langle z | \varphi_k \rangle_{\mathfrak{h}} \mathbb{1}_{B_k}(\mathbf{x}) \\ &= \sum_{m_1,m_2,m_3,m_4=1}^{M} \langle z | \eta_{m_1}^+ - \eta_{m_2}^- + i(\xi_{m_3}^+ - \xi_{m_4}^-) \rangle_{\mathfrak{h}} \mathbb{1}_{B_{\jmath(m_1,m_2,m_3,m_4)}}(\mathbf{x}). \end{aligned} \tag{2.29}$$

By construction,

$$\|\mathcal{F}(z) - \mathcal{F}_M(z)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \le 4K\|z\|_{\mathfrak{h}}/M, \tag{2.30}$$

and thus the convergence is proved. ∎

Corollary 2.8. *The approximating function $\mathcal{F}_M(z)$ can be rewritten as*

$$\mathcal{F}_M(z) = \sum_{j=1}^{N} \langle z | \lambda_{M,1}(\mathbf{x}_j) \rangle_{\mathfrak{h}} \cdots \langle z | \lambda_{M,\ell}(\mathbf{x}_j) \rangle_{\mathfrak{h}} \langle \lambda_{M,\ell+1}(\mathbf{x}_j) | z \rangle_{\mathfrak{h}} \cdots \langle \lambda_{M,\ell+m}(\mathbf{x}_j) | z \rangle_{\mathfrak{h}}, \tag{2.31}$$

where $\lambda_{M,j} \in L^\infty(\mathbb{R}^d; \mathfrak{h})$, $j \in \{1, \dots, \ell + m\}$, and

$$\lim_{M \to +\infty} \|\lambda_j - \lambda_{M,j}\|_{L^\infty(\mathbb{R}^d;\mathfrak{h})} = 0. \tag{2.32}$$

Proof. Again, it is sufficient to prove the corollary for $N = 1$, $n = 1$, $m = 0$, the other cases being direct consequences. The function λ_M approximating λ is defined in (2.25) in the proof of Lemma 2.7, i.e.,

$$\lambda_M(\mathbf{x}) := \sum_{k=1}^{J(M)} \varphi_k \mathbb{1}_{B_k}(\mathbf{x}).$$

From the same proof it also follows that, for all $z \in \mathfrak{h}$ and all $\mathbf{x} \in \mathbb{R}^d$,

$$|\langle z | \lambda(\mathbf{x}) - \lambda_M(\mathbf{x}) \rangle_{\mathfrak{h}}| \le 4K\|z\|_{\mathfrak{h}}/M.$$

Therefore,

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d} \|\lambda(\mathbf{x}) - \lambda_M(\mathbf{x})\|_{\mathfrak{h}} = \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d} \sup_{\|z\|_{\mathfrak{h}}=1} |\langle z|\lambda(\mathbf{x}) - \lambda_M(\mathbf{x})\rangle_{\mathfrak{h}}| \le 4K/M,$$

and the convergence is proved. ■

Proof of Proposition 2.6. Let us prove (2.22). Let us approximate $\mathcal{F}(z)$ with $\mathcal{F}_M(z)$, as dictated by Lemma 2.7. The advantage of $\mathcal{F}_M(z)$ is that its dependence on the z and $\mathbf{x}$ variables is separated, and thus its Wick quantization is a finite sum of tensor products of operators:

$$\begin{aligned}\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon}(\mathcal{F}_M) &= \sum_{j=1}^{N}\sum_{k=1}^{J(M)} \mathbb{1}_{B_k}(\mathbf{x}_j) \otimes a^{\dagger}_{\varepsilon}(\varphi_{k,1})\cdots a^{\dagger}_{\varepsilon}(\varphi_{k,\ell})a_{\varepsilon}(\varphi_{k,\ell+1})\cdots a_{\varepsilon}(\varphi_{k,\ell+m})\\ &= \sum_{j=1}^{N} a^{\dagger}_{\varepsilon}(\lambda_{M,1}(\mathbf{x}_j))\cdots a^{\dagger}_{\varepsilon}(\lambda_{M,\ell}(\mathbf{x}_j))a_{\varepsilon}(\lambda_{M,\ell+1}(\mathbf{x}_j))\cdots a_{\varepsilon}(\lambda_{M,\ell+m}(\mathbf{x}_j)). \qquad (2.33)\end{aligned}$$

Next we exploit the linearity of Wick quantization to split

$$\begin{aligned}\mathrm{Tr}\big(\Gamma_{\varepsilon_n} T\,\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F})S(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big) &= \mathrm{Tr}\big(\Gamma_{\varepsilon_n} T\,\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F}-\mathcal{F}_M)S(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big)\\ &\quad + \mathrm{Tr}\big(\Gamma_{\varepsilon_n} T\,\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F}_M)S(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big). \qquad (2.34)\end{aligned}$$

The first term on the right hand side can be estimated using well-known estimates for creation and annihilation operators, the hypothesis on the expectation of the number operator, and Corollary 2.8:

$$\begin{aligned}\big|\mathrm{Tr}\big(\Gamma_{\varepsilon_n} T\,\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F}-\mathcal{F}_M)S(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big)\big| &\le N\|\mathcal{B}\|\|\mathcal{T}\|\|\mathcal{S}\|\,\mathrm{Tr}(\Gamma_{\varepsilon_n}\mathrm{d}\mathcal{G}_{\varepsilon}(1)^{(\ell+m)/2})\\ &\quad\times\sum_{p=1}^{\ell+m}\|\lambda_1\|_{L^\infty(\mathbb{R}^d,\mathfrak{h})}\cdots\|\lambda_p-\lambda_{M,p}\|_{L^\infty(\mathbb{R}^d,\mathfrak{h})}\cdots\|\lambda_{M,\ell+m}\|_{L^\infty(\mathbb{R}^d,\mathfrak{h})}\\ &\le CN(\ell+m)\max_{p\in\{1,\dots,\ell+m\}}\|\lambda_p-\lambda_{M,p}\|_{L^\infty(\mathbb{R}^d;\mathfrak{h})},\end{aligned}$$

where we have used the fact that the $\|\lambda_{M,p}\|_{L^\infty(\mathbb{R}^d,\mathfrak{h})}$ are all uniformly bounded with respect to M by (2.32). The right hand side of the above expression then converges to zero as $M\to+\infty$ by Corollary 2.8, uniformly in ε_n.

Let us now discuss the limit as $n\to+\infty$ of the second term on the right hand side of (2.34): for any $\mathcal{B}\in\mathcal{L}^\infty(L^2(\mathbb{R}^{Nd}))$, using the first identity of (2.33), we obtain

$$\begin{aligned}&\mathrm{Tr}\big(\Gamma_{\varepsilon_n} T\,\mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F}_M)S(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big)\\ &\quad=\sum_{j=1}^{N}\sum_{k=1}^{J(M)}\mathrm{tr}_{\mathcal{H}}\big(\mathrm{tr}_{\mathcal{K}_\varepsilon}\big(\Gamma_{\varepsilon_n}a^{\dagger}_{\varepsilon_n}(\varphi_{k,1})\cdots a_{\varepsilon}(\varphi_{k,\ell+m})W_{\varepsilon_n}(\eta)\big)\mathbb{1}_{B_k}(\mathbf{x}_j)\mathcal{S}\mathcal{B}\mathcal{T}\big).\end{aligned}$$

Now, on the one hand we know that $\Gamma_{\varepsilon_n} \to \mathfrak{m}$ by Proposition 2.3, and on the other hand

$$a^\dagger_{\varepsilon_n}(\varphi_{k,1}) \cdots a_\varepsilon(\varphi_{k,\ell+m}) = \mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\langle z|\varphi_{k,1}\rangle_{\mathfrak{h}} \cdots \langle \varphi_{k,\ell+m}|z\rangle_{\mathfrak{h}}),$$

where the scalar symbol on the right hand side is polynomial and cylindrical. Therefore, since $\mathbb{1}_{B_k}(\mathbf{x}_j)\mathcal{S}\mathcal{B}\mathcal{T} \in \mathcal{L}^\infty(L^2(\mathbb{R}^{Nd}))$, by the quasi-classical analogue of [6, Theorem 6.13],

$$\begin{aligned}&\lim_{n\to+\infty} \mathrm{Tr}\big(\Gamma_{\varepsilon_n} T \mathrm{Op}^{\mathrm{Wick}}_{\varepsilon_n}(\mathcal{F}_M) S(\mathcal{B} \otimes W_{\varepsilon_n}(\eta))\big)\\ &= \sum_{j=1}^{N} \sum_{k=1}^{J(M)} \mathrm{tr}_{\mathcal{H}}\bigg(\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, e^{2i\,\mathrm{Re}\langle \eta|z\rangle_{\mathfrak{h}}} \langle z|\varphi_{k,1}\rangle_{\mathfrak{h}} \cdots \langle \varphi_{k,\ell+m}|z\rangle_{\mathfrak{h}} \gamma_{\mathfrak{m}}(z) \mathbb{1}_{B_k}(\mathbf{x}_j) \mathcal{S}\mathcal{B}\mathcal{T}\bigg).\end{aligned}$$

The proof is then concluded by taking the limit as $M \to \infty$ of the last expression, which by dominated convergence yields the sought result. ∎

3. The microscopic model

Our aim is to study systems of non-relativistic particles in interaction with radiation. As discussed previously, the techniques developed in this paper allow one to study some well-known classes of explicit models (Nelson, polaron, Pauli–Fierz). Here we carry out the detailed analysis only for the simplest example, the Nelson model, in order to convey the general strategy without too many technical details. The main adaptations needed for the polaron and Pauli–Fierz systems are outlined in §7.

Let $\mathcal{H} \otimes \mathcal{K}_\varepsilon = L^2(\mathbb{R}^{dN}) \otimes \mathcal{G}_\varepsilon(L^2(\mathbb{R}^d))$ be the Hilbert space of the theory. Then the Nelson Hamiltonian H_ε is explicitly given by

$$H_\varepsilon = K_0 + \nu(\varepsilon)\mathrm{d}\Gamma_\varepsilon(\omega) + \sum_{j=1}^{N} a^\dagger_\varepsilon(\lambda(\mathbf{x}_j)) + a_\varepsilon(\lambda(\mathbf{x}_j)), \tag{3.1}$$

where $K_0 = \mathcal{K}_0 \otimes 1$ is the part of the Hamiltonian acting on the particles alone, such that $\mathcal{K}_0$ is self-adjoint on $\mathcal{D}(\mathcal{K}_0) \subset L^2(\mathbb{R}^{dN})$, $\nu(\varepsilon) > 0$ is a quasi-classical scaling factor to be discussed in detail below, ω is the operator on $L^2(\mathbb{R}^d)$ acting as multiplication by the positive dispersion relation of the field $\omega(\mathbf{k})$, and $\lambda \in L^\infty(\mathbb{R}^d; L^2(\mathbb{R}^d)) =: L^\infty_{\mathbf{x}} L^2_{\mathbf{k}}$ is the interaction's form factor. In addition, let us define the set of vectors with a finite number of field's excitations $C_0^\infty(\mathrm{d}\mathcal{G}_\varepsilon(1))$:

$$\begin{aligned}C_0^\infty(\mathrm{d}\mathcal{G}_\varepsilon(1)) = \{\psi \in L^2(\mathbb{R}^{Nd}) \otimes \mathcal{G}_\varepsilon(\mathfrak{h}) \mid \exists M \in \mathbb{N} : a_\varepsilon(f_1) \cdots a_\varepsilon(f_{M'})\psi = 0,&\\ \forall M' > M, \forall \{f_j\}_{j=1}^{M'} \subset L^2(\mathbb{R}^d)\}.&\end{aligned} \tag{3.2}$$

The question of self-adjointness of H_ε has already been addressed in the literature and indeed the following proposition holds:

Proposition 3.1 (Self-adjointness of H_ε [21, Theorem 3.1]). *The operator H_ε is essentially self-adjoint on $D(K_0) \cap D(\mathrm{d}\mathcal{G}_\varepsilon(\omega)) \cap C_0^\infty(\mathrm{d}\mathcal{G}_\varepsilon(1))$.*

Therefore, there exists a unitary evolution generated by H_ε,

$$U_\varepsilon(t) = e^{-itH_\varepsilon}. \tag{3.3}$$

Now for any normalized density matrix $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$, we denote by $\Gamma_\varepsilon(t)$ its unitary evolution by means of $U_\varepsilon(t)$, i.e.,

$$\Gamma_\varepsilon(t) = U_\varepsilon(t)\Gamma_\varepsilon U_\varepsilon^\dagger(t). \tag{3.4}$$

The main aim of this paper is to characterize the asymptotic behavior as $\varepsilon \to 0$ of

$$\gamma_\varepsilon(t) := \mathrm{tr}_{\mathcal{K}_\varepsilon}(\Gamma_\varepsilon(t)) = \mathrm{tr}_{\mathcal{G}_\varepsilon(L^2(\mathbb{R}^d))}(\Gamma_\varepsilon(t)). \tag{3.5}$$

As stated in Definition 1.1 and characterized in Proposition 2.6, the quasi-classical limit of a sufficiently regular state is determined by the weak convergence of its vector-valued non-commutative Fourier transform $\widehat{\Gamma}_\varepsilon(t) : \mathfrak{h} \to \mathcal{L}^1(\mathcal{H})$ defined in (1.19):

$$\eta \mapsto [\widehat{\Gamma}_\varepsilon(t)](\eta) := \mathrm{tr}_{\mathcal{K}_\varepsilon}(\Gamma_\varepsilon(t)W_\varepsilon(\eta)) = \mathrm{tr}_{\mathcal{K}_\varepsilon}\big(\Gamma_\varepsilon(t)e^{i(a_\varepsilon^\dagger(\eta)+a_\varepsilon(\eta))}\big).$$

Note that consequently $\gamma_\varepsilon(t) = [\widehat{\Gamma}_\varepsilon(t)](0)$.

The regularity of the state is given by (2.12), which should be satisfied at any time. It is therefore necessary to ensure a proper propagation in time of such a regularity. An estimate of that kind is however readily available for the Nelson model with cutoff:

Proposition 3.2 (Regularity propagation [20, Proposition 4.2]). *For any $\varepsilon > 0$, $t \in \mathbb{R}$ and $\delta \in \mathbb{R}$,*

$$\begin{aligned}
&\mathrm{Tr}\big(\Gamma_\varepsilon(t)(\mathrm{d}\mathcal{G}_\varepsilon(1) + N^2 + \varepsilon)^\delta\big) \\
&\qquad \le e^{c_{\delta/2}(\varepsilon)\sqrt{\varepsilon}|\delta|\,|t|\,\|\lambda\|_{L^\infty_x L^2_k}}\,\mathrm{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + N^2 + \varepsilon)^\delta\big), && (3.6)\\
&\mathrm{Tr}\,\big|\Gamma_\varepsilon(t)(\mathrm{d}\mathcal{G}_\varepsilon(1) + N^2 + \varepsilon)^\delta\big| \\
&\qquad \le e^{c_{\delta}(\varepsilon)\sqrt{\varepsilon}|\delta|\,|t|\,\|\lambda\|_{L^\infty_x L^2_k}}\,\mathrm{Tr}\,\big|\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + N^2 + \varepsilon)^\delta\big|, && (3.7)
\end{aligned}$$

where $c_\delta(\varepsilon) := \max\{2 + \varepsilon, 1 + (1 + \varepsilon)^\delta\}$.

Since the exponential in the above inequality is bounded uniformly with respect to $\varepsilon \in (0, 1)$, it follows that the bound (2.12) is satisfied by the state at any time with a suitable time-dependent constant, provided it is satisfied by the state at $t = 0$: using the fact that, for any $\delta \in \mathbb{R}^+$,

$$\begin{aligned}
(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta \le (\mathrm{d}\mathcal{G}_\varepsilon(1) + N^2 + \varepsilon)^\delta &\le (\mathrm{d}\mathcal{G}_\varepsilon(1) + N^2 + 1)^\delta \\
&\le (N^2 + 1)^\delta(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta,
\end{aligned}$$

we can use (3.6) to obtain

$$\operatorname{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^\delta\big) \leq C_\delta \implies \\ \operatorname{Tr}\big(\Gamma_\varepsilon(t)(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^\delta\big) \leq C_\delta(N^2+1)^\delta e^{c_{\delta/2}(1)|\delta|\,|t|\,\|\lambda\|_{L^\infty_{\mathbf{x}}L^2_{\mathbf{k}}}} \leq C(\delta,t), \tag{3.8}$$

which guarantees that the a priori bound (2.12) is preserved by the time evolution.

In analogy with the dynamical semiclassical limit for bosonic field theories (see, e.g., [3,4,9]), the quasi-classical dynamics is characterized by studying the limit $\varepsilon \to 0$ of the integral equation of evolution for the microscopic system. Let us sketch the main ideas. Consider the family of states

$$\{\Gamma_\varepsilon(t)\}_{\varepsilon\in(0,1),\,t\in\mathbb{R}}$$

at time $t = 0$, satisfying the bound (2.12). Then for each fixed $t \in \mathbb{R}$, there exists a subsequence $\varepsilon_n \to 0$ such that $\Gamma_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{m}_t$ in the sense of Definition 1.1 by Propositions 2.3 and 3.2. In the next section we prove that it is actually possible to extract a *common subsequence* $\varepsilon_{n_k} \to 0$ such that *for all* $t \in \mathbb{R}$,

$$\Gamma_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \mathfrak{m}_t.$$

Hence one only needs to characterize the map $t \mapsto \mathfrak{m}_t$, and this is done by studying the associated transport equation, obtained by passing to the limit in the microscopic integral equation of evolution. Let us provide some intuition on that strategy. For later convenience let us pass to the interaction representation and set

$$\widetilde{\Gamma}_\varepsilon(t) := e^{-i\nu(\varepsilon)t\mathrm{d}\mathcal{G}_\varepsilon(\omega)}\Gamma_\varepsilon(t)e^{i\nu(\varepsilon)t\mathrm{d}\mathcal{G}_\varepsilon(\omega)}. \tag{3.9}$$

Then the microscopic evolution can be rewritten as an integral equation, using Duhamel's formula:

$$\widetilde{\Gamma}_\varepsilon(t) = \Gamma_\varepsilon - i\int_0^t \mathrm{d}\tau\,[\widetilde{H}_\varepsilon(\tau), \widetilde{\Gamma}_\varepsilon(\tau)], \tag{3.10}$$

where

$$\widetilde{H}_\varepsilon(t) := e^{-i\nu(\varepsilon)t\mathrm{d}\mathcal{G}_\varepsilon(\omega)}(H_\varepsilon - \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))e^{i\nu(\varepsilon)t\mathrm{d}\mathcal{G}_\varepsilon(\omega)}. \tag{3.11}$$

In addition, $H_\varepsilon - \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega)$ is the Wick quantization of an operator-valued symbol $\mathcal{K}_0 + \mathcal{V}(z)$. Therefore, the quasi-classical analysis developed in §2 suggests that the integral equation (3.10) converges, as $\varepsilon \to 0$, to an equation for the measure $\widetilde{\mathfrak{m}}_t$, obtained by replacing

$$\widetilde{\Gamma}_\varepsilon(t) \rightsquigarrow \widetilde{\mathfrak{m}}_t, \\ H_\varepsilon - \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega) \rightsquigarrow \mathcal{K}_0 + \mathcal{V}(z),$$

and substituting the quantum flow $e^{-i\nu(\varepsilon)t\mathrm{d}\mathcal{G}_\varepsilon(\omega)}$ in the phase space $\mathfrak{h} = L^2(\mathbb{R}^3)$ by its classical counterpart, i.e.,

$$z \mapsto e^{-i\nu t\omega}z, \quad \forall z \in L^2(\mathbb{R}^3). \tag{3.12}$$

In conclusion, we get the equation

$$\mathrm{d}\tilde{\mathfrak{m}}_t(z) = \mathrm{d}\tilde{\mathfrak{m}}(z) - i \int_0^t \mathrm{d}\tau\, \mathrm{d}\mu_{\tilde{\mathfrak{m}}_\tau}(z)\, [\mathcal{K}_0 + \mathcal{V}(e^{-i\nu t\omega} z), \gamma_{\tilde{\mathfrak{m}}_\tau}(z)], \tag{3.13}$$

and the classical measure $\mathfrak{m}_t$ associated with the original state $\Gamma_\varepsilon(t)$ is simply given by the *pushforward* of $\tilde{\mathfrak{m}}_t$ through the flow (3.12), i.e.,

$$\Gamma_\varepsilon(t) \rightsquigarrow \mathfrak{m}_t = (e^{-i\nu t\omega})_\star \tilde{\mathfrak{m}}_t. \tag{3.14}$$

Such an equation is the integral form of a Liouville-type equation. Once the convergence of the microscopic to the quasi-classical integral equation has been established (see §4), the crucial point is to prove that (3.13) has a unique solution that satisfies some properties, given by the a priori information that we have on the quasi-classical measure (see §5). As a final step (§6), we show that the convergence is in fact at any time $t \geq 0$ along the *same* subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Let us remark that in order to make this heuristic strategy rigorous, some technical modifications are necessary, in particular it is necessary to pass to the full interaction representation.

We conclude the section with the rigorous derivation of the microscopic integral evolution equation for the Fourier transform of $\Gamma_\varepsilon(t)$. By definition, for any $\eta \in \mathfrak{h}$, the Fourier transform $[\widehat{\Gamma}_\varepsilon(t)](\eta)$ is a reduced microscopic complex state for the particles, and therefore if Γ_ε is regular enough, its time evolution can be described by means of the microscopic generator H_ε. It is technically convenient to use the evolved state in the interaction picture, i.e.,

$$\Upsilon_\varepsilon(t) := e^{it(K_0+\nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))}\Gamma_\varepsilon(t) e^{-it(K_0+\nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))}, \tag{3.15}$$

in place of $\Gamma_\varepsilon(t)$, and therefore study the integral equation for $\widehat{\Upsilon}_\varepsilon(t)$.

Remark 3.3 (Regularity propagation for Υ_ε). Since $e^{it(K_0+\nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))}$ commutes with $\mathrm{d}\mathcal{G}_\varepsilon(1)$, one can easily realize that the results stated in Proposition 3.2, and consequently the bound propagation in (3.8), also hold true for the density matrix $\Upsilon_\varepsilon(t)$ in the interaction picture with the same constants.

Lemma 3.4. *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be such that*

$$\mathrm{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{1/2}\big) \leq C.$$

Then, for any $s, t \in \mathbb{R}$,

$$\begin{aligned}[\widehat{\Upsilon}_\varepsilon(t)](\eta) &= [\widehat{\Upsilon}_\varepsilon(s)](\eta) \\ &\quad - i \sum_{j=1}^N \int_s^t \mathrm{d}\tau\, e^{i\tau\mathcal{K}_0}\, \mathrm{tr}_{\mathcal{K}_\varepsilon}\big([\varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j)), \Upsilon_\varepsilon(\tau)] W_\varepsilon(\eta)\big) e^{-i\tau\mathcal{K}_0},\end{aligned} \tag{3.16}$$

weakly in $\mathcal{L}^1(L^2(\mathbb{R}^{Nd}))$, where $\varphi_\varepsilon(\cdot) = a_\varepsilon^\dagger(\cdot) + a_\varepsilon(\cdot)$ is the Segal field.

Proof. The proposition is an adaptation of [3, Proposition 3.5], and the proof follows accordingly. The differences here are only the presence of an arbitrary bounded particle observable, and that the Weyl operator acts only on the field's degrees of freedom. Therefore, we omit the details. ∎

4. The quasi-classical limit of time evolved states

In this section we focus on the quasi-classical limit as $\varepsilon \to 0$ of the Fourier transform $\widehat{\Upsilon}_\varepsilon(t)$ of time evolved states in the interaction picture. The first and most relevant step is the proof that it is possible to extract a common subsequence for the convergence of $\widehat{\Upsilon}_\varepsilon(t)$ at any time (Proposition 4.3), which in turn follows from the uniform equicontinuity of $\widehat{\Upsilon}_\varepsilon$ (Proposition 4.2). Finally, we show (Proposition 4.5) that the limit measure satisfies the transport equation of Lemma 3.4.

Let us start with a preparatory lemma.

Lemma 4.1 ([6, Lemma 3.1]). *For any $0 < \delta \leq 1/2$, there exists a finite constant c_δ such that for all $\eta, \xi \in \mathfrak{h}$,*

$$\|(W_\varepsilon(\eta) - W_\varepsilon(\xi))(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-\delta}\|_{\mathcal{B}(\mathcal{K}_\varepsilon)} \leq c_\delta(\min\{\|\eta\|_{\mathfrak{h}}^{2\delta}, \|\xi\|_{\mathfrak{h}}^{2\delta}\}+1)\|\eta-\xi\|_{\mathfrak{h}}^{2\delta}. \tag{4.1}$$

We are now able to prove uniform equicontinuity of $[\widehat{\Upsilon}_\varepsilon(\cdot)](\cdot)$.

Proposition 4.2 (Equicontinuity of $\widehat{\Upsilon}_\varepsilon$). *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be such that*

$$\mathrm{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{1/2}\big) \leq C.$$

Then $[\widehat{\Upsilon}_\varepsilon(\cdot)](\cdot) : \mathbb{R} \times \mathfrak{h} \to \mathcal{L}^1(\mathcal{H})$ is uniformly equicontinuous with respect to $\varepsilon \in (0, 1)$ on bounded subsets of $\mathbb{R} \times \mathfrak{h}$ if we endow $\mathcal{L}^1(\mathcal{H})$ with the weak-$*$ topology.

Proof. Let us fix $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$ and $(t, \eta), (s, \xi) \in \mathbb{R} \times \mathfrak{h}$ with $0 \leq s \leq t$. Then

$$\begin{aligned}
&\big|\mathrm{tr}_{\mathcal{H}}\big[\big([\widehat{\Upsilon}_\varepsilon(t)](\eta) - [\widehat{\Upsilon}_\varepsilon(s)](\xi)\big)\mathcal{B}\big]\big| \\
&\qquad\leq \big|\mathrm{tr}_{\mathcal{H}}\big[\big([\widehat{\Upsilon}_\varepsilon(t)](\eta) - [\widehat{\Upsilon}_\varepsilon(s)](\eta)\big)\mathcal{B}\big]\big| \\
&\qquad\quad + \big|\mathrm{tr}_{\mathcal{H}}\big[\big([\widehat{\Upsilon}_\varepsilon(s)](\eta) - [\widehat{\Upsilon}_\varepsilon(s)](\xi)\big)\mathcal{B}\big]\big| =: (I) + (II).
\end{aligned} \tag{4.2}$$

Let us consider the two terms separately. Making use of Lemma 3.4, we obtain

$$(I) \leq \sum_{j=1}^N \int_s^t \mathrm{d}\tau \, \big|\mathrm{Tr}\big([\varphi_\varepsilon(\lambda(\mathbf{x}_j)), \Gamma_\varepsilon(\tau)](\widetilde{\mathcal{B}}(\tau) \otimes W_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\eta))\big)\big|,$$

where $\widetilde{\mathcal{B}}(\tau) := e^{-i\tau\mathcal{K}_0}\mathcal{B}e^{i\tau\mathcal{K}_0}$. Therefore,

$$\begin{aligned}
(I) \leq{}& 2N\|\mathcal{B}\| \, \|(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{-1/4}\varphi_\varepsilon(\lambda(\cdot))(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{-1/4}\|_{\mathcal{B}(\mathcal{H}\otimes\mathcal{K}_\varepsilon)} \\
&\times \int_s^t \mathrm{d}\tau \, \|(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{1/4} W_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\eta)(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{-1/4}\|_{\mathcal{B}(\mathcal{K}_\varepsilon)} \\
&\qquad\times \mathrm{Tr}\big(\Gamma_\varepsilon(\tau)(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{1/2}\big)
\end{aligned}$$

where we have used the identity

$$\|(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-1/4}\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-1/4}\|_{\mathcal{L}^1(\mathcal{H})} = \mathrm{Tr}\big(\Gamma_\varepsilon(\tau)(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{1/2}\big). \tag{4.3}$$

Next, we apply [3, Corollary 6.2 (ii)] and (3.8), which follows from Proposition 3.2, to deduce

$$\begin{aligned}(I) &\leq C C_{1/2} N(N^2+1)^{1/2}\|\lambda\|_{L^\infty_{\mathrm{x}}L^2_{\mathrm{k}}}\|\mathcal{B}\|\int_s^t \mathrm{d}\tau\ \exp\{\tfrac{3}{2}\|\lambda\|_{L^\infty_{\mathrm{x}}L^2_{\mathrm{k}}}|\tau|\}\\ &\leq C\big|\exp\{\tfrac{3}{2}\|\lambda\|_{L^\infty_{\mathrm{x}}L^2_{\mathrm{k}}}t\} - \exp\{\tfrac{3}{2}\|\lambda\|_{L^\infty_{\mathrm{x}}L^2_{\mathrm{k}}}s\}\big|.\end{aligned} \tag{4.4}$$

The second term (II) is bounded by using again Proposition 3.2 and Lemma 4.1, and the fact that $e^{it(K_0+\nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega))}$ commutes with $\mathrm{d}\mathcal{G}_\varepsilon(1)$:

$$\begin{aligned}(II) &\leq \|\mathcal{B}\|\ \|(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-1/4}(W_\varepsilon(\eta)-W_\varepsilon(\xi))(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-1/4}\|\\ &\qquad\qquad\qquad\qquad \times \mathrm{Tr}\big(\Upsilon_\varepsilon(s)(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{1/2}\big)\\ &\leq \|\mathcal{B}\|\|(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-1/4}(W_\varepsilon(\eta)-W_\varepsilon(\xi))(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-1/4}\|\,\mathrm{Tr}\big(\Gamma_\varepsilon(s)(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{1/2}\big)\\ &\leq c_{1/4}C_{1/2}(N^2+1)^{1/2}\|\mathcal{B}\|(\min\{\|\eta\|_{\mathfrak{h}}^{1/2},\|\xi\|_{\mathfrak{h}}^{1/2}\}+1)e^{\frac{3}{2}\|\lambda\|_{L^\infty_{\mathrm{x}}L^2_{\mathrm{k}}}|s|}\|\eta-\xi\|_{\mathfrak{h}}^{1/2}.\end{aligned}$$

This concludes the proof. ∎

By means of Proposition 4.2, we are now in a position to prove the existence of a common subsequence, convergent for all times.

Proposition 4.3 (Existence of a converging subsequence). *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H}\otimes\mathcal{K}_\varepsilon)$ be such that there exists $\delta \geq 1/2$ satisfying*

$$\mathrm{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^\delta\big) \leq C.$$

Then, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$, there exists a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_{n_k}\to 0$ and a family $\{\mathfrak{n}_t\}_{t\in\mathbb{R}}$ of state-valued probability measures indexed by time such that for all $t\in\mathbb{R}$,

$$\Upsilon_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \mathfrak{n}_t. \tag{4.5}$$

Furthermore, for any $T>0$, there exists $C(T)>0$ such that, for any $t\in[-T,T]$ and any $\delta'\leq\delta$,

$$\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{n}_t}(z)\,(\|z\|_{\mathfrak{h}}^2+1)^{\delta'} \leq C(T). \tag{4.6}$$

Proof. Let $E := \{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ be a dense countable subset of $\mathbb{R}$, and let $\varepsilon_n \to 0$. Using a diagonal extraction argument, and Propositions 2.3 and 3.2 (see also Remark 3.3 and (3.8)), there exists a subsequence $\varepsilon_{n_k}\to 0$ such that for all $t_j\in E$,

$$\Upsilon_{\varepsilon_{n_k}}(t_j) \xrightarrow[k\to+\infty]{} \mathfrak{n}_{t_j}.$$

In addition, since $\|[\widehat{\Upsilon}_\varepsilon(t_j)](\eta)\|_{\mathcal{L}^1(\mathcal{H})} \leq 1$ for any $\eta \in \mathfrak{h}$ and $t_j \in E$, it follows that $\|\widehat{\mathfrak{n}}_{t_j}(\eta)\|_{\mathcal{L}^1(\mathcal{H})} \leq 1$, by Banach–Alaoglu's theorem. Furthermore, by Proposition 4.2, for any $t_j, t_\ell \in E$ and any $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$,

$$\left|\mathrm{tr}_{\mathcal{H}}\left[\left([\widehat{\Upsilon}_{\varepsilon_{n_k}}(t_j) - \widehat{\Upsilon}_{\varepsilon_{n_k}}(t_\ell)](\eta)\right)\mathcal{B}\right]\right| \leq C\left|e^{c|t_j|} - e^{c|t_\ell|}\right|,$$

where the constants on the right hand side are independent of k. Therefore, we can take the limit as $k \to +\infty$ of the above inequality, obtaining, for all $\mathcal{B} \in \mathcal{L}^\infty(\mathcal{H})$,

$$|\,\mathrm{tr}_{\mathcal{H}}[(\widehat{\mathfrak{n}}_{t_j}(\eta) - \widehat{\mathfrak{n}}_{t_\ell}(\eta))\mathcal{B}]| \leq C|e^{c|t_j|} - e^{c|t_\ell|}|. \tag{4.7}$$

Now, let $t \in \mathbb{R}$ be arbitrary. By density of $E \subset \mathbb{R}$, there exists a sequence $\{t_j\}_{j\in\mathbb{N}}$ of times in E such that $t_j \to t$. It follows that, for any $\eta \in \mathfrak{h}$, $\{\widehat{\mathfrak{n}}_{t_j}(\eta)\}_{j\in\mathbb{N}}$ is a weak-$*$ Cauchy sequence in the ultraweakly compact unit ball of the uniform space $\mathcal{L}^1(\mathcal{H})$. Thus, it converges when $t_j \to t$. Hence, we define

$$\widehat{\mathfrak{n}}_t(\eta) := \operatorname*{w\text{-}lim}_{j\to+\infty} \widehat{\mathfrak{n}}_{t_j}(\eta), \tag{4.8}$$

where the limit is meant in the weak-$*$ topology. For any $t \in \mathbb{R}$, $\eta \mapsto \widehat{\mathfrak{n}}_t(\eta)$ is an ultraweakly continuous function such that

- $\|\widehat{\mathfrak{n}}_t(0)\|_{\mathcal{L}^1(\mathcal{H})} = 1$;
- $\eta \mapsto \widehat{\mathfrak{n}}_t(\eta)$ is a function of completely positive type (see, e.g., [23, Definition A.7]).

Therefore, by Bochner's theorem for cylindrical vector measures [23, Theorem A.17], $\widehat{\mathfrak{n}}_t$ is the Fourier transform of a unique state-valued cylindrical probability measure $\mathfrak{n}_t$.

Furthermore, by approximating $\Upsilon_{\varepsilon_{n_k}}(t)$ with $\Upsilon_{\varepsilon_{n_k}}(t_j)$ and using the uniform equicontinuity of the non-commutative Fourier transform, one can prove that

$$\Upsilon_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \mathfrak{n}_t.$$

Here, we have used Proposition 2.6 to lift the convergence from the weak-$*$ to the weak topology. This in particular implies that $\mathfrak{n}_t$ is a probability Radon measure on $\mathfrak{h}$, because it is a Wigner measure of $\Upsilon_\varepsilon(t)$, satisfying the hypotheses of Proposition 2.3, thanks to Proposition 3.2.

To summarize, we have defined the common subsequence, and the family of state-valued probability measures obtained in the limit at any time. The last inequality (4.6) is finally proved by again combining Propositions 2.3 and 3.2. ∎

Once rewritten for the density matrix $\Gamma_\varepsilon(t)$, the result of Proposition 4.3 reads as follows:

Corollary 4.4. *If* $\lim_{\varepsilon\to0}\varepsilon\nu(\varepsilon) = \nu \in \mathbb{R}$, *then, under the hypotheses of Proposition 4.3, there exists a common subsequence* $\{n_k\}_{k\in\mathbb{N}}$ *such that, for any* $t \in \mathbb{R}$,

$$\Gamma_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \mathfrak{m}_t := e^{-it\mathcal{K}_0}((e^{-it\nu\omega})_\star\mathfrak{n}_t)e^{it\mathcal{K}_0}, \tag{4.9}$$

where $(e^{-it\nu\omega})_{\star}\mathfrak{n}_t$ *is the measure obtained by pushing forward* $\mathfrak{n}_t$ *by means of the unitary map* $e^{-it\nu\omega} : \mathfrak{h} \to \mathfrak{h}$*. Furthermore, for any* $T > 0$*, any* $t \in [-T, T]$ *and any* $\delta' \leq \delta$*,*

$$\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}_t}(z) \left(\|z\|_{\mathfrak{h}}^2 + 1\right)^{\delta'} \leq C(T), \tag{4.10}$$

where $C(T)$ *is as in* (4.6)*.*

Proof. The result trivially follows from Proposition 4.3 by identifying $e^{it\mathcal{K}_0}\mathcal{B}e^{-it\mathcal{K}_0}$ with $\mathcal{B} \in \mathcal{B}(\mathcal{H})$, as the bounded operator for the weak convergence, and using a very general result for linear symplectic maps, and their quantization as maps on algebras of canonical commutation relations [23, Proposition 6.1]. ∎

Thus, we have obtained a common convergent subsequence, and a map $t \mapsto \mathfrak{n}_t$ of quasi-classical Wigner measures. The next step is to characterize that dynamical map explicitly by means of a transport equation, and study the uniqueness properties of the latter. To do that, we study the convergence of the integral equation provided in Lemma 3.4.

Proposition 4.5 (Transport equation for $\mathfrak{n}(t)$). *Under the assumptions of Proposition* 4.3*, the family* $\{\mathfrak{n}_t\}_{t\in\mathbb{R}}$ *of state-valued probability measures as in* (4.5) *satisfies in the weak sense, i.e., when tested against any* $\mathcal{B} \in \mathcal{B}(\mathcal{H})$*, the integral equation*

$$\widehat{\mathfrak{n}}_t(\eta) = \widehat{\mathfrak{n}}_s(\eta) - i \int_s^t \mathrm{d}\tau \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{n}_\tau}(z) \left[\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)\right] e^{2i\,\mathrm{Re}\,\langle \eta | z\rangle_{\mathfrak{h}}}, \tag{4.11}$$

indexed by $\eta \in \mathfrak{h}$*, where*

$$\widetilde{\mathcal{V}}_\tau(\cdot) := e^{i\tau\mathcal{K}_0}\mathcal{V}(\cdot)e^{-i\tau\mathcal{K}_0} = \sum_{j=1}^{N} e^{i\tau\mathcal{K}_0} 2\,\mathrm{Re}\,\langle \lambda(\mathbf{x}_j)|\cdot\rangle_{\mathfrak{h}} e^{-i\tau\mathcal{K}_0} \tag{4.12}$$

is meant as a map from $\mathfrak{h}$ *to* $\mathcal{B}(\mathcal{H})$*.*

Proof. The existence of a common subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, $\varepsilon_{n_k} \to 0$, such that (4.5) holds true is guaranteed by Proposition 4.3. Let us now fix $s, t \in \mathbb{R}$; given the convergence along the subsequence at any time, it is possible to let $k \to \infty$ separately in all terms of the microscopic integral equation of evolution given in Lemma 3.4, traced against an arbitrary operator $\mathcal{B} \in \mathcal{B}(\mathcal{H})$.

For the integral term (second term on the right hand side of (3.16)), we make use of Propositions 2.6 and 3.2, where the latter is used to prove that $\Upsilon_\varepsilon(\tau)$ satisfies the hypotheses of the former for all $\tau \in [s, t]$, using $e^{-i\tau\mathcal{K}_0}\mathcal{B}e^{i\tau\mathcal{K}_0}$ as test operators. In order to do that, it is necessary to take the limit within the time integral. That is possible thanks to a dominated convergence argument, which makes use of the regularity assumption on Γ_ε: for any bounded operator $\mathcal{B}$, consider the integrand function

$$I(\tau) := \sum_{j=1}^{N} \mathrm{Tr}\Big(\big[\varphi_\varepsilon(e^{-i\tau\varepsilon\nu(\varepsilon)\omega}\lambda(\mathbf{x}_j)), \Upsilon_\varepsilon(\tau)\big]\big(e^{-i\tau\mathcal{K}_0}\mathcal{B}e^{i\tau\mathcal{K}_0} \otimes W_\varepsilon(\eta)\big)\Big).$$

Its absolute value is bounded, using standard Fock space estimates, as

$$|I(\tau)| \leq 2N\|\lambda\|_{L^\infty(\mathbb{R}^d;,\mathfrak{h})}\|\mathcal{B}\|\operatorname{Tr}\big(\Upsilon_\varepsilon(\tau)(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{1/2}\big).$$

Using Proposition 3.2 (see (3.8) and Remark 3.3) and the regularity assumption on Γ_ε, it follows that the right hand side of the above expression is uniformly bounded by a finite constant. Hence, $I(\tau)$ is integrable on any finite interval $[s,t]$, uniformly in ε. ■

5. Uniqueness for the quasi-classical transport equation

In this section we study the properties of the transport equation for state-valued measures obtained in Proposition 4.5 as the quasi-classical limit of the microscopic evolution of states.

The first technical point is discussed in Lemma 5.1 below, where it is proven that it is possible to exchange freely the two integrals of the aforementioned equation, which reads

$$\mathrm{d}\mathfrak{n}_t(z) = \mathrm{d}\mathfrak{n}(z) - i\int_s^t \mathrm{d}\tau\, \mathrm{d}\mu_{\mathfrak{n}_\tau}(z)\,[\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)], \tag{5.1}$$

or equivalently, using the Radon–Nikodým decomposition $\mathfrak{n}_t = (\mu_{\mathfrak{n}_t}, \gamma_{\mathfrak{n}_t}(z))$,

$$\gamma_{\mathfrak{n}_t}(z)\mathrm{d}\mu_{\mathfrak{n}_t}(z) = \gamma_{\mathfrak{n}_s}(z)\mathrm{d}\mu_{\mathfrak{n}_s}(z) - i\int_s^t \mathrm{d}\tau\, \mathrm{d}\mu_{\mathfrak{n}_\tau}(z)\,[\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)]. \tag{5.2}$$

Let us discuss the Bochner integrability of $[\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)]$ and justify the above statement.

Lemma 5.1. *Let $\{\mathfrak{n}_t\}_{t\in\mathbb{R}}$ be the family of state-valued measures as in Proposition 4.3. Then $[\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z), \gamma_{\mathfrak{n}_t}(z)]$ is Bochner $\mu_{\mathfrak{n}_t}$-integrable for any $t \in \mathbb{R}$, and the norm of the integral is uniformly bounded with respect to t on compact sets.*

Proof. By (4.6), we immediately see that

$$\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{n}_t}(z)\|z\|_{\mathfrak{h}} \leq C(t) \tag{5.3}$$

for all $t \in \mathbb{R}$ and some $C(t) < +\infty$. Moreover, for $\mu_{\mathfrak{n}_t}$-almost all $z \in \mathfrak{h}$, $\|\gamma_{\mathfrak{n}_t}(z)\|_{\mathcal{L}^1(\mathcal{H})} = 1$, so that

$$\|[\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z), \gamma_{\mathfrak{n}_t}(z)]\|_{\mathcal{L}^1(\mathcal{H})} \leq 2\|\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z)\| \leq 4N\|\lambda\|_{L^\infty(\mathbb{R}^d;\mathfrak{h})}\|z\|_{\mathfrak{h}},$$

which implies the result via (5.3). ■

From now on, we assume that we are considering a solution $t \mapsto \mathfrak{n}_t$ that satisfies (4.11). Let us introduce some terminology: a family of measures $t \mapsto \mathfrak{n}_t$ solving (4.11) in Proposition 4.5 for all $\eta \in \mathfrak{h}$ is called a *weak* or *weak-$*$ Fourier solution* if (4.11) holds true when tested against bounded or compact operators, respectively. Note that every weak or weak-$*$ Fourier solution is also a weak or weak-$*$ solution of (5.2), respectively, where the latter are solutions obtained by testing with smooth cylindrical scalar functions instead of Fourier characters. Let us further specify these last features. We first have to properly define the set of test cylindrical functions.

Definition 5.2 (Cylindrical functions). A function $f : \mathfrak{h} \to \mathbb{C}$ is a smooth and compactly supported *cylindrical function* over $\mathbb{P}\mathfrak{h}$, where $\mathbb{P}$ is an orthogonal projector and $\dim \mathbb{P}\mathfrak{h} < \infty$, if there exists $g \in C_0^\infty(\mathbb{P}\mathfrak{h})$ such that for all $z \in \mathfrak{h}$,

$$f(z) = g(\mathbb{P}z).$$

We denote by $C^\infty_{0,\mathrm{cyl}}(\mathfrak{h})$ the set of all smooth cylindrical functions.

Now, let $f \in C^\infty_{0,\mathrm{cyl}}(\mathfrak{h})$ and let $\hat{f} : \mathfrak{h} \to \mathbb{C}$ be its Fourier transform, also cylindrical over $\mathbb{P}\mathfrak{h}$, defined as

$$\hat{f}(\xi) = \int_{\mathbb{P}\mathfrak{h}} \mathrm{d}\mathbb{P}z \, e^{-2\pi i \operatorname{Re}\langle \xi | \mathbb{P}z \rangle} f(z) = \int_{\mathbb{P}\mathfrak{h}} \mathrm{d}\mathbb{P}z \, e^{-2\pi i \operatorname{Re}\langle \xi | \mathbb{P}z \rangle} g(\mathbb{P}z) = \hat{g}(\mathbb{P}\xi), \tag{5.4}$$

where $\mathrm{d}\mathbb{P}z$ stands for the Lebesgue measure on $\mathbb{P}\mathfrak{h}$. By testing (4.11) against a cylindrical function $\hat{g}(\mathbb{P}\xi)$, we get

$$\begin{aligned}\int_{\mathbb{P}\mathfrak{h}} \mathrm{d}\mathbb{P}\xi \, \hat{g}(\mathbb{P}\xi)\hat{\mathfrak{n}}_t(\mathbb{P}\xi) &= \int_{\mathbb{P}\mathfrak{h}} \mathrm{d}\mathbb{P}\xi \, \hat{g}(\mathbb{P}\xi)\hat{\mathfrak{n}}_s(\mathbb{P}\xi) \\ &\quad - i \int_{\mathbb{P}\mathfrak{h}} \mathrm{d}\mathbb{P}\xi \, \hat{g}(\mathbb{P}\xi) \int_s^t \mathrm{d}\tau \int_{\mathfrak{h}} \mathrm{d}\mathfrak{n}_\tau(z) \, [\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)] e^{2\pi i \operatorname{Re}\langle \mathbb{P}\xi | z \rangle_{\mathfrak{h}}}.\end{aligned}$$

Hence, it follows that, for any $f \in C^\infty_{0,\mathrm{cyl}}(\mathfrak{h})$,

$$\int_{\mathfrak{h}} \mathrm{d}\mathfrak{n}_t(z) \, f(z) = \int_{\mathfrak{h}} \mathrm{d}\mathfrak{n}_s(z) \, f(z) - i \int_s^t \mathrm{d}\tau \int_{\mathfrak{h}} \mathrm{d}\mathfrak{n}_\tau(z) \, f(z) [\widetilde{\mathcal{V}}_\tau(e^{-i\tau\nu\omega}z), \gamma_{\mathfrak{n}_\tau}(z)]. \tag{5.5}$$

Now, fix $s \in \mathbb{R}$ as the initial time, and the corresponding $\mathfrak{n}_s \equiv \mathfrak{n}$ as the initial datum. Then the following map $t \mapsto \mathfrak{n}_t$ is easily checked to be both a weak and weak-$*$ solution of (5.2):

$$t \mapsto (\mu_{\mathfrak{n}_t}, \gamma_{\mathfrak{n}_t}(z)) := (\mu_{\mathfrak{n}}, \widetilde{\mathcal{U}}_{t,s}(z)\gamma_{\mathfrak{n}}(z)\widetilde{\mathcal{U}}^\dagger_{t,s}(z)), \tag{5.6}$$

where $\widetilde{\mathcal{U}}_{t,s}(z)$ is the two-parameter unitary group on $\mathcal{H}$ generated by the time-dependent generator $\widetilde{\mathcal{V}}_\tau(e^{-it\nu\omega}z) \in \mathcal{L}(L^2)$. Note that this evolution two-parameter group exists for all $z \in \mathfrak{h}$ and $t \in \mathbb{R}$, since the $\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z)$ are bounded operators on $\mathcal{H}$ (see, e.g., [49]). Furthermore, the solution given by (5.6) satisfies (5.3) at all times, provided the inequality is satisfied by the initial datum.

It just remains to prove the solution in (5.6) is actually *unique*. This of course might depend on the notion of solution we adopt, but proving weak-$*$ uniqueness, we also get uniqueness for stronger solutions (weak, Fourier weak-$*$, and Fourier weak). As a matter of fact, the proof of uniqueness is actually independent of the notion of solution considered.

Proposition 5.3 (Uniqueness for the transport equation for $\mathfrak{n}_t$). *Let $s \in \mathbb{R}$ be the fixed initial time, and let $\mathfrak{n}_s \equiv \mathfrak{n} \in \mathcal{M}(\mathfrak{h}; \mathcal{H})$ be a Borel state-valued measure such that*

$$\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{n}}(z) \, \|z\|_{\mathfrak{h}} < C.$$

Then the integral transport equation (5.1) admits a unique weak-$*$ solution $\mathfrak{n}_t$ that satisfies (5.3), defined by its norm Radon–Nikodým decomposition

$$(\mu_{\mathfrak{n}_t}, \gamma_{\mathfrak{n}_t}(z)) = (\mu_{\mathfrak{n}}, \widetilde{\mathcal{U}}_{t,s}(z)\gamma_{\mathfrak{n}}(z)\widetilde{\mathcal{U}}^{\dagger}_{t,s}(z)). \tag{5.7}$$

This solution is continuous and differentiable on every Borel set in the strong topology of $\mathcal{L}^1(\mathcal{H})$ and its derivative $\partial_t \mathfrak{n}_t$ is a self-adjoint but in general not positive state-valued measure.

Proof. Any weak solution $\mathfrak{n}_t$ of the transport equation (5.1) or (5.2) satisfying (5.3) is continuous and can be weakly differentiated with respect to time on Borel sets. However, given the structure of equation (5.2), it is easy to realize that the derivative actually exists in the strong topology of $\mathcal{L}^1(\mathcal{H})$ and reads

$$\mathrm{d}\partial_t \mathfrak{n}_t(z) = -i[\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z), \gamma_{\mathfrak{n}_t}(z)]\mathrm{d}\mu_{\mathfrak{n}_t}(z). \tag{5.8}$$

To prove uniqueness, suppose that $\mathfrak{n}_t$ is a solution satisfying (5.3). Since we already know that (5.6) solves the equation, it is sufficient to prove that $\mathfrak{n}_t$ admits the Radon–Nikodým decomposition (5.6) (recall Proposition 2.2). In order to do that, let us set

$$\mathrm{d}\tilde{\mathfrak{n}}_t(z) := \widetilde{\mathcal{U}}^{\dagger}_{t,s}(z)\gamma_{\mathfrak{n}_t}(z)\widetilde{\mathcal{U}}_{t,s}(z)\mathrm{d}\mu_{\mathfrak{n}_t}(z),$$

so that, using (5.2) once more, we get

$$\begin{aligned} i\,\mathrm{d}\partial_t \tilde{\mathfrak{n}}_t(z) = {}& \widetilde{\mathcal{U}}^{\dagger}_{t,s}(z)[\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z), \gamma_{\mathfrak{n}_t}(z)]\widetilde{\mathcal{U}}_{t,s}(z)\mathrm{d}\mu_{\mathfrak{n}_t}(z) \\ & - \widetilde{\mathcal{U}}^{\dagger}_{t,s}(z)[\widetilde{\mathcal{V}}_t(e^{-it\nu\omega}z), \gamma_{\mathfrak{n}_t}(z)]\widetilde{\mathcal{U}}_{t,s}(z)\mathrm{d}\mu_{\mathfrak{n}_t}(z) = 0. \end{aligned}$$

Hence, $\tilde{\mathfrak{n}}_t = \tilde{\mathfrak{n}}_s = \mathfrak{n}$. Therefore, $\mathfrak{n}_t$ has indeed the norm Radon–Nikodým decomposition (5.6). ■

6. Putting it all together: Proof of Theorem 1.6

It is now possible to combine the results obtained in §§2 to 5, and thus prove Theorem 1.6. We first state and prove the result for the evolution in the interaction picture and under a stronger assumption on the initial datum, and then complete the proof by relaxing it and going back to the evolution for $\Gamma_\varepsilon(t)$.

Proposition 6.1 (Quasi-classical evolution in the interaction picture). *Let $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ be such that there exists $\delta > 1/2$ satisfying*

$$\mathrm{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^{\delta}\big) \le C.$$

If $\Gamma_{\varepsilon_n} \to \mathfrak{m}$, *then*

$$\Upsilon_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{n}_t, \quad \forall t \in \mathbb{R}, \tag{6.1}$$

where

$$\mathfrak{n}_t = (\mu_{\mathfrak{m}}, \widetilde{\mathcal{U}}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\widetilde{\mathcal{U}}^{\dagger}_{t,0}(z)).$$

Proof. By Proposition 4.3, there exists a common subsequence $\varepsilon_{n_k} \to 0$ such that for all $t \in \mathbb{R}$,

$$\Upsilon_{\varepsilon_{n_k}}(t) \xrightarrow[k\to+\infty]{} \mathfrak{n}_t.$$

Clearly, $\mathfrak{n}_0 = \mathfrak{m}$. Moreover, by Proposition 4.5, $\mathfrak{n}_t$ is also a weak solution of (5.2) satisfying Lemma 5.1 and (5.3). The weak solution of (5.2) satisfying (5.3) is however unique by Proposition 5.3, and therefore $\mathfrak{n}_t$ has the Radon–Nikodým decomposition

$$(\mu_{\mathfrak{m}}, \widetilde{\mathcal{U}}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\widetilde{\mathcal{U}}^{\dagger}_{t,0}(z)).$$

We now show that the convergence holds at any time along the original subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Let us take a convergent subsequence of $\Upsilon_{\varepsilon_n}(t)$ at an arbitrary time t, i.e.,

$$\Upsilon_{\varepsilon_{n_j}}(t) \xrightarrow[j\to+\infty]{} \mathfrak{n}'_t. \tag{6.2}$$

Then, by convergence at time $t = 0$, we immediately see that

$$\Upsilon_{\varepsilon_{n_j}}(0) \xrightarrow[j\to+\infty]{} \mathfrak{m}.$$

Hence, we can proceed as in the proof of Proposition 4.3 and extract a subsequence $\{\varepsilon_{n_{j_k}}\}_{k\in\mathbb{N}}$ of $\{\varepsilon_{n_j}\}_{j\in\mathbb{N}}$ such that we have convergence at any time. Furthermore, again by Proposition 4.5, the limit points $\mathfrak{n}'_t$ are weak solutions of the transport equation. Therefore, by uniqueness of the solution, $\mathfrak{n}'_t = \mathfrak{n}_t$. Hence, all the convergent subsequences of $\Upsilon_{\varepsilon_n}(t)$ have the same limit point, which implies that $\Upsilon_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{n}_t$. ■

The analogue for Γ_ε of Proposition 6.1 is

Corollary 6.2. *Under the hypotheses of Proposition* 6.1, *for any* $t \in \mathbb{R}$,

$$\Gamma_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{m}_t = e^{-it\mathcal{K}_0}((e^{-it\nu\omega})_\star \mathfrak{n}_t)e^{it\mathcal{K}_0}. \tag{6.3}$$

Proof. In view of Proposition 6.1, this is a direct consequence of Corollary 4.4. ■

The proof of Theorem 1.6 is almost complete; it remains only to extend the result to states $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ satisfying the weaker condition that there exist $\delta > 0$ and $C < +\infty$ such that

$$\operatorname{Tr}\big(\Gamma_\varepsilon(\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta\big) \le C. \tag{6.4}$$

This is done by standard approximation techniques, using an argument originally proposed in [8, §2] (see also [3, §4.5]). Let us briefly reproduce the key ideas here. Let Γ_ε satisfy (6.4), and define

$$\Gamma_\varepsilon^{(r)} := \frac{\chi_r(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)\Gamma_\varepsilon\chi_r(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)}{\mathrm{Tr}(\chi_r(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)\Gamma_\varepsilon\chi_r(\mathrm{d}\mathcal{G}_\varepsilon(1)+1))} \in \mathcal{L}^1_{+,1}(\mathcal{H}\otimes\mathcal{K}_\varepsilon), \tag{6.5}$$

where $r>0$ and $\chi_r(\cdot)=\chi(\cdot/r)$, $\chi\in C_0^\infty(\mathbb{R})$, with $0\le\chi\le 1$ and $\chi=1$ in a neighborhood of zero. By functional calculus and (6.4), for any $t\in\mathbb{R}$,

$$\|\Gamma_\varepsilon(t)-\Gamma_\varepsilon^{(r)}(t)\|_{\mathcal{L}^1_{+,1}(\mathcal{H}\otimes\mathcal{K}_\varepsilon)} = \|\Gamma_\varepsilon-\Gamma_\varepsilon^{(r)}\|_{\mathcal{L}^1_{+,1}(\mathcal{H}\otimes\mathcal{K}_\varepsilon)} = o_r(1) \quad \text{as } r\to\infty,$$

uniformly in $\varepsilon\in(0,1)$. In addition, $\Gamma_\varepsilon^{(r)}$ satisfies the assumptions of Proposition 6.1. Suppose now that $\Gamma_{\varepsilon_n}\to\mathfrak{m}$, and for all $r>0$, let $\varepsilon_{n_k(r)}\to 0$ be a subsequence and $\mathfrak{m}^{(r)}$ a state-valued measure such that $\Gamma^{(r)}_{\varepsilon_{n_k(r)}}\to\mathfrak{m}^{(r)}$. Then, by Corollary 6.2, $\Gamma^{(r)}_{\varepsilon_{n_k(r)}}(t)\to\mathfrak{m}_t^{(r)}$ for any $t\in\mathbb{R}$, where the latter is defined by Theorem 1.6 with $\mathfrak{m}^{(r)}$ in place of $\mathfrak{m}$. Finally, let us extract a subsequence $\varepsilon_{n_{k_\ell}(r,t)}\to 0$ such that $\Gamma_{\varepsilon_{n_{k_l}(r,t)}}(t)\to\boldsymbol{\nu}_t$. By adapting the argument in [8, Proposition 2.10] to state-valued measures, we find that for any fixed $t\in\mathbb{R}$,

$$\int_{\mathfrak{h}} \mathrm{d}|\mu_{\boldsymbol{\nu}_t}-\mu_{\mathfrak{m}_t^{(r)}}| = o_r(1),$$

where $|\mu_{\boldsymbol{\nu}_t}-\mu_{\mathfrak{m}_t^{(r)}}|$ is the scalar measure in the norm Radon–Nikodým decomposition of the total variation of the signed state-valued measure $\boldsymbol{\nu}_t-\mathfrak{m}_t^{(r)}$ [23, §A.3], i.e., the sum of its positive and negative parts. Hence, denoting by $\mathfrak{m}_t$ the measure appearing in Theorem 1.6, we have

$$\int_{\mathfrak{h}} \mathrm{d}|\mu_{\boldsymbol{\nu}_t}-\mu_{\mathfrak{m}_t}| \le \int_{\mathfrak{h}} \mathrm{d}|\mu_{\boldsymbol{\nu}_t}-\mu_{\mathfrak{m}_t^{(r)}}| + \int_{\mathfrak{h}} \mathrm{d}|\mu_{\mathfrak{m}_t^{(r)}}-\mu_{\mathfrak{m}_t}| = o_r(1).$$

Therefore, $\boldsymbol{\nu}_t\equiv\mathfrak{m}_t$. Since any subsequence extraction yields the same result, it follows that, for all $t\in\mathbb{R}$, $\Gamma_{\varepsilon_n}(t)\to\mathfrak{m}_t$, thus concluding the proof of Theorem 1.6.

Corollary 1.14 is then a trivial application of the definition of w-$*$ convergence and so we omit the proof. It only remains to prove Corollary 1.7 and Theorem 1.16.

Proof of Corollary 1.7. We first of all remark that the quasi-classical evolution $t\mapsto\mathfrak{m}_t$ preserves the mass, i.e.,

$$\forall t,t'\in\mathbb{R},\quad \|\mathfrak{m}_t(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})} = \|\mathfrak{m}_{t'}(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})}.$$

Therefore, for the first part of the statement, it suffices to prove that, under assumption (A2) or (A2′), $\|\mathfrak{m}(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})}=1$.

In the case of assumption (A2), we can actually show that quasi-classical convergence can be lifted to bounded quasi-classical convergence. In fact, let $\mathcal{B}\in\mathcal{B}(\mathcal{H})$ and consider

$$\mathrm{tr}_{\mathcal{H}}(\hat{\Gamma}_\varepsilon(\eta)\mathcal{B}) = \mathrm{tr}_{\mathcal{H}}(\hat{\Gamma}_{\varepsilon,A}(\eta)\mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2}),$$

where

$$\hat{\Gamma}_{\varepsilon,A}(\eta) = \mathrm{tr}_{\mathcal{K}_\varepsilon}(A^{1/2}\Gamma_\varepsilon A^{1/2} W_\varepsilon(\eta))$$

is the Fourier transform of the state $A^{1/2}\Gamma_\varepsilon A^{1/2} \in \mathcal{L}^1_+(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ by (A2). Therefore, on the one hand, by Proposition 2.5, if $\Gamma_{\varepsilon_n} \to \mathfrak{m}$, then

$$A^{1/2}\Gamma_\varepsilon A^{1/2} \to \mathcal{A}^{1/2}\mathfrak{m}\mathcal{A}^{1/2};$$

on the other hand, since $\mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2} \in \mathcal{L}^\infty(\mathcal{H})$, Theorem 1.6 guarantees that

$$\begin{aligned} \mathrm{tr}_{\mathcal{H}}(\hat{\Gamma}_{\varepsilon_n}(\eta)\mathcal{B}) &= \mathrm{tr}_{\mathcal{H}}(\hat{\Gamma}_{\varepsilon_n,A}(\eta)\mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2}) \\ &\xrightarrow[n\to+\infty]{} \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}\, e^{2i\,\mathrm{Re}\langle \eta | z\rangle_{\mathfrak{h}}}\, \mathrm{tr}_{\mathcal{H}}(\mathcal{A}^{1/2}\gamma_{\mathfrak{m}}(z)\mathcal{A}^{1/2}\mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2}) \\ &= \mathrm{tr}_{\mathcal{H}}(\hat{\mathfrak{m}}(\eta)\mathcal{B}). \end{aligned} \tag{6.6}$$

Choosing $\eta = 0$ and $\mathcal{B} = \mathbb{1}$ one concludes that $\|\mathfrak{m}(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})} = 1$.

The proof assuming (A2$'$) uses a dominated convergence argument. Let us denote by $\{\mathcal{P}_k\}_{k\in\mathbb{N}} \subset \mathcal{L}^\infty(\mathcal{H})$ the projections onto the eigenspaces of $\mathfrak{m}(\mathfrak{h})$, i.e.,

$$\mathfrak{m}(\mathfrak{h}) = \sum_{k\in\mathbb{N}} m_k \mathcal{P}_k,$$

$m_k \in \mathbb{R}^+$ being the associated eigenvalues. Then, for any $k \in \mathbb{N}$, we can apply Theorem 1.6 to get

$$\mathrm{Tr}(\Gamma_{\varepsilon_n} P_k) = \mathrm{tr}_{\mathcal{H}}(\gamma_{\varepsilon_n}\mathcal{P}_k) \xrightarrow[n\to+\infty]{} \mathrm{tr}_{\mathcal{H}}(\mathfrak{m}(\mathfrak{h})\mathcal{P}_k) = \alpha_k m_k, \tag{6.7}$$

where $\alpha_k \in \mathbb{N}$ is the multiplicity of the eigenvalue m_k of $\mathfrak{m}(\mathfrak{h})$. Hence, by (A2$'$),

$$\begin{aligned} 1 = \mathrm{Tr}(\Gamma_\varepsilon) = \mathrm{tr}_{\mathcal{H}}(\gamma_\varepsilon) &= \sum_{k\in\mathbb{N}} \mathrm{tr}_{\mathcal{H}}(\gamma_\varepsilon\mathcal{P}_k) \\ &\leq \sum_{k\in\mathbb{N}} \mathrm{tr}_{\mathcal{H}}(\gamma\mathcal{P}_k) = \mathrm{tr}_{\mathcal{H}}(\gamma) < +\infty. \end{aligned}$$

Therefore, by dominated convergence,

$$\begin{aligned} 1 &= \lim_{n\to+\infty} \mathrm{Tr}(\Gamma_{\varepsilon_n}) = \lim_{n\to+\infty} \sum_{k\in\mathbb{N}} \mathrm{tr}_{\mathcal{H}}(\gamma_{\varepsilon_n}\mathcal{P}_k) = \sum_{k\in\mathbb{N}} \lim_{n\to+\infty} \mathrm{tr}_{\mathcal{H}}(\gamma_{\varepsilon_n}\mathcal{P}_k) \\ &= \sum_{k\in\mathbb{N}} \alpha_k m_k = \|\mathfrak{m}(\mathfrak{h})\|_{\mathcal{L}^1(\mathcal{H})}. \end{aligned}$$

∎

Proof of Theorem 1.16. First of all, assumption (A_δ) is propagated in time by means of Proposition 3.2. In addition, the measure $\mathfrak{m}_t$ is characterized by Theorem 1.6 at any time $t \in \mathbb{R}$. Finally, the convergence of the expectations of the Wick quantizations of symbols $\mathcal{F} \in \mathcal{S}_{\ell,m}$, under condition ($A_\delta$), is given by Proposition 2.6. Combining the above ingredients proves Theorem 1.16. ∎

7. Technical modifications for Pauli–Fierz and polaron models

Theorem 1.6 is stated not only for the regularized Nelson model, but also for the Pauli–Fierz and polaron models. The strategy of the proof for these cases is identical to the one followed above for the Nelson model. However, one has to overcome some technical difficulties related to the fact that such models are “more singular”. In particular, the major difficulty is due to the presence of terms of type $\nabla \cdot a_\varepsilon^\#(\boldsymbol{\lambda}(\mathbf{x}))$ and their adjoints in the microscopic Hamiltonian H_ε. In this connection, one needs to propagate in time some further regularity of quantum states, in addition to what is done in Proposition 3.2 for the Nelson model. Finally, some care has to be taken in defining the effective limit dynamics $\mathcal{U}_{t,s}(z)$. We comment below on the technical adaptations needed to take care of such difficulties.

7.1. Quasi-classical analysis of gradient terms

In order to deal with terms of the form $\nabla \cdot a_\varepsilon^\#(\boldsymbol{\lambda}(\mathbf{x}))$ with $\boldsymbol{\lambda} \in L^\infty(\mathbb{R}^d; \mathfrak{h}^d)$, one needs to extend the convergence proven in Proposition 2.6 to such observables. This is done in two steps: first, it is possible to restrict the set of test observables using the set $\mathfrak{K}$ defined in Lemma 2.4, for it separates points, and then prove that with that restriction the expectation values indeed converge (Proposition 7.1). In particular, Lemma 2.4 is used below for the convergence of gradient terms, to solve possible domain ambiguities whenever the gradient acts on the test operator: we end up with a form of the integral transport equation for the measure that holds only when tested with particle observables in $\mathfrak{K}$ (recall (2.16)), setting $\mathcal{T} = \mathcal{K}_0$, where $\mathcal{K}_0$ is the self-adjoint free particle Hamiltonian. With such testing it still makes sense to study uniqueness of the solution, since the aforementioned set separates points.

Let us now consider the convergence of the expectation value of the gradient term. Let us recall that $a_\varepsilon^\#(f)$ stands for either $a_\varepsilon(f)$ or $a_\varepsilon^\dagger(f)$, and correspondingly $\langle f|z\rangle_{\mathfrak{h}}^\#$ stands for either $\langle f|z\rangle_{\mathfrak{h}}$ or $\langle z|f\rangle_{\mathfrak{h}}$. Let us recall that in all the concrete models considered, $\mathcal{K}_0 \geq p > -\infty$, and

$$|\nabla|(\mathcal{K}_0 + 1 - p)^{-1/2} \in \mathcal{B}(\mathcal{H}). \tag{7.1}$$

Proposition 7.1 (Convergence of expectation values of gradient terms). *Let* $\Gamma_\varepsilon \in \mathcal{L}^1_{+,1}(\mathcal{H} \otimes \mathcal{K}_\varepsilon)$ *be such that there exists* $\delta > 1$ *such that*

$$\mathrm{Tr}\big(\Gamma_\varepsilon(K_0 - p + (\mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta)\big) \leq C. \tag{7.2}$$

If $\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m}$, *then for any* $\mathcal{B} \in \mathfrak{K}$, *any* $\alpha, \beta \in \mathbb{R}$ *and all* $\eta \in \mathfrak{h}$,

$$\begin{aligned}\lim_{n\to+\infty} \mathrm{Tr}\Big(\Gamma_{\varepsilon_n}\big(\alpha\nabla \cdot a_{\varepsilon_n}^\#(\boldsymbol{\lambda}(\mathbf{x})) + \beta a_{\varepsilon_n}^\#(\boldsymbol{\lambda}(\mathbf{x})) \cdot \nabla\big)(\mathcal{B} \otimes W_{\varepsilon_n}(\eta))\Big) \\ = \mathrm{tr}_{\mathcal{H}}\bigg(\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z)\big(\alpha\nabla \cdot \langle\boldsymbol{\lambda}(\mathbf{x})|z\rangle_{\mathfrak{h}}^\# + \beta\langle\boldsymbol{\lambda}(\mathbf{x})|z\rangle_{\mathfrak{h}}^\# \cdot \nabla\big)e^{2i\,\mathrm{Re}\langle\eta|z\rangle_{\mathfrak{h}}}\mathcal{B}\bigg),\end{aligned} \tag{7.3}$$

$$\lim_{n\to+\infty} \operatorname{Tr}\Big(\big(\alpha\nabla\cdot a^{\#}_{\varepsilon_n}(\boldsymbol{\lambda}(\mathbf{x})) + \beta a^{\#}_{\varepsilon_n}(\boldsymbol{\lambda}(\mathbf{x}))\cdot\nabla\big)\Gamma_{\varepsilon_n}(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\Big)$$
$$= \operatorname{tr}_{\mathcal{H}}\bigg(\bigg(\int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\,\big(\alpha\nabla\cdot\langle\boldsymbol{\lambda}(\mathbf{x})|z\rangle^{\#}_{\mathfrak{h}} + \beta\langle\boldsymbol{\lambda}(\mathbf{x})|z\rangle^{\#}_{\mathfrak{h}}\cdot\nabla\big)e^{2i\operatorname{Re}\langle\eta|z\rangle_{\mathfrak{h}}}\gamma_{\mathfrak{m}}(z)\mathcal{B}\bigg). \tag{7.4}$$

Proof. We prove the result for $\Gamma_\varepsilon(\nabla\cdot a_\varepsilon(\boldsymbol{\lambda}(\mathbf{x})) + a_\varepsilon(\boldsymbol{\lambda}(\mathbf{x}))\cdot\nabla)$, the other cases being perfectly analogous.

First of all, we observe that $(K_0-p)^{1/2}\Gamma_{\varepsilon_n}(K_0-p)^{1/2}$ is a positive operator and we can consider its quasi-classical convergence as $n\to+\infty$: by Proposition 2.5,

$$(K_0-p)^{1/2}\Gamma_{\varepsilon_n}(K_0-p)^{1/2} \xrightarrow[n\to+\infty]{} (K_0-p)^{1/2}\mathfrak{m}(K_0-p)^{1/2}. \tag{7.5}$$

The term $\Gamma_\varepsilon a_\varepsilon(\boldsymbol{\lambda}(\mathbf{x}))\cdot\nabla$, in which the gradient acts directly on $\mathcal{B}$, converges by Proposition 2.6, since $\partial_j\mathcal{B}\in\mathcal{L}^\infty(\mathcal{H})$ for all $j=1,\dots,d$ and $\mathcal{B}\in\mathcal{L}^\infty(\mathcal{H})$.

It remains to discuss the term $\Gamma_\varepsilon\nabla\cdot a_\varepsilon(\boldsymbol{\lambda}(\mathbf{x}))$. This term requires suitable approximations. First of all, let us approximate each operator-valued symbol

$$F_j^{(\boldsymbol{\lambda})}(z) := \langle\lambda_j(\mathbf{x})|z\rangle_{\mathfrak{h}}, \quad j=1,\dots,d,$$

by means of Lemma 2.7, and denote its approximation by $F^{(\boldsymbol{\lambda})}_{j,M}$. It follows that, using estimates analogous to the ones used in the proof of Proposition 2.6,

$$\bigg|\sum_{j=1}^{d}\operatorname{Tr}\big(\Gamma_\varepsilon\partial_j[a_\varepsilon(\lambda_\mu(\mathbf{x})) - \operatorname{Op}^{\mathrm{Wick}}_\varepsilon(F^{(\boldsymbol{\lambda})}_{j,M})](\mathcal{B}\otimes W_\varepsilon(\eta))\big)\bigg| \le C\|\mathcal{B}\|\sum_{j=1}^{d}\|F^{(\boldsymbol{\lambda})}_j - F^{(\boldsymbol{\lambda})}_{j,M}\|,$$

and the right hand side does not depend on ε, and converges to zero as $M\to+\infty$. In addition, let us recall that the symbol $F^{(\boldsymbol{\lambda})}_{j,M}$ has the form

$$F^{(\boldsymbol{\lambda})}_{j,M} = \sum_{k=1}^{J(M)}\langle\varphi_{j,k}|z\rangle_{\mathfrak{h}}\mathbb{1}_{B_k}(\mathbf{x}),$$

where $J(M)\in\mathbb{N}$, $\varphi_{j,k}\in\mathfrak{h}$, and $B_k\subseteq\mathbb{R}^d$ is a Borel set. Let us consider the convergence as $\varepsilon_n\to 0$ of each term of the above sums separately, for M fixed. In other words, let us consider the convergence of

$$\operatorname{Tr}\big(\Gamma_{\varepsilon_n}\partial_j a_{\varepsilon_n}(\varphi_{j,k})\mathbb{1}_{B_k}(\mathbf{x})(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big)$$
$$= \operatorname{tr}_{\mathcal{H}}\big\{\operatorname{tr}_{\mathcal{K}_\varepsilon}[\Gamma_{\varepsilon_n}a_{\varepsilon_n}(\varphi_{j,k})W_{\varepsilon_n}(\eta)]\partial_j\mathbb{1}_{B_k}(\mathbf{x})\mathcal{B}\big\}.$$

The operator $a_\varepsilon(\varphi_{\mu,k})W_\varepsilon(\eta)$ is the product of the Weyl quantizations of two cylindrical albeit not compactly supported symbols, over the complex Hilbert subspace spanned by $\varphi_{j,k}$ and η. Therefore, by finite-dimensional pseudodifferential calculus, for all M there exists a smooth compactly supported scalar symbol $F^{(j,k,\eta)}_\sigma\in C^\infty_{0,\mathrm{cyl}}(\mathfrak{h})$ such that, for any $\delta'>1/2$,

$$\|[a_\varepsilon(\varphi_{j,k})W_\varepsilon(\eta) - \operatorname{Op}^{\mathrm{Weyl}}_\varepsilon(F^{(j,k,\eta)}_\sigma)](\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-\delta'}\|_{\mathcal{B}(\mathcal{K}_\varepsilon)} = o_\sigma(1) + o_\varepsilon(1). \tag{7.6}$$

Hence, the Cauchy–Schwarz inequality, (7.2) and (7.1) yield

$$\begin{aligned}
&\big|\mathrm{Tr}\big(\Gamma_{\varepsilon_n}\partial_j\big(a_{\varepsilon_n}(\varphi_{j,k})-\mathrm{Op}_\varepsilon^{\mathrm{Weyl}}(F_\sigma^{(j,k,\eta)})\big)\mathbb{1}_{B_k}(\mathbf{x})(\mathcal{B}\otimes W_{\varepsilon_n}(\eta))\big)\big|\\
&\quad\le C\|\mathcal{B}\|\,\|[a_\varepsilon(\varphi_{j,k})W_\varepsilon(\eta)-\mathrm{Op}_\varepsilon^{\mathrm{Weyl}}(F_\sigma^{(j,k,\eta)})](\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-\delta'}\|_{\mathcal{B}(\mathcal{K}_\varepsilon\otimes\mathcal{H})}\\
&\qquad\qquad\times\big|\mathrm{tr}\big((\mathcal{K}_0+1-p)^{1/2}(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{\delta'}\Gamma_{\varepsilon_n}\big)\big|\\
&\quad\le C\|[a_\varepsilon(\varphi_{j,k})W_\varepsilon(\eta)-\mathrm{Op}_\varepsilon^{\mathrm{Weyl}}(F_\sigma^{(j,k,\eta)})](\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{-\delta'}\|_{\mathcal{B}(\mathcal{K}_\varepsilon\otimes\mathcal{H})}\\
&\qquad\qquad\times\big|\mathrm{tr}\big[\big(\mathcal{K}_0+1-p+(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^{2\delta'}\big)\Gamma_{\varepsilon_n}\big]\big|=o_\sigma(1)+o_\varepsilon(1).
\end{aligned}\tag{7.7}$$

In addition, for all $\delta'>1/2$,

$$\big(\langle\varphi_{j,k}|z\rangle_{\mathfrak{h}}e^{2i\,\mathrm{Re}\,\langle\eta|z\rangle_{\mathfrak{h}}}-F_\sigma^{(j,k,\eta)}(z)\big)(\|z\|_{\mathfrak{h}}^2+1)^{-\delta'}=o_\sigma(1)\tag{7.8}$$

uniformly in $z\in\mathfrak{h}$. We can now take the limit as $\varepsilon_n\to0$ of the remaining term

$$\begin{aligned}
&\mathrm{Tr}\big(\Gamma_{\varepsilon_n}\partial_j\mathrm{Op}_\varepsilon^{\mathrm{Weyl}}(F_\sigma^{(j,k,\eta)})\mathbb{1}_{B_k}(\mathbf{x})B\big)\\
&\quad=\mathrm{tr}_{\mathcal{H}}\Big(\mathrm{tr}_{\Gamma_{\mathrm{s}}}\big((K_0-p)^{1/2}\Gamma_{\varepsilon_n}(K_0-p)^{1/2}\mathrm{Op}_\varepsilon^{\mathrm{Weyl}}(F_\sigma^{(j,k,\eta)})\big)(K_0-p)^{-1/2}\partial_j\mathbb{1}_{B_k}(\mathbf{x})\\
&\qquad\qquad\times\mathcal{B}(K_0-p)^{-1/2}\Big).
\end{aligned}$$

Since the symbol $F_\sigma^{(j,k,\eta)}$ is in $C_{0,\mathrm{cyl}}^\infty(\mathfrak{h})$, this converges to (see [23] for further details)

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{H}}\Big(\int_{\mathfrak{h}}\mathrm{d}\mu_{\mathfrak{m}}(z)\,(K_0-p)^{1/2}\gamma_{\mathfrak{m}}(z)(K_0-p)^{1/2}F_\sigma^{(j,k,\eta)}(z)(K_0-p)^{-1/2}\partial_j\mathbb{1}_{B_k}(\mathbf{x})\\
&\qquad\qquad\times\mathcal{B}(K_0-p)^{-1/2}\Big)\\
&\quad=\mathrm{tr}_{\mathcal{H}}\Big(\int_{\mathfrak{h}}\mathrm{d}\mu_{\mathfrak{m}}(z)\,\gamma_{\mathfrak{m}}(z)(K_0-p)^{1/2}F_\sigma^{(j,k,\eta)}(z)(K_0-p)^{-1/2}\partial_j\mathbb{1}_{B_k}(\mathbf{x})\mathcal{B}\Big).
\end{aligned}$$

The limit as $\sigma\to+\infty$ can then be taken by dominated convergence, at fixed M, thanks to the uniform bound for $\delta>1$,

$$\int_{\mathfrak{h}}\mathrm{d}\mu_{\mathfrak{m}}(z)\,\big(\mathrm{tr}_{\mathcal{H}}(\gamma_{\mathfrak{m}}(z)(K_0-p))+(\|z\|_{\mathfrak{h}}^2+1)^{\delta}\big)\le C,\tag{7.9}$$

which also allows us to take the limit as $M\to+\infty$. ■

7.2. *Propagation estimates and the pull-through formula*

In this section we discuss the so-called pull-through formula, needed to characterize the dynamics in the quasi-classical limit for the polaron model; as we are going to see, the pull-through formula is key to propagate the a priori bounds on the initial state at later times. The formula holds for the massive Nelson and the polaron model, therefore H_ε in this section stands for any of the Hamiltonians defined above, although it is not needed for the Nelson model with ultraviolet cutoff, as considered in this paper. Indeed, in that case, one can simply use the propagation estimates of Proposition 3.2, valid also in the massless case.

Before discussing the formula, let us remark that the Pauli–Fierz and polaron Hamiltonians are self-adjoint and bounded from below. There is an extensive literature concerning the self-adjointness of the Pauli–Fierz Hamiltonian (see, e.g., [21, 37–39, 45, 52] and references therein), which, under our assumptions, is self-adjoint on $\mathcal{D}(K_0) \cap \mathcal{D}(\mathrm{d}\mathcal{G}_\varepsilon(\omega))$. The polaron Hamiltonian is also self-adjoint [27, 35], but its domain of self-adjointness is not explicitly characterized. On the other hand, its form domain is known, and it coincides with the form domain of $K_0 + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(1)$.

We do not prove the pull-through formula, since it is discussed in detail for the massive renormalized Nelson model in [1], and its independence of the semiclassical parameter has been shown in [4]. The models we consider here are "contained" in the massive renormalized Nelson model, in the sense that all the terms in the Hamiltonians contained here are parts of or are analogous to some parts of the renormalized Nelson Hamiltonian. Therefore, they have already been discussed in the aforementioned papers.

Proposition 7.2 (Pull-through formula). *There exist finite constants a, b, independent of ε, such that for any $\varepsilon \in (0,1)$ and any $\Psi_\varepsilon \in \mathcal{D}(H_\varepsilon)$,*

$$\|\mathrm{d}\mathcal{G}_\varepsilon(1)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon} \leq \frac{a}{\nu(\varepsilon)}\|(H_\varepsilon + b)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon}. \tag{7.10}$$

To study the quasi-classical limit of the (massless) Pauli–Fierz model, we cannot use the pull-through formula; we use instead the following propagation result (see [5] for a detailed proof).

Proposition 7.3 (Propagation estimate). *Let H_ε be the Pauli–Fierz Hamiltonian with either $\nu(\varepsilon) = 1$ or $\nu(\varepsilon) = 1/\varepsilon$. Then there exist finite constants C_1, C_2, independent of ε, such that for any $\varepsilon \in (0,1)$, any $\Psi_\varepsilon \in \mathcal{D}(K_0) \cap \mathcal{D}(\mathrm{d}\mathcal{G}_\varepsilon(\omega)) \cap \mathcal{D}(\mathrm{d}\mathcal{G}_\varepsilon(1))$ and any $t \in \mathbb{R}$,*

$$\begin{aligned}&\|\mathrm{d}\mathcal{G}_\varepsilon(1)e^{-itH_\varepsilon}\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon}\\ &\qquad \leq C_1[\|\mathrm{d}\mathcal{G}_\varepsilon(1)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon} + \|(K_0 + \mathrm{d}\mathcal{G}_\varepsilon(\omega) + 1)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon}]e^{C_2|t|}.\end{aligned} \tag{7.11}$$

In addition, there exist finite constants $c, C > 0$, independent of ε, such that for any $\varepsilon \in (0,1)$ and any $\Psi_\varepsilon \in \mathcal{D}(H_\varepsilon) = \mathcal{D}(K_0) \cap \mathcal{D}(\mathrm{d}\mathcal{G}_\varepsilon(\omega))$,

$$\begin{aligned}&c\|(H_\varepsilon + 1)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon}\\ &\qquad \leq \|(K_0 + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega) + 1)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon} \leq C\|(H_\varepsilon + 1)\Psi_\varepsilon\|_{\mathcal{H}\otimes\mathcal{K}_\varepsilon}.\end{aligned} \tag{7.12}$$

Let us now outline in more detail how one can use the pull-through formula in the adaptations of the arguments to cover the polaron model. The main technicality is the propagation of the a priori bound and regularity of the state. This can be achieved by a direct application of Proposition 7.2: one can simply restrict the proof of Theorem 1.6 to states satisfying

$$\mathrm{Tr}\big(\Gamma_\varepsilon((K_0 + \mathrm{d}\mathcal{G}_\varepsilon(\omega) + 1)^2 + \mathrm{d}\mathcal{G}_\varepsilon(1)^2)\big) \leq C, \tag{7.13}$$

$$\mathrm{Tr}(\Gamma_\varepsilon H_\varepsilon^2) \leq C\nu(\varepsilon)^2, \tag{7.14}$$

for any $\varepsilon \in (0,1)$. Let us remark that the regularity assumptions above are not propagated in time as they are, but they are rather used to control the following expectations at any time $t \in \mathbb{R}$:

- $\mathrm{Tr}(\Gamma_\varepsilon(t) K_0)$;
- $\mathrm{Tr}(\Gamma_\varepsilon(t)(\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^2)$.

The first expectation is bounded uniformly in ε as in [14, Lemma 3.4], using the assumption (7.13). The second expectation is bounded using Proposition 7.2 and assumption (7.14). Once the bounds for the two quantities above are established at any time, it is possible to use Proposition 7.1 for the quasi-classical convergence of the interaction terms appearing in the integral equation. The result is then extended to general states satisfying (A1) by means of the procedure outlined in §6.

For the Pauli–Fierz model one proceeds similarly, using Proposition 7.3 instead of the pull-through formula. Theorem 1.6 is first proved for initial states such that

$$\mathrm{Tr}\big(\Gamma_\varepsilon(K_0 + \mathrm{d}\mathcal{G}_\varepsilon(\omega) + (\mathrm{d}\mathcal{G}_\varepsilon(1)+1)^2)\big) \le C \tag{7.15}$$

for $\varepsilon \in (0,1)$. The needed regularity of the expectation of the number operator at any time is then obtained thanks to Proposition 7.3. To bound the free particle part, one proceeds as for the polaron model in [14, Lemma 3.4], the only difference being that instead of using KLMN-smallness, which would be true only for small values of the particles' charge, one uses again the number estimate of Proposition 7.3 to close the argument (see [47] for additional details). Therefore, it is possible to apply Proposition 7.1 to get the quasi-classical convergence of the gradient terms appearing in the integral equation, and Proposition 2.6 to get the convergence of quadratic terms in the creation and annihilation operators. The proof can then be completed exactly as for the polaron model.

7.3. Quasi-classical evolution

In this section we briefly discuss the well-posedness of the effective evolution equation for the polaron and Pauli–Fierz models with $\nu(\varepsilon) = 1/\varepsilon$. In fact, when $\nu(\varepsilon) = 1$ or $\nu = 0$, the generator of $\mathcal{U}_{t,s}(z)$, $\mathcal{K}_0 + \mathcal{V}(z)$, does not depend on time and therefore the evolution is defined by Stone's theorem, and the interaction picture $\widetilde{\mathcal{U}}_{t,s}(z)$ is weakly generated by $t \mapsto \widetilde{\mathcal{V}}_t(z)$.

If however $\nu(\varepsilon) = 1/\varepsilon$, we need to prove the existence of a two-parameter group $(\mathcal{U}_{t,s}(z))_{t,s\in\mathbb{R}}$ of unitary operators satisfying

$$\begin{aligned} i\partial_t \mathcal{U}_{t,s}(z)\psi_s &= (\mathcal{K}_0 + \mathcal{V}_t(z))\mathcal{U}_{t,s}(z)\psi_s, \\ i\partial_s \mathcal{U}_{t,s}(z)\psi_s &= -\mathcal{U}_{t,s}(z)(\mathcal{K}_0 + \mathcal{V}_s(z))\psi_s, \\ \mathcal{U}_{s,s}(z)\psi_s &= \psi_s, \end{aligned}$$

for any $\psi \in \mathcal{D}(\mathcal{K}_0)$, where $\mathcal{D}(\mathcal{K}_0)$ is also the domain of self-adjointness for $\mathcal{K}_0 + \mathcal{V}_t(z)$ for all $t \in \mathbb{R}$.

For the Pauli–Fierz model, it is easy to prove that the map $t \mapsto \mathcal{K}_0 + \mathcal{V}_t(z)$ is strongly continuously differentiable on $\mathcal{D}(\mathcal{K}_0)$. Therefore, a result of [50] guarantees the existence of $(\mathcal{U}_{t,s}(z))_{t,s\in\mathbb{R}}$. Again, $\widetilde{\mathcal{U}}_{t,s}(z)$ is then defined by $\widetilde{\mathcal{U}}_{t,s}(z) = e^{it\mathcal{K}_0}\mathcal{U}_{t,s}(z)e^{-is\mathcal{K}_0}$, and it is weakly generated by $\widetilde{\mathcal{V}}_t(z)$. For the polaron model, on the other hand, the existence of the quasi-classical dynamics follows from a general result concerning the evolution generated by time-dependent closed quadratic forms with a time-independent common core, proved, e.g., in [51, Theorem II.27 & Corollary II.28].

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