

Optimal distributed control of a stochastic Cahn-Hilliard equation *

LUCA SCARPA

E-mail: luca.scarpa@univie.ac.at

Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

Abstract

We study an optimal distributed control problem associated to a stochastic Cahn-Hilliard equation with a classical double-well potential and Wiener multiplicative noise, where the control is represented by a source-term in the definition of the chemical potential. By means of probabilistic and analytical compactness arguments, existence of an optimal control is proved. Then the linearized system and the corresponding backward adjoint system are analysed through monotonicity and compactness arguments, and first-order necessary conditions for optimality are proved.

AMS Subject Classification: 35K55, 35R60, 60H15, 80A22, 82C26.

Key words and phrases: stochastic Cahn-Hilliard equation, phase separation, optimal control, linearized state system, adjoint state system, first-order optimality conditions.

1 Introduction

The pure Cahn-Hilliard equation on a smooth bounded domain $D \subset \mathbb{R}^N$, $N = 2, 3$, can be written in its simplest form as

$$\partial_t y - \Delta w = 0, \quad w = -\Delta y + \Psi'(y) - u \quad \text{in } (0, T) \times D,$$

where $T > 0$ is a fixed final time, y and w denote the order parameter and the chemical potential of the system, respectively, and u represents a given distributed source term. Furthermore, Ψ' is the derivative of a so-called double-well potential Ψ , which may be seen

*This paper was funded by Vienna Science and Technology Fund (WWTF) through Project MA14-009.

as the sum of a convex function and a concave quadratic perturbation: typical examples of Ψ which are relevant in applications are discussed in [18]. Usually, in order to ensure the conservation of the mean on D , the equation is complemented by homogenous Neumann conditions for both y and w , and a given initial value, namely

$$\partial_{\mathbf{n}}y = \partial_{\mathbf{n}}w = 0 \quad \text{in } (0, T) \times \partial D, \quad y(0) = y_0 \quad \text{in } D,$$

where \mathbf{n} denotes the outward normal unit vector on ∂D .

The Cahn-Hilliard equation was originally introduced in [7] (see also [31, 32, 49]) to capture the spinodal decomposition phenomenon occurring in a phase-separation of a binary metallic alloy. The mathematical literature on the deterministic Cahn-Hilliard equation has been widely developed in the last years, especially in much more general settings as the presence of viscosity terms and dynamic boundary conditions: in this direction we mention, among all, the contributions (as well as the references therein) [4, 8–10, 13, 14, 18, 37, 48, 57] on the well-posedness of the system, and [15, 21, 38] on asymptotic behaviour of the solutions. Optimal distributed and boundary control problems have been studied in the context of Allen-Cahn and Cahn-Hilliard equations in the works [11, 12, 16, 17, 19, 20, 42, 54].

More recently, in order to account also for the random vibrational movements at a microscopic level in the system, which may be of magnetic, electronic or configurational nature, the equation has been modified by adding a cylindrical Wiener process W (see [23, 46]). This has resulted in the well-accepted version of the stochastic Cahn-Hilliard equation

$$dy - \Delta w dt = B(y) dW \quad \text{in } (0, T) \times D =: Q, \quad (1.1)$$

$$w = -\Delta y + \Psi'(y) - u \quad \text{in } (0, T) \times D, \quad (1.2)$$

$$\partial_{\mathbf{n}}y = \partial_{\mathbf{n}}w = 0 \quad \text{in } (0, T) \times \partial D =: \Sigma, \quad (1.3)$$

$$y(0) = y_0 \quad \text{in } D, \quad (1.4)$$

where B is a stochastically integrable operator with respect to W . The mathematical literature on the stochastic Cahn-Hilliard and Allen-Cahn equations is significantly less developed. Let us mention the works [24, 25, 30, 55] dealing with existence, uniqueness and regularity for the pure equation, and [41, 45, 55] for an analysis of the viscous case in terms of well-posedness, regularity and vanishing viscosity limit. We point out for completeness also the contributions [1] for a study of a stochastic Cahn-Hilliard equation with unbounded noise, and [26, 27, 39] dealing with stochastic Cahn-Hilliard equations with reflections. The reader can refer also to [3, 51] for the context of stochastic Allen-Cahn equations, and [33] for a study of a diffuse interface model with thermal fluctuations.

While the literature on stochastic optimal control problems is widely developed, we are not aware of any result dealing with controllability of the stochastic Cahn-Hilliard equation. The main novelty of the present contribution is to provide a first study in this direction, and represents a starting point for the study of optimal control problems associated to the wide class of more general phase-field models with stochastic perturbation. Optimal control problems have been studied in the stochastic case especially in connection with the stochastic maximum principle: the reader can refer to [61] for a general treatment on the subject. Let us mention the works [36, 50] dealing with stochastic

maximal principle for nonlinear SPDEs with dissipative drift, [29] for optimal control of stochastic evolution equations in Hilbert spaces, and [35] for a study on optimal control of SPDEs with control contained both in the drift and the diffusion. Let us also point out [53] for a stochastic optimal control problem on infinite time horizon, [52] on ergodic maximum principle, and [6] for a study on optimal relaxed controls of dissipative SPDEs. Stochastic optimal control problems have also been considered in [2] in the context of the Schrödinger equation.

In the present contribution we are interested in studying a distributed optimal control problem associated to the stochastic pure Cahn-Hilliard equation, where the control is the source term u in the definition of the chemical potential and the cost functional is of standard quadratic tracking-type. More precisely, we want to minimize

$$J(y, u) := \frac{\alpha_1}{2} \mathbb{E} \int_Q |y - x_Q|^2 + \frac{\alpha_2}{2} \mathbb{E} \int_D |y(T) - x_T|^2 + \frac{\alpha_3}{2} \mathbb{E} \int_Q |u|^2, \quad (1.5)$$

subject to the state equation (1.1)–(1.4) and a control constraint on $u \in \mathcal{U}$, where \mathcal{U} is a suitable convex closed subset of $L^2(\Omega \times Q)$ which will be specified in Section 2 below. Here, $\alpha_1, \alpha_2, \alpha_3$ are nonnegative constants, x_Q and x_T are given functions in $L^2(\Omega \times Q)$ and $L^2(\Omega \times D)$, respectively. The main results of this work are the existence of a relaxed optimal control and the proof of first-order necessary conditions for optimality.

The first step of our analysis consists in studying the control-to-state mapping. In particular, we show that for every admissible control $u \in \mathcal{U}$, the state system (1.1)–(1.4) admits a unique solution y , and the map $S : u \mapsto y$ is Lipschitz-continuous in some suitable spaces. Consequently, the cost functional J can be expressed in a reduced form only in terms of the control u , i.e. introducing the reduced cost functional \tilde{J} as

$$\tilde{J}(u) := J(S(u), u), \quad u \in \mathcal{U}.$$

At this point, in the deterministic setting the most natural necessary condition for optimality of $\bar{u} \in \mathcal{U}$ would read

$$D\tilde{J}(\bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U},$$

where $D\tilde{J}$ represents the derivative of \tilde{J} at least in the sense of Gâteaux. In this direction, the classical approach consists in showing that the map S is Fréchet-differentiable, hence so is \tilde{J} by the usual chain rule for Fréchet-differentiable functions, and to characterize the derivative $DS(\bar{u})$ as the solution of a suitable linearized system. In the context of Cahn-Hilliard equations with possibly degenerate potentials (for example if Ψ is the double-well logarithmic potential), the Fréchet differentiability of the control-to-state mapping is usually obtained by requiring sufficient conditions in the box constraint for u , ensuring at least that $\Psi''(\bar{y}) \in L^\infty(Q)$, where $\bar{y} := S(\bar{u})$ and Ψ'' is the second derivative of Ψ (for example that \mathcal{U} is contained in a closed ball in $L^\infty(Q)$).

However, if we add a stochastic perturbation in the equation, under reasonable assumptions on the data it is *not* possible to prove that $\Psi''(\bar{y})$ is uniformly bounded in $L^\infty(\Omega \times Q)$, even if we add a constraint on the L^∞ -norm in the definition of the admissible controls. This behaviour gives rise to several nontrivial difficulties: among all, it is not true *a priori* that the control-to-state map S is Fréchet-differentiable in some space.

This issue is usually overcome in the stochastic setting using specific time-variations on the control (the so-called “spike-variation” technique). In our case, however, we are able to avoid such procedure by analysing explicitly the linearized system. More specifically, we prove that the linearized system admits a unique variational solution by means of compactness and monotonicity arguments. Then, we show that the control-to-state mapping is Gâteaux differentiable in a suitable weak sense, and that the (weak) Gâteaux derivative of S can still be identified as the unique solution z to the linearized system. Performing usual first-order variations around a fixed optimal control \bar{u} , we then prove that the weak Gâteaux-differentiability is enough to ensure first-order necessary conditions for optimality.

The second main issue that we tackle in this work consist in removing the dependence on z in the first-order necessary conditions by studying the adjoint problem. As it is well-known, in the stochastic framework the adjoint problem becomes a backward stochastic partial differential equation (BSPDE) of the form

$$\tilde{p} = -\Delta p \quad \text{in } Q, \quad (1.6)$$

$$-dp - \Delta \tilde{p} dt + \Psi''(y)\tilde{p} dt = \alpha_1(y - x_Q) dt + DB(y)^*q dt - q dW \quad \text{in } Q, \quad (1.7)$$

$$\partial_{\mathbf{n}}p = \partial_{\mathbf{n}}\tilde{p} = 0 \quad \text{in } \Sigma, \quad (1.8)$$

$$p(T) = \alpha_2(\bar{y}(T) - x_T) \quad \text{in } D, \quad (1.9)$$

where the unknown is the triple (p, \tilde{p}, q) . Since $\Psi''(y)$ does *not* belong to $L^\infty(\Omega \times Q)$, as we have pointed out above, the adjoint problem cannot be framed in any available existence theory for BSPDEs, and is absolutely nontrivial and interesting on its own. Through a suitable approximation involving a truncation on Ψ'' and a passage to the limit, we show existence and uniqueness of a solution to the adjoint problem. Furthermore, we prove a suitable duality relation between z and \tilde{p} , which allows us to express the first-order optimality conditions only in terms of \tilde{p} and \bar{u} in a much more simplified form.

The paper is organized as follows. In Section 2 we fix the assumptions, the general setting of the work and the main results. Section 3 contains the proof of well-posedness of the state system. In Section 4 we prove that a relaxed optimal control always exists, using Prokhorov and Skorokhod theorems and natural lower semicontinuity results. In Section 5 we study the control-to-state map: we show that it is well-defined and differentiable in a certain weak sense, and we identify its (weak) derivative as the unique solution to the linearized problem. Finally, in Section 6 we study the adjoint problem and prove the first-order necessary conditions for optimality.

2 Main results

Throughout the paper $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ denotes a filtered probability space satisfying the usual conditions, with $T > 0$ fixed, and W is a cylindrical Wiener process on a separable Hilbert space U . The progressive σ -algebra on $\Omega \times [0, T]$ is denoted by \mathcal{P} . Furthermore, $D \subset \mathbb{R}^N$, with $N = 2, 3$, is a smooth bounded connected domain, and we use the notation $Q := (0, T) \times D$ and $Q_t := (0, t) \times D$ for every $t \in (0, T)$.

For every Hilbert spaces E_1 and E_2 we denote by $\mathcal{L}(E_1, E_2)$ and $\mathcal{L}^2(E_1, E_2)$ the spaces of linear continuous and Hilbert-Schmidt operators from E_1 to E_2 , respectively.

The symbols for norms and dualities are $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively, with a sub-script indicating the specific spaces in consideration. We shall use the symbols \rightarrow , \rightharpoonup and $\overset{*}{\rightharpoonup}$ to denote strong, weak, and weak* convergences, respectively. For any Banach space E and $p \in [1, +\infty]$ we shall use the symbols $L^p(\Omega; E)$ and $L^p(0, T; E)$ for the usual spaces of Bochner-integrable functions, and the symbols $C^0([0, T]; E)$ and $C_w^0([0, T]; E)$ for the spaces of continuous functions from $[0, T]$ to E endowed with the norm topology or weak topology, respectively. If $p, q \in [1, +\infty)$ we shall denote by $L^p_{\mathcal{P}}(\Omega; L^q(0, T; E))$ the space of E -valued progressively measurable processes X such that $\mathbb{E} \left(\int_0^T \|X(s)\|_E^q ds \right)^{p/q} < +\infty$.

We define the functional spaces

$$H := L^2(D), \quad V := H^1(D), \quad Z := \{\varphi \in H^2(D) : \partial_{\mathbf{n}}\varphi = 0 \text{ a.e. on } \partial D\},$$

endowed with their natural norms. In the sequel H is identified to H^* , so that (V, H, V^*) is a Hilbert triplet, with dense, continuous and compact inclusions. The Laplace operator with Neumann homogeneous conditions will be intended in the usual variational way as the operator

$$-\Delta : V \rightarrow V^*, \quad \langle -\Delta x, \varphi \rangle_V := \int_D \nabla x \cdot \nabla \varphi, \quad x, \varphi \in V,$$

or

$$-\Delta : H \rightarrow Z^*, \quad \langle -\Delta x, \varphi \rangle_Z := - \int_D x \Delta \varphi, \quad x \in H, \varphi \in Z.$$

We recall also that in the context of Cahn-Hilliard equations it is useful to introduce the operator \mathcal{N} as the inverse of $-\Delta$ restricted to the subspace of null-mean elements in V . More specifically, if we denote $x_D := \frac{1}{|D|} \langle x, 1 \rangle_V$ for any $x \in V^*$, by the Poincaré inequality we know that

$$-\Delta : \{x \in V : x_D = 0\} \rightarrow \{x \in V^* : x_D = 0\}$$

is an isomorphism, hence its inverse \mathcal{N} is well-defined. Furthermore, it is well-known (see [18, pp. 979-980]) that

$$x \mapsto \|x\|_* := \|\nabla \mathcal{N}(x - x_D)\|_H + |y_D|, \quad x \in V^*,$$

defines a norm on V^* , equivalent to the usual one, such that

$$\forall \sigma > 0, \quad \exists C_\sigma > 0 : \quad \|x\|_H^2 \leq \sigma \|\nabla x\|_H^2 + C_\sigma \|x\|_*^2 \quad \forall x \in V_1, \quad (2.1)$$

and

$$\langle \partial_t x(t), \mathcal{N}x(t) \rangle_{V_1} = \frac{1}{2} \frac{d}{dt} \|\nabla \mathcal{N}x(t)\|_H^2 \quad \text{for a.e. } t \in (0, T)$$

for every $x \in H^1(0, T; V^*)$ with $x_D = 0$ almost everywhere in $(0, T)$. We shall denote for simplicity

$$H_0 := \{x \in H : x_D = 0\}.$$

The following assumptions on the data of the problem will be in force throughout:

(A1) $\Psi \in C^2(\mathbb{R}, \mathbb{R}_+)$;

(A2) there exist $c_1, c_2 > 0$ such that, for every $r \in \mathbb{R}$,

$$\Psi''(r) \geq -c_1, \quad |\Psi''(r)| \leq c_2(1 + |r|^2), \quad |\Psi'(r)| \leq c_2(1 + \Psi(r)).$$

(A3) $y_0 \in L^{12}(\Omega, \mathcal{F}_0; H) \cap L^6(\Omega, \mathcal{F}_0; V)$ and $\Psi(y_0) \in L^3(\Omega; L^1(D))$;

(A4) $B : [0, T] \times H \rightarrow \mathcal{L}^2(U, V)$ is measurable and there exists a constant $L_B > 0$ such that, for every $t \in [0, T]$,

$$\begin{aligned} \|B(t, x_1) - B(t, x_2)\|_{\mathcal{L}^2(U, H)} &\leq L_B \|x_1 - x_2\|_H \quad \forall x_1, x_2 \in H, \\ \|B(t, x)\|_{\mathcal{L}^2(U, H)} &\leq L_B(1 + \|x\|_H) \quad \forall x \in H, \\ \|B(t, x)\|_{\mathcal{L}^2(U, V)} &\leq L_B(1 + \|x\|_V) \quad \forall x \in V. \end{aligned}$$

If B is genuinely of multiplicative type, i.e. if it is not constant in the last variable, we further assume that the image of B is contained in $\mathcal{L}^2(U, H_0)$.

(A5) for every $t \in [0, T]$, the operator $B(t, \cdot) : H \rightarrow \mathcal{L}^2(U, H)$ is of class C^1 .

Remark 2.1. Let us comment on assumptions (A1)–(A5). In order for the state system (1.1)–(1.4) to be well-posed, one can require less stringent assumptions on the data (see for example [55, 56]). However, in order to study the linearized and the adjoint problems, one needs some further regularity on the solution y to the state equation, and for this reason (A1)–(A5) are in order. Let us point out that by the hypotheses (A1)–(A2) we can decompose Ψ' as the sum of a continuous increasing function and a Lipschitz-continuous function as $\Psi'(r) = (\Psi'(r) + c_1 r) - c_1 r$, $r \in \mathbb{R}$. Furthermore, note that (A3)–(A4) are trivially satisfied for example when $y_0 \in V$ is nonrandom with $\Psi(y_0) \in L^1(D)$ and $B \in \mathcal{L}^2(U, V)$ is time-independent and of additive type. The reason why we assume existence of higher moments on y_0 might not be intuitive at this level and will be clarified later: let us mention that these assumptions will be needed to solve the linearized system and the adjoint problem, and that the hypothesis on the moment of order 6 is “optimal” in this sense. In case of multiplicative noise, assumption (A4) corresponds to usual boundedness and Lipschitz-continuity conditions on B , and the differentiability assumption (A5) is needed in order to analyse the linearized system. In particular, (A4)–(A5) imply that

$$\|DB(t, x)\|_{\mathcal{L}(H; \mathcal{L}^2(U, H))} \leq L_B \quad \forall (t, x) \in [0, T] \times H.$$

We define the set of admissible controls \mathcal{U} as

$$\mathcal{U} := \left\{ u \in L_{\mathcal{P}}^{12}(\Omega; L^2(0, T; H)) \cap L_{\mathcal{P}}^6(\Omega; L^2(0, T; V)) : \|u\|_{L^{12}(\Omega; L^2(0, T; H)) \cap L^6(\Omega; L^2(0, T; V))} \leq C_0 \right\},$$

where $C_0 > 0$ and $s \in (0, 1/2)$ are fixed constants. It will be useful to introduce also the bigger set

$$\mathcal{U}' := \left\{ u \in L_{\mathcal{P}}^{12}(\Omega; L^2(0, T; H)) \cap L_{\mathcal{P}}^6(\Omega; L^2(0, T; V)) : \|u\|_{L^{12}(\Omega; L^2(0, T; H)) \cap L^6(\Omega; L^2(0, T; V))} < 2C_0 \right\},$$

which is open and bounded in $L^2_{\mathcal{P}}(\Omega; L^2(0, T; H)) \cap L^6_{\mathcal{P}}(\Omega; L^2(0, T; V))$, and $\mathcal{U} \subset \mathcal{U}'$. Moreover, we define the cost functional

$$J : L^2_{\mathcal{P}}(\Omega; C^0([0, T]; H)) \times L^2_{\mathcal{P}}(\Omega; L^2(0, T; H)) \rightarrow \mathbb{R}_+,$$

$$J(y, u) := \frac{\alpha_1}{2} \mathbb{E} \int_Q |y - x_Q|^2 + \frac{\alpha_2}{2} \mathbb{E} \int_D |y(T) - x_T|^2 + \frac{\alpha_3}{2} \mathbb{E} \int_Q |u|^2,$$

where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ are fixed constants and

$$\alpha_1 x_Q \in L^6_{\mathcal{P}}(\Omega; L^6(0, T; H)), \quad \alpha_2 x_T \in L^6(\Omega, \mathcal{F}_T; V).$$

Remark 2.2. The choices $\alpha_2 x_T \in L^6(\Omega; V)$ and $\alpha_1 x_Q \in L^6(\Omega; L^6(0, T; H))$ might look unnatural to the reader at this level, due to the form of the cost functional. However, this will be necessary in order to solve the adjoint system. Let us mention that conditions of this type are not new in literature of optimal control problems: see for example [22] for an analogous assumption in the context of the Allen-Cahn equation.

As we have anticipated in Section 1, we are interested in minimizing $J(y, u)$ subject to the constraint $u \in \mathcal{U}$ and the state system (1.1)–(1.4). We shall call optimal pair any couple (y, u) with $u \in \mathcal{U}$ satisfying (1.1)–(1.4) and minimizing the cost functional J .

Under the hypotheses (A1)–(A4), we can prove that the state system is well-posed for every admissible control, and that the map $u \mapsto y$ is well-defined and Lipschitz-continuous. These results are summarized in the following theorem.

Theorem 2.1. *Assume (A1)–(A4). Then for every $u \in \mathcal{U}'$ there exists a unique pair (y, w) with*

$$y \in L^{12}_{\mathcal{P}}(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z)), \quad (2.2)$$

$$y \in L^6_{\mathcal{P}}(\Omega; L^\infty(0, T; V)) \cap L^3(\Omega; L^2(0, T; H^3(D))), \quad (2.3)$$

$$y - \int_0^\cdot B(s, y(s)) dW(s) \in L^6(\Omega; H^1(0, T; V^*)), \quad (2.4)$$

$$w \in L^3_{\mathcal{P}}(\Omega; L^2(0, T; V)), \quad \Psi'(y) \in L^3_{\mathcal{P}}(\Omega; L^2(0, T; V)), \quad (2.5)$$

such that $y(0) = y_0$ and, for every $\varphi \in V$, for almost every $t \in (0, T)$, \mathbb{P} -almost surely,

$$\left\langle \partial_t \left(y - \int_0^\cdot B(s, y(s)) dW(s) \right) (t), \varphi \right\rangle_V + \int_D \nabla w(t) \cdot \nabla \varphi = 0, \quad (2.6)$$

$$\int_D w(t) \varphi = \int_D \nabla y(t) \cdot \nabla \varphi + \int_D \Psi'(y(t)) \varphi - \int_D u(t) \varphi. \quad (2.7)$$

Moreover, there exists a constant $M' > 0$, only depending on y_0, C_0, c_1, c_2, L_B and Q , such that, for every $u \in \mathcal{U}'$ and for any respective state (y, w) satisfying (2.2)–(2.7),

$$\|y\|_{L^{12}(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z))} \leq M', \quad (2.8)$$

$$\|y\|_{L^6(\Omega; L^\infty(0, T; V)) \cap L^3(\Omega; L^2(0, T; H^3(D)))} \leq M', \quad (2.9)$$

$$\|w\|_{L^3(\Omega; L^2(0, T; V))} + \|\Psi'(y)\|_{L^3(\Omega; L^2(0, T; V))} \leq M'. \quad (2.10)$$

Finally, there exists a constant $M > 0$, only depending on y_0 , C_0 , c_1 , c_2 , L_B and Q , such that, for any $u_1, u_2 \in \mathcal{U}'$ and for any respective pairs (y_1, w_1) , (y_2, w_2) satisfying (2.2)–(2.7), it holds

$$\|y_1 - y_2\|_{L^6(\Omega; C^0([0, T]; V^*)) \cap L^6(\Omega; L^2(0, T; V))} \leq M \|u_1 - u_2\|_{L^6(\Omega; L^2(0, T; V^*))}, \quad (2.11)$$

$$\|y_1 - y_2\|_{L^2(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z))} \leq M \|u_1 - u_2\|_{L^6(\Omega; L^2(0, T; H))}. \quad (2.12)$$

By Theorem 2.1, it is clear that uniqueness of y holds for the state system. Consequently, it is well-defined the control-to-state map

$$S : \mathcal{U}' \rightarrow L^6(\Omega; C^0([0, T]; H) \cap L^\infty(0, T; V) \cap L^2(0, T; Z)),$$

which is Lipschitz-continuous in the sense specified in (2.11)–(2.12). This allows us to introduce the reduced cost functional as

$$\tilde{J} : \mathcal{U}' \rightarrow \mathbb{R}_+, \quad \tilde{J}(u) := J(S(u), u), \quad u \in \mathcal{U}'.$$

The optimal control problem is thus equivalent to minimizing \tilde{J} over $\mathcal{U} \subset \mathcal{U}'$. The following definitions of optimal control are very natural.

Definition 2.3. *An optimal control is an element $u \in \mathcal{U}$ such that*

$$\tilde{J}(u) \leq \tilde{J}(v) \quad \forall v \in \mathcal{U}.$$

A relaxed optimal control is a family

$$((\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t \in [0, T]}, \mathbb{P}^*), W^*, x_Q^*, x_T^*, y_0^*, u^*, y^*, w^*),$$

where $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t \in [0, T]}, \mathbb{P}^*)$ is a filtered probability space satisfying the usual conditions, W^* is a $(\mathcal{F}_t^*)_t$ -cylindrical Wiener process with values in U , X_Q^* is a $(\mathcal{F}_t^*)_t$ -progressively measurable $L^2(0, T; H)$ -valued process with the same law of x_Q , x_T^* is a \mathcal{F}_T^* -measurable H -valued random variable with the same law of x_T , y_0^* is a \mathcal{F}_0^* -measurable random variable with the same law of y_0 , u^* is a process in the set \mathcal{U}^* (defined as \mathcal{U} replacing Ω with Ω^*), (y^*, w^*) is the unique solution to the system (2.2)–(2.7) on Ω^* with respect to the data (W^*, y_0^*, u^*) , and such that

$$\tilde{J}^*(u^*) := \frac{\alpha_1}{2} \mathbb{E}^* \int_Q |y^* - x_Q^*|^2 + \frac{\alpha_2}{2} \mathbb{E}^* \int_D |y^*(T) - x_T^*|^2 + \frac{\alpha_3}{2} \mathbb{E}^* \int_Q |u^*|^2 \leq \inf_{v \in \mathcal{U}} \tilde{J}(v).$$

The first main result that we prove concerns with the existence of a relaxed optimal control. Note however that existence of (strong) optimal controls is nontrivial, since the minimization problem is not convex, hence uniqueness of optimal controls may fail. In case of uniqueness of optimal controls, existence of a strong optimal control can be proved for example by a well-known criterion on convergence in probability due to Gyöngy–Krylov (see [40, Lem. 1.1], [43, Prop. 4.16], and [2, Def. 2.4 and Thm. 2.5]).

Theorem 2.2. *Assume (A1)–(A4). Then there exists a relaxed optimal control.*

We focus now on the necessary conditions for optimality. As we have anticipated, the first step consists in showing that S is Gâteaux-differentiable in certain weak-sense and to characterize its weak derivative as the unique solution of a linearized system. The following proposition ensures that the linearized system is well-posed in a suitable variational sense.

Proposition 2.4. *Assume (A1)–(A5). Then, for every $u \in \mathcal{U}'$ and for every $h \in L^6_{\mathcal{P}}(\Omega; L^2(0, T; H))$, setting $y := S(u)$, there exists a unique pair (z_h, μ_h) with*

$$z_h \in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z)) , \quad (2.13)$$

$$z_h - \int_0^\cdot DB(s, y(s))z_h(s) dW(s) \in L^2(\Omega; H^1(0, T; Z^*)) , \quad (2.14)$$

$$\mu_h \in L^2_{\mathcal{P}}(\Omega; L^2(0, T; H)) , \quad (2.15)$$

such that $z_h(0) = 0$ and, for every $\varphi \in Z$, for almost every $t \in (0, T)$, \mathbb{P} -almost surely,

$$\left\langle \partial_t \left(z_h(t) - \int_0^t DB(s, y(s))z_h(s) dW(s) \right), \varphi \right\rangle_Z - \int_D \mu_h(t) \Delta \varphi = 0 , \quad (2.16)$$

$$\int_D \mu_h(t) \varphi = \int_D \nabla z_h(t) \cdot \nabla \varphi + \int_D \Psi''(y(t))z_h(t) \varphi - \int_D h(t) \varphi . \quad (2.17)$$

Remark 2.5. Let us point out that (2.13)–(2.17) is the weak formulation of the linearized system, which can be obtained formally differentiating the state system (1.1)–(1.4) with respect to u in the direction h , i.e.

$$\begin{aligned} dz_h - \Delta \mu_h dt &= DB(y)z_h dW && \text{in } (0, T) \times D , \\ \mu_h &= -\Delta z_h + \Psi''(y)z_h - h && \text{in } (0, T) \times D , \\ \partial_{\mathbf{n}} z_h &= \partial_{\mathbf{n}} \mu_h = 0 && \text{in } (0, T) \times \partial D , \\ z_h(0) &= 0 && \text{in } D . \end{aligned}$$

We are now able to give a characterization of the weak Gâteaux derivative of S in terms of the unique solution to the linearized system (2.13)–(2.17).

Theorem 2.3. *Assume (A1)–(A5). Then the control-to-state map S is weakly Gâteaux-differentiable from \mathcal{U}' to $L^2(\Omega; C^0([0, T]; H)) \cap L^2(\Omega; L^2(0, T; Z))$ in the following sense: for every $u, h \in \mathcal{U}'$, as $\varepsilon \searrow 0$,*

$$\begin{aligned} \frac{S(u + \varepsilon h) - S(u)}{\varepsilon} &\rightarrow z_h && \text{in } L^p(\Omega; L^2(0, T; V)) \quad \forall p \in [1, 2) , \\ \frac{S(u + \varepsilon h) - S(u)}{\varepsilon} &\rightarrow z_h && \text{in } L^2(\Omega; L^2(0, T; Z)) , \\ \frac{S(u + \varepsilon h)(t) - S(u)(t)}{\varepsilon} &\rightarrow z_h(t) && \text{in } L^2(\Omega; H) \quad \forall t \in [0, T] , \end{aligned}$$

where z_h is the unique solution to the linearized system (2.13)–(2.17).

The first natural necessary optimality condition is collected in the following result.

Theorem 2.4. *Assume (A1)–(A5), let $\bar{u} \in \mathcal{U}$ be an optimal control and $\bar{y} := S(\bar{u})$ be the respective optimal state. Then*

$$\alpha_1 \mathbb{E} \int_Q (\bar{y} - x_Q) z_{v-\bar{u}} + \alpha_2 \mathbb{E} \int_D (\bar{y}(T) - x_T) z_{v-\bar{u}}(T) + \alpha_3 \mathbb{E} \int_Q \bar{u}(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U} ,$$

where $z_{v-\bar{u}}$ is the unique solution to (2.13)–(2.17) with respect to the choice $h := v - \bar{u}$.

The last result that we present is an alternative formulation of the first-order necessary conditions for optimality which does not involve the solution z to the linearized problem, but the solution to the corresponding adjoint problem. In this sense, the advantage is that the resulting variational inequality that we obtain is much simpler to interpret. The following proposition states that the adjoint problem is well-posed in a suitable variational sense.

Proposition 2.6. *Assume (A1)–(A5). Then for every $u \in \mathcal{U}'$, setting $y := S(u)$, there exists a triple of processes (p, \tilde{p}, q) , with*

$$p \in C_w^0([0, T]; L^2(\Omega; V)) \cap L_{\mathcal{P}}^2(\Omega; L^2(0, T; Z \cap H^3(D))), \quad (2.18)$$

$$\tilde{p} \in C_w^0([0, T]; L^6(\Omega; V^*)) \cap L_{\mathcal{P}}^6(\Omega; L^2(0, T; V)), \quad (2.19)$$

$$q \in L_{\mathcal{P}}^2(\Omega; L^2(0, T; \mathcal{L}^2(U, V))), \quad (2.20)$$

such that, for every $\varphi \in Z$, \mathbb{P} -almost surely and for every $t \in [0, T]$,

$$\int_D \tilde{p}(t)\varphi = \int_D \nabla p(t) \cdot \nabla \varphi, \quad (2.21)$$

$$\begin{aligned} & \int_D p(t)\varphi - \int_t^T \int_D \tilde{p}(s)\Delta\varphi ds + \int_t^T \int_D \Psi''(y)\tilde{p}(s)\varphi ds \\ &= \alpha_2 \int_D (\bar{y}(T) - x_T)\varphi + \alpha_1 \int_t^T \int_D (y - x_Q)(s)\varphi ds \\ &+ \int_t^T (DB(s, y(s))^* q(s), \varphi)_H ds - \int_D \left(\int_t^T q(s) dW(s) \right) \varphi. \end{aligned} \quad (2.22)$$

Moreover, if (p_1, \tilde{p}_1, q_1) and (p_2, \tilde{p}_2, q_2) are two solutions to (2.18)–(2.22), then

$$p_1 - (p_1)_D = p_2 - (p_2)_D, \quad \tilde{p}_1 = \tilde{p}_2.$$

Our last result is a simplified version of the first-order necessary optimality conditions which do not involve the solution z to the linearized system, but the unique solution \tilde{p} to the adjoint problem instead.

Theorem 2.5. *Assume (A1)–(A5), let $\bar{u} \in \mathcal{U}$ be an optimal control and let $\bar{y} := S(\bar{u})$ be the respective optimal state. Then the following variational inequality holds:*

$$\mathbb{E} \int_Q (\tilde{p} + \alpha_3 \bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U},$$

where \tilde{p} is the unique second solution component satisfying (2.18)–(2.22) with respect to (\bar{u}, \bar{y}) . In particular, if $\alpha_3 > 0$, then \bar{u} is the orthogonal projection of the point $-\frac{\tilde{p}}{\alpha_3}$ on the closed convex set \mathcal{U} in the Hilbert space $L^2(\Omega; L^2(0, T; H))$.

Remark 2.7. The form of the cost functional J considered in this paper is of standard quadratic tracking-type and is widely used in optimal control theory. However, let us point out that this choice is a particular case of the more general class of nonlinear performances

$$J(y, u) = \mathbb{E} \int_0^T F_Q(t, y(t), u(t)) dt + \mathbb{E} F_T(y(T)),$$

where $F_Q : [0, T] \times H \times H \rightarrow \mathbb{R}$ and $F_T : H \rightarrow \mathbb{R}$ are $\mathcal{B}([0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$ -measurable and $\mathcal{B}(H)$ -measurable, respectively. The techniques used here can also be adapted to deal with such more general situations. For example, one can show existence of relaxed optimal controls under very natural lower semicontinuity assumptions on F_Q and F_T . Furthermore, requiring that $F(t, \cdot, \cdot) : H \times H \rightarrow \mathbb{R}$ and $F_T : H \rightarrow \mathbb{R}$ are Fréchet-differentiable for every $t \in [0, T]$, with

$$\begin{aligned} \|D_y F_Q(t, y, u)\|_H + \|D_u F_Q(t, y, u)\|_H &\lesssim 1 + \|y\|_H + \|u\|_H \\ \|D_y F_T(y)\|_H &\lesssim 1 + \|y\|_H \end{aligned}$$

for every $(t, y, u) \in [0, T] \times H \times H$, necessary conditions for optimality can be also studied. Note however that in the case of nonlinear performance, the resulting variational inequality in Theorem 2.5 would not give a characterization of the optimal controls in terms of orthogonal projection on \mathcal{U} .

3 Well-posedness of the state system

This section is devoted to the proof of Theorem 2.1, ensuring the the state system is well-posed.

For any $\lambda > 0$, we consider the approximated problem

$$\begin{cases} dy_\lambda - \Delta w_\lambda dt = B(y_\lambda) dW & \text{in } (0, T) \times D, \\ w_\lambda = -\Delta y_\lambda + \Psi'_\lambda(y_\lambda) - u & \text{in } (0, T) \times D, \\ \partial_{\mathbf{n}} y_\lambda = \partial_{\mathbf{n}} w_\lambda = 0 & \text{in } (0, T) \times \partial D, \\ y_\lambda(0) = y_0 & \text{in } D, \end{cases}$$

where Ψ'_λ is a Lipschitz-continuous smooth Yosida-type approximation of Ψ' . The classical variational theory ensures the existence and uniqueness of an approximated solution $y_\lambda \in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z))$. Arguing as in [55, 56], Itô's formula for the square of the H -norm and the linear growth condition on B in assumption (A4) yields, together with the Gronwall lemma, that

$$\|y_\lambda\|_{L^2(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z))} \leq M', \quad (3.1)$$

where the constant $M' > 0$ only depends on y_0, C_0, c_1, c_2, L_B and Q . Furthermore, writing Itô's formula for the free-energy functional (see again [55]) yields

$$\begin{aligned} &\frac{1}{2} \int_D |\nabla y_\lambda(t)|^2 + \int_D \Psi_\lambda(y_\lambda(t)) + \int_{Q_t} \nabla w_\lambda \cdot \nabla(w_\lambda + u) \\ &= \frac{1}{2} \int_D |\nabla y_0|^2 + \int_D \Psi_\lambda(y_0) + \int_0^t ((w_\lambda + u)(s), B(s, y_\lambda(s)))_H dW(s) \\ &\quad + \frac{1}{2} \int_0^t \|\nabla B(s, y_\lambda(s))\|_{\mathcal{L}^2(U, H)}^2 ds + \frac{1}{2} \sum_{k=0}^\infty \int_{Q_t} \Psi''_\lambda(y_\lambda) |B(\cdot, y_\lambda) e_k|^2. \end{aligned} \quad (3.2)$$

We want now to take power 3 at both sides, and then supremum in time and expectations. Note that the trace term on the right-hand side can be estimated thanks to the Hölder

inequality, the Sobolev embedding $V \hookrightarrow L^6(D)$ and the growth conditions (A2) and (A4) on Ψ'' and B , as

$$\begin{aligned} & \sum_{k=0}^{\infty} \int_{Q_t} \Psi''_{\lambda}(y_{\lambda}) |B(\cdot, y_{\lambda}) e_k|^2 \\ & \lesssim \|B(\cdot, y_{\lambda})\|_{L^2(0,t; \mathcal{L}^2(U,H))}^2 + \int_0^t \|y_{\lambda}(s)\|_{L^4(D)}^2 \|B(\cdot, y_{\lambda})\|_{\mathcal{L}^2(U,V)}^2 ds \\ & \lesssim 1 + \|y_{\lambda}\|_{L^2(0,T;V)}^2 + \|y_{\lambda}\|_{L^4(0,T;V)}^4. \end{aligned}$$

Since by interpolation we have

$$\|y_{\lambda}\|_{L^4(0,t;V)} \lesssim \|y_{\lambda}\|_{L^{\infty}(0,T;H) \cap L^2(0,T;Z)},$$

the right-hand side is uniformly bounded in $L^3(\Omega)$ by the estimate (3.1). Furthermore, for the stochastic integral we note that

$$(w_{\lambda} + u, B)_H = (w_{\lambda} - (w_{\lambda})_D, B)_H + |D|(w_{\lambda})_D B_D + (u, B).$$

The Burkholder-Davis-Gundy, Young, and Poincaré-Wirtinger inequalities imply, together with assumption (A4), that

$$\begin{aligned} & \mathbb{E} \sup_{r \in [0,t]} \left| \int_0^r ((w_{\lambda} + u)(s), B(s, y_{\lambda}(s)))_H dW(s) \right|^3 \\ & \lesssim \mathbb{E} \left(\int_0^t (\|(w_{\lambda} - (w_{\lambda})_D)(s)\|_H^2 + \|u(s)\|_H^2) \|B(s, y_{\lambda}(s))\|_{\mathcal{L}^2(U,H)}^2 ds \right)^{3/2} \\ & \quad + \mathbb{E} \left(\int_0^t |(w_{\lambda}(s))_D|^2 \|B_D(s, y_{\lambda}(s))\|_{\mathcal{L}^2(U,\mathbb{R})}^2 ds \right)^{3/2}. \end{aligned}$$

Now, in case of multiplicative noise we have that $B_D = 0$ by (A4) and the second term on the right-hand side vanishes, so that we can continue the estimate by

$$\begin{aligned} & \mathbb{E} \left[\|B(\cdot, y_{\lambda})\|_{L^{\infty}(0,T; \mathcal{L}^2(U,H))}^3 \left(\|w_{\lambda} - (w_{\lambda})_D\|_{L^2(0,t;H)}^3 + \|u\|_{L^2(0,T;H)}^3 \right) \right] \\ & \lesssim 1 + \delta \|\nabla w_{\lambda}\|_{L^6(\Omega; L^2(0,T;H))}^6 + C_{\delta} \|y_{\lambda}\|_{L^6(\Omega; C^0([0,T];H))}^6 + \|u\|_{L^6(\Omega; L^2(0,T;H))}^6. \end{aligned}$$

In case of additive noise, on the right-hand side we obtain the further contribution

$$\|B\|_{L^{\infty}(0,T; \mathcal{L}^2(U,V^*))}^3 \|(w_{\lambda})_D\|_{L^3(\Omega; L^2(0,t))}^3 \lesssim t^{3/2} \|(w_{\lambda})_D\|_{L^3(\Omega; L^{\infty}(0,t))}^3.$$

We go back now to (3.2), and take power 3, supremum in $t \in [0, T_0]$ for a certain $T_0 \in (0, T)$ and expectations. Since $w_{\lambda} = -\Delta y_{\lambda} + \Psi'_{\lambda}(y_{\lambda}) - u$, by (A2) the term involving $\Psi_{\lambda}(y_{\lambda})$ on the left-hand side of (3.2) yields a bound from below for $(w_{\lambda})_D$ in $L^3(\Omega; L^{\infty}(0, T_0))$: hence, choosing $\delta > 0$ and T_0 small enough, rearranging the terms, and using the Gronwall lemma yields

$$\begin{aligned} & \|y_{\lambda}\|_{L^6(\Omega; L^{\infty}(0,T_0;V))}^6 + \|\Psi_{\lambda}(y_{\lambda})\|_{L^3(\Omega; L^{\infty}(0,T_0;L^1(D)))}^6 + \|\nabla w_{\lambda}\|_{L^6(\Omega; L^2(0,T_0;H))}^6 \\ & \lesssim 1 + \|y_{\lambda}\|_{L^{12}(\Omega; C^0([0,T];H) \cap L^2(0,T;Z))}^{12} + \|u\|_{L^6(\Omega; L^2(0,T;V))}^6, \end{aligned}$$

where the implicit constant is independent of λ . Taking these remarks into account, noting that $\|w_\lambda\|_V \lesssim \|\nabla w_\lambda\|_H + |(w_\lambda)_D|$, using (3.1) and the fact that $u \in \mathcal{U}'$, we infer by using a classical patching-in-time technique that

$$\|y_\lambda\|_{L^6(\Omega;L^\infty(0,T;V))} + \|w_\lambda\|_{L^3(\Omega;L^2(0,T;V))} \leq M'. \quad (3.3)$$

Now, by comparison in the equation for the chemical potential we deduce that

$$\|\Psi'_\lambda(y_\lambda)\|_{L^3(\Omega;L^2(0,T;H))} \leq M',$$

while by (A2), the embedding $V \hookrightarrow L^6(D)$, and interpolation we have

$$\|\nabla \Psi'(y_\lambda)\|_{L^2(0,T;H)} = \|\Psi''_\lambda(y_\lambda)\nabla y_\lambda\| \lesssim 1 + \|y_\lambda\|_{L^\infty(0,T;V)} \|y_\lambda\|_{L^\infty(0,T;H) \cap L^2(0,T;Z)}^2,$$

so that by (3.1)–(3.3)

$$\|\Psi'_\lambda(y_\lambda)\|_{L^3(\Omega;L^2(0,T;V))} \leq M'.$$

By elliptic regularity we infer then also

$$\|\Psi'(y_\lambda)\|_{L^3(\Omega;L^2(0,T;V))} + \|y_\lambda\|_{L^3(\Omega;L^2(0,T;H^3(D)))} \leq M'. \quad (3.4)$$

It is now a standard matter to pass to the limit as $\lambda \searrow 0$ in the approximated problem, and recover (2.2)–(2.10): for further details we refer to [55, 56].

It only remains to prove the continuous dependence property (2.12), as (2.11) has already been proved in [56]. To this end, note that

$$\begin{aligned} d(y_1 - y_2) - \Delta(w_1 - w_2) dt &= (B(t, y_1) - B(t, y_2)) dW, \\ w_1 - w_2 &= -\Delta(y_1 - y_2) + \Psi'(y_1) - \Psi'(y_2) - (u_1 - u_2), \end{aligned}$$

so that Itô's formula for the square of the H -norm yields

$$\begin{aligned} &\frac{1}{2} \|(y_1 - y_2)(t)\|_H^2 + \int_{Q_t} |\Delta(y_1 - y_2)|^2 \\ &= \int_{Q_t} (\Psi'(y_1) - \Psi'(y_2)) \Delta(y_1 - y_2) - \int_{Q_t} (u_1 - u_2) \Delta(y_1 - y_2) \\ &\quad + \frac{1}{2} \|B(\cdot, y_1) - B(\cdot, y_2)\|_{L^2(0,t;\mathcal{L}^2(U,H))}^2 \\ &\quad + \int_0^t ((y_1 - y_2)(s), B(s, y_1(s)) - B(s, y_2(s)))_H dW(s). \end{aligned}$$

Using the Burkholder-Davis-Gundy and Young inequalities, and employing the Lipschitz continuity of B , we have

$$\begin{aligned} &\|y_1 - y_2\|_{L^2(\Omega;C^0([0,t];H))}^2 + \|\Delta(y_1 - y_2)\|_{L^2(\Omega;L^2(0,t;H))}^2 \\ &\lesssim \mathbb{E} \int_Q |u_1 - u_2|^2 + \mathbb{E} \int_Q |\Psi'(y_1) - \Psi'(y_2)|^2 + \|y_1 - y_2\|_{L^2(\Omega;L^2(0,t;H))}^2 \\ &\quad + \delta \|y_1 - y_2\|_{L^2(\Omega;C^0([0,t];H))}^2 + C_\delta \|y_1 - y_2\|_{L^2(\Omega;L^2(0,t;H))}^2, \end{aligned}$$

for every $\delta > 0$ and a suitable constant $C_\delta > 0$. By the mean-value theorem, assumption (A2), the Hölder inequality and the embedding $V \hookrightarrow L^6(D)$,

$$\begin{aligned} \mathbb{E} \int_Q |\Psi'(y_1) - \Psi'(y_2)|^2 &\lesssim \mathbb{E} \int_Q (1 + |y_1|^4 + |y_2|^4) |y_1 - y_2|^2 \\ &\lesssim \mathbb{E} \int_0^T \left(1 + \|y_1(s)\|_{L^6(D)}^4 + \|y_2(s)\|_{L^6(D)}^4\right) \|(y_1 - y_2)(s)\|_{L^6(D)}^2 ds \\ &\lesssim \mathbb{E} \left(1 + \|y_1\|_{L^\infty(0,T;V)}^4 + \|y_2\|_{L^\infty(0,T;V)}^4\right) \|y_1 - y_2\|_{L^2(0,T;V)}^2 \\ &\lesssim \left(1 + \|y_1\|_{L^6(\Omega;L^\infty(0,T;V))}^4 + \|y_2\|_{L^6(\Omega;L^\infty(0,T;V))}^4\right) \|y_1 - y_2\|_{L^6(\Omega;L^2(0,T;V))}^2. \end{aligned}$$

Hence, (2.12) follows rearranging the terms and using (2.11) and (2.9).

4 Existence of a relaxed optimal control

This section is devoted to the proof of Theorem 2.2: we show that a relaxed optimal control always exists.

Let $(u_n)_n \subset \mathcal{U}$ be a minimizing sequence for the reduced cost functional \tilde{J} , and set (y_n, w_n) as the respective solution to (2.2)–(2.7). Then, by definition of \mathcal{U} and the uniform estimates (2.8)–(2.10), we deduce that there exists a positive constant c , independent of n , such that

$$\begin{aligned} \|u_n\|_{L^{12}(\Omega;L^2(0,T;H)) \cap L^6(\Omega;L^2(0,T;V))} &\leq C_0, \\ \|y_n\|_{L^{12}(\Omega;C^0([0,T];H) \cap L^2(0,T;Z)) \cap L^6(\Omega;L^\infty(0,T;V)) \cap L^3(\Omega;L^2(0,T;H^3(D)))} &\leq c, \\ \|w_n\|_{L^3(\Omega;L^2(0,T;V))} + \|\Psi'(y_n)\|_{L^3(\Omega;L^2(0,T;V))} &\leq c. \end{aligned}$$

Recalling hypothesis (A4), we also deduce that

$$\|B(\cdot, y_n)\|_{L^6(\Omega;L^\infty(0,T;\mathcal{L}^2(U,V))} \leq c.$$

Hence, by [34, Lem. 2.1], for every $s \in (0, 1/2)$, there exists $c_s > 0$, independent of n , such that

$$\|B(\cdot, y_n) \cdot W\|_{L^6(\Omega;W^{s,6}(0,T;V))} \leq c_s,$$

where we have used the classical notation \cdot for the stochastic integral. Since $1 - \frac{1}{2} > s - \frac{1}{6}$, we have that $H^1(0, T; V^*) \hookrightarrow W^{s,6}(0, T; V^*)$ by the Sobolev embeddings, and by comparison in (2.6) we infer that

$$\|y_n\|_{L^6(\Omega;W^{s,6}(0,T;V^*))} \leq c_s.$$

Let us define now π_{y_n} as the law of y_n on $C^0([0, T]; H) \cap L^2(0, T; Z)$ and show that $(\pi_{y_n})_n$ is tight. Fixing now $s \in (1/6, 1/2)$ so that $6s > 1$, by [58, Sec. 8, Cor. 4–5] we have the compact inclusions

$$\begin{aligned} L^2(0, T; H^3(D) \cap Z) \cap W^{s,6}(0, T; V^*) &\xhookrightarrow{c} L^2(0, T; Z), \\ L^\infty(0, T; V) \cap W^{s,6}(0, T; V^*) &\xhookrightarrow{c} C^0([0, T]; H). \end{aligned}$$

If we define the space

$$\mathcal{W} := L^\infty(0, T; V) \cap L^2(0, T; H^3(D) \cap Z) \cap W^{s,6}(0, T; V^*),$$

we deduce that $\mathcal{W} \xrightarrow{c} C^0([0, T]; H) \cap L^2(0, T; Z)$ compactly and also the estimate

$$\|y_n\|_{L^3(\Omega; \mathcal{W})} \leq c.$$

This ensures by a standard argument that $(\pi_{y_n})_n$ is tight on $C^0([0, T]; H) \cap L^2(0, T; Z)$. Indeed, if B_R denotes the closed ball of radius $R > 0$ in \mathcal{W} , for any $R > 0$, we have that B_R is compact in $C^0([0, T]; H) \cap L^2(0, T; Z)$, and by Markov's inequality

$$\pi_{y_n}(B_R^c) = \mathbb{P} \{ \|y_n\|_{\mathcal{W}}^3 > R^3 \} \leq \frac{1}{R^3} \|y_n\|_{L^3(\Omega; \mathcal{W})}^3 \leq \frac{c^3}{R^3} \quad \forall n \in \mathbb{N},$$

from which the tightness of $(\pi_{y_n})_n$. Similarly, by [58, Sec. 8, Cor. 4–5] we also have the compact inclusion $W^{s,6}(0, T; V) \xrightarrow{c} C^0([0, T]; H)$, so that an entirely analogous argument yields that the laws of $(B(\cdot, y_n) \cdot W)_n$ on the space $C^0([0, T]; H)$ are tight. Moreover, denoting by $L_w^2(0, T; V)$ the space $L^2(0, T; V)$ endowed with its weak topology, it is clear that the laws of $(u_n)_n$ on $L_w^2(0, T; V)$ are tight.

Now, taking into account the remarks above, we deduce in particular that the septuple $(y_n, u_n, B(\cdot, y_n) \cdot W, W, y_0, x_Q, x_T)_n$ is tight on the space

$$C^0([0, T]; H) \times L_w^2(0, T; V) \times C^0([0, T]; H) \times C^0([0, T]; U) \times V \times L^2(0, T; H) \times H.$$

By Skorokhod theorem (see [44, Thm. 2.7] and [59, Thm. 1.10.4, Add. 1.10.5]) and the Jakubowski-Skorokhod version (see [5, Thm. 2.7.1]), there is a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, a sequence of maps $(\phi_n)_n$, where $\phi_n : (\Omega^*, \mathcal{F}^*) \rightarrow (\Omega, \mathcal{F})$ are measurable and satisfy $\mathbb{P} = \mathbb{P}^* \circ \phi_n^{-1}$ for every $n \in \mathbb{N}$, and measurable random variables $(y^*, u^*, I^*, W^*, y_0^*, x_Q^*, x_T^*)$ defined on $(\Omega^*, \mathcal{F}^*)$ with values in

$$C^0([0, T]; H) \times L^2(0, T; V) \times C^0([0, T]; H) \times C^0([0, T]; U) \times V \times L^2(0, T; H) \times H,$$

such that

$$\begin{aligned} y_n^* &:= y_n \circ \phi_n \rightarrow y^* && \text{in } C^0([0, T]; H) \quad \mathbb{P}^*\text{-a.s.}, \\ u_n^* &:= u_n \circ \phi_n \rightarrow u^* && \text{in } L^2(0, T; V) \quad \mathbb{P}^*\text{-a.s.}, \\ I_n^* &:= (B(\cdot, y_n) \cdot W) \circ \phi_n \rightarrow I^* && \text{in } C^0([0, T]; H) \quad \mathbb{P}^*\text{-a.s.}, \\ W_n^* &:= W \circ \phi_n \rightarrow W^* && \text{in } C^0([0, T]; U) \quad \mathbb{P}^*\text{-a.s.}, \\ y_{0,n}^* &:= y_0 \circ \phi_n \rightarrow y_0^* && \text{in } V \quad \mathbb{P}^*\text{-a.s.}, \\ x_{Q,n}^* &:= x_Q \circ \phi_n \rightarrow x_Q^* && \text{in } L^2(0, T; H) \quad \mathbb{P}^*\text{-a.s.}, \\ x_{T,n}^* &:= x_T \circ \phi_n \rightarrow x_T^* && \text{in } H \quad \mathbb{P}^*\text{-a.s.} \end{aligned}$$

Since the sequence $(y_0, x_Q, x_T)_n$ is constant, it is clear that the laws of (y_0^*, x_Q^*, x_T^*) and (y_0, x_Q, x_T) coincide. Moreover, setting $w_n^* := w_n \circ \phi_n$, since the maps $(\phi_n)_n$ preserve the laws, we readily deduce that

$$\begin{aligned} \|y_n^*\|_{L^{12}(\Omega^*; C^0([0, T]; H) \cap L^2(0, T; Z) \cap L^6(\Omega^*; L^\infty(0, T; V)) \cap L^3(\Omega^*; L^2(0, T; H^3(D))))} &\leq c, \\ \|u_n^*\|_{L^{12}(\Omega^*; L^2(0, T; H)) \cap L^6(\Omega^*; L^2(0, T; V))} &\leq C_0, \\ \|I_n^*\|_{L^6(\Omega^*; W^{s,6}(0, T; V))} &\leq c_s, \\ \|w_n^*\|_{L^3(\Omega^*; L^2(0, T; V))} + \|\Psi'(y_n^*)\|_{L^3(\Omega^*; L^2(0, T; V))} &\leq c, \end{aligned}$$

hence in particular that

$$\begin{aligned} y^* &\in L^{12}(\Omega^*; C^0([0, T]; H) \cap L^2(0, T; Z)), \\ y^* &\in L^6(\Omega^*; L^\infty(0, T; V) \cap L^3(\Omega^*; L^2(0, T; H^3(D))), \\ u^* &\in \mathcal{U}^*, \\ I^* &\in L^6(\Omega^*; W^{s,6}(0, T; V)) \end{aligned}$$

and

$$\begin{aligned} y_n^* &\rightarrow y^* && \text{in } L^p(\Omega^*; C^0([0, T]; H))^2 \quad \forall p \in [1, 12), \\ y_n^* &\rightharpoonup y^* && \text{in } L^{12}(\Omega^*; L^2(0, T; Z)) \cap L^3(\Omega^*; L^2(0, T; H^3(D))), \\ u_n^* &\rightharpoonup u^* && \text{in } L^{12}(\Omega^*; L^2(0, T; H)) \cap L^6(\Omega^*; L^2(0, T; V)), \\ w_n^* &\rightharpoonup w^* && \text{in } L^6(\Omega^*; L^2(0, T; V)), \\ \Psi'(y_n^*) &\rightharpoonup \xi^* && \text{in } L^3(\Omega^*; L^2(0, T; V)), \end{aligned}$$

for some

$$w^* \in L^6(\Omega^*; L^2(0, T; V)), \quad \xi^* \in L^3(\Omega^*; L^2(0, T; V)).$$

The strong-weak closure of the maximal monotone operator $r \mapsto \Psi'(r) + c_1 r$, $r \in \mathbb{R}$, ensures that $\xi^* = \Psi'(y^*)$ almost everywhere. Moreover, by (A4) we have that

$$B(\cdot, y_n^*) \rightarrow B(\cdot, y^*) \quad \text{in } L^p(\Omega^*; L^2(0, T; \mathcal{L}^2(U, H))) \quad \forall p \in [1, 12).$$

Now, defining the filtrations $(\mathcal{F}_{n,t}^*)_{t \in [0, T]}$ and $(\mathcal{F}_t^*)_{t \in [0, T]}$ as

$$\mathcal{F}_{n,t}^* := \sigma(W_n^*(s))_{s \in [0, t]}, \quad \mathcal{F}_t^* := \sigma(y^*(s), u^*(s), I^*(s), W^*(s))_{s \in [0, t]}, \quad t \in [0, T],$$

using classical representation theorems for martingales (see for example the arguments in [60, § 4]) it is possible to show that W_n^* is a $(\mathcal{F}_{n,t}^*)_t$ -cylindrical Wiener process on U , W^* is a $(\mathcal{F}_t^*)_t$ -cylindrical Wiener process on U , and that

$$I_n^* = \int_0^\cdot B(s, y_n^*(s)) dW_n^*(s), \quad I^* = \int_0^\cdot B(s, y^*(s)) dW^*(s).$$

Since (y_n^*, w_n^*) satisfies the variational formulation (2.6)–(2.7) on the space Ω^* with respect to $(y_{0,n}^*, u_n^*)$, passing to the weak limit it follows that (y^*, w^*) is the unique solution to (2.2)–(2.7) on the probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ corresponding to (y_0^*, u^*) . Using the weak lower semicontinuity of J , the fact that ϕ_n preserves the law for every n , and the definition of the minimizing sequence $(u_n)_n$, we deduce that

$$\begin{aligned} \tilde{J}^*(u^*) &= \frac{\alpha_1}{2} \mathbb{E}^* \int_Q |y^* - x_Q^*|^2 + \frac{\alpha_2}{2} \mathbb{E}^* \int_D |y^*(T) - x_T^*|^2 + \frac{\alpha_3}{2} \mathbb{E}^* \int_Q |u^*|^2 \\ &\leq \liminf_{n \rightarrow \infty} \frac{\alpha_1}{2} \mathbb{E}^* \int_Q |y_n^* - x_{Q,n}^*|^2 + \frac{\alpha_2}{2} \mathbb{E}^* \int_D |y_n^*(T) - x_{T,n}^*|^2 + \frac{\alpha_3}{2} \mathbb{E}^* \int_Q |u_n^*|^2 \\ &= \liminf_{n \rightarrow \infty} \tilde{J}(u_n) = \lim_{n \rightarrow \infty} \tilde{J}(u_n) = \inf_{v \in \mathcal{U}} \tilde{J}(v), \end{aligned}$$

so that u^* is a relaxed optimal control.

5 The control-to-state map

In this section we study the Gâteaux differentiability of the control-to-state map and we prove the first version of first-order necessary conditions for optimality.

5.1 Existence-uniqueness of the linearized system

We prove here Proposition 2.4. Let $u \in \mathcal{U}'$ be given, set $y := S(u)$, and let $h \in L^6_{\mathcal{P}}(\Omega; L^2(0, T; H))$. We show that the linearized system (2.13)–(2.15) admits a unique solution z_h .

Uniqueness. For $i = 1, 2$, let

$$(z_h^i, \mu_h^i) \in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z)) \times L^2_{\mathcal{P}}(\Omega; L^2(0, T; H)),$$

such that (z_h^i, μ_h^i) satisfy (2.16)–(2.17). Then we have, in the variational sense in the triple (Z, H, Z^*) ,

$$\begin{aligned} d(z_h^1 - z_h^2) - \Delta(\mu_h^1 - \mu_h^2) dt &= DB(y) (z_h^1 - z_h^2) dW && \text{in } (0, T) \times D, \\ \mu_h^1 - \mu_h^2 &= -\Delta(z_h^1 - z_h^2) + \Psi''(y)(z_h^1 - z_h^2) && \text{in } (0, T) \times D, \\ \partial_{\mathbf{n}}(z_h^1 - z_h^2) &= \partial_{\mathbf{n}}(\mu_h^1 - \mu_h^2) = 0 && \text{in } (0, T) \times \partial D, \\ (z_h^1 - z_h^2)(0) &= 0 && \text{in } D. \end{aligned}$$

Integrating on D the first equation, it follows from (A4) that $(z_h^1 - z_h^2)_D = 0$. Hence, Itô's formula for the square of the V^* -norm yields

$$\begin{aligned} &\frac{1}{2} \|\nabla \mathcal{N}(z_h^1 - z_h^2)(t)\|_H^2 + \int_{Q_t} |\nabla(z_h^1 - z_h^2)|^2 + \int_{Q_t} \Psi''(y) |(z_h^1 - z_h^2)|^2 \\ &= \frac{1}{2} \int_0^t \text{Tr} (DB(s, y(s)) (z_h^1 - z_h^2)(s)^* \circ \mathcal{N} \circ DB(s, y(s)) (z_h^1 - z_h^2)(s)) ds \\ &+ \int_0^t (\mathcal{N}(z_h^1 - z_h^2)(s), DB(s, y(s)) (z_h^1 - z_h^2)(s))_H dW(s). \end{aligned}$$

Now, by the uniform boundedness of DB , the first term on the right-hand side is bounded by

$$\begin{aligned} &\frac{1}{2} \|DB(\cdot, y)(z_h^1 - z_h^2)\|_{L^2(0, t; \mathcal{L}^2(U, V^*))}^2 \lesssim_{L_B} 1 + \|z_h^1 - z_h^2\|_{L^2(0, t; H)}^2 \\ &\leq \delta \|\nabla(z_h^1 - z_h^2)\|_{L^2(0, t; H)}^2 + C_\delta \|z_h^1 - z_h^2\|_{L^2(0, t; V^*)}^2 \end{aligned}$$

for every $\delta > 0$ and a certain $C_\delta > 0$, while the second term on the right-hand side can be estimated using the Burkholder-Davis-Gundy and Young inequalities as

$$\begin{aligned} &\mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r (\mathcal{N}(z_h^1 - z_h^2)(s), DB(s, y(s)) (z_h^1 - z_h^2)(s))_H dW(s) \right| \\ &\lesssim \delta \|z_h^1 - z_h^2\|_{L^2(\Omega; C^0([0, t]; V^*))}^2 + C_\delta \|DB(\cdot, y) (z_h^1 - z_h^2)\|_{L^2(\Omega; L^2(0, t; \mathcal{L}^2(U, H)))}^2 \\ &\lesssim \delta \|z_h^1 - z_h^2\|_{L^2(\Omega; C^0([0, t]; V^*))}^2 + \delta \|\nabla(z_h^1 - z_h^2)\|_{L^2(\Omega; L^2(0, t; H))}^2 \\ &+ C_\delta \|z_h^1 - z_h^2\|_{L^2(\Omega; L^2(0, t; V^*))}^2. \end{aligned}$$

Furthermore, since $\Psi'' \geq -c_1$, choosing δ sufficiently small and rearranging the terms yields, the Young inequality,

$$\begin{aligned} \frac{1}{2} \|z_h^1 - z_h^2\|_{L^2(\Omega; C^0([0,t]; H))}^2 + \mathbb{E} \int_{Q_t} |\nabla(z_h^1 - z_h^2)|^2 &\lesssim c_1 \mathbb{E} \int_{Q_t} |z_h^1 - z_h^2|^2 \\ &\lesssim \sigma \mathbb{E} \int_{Q_t} |\nabla(z_h^1 - z_h^2)|^2 + \tilde{c}_\sigma \mathbb{E} \int_0^t \|\nabla \mathcal{N}(z_h^1 - z_h^2)(s)\|_H^2 ds \end{aligned}$$

for every $\sigma > 0$ and a certain $\tilde{c}_\sigma > 0$. Taking σ small enough, the Gronwall lemma yields then $z_h^1 = z_h^2$, hence also $\mu_h^1 = \mu_h^2$ by comparison in the system, from which uniqueness.

Approximation. Let us focus on existence. To this end, we consider the approximation

$$\begin{aligned} dz_h^n - \Delta \mu_h^n dt &= DB(y) z_h^n dW && \text{in } (0, T) \times D, \\ \mu_h^n &= -\Delta z_h^n + \Psi_n''(y) z_h^n - h && \text{in } (0, T) \times D, \\ \partial_{\mathbf{n}} z_h^n &= \partial_{\mathbf{n}} \mu_h^n = 0 && \text{in } (0, T) \times \partial D, \\ z_h^n(0) &= 0 && \text{in } D, \end{aligned}$$

where $\Psi_n'' := T_n \circ \Psi''$ and $T_n : \mathbb{R} \rightarrow \mathbb{R}$ is the truncation operator at level n . i.e.

$$T_n(r) := \begin{cases} n & \text{if } r > n, \\ r & \text{if } |r| \leq n, \\ -n & \text{if } r < -n, \end{cases} \quad r \in \mathbb{R}.$$

Since $\Psi_n''(y) \in L^\infty(\Omega \times Q)$, it is not difficult to check that such approximated problem admits a unique solution (z_h^n, μ_h^n) satisfying (2.13)–(2.17) with Ψ_n'' instead of Ψ'' . Indeed, one can reformulate the problem in the variational triple (Z, H, Z^*) as

$$dz_h^n + A_n z_h^n dt = DB(t, y) z_h^n dW, \quad z_h^n(0) = 0,$$

where $A_n : \Omega \times [0, T] \times Z \rightarrow Z^*$ is given by

$$\langle A_n(\omega, t, x), \varphi \rangle_Z := \int_D \Delta x \Delta \varphi - \int_D \Psi_n''(y(\omega, t)) x \Delta \varphi + \int_D h(\omega, t) \Delta \varphi,$$

for $(\omega, t) \in \Omega \times [0, T]$ and $x, \varphi \in Z$. Since $\Psi_n''(y) \in L^\infty(\Omega \times Q)$, it is not difficult to check that A_n is progressively measurable, weakly monotone, weakly coercive and linearly bounded. Moreover, it is clear that the operator

$$x \mapsto DB(t, y(\omega, t))x, \quad x \in H,$$

is Lipschitz-continuous and linearly bounded from H to $\mathcal{L}^2(U, H)$, uniformly on $\Omega \times [0, T]$. Hence, the approximated problem admits a unique solution z_h^n such that, setting $\mu_h^n := -\Delta z_h^n + \Psi_n''(y) z_h^n - h$, conditions (2.13)–(2.17) are satisfied with Ψ_n'' .

Uniform estimates. Let us now prove uniform estimates independently of n and pass to the limit as $n \rightarrow \infty$. Noting that $(z_h^n)_D = 0$ by (A4), Itô's formula for the square of

the V^* -norm yields

$$\begin{aligned} & \frac{1}{2} \|\nabla \mathcal{N}(z_h^n)(t)\|_H^2 + \int_{Q_t} |\nabla z_h^n|^2 + \int_{Q_t} \Psi''(y) |z_h^n|^2 = \int_{Q_t} h z_h^n \\ & + \frac{1}{2} \int_0^t \text{Tr} (DB(s, y(s)) z_h^n(s)^* \circ \mathcal{N} \circ DB(s, y(s)) z_h^n(s)) ds \\ & + \int_0^t (\mathcal{N}(z_h^n)(s), DB(s, y(s)) z_h^n(s))_H dW(s). \end{aligned}$$

for every $t \in [0, T]$, \mathbb{P} -almost surely. Since $\Psi'' \geq -c_1$ implies that $\Psi''_n \geq -c_1$ for every $n \in \mathbb{N}$, by the Young inequality we have

$$\begin{aligned} & \frac{1}{2} \|\nabla \mathcal{N}(z_h^n)(t)\|_H^2 + \int_{Q_t} |\nabla z_h^n|^2 \leq \frac{1}{2} \int_{Q_t} |h|^2 + \left(\frac{1}{2} + c_1\right) \int_{Q_t} |z_h^n|^2 \\ & + \frac{1}{2} \int_0^t \text{Tr} (DB(s, y(s)) z_h^n(s)^* \circ \mathcal{N} \circ DB(s, y(s)) z_h^n(s)) ds \\ & + \int_0^t (\mathcal{N}(z_h^n)(s), DB(s, y(s)) z_h^n(s))_H dW(s) ., \end{aligned}$$

where, by the properties of \mathcal{N} ,

$$\int_{Q_t} |z_h^n|^2 \leq \delta \int_{Q_t} |\nabla z_h^n|^2 + C_\delta \int_0^t \|\nabla \mathcal{N} z_h^n(s)\|_H^2 ds$$

for every $\delta > 0$ and a positive constant $C_\delta > 0$. Hence, choosing δ sufficiently small, taking power 3, supremum in time and then expectations, arguing on the right-hand side exactly as in the proof of uniqueness in Section 5.1, we deduce that

$$\|z_h^n\|_{L^6(\Omega; C^0([0, T]; V^*) \cap L^2(0, T; V))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; H))} \quad \forall n \in \mathbb{N}. \quad (5.1)$$

Now we write Itô's formula for the square of the H -norm, getting

$$\begin{aligned} & \frac{1}{2} \|z_h^n(t)\|_H^2 + \int_{Q_t} |\Delta z_h^n|^2 = \int_{Q_t} \Psi''_n(y) z_h^n \Delta z_h^n - \int_{Q_t} h \Delta z_h^n \\ & + \int_0^t \|DB(s, y(s)) z_h^n(s)\|_{\mathcal{L}^2(U, H)}^2 ds + \int_0^t (z_h^n(s), DB(s, y(s)) z_h^n(s))_H dW(s) \end{aligned}$$

for every $t \in [0, T]$, \mathbb{P} -almost surely. We proceed now similarly to the previous estimate, taking supremum in time and expectations. Using the boundedness of DB together with the Burkholder-Davis-Gundy and Young inequalities on the right-hand side we have in particular that

$$\begin{aligned} & \mathbb{E} \int_0^t \|DB(s, y(s)) z_h^n(s)\|_{\mathcal{L}^2(U, H)}^2 ds + \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r (z_h^n(s), DB(s, y(s)) z_h^n(s)) dW(s) \right| \\ & \lesssim \delta \mathbb{E} \|z_h^n\|_{C^0([0, T]; H)}^2 + C_\delta \mathbb{E} \|z_h^n\|_{L^2(0, t; H)}^2 \end{aligned}$$

for every $\delta > 0$ and a certain $C_\delta > 0$. Choosing δ sufficiently small we infer that

$$\begin{aligned} & \|z_h^n\|_{L^2(\Omega; C^0([0, T]; H))}^2 + \|\Delta z_h^n\|_{L^2(\Omega; L^2(0, T; H))}^2 \\ & \lesssim \|h\|_{L^2(\Omega; L^2(0, T; H))}^2 + \mathbb{E} \int_Q |\Psi''_n(y) z_h^n|^2 + \|z_h^n\|_{L^2(\Omega; L^2(0, t; H))}^2 , \end{aligned}$$

where by (A2), the Hölder inequality and the Sobolev embedding $V \hookrightarrow L^6(D)$ and (5.1),

$$\begin{aligned} \mathbb{E} \int_Q |\Psi_n''(y) z_h^n|^2 &\lesssim_{c_2} \mathbb{E} \int_Q (1 + |y|^4) |z_h^n|^2 \\ &\leq \mathbb{E} \int_0^T \|1 + |y(s)|^4\|_{L^{3/2}(D)} \| |z_h^n(s)|^2 \|_{L^3(D)} ds \\ &\lesssim \mathbb{E} (1 + \|y\|_{L^\infty(0,T;V)}^4) \|z_h^n\|_{L^2(0,T;V)}^2 \\ &\leq (1 + \|y\|_{L^6(\Omega;L^\infty(0,T;V))}^4) \|h\|_{L^6(\Omega;L^2(0,T;H))}^2. \end{aligned}$$

The estimate (2.9) yields then, thanks to the Gronwall lemma,

$$\|z_h^n\|_{L^2(\Omega;C^0([0,T];H) \cap L^2(0,T;Z))} \lesssim \|h\|_{L^6(\Omega;L^2(0,T;H))} \quad \forall n \in \mathbb{N}. \quad (5.2)$$

By comparison in the equations we also deduce that

$$\|\mu_h^n\|_{L^2(\Omega;L^2(0,T;H))} \lesssim \|h\|_{L^6(\Omega;L^2(0,T;H))} \quad \forall n \in \mathbb{N}. \quad (5.3)$$

Passage to the limit. By the estimates (5.1)–(5.3), we deduce that there are

$$z_h \in L^2(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z)), \quad \mu_h \in L^2(\Omega; L^2(0, T; H)),$$

such that, as $n \rightarrow \infty$,

$$z_h^n \rightharpoonup z_h \quad \text{in } L^2(\Omega; L^2(0, T; Z)), \quad \mu_h^n \rightharpoonup \mu_h \quad \text{in } L^2(\Omega; L^2(0, T; H)).$$

Moreover, since $DB(\cdot, y) \in \mathcal{L}(H; \mathcal{L}^2(U, H))$ and $DB(\cdot, y)^* \in \mathcal{L}(\mathcal{L}^2(U, H); H)$, the boundedness of DB ensures that for every $\varphi \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))$ we have that $DB(\cdot, y)^* \varphi \in L^2(\Omega; L^2(0, T; H))$: hence, the weak convergence of $(z_h^n)_n$ readily implies that

$$\begin{aligned} \mathbb{E} \int_0^T (DB(s, y(s)) z_h^n(s), \varphi(s))_{\mathcal{L}^2(U, H)} ds &= \mathbb{E} \int_0^T (z_h^n(s), DB(s, y(s))^* \varphi(s))_H ds \\ &\rightarrow \mathbb{E} \int_0^T (z_h(s), DB(s, y(s))^* \varphi(s))_H ds = \mathbb{E} \int_0^T (DB(s, y(s)) z_h(s), \varphi(s))_{\mathcal{L}^2(U, H)} ds. \end{aligned}$$

Since φ is arbitrary we infer that

$$DB(\cdot, y) z_h^n \rightharpoonup B(\cdot, y) z_h \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H))),$$

hence also, by the linearity and continuity of the stochastic intragrals,

$$\int_0^\cdot DB(s, y(s)) z_h^n(s) dW(s) \rightharpoonup \int_0^\cdot DB(s, y(s)) z_h(s) dW(s) \quad \text{in } L^2(\Omega; L^2(0, T; H)).$$

It is clear then that these convergences are enough to pass to the limit in the variational formulation (2.16)–(2.17), except for the term $\Psi_n''(y) z_h^n$: let us analyse it explicitly. To this end, note that since Ψ'' has quadratic growth and $y \in L^6(\Omega; L^\infty(0, T; L^6(D)))$, we have in particular that $\Psi''(y) \in L^3(\Omega \times (0, T) \times D)$, hence also

$$\Psi_n''(y) \rightarrow \Psi''(y) \quad \text{in } L^3(\Omega \times (0, T) \times D).$$

The weak convergence of $(z_h^n)_n$ implies then that

$$\Psi_n''(y) z_h^n \rightharpoonup \Psi''(y) z_h \quad \text{in } L^{6/5}(\Omega \times (0, T) \times D).$$

Hence, letting $n \rightarrow \infty$ in (2.16)–(2.17) we deduce that (z_h, μ_h) satisfies the variational formulation of the linearized system.

5.2 Weak differentiability of the control-to-state map

We show here that the map S is weakly Gâteaux-differentiable in the sense specified by Theorem 2.3, and that its weak derivative is the unique solution z_h to (2.13)–(2.17).

Let $u, h \in \mathcal{U}'$ and fix $\varepsilon_0 > 0$ sufficiently small such that $u + \varepsilon h \in \mathcal{U}'$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Set also $y := S(u)$ and $y_h^\varepsilon := S(u + \varepsilon h)$ for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, and let z_h be the unique solution to the linearized system given by Proposition 2.4. Then we have, in the variational triple (Z, H, Z^*) ,

$$\begin{aligned} d \left(\frac{y_h^\varepsilon - y}{\varepsilon} \right) - \Delta \left(\frac{w_h^\varepsilon - w}{\varepsilon} \right) dt &= \frac{B(y_h^\varepsilon) - B(y)}{\varepsilon} dW, \\ \frac{w_h^\varepsilon - w}{\varepsilon} &= -\Delta \left(\frac{y_h^\varepsilon - y}{\varepsilon} \right) + \frac{\Psi'(y_h^\varepsilon) - \Psi'(y)}{\varepsilon} - h. \end{aligned}$$

By the continuous dependence properties (2.11)–(2.12) we have that

$$\left\| \frac{y_h^\varepsilon - y}{\varepsilon} \right\|_{L^6(\Omega; C^0([0, T]; V^*) \cap L^2(0, T; V))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; V^*))} \quad (5.4)$$

and

$$\left\| \frac{y_h^\varepsilon - y}{\varepsilon} \right\|_{L^2(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; H))}. \quad (5.5)$$

Furthermore, the mean-value theorem and the fact that Ψ'' has quadratic growth implies, by the Hölder inequality and the continuous embedding $V \hookrightarrow L^6(D)$,

$$\begin{aligned} \mathbb{E} \int_Q \left| \frac{\Psi'(y_h^\varepsilon) - \Psi'(y)}{\varepsilon} \right|^2 &\leq \mathbb{E} \int_Q \int_0^1 |\Psi''(y + \sigma(y_h^\varepsilon - y))|^2 \left| \frac{y_h^\varepsilon - y}{\varepsilon} \right|^2 d\sigma \\ &\lesssim \mathbb{E} \int_Q (1 + |y|^4 + |y_h^\varepsilon|^4) \left| \frac{y_h^\varepsilon - y}{\varepsilon} \right|^2 \\ &\lesssim \left(1 + \|y\|_{L^6(\Omega; L^\infty(0, T; V))}^4 + \|y_h^\varepsilon\|_{L^6(\Omega; L^\infty(0, T; V))}^4 \right) \left\| \frac{y_h^\varepsilon - y}{\varepsilon} \right\|_{L^6(\Omega; L^2(0, T; V))}^2, \end{aligned}$$

so that (2.9) and (5.4) imply that

$$\left\| \frac{\Psi'(y_h^\varepsilon) - \Psi'(y)}{\varepsilon} \right\|_{L^2(\Omega; L^2(0, T; H))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; H))}. \quad (5.6)$$

Moreover, the Lipschitz-continuity of B and (5.5) ensures that

$$\left\| \frac{B(\cdot, y_h^\varepsilon) - B(\cdot, y)}{\varepsilon} \right\|_{L^2(\Omega; C^0([0, T]; \mathcal{L}^2(U, H)))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; H))},$$

from which

$$\left\| \int_0^\cdot \frac{B(s, y_h^\varepsilon(s)) - B(s, y(s))}{\varepsilon} dW(s) \right\|_{L^2(\Omega; C^0([0, T]; H))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; H))}. \quad (5.7)$$

Hence by comparison in the equation we also have that

$$\left\| \frac{w_h^\varepsilon - w}{\varepsilon} \right\|_{L^2(\Omega; L^2(0, T; H))} \lesssim \|h\|_{L^6(\Omega; L^2(0, T; V))}. \quad (5.8)$$

Let us pass to the limit as $\varepsilon \searrow 0$. From the estimates (5.4)–(5.8) we deduce that there are

$$z_h \in L^2(\Omega; L^\infty(0, T; H) \cap L^2(0, T; Z)), \quad \mu_h \in L^2_p(\Omega; L^2(0, T; H))$$

such that, as $\varepsilon \searrow 0$,

$$\frac{y_h^\varepsilon - y}{\varepsilon} \rightharpoonup z_h \quad \text{in } L^2(\Omega; L^p(0, T; H) \cap L^2(0, T; Z)) \quad \forall p \in [1, +\infty), \quad (5.9)$$

$$\frac{w_h^\varepsilon - w}{\varepsilon} \rightharpoonup \mu_h \quad \text{in } L^2(\Omega; L^2(0, T; H)). \quad (5.10)$$

Moreover, note that

$$\begin{aligned} & \frac{\Psi'(y_h^\varepsilon) - \Psi'(y)}{\varepsilon} - \Psi''(y)z_h \\ &= \frac{\Psi'(y_h^\varepsilon) - \Psi'(y) - \Psi''(y)(y_h^\varepsilon - y)}{\varepsilon} + \Psi''(y) \left(\frac{y_h^\varepsilon - y}{\varepsilon} - z_h \right) \\ &= \int_0^1 (\Psi''(y + r(y_h^\varepsilon - y)) - \Psi''(y)) \frac{y_h^\varepsilon - y}{\varepsilon} dr + \Psi''(y) \left(\frac{y_h^\varepsilon - y}{\varepsilon} - z_h \right). \end{aligned}$$

Since $\Psi''(y) \in L^3(\Omega \times (0, T) \times D)$, the weak convergences proved above imply that

$$\Psi''(y) \left(\frac{y_h^\varepsilon - y}{\varepsilon} - z_h \right) \rightharpoonup 0 \quad \text{in } L^{6/5}(\Omega \times (0, T) \times D).$$

Let us show that also the first term goes to 0. To this end, note that since

$$\|y_h^\varepsilon - y\|_{L^2(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z))} \lesssim \varepsilon \|h\|_{L^6(\Omega; L^2(0, T; H))} \rightarrow 0,$$

by continuity of Ψ'' , we have, along a subsequence,

$$\Psi''(y + r(y_h^\varepsilon - y)) - \Psi''(y) \rightarrow 0 \quad \forall r \in [0, 1], \quad \text{a.e. in } \Omega \times (0, T) \times D.$$

Moreover, since Ψ'' has quadratic growth, we deduce that

$$\left| \int_0^1 (\Psi''(y + r(y_h^\varepsilon - y)) - \Psi''(y)) dr \right| \lesssim 1 + |y_h^\varepsilon|^2 + |y|^2,$$

where the right hand side is bounded in $L^3(\Omega \times (0, T) \times D)$ because y and $(y_h^\varepsilon)_\varepsilon$ are bounded in $L^6(\Omega; L^\infty(0, T; V))$ by Theorem 2.1 and $V \hookrightarrow L^6(D)$. Consequently,

$$\int_0^1 (\Psi''(y + r(y_h^\varepsilon - y)) - \Psi''(y)) dr \rightarrow 0 \quad \text{in } L^p(\Omega \times (0, T) \times D) \quad \forall p \in [2, 3).$$

In particular, we deduce that

$$\int_0^1 (\Psi''(y + r(y_h^\varepsilon - y)) - \Psi''(y)) \frac{y_h^\varepsilon - y}{\varepsilon} dr \rightharpoonup 0 \quad \text{in } L^p(\Omega \times (0, T) \times D)$$

for all $p \in [1, 6/5)$, from which

$$\frac{\Psi'(y_h^\varepsilon) - \Psi'(y)}{\varepsilon} \rightharpoonup \Psi''(y)z_h \quad \text{in } L^p(\Omega \times (0, T) \times D) \quad \forall p \in [1, 6/5). \quad (5.11)$$

Let us show the convergence of the stochastic integrals. To this end, note that

$$\begin{aligned} & \frac{B(\cdot, y_h^\varepsilon) - B(\cdot, y)}{\varepsilon} - DB(\cdot, y)z_h \\ &= \frac{B(\cdot, y_h^\varepsilon) - B(\cdot, y) - DB(\cdot, y)(y^\varepsilon - y)}{\varepsilon} + DB(\cdot, y) \left(\frac{y_h^\varepsilon - y}{\varepsilon} - z_h \right) \\ &= \int_0^1 (DB(\cdot, y + r(y_h^\varepsilon - y)) - DB(\cdot, y)) \frac{y_h^\varepsilon - y}{\varepsilon} dr + DB(\cdot, y) \left(\frac{y_h^\varepsilon - y}{\varepsilon} - z_h \right). \end{aligned}$$

The weak convergences proved above, the linearity of $DB(\cdot, y)$, the boundedness of DB and the dominated convergence theorem yields

$$DB(\cdot, y) \left(\frac{y_h^\varepsilon - y}{\varepsilon} - z_h \right) \rightharpoonup 0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H))).$$

Moreover, since $DB(t, \cdot) \in C^0(H; \mathcal{L}(H, \mathcal{L}^2(U, H)))$ by assumption (A5), recalling also that $y_h^\varepsilon \rightarrow y$ in $L^2(\Omega; C^0([0, T]; H))$, by the dominated convergence theorem we have that

$$\int_0^1 (DB(\cdot, y + r(y_h^\varepsilon - y)) - DB(\cdot, y)) dr \rightarrow 0 \quad \text{in } L^p(\Omega; L^p(0, T; \mathcal{L}(H; \mathcal{L}^2(U, H))))$$

for every $p \in [2, \infty)$. Since $\frac{y_h^\varepsilon - y}{\varepsilon} \rightharpoonup z_h$ in $L^2(\Omega; L^p(0, T; H))$, we deduce in particular that

$$\int_0^1 (DB(\cdot, y + r(y_h^\varepsilon - y)) - DB(\cdot, y)) \frac{y_h^\varepsilon - y}{\varepsilon} dr \rightharpoonup 0 \quad \text{in } L^p(\Omega; L^2(0, T; \mathcal{L}^2(U, H))),$$

Consequently, taking this information into account, we have

$$\frac{B(\cdot, y_h^\varepsilon) - B(\cdot, y)}{\varepsilon} \rightharpoonup DB(\cdot, y)z_h \quad \text{in } L^p(\Omega; L^2(0, T; \mathcal{L}^2(U, H))) \quad \forall p \in [1, 2),$$

from which

$$\int_0^\cdot \frac{B(s, y_h^\varepsilon(s)) - B(s, y(s))}{\varepsilon} dW(s) \rightharpoonup \int_0^\cdot DB(s, y(s))z_h(s) dW(s) \quad (5.12)$$

in $L^p(\Omega; L^2(0, T; H))$ for all $p \in [1, 2)$.

Hence, letting $\varepsilon \rightarrow 0$ in the variational formulation we deduce that (z_h, μ_h) satisfy the linearized system (2.13)–(2.17). Since we have already proved uniqueness for such system in the previous section, we deduce that (z_h, μ_h) is the unique solution to (2.13)–(2.17).

5.3 First-order necessary conditions for optimality

We prove here the version of first-order necessary optimality conditions contained in Theorem 2.4.

Let $\bar{u} \in \mathcal{U}$ be an optimal control and set $\bar{u} := S(\bar{u})$. For every $v \in \mathcal{U}$ let us define $h := v - u$, and $y_h^\varepsilon := S(\bar{u} + \varepsilon h)$ for every $\varepsilon > 0$. Since \mathcal{U} is convex, we have that $u + \varepsilon(v - u) \in \mathcal{U}$ for all $\varepsilon \in [0, 1]$: hence, by definition of optimal control we have that $\tilde{J}(\bar{u}) \leq \tilde{J}(\bar{u} + \varepsilon h)$, which may be rewritten

$$J(\bar{y}, \bar{u}) \leq \frac{\alpha_1}{2} \mathbb{E} \int_Q |y_h^\varepsilon - x_Q|^2 + \frac{\alpha_2}{2} \int_D |y_h^\varepsilon(T) - x_T|^2 + \frac{\alpha_3}{2} \mathbb{E} \int_Q |\bar{u} + \varepsilon h|^2.$$

Using the definition of $J(\bar{y}, \bar{u})$ and rearranging the terms we have

$$\begin{aligned} 0 &\leq \frac{\alpha_1}{2} \mathbb{E} \int_Q (|y_h^\varepsilon|^2 - |\bar{y}|^2 - 2(y_h^\varepsilon - \bar{y})x_Q) \\ &+ \frac{\alpha_2}{2} \mathbb{E} \int_D (|y_h^\varepsilon(T)|^2 - |\bar{y}(T)|^2 - 2(y_h^\varepsilon - \bar{y})(T)x_T) + \frac{\alpha_3}{2} \mathbb{E} \int_Q (\varepsilon^2 |h|^2 + 2\varepsilon \bar{u}h). \end{aligned}$$

Since the functions $x \mapsto \mathbb{E} \int_Q |x|^2$ and $x \mapsto \mathbb{E} \int_D |x|^2$ are Fréchet-differentiable in $L^2(\Omega \times Q)$ and $L^2(\Omega \times D)$, respectively, dividing by ε we get

$$\begin{aligned} 0 &\leq \alpha_1 \mathbb{E} \int_Q \left(\int_0^1 (\bar{y} + \sigma(y_h^\varepsilon - \bar{y})) d\sigma - x_Q \right) \frac{y_h^\varepsilon - \bar{y}}{\varepsilon} \\ &+ \alpha_2 \mathbb{E} \int_D \left(\int_0^1 (\bar{y} + \sigma(y_h^\varepsilon - \bar{y}))(T) d\sigma - x_T \right) \frac{y_h^\varepsilon - \bar{y}}{\varepsilon}(T) \\ &+ \alpha_3 \mathbb{E} \int_Q \bar{u}h + \frac{\alpha_3}{2} \varepsilon \|h\|_{L^2(\Omega \times Q)}^2. \end{aligned}$$

Since $\bar{u} + \varepsilon h \rightarrow \bar{u}$ in $L^6(\Omega; L^2(0, T; V))$ as $\varepsilon \searrow 0$, we deduce from (2.11)–(2.12), the definition of \mathcal{U} and the dominated convergence theorem that

$$\begin{aligned} \int_0^1 (\bar{y} + \sigma(y_h^\varepsilon - \bar{y})) d\sigma - x_Q &\rightarrow \bar{y} - x_Q \quad \text{in } L^6(\Omega; L^2(0, T; H)), \\ \int_0^1 (\bar{y} + \sigma(y_h^\varepsilon - \bar{y}))(T) d\sigma - x_T &\rightarrow \bar{y}(T) - x_T \quad \text{in } L^2(\Omega; H). \end{aligned}$$

Furthermore, by Theorem 2.3 we know that

$$\begin{aligned} \frac{y_h^\varepsilon - \bar{y}}{\varepsilon} &\rightarrow z_h \quad \text{in } L^{6/5}(\Omega; L^2(0, T; H)), \\ \frac{y_h^\varepsilon - \bar{y}}{\varepsilon}(T) &\rightarrow z_h(T) \quad \text{in } L^2(\Omega; H), \end{aligned}$$

so that letting $\varepsilon \searrow 0$ in the last inequality Theorem 2.4 is proved.

6 The adjoint problem

In this section we study the adjoint problem (1.6)–(1.9) in terms of existence and uniqueness of solutions. Moreover, we prove the refined version of first-order necessary optimality conditions contained in Theorem 2.5.

6.1 Existence-uniqueness of the adjoint problem

We prove here Proposition 2.6. Let $u \in \mathcal{U}'$ and $y := S(u)$.

Uniqueness. First of all we prove uniqueness of solutions. Let (p_i, \tilde{p}_i, q_i) satisfy (2.18)–(2.22) for $i = 1, 2$: taking the difference of the respective equations we have, setting $p := p_1 - p_2$, $\tilde{p} := \tilde{p}_1 - \tilde{p}_2$, and $q := q_1 - q_2$,

$$-dp - \Delta \tilde{p} dt + \Psi''(y) \tilde{p} dt = DB(y)^* q dt - q dW, \quad \tilde{p} = -\Delta p.$$

Itô's formula for $\frac{1}{2} \|\nabla p\|_H^2$ yields then

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \|\nabla p(t)\|_H^2 + \mathbb{E} \int_t^T \int_D |\nabla \tilde{p}(s)|^2 ds + \mathbb{E} \int_t^T \int_D \Psi''(y(s)) |\tilde{p}(s)|^2 \\ & + \frac{1}{2} \mathbb{E} \int_t^T \|\nabla q(s)\|_{\mathcal{L}^2(U, H)}^2 ds = \mathbb{E} \int_t^T (q(s), DB(s, y(s)) \tilde{p}(s))_{\mathcal{L}^2(U, H)} ds. \end{aligned}$$

Recalling assumption (A4), we have that $DB(\cdot, y) \tilde{p} \in \mathcal{L}^2(U, H_0)$, so that

$$(q, DB(\cdot, y) \tilde{p})_{\mathcal{L}^2(U, H)} = (q - q_D, DB(\cdot, y) \tilde{p})_{\mathcal{L}^2(U, H)}.$$

Taking into account (A2) and noting that $\tilde{p}_D = 0$, we get, by the Young and Poincaré inequalities and (2.1),

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \|\nabla p(t)\|_H^2 + \mathbb{E} \int_t^T \int_D |\nabla \tilde{p}(s)|^2 ds + \frac{1}{2} \mathbb{E} \int_t^T \|\nabla q(s)\|_{\mathcal{L}^2(U, H)}^2 ds \\ & \leq c_1 \mathbb{E} \int_t^T \int_D |\tilde{p}(s)|^2 ds + L_B \mathbb{E} \int_t^T \|(q - q_D)(s)\|_{\mathcal{L}^2(U, H)} (1 + \|\tilde{p}(s)\|_H) ds \\ & \leq \sigma \mathbb{E} \int_t^T \int_D |\nabla \tilde{p}(s)|^2 ds + C_\sigma \mathbb{E} \int_t^T \|\nabla \mathcal{N} \tilde{p}(s)\|_H^2 ds + \sigma \mathbb{E} \int_t^T \|\nabla q(s)\|_{\mathcal{L}^2(U, H)}^2 ds \\ & \lesssim \sigma \mathbb{E} \int_t^T \int_D |\nabla \tilde{p}(s)|^2 ds + \sigma \mathbb{E} \int_t^T \|\nabla q(s)\|_{\mathcal{L}^2(U, H)} + C_\sigma \mathbb{E} \int_t^T \|\nabla p(s)\|_H^2 ds \end{aligned}$$

for every $\sigma > 0$ for a certain $C_\sigma > 0$. Choosing σ sufficiently small and applying the Gronwall lemma yields then $\nabla \tilde{p} = 0$, from which $\tilde{p} = 0$ since $\tilde{p}_D = 0$. Since $\tilde{p} = -\Delta p$, we infer that $-\Delta p = 0$, from which $p_1 - (p_1)_D = p_2 - (p_2)_D$.

Approximation. Let us prove now existence of solution to the BSPDE (1.6)–(1.9). We perform the same approximation that we used for the linearized system in Section 5.1, and we consider for every $n \in \mathbb{N}$ the approximated problem

$$\begin{aligned} & \tilde{p}_n = -\Delta p_n && \text{in } Q, \\ -dp_n - \Delta \tilde{p}_n dt + \Psi''_n(y) \tilde{p}_n dt &= \alpha_1(y - x_Q) dt + DB(y)^* q_n dt - q_n dW && \text{in } Q, \\ \partial_{\mathbf{n}} p_n &= \partial_{\mathbf{n}} \tilde{p}_n = 0 && \text{in } \Sigma, \\ p_n(T) &= \alpha_2(y(T) - x_T) && \text{in } D, \end{aligned}$$

where $\Psi''_n := T_n \circ \Psi''$ and $T_n : \mathbb{R} \rightarrow \mathbb{R}$ is the truncation operator at level n . The variational

formulation of the approximated problem is given by

$$\begin{aligned} & \int_D p_n(t)\varphi + \int_t^T \int_D \Delta p_n(s)\Delta\varphi ds - \int_t^T \int_D \Psi_n''(y(s))\Delta p_n(s)\varphi ds \\ &= \alpha_2 \int_D (y(T) - x_T)\varphi + \alpha_1 \int_t^T \int_D (y - x_Q)(s)\varphi ds \\ &+ \int_t^T (DB(s, y(s))^* q_n(s), \varphi)_H ds - \int_D \left(\int_t^T q_n(s) dW(s) \right) \varphi \end{aligned}$$

for every $\varphi \in Z$, \mathbb{P} -almost surely, for every $t \in [0, T]$. Hence, we introduce the operator $A_n^* : \Omega \times [0, T] \times Z \rightarrow Z^*$ as

$$\langle A_n^*(\omega, t, x), \varphi \rangle_Z := \int_D \Delta x \Delta \varphi - \int_D \Psi_n''(y(\omega, t)) \Delta x \varphi$$

and note that since $\Psi_n''(y) \in L^\infty(\Omega \times Q)$, then A_n^* is progressively measurable, weakly monotone, weakly coercive and linearly bounded. Moreover, the operator $DB(\cdot, y)^*$ is uniformly bounded in $\Omega \times [0, T]$ by (A4). Hence, by the classical variational approach to BSPDEs (see [28, Sec. 3]) the approximated problem admits a unique solution (p_n, \tilde{p}_n, q_n) with

$$\begin{aligned} p_n &\in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; H)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0, T; Z)), \\ \tilde{p}_n &\in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; Z^*)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0, T; H)), \\ q_n &\in L^2_{\mathcal{P}}(\Omega; L^2(0, T; \mathcal{L}^2(U, H))). \end{aligned}$$

Moreover, by assumption we have $\alpha_2 x_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; V)$, while by Theorem 2.1 we know that $y \in L^2(\Omega; C^0([0, T]; H) \cap L^\infty(0, T; V))$, so that y is weakly continuous in V and $y(T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; V)$. Consequently, we have that $\alpha_2(y(T) - x_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; V)$, and this ensures a further regularity on (p_n, \tilde{p}_n, q_n) , namely

$$\begin{aligned} p_n &\in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; V)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0, T; H^3(D))), \\ \tilde{p}_n &\in L^2_{\mathcal{P}}(\Omega; C^0([0, T]; V^*)) \cap L^2_{\mathcal{P}}(\Omega; L^2(0, T; V)), \\ q_n &\in L^2_{\mathcal{P}}(\Omega; L^2(0, T; \mathcal{L}^2(U, V))). \end{aligned}$$

In order to prove this, one should perform a further approximation on the problem depending on a further parameter (let us say k , for example), write Itô's formula for $\frac{1}{2} \|\nabla p_n^k\|_H^2$ and then pass to the limit as $k \rightarrow \infty$. Since the procedure is quite standard, to avoid heavy notations we shall proceed formally writing Itô's formula for $\frac{1}{2} \|\nabla p_n\|_H^2$: this reads

$$\begin{aligned} & \frac{1}{2} \|\nabla p_n(t)\|_H^2 + \int_t^T \int_D |\nabla \Delta p_n(s)|^2 ds + \int_t^T \int_D \Psi_n''(y(s)) |\Delta p_n(s)|^2 ds \\ &+ \frac{1}{2} \int_t^T \|\nabla q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds - \int_t^T (\Delta p_n(s), q_n(s))_H dW(s) \\ &= \frac{\alpha_2^2}{2} \|\nabla(y(T) - x_T)\|_H^2 - \alpha_1 \int_t^T \int_D (y - x_Q)(s) \Delta p_n(s) ds \\ &+ \int_t^T (DB(s, y(s))^* q_n(s), \tilde{p}_n(s))_H ds. \end{aligned} \tag{6.1}$$

Since $\Psi''_n \in L^\infty(\Omega \times Q)$ (recall that here n is fixed) and we already know that $p_n \in L^2(\Omega; L^2(0, T; Z))$, the desired regularity follows by a classical procedure based on the Burkholder-Davis-Gundy inequality. The regularity for \tilde{p}_n follows then by comparison.

First estimate. We now prove uniform estimates independently of n and pass to the limit as $n \rightarrow \infty$. First of all, taking expectations in (6.1), and performing the same computations as in the proof of uniqueness at the beginning of Section 6.1 yields

$$\begin{aligned} & \mathbb{E} \|\tilde{p}_n(t)\|_{V^*}^2 + \mathbb{E} \int_t^T \int_D |\nabla \tilde{p}_n(s)|^2 ds + \mathbb{E} \int_t^T \|\nabla q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds \\ & \lesssim_{c_1, L_B} \|y(T)\|_{L^2(\Omega; V)}^2 + \|\alpha_2 x_T\|_{L^2(\Omega; V)}^2 + \alpha_1^2 \mathbb{E} \int_Q |y - x_Q|^2 \\ & \quad + \mathbb{E} \int_t^T \int_D |\tilde{p}_n(s)|^2 ds + \mathbb{E} \int_t^T \|\nabla q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds. \end{aligned}$$

Recalling that $\|\nabla p_n\|_H = \|\nabla \mathcal{N} \tilde{p}_n\|_H \lesssim \|\tilde{p}_n\|_{V^*}$, using the compactness inequality (2.1) on the right-hand side yields, by the Gronwall lemma,

$$\|\nabla q_n\|_{L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))} \leq c.$$

Hence, going back again in Itô's formula (6.1), we now take supremum in time and then expectations: we estimate the stochastic integral using the Burkholder-Davis Gundy inequality and integration by parts as (see. e.g. [47, Lem. 4.3])

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} \left| \int_t^T (\Delta p_n(s), q_n(s))_H dW(s) \right| \\ & \lesssim \varepsilon \mathbb{E} \|\nabla p_n\|_{C^0([0, T]; H)}^2 + C_\varepsilon \mathbb{E} \|\nabla q_n\|_{L^2(0, T; \mathcal{L}^2(U, H))}^2 \end{aligned}$$

for every $\varepsilon > 0$, so that choosing ε sufficiently small, rearranging the terms and recalling the estimate just proved on $(\nabla q_n)_n$ yields, for a positive constant c independent of n ,

$$\|\tilde{p}_n\|_{L^2(\Omega; C^0([0, T]; V^*)) \cap L^2(\Omega; L^2(0, T; V))} + \|\nabla q_n\|_{L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))} \leq c. \quad (6.2)$$

Second estimate. We write Itô's formula for $(\frac{1}{2} \|\nabla p_n\|_H^2)^3$, getting

$$\begin{aligned} & \frac{1}{8} \|\nabla p_n(t)\|_H^6 + \frac{3}{4} \int_t^T \|\nabla p_n(s)\|_H^4 \|\nabla \tilde{p}_n(s)\|_H^2 ds \\ & + \frac{3}{4} \int_t^T \|\nabla p_n(s)\|_H^4 \int_D \Psi''(y) |\tilde{p}_n(s)|^2 ds + \frac{3}{8} \int_t^T \|\nabla p_n(s)\|_H^4 \|\nabla q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds \\ & + \frac{3}{2} \int_t^T \|\nabla p_n(s)\|_H^2 \|\langle -\Delta p_n(s), q_n(s) \rangle_V\|_{\mathcal{L}^2(U, \mathbb{R})}^2 ds \\ & - \frac{3}{4} \int_t^T \|\nabla p_n(s)\|_H^4 (\Delta p_n(s), q_n(s))_H dW(s) \\ & = \frac{1}{8} \|\alpha_2 \nabla(y(T) - x_T)\|_H^6 + \frac{3}{4} \alpha_1 \int_t^T \int_D \|\nabla p_n(s)\|_H^4 (y - x_Q)(s) \tilde{p}_n(s) ds \\ & + \frac{3}{4} \int_t^T \|\nabla p_n(s)\|_H^4 (q_n(s), DB(s, y(s)) \tilde{p}_n(s))_{\mathcal{L}^2(U, H)} ds, \end{aligned} \quad (6.3)$$

By the Hölder and Young inequalities and the definition of \tilde{p}_n , for all $\delta > 0$ we have

$$\begin{aligned} & \alpha_1 \int_t^T \int_D \|\nabla p_n(s)\|_H^4 (y - x_Q)(s) \tilde{p}_n(s) ds \\ & \leq \alpha_1 \int_t^T \|\nabla p_n(s)\|_H^4 \|\tilde{p}_n(s)\|_H \|(y - x_Q)(s)\|_H ds \\ & \leq \delta \int_t^T \|\nabla p_n(s)\|_H^4 \|\nabla \tilde{p}_n(s)\|_H^2 ds + \|\alpha_1(y - x_Q)\|_{L^6(0,T;H)}^6 + C_\delta \int_t^T \|\nabla p_n(s)\|_H^6 ds. \end{aligned}$$

Moreover, recalling that $\Psi'' \geq -c_1$ and $\|\tilde{p}_n\|_{V^*} \lesssim \|\nabla p_n\|_H$, for every $\delta > 0$ we have

$$\begin{aligned} & - \mathbb{E} \int_t^T \|\nabla p_n(s)\|_H^4 \int_D \Psi''(y(s)) |\tilde{p}_n(s)|^2 ds \leq c_1 \mathbb{E} \int_t^T \|\nabla p_n(s)\|_H^4 \|\tilde{p}_n(s)\|_H^2 ds \\ & \leq \delta \mathbb{E} \int_t^T \|\nabla p_n(s)\|_H^4 \|\nabla \tilde{p}_n(s)\|_H^2 ds + C_\delta \int_t^T \mathbb{E} \|\nabla p_n(s)\|_H^6 ds. \end{aligned}$$

Similarly, by assumptions (A4)–(A5), recalling that $(DB(\cdot, y)\tilde{p}_n)_D = 0$ and writing $q = q - q_D + q_D$, and arguing as in the the proof of (6.2) we have

$$\begin{aligned} & \mathbb{E} \int_t^T \|\nabla p_n(s)\|_H^4 (q_n(s), DB(s, y(s))\tilde{p}_n(s))_{\mathcal{L}^2(U,H)} ds \\ & \lesssim_{L_B} 1 + \delta \mathbb{E} \int_t^T \|\nabla p_n(s)\|_H^4 \|\nabla q_n(s)\|_{\mathcal{L}^2(U,H)}^2 ds \\ & \quad + \delta \mathbb{E} \int_t^T \|\nabla p_n(s)\|_H^4 \|\nabla \tilde{p}_n(s)\|_H^2 ds + C_\delta \int_t^T \|\nabla p_n(s)\|_H^6 ds. \end{aligned}$$

Hence, taking expectations in (6.3), recalling the assumptions on x_T and x_Q and that $y \in L^6(\Omega; L^\infty(0, T; V))$, choosing $\delta > 0$ sufficiently small, the Gronwall lemma yields

$$\|\nabla p_n\|_{C^0([0,T];L^6(\Omega;H))}^6 + \mathbb{E} \int_0^T \|\nabla p_n(s)\|_H^4 \|\nabla q_n(s)\|_{\mathcal{L}^2(U,H)}^2 ds \leq c.$$

At this point, we go back to (6.3), take supremum in time and then expectations: estimating the stochastic integral through the Burkholder-Davis-Gundy inequality as

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0,T]} \left| \int_t^T \|\nabla p_n(s)\|_H^4 (\Delta p_n(s), q_n(s))_H dW(s) \right| \\ & \lesssim \mathbb{E} \left(\int_0^T \|\nabla p_n(s)\|_H^{10} \|\nabla q_n(s)\|_{\mathcal{L}^2(U,H)}^2 ds \right)^{1/2} \\ & \leq \mathbb{E} \|\nabla p_n\|_{C^0([0,T];H)}^3 \left(\int_0^T \|\nabla p_n(s)\|_H^4 \|\nabla q_n(s)\|_{\mathcal{L}^2(U,H)}^2 ds \right)^{1/2} \\ & \leq \delta \mathbb{E} \|\nabla p_n\|_{C^0([0,T];H)}^6 + C_\delta \mathbb{E} \int_0^T \|\nabla p_n(s)\|_H^4 \|\nabla q_n(s)\|_{\mathcal{L}^2(U,H)}^2 ds. \end{aligned}$$

Choosing $\delta > 0$, rearranging the terms and taking into account the estimate already proved, we get

$$\|\nabla p_n\|_{L^6(\Omega;C^0([0,T];H))}^6 + \mathbb{E} \int_0^T \|\nabla p_n(s)\|_H^4 \|\nabla q_n(s)\|_{\mathcal{L}^2(U,H)}^2 ds \leq c. \quad (6.4)$$

Finally, we go back to (6.1), take the 3rd-power and then expectations: using again (A2) and the Young inequality we get

$$\begin{aligned} & \mathbb{E} \sup_{r \in [t, T]} \|\nabla p_n(r)\|_H^6 + \mathbb{E} \|\nabla \tilde{p}_n\|_{L^2(t, T; H)}^6 + \mathbb{E} \|\nabla q_n\|_{L^2(t, T; \mathcal{L}^2(U, H))}^6 \\ & \lesssim_{c_1, L_B} \mathbb{E} \|\alpha_2 \nabla(y(T) - x_T)\|_H^6 + \mathbb{E} \|\tilde{p}_n\|_{L^2(t, T; H)}^6 + \mathbb{E} \|\alpha_1(y - x_Q)\|_{L^2(0, T; H)}^6 \\ & \quad + \mathbb{E} \sup_{r \in [0, T]} \left| \int_r^T (\Delta p_n(s), q_n(s))_H dW(s) \right|^3, \end{aligned}$$

where the last term is estimated thanks to the Burkholder-Davis-Gundy inequality by

$$\begin{aligned} & \mathbb{E} \left(\int_t^T \|\nabla p_n(s)\|_H^2 \|\nabla q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds \right)^{3/2} \\ & \leq \sigma \mathbb{E} \|\nabla q_n\|_{L^2(t, T; \mathcal{L}^2(U, H))}^6 + C_\sigma \mathbb{E} \|\nabla p_n\|_{C^0([0, T; H])}^6 \end{aligned}$$

for every $\sigma > 0$ and for a certain $C_\sigma > 0$. Hence, noting also that

$$\mathbb{E} \|\tilde{p}_n\|_{L^2(t, T; H)}^6 \leq \sigma \mathbb{E} \|\nabla \tilde{p}_n\|_{L^2(t, T; H)}^6 + \tilde{C}_\sigma \mathbb{E} \|\nabla p_n\|_{L^2(t, T; H)}^6$$

for a certain $\tilde{C}_\sigma > 0$, choosing $\sigma > 0$ sufficiently small, rearranging the terms and taking (6.4) into account, by the Gronwall lemma we deduce that

$$\|\tilde{p}_n\|_{L^6(\Omega; L^2(0, T; V))} + \|\nabla q_n\|_{L^6(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))} \leq c. \quad (6.5)$$

Third estimate. We write Itô's formula for $\frac{1}{2} \|p_n\|_H^2$, getting for every $t \in [0, T]$, \mathbb{P} -a.s.

$$\begin{aligned} & \frac{1}{2} \|p_n(t)\|_H^2 + \int_t^T \int_D |\nabla p_n(s)|^2 ds + \int_t^T \int_D \Psi_n''(y(s)) \tilde{p}_n(s) p_n(s) ds \\ & \quad + \frac{1}{2} \int_t^T \|q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds + \int_t^T (p_n(s), q_n(s))_H dW(s) \\ & = \frac{\alpha_2^2}{2} \|y(T) - x_T\|_H^2 + \alpha_1 \int_t^T \int_D (y - x_Q)(s) p_n(s) ds \\ & \quad + \int_t^T (q_n(s), DB(s, y(s)) p_n(s))_{\mathcal{L}^2(U, H)} ds. \end{aligned}$$

Taking expectations, using the Young inequality, the boundedness of DB and the estimates (6.2)–(6.5), we infer that, for every $t \in [0, T]$,

$$\mathbb{E} \|p_n(t)\|_H^2 + \mathbb{E} \int_t^T \int_D |\nabla p_n(s)|^2 ds + \mathbb{E} \int_t^T \|q_n(s)\|_{\mathcal{L}^2(U, H)}^2 ds \lesssim 1 + \mathbb{E} \int_Q |\Psi_n''(y) \tilde{p}_n|^2,$$

where the implicit constant is independent of n . Now, by (A2) and the Hölder and Young inequalities, we have

$$\begin{aligned} \mathbb{E} \int_Q |\Psi_n''(y) \tilde{p}_n|^2 & \lesssim \mathbb{E} \int_Q |\tilde{p}_n|^2 + \mathbb{E} \int_Q |y|^4 |\tilde{p}_n|^2 \\ & \lesssim \mathbb{E} \int_Q |\tilde{p}_n|^2 + \mathbb{E} \|y\|_{L^\infty(0, T; V)}^4 \|\tilde{p}_n\|_{L^2(0, T; V)}^2 \\ & \leq \mathbb{E} \|\tilde{p}_n\|_{L^2(0, T; H)}^2 + \mathbb{E} \|y\|_{L^\infty(0, T; V)}^6 + \mathbb{E} \|\tilde{p}_n\|_{L^2(0, T; V)}^6, \end{aligned}$$

so that by (6.2) and (6.5) we get

$$\|q_n\|_{L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))} + \|\Psi''_n(y)\tilde{p}_n\|_{L^2(\Omega; L^2(0, T; H))} \leq c. \quad (6.6)$$

Now we go back to Itô's formula for $\frac{1}{2}\|p_n\|_H^2$: instead of taking expectations straight away, we take at first supremum in time and then expectations, getting Performing the usual computation as before using the Young inequality we get, for every $t \in [0, T]$,

$$\mathbb{E} \sup_{r \in [t, T]} \|p_n(t)\|_H^2 \lesssim 1 + \mathbb{E} \sup_{r \in [t, T]} \left| \int_r^T (p_n(s), q_n(s))_H dW(s) \right|.$$

The Burkholder-Davis-Gundy and Young inequalities ensure that, for every $\delta > 0$,

$$\mathbb{E} \sup_{r \in [t, T]} \left| \int_r^T (p_n(s), q_n(s))_H dW(s) \right| \leq \delta \mathbb{E} \sup_{r \in [t, T]} \|p_n(t)\|_H^2 + C_\delta \mathbb{E} \|q_n\|_{L^2(t, T; \mathcal{L}^2(U, H))}^2,$$

so that choosing δ sufficiently small and using (6.6) we infer that

$$\|p_n\|_{L^2(\Omega; C^0([0, T]; H))} \leq c. \quad (6.7)$$

Passage to the limit. We deduce that there is (p, \tilde{p}, q) with

$$\begin{aligned} p &\in L^\infty(0, T; L^2(\Omega; V)) \cap L^2_p(\Omega; L^2(0, T; Z \cap H^3(D))), \\ \tilde{p} &\in L^\infty(0, T; L^6(\Omega; V^*)) \cap L^6_p(\Omega; L^2(0, T; V)), \\ q &\in L^2_p(\Omega; L^2(0, T; \mathcal{L}^2(U, V))), \end{aligned}$$

such that $\tilde{p} = -\Delta p$ and

$$\begin{aligned} p_n &\overset{*}{\rightharpoonup} p && \text{in } L^\infty(0, T; L^2(\Omega; V)) \cap L^2(\Omega; L^2(0, T; Z \cap H^3(D))), \\ \tilde{p}_n &\overset{*}{\rightharpoonup} \tilde{p} && \text{in } L^\infty(0, T; L^6(\Omega; V^*)) \cap L^6(\Omega; L^2(0, T; V)), \\ q_n &\rightharpoonup q && \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, V))). \end{aligned}$$

Now, we know from [34, Lem. 2.1] that the stochastic integral operator is linear continuous (hence also weakly continuous) from the space $L^2_p(\Omega; L^2(0, T; \mathcal{L}^2(U, V)))$ to the space $L^2(\Omega; W^{s,2}(0, T; V))$: consequently, we deduce that

$$\int_0^\cdot q_n(s) dW(s) \rightharpoonup \int_0^\cdot q(s) dW(s) \quad \text{in } L^2(\Omega; W^{s,2}(0, T; V)).$$

Finally, by (A2), the embedding $V \hookrightarrow L^6(D)$ and the fact that $y \in L^6(\Omega; L^\infty(0, T; V))$ it is immediate to check that $\Psi''(y) \in L^3(\Omega; L^\infty(0, T; L^3(D)))$, so in particular

$$\Psi''_n(y) \rightarrow \Psi''(y) \quad \text{in } L^3(\Omega \times Q).$$

Hence, since by the convergences of $(\tilde{p}_n)_n$ we have $\tilde{p}_n \rightharpoonup \tilde{p}$ in $L^2(\Omega \times Q)$, so that

$$\Psi''_n(y)\tilde{p}_n \rightharpoonup \Psi''(y)\tilde{p} \quad \text{in } L^{6/5}(\Omega \times Q).$$

Similarly, it is a standard matter to check that the weak convergence of $(q_n)_n$ and the boundedness of DB imply

$$DB(\cdot, y)^* q_n \rightharpoonup DB(\cdot, y)^* q \quad \text{in } L^2(\Omega; L^2(0, T; H)).$$

Hence, passing to the weak limit as $n \rightarrow \infty$, we get that (p, \tilde{p}, q) is a solution to the (2.18)–(2.22). Finally, note the extra regularities $p \in C_w^0([0, T]; L^2(\Omega; V))$ and $\tilde{p} \in C_w^0([0, T]; L^6(\Omega; V^*))$ follow a posteriori by comparison in the limit equation.

6.2 Duality and conclusion

In this final section we prove the last Theorem 2.5 containing the simpler version of first-order necessary conditions on optimality. The main idea is to remove the dependence on z in the variational inequality of Theorem 2.4 by using the adjoint problem and a suitable duality relation between z and \tilde{p} .

Let then $\bar{u} \in \mathcal{U}$ be an optimal control and $\bar{y} := S(\bar{u})$ be the corresponding solution to the state equation. Then we know that the adjoint problem admits a solution (p, \tilde{p}, q) solving (2.18)–(2.22), where \tilde{p} is uniquely determined. Let $v \in \mathcal{U}$ be arbitrary and set $h := v - \bar{u}$: the main point is to prove the duality relation

$$\alpha_1 \mathbb{E} \int_Q (\bar{y} - x_Q) z_h + \alpha_2 \mathbb{E} \int_D (\bar{y}(T) - x_T) z_h(T) = \mathbb{E} \int_Q \tilde{p} h.$$

If we are able to prove such duality result, then it is clear that Theorem 2.5 follows directly from Theorem 2.4.

Let $(z_h^n)_n$ and $(p_n, \tilde{p}_n, q_n)_n$ be the approximated solutions introduced in Sections 5.1 and 6.1: then we have

$$\begin{aligned} z_h^n &\in L^2(\Omega; C^0([0, T]; H) \cap L^2(0, T; Z)), \\ \tilde{p}_n &\in C^0([0, T]; L^2(0, T; V^*)) \cap L^2(\Omega; L^2(0, T; V)), \end{aligned}$$

with $z_h^n(0) = 0$, $p_n(T) = \alpha_2(\bar{y}(T) - x_T)$, and

$$\begin{aligned} dz_h^n - \Delta(-\Delta z_h^n + \Psi_n''(\bar{y})z_h^n - h) dt &= DB(\bar{y})z_h^n dW, \\ -dp_n - \Delta\tilde{p}_n dt + \Psi_n''(\bar{y})\tilde{p}_n dt &= \alpha_1(\bar{y} - x_Q) dt + D(\bar{y})^*q_n dt - q_n dW, \end{aligned}$$

where the equations are intended in the Hilbert triplet (Z, H, Z^*) . We deduce in particular that

$$d(z_h^n, p_n)_H = (z_h^n, dp_n)_H + \langle dz_h^n, p_n \rangle_Z + (DB(\bar{y})z_h^n, q_n)_{\mathcal{L}^2(U, H)} dt,$$

where

$$(z_h^n, p_n)_H(0) = 0, \quad (z_h^n, p_n)_H(T) = \alpha_2 \int_D (\bar{y}(T) - x_T) z_h^n(T).$$

Writing Itô's formula for $(z_h^n, p_n)_H$ yields then

$$\begin{aligned} \alpha_2 \mathbb{E} \int_D (\bar{y}(T) - x_T) z_h^n(T) &= -\mathbb{E} \int_Q \Delta z_h^n \tilde{p}_n + \mathbb{E} \int_Q \Psi_n''(\bar{y}) \tilde{p}_n z_h^n - \alpha_1 \mathbb{E} \int_Q (\bar{y} - x_Q) z_h^n \\ &- \mathbb{E} \int_0^T (DB(s, \bar{y}(s))^* q_n(s), z_h^n(s))_H ds - \mathbb{E} \int_Q \Delta z_h^n \Delta p_n + \mathbb{E} \int_Q \Psi_n''(\bar{y}) z_h^n \Delta p_n \\ &- \mathbb{E} \int_Q h \Delta p_n + \mathbb{E} \int_0^T (DB(s, \bar{y}(r)) z_h^n(s), q_n(s))_{\mathcal{L}^2(U, H)} ds, \end{aligned}$$

from which, recalling the definition of $DB(\cdot, \bar{y})^*$ and that $-\Delta p_n = \tilde{p}_n$,

$$\alpha_1 \mathbb{E} \int_Q (\bar{y} - x_Q) z_h^n + \alpha_2 \int_D (\bar{y}(T) - x_T) z_h^n(T) = \mathbb{E} \int_Q h \tilde{p}_n \quad \forall n \in \mathbb{N}.$$

The thesis now follows letting $n \rightarrow \infty$.

References

- [1] D. C. ANTONOPOULOU, G. KARALI, AND A. MILLET, *Existence and regularity of solution for a stochastic Cahn-Hilliard/Allen-Cahn equation with unbounded noise diffusion*, J. Differential Equations, 260 (2016), pp. 2383–2417.
- [2] V. BARBU, M. RÖCKNER, AND D. ZHANG, *Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise*, Ann. Probab., 46 (2018), pp. 1957–1999.
- [3] C. BAUZET, E. BONETTI, G. BONFANTI, F. LEBON, AND G. VALLET, *A global existence and uniqueness result for a stochastic Allen-Cahn equation with constraint*, Math. Methods Appl. Sci., 40 (2017), pp. 5241–5261.
- [4] E. BONETTI, P. COLLI, L. SCARPA, AND G. TOMASSETTI, *A doubly nonlinear Cahn-Hilliard system with nonlinear viscosity*, Commun. Pure Appl. Anal., 17 (2018), pp. 1001–1022.
- [5] D. Breit, E. Feireisl, and M. Hofmanová. *Stochastically forced compressible fluid flows*, volume 3 of *De Gruyter Series in Applied and Numerical Mathematics*. De Gruyter, Berlin, 2018.
- [6] Z. A. BRZEŹNIAK AND R. SERRANO, *Optimal relaxed control of dissipative stochastic partial differential equations in Banach spaces*, SIAM J. Control Optim., 51 (2013), pp. 2664–2703.
- [7] J. W. CAHN AND J. E. HILLIARD, *Free energy of a nonuniform system. i. interfacial free energy*, The Journal of Chemical Physics, 28 (1958), pp. 258–267.
- [8] L. CHERFILS, S. GATTI, AND A. MIRANVILLE, *A variational approach to a Cahn-Hilliard model in a domain with nonpermeable walls*, J. Math. Sci. (N.Y.), 189 (2013), pp. 604–636.
- [9] L. CHERFILS, A. MIRANVILLE, AND S. ZELIK, *The Cahn-Hilliard equation with logarithmic potentials*, Milan J. Math., 79 (2011), pp. 561–596.
- [10] L. CHERFILS AND M. PETCU, *A numerical analysis of the Cahn-Hilliard equation with non-permeable walls*, Numer. Math., 128 (2014), pp. 517–549.
- [11] P. COLLI, M. H. FARSHBAF-SHAKER, G. GILARDI, AND J. SPREKELS, *Optimal boundary control of a viscous Cahn-Hilliard system with dynamic boundary condition and double obstacle potentials*, SIAM J. Control Optim., 53 (2015), pp. 2696–2721.
- [12] P. COLLI, M. H. FARSHBAF-SHAKER, AND J. SPREKELS, *A deep quench approach to the optimal control of an Allen-Cahn equation with dynamic boundary conditions and double obstacles*, Appl. Math. Optim., 71 (2015), pp. 1–24.
- [13] P. COLLI AND T. FUKAO, *Cahn-Hilliard equation with dynamic boundary conditions and mass constraint on the boundary*, J. Math. Anal. Appl., 429 (2015), pp. 1190–1213.

- [14] P. COLLI AND T. FUKAO, *Equation and dynamic boundary condition of Cahn-Hilliard type with singular potentials*, *Nonlinear Anal.*, 127 (2015), pp. 413–433.
- [15] P. COLLI AND T. FUKAO, *Nonlinear diffusion equations as asymptotic limits of Cahn-Hilliard systems*, *J. Differential Equations*, 260 (2016), pp. 6930–6959.
- [16] P. COLLI, G. GILARDI, P. PODIO-GUIDUGLI, AND J. SPREKELS, *Distributed optimal control of a nonstandard system of phase field equations*, *Contin. Mech. Thermodyn.*, 24 (2012), pp. 437–459.
- [17] P. COLLI, G. GILARDI, AND J. SPREKELS, *Analysis and optimal boundary control of a nonstandard system of phase field equations*, *Milan J. Math.*, 80 (2012), pp. 119–149.
- [18] P. COLLI, G. GILARDI, AND J. SPREKELS, *On the Cahn-Hilliard equation with dynamic boundary conditions and a dominating boundary potential*, *J. Math. Anal. Appl.*, 419 (2014), pp. 972–994.
- [19] P. COLLI, G. GILARDI, AND J. SPREKELS, *A boundary control problem for the pure Cahn-Hilliard equation with dynamic boundary conditions*, *Adv. Nonlinear Anal.*, 4 (2015), pp. 311–325.
- [20] P. COLLI, G. GILARDI, AND J. SPREKELS, *A boundary control problem for the viscous Cahn-Hilliard equation with dynamic boundary conditions*, *Appl. Math. Optim.*, 73 (2016), pp. 195–225.
- [21] P. COLLI AND L. SCARPA, *From the viscous Cahn-Hilliard equation to a regularized forward-backward parabolic equation*, *Asymptot. Anal.*, 99 (2016), pp. 183–205.
- [22] P. COLLI AND J. SPREKELS, *Optimal control of an Allen-Cahn equation with singular potentials and dynamic boundary condition*, *SIAM J. Control Optim.*, 53 (2015), pp. 213–234.
- [23] H. COOK, *Brownian motion in spinodal decomposition*, *Acta Metallurgica*, 18 (1970), pp. 297 – 306.
- [24] F. CORNALBA, *A nonlocal stochastic Cahn-Hilliard equation*, *Nonlinear Anal.*, 140 (2016), pp. 38–60.
- [25] G. DA PRATO AND A. DEBUSSCHE, *Stochastic Cahn-Hilliard equation*, *Nonlinear Anal.*, 26 (1996), pp. 241–263.
- [26] A. DEBUSSCHE AND L. GOUDENÈGE, *Stochastic Cahn-Hilliard equation with double singular nonlinearities and two reflections*, *SIAM J. Math. Anal.*, 43 (2011), pp. 1473–1494.
- [27] A. DEBUSSCHE AND L. ZAMBOTTI, *Conservative stochastic Cahn-Hilliard equation with reflection*, *Ann. Probab.*, 35 (2007), pp. 1706–1739.
- [28] K. DU AND Q. MENG, *A revisit to W_2^n -theory of super-parabolic backward stochastic partial differential equations in \mathbb{R}^d* , *Stochastic Process. Appl.*, 120 (2010), pp. 1996–2015.

- [29] K. DU AND Q. MENG, *A maximum principle for optimal control of stochastic evolution equations*, SIAM J. Control Optim., 51 (2013), pp. 4343–4362.
- [30] N. ELEZOVIĆ AND A. MIKELIĆ, *On the stochastic Cahn-Hilliard equation*, Nonlinear Anal., 16 (1991), pp. 1169–1200.
- [31] C. M. ELLIOTT AND Z. SONGMU, *On the Cahn-Hilliard equation*, Arch. Rational Mech. Anal., 96 (1986), pp. 339–357.
- [32] C. M. ELLIOTT AND A. M. STUART, *Viscous Cahn-Hilliard equation. II. Analysis*, J. Differential Equations, 128 (1996), pp. 387–414.
- [33] E. FEIREISL AND M. PETCU, *A diffuse interface model of a two-phase flow with thermal fluctuations*, ArXiv e-prints, (2018).
- [34] F. FLANDOLI AND D. GATAREK, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields, 102 (1995), pp. 367–391.
- [35] M. FUHRMAN, Y. HU, AND G. TESSITORE, *Stochastic maximum principle for optimal control of SPDEs*, Appl. Math. Optim., 68 (2013), pp. 181–217.
- [36] M. FUHRMAN AND C. ORRIERI, *Stochastic maximum principle for optimal control of a class of nonlinear SPDEs with dissipative drift*, SIAM J. Control Optim., 54 (2016), pp. 341–371.
- [37] G. GILARDI, A. MIRANVILLE, AND G. SCHIMPERNA, *On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions*, Commun. Pure Appl. Anal., 8 (2009), pp. 881–912.
- [38] G. GILARDI, A. MIRANVILLE, AND G. SCHIMPERNA, *Long time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions*, Chin. Ann. Math. Ser. B, 31 (2010), pp. 679–712.
- [39] L. GOUDENÈGE, *Stochastic Cahn-Hilliard equation with singular nonlinearity and reflection*, Stochastic Process. Appl., 119 (2009), pp. 3516–3548.
- [40] I. GYÖNGY AND N. KRYLOV, *Existence of strong solutions for Itô's stochastic equations via approximations*, Probab. Theory Related Fields, 105 (1996), pp. 143–158.
- [41] C. HAO AND G. WANG, *Well-posedness for the stochastic viscous Cahn-Hilliard equation*, J. Nonlinear Convex Anal., 18 (2017), pp. 2219–2228.
- [42] M. HINTERMÜLLER AND D. WEGNER, *Distributed optimal control of the Cahn-Hilliard system including the case of a double-obstacle homogeneous free energy density*, SIAM J. Control Optim., 50 (2012), pp. 388–418.
- [43] M. HOFMANOVÁ, *Degenerate parabolic stochastic partial differential equations*, Stochastic Process. Appl., 123 (2013), pp. 4294–4336.
- [44] N. IKEDA AND S. WATANABE, *Stochastic differential equations and diffusion processes*, vol. 24 of North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second ed., 1989.

- [45] X. JU, H. WANG, D. LI, AND J. DUAN, *Global mild solutions and attractors for stochastic viscous Cahn-Hilliard equation*, Abstr. Appl. Anal., (2011), pp. Art. ID 670786, 22.
- [46] D. LEE, J.-Y. HUH, D. JEONG, J. SHIN, A. YUN, AND J. KIM, *Physical, mathematical, and numerical derivations of the Cahn-Hilliard equation*, Computational Materials Science, 81 (2014), pp. 216 – 225.
- [47] C. MARINELLI AND L. SCARPA. *A variational approach to dissipative SPDEs with singular drift*, Ann. Probab., 46 (2018), pp. 1455–1497.
- [48] A. MIRANVILLE AND G. SCHIMPERNA, *On a doubly nonlinear Cahn-Hilliard-Gurtin system*, Discrete Contin. Dyn. Syst. Ser. B, 14 (2010), pp. 675–697.
- [49] A. NOVICK-COHEN, *On the viscous Cahn-Hilliard equation*, in Material instabilities in continuum mechanics (Edinburgh, 1985–1986), Oxford Sci. Publ., Oxford Univ. Press, New York, 1988, pp. 329–342.
- [50] C. ORRIERI, *A stochastic maximum principle with dissipativity conditions*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 5499–5519.
- [51] C. ORRIERI AND L. SCARPA, *Singular stochastic Allen-Cahn equations with dynamic boundary conditions*, J. Differential Equations, 266 (2019), pp. 4624–4667.
- [52] C. ORRIERI, G. TESSITORE, AND P. VEVERKA, *Ergodic maximum principle for stochastic systems*, Appl. Math. Optim., (2017).
- [53] C. ORRIERI AND P. VEVERKA, *Necessary stochastic maximum principle for dissipative systems on infinite time horizon*, ESAIM Control Optim. Calc. Var., 23 (2017), pp. 337–371.
- [54] E. ROCCA AND J. SPREKELS, *Optimal distributed control of a nonlocal convective Cahn-Hilliard equation by the velocity in three dimensions*, SIAM J. Control Optim., 53 (2015), pp. 1654–1680.
- [55] L. SCARPA, *On the stochastic Cahn-Hilliard equation with a singular double-well potential*, Nonlinear Anal., 171 (2018), pp. 102–133.
- [56] L. SCARPA, *The stochastic viscous Cahn-Hilliard equation: well-posedness, regularity and vanishing viscosity limit*, ArXiv e-prints, (2018).
- [57] L. SCARPA, *Existence and uniqueness of solutions to singular Cahn-Hilliard equations with nonlinear viscosity terms and dynamic boundary conditions*, J. Math. Anal. Appl., 469 (2019), pp. 730 – 764.
- [58] J. SIMON, *Compact sets in the space $L^p(0, T; B)$* , Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.
- [59] A. W. VAN DER VAART AND J. A. WELLNER, *Weak convergence and empirical processes*, Springer Series in Statistics, Springer-Verlag, New York, 1996.

- [60] G. Vallet and A. Zimmermann. Well-posedness for a pseudomonotone evolution problem with multiplicative noise. *J. Evol. Equ.*, 19:153–202, 2019.
- [61] J. YONG AND X. Y. ZHOU, *Stochastic controls*, vol. 43 of Applications of Mathematics (New York), Springer-Verlag, New York, 1999.