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To cite this article: David Faraci and Claudia Comi 2021 *J. Phys.: Conf. Ser.* **2015** 012038

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Asymptotic homogenization of metamaterials elastic plates

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Abstract. The asymptotic homogenization technique is applied to evaluate the effective properties of thin plates with periodic heterogeneity. The effect of shear deformation in the homogenization process is evidenced and the role of cell slenderness, besides that of the plate, is clarified by several numerical analyses.

1. Introduction

The numerical simulations of thin plates with periodic heterogeneity, or metamaterials plates, have a high computational burden, especially when the size of the heterogeneity is small if compared with the global dimension of the plate. In the elastic field, asymptotic homogenization have been effectively and widely employed to obtain equivalent properties of periodic solids, both in statics and dynamics [1, 5].

For plates the procedure can be developed starting from a three-dimensional formulation [3, 4, 7, 9]. This leads to local 3D cell problems, which can be treated in a simplified manner if the unit cell can be modelled as a plate. A possible way to proceed is introducing in the 3D cell problems the kinematic constraints of plate theories [8, 9].

In this work, following a different path, we develop the homogenization procedure directly from the structural model, as done originally in [6] for the Kirchhoff-Love plate theory, and in [2] for the Mindlin-Reissner formulation and we apply it to several problems to evidence the role of shear deformation in the numerical results when the typical dimension of the heterogeneity decreases and hence the cell slenderness decreases.

2. Two-scale asymptotic homogenization

Let us consider a heterogeneous plate of mid-surface Ω characterized by an in-plane periodicity, with unit cell Y^ϵ , as depicted e.g. in Figure 1 (left), made of linear elastic isotropic materials. The typical dimension of the unit cell is denoted by $a^\epsilon = \epsilon a$, being ϵ a small parameter. Besides the macroscopic in-plane coordinate $\mathbf{x} = (x_1, x_2)$, according to the two-scale asymptotic method, we introduce the fast variable $\mathbf{y} = \mathbf{x}/\epsilon$, which lives in the in-plane re-scaled unit cell $Y = Y^\epsilon/\epsilon$, of size a , as shown e.g. in Figure 1 (middle). We assume that the thickness of the plate does not scale with ϵ , i.e. $t^\epsilon(\mathbf{x}) = t(\mathbf{x}/\epsilon)$. Similarly, we consider the transversal load $p^\epsilon(\mathbf{x}) = p(\mathbf{x}/\epsilon)$ and the periodically varying Young's modulus and Poisson's ratio, $E^\epsilon(\mathbf{x}) = E(\mathbf{x}/\epsilon)$ and $\nu^\epsilon(\mathbf{x}) = \nu(\mathbf{x}/\epsilon)$ respectively, independent from ϵ .

The homogenized (effective) properties of the plate are obtained by analyzing the asymptotic behaviour as $\epsilon \rightarrow 0$.



2.1. Homogenization of Kirchhoff-Love plates

The kinematic of a Kirchhoff-Love (K-L) plate is completely described by the out of plane displacement w^ϵ , while the equilibrium requires that:

$$\operatorname{div} \operatorname{div} (-\mathbb{B}^\epsilon : \nabla \odot \nabla w^\epsilon) + p^\epsilon = 0 \quad \text{in } \Omega \quad \text{with} \quad \mathbb{B}^\epsilon = \frac{E^\epsilon t^{\epsilon 3}}{12(1-\nu^\epsilon 2)} [(1-\nu^\epsilon)\mathbb{I} + \nu^\epsilon \mathbf{I} \otimes \mathbf{I}] \quad (1)$$

\mathbb{B}^ϵ is the bending stiffness tensor, div is the in-plane divergence operator and \odot denotes the symmetric part of the tensorial product. The solution of the problem can be searched in the form of the asymptotic expansion:

$$w^\epsilon(\mathbf{x}) = w^0(\mathbf{x}, \mathbf{x}/\epsilon) + \epsilon w^1(\mathbf{x}, \mathbf{x}/\epsilon) + \epsilon^2 w^2(\mathbf{x}, \mathbf{x}/\epsilon) + \epsilon^3 w^3(\mathbf{x}, \mathbf{x}/\epsilon) + \dots \quad (2)$$

where the fields $w^i(\mathbf{x}, \mathbf{y})$ are defined on $\Omega \times \mathbb{R}^2$ and are Y -periodic with respect the fast variable. Substituting (2) into (1) one obtains a sequence of differential problems from which is possible to obtain the effective homogenized bending stiffness of K-L plate, which components can be computed as:

$$B_{ijhk}^0 = \frac{1}{|Y|} \int_Y (\mathbf{e}_i \odot \mathbf{e}_j - \nabla_{\mathbf{y}} \odot \nabla_{\mathbf{y}} \alpha^{ij}) : \mathbb{B} : (\mathbf{e}_h \odot \mathbf{e}_k - \nabla_{\mathbf{y}} \odot \nabla_{\mathbf{y}} \alpha^{hk}) \, d\mathbf{y} \quad i, j, h, k \in \{1, 2\} \quad (3)$$

where \mathbf{e}_i is the unit vector in the i -th direction and $|Y|$ is the area of the re-scaled unit cell. The functions $\alpha^{ij}(\mathbf{y})$ in (3) are the Y -periodic solutions of the cell problems:

$$\operatorname{div}_{\mathbf{y}} \operatorname{div}_{\mathbf{y}} [\mathbb{B} : (-\nabla_{\mathbf{y}} \odot \nabla_{\mathbf{y}} \alpha^{ij} + \mathbf{e}_i \odot \mathbf{e}_j)] = 0 \quad \text{in } Y \quad (4)$$

2.2. Homogenization of Mindlin-Reissner plates

In the Mindlin-Reissner (M-R) formulation, the kinematic of the plate is described by the out of plane displacement w^ϵ and the rotation φ^ϵ of the thickness segments normal to the mid-surface. The equilibrium conditions are given by:

$$\operatorname{div} [\mathbf{C}^\epsilon \cdot (\nabla w^\epsilon - \varphi^\epsilon)] + p^\epsilon = 0 \quad \text{and} \quad \operatorname{div} (-\mathbb{B}^\epsilon : \nabla \varphi^\epsilon) = \mathbf{C}^\epsilon \cdot (\nabla w^\epsilon - \varphi^\epsilon) \quad \text{in } \Omega \quad (5)$$

where \mathbb{B}^ϵ is the plate bending stiffness defined in (1), while $\mathbf{C}^\epsilon = 5/6 G^\epsilon \mathbf{I}$ is the plate shear stiffness tensor, G^ϵ being the shear modulus of the material. In addition to the expansion (2) adopted in the previous subsection, we assume the following asymptotic expansions of the rotation:

$$\varphi^\epsilon(\mathbf{x}) = \varphi^0(\mathbf{x}, \mathbf{x}/\epsilon) + \epsilon \varphi^1(\mathbf{x}, \mathbf{x}/\epsilon) + \epsilon^2 \varphi^2(\mathbf{x}, \mathbf{x}/\epsilon) + \dots \quad (6)$$

where the fields $\varphi^i(\mathbf{x}, \mathbf{y})$ are defined on $\Omega \times \mathbb{R}^2$ and are Y -periodic with respect the second variable. Substituting (2,6) in equation (5) one obtains the effective bending and shear stiffnesses of M-R plate, which components are:

$$B_{ijhk}^0 = \frac{1}{|Y|} \int_Y (\mathbf{e}_i \odot \mathbf{e}_j - \nabla_{\mathbf{y}} \beta^{ij}) : \mathbb{B} : (\mathbf{e}_h \odot \mathbf{e}_k - \nabla_{\mathbf{y}} \beta^{hk}) \, d\mathbf{y} \quad (7)$$

$$C_{ij}^0 = \frac{1}{|Y|} \int_Y (\mathbf{e}_i + \nabla_{\mathbf{y}} \gamma^i) \cdot \mathbf{C} \cdot (\mathbf{e}_j + \nabla_{\mathbf{y}} \gamma^j) \, d\mathbf{y}$$

where $\beta^{ij}(\mathbf{y})$ and $\gamma^i(\mathbf{y})$ are the Y -periodic solutions, respectively, of the two cell problems:

$$\operatorname{div}_{\mathbf{y}} [\mathbb{B} : (-\nabla_{\mathbf{y}} \beta^{ij} + \mathbf{e}_i \odot \mathbf{e}_j)] = \mathbf{0} \quad \text{and} \quad \operatorname{div}_{\mathbf{y}} [\mathbf{C} \cdot (\nabla_{\mathbf{y}} \gamma^i + \mathbf{e}_i)] = 0 \quad \text{in } Y \quad (8)$$

The presented K-L and M-R homogenization are not able to take into account the real slenderness of the unit cell, since they are derived with an in-plane re-scaling. Better results are obtained by scaling also the thickness ($t^\epsilon(\mathbf{x}) = \epsilon t(\mathbf{x}/\epsilon)$) in the cell problems as in the refined M-R (R-M-R) homogenization proposed in [9].

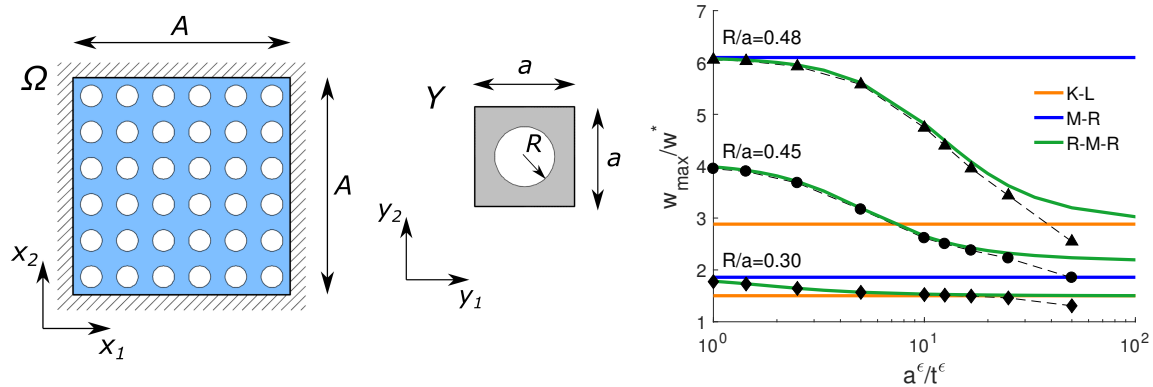


Figure 1: Periodic perforated plate with circular holes (left); re-scaled unit cell (middle); maximum deflection versus a^ϵ/t^ϵ for the holed plate (markers) and the homogenized one (continuous lines), (right).

3. Numerical analyses

To validate the above homogenized formulations and to set the level of approximation for different values of ϵ , two reference problems are considered. The results obtained of homogenized plates, with stiffness evaluated through (3) or (7), are compared with those obtained with finite element analyses on the actual heterogeneous plates.

To verify the convergence of the homogenization solution as $\epsilon \rightarrow 0$, and to investigate the effect of the unit cell slenderness, different analyses are conducted on plates with increasing numbers of cells keeping fix the global dimensions of the plate and its thickness t^ϵ .

3.1. Periodic perforated plate

As the first example, we consider a square plate, of side A and constant thickness $t = A/100$, with cylindrical holes of radius $R^\epsilon = \epsilon R$, periodically distributed in a square lattice of side a^ϵ , see Figure 1 (left). In the mid-plane of the plate, the re-scaled unit cell, shown in Figure 1 (middle), is hence a square of side a with a circular hole of radius R . The plate is subject to a uniform transversal load p and it is simply supported on its boundary.

Figure 1 (right) shows the maximum displacement w_{max} numerically computed on the holed M-R plates (markers) and the same quantity evaluated through homogenization (continuous line), for different values of R/a , as a function of the unit cell slenderness a^ϵ/t^ϵ . Green curves are obtained considering the R-M-R homogenization, while the blue and the orange ones refer to the M-R and K-L ones (for clarity these latter are only shown for $R/a = 0.3$ and 0.48). All results are normalized with the maximum displacement w^* of the uniform plate without holes ($R/a \rightarrow 0$). The results obtained with R-M-R homogenization are very close to those of the real holed plate, except when there are few cells (i.e. for high values of a^ϵ/t^ϵ). This is due to the lack of separation between macro and micro-scale, which is a fundamental hypothesis for the validity of the homogenization technique. K-L and M-R homogenization, conversely, can be used only for large and small values of the cell slenderness (respectively). In particular as the size of the holes increases (compare the curves $R/a = 0.3$ and 0.48) the shear deformation of the unit cell becomes relevant. Hence, the K-L solution significantly differ from the R-M-R one even for very slender cells.

Figure 2 shows the deformed configurations of the holed plate and of the equivalent homogeneous one for $a^\epsilon/t^\epsilon = 10$ and $R/a = 0.45$. Even though the number of cells is small (10×10), the homogenized approach is very accurate.

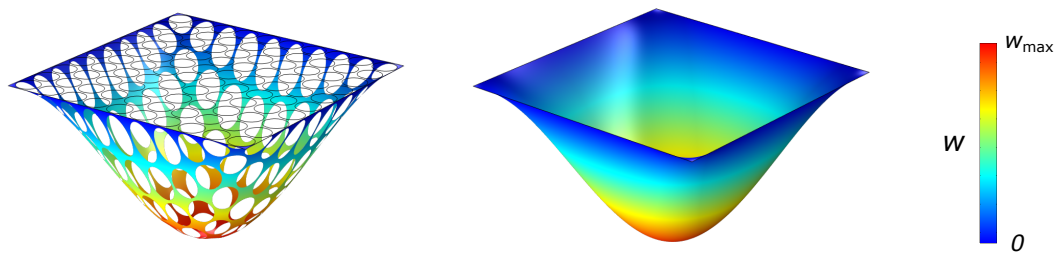


Figure 2: Contour of deflection on deformed shapes for the holed plate (left) and the R-M-R homogenized one (right), for $a^\epsilon/t^\epsilon = 10$ and $R/a = 0.45$.

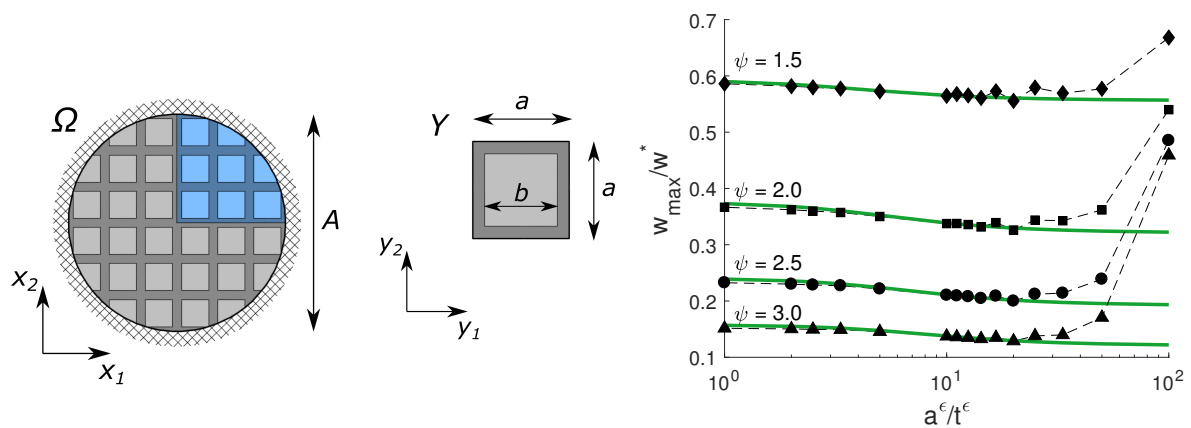


Figure 3: Two-way ribbed plate with a square inclusion (left); re-scaled unit cell (middle); maximum deflection versus a^ϵ/t^ϵ for the real problem (markers) and the R-M-R homogenized one (continuous lines), (right).

3.2. Two-ways ribbed plate

We consider a circular two-ways ribbed plate of diameter A made of a homogeneous material, as shown in Figure 3 (left). We denote by $t^\epsilon = A/100$ the thickness of the plate and, as in waffle slabs, we consider a reinforcement with a square pattern (shown in dark grey in Figure 3) of thickness ψt^ϵ , with $\psi \geq 1$, and width $a^\epsilon - b^\epsilon = 0.25a^\epsilon$. The re-scaled unit cell (Figure 3, middle) is a square of side a . The plate is clamped on its boundary and is loaded with a uniform transversal load p on the top-right quarter (the light blue region in the figure).

The maximum deflection w_{max} numerically obtained accounting for the real geometry with reinforcements and the same quantity obtained on the homogenized plate are compared in Figure 3 (right) for some values of ψ . The values are normalized with w^* , which is the maximum displacement for the case of uniform thickness plate ($\psi = 1$). As for the previous example, for few cells (i.e. for high ratios a^ϵ/t^ϵ) the R-M-R homogenized solution (continuous line) differs from the real one (markers) due to the lack of scale separation. For small values of a^ϵ/t^ϵ the shear deformability of the unit cell is no more negligible and seems rather independent on ψ .

Figure 4 shows the contour plots of the displacement w (Figure 4a and b) and of the bending moment M_{11} (Figure 4c and d) for $\psi = 2$ with 3 and 10 cells along the diameter. The first row refers to the analysis with the real geometry, the second refers to the homogenized plate. With few cells (Figure 4a), the displacement of the homogenized plate significantly differs from the real one, while with more cells (Figure 4b) the two solutions are practically equal. On the contrary, even with a lot of cells, the solution of the homogenized plate provides only average values of the bending moments, therefore it is unable

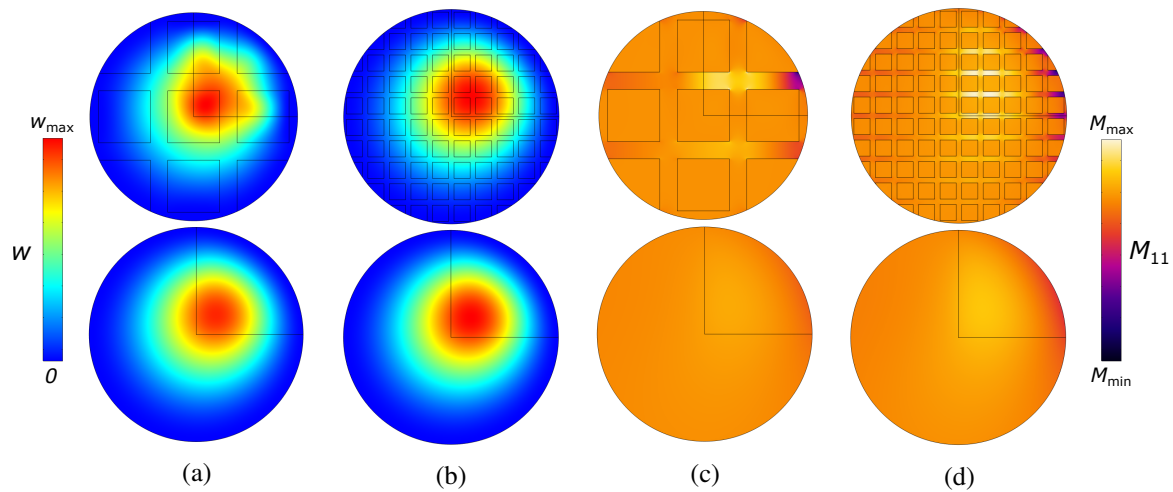


Figure 4: Top: analysis on real geometry; bottom: analysis on homogenized plate. Contours of displacement w (left) and bending moment M_{11} (right) for the cases $a^\epsilon/t^\epsilon = 33.33$ (a,c) and $a^\epsilon/t^\epsilon = 10$ (b,d) with $\psi = 2$.

to capture their maxima and minima (see Figure 4d). However, peak values can be taken into account through the so-called stress localization tensor, which is not considered in the present work.

4. Conclusions

The presented homogenization methods can be effectively adopted to simulate elastic periodic metamaterial plates when a proper scale separation (at least one order of magnitude between the lattice and the global dimensions) is guaranteed. The methods gives good results also when the load is non-uniform and the boundary of the plate does not match the periodicity of the lattice. However, one should select carefully the homogenization technique to adopt. In particular, even for very thin plates, when the heterogeneity dimension a^ϵ is of the same order of the plate thickness t^ϵ , the shear deformation plays a significant role in the homogenization and the K-L theory cannot be used. Since, on the other hand, the separation of scales requires small values of a^ϵ , the homogenized K-L formulation has little practical use. Conversely, the homogenized R-M-R formulation, can be effectively applied in a wide range of geometric parameters, as shown by the examples here considered.

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