Elastic three-dimensional metaframe for directional wave filtering and polarization control: Supplementary Material

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To aid insight into the underlying physics of the elastic metaframe, described in the main text, we give a detailed description of the analytical and numerical models.

I. SUPPLEMENTARY NOTE 1 - EQUIVALENT PROPERTIES OF THE RESONATOR

The aim of this section is to show how the equivalent stiffness and participating mass of the resonators are evaluated. Such parameters are employed in the main text to compute the dispersion relation of the system through a simplified model. We simplify the resonators by using a 2 degrees of freedom lumped model. The elastic connection is modeled using Timoshenko beam theory, while the tip sphere is considered as a rigid mass.

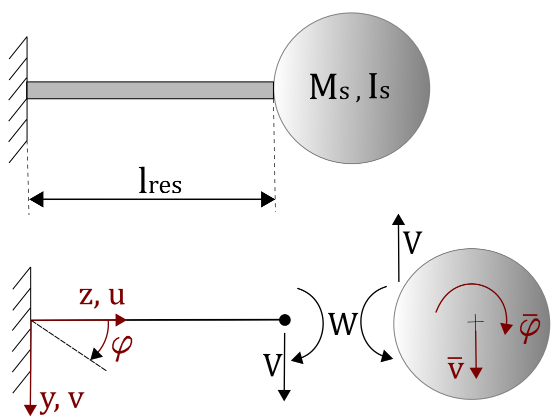

FIG. S1. Schematic of the resonators attached to the elastic frame.

Assuming a negligible mass of the beam with respect to the sphere, the governing equilibrium equations for the beam are:

$$\begin{cases} \dfrac{d^2 v}{dz^2} - \dfrac{d\varphi}{dz} = 0 \\ EI\dfrac{d^2\varphi}{dz^2} + GA^*\left(\dfrac{dv}{dz} - \varphi\right) = 0 \end{cases} \tag{S1}$$

where $\bar{v}$ and $\bar{\varphi}$ are the translation and rotation of the mass respectively, with kinematic and static boundary conditions:

$$\begin{cases} v\big|_{z=0} = 0 \\ \varphi\big|_{z=0} = 0 \\ \dfrac{d\varphi}{dz}\Big|_{z=l} = -\dfrac{W}{EI} \\ \left(\dfrac{dv}{dz} - \varphi\right)\Big|_{z=l} = \dfrac{V}{GA^*} \end{cases} \tag{S2}$$

where E is the material Young modulus, I the second moment of area, A^* the shear area and G the shear modulus. By enforcing the boundary conditions we get:

$$\begin{aligned} v &= -\frac{V}{6EI}z^3 + \left(\frac{Vl}{2EI} - \frac{W}{2EI}\right)z^2 + \frac{V}{GA^*}z \\ \varphi &= -\frac{V}{2EI}z^2 + \left(\frac{Vl}{EI} - \frac{W}{EI}\right)z \end{aligned} \tag{S3}$$

The equilibrium equations for the rigid sphere are:

$$\begin{cases} M_s\dfrac{d^2\bar{v}}{dt^2} + V = 0 \\ I_s\dfrac{d^2\bar{\varphi}}{dt^2} + M_sR\dfrac{d^2\bar{v}}{dt^2} + W = 0 \end{cases} \tag{S4}$$

being $\bar{\varphi} = \varphi(l)$ and $\bar{v} = v(l) + \varphi(l)R$, with R the sphere radius. By expressing the unknown static quantities V and W in terms of $\varphi(l)$ and $v(l)$ from Eq. S3, we write Eq. S4 as:

$$\begin{cases} M_s\dfrac{d^2v(l)}{dt^2} + M_sR\dfrac{d^2\varphi(l)}{dt^2} + K_v v(l) - \dfrac{K_v l}{2}\varphi(l) = 0 \\ M_sR\dfrac{d^2v(l)}{dt^2} + \left(I_s + M_sR^2\right)\dfrac{d^2\varphi(l)}{dt^2} - \dfrac{K_v l}{2}v(l) + K_\varphi\varphi(l) = 0 \end{cases} \tag{S5}$$

with $K_v = (12EI/l^3)/(1 + 12EI/GA^*l^2)$ and $K_\varphi = (3EI/l)/(1 + 12EI/GA^*l^2) + EI/l$. Imposing harmonic solutions of the form $v(l) = \tilde{V}e^{i\omega t}$ and $\varphi(l) = \tilde{\Phi}e^{i\omega t}$, the following eigenvalue problem is obtained:

$$\left\{ \begin{bmatrix} M_s & M_sR \\ M_sR & (I_s + M_sR^2) \end{bmatrix}\omega^2 - \begin{bmatrix} K_v & -\dfrac{K_v l}{2} \\ -\dfrac{K_v l}{2} & K_\varphi \end{bmatrix} \right\} \begin{bmatrix} \tilde{V} \\ \tilde{\Phi} \end{bmatrix} = 0 \tag{S6}$$

By solving the eigenvalue problem, we finally get the resonance frequencies of the resonators:

$$\omega^2 = \frac{2M_sR\frac{K_v l}{2} + K_v(I_s + M_sR^2) + K_\varphi M_s}{2M_sI_s} \pm \frac{\sqrt{\begin{Bmatrix} [2M_sR\frac{K_v l}{2} + K_v(I_s + M_sR^2) - K_\varphi M_s]^2 + \\ +4K_vK_\varphi M_s^2R^2 + 4\frac{K_v l}{2}M_s(\frac{K_v l}{2}I_s + 2M_sRK_\varphi) \end{Bmatrix}}}{2M_sI_s} \tag{S7}$$

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and the eigenvectors $\bar{V}$ and $\bar{\Phi}$ corresponding to the obtained eigenvalues. By doing so, we finally get the modal stiffness and modal mass:

$$\begin{cases} k_{mod} = K_v \bar{V}^2 - 2\frac{k_v l}{2}\bar{V}\bar{\Phi} + K_\varphi \bar{\Phi}^2 \\ m_{mod} = M_s \bar{V}^2 + 2M_s R \bar{V}\bar{\Phi} + (I_s + M_s R^2)\bar{\Phi}^2 \end{cases} \tag{S8}$$

and the modal participation factor:

$$\Gamma = \frac{M_s \bar{V} + M_s R \bar{\Phi}}{m_{mod}} \tag{S9}$$

Finally, the resonating mass is defined as:

$$m_{res} = m_{mod}\Gamma^2 \tag{S10}$$

II. SUPPLEMENTARY NOTE 2 - ANALYTICAL BANDGAP PREDICTION

We model the metaframe using a simple spring-mass chain, as depicted in Fig. S2. The stiffness k of the main structure is the axial or flexural stiffness of the frame depending on the polarization of the wave, while for the resonators we consider only the flexural stiffness, since the axial stiffness is assumed infinite. m represent the lumped mass of the frame.

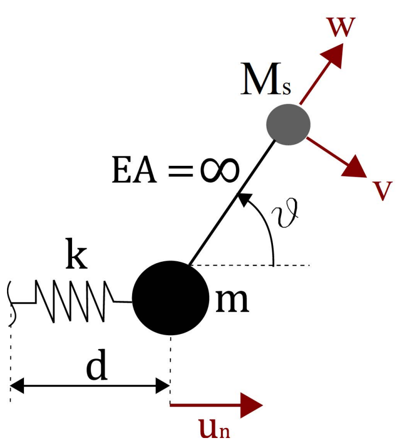

FIG. S2. Schematic of the lumped spring-mass chain model used to predict the local resonance bandgap.

The equation governing the movement of the primary spring-mass chain is given by :

$$m\ddot{u}_n = k(u_{n+1} - u_n) + k(u_{n-1} - u_n) + \\ + \sum_j V_j \sin(\theta_j) + \sum_j H_j \cos(\theta_j) \tag{S11}$$

where V_j and H_j are the transverse and axial base reaction, respectively, for the j-th resonator attached to each node of the spring-mass chain.

Each resonator is subject to the base motion, projected in the transverse and axial directions:

$$\begin{aligned} v_{bj} &= u_n \sin(\theta_j) \\ u_{bj} &= u_n \cos(\theta_j) \end{aligned} \tag{S12}$$

By assuming infinite axial stiffness, one gets:

$$H_j = -M_s \ddot{u}_n \cos(\theta_j) \tag{S13}$$

The transverse reaction is obtained on the basis of the analysis of the resonator. Eq. S5 is written in matrix form, with the addition of the base motion along the transverse direction only $\mathbf{U}_b = \mathbf{T} u_n \sin\theta_j$; $\mathbf{T} = [1 \quad 0]^T$ is the projecting vector.

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = -\mathbf{M}\ddot{\mathbf{U}}_b \tag{S14}$$

The reaction force is given by:

$$V_j = -\mathbf{T}^T \mathbf{M}(\ddot{\mathbf{U}} + \ddot{\mathbf{U}}_b) \tag{S15}$$

The modal projection is adopted $\mathbf{U} = \mathbf{\Psi} q$, using the first dominant mode of the resonator and the relevant eigenvector Ψ as computed in the previous Section. As a consequence, one gets:

$$V_j = -m_{mod}\Gamma \ddot{q} - M_s \ddot{u}_n \sin(\theta_j) \tag{S16}$$

The solution is sought in the form $u_n = \tilde{u} e^{i(\kappa x_n - \omega t)}$, so that the modal response of the resonator reads:

$$q = \frac{\omega^2}{\omega_R^2 - \omega^2}\Gamma \sin\theta_j \tilde{u} e^{i(\kappa x_n - \omega t)} \tag{S17}$$

After some algebra, the equation of motion S11 becomes:

$$-m\omega^2 \tilde{u} = k(e^{i\kappa d} + e^{-i\kappa d} - 2) + \\ + \omega^2 \left(m_{res}\frac{\omega^2}{\omega_R^2 - \omega^2} + M_s \right) \sum_j \sin^2(\theta_i)\tilde{u} + \omega^2 M_s \sum_j \cos^2(\theta_j)\tilde{u} \tag{S18}$$

The second addend in the r.h.s can be modified by defining the non-resonating mass as $m_{nres} = M_s - m_{res}$. Then, the dispersion relation is written as follows:

$$\cos(\kappa d) = 1 - \frac{\omega^2}{2k}\left[m + m_{nres}\sum_j \sin^2(\theta_j) + M_s \sum_j \cos^2(\theta_j) \right] + \\ - \frac{\omega^2}{2k}\frac{\omega_R^2}{\omega_R^2 - \omega^2} m_{res} \sum_j \sin^2(\theta_j) \tag{S19}$$

The dispersive behavior of the spring-mass chain is dominated by the effective values of non-resonating mass and resonating mass. The former value is given by the mass of the frame added to the non-resonating mass for flexural motion of the resonator and to the axial contribution:

$$M = m + m_{nres}\sum_j \sin^2(\theta_j) + M_s \sum_j \cos^2(\theta_j) \tag{S20}$$

The effective resonating mass is:

$$m_R = m_{res}\sum_j \sin^2(\theta_j) \tag{S21}$$

For the specific geometry considered in this paper, one finds that $\sum_j \sin^2(\theta_j) = 5.33$ and $\sum_j \cos^2(\theta_j) = 2.67$.

By defining $\Omega^2 = k/M$ and introducing the non-dimensional quantities $\bar{\omega}^2 = \omega^2/\Omega^2$, $\bar{\omega}_R^2 = \omega_R^2/\Omega^2$, $\mu = \kappa d$, the dispersion relation is:

$$\cos(\mu) = 1 - \frac{\bar{\omega}^2}{2}\left[1 + \frac{m_R}{M}\frac{\bar{\omega}_R^2}{\bar{\omega}_R^2 - \bar{\omega}^2}\right] \tag{S22}$$

Evanescent are supported if (i) $\cos(\mu) \geq 1$ and (ii) $\cos(\mu) \leq -1$. Condition (i) gives:

- if $\bar{\omega}^2 < \bar{\omega}_R^2 \Rightarrow \nexists\ \bar{\omega}$
- if $\bar{\omega}^2 > \bar{\omega}_R^2 \Rightarrow \bar{\omega}_R^2 < \bar{\omega}^2 < \bar{\omega}_R^2\left[1 + \frac{m_R}{M}\right]$

Condition (ii) gives:

- if $\bar{\omega}^2 < \bar{\omega}_R^2 \Rightarrow \bar{\omega}_1^2 < \bar{\omega}^2 < \bar{\omega}_R^2$
- if $\bar{\omega}^2 > \bar{\omega}_R^2 \Rightarrow \bar{\omega}^2 > \bar{\omega}_2^2 > \bar{\omega}_R^2$

where:

$$\bar{\omega}_1^2 = \frac{4 + \bar{\omega}_R^2(1 + \frac{m_R}{M}) - \sqrt{[4 + \bar{\omega}_R^2(1 + \frac{m_R}{M})]^2 - 16\bar{\omega}_R^2}}{2}$$
$$\bar{\omega}_2^2 = \frac{4 + \bar{\omega}_R^2(1 + \frac{m_R}{m}) + \sqrt{[4 + \bar{\omega}_R^2(1 + \frac{m_R}{M})]^2 - 16\bar{\omega}_R^2}}{2} \tag{S23}$$

In conclusion, evanescent waves exist in the range $[\bar{\omega}_1, \bar{\omega}_R\sqrt{1 + m_R/M}]$ and $[\bar{\omega}_2, +\infty)$.

We sum up in table S1 the main elastic and inertial properties of the resonator and the main structure used to obtain the analytical dispersion curves. The last row contains the effective non-resonating and resonating mass, computed on the basis of the aforementioned definition and of the geometry of the considered metaframe.

	Frame	Resonators
k_{axial} $[N/mm]$	507.520	∞
$k_{flexural}$ $[N/mm]$	3.248	34.412
$m_{no-res./res.}$ $[g]$	20.471	21.297

TABLE S1. Elastic and inertial parameters adopted in the analytical lumped model.

III. SUPPLEMENTARY NOTE 3 - ADDITIONAL FEM ANALYSES

To validate the size-independent behaviour of the proposed elastic metaframe, we perform FEM analyses on larger samples, with 6 and 9 cells along the propagation direction of the wave. Fig. S3 shows the response of the metaframe for increasing number of cells along the wave propagation direction. It can be noticed that increasing the number of cells, transverse waves still are much more attenuated than longitudinal waves. Such behaviour totally complies with the imaginary part of the dispersion relation, which shows a different attenuation for transverse and longitudinal waves.

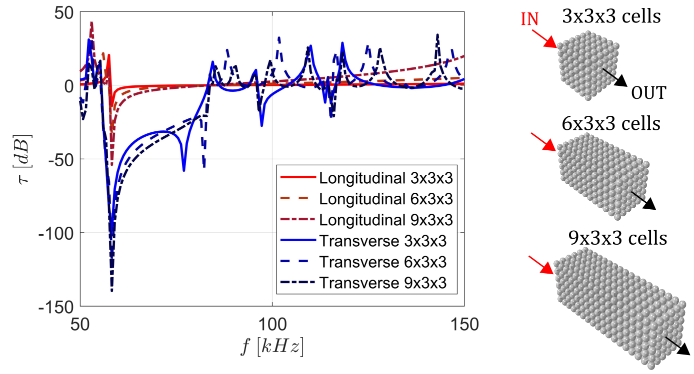

FIG. S3. Transmission analyses for 3, 6 and 9 cells along the wave propagation direction. Even increasing the number of cells, a remarkably different behaviour for longitudinal and transverse waves is observed.

IV. SUPPLEMENTARY NOTE 4 - NUMERICAL FEM MODEL

The numerical FEM model is carried out using the commercial software Abaqus®. The metaframe is modeled as a deformable three-dimensional structure which is discretized through a free mesh made of 10-node quadratic tetraedra (C3D10). The dispersion curves are obtained through a modal analysis of the unit cell complemented with a Matlab code to apply Bloch-Floquet boundary conditions. Transmission spectra are obtained performing steady-state dynamics direct analyses, in which the viscoelastic model is introduced by means of Prony series relaxation functions, defined by the coefficients g_i and τ_i, that Abaqus® relates to the storage $G_s(\omega)$ and loss moduli $G_l(\omega)$.[1]

REFERENCES

[1]M. Smith, ABAQUS/Standard 2018 User's Manual, Dassault Systèmes Simulia Corp., USA (2018)