Available online at www.sciencedirect.com

ScienceDirect

Procedia CIRP 103 (2021) 20–25

www.elsevier.com/locate/procedia

9th CIRP Global Web Conference – Sustainable, resilient, and agile manufacturing and service operations : Lessons from COVID-19

A Markov Chain model for the performance evaluation of manufacturing lines with general processing times

Salvatore Scrivano[a]*, Tullio Tolio[a]

[a]Department of Mechanical Engineering, Politecnico di Milano, via La Masa 1, 20156 Milano, Italy

* Corresponding author. Tel.: +39 02 2399 8556. *E-mail address:* salvatore.scrivano@polimi.it

Abstract

This paper presents a Markov Chain approximation to model stations in manufacturing lines with general distributed processing times. The proposed Markov Chain approximation enables the use of continuous flow models for the performance evaluation of serial lines with finite buffers and mixed manual – automated operations. Each station in the line can consist of a highly automated machine with deterministic processing times, or of a human operator performing manual operations with general distributed processing times. Stations with random processing times are modelled through a continuous time – discrete state Markov Chain characterized by an operational state with a deterministic processing time, and by an auxiliary down state used to stochastically dilate the overall completion time of a part on the station. The Markov Chain parameters are defined through moments fitting of the probability distribution of the processing time of the original station. The resulting Markov Chain represents the behavior of the station in isolation and is then used as input in the decomposition techniques, based on continuous flow models, for the performance evaluation of serial lines. The model has been applied in the analysis of the production performances of a real assembly line.

Peer-review under responsibility of the scientific committee of the 9th CIRP Global Web Conference – Sustainable, resilient, and agile manufacturing and service operations : Lessons from COVID-19 (CIRPe 2021)

Keywords: Manufacturing systems; Performance evaluation; Continuous flow; Processing times; General distributions;

1. Introduction

Different methods and techniques have been developed by researcher to evaluate the performances of asynchronous manufacturing lines with finite buffers. Analytical methods are one of the formulated solutions. Analytical methods have been promoted to be faster and to provide greater insights to the dynamics of the manufacturing systems than other evaluation methods [1], such as simulation models. One of the possible ways in which analytical models can be classified is according to the discrete or continuous flow of material used to model the original discrete flow of parts in the manufacturing system under analysis. Discrete flow models allow to perfectly mimics the discrete nature of parts in the line. Moreover, they allow to deal with asynchronous stations characterized by random processing times. If processing times are exponentially distributed, Markov Chain theory can be applied to model the system or subsystems of the original line [2]. In [3] a decomposition technique for flow lines with exponential parallel machines is presented. Continuous and discrete phase-type (PH) distributions have been used to fit processing times and to extend the applicability of discrete flow models to system with non-exponential processing times, such as in [4], [5] and [6]. In [7] an analytical method is defined for multiproduct systems with a single non-exponential machine and M dedicated buffers. On the other hand, continuous flow

Peer-review under responsibility of the scientific committee of the 9th CIRP Global Web Conference – Sustainable, resilient, and agile manufacturing and service operations : Lessons from COVID-19 (CIRPe 2021)
10.1016/j.procir.2021.10.002

models consider quite naturally deterministic processing times and asynchronous behaviours which are typically of real applications characterized by highly automated manufacturing lines. Some of the works proposed to evaluate the performances of systems with deterministic processing times are the continuous flow models for a two-stage line in [8] and [9], and the decomposition models of multi-stage lines in [10] and [11]. However, real manufacturing lines are often characterized by some stations with deterministic processing times and other stations with general distributed processing times. For these types of system, the current literature does not provide analytical methods for the performance evaluation.

In this work, a continuous time – discrete state Markov Chain is proposed to model the behaviour of stations with random processing times. The processing time of the modelled station and the parameters of the related Markov Chain are defined through the moments fitting of the probability distribution of the random processing time of the original station in the line. The proposed model of a single station can be used as input for some of the continuous time – continuous flow models presented in the literature for manufacturing lines with deterministic processing times. The aim is to exploit these analytical methods also for lines with mixed deterministic – stochastic processing times and so to reduce the gap in the literature about these real manufacturing lines. An example of this type of systems can be found in many disassembly lines of end-of-life products. In these lines, operations requiring high flexibility because of the product condition variability are performed by human operators. The human element and the product condition variability bring to stochastic processing times of these operations. On the other hand, more standardized operations can be automatized and so performed by automated machines, for which the processing times can be assumed deterministic. In the following Section the methodology is presented for both a perfectly reliable station and an unreliable station. In Section 3 the proposed model has been used in a decomposition model for serial lines and numerical results are shown. In Section 4 a real industrial case with the performance evaluation of an assembly line is presented. In Section 5 conclusions and future developments are provided.

2. Methodology

The proposed methodology decomposes the operational state U of a station, characterized by a random processing time CT, in two fictional modelled states: an operational state U* with a deterministic processing time CT_{U^*}, and an auxiliary down state D* that is used to stochastically dilate the overall completion time of a part on the modelled station. Times to transition between U* and D* are set exponentially distributed. Therefore, the station is modelled as a continuous time – discrete state Markov Chain, with a fictional failure rate p^* from U* to D* and a fictional repair rate r^* from D* to U*. The values of the transition rates p^* and r^*, and of the deterministic processing rate μ^* (with $\mu^* = 1/CT_{U^*}$) of the fictional operational state U* are defined through the moments fitting of the original processing time CT. More precisely, the values of the above parameters are obtained by equalling the first three moments of the probability distribution of the completion time of a single part on the modelled station, to those of the processing time CT of the original one, such that:

$$\begin{cases} \mathrm{E}[CT^*] = \mathrm{E}[CT] \\ \mathrm{E}\left[CT^{*2}\right] = \mathrm{E}[CT^2] \\ \mathrm{E}\left[CT^{*3}\right] = \mathrm{E}[CT^3] \end{cases} \tag{1}$$

with CT^* the completion time of a single part on the modelled station.

The values of $\mathrm{E}[CT]$, $\mathrm{E}[CT^2]$ and $\mathrm{E}[CT^3]$ are known since they can be directly computed from the theoretical or empirical distribution of CT. The modelled completion time of one part CT^* is a stochastic variable composed by the deterministic processing time CT_{U^*} and by the duration of all the fictional failures F^* occurring during the operational time CT_{U^*}. Since the occurrence of a single fictional failure corresponds to a transition from the state U* to the state D*, the number of fictional failures F^* occurring during the operational time CT_{U^*} is a random variable with Poisson distribution and expected value equal to p^*/μ^*. The duration of a single fictional failure is defined as TTR_f since it corresponds to the time to repair from D* to U*. Consequently, TTR_f follows the exponential distribution with expected value $1/r^*$. Therefore, it is possible to define the modelled completion time of one part as:

$$CT^* = CT_{U^*} + \sum_{f=0}^{F^*} TTR_f \tag{2}$$

with TTR_0 the time to repair when no failures occur, that is $TTR_0 = 0$. At this point, it is possible to compute the first three moments $\mathrm{E}[CT^*]$, $\mathrm{E}\left[CT^{*2}\right]$ and $\mathrm{E}\left[CT^{*3}\right]$ through the three unknowns μ^*, p^* and r^*, such as:

$$\mathrm{E}[CT^*] = \frac{1}{\mu^*} + \frac{p^*}{\mu^*}\cdot\frac{1}{r^*} \tag{3}$$

$$\mathrm{E}\left[CT^{*2}\right] = \frac{1}{\mu^{*2}} + \frac{p^*}{\mu^*\cdot r^*}\cdot\left(\frac{2}{\mu^*} + \frac{2}{r^*} + \frac{p^*}{\mu^*\cdot r^*}\right) \tag{4}$$

$$\mathrm{E}\left[CT^{*3}\right] = \frac{1}{\mu^{*3}} + 3\cdot\frac{p^*}{\mu^*\cdot r^*}\cdot\left(\frac{1}{\mu^{*2}} + \frac{1}{\mu^*\cdot r^*}\cdot\left(2+\frac{p^*}{\mu^*}\right) + \frac{2}{r^{*2}} + +2\cdot\frac{p^*}{\mu^*\cdot r^{*2}} + \frac{1}{3}\cdot\left(\frac{p^*}{\mu^*\cdot r^*}\right)^2\right) \tag{5}$$

The demonstrations of equations (3), (4) and (5) are provided in Appendix A. By substituting (3), (4) and (5) in (1) and solving the system of equations, the unknown parameters μ^*, p^* and r^* result:

$$\begin{cases} \mu^* = \frac{2\cdot A}{B} \\ p^* = \frac{9\cdot C^3}{A\cdot B} \\ r^* = \frac{3\cdot C}{A} \end{cases} \tag{6}$$

with

$$A = 2\cdot\mathrm{E}[CT]^3 - 3\cdot\mathrm{E}[CT]\cdot\mathrm{E}[CT^2] + \mathrm{E}[CT^3] \tag{7}$$

$$B = \mathrm{E}[CT]^4 + 2\cdot\mathrm{E}[CT]\cdot\mathrm{E}[CT^3] - 3\cdot\mathrm{E}[CT^2]^2 \tag{8}$$

$$C = \mathrm{E}[CT^2] - \mathrm{E}[CT]^2 \tag{9}$$

Since μ^* is a production rate and p^* and r^* are the transition rates of a Markov Chain, the values in (6) must be positive. Therefore, by imposing the results in (6) greater than zero, the following set of constraints about the moments of the original processing time CT is obtained.

$$\begin{cases} E[CT] > 0 \\ E[CT^2] - E[CT]^2 > 0 \\ E[CT^3] > \frac{3 \cdot E[CT^2]^2 - E[CT]^4}{2 \cdot E[CT]} \end{cases} \quad (10)$$

The first constraint in (10) is verified for any probability distributions, since CT is a processing time and so it cannot be negative or null. Also the second constraint is verified for any distribution. Indeed, it means that the variance of CT must be greater than zero, that is CT cannot be deterministic. If CT were deterministic, it could be directly taken as input in one of the models proposed in [8] or [9] without any further manipulation. On the other hand, the last constraint in (10) limits the set of theoretical and empirical distribution of CT to which the model in this work can be applied. Therefore, before applying the method, the third constraint in (10) must be verified. However, this constraint is always verified for processing times exponentially distributed.

2.1. Unreliable station

Considering a general unreliable station characterized by multiple up states and multiple down states, with one or more operational states having random processing times, a further step must be added to the methodology presented above.

The unreliable station can be defined by the following elements: a set of up states $\mathrm{U} = \{\mathrm{U}_1, \ldots, \mathrm{U}_i, \ldots, \mathrm{U}_I\}$, with each state U_i characterized by a deterministic or stochastic processing time CT_i and a processing rate $\mu_i = 1/CT_i$; a set of down states $\mathrm{D} = \{\mathrm{D}_1, \ldots, \mathrm{D}_j, \ldots, \mathrm{D}_J\}$; a set of transition rates $p = \{p_1, \ldots, p_k, \ldots, p_K\}$ describing the operation dependent transitions among states; a set of transition rates $r = \{r_1, \ldots, r_l, \ldots, r_L\}$ describing the time dependent transitions among states.

Assuming times to transition between states exponentially distributed, the station can be represented through a Markov Chain, such as the example provided in Fig. 1. (a). If one or more times to transition are not exponentially distributed, continuous PH distributions can be used to approximate them, and the resulting Markov Chain can be taken as input for the following steps of the proposed method.

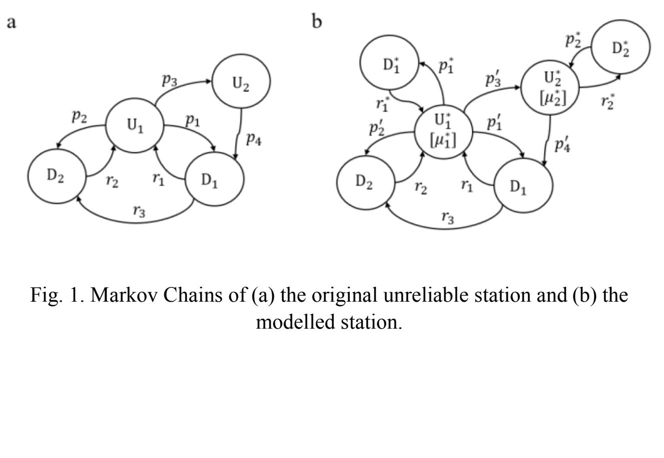

Fig. 1. Markov Chains of (a) the original unreliable station and (b) the modelled station.

The first step for modelling an unreliable station corresponds to the methodology described above: each state U_i with random processing time CT_i is divided in two fictional states U_i^* and D_i^*, and the related parameters μ_i^*, p_i^* and r_i^* are computed according to (6). It implies again that for each random processing time CT_i the set of constraints in (10) is not violated. Fig. 1. (b) shows the resulting Markov Chain if the processing times of states U_1 and U_2 of the original Markov Chain in Fig. 1. (a) are stochastic.

The second step consists in redefining the values of each transition rate p_k exiting from each original state U_i which has been divided in U_i^* and D_i^*. Indeed, p_k represents the rate of an operation dependent transition that can stochastically occur each time the station is in the state U_i. If U_i is divided in U_i^* and D_i^*, the operation dependent transition can occur only when the modelled station is in the state U_i^*, since D_i^* is not an operational state. Therefore, a new value p_k' must be defined such that the original probability flow related to this transition is conserved, that is:

$$\Pi(\mathrm{U}_i^*) \cdot p_k' = \Pi(\mathrm{U}_i) \cdot p_k \quad (11)$$

with $\Pi(\mathrm{U}_i^*)$ and $\Pi(\mathrm{U}_i)$ respectively the probability of the modelled state U_i^* and the original state U_i. Since U_i has been divided in U_i^* and D_i^*, the probability $\Pi(\mathrm{U}_i)$ can be written as:

$$\Pi(\mathrm{U}_i) = \Pi(\mathrm{U}_i^*) + \Pi(\mathrm{D}_i^*) \quad (12)$$

Moreover, since D_i^* is reachable only from U_i^* with rate p_i^*, and the only reachable state from D_i^* is U_i^* with rate r_i^*, the probability $\Pi(\mathrm{D}_i^*)$ is proportional to $\Pi(\mathrm{U}_i^*)$, such as:

$$\frac{\Pi(\mathrm{D}_i^*)}{\Pi(\mathrm{U}_i^*)} = \frac{p_i^*}{r_i^*} \quad (13)$$

By substituting (12) and (13) in (11), it results:

$$p_k' = \left(1 + \frac{p_i^*}{r_i^*}\right) \cdot p_k \quad (14)$$

Therefore, the new value of the transition rate p_k' depends only on the original rate p_k and on the values of p_i^* and r_i^* computed in the previous step.

3. Numerical results

The accuracy of the method has been tested by using this model as input in the decomposition technique of [11] for the evaluation of a set of three-machine two-buffer lines (3M-2B), four-machine three-buffer lines (4M-3B) and five-machine four-buffer lines (5M-4B). The decomposition technique of [11] decomposes a serial line in smaller lines, called building blocks, composed by two machines and one buffer and for which an exact evaluation of the performances is provided by [13]. The different building blocks are related one each other through a set of equations which guarantee the convergence of the results. In the building blocks, the behaviour of each machine is modelled through the approach proposed in this work. The aim is to demonstrate that through the proposed Markov Chain model it is possible to apply a continuous flow model to the performance evaluation of serial lines with finite buffers and mixed deterministic – exponentially distributed processing times. The results are compared with those of a DES model with discrete material flow, developed by using Simulink, over 40 cases for each type of lines. For each test case, five replicates of the simulation experiment are carried out. Every replicate j is 1000000 time units long, with a warm-up period of 200000 time units. Each machine of the line has been randomly generated and it can have deterministic processing time $CT = 1 / \mu$ with probability 0.5, or exponential processing time with expected value $\text{E}[CT] = 1 / \mu$ with probability 0.5. For each machine, the parameter μ has been randomly generated from the continuous uniform distribution $U(1, 5)$. Machines with deterministic processing times are unreliable. Machines with exponential processing times can be unreliable with probability 0.5, or perfectly reliable with probability 0.5. Unreliable machines have a single up state and a single down state, with time to failure and time to repair exponentially distributed and failure rate and repair rate randomly generated as $p{\sim}U(0.001, 0.02)$ and $r{\sim}U(0.01, 0.2)$. The capacity $\text{N}\{i\}$ of each buffer $\text{B}\{i\}$ has been randomly generated from the discrete uniform distribution in the interval $[1,\ 10]$. For each original machine assumed with exponential processing time, the parameters μ^*, p^*, r^* and p' of the corresponding modelled machine have been computed through equations (6) and (14), starting from the original parameters μ and p, and they result:

$$\begin{cases} \mu^* = 4 \cdot \mu \\ p^* = \frac{9}{2} \cdot \mu \\ r^* = \frac{3}{2} \cdot \mu \end{cases} \qquad p' = \left(1 + \frac{p^*}{r^*}\right) \cdot p = 4 \cdot p \tag{15}$$

For each case, the steady state throughput th and the total average inventories $\bar{n}$ have been evaluated through both the simulation and the decomposition model, and the percentage errors $\epsilon(th)$ and $\epsilon(\bar{n})$ have been computed as:

$$\epsilon(th) = \frac{|th^{\text{Sim}} - th^{\text{Dec}}|}{th^{\text{Sim}}} \cdot 100, \quad th^{\text{Sim}} = \frac{\sum_{j=1}^{5} th_j^{Sim}}{5} \tag{16}$$

$$\epsilon(\bar{n}) = \frac{|\bar{n}^{\text{Sim}} - \bar{n}^{\text{Dec}}|}{\sum_i N\{i\}} \cdot 100, \quad \bar{n}^{\text{Sim}} = \frac{\sum_{j=1}^{5} \bar{n}_j^{Sim}}{5} \tag{17}$$

Table 1. shows the mean and maximum values of the percentage errors for the three set of test systems. Errors are very low and quite similar to those of decomposition techniques proposed in the literature for asynchronous lines with deterministic processing times. Therefore, the Markov Chain model presented in this work can be exploited to apply continuous flow models to lines with mixed deterministic – stochastic processing times, without reducing the accuracy of the evaluation models.

Table 1. Results of the test systems.

Line	mean $\epsilon(th)$	max $\epsilon(th)$	mean $\epsilon(\bar{n})$	max $\epsilon(\bar{n})$
3M-2B	0.2489	0.6431	0.8307	4.4766
4M-3B	0.4243	1.0131	2.0072	7.6404
5M-4B	0.6784	1.5650	2.6900	4.7070

4. Industrial case

The Markov Chain model of this paper has been used to approximate the behaviour of manual operations in a real assembly line and then to evaluate the performances of the system through the continuous flow model of [11]. The aim of this analysis is to quantify the blocking and starvation phenomena in the line because of the asynchrony and the stochastic processing times of the stations. The system is an assembly line of a company producing industrial switches. The line is composed by six stations performing the product customization according to the customer orders. The customization phase is mainly characterized by the wiring and coil assembly and few other specific activities. The product customization is required for all the types of industrial switches produced by the company, which are pre-assembled in dedicated lines. For this reason, the assembly line of interest is almost never starved of parts from the dedicated lines. Finished parts are then stocked in storage area, wating for the packaging and the order preparation. The storage area is big enough to do not block the assembly line. Parts in the assembly line are handled between stations by a conveyor, through a fixed routing from the first station to the last one. The conveyor operates also as an intermediate buffer between stations, with capacity always equal to two parts between consecutive working stations. According to the definition in [12], the blocking rule applied by the operators in the working stations is classified as the blocking after service (BAS) rule. The historical data collected in the plant over one year show that stations can be assumed perfectly reliable and with stochastic processing times. In order to evaluate the line performances, each station i has been modelled as a single up – single down machine $\text{M}\{i\}$, through the Markov Chain approximation of this paper. Table 2. reports the empirical moments computed from the historical processing times of each station (original values have been transformed for confidentiality reasons) and the estimated parameters of the related Markov Chain according to set of equations (6). The constraints in (10) are verified for the empirical moments of all the stations in the line. The parameters of Table 2. have been used in the continuous flow models of [11] and the results are shown in Table 3. The performances of interest are the steady state throughput th of the line expressed as parts/time units, and the average number of parts $\bar{n}\{i\}$ in each buffer $\text{B}\{i\}$ between stations $\text{M}\{i\}$ and $\text{M}\{i+1\}$. Table 4. provides a detailed analysis on the probabilities of each station (rows) to be blocked or starved by any other station (columns). Elements above the diagonal are blocking probabilities, elements below the diagonal are starvation probabilities. It results that $\text{M}\{1\}$ is the bottleneck of the line since it is the main source of limitation for the other stations. To increase the throughput of the line, the company can reduce the impact of $\text{M}\{1\}$ on the other stations by increasing the capacity $\text{N}\{1\}$ of the buffer $\text{B}\{1\}$. Another alternative is to increase the availability of the bottleneck $\text{M}\{1\}$ by reducing its main cause of inefficiency, that is the blocking caused by the station $\text{M}\{3\}$. In this case, an increase in the buffer capacity $\text{N}\{1\}$ would also reduce the propagation from $\text{M}\{2\}$ to $\text{M}\{1\}$ of the blocking caused by $\text{M}\{3\}$. On the other hand, an increase in the capacity $\text{N}\{2\}$ of the buffer $\text{B}\{2\}$ would directly reduce the blocking probability of M{2} caused by M{3}, and consequently the blocking probability of the bottleneck M{1}. Figure 2. shows the percentage variation of the throughput of the line as a function of the buffer capacities N{1} and N{2}. Because of space constraints in the plant, the buffer capacities vary up to a maximum of 6 buffer slots. The slope of the resulting surface shows that an increase in the buffer capacity N{1} should be possibly preferred to an increase in N{2}, since it would result in a bigger increase in the line throughput.

Table 2. Empirical moments of original stations and parameters of modelled stations.

Station	$E[CT]$	$E[CT^2]$	$E[CT^3]$	μ^*	p^*	r^*
M{1}	10.09	$111 \cdot 10^3$	$723 \cdot 10^5$	0.13	0.00015	0.0005
M{2}	4.94	$3 \cdot 10^3$	$2 \cdot 10^5$	0.23	0.00050	0.0042
M{3}	8.99	$42 \cdot 10^3$	$130 \cdot 10^5$	0.14	0.00028	0.0010
M{4}	7.21	$16 \cdot 10^3$	$53 \cdot 10^5$	0.15	0.00010	0.0009
M{5}	5.66	$14 \cdot 10^3$	$48 \cdot 10^5$	0.20	0.00011	0.0009
M{6}	5.18	$11 \cdot 10^3$	$35 \cdot 10^5$	0.21	0.00009	0.0009

Table 3. As-is performances of the assembly line.

th	$\bar{n}\{1\}$	$\bar{n}\{2\}$	$\bar{n}\{3\}$	$\bar{n}\{4\}$	$\bar{n}\{5\}$
0.07	0.615	0.934	0.264	0.152	0.065

Table 4. Blocking and starvation probabilities between each couple of stations

Prob [%]	M{1}	M{2}	M{3}	M{4}	M{5}	M{6}
M{1}		3.53	13.64	4.65	3.99	2.84
M{2}	37.38		15.98	4.76	4.09	2.90
M{3}	19.80	5.09		0.14	4.16	2.96
M{4}	18.73	4.47	18.80		4.28	3.05
M{5}	18.46	4.23	15.83	18.55		3.14
M{6}	18.29	4.06	15.09	12.67	13.42	

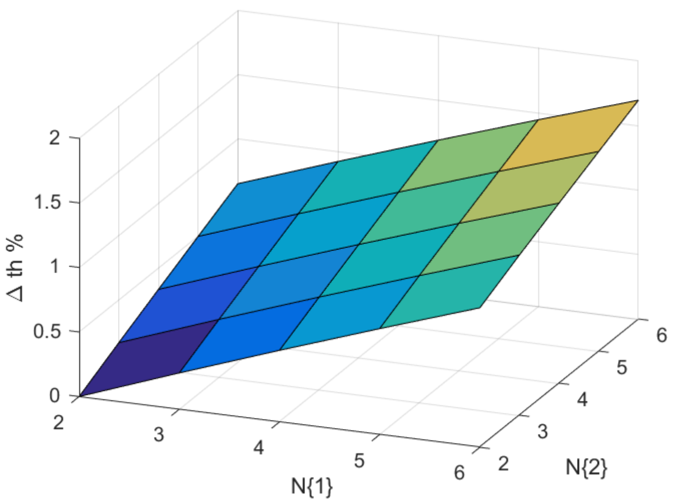

Fig. 2. Percentage variation of the throughput of the line as a function of the capacities of the first and second buffers.

5. Conclusion

This work has proposed a continuous time – discrete state Markov Chain model that approximates the behaviour of stations with general distributed processing times. This approximation allows the employment of continuous flow analytical model to evaluate the performance of asynchronous manufacturing lines with mixed deterministic – stochastic processing times. The numerical results have shown the accuracy of the model, while the industrial case has shown the usefulness of the model for the evaluation of blocking and starvation propagations in a real assembly line. Future research will focus on the study of different Markov Chain structures in order to increase the set of probability distributions of processing times that is possible to fit through this approach.

Appendix A.

In this appendix, the demonstrations of equations (3), (4) and (5) are provided. The following demonstrations are based of the definitions in Section 2, which are: the processing time CT_{U^*} in the modelled state U* is deterministic and equal to $1/\mu^*$; the number of fictional failures F^* occurring during the time period CT_{U^*} follows the Poisson distribution with expected value p^*/μ^*; the time to repair TTR_f of a fictional failure is exponentially distributed with expected value $1/r^*$.

A.1. First moment demonstration

Starting from the definition of CT^* provided in (2), the expected value of CT^* in (3) is computed as follow:

$$
\begin{aligned}
&E[CT^*] = E\left[CT_{U^*} + \textstyle\sum_{f=0}^{F^*} TTR_f\right] = \\
&= E[CT_{U^*}] + E\left[\textstyle\sum_{f=0}^{F^*} TTR_f\right] = \\
&= E[CT_{U^*}] + E[F^*] \cdot E\left[TTR_f\right] = \frac{1}{\mu^*} + \frac{p^*}{\mu^*} \cdot \frac{1}{r^*} \qquad (18)
\end{aligned}
$$

A.2. Second moment demonstration

In order to demonstrate the result of equation (4), the explicit definition of CT^* in (2) is substituted in the quantity $E\left[{CT^*}^2\right]$, and then it is solved as follow:

$$
\begin{aligned}
&E\left[{CT^*}^2\right] = E\left[\left(CT_{U^*} + \textstyle\sum_{f=0}^{F^*} TTR_f\right)^2\right] = \\
&= E\left[(CT_{U^*})^2 + 2 \cdot CT_{U^*} \cdot \textstyle\sum_{f=0}^{F^*} TTR_f + \left(\sum_{f=0}^{F^*} TTR_f\right)^2\right] = \\
&= E[(CT_{U^*})^2] + 2 \cdot E[CT_{U^*}] \cdot E\left[\textstyle\sum_{f=0}^{F^*} TTR_f\right] + \\
&+ E\left[\left(\textstyle\sum_{f=0}^{F^*} TTR_f\right)^2\right] \qquad (19)
\end{aligned}
$$

The only term to be defined in (19) is $E\left[\left(\sum_{f=0}^{F^*} TTR_f\right)^2\right]$. By solving the square of this sum, it is equal to:

$$
\begin{aligned}
&E\left[\left(\textstyle\sum_{f=0}^{F^*} TTR_f\right)^2\right] = E[F^*] \cdot E\left[{TTR_f}^2\right] + \\
&+ 2 \cdot E\left[\binom{F^*}{2}, F^* \geq 2\right] \cdot \left(E\left[TTR_f\right] \cdot E[TTR_k]\right)_{f \neq k} \qquad (20)
\end{aligned}
$$

with $E\left[TTR_f\right]$ and $E[TTR_k]$ the expected times to repair of two different fictional failures among the F^* fictional failures occurred in the time period CT_{U^*}, while $E\left[\binom{F^*}{2}, F^* \geq 2\right]$ is the expected number of combinations of two different fictional failures among F^*, when F^* is not lower than two. The two quantities which have not been yet defined in (20) are $E\left[TTR_f{}^2\right]$ and $E\left[\binom{F^*}{2}, F^* \geq 2\right]$, and they are computed as follow:

$$E\left[TTR_f{}^2\right] = V\left[TTR_f\right] + E\left[TTR_f\right]^2 = \frac{2}{r^{*2}} \quad (21)$$

$$E\left[\binom{F^*}{2}, F^* \geq 2\right] = \sum_{F^*=2}^{+\infty} \frac{F^*!}{2!\cdot(F^*-2)!} \cdot \frac{\left(\frac{p^*}{\mu^*}\right)^{F^*} \cdot e^{-\left(\frac{p^*}{\mu^*}\right)}}{F^*!} = \frac{\left(\frac{p^*}{\mu^*}\right)^2}{2} \quad (22)$$

By substituting (21) and (22) in (20) and then in (19), the demonstration of the second moment is:

$$E\left[CT^{*2}\right] = \frac{1}{\mu^{*2}} + 2 \cdot \frac{p^*}{\mu^{*2}} \cdot \frac{1}{r^*} + \frac{p^*}{\mu^*} \cdot \frac{2}{r^{*2}} + 2 \cdot \frac{1}{r^{*2}} \cdot \frac{\left(\frac{p^*}{\mu^*}\right)^2}{2} = \frac{1}{\mu^{*2}} + \frac{p^*}{\mu^* \cdot r^*} \cdot \left(\frac{2}{\mu^*} + \frac{2}{r^*} + \frac{p^*}{\mu^* \cdot r^*}\right) \quad (23)$$

A.3. Third moment demonstration

Similarly, for the third moment demonstration the quantity in (2) is substituted in $E\left[CT^{*3}\right]$ of (5) and the cube of the polynomial is solved as follow:

$$E\left[CT^{*3}\right] = E\left[\left(CT_{U^*} + \sum_{f=0}^{F^*} TTR_f\right)^3\right] = E\left[(CT_{U^*})^3\right] + 3 \cdot E\left[(CT_{U^*})^2\right] \cdot E\left[\sum_{f=0}^{F^*} TTR_f\right] + 3 \cdot E\left[CT_{U^*}\right] \cdot E\left[\left(\sum_{f=0}^{F^*} TTR_f\right)^2\right] + E\left[\left(\sum_{f=0}^{F^*} TTR_f\right)^3\right] \quad (24)$$

In order to solve (24), the cube of the polynomial in $E\left[\left(\sum_{f=0}^{F^*} TTR_f\right)^3\right]$ is computed, and its expected value results:

$$E\left[\left(\sum_{f=0}^{F^*} TTR_f\right)^3\right] = E[F^*] \cdot E\left[TTR_f{}^3\right] + 3 \cdot E\left[F^*(F^*-1), F^* \geq 2\right] \cdot \left(E\left[TTR_f^2\right] \cdot E\left[TTR_k\right]\right)_{f\neq k} + 6 \cdot E\left[\binom{F^*}{3}, F^* \geq 3\right] \cdot \left(E\left[TTR_f\right] \cdot E\left[TTR_k\right] \cdot E\left[TTR_h\right]\right)_{f\neq k\neq h} \quad (25)$$

with: $E\left[F^*(F^*-1), F^* \geq 2\right]$ the expected number of 2-permutations of F^*, when at least two fictional failures F^* occur; $E\left[\binom{F^*}{3}, F^* \geq 3\right]$ the expected number of combinations of three different fictional failures among F^*, when F^* is not lower than three; $E\left[TTR_f\right]$, $E\left[TTR_k\right]$ and $E\left[TTR_h\right]$ the expected times to repair of three different fictional failures among F^*. To solve (25), the values of $E\left[TTR_f{}^3\right]$, $E\left[F^*(F^*-1), F^* \geq 2\right]$ and $E\left[\binom{F^*}{3}, F^* \geq 3\right]$ are computed as follow:

$$E\left[TTR_f{}^3\right] = \frac{6}{r^{*3}} \quad (26)$$

$$E\left[F^* \cdot (F^*-1), F^* \geq 2\right] = \sum_{F^*=2}^{+\infty} F^* \cdot (F^*-1) \cdot \frac{\left(\frac{p^*}{\mu^*}\right)^{F^*} \cdot e^{-\left(\frac{p^*}{\mu^*}\right)}}{F^*!} = \left(\frac{p^*}{\mu^*}\right)^2 \quad (27)$$

$$E\left[\binom{F^*}{3}, F^* \geq 3\right] = \sum_{F^*=3}^{+\infty} \frac{F^*!}{3!\cdot(F^*-3)!} \cdot \frac{\left(\frac{p^*}{\mu^*}\right)^{F^*} \cdot e^{-\left(\frac{p^*}{\mu^*}\right)}}{F^*!} = \frac{\left(\frac{p^*}{\mu^*}\right)^3}{6} \quad (28)$$

By substituting (26), (27) and (28) in (25) and then in (24), the demonstration of the third moment is:

$$E\left[CT^{*3}\right] = \frac{1}{\mu^{*3}} + 3 \cdot \frac{1}{\mu^{*2}} \cdot \frac{p^*}{\mu^*} \cdot \frac{1}{r^*} + 3 \cdot \frac{1}{\mu^*} \cdot \left(\frac{p^*}{\mu^*} \cdot \frac{2}{r^{*2}} + \frac{1}{r^{*2}} \cdot \left(\frac{p^*}{\mu^*}\right)^2\right) + \frac{p^*}{\mu^*} \cdot \frac{6}{r^{*3}} + 3 \cdot \left(\frac{p^*}{\mu^*}\right)^2 \cdot \frac{2}{r^{*3}} + \left(\frac{p^*}{\mu^*}\right)^3 \cdot \frac{1}{r^{*3}} = \frac{1}{\mu^{*3}} + 3 \cdot \frac{p^*}{\mu^* \cdot r^*} \cdot \left(\frac{1}{\mu^{*2}} + \frac{1}{\mu^* \cdot r^*} \cdot \left(2 + \frac{p^*}{\mu^*}\right) + \frac{2}{r^{*2}} + 2 \cdot \frac{p^*}{\mu^* \cdot r^{*2}} + \frac{1}{3} \cdot \left(\frac{p^*}{\mu^* \cdot r^*}\right)^2\right) \quad (29)$$

References

[1] Colledani, M., Ekvall, M., Lundholm, T., Moriggi, P., Polato, A., Tolio, T., 2010. Analytical methods to support continuous improvements at Scania. International Journal of Production Research, 48(7), 1913–1945

[2] Gershwin, S. B., 1994. Manufacturing systems engineering. Prentice Hall, 93–112

[3] Diamantidis, A., Lee, J. H., Papadopoulos, C. T., Li, J., Heavey, C., 2020. Performance evaluation of flow lines with non-identical and unreliable parallel machines and finite buffers. International Journal of Production Research, 58(13), 3881–3904

[4] Altiok, T., 1989. Approximate analysis of queues in series with phase-type service times and blocking. Operations Research, 37(4), 601–610

[5] Helber, S., 2005. Analysis of flow lines with Cox-2-distributed processing times and limited buffer capacity. OR Spectrum, 27(2–3), 221–242

[6] Tancrez, J.-S., 2020. A decomposition method for assembly/disassembly systems with blocking and general distributions. Flexible Services and Manufacturing Journal, 32(2), 272–296

[7] Kang, N., Zheng, L., Li, J., 2015. Analysis of multi-product manufacturing systems with arbitrary processing times. International Journal of Production Research, 53(3), 983–1001

[8] Tan, B., Gershwin, S., 2009. Analysis of a general markovian two-stage continuous-flow production system with a finite buffer. International Journal of Production Economics, 120, 327–339

[9] Tolio, T., Ratti, A., 2018. Performance evaluation of two machine lines with generalized thresholds. International Journal of Production Research, 56(1–2), 926–949

[10] Colledani, M., Gershwin, S., 2013. A decomposition method for approximate evaluation of continuous flow multi-stage lines with general Markovian machines. Ann Oper Res 209, 5–40

[11] Magnanini, M. C., 2019. Stochastic approximate analytical methods integrating control mechanisms for evaluating the performance of asynchronous manufacturing systems (Doctoral dissertation, Italy)

[12] Dallery, Y., Gershwin, S.B., 1992. Manufacturing flow line systems: a review of models and analytical results. Queueing Syst 12, 3–94.

[13] Magnanini, M. C., Tolio, T., 2021. Performanc evaluation of asynchronous two-stage manufacturing lines fabricating discrete parts. CIRP Journal of Manufacturing Science and Technology, 33, 488–505