# AN EXTENDED VARIATIONAL THEORY FOR NONLINEAR EVOLUTION EQUATIONS VIA MODULAR SPACES 

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#### Abstract

We propose an extension of the classical variational theory of evolution equations that accounts for dynamics also in possibly non-reflexive and non-separable spaces. The pivoting point is to establish a novel variational structure, based on abstract modular spaces associated to a given convex function. Firstly, we show that the new variational triple is suited for framing the evolution, in the sense that a novel duality paring can be introduced and a generalised computational chain rule holds. Secondly, we prove well-posedness in an extended variational sense for evolution equations, without relying on any reflexivity assumption and any polynomial requirement on the nonlinearity. Finally, we discuss several important applications that can be addressed in this framework: these cover, but are not limited to, equations in Musielak-Orlicz-Sobolev spaces, such as variable exponent, Orlicz, weighted Lebesgue, and double-phase spaces.


## 1. Introduction

In this paper we deal with evolution equations on a Hilbert space $H$ in the form

$$
\begin{equation*}
\partial_{t} u+A(u) \ni f, \quad u(0)=u_{0}, \tag{1.1}
\end{equation*}
$$

where $A:=\partial \varphi$ is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi: H \rightarrow(-\infty,+\infty]$. Moreover, $f:(0, T) \rightarrow H$ is a prescribed forcing term, with $T>0$ being the final reference time, and the initial datum $u_{0}$ is given in $H$.

From the mathematical perspective, nonlinear evolution equations in Hilbert or Banach spaces have been extensively investigated in the last decades. Starting from the pioneering literature of the 70 s, for which we refer the reader to Barbu [6, Ch. 4], their study represents nowadays one of the most flourishing fields of modern mathematical analysis, with applications ranging from partial differential equations to functional analysis. In this regard, we point out the contributions [ $20,21,26,68,69]$ on doubly nonlinear evolution equations, and $[3,4,57]$ on variational principles, as well as the references therein. Well-posedness for equations in the form (1.1) has been tackled in several frameworks, and various concepts of solutions have been proposed. The specific regularity of the solutions strongly depends on the assumptions on the initial datum $u_{0}$ and the forcing term $f$.

The most classical (and the strongest) assumption on the initial datum is that $u_{0} \in D(A)$, where $D(A)$ stands for the effective domain of the maximal monotone operator $A$ on $H$. In this framework, existence of strong solutions for (1.1) has been thoroughly studied in the Hilbert case in relation to the nonlinear extension of the celebrated Hille-Yosida theory: this was first accomplished by Kōmura [43] and then the result was well reviewed in the monograph by Brezis [13], not necessarily requiring $A$ to be cyclically monotone. More in detail, the corresponding homogeneous equation (i.e. with $f=0$ ) admits a unique strong solution $u$, in the sense that $u \in W^{1, \infty}(0, T ; H)$ and the differential inclusion (1.1) is satisfied almost everywhere on $(0, T)$. In the nonhomogeneous case, existence of strong solutions in $W^{1, \infty}(0, T ; H)$ is ensured if the forcing term satisfies at least $f \in B V(0, T ; H)$. In the special case of subdifferential operators $A=\partial \varphi$, existence of strong solutions in $W^{1, p}(0, T ; H)$, for $p \in[2,+\infty)$, is ensured if $f \in L^{p}(0, T ; H)$. Here, the proof can be

[^0]based on a regularisation and passage-to-the-limit procedure based on a Yosida-type approximation of the operator $A$.

However, in several situations the assumption that the initial datum belongs to the domain of $A$ is too restrictive. In this direction, weaker concepts of solution have been proposed in order to give appropriate meaning to equation (1.1) in more general frameworks. In this direction, the initial datum can be required to satisfy only $u_{0} \in \overline{D(A)}$, the closure in $H$ of the domain of $A$, and the forcing term is only taken as $f \in L^{1}(0, T ; H)$. Under these condition, in Brezis [13] existence and uniqueness of a weak solution $u \in C^{0}([0, T] ; H)$ is proved, in the sense that there exist some sequences of data $\left(u_{0, n}, f_{n}\right)_{n} \subset D(A) \times L^{1}(0, T ; H)$ and a sequence of respective strong solutions $\left(u_{n}\right)_{n} \in W^{1,1}(0, T ; H)$, satisfying (1.1) for every $n \in \mathbb{N}$, and such that, as $n \rightarrow \infty, u_{0, n} \rightarrow u_{0}$ in $H$, $f_{n} \rightarrow f$ in $L^{1}(0, T ; H)$, and $u_{n} \rightarrow u$ in $C^{0}([0, T] ; H)$. This concept of weak solution is certainly satisfactory, as it allows to give sense to the equation in more general settings, and it generalises the notion of strong solution. Furthermore, whenever $A$ is cyclically monotone with $A=\partial \varphi$, every weak solution is also a strong solution as soon as $u_{0} \in D(\partial \varphi)$ and $f \in L^{2}(0, T ; H)$, for example.

Alternative concepts of weak solutions have been proposed and studied, also for evolutions in Banach spaces, especially relying on the notion of mild and integral solutions. In this direction, the reader can refer to the classical pioneering works by Bénilan [10], Bénilan \& Brezis [11], Crandall \& Evans [22], Crandall \& Liggett [23], Crandall \& Pazy [24, 25], Kato [40, 41], and Komura [44, 45].

One of the main drawbacks of the above-mentioned notion of weak solution is that the differential inclusion $f-\partial_{t} u \in A(u)$ is not explicitly specified. More precisely, for weak solutions $u$ the meaning of the differential inclusion is somehow passed by: although $u$ is obtained as limit of suitable approximations (strong solutions, time discretisation, etc.), each one solving the differential inclusion almost everywhere, it is not granted that $u$ itself satisfies (1.1) in some sense on $(0, T)$. This issue has been successfully overcome by proposing a further concept of weak solution for the problem, for which the differential inclusion could be made explicit in some appropriate sense: it is the case of the well-celebrated variational theory introduced by Lions [51]. The main idea behind this is to introduce, apart from the given Hilbert space $H$, a further Banach space $V$, which is assumed to be separable, reflexive, and continuously and densely embedded in $H$. If one identifies $H$ with its dual $H^{*}$, then it is possible to work on the variational triplet $V \hookrightarrow H \hookrightarrow V^{*}$. The maximal monotone operator $A$ is then considered as an operator $A$ : $V \rightarrow 2^{V^{*}}$ satisfying some natural coercivity and boundedness conditions in the form

$$
\begin{equation*}
\langle y, \varphi\rangle_{V^{*}, V} \geq c\|\varphi\|^{p}, \quad\|y\|_{V^{*}}^{\frac{p}{p-1}} \leq C\left(1+\|\varphi\|_{V}^{p}\right) \quad \forall \varphi \in V, \quad \forall y \in A(\varphi) \tag{1.2}
\end{equation*}
$$

where $p>1$ is a given constant. In this framework, for every initial datum $u_{0} \in H$ and forcing term $f \in L^{1}(0, T ; H)$ there exists a unique variational solution $u$, in the sense that $u \in W^{1, \frac{p}{p-1}}\left(0, T ; V^{*}\right) \cap$ $L^{p}(0, T ; V)$ satisfies the differential inclusion (1.1) as an equality in $V^{*}$, almost everywhere on $(0, T)$. Let us stress that variational solutions are effective in rendering the differential inclusion explicitly for $u$, and not only as limit of suitable approximations: indeed, one is actually able to prove that the inclusion $f-\partial_{t} u \in A(u)$ holds in $V^{*}$, almost everywhere on $(0, T)$.

The main downside of the classical variational theory is that the operator $A$ needs to possess relatively nice polynomial behaviour. While on the one hand this is certainly enough to cover several classes of evolution equations, such as reaction-diffusion and $p$-Laplace equations, on the other hand numerous important applications are left out. This happens in particular when the operator $A$ fails to satisfy the coercivity-boundedness polynomial conditions (1.2) for a certain exponent $p$. Quite common $A$ is actually needed to be coercive and bounded in some spaces, but the respective exponents are different: this is very classical, for example, for $p(x)$-Laplace and double-phase equations, as well as equations in weighted Sobolev spaces. Alternatively, it may happen that $A$ is coercive and bounded as required, but the natural space $V$ associated to it is not reflexive or not separable: this is the case, among many others, of reaction-diffusion equations with singular (i.e. superpolynomial) potentials. In all these pathological scenarios, if the initial datum
only satisfies $u_{0} \in H$ an appropriate concept of variational solution is not known so far: one is only able to obtain strong solutions to (1.1) by forcing the initial datum to belong to $D(A)$.

These issues naturally call for an extension of the classical variational theory, in order to establish, in some appropriate variational sense, weak well-posedness of evolution equations in the form (1.1) with general initial datum $u_{0} \in H$, also when assumption (1.2) is not satisfied. This is the main focus of the present paper.

The idea is to abandon the introduction of the Banach space $V$, and to work instead in the modular spaces naturally associated to $\varphi$. Indeed, one can define the small and large modular spaces as, respectively,

$$
\begin{array}{lll}
L_{\varphi}:=\{v \in H: \exists \alpha>0: & \varphi(\alpha v)<+\infty\}, \\
E_{\varphi}:=\{v \in H: \forall \alpha>0: & \varphi(\alpha v)<+\infty\} .
\end{array}
$$

As $E_{\varphi}$ is generally dense in $H$ (details are given in Section 3 below), by identifying $H$ with its dual one has the natural variational structure

$$
E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow H \hookrightarrow E_{\varphi}^{*} .
$$

In the classical variational theory, one is implicitly assuming that $E_{\varphi}=L_{\varphi}=V$ is separable and reflexive: this is satisfied only in very specific situations, for example when $\varphi$ satisfy some suitable $\Delta_{2}$ and $\nabla_{2}$ conditions (details are given in Section 6 below). In general, however, the spaces $E_{\varphi}$ and $L_{\varphi}$ are different and not necessarily reflexive. The pivoting idea of the entire work is to work in the variational triplet

$$
L_{\varphi} \hookrightarrow H \hookrightarrow E_{\varphi}^{*} .
$$

Beyond the reflexivity issue, a further problem is that $E_{\varphi}^{*}$ is not the dual of $L_{\varphi}$, hence no duality pairing is in principle defined between $E_{\varphi}^{*}$ and $L_{\varphi}$. The preliminary step is then to show that, nonetheless, it is possible to define a new duality $[\cdot, \cdot]$ between $E_{\varphi}^{*}$ and $L_{\varphi}$ generalising the scalar product of $H$. This guarantees indeed that the triple ( $L_{\varphi}, H, E_{\varphi}^{*}$ ), despite being unconventional in this sense, is suited for framing the evolution problem in a variational way: the solution $u$ is expected to be $L_{\varphi}$-valued, and the differential inclusion (1.1) will be intended in $E_{\varphi}^{*}$.
The first main result of the paper is a fundamental computational tool collected in Theorem 3.6 below, establishing that the novel variational triplet ( $L_{\varphi}, H, E_{\varphi}^{*}$ ) endowed with the novel duality pairing $[\cdot, \cdot]$ satisfies the well-known "chain rule" formula for the square of the $H$-norm. This is highly nontrivial, since the spaces $L_{\varphi}$ and $E_{\varphi}^{*}$ are not reflexive and separable in general, hence the classical results do not apply. In particular, the non-separability of the spaces in play forces to introduce a different notion of measurability for vector-valued functions, as the classical strong measurability in the Bochner sense is too restrictive in this framework. The chain rule is proved using an elliptic-in-time regularisation by convolution and passage to the limit, where a key role is played by an abstract version of the Jensen inequality proved by Haase [34].

The second main result of the paper is contained in Theorems 3.7-3.8 and establishes the variational well-posedness of equation (1.1) is the new variational setting ( $L_{\varphi}, H, E_{\varphi}^{*}$ ). The structure of the prof is based on a Yosida-type approximation on $\partial \varphi$ and passage to the limit. Let us stress that due to the lack of reflexivity several compactness issues arise, especially in the direction of identifying the nonlinearity $\partial \varphi$ at the limit.

The main novelty of this paper is to provide an extended variational structure that allows to frame also singular evolution equations in possibly non-reflexive spaces. This is fundamental as it provides a unifying variational framework for a wide variety of problems, such as equations in Musielak-Sobolev-Orlicz spaces, that so far have been studied independently by hand. In this regard, Musielak-Orlicz spaces and their special cases are receiving much attention at the present time. A survey of nonlinear PDEs in Musielak-Orlicz spaces is presented in Chlebicka [17], especially in the cases of variable exponent, Orlicz, weighted Lebesgue, and double-phase spaces. Further contributions on Musielak-Orlicz-Sobolev spaces are also given in the recent articles [15, 19, 31]
and [36,62, 70]. Such spaces were introduced in the late 50 s in the works of Musielak and Orlicz $[59,60]$. The theory has then been viewed as a special case of a more general approach, based on modular spaces: the main idea is to consider a convex functional (a so-called modular) on a vector space instead of the usual integrals of convex functions involved in Musielak-Orlicz spaces. This approach is widely employed already in several fields, such as approximation and interpolation theory (e.g. $[1,39,46,71]$ ) and in operator theory (e.g. [8, 42, 63,65]). Also, metric space theory on such spaces is developed (see Chistyakov [16]). Recent results concerning existence of solutions to parabolic equations in Orlicz spaces have been obtained in $[9,18,19,29,30,33,72]$. For a highly detailed presentation of the existing literature we refer specifically to [17, § 3].

The importance of the extended variational approach introduced in this work is extremely evident in all those scenarios where existence of strong solutions is out of reach, due to some specific pathological structure of the problems themselves. In this regard, a special mention goes to nonlinear evolution equations with random forcing. Indeed, in the stochastic setting existence of analytically strong solutions may be severely problematic if the nonlinearity $A$ is too singular, due to the presence of extra second order contributions in the energy balance (see for example Gess [32] for the case of sub-homogenous potential). Consequently, forcing the initial datum to belong to $D(A)$ does not help in this case, and it is fundamental to have at hand a valid wellposedness theory in a variational sense, in order to give appropriate meaning to possibly singular evolution equations with general $u_{0} \in H$. For these reasons, we believe that the current work represents also a preliminary step in the direction of building a generalised variational theory for stochastic evolution equations. The variational theory for SPDEs was originally introduced by Pardoux [64] and Krylov \& Rozovskiĭ [47] (see also [52] and the references therein) under the classical reflexivity-separability conditions on the space $V$ and under the polynomial requirements (1.2) on the nonlinearity. Some first contributions in the spirit of Orlicz spaces have been given so far only in very special cases, namely in Barbu \& Da Prato \& Röckner [7, Ch. 4] for the stochastic porous media equation, in $[5,55,56,61]$ for semilinear stochastic equations, in [53,54] for stochastic divergence-form equations, and in $[66,67]$ for the stochastic Cahn-Hilliard equation. Nonetheless, a general extended variational theory for stochastic evolution equations taking into account possibly non-reflexive spaces and non-polynomial nonlinearities is missing: in this direction, the present work represents a valuable candidate for obtaining an extension to the stochastic case, which is itself currently in preparation.

Finally, let us briefly present the structure of the paper. In Section 2 we collect some preliminary general results on modular spaces, while in Section 3 we introduced the novel variational setting and state our main results. Section 4 contains then the proof of the generalised chain rule, and Section 5 is focused on the proof of well-posedness. Eventually, in Section 6 we collect several important applications that can be treated in this framework.

## 2. Preliminaries on modular spaces

In this section, we recall the main definitions and properties concerning the theory of modular spaces, and we prove some preliminary abstract results that will be useful in the sequel. For the details on the theory of modular spaces, we refer the reader to Musielak [58].
Let $X$ be a real Banach space with dual $X^{*}$. The norm in $X$ and the duality between $X^{*}$ and $X$ will be denoted by the symbols $\|\cdot\|_{X}$ and $\langle\cdot, \cdot\rangle_{X^{*}, X}$, respectively.
Definition 2.1. A convex semi-modular on $X$ is a convex functional $\varphi: X \rightarrow[0, \infty]$ satisfying the following conditions:

- $\varphi(0)=0$,
- if $\varphi(\alpha x)=0$ for all $\alpha>0$, then $x=0$;
- $\varphi(-x)=\varphi(x)$ for all $x \in X$.

If also $\varphi(x)=0$ iff $x=0$, then $\varphi$ is called convex modular.
In this section, $\varphi$ is a lower semicontinuous convex semi-modular on $X$. It is possible to naturally associate to $\varphi$ the modular spaces

$$
\begin{array}{ll}
L_{\varphi}:=\{x \in X: \exists \alpha>0: & \varphi(\alpha x)<+\infty\} \\
E_{\varphi}:=\{x \in X: \forall \alpha>0: & \varphi(\alpha x)<+\infty\} \tag{2.2}
\end{array}
$$

It is not difficult to check that $E_{\varphi}$ and $L_{\varphi}$ are real linear spaces, with

$$
E_{\varphi} \subset L_{\varphi} \subset X
$$

Furthermore, we set

$$
\begin{equation*}
\|x\|_{\varphi}:=\inf \{\lambda>0: \quad \varphi(x / \lambda) \leq 1\}, \quad x \in L_{\varphi} \tag{2.3}
\end{equation*}
$$

It is well-known (see for example [58, Chapter I]) that $\|\cdot\|_{\varphi}$ defines a norm on $E_{\varphi}$ and $L_{\varphi}$, called the Luxemburg norm, so that $\left(E_{\varphi},\|\cdot\|_{\varphi}\right)$ and $\left(L_{\varphi},\|\cdot\|_{\varphi}\right)$ are linear normed spaces. From the definition of $\|\cdot\|_{\varphi}$, the properties collected in the following Lemma are well-known: we refer for example to Musielak [58, Thms. 1.5-1.6, Lem. 2.4].

Lemma 2.2. The following properties hold:
(1) for every $x_{1}, x_{2} \in L_{\varphi}$, if $\varphi\left(\alpha x_{1}\right) \leq \varphi\left(\alpha x_{2}\right)$ for all $\alpha>0$, then $\left\|x_{1}\right\|_{\varphi} \leq\left\|x_{2}\right\|_{\varphi}$;
(2) for every $x \in L_{\varphi}$ with $\|x\|_{\varphi}<1$, it holds $\varphi(x) \leq\|x\|_{\varphi}$;
(3) for every $x \in L_{\varphi}$ with $\|x\|_{\varphi}>1$, it holds $\varphi(x) \geq\|x\|_{\varphi}$;
(4) for every sequence $\left(x_{n}\right)_{n} \subset L_{\varphi}$ and $x \in L_{\varphi}$, it holds

$$
\lim _{n \rightarrow \infty}\left\|x_{n}-x\right\|_{\varphi}=0 \quad \text { iff } \quad \lim _{n \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x\right)\right)=0 \quad \forall \alpha>0
$$

(5) for every sequence $\left(x_{n}\right)_{n} \subset L_{\varphi}$, it holds

$$
\lim _{n, k \rightarrow \infty}\left\|x_{n}-x_{k}\right\|_{\varphi}=0 \quad \text { iff } \quad \lim _{n, k \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{k}\right)\right)=0 \quad \forall \alpha>0
$$

Since we are interested in applications to evolutionary PDEs in modular spaces, we look now for sufficient conditions on $\varphi$ ensuring that $\left(E_{\varphi},\|\cdot\|_{\varphi}\right)$ and $\left(L_{\varphi},\|\cdot\|_{\varphi}\right)$ are actually Banach spaces. In the direction, we have the following result.
Proposition 2.3. Suppose that there exists a strictly increasing function $\rho:[0,+\infty) \rightarrow[0,+\infty)$ with $\rho(0)=0$ such that

$$
\begin{equation*}
\varphi(x) \geq \rho\left(\|x\|_{X}\right) \quad \forall x \in X \tag{2.4}
\end{equation*}
$$

Then, $\left(E_{\varphi},\|\cdot\|_{\varphi}\right)$ and $\left(L_{\varphi},\|\cdot\|_{\varphi}\right)$ are Banach spaces, and the following inclusions are continuous:

$$
E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow X .
$$

Proof. Step 1. Firstly, we prove that $\left(E_{\varphi},\|\cdot\|_{\varphi}\right)$ is a Banach space. Let $\left(x_{n}\right)_{n} \subset E_{\varphi}$ be a Cauchy sequence: then, there exists an index $\bar{m} \in \mathbb{N}$ such that, for every $n, k \geq \bar{m}$ we have $\left\|x_{n}-x_{k}\right\|_{\varphi}<1$. Consequently, by Lemma 2.2 (2) and the assumption (2.4) we have

$$
\rho\left(\left\|x_{n}-x_{k}\right\|_{X}\right) \leq \varphi\left(x_{n}-x_{k}\right) \leq\left\|x_{n}-x_{k}\right\|_{\varphi} \quad \forall n, k \geq \bar{m}
$$

In particular, it follows that $\left(x_{n}\right)_{n}$ is a Cauchy sequence in $X$. By completeness of $X$, there exists $x \in X$ such that $x_{n} \rightarrow x$ in $X$, hence also $\alpha x_{n} \rightarrow \alpha x$ in $X$ for all $\alpha>0$. Let now $\alpha>0$ be arbitrary: since $\left(x_{n}\right)_{n}$ is Cauchy in $E_{\varphi} \subset L_{\varphi}$, by Lemma 2.2 (5) we have that

$$
\lim _{n, k \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{k}\right)\right)=0
$$

so that there exists $\bar{m}_{\alpha} \in \mathbb{N}$ such that

$$
\varphi\left(\alpha\left(x_{n}-x_{k}\right)\right) \leq 1 \quad \forall n, k \geq \bar{m}_{\alpha}
$$

By lower semicontinuity and convexity of $\varphi$ in $X$, since $x_{\bar{m}_{\alpha}} \in E_{\varphi}$ we deduce that

$$
\begin{aligned}
\varphi\left(\frac{\alpha}{2} x\right) & \leq \liminf _{n \rightarrow \infty} \varphi\left(\frac{\alpha}{2} x_{n}\right) \\
& =\liminf _{n \rightarrow \infty} \varphi\left(\frac{1}{2} \alpha\left(x_{n}-x_{\bar{m}_{\alpha}}\right)+\frac{1}{2} \alpha x_{\bar{m}_{\alpha}}\right) \\
& \leq \frac{1}{2} \liminf _{n \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{\bar{m}_{\alpha}}\right)\right)+\frac{1}{2} \varphi\left(\alpha x_{\bar{m}_{\alpha}}\right) \\
& \leq 1+\frac{1}{2} \varphi\left(\alpha x_{\bar{m}_{\alpha}}\right)<+\infty
\end{aligned}
$$

Hence, $\varphi\left(\frac{\alpha}{2} x\right)<+\infty$ for all $\alpha>0$, from which $x \in E_{\varphi}$. Moreover, again by lower semicontinuity of $\varphi$, for every $\alpha>0$ we have

$$
\varphi\left(\alpha\left(x_{n}-x\right)\right) \leq \liminf _{k \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{k}\right)\right), \quad \forall n \in \mathbb{N}
$$

which yields

$$
\limsup _{n \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x\right)\right) \leq \limsup _{n, k \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{k}\right)\right)=0
$$

Hence $x_{n} \rightarrow x$ in $E_{\varphi}$ by Lemma 2.2 (4). This shows that $\left(E_{\varphi},\|\cdot\|_{\varphi}\right)$ is a Banach space.
Step 2. We prove now that also $\left(L_{\varphi},\left\|_{\cdot}\right\|_{\varphi}\right)$ is a Banach space. Let $\left(x_{n}\right)_{n} \subset L_{\varphi}$ be a Cauchy sequence. Arguing exactly as in Step 1 we deduce that there exists $x \in X$ such that $x_{n} \rightarrow x$ in $X$ : we have to show that $x \in L_{\varphi}$ and $x_{n} \rightarrow x$ in $L_{\varphi}$. To this end, note that since $\left(x_{n}\right)_{n}$ is Cauchy in $L_{\varphi}$, by the triangular inequality the real sequence $\left(\left\|x_{n}\right\|_{\varphi}\right)_{n}$ is Cauchy in $\mathbb{R}$, so that there exists $\lambda \geq 0$ such that $\lambda_{n}:=\left\|x_{n}\right\|_{\varphi} \rightarrow \lambda$ as $n \rightarrow \infty$. Now, if $\lambda=0$, then $x=0 \in L_{\varphi}$ and $x_{n} \rightarrow 0$ in $L_{\varphi}$, so the conclusion is trivial. Let us suppose that $\lambda>0$ : then $\lambda_{n}>\lambda / 2$ for every $n$ sufficiently large. It follows that

$$
\limsup _{n \rightarrow \infty}\left\|\frac{x_{n}}{\lambda_{n}}-\frac{x}{\lambda}\right\|_{X} \leq \frac{2}{\lambda} \limsup _{n \rightarrow \infty}\left\|x_{n}-x\right\|_{X}+\|x\|_{X} \limsup _{n \rightarrow \infty}\left|\frac{1}{\lambda_{n}}-\frac{1}{\lambda}\right|=0
$$

hence $x_{n} / \lambda_{n} \rightarrow x / \lambda$ in $X$. By lower semicontinuity of $\varphi$ and definition of $\lambda_{n}$ we have then

$$
\varphi(x / \lambda) \leq \liminf _{n \rightarrow \infty} \varphi\left(x_{n} / \lambda_{n}\right) \leq 1
$$

which implies that $x \in L_{\varphi}$ and $\|x\|_{\varphi} \leq \lambda$. Finally, let $\alpha>0$ be arbitrary: then, again by lower semicontinuity of $\varphi$ we have

$$
\varphi\left(\alpha\left(x_{n}-x\right)\right) \leq \liminf _{k \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{k}\right)\right), \quad \forall n \in \mathbb{N}
$$

which yields

$$
\limsup _{n \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x\right)\right) \leq \limsup _{n, k \rightarrow \infty} \varphi\left(\alpha\left(x_{n}-x_{k}\right)\right)=0
$$

Since $\alpha>0$ is arbitrary, we have $x_{n} \rightarrow x$ in $L_{\varphi}$ by Lemma 2.2 (4). This shows that $\left(L_{\varphi},\|\cdot\|_{\varphi}\right)$ is a Banach space.
Step 3. We prove here the continuous inclusions $E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow X$. We already know that $E_{\varphi} \subset L_{\varphi} \subset X$ as inclusions of sets. Moreover, the continuous inclusion $E_{\varphi} \hookrightarrow L_{\varphi}$ is trivial since

$$
\|x\|_{E_{\varphi}}=\|x\|_{L_{\varphi}}=\|x\|_{\varphi} \quad \forall x \in E_{\varphi}
$$

Let now $x \in L_{\varphi}$ and $\lambda>0$ such that $\varphi(x / \lambda) \leq 1$. Then by assumption on $\varphi$ we have

$$
\rho\left(\|x / \lambda\|_{X}\right) \leq \varphi(x / \lambda) \leq 1
$$

from which $\|x\|_{X} \leq \rho^{-1}(1) \lambda$, where $\rho^{-1}$ denotes the generalised inverse of $\rho$. By arbitrariness of $\lambda$ and definition of $\|\cdot\|_{\varphi}$, we have then

$$
\|x\|_{X} \leq \rho^{-1}(1)\|x\|_{\varphi} \quad \forall x \in L_{\varphi}
$$

as required.

Proposition 2.3 ensures then that for a lower semicontinuous convex semi-modular $\varphi$ satisfying condition (2.4), the respective modular spaces ( $E_{\varphi},\|\cdot\| \|_{\varphi}$ ) and ( $L_{\varphi},\|\cdot\|_{\varphi}$ ) are actually Banach spaces. The next main issue in studying the variational structure of $E_{\varphi}$ and $L_{\varphi}$ concerns density properties. In this direction, although the inclusion $E_{\varphi} \hookrightarrow X$ and $L_{\varphi} \hookrightarrow X$ may be dense in the majority of applications, let us stress that the inclusion $E_{\varphi} \hookrightarrow L_{\varphi}$ is not necessarily dense in general: see Section 6 for details. This calls for the introduction of a weaker concept of convergence in $L_{\varphi}$, namely the so-called modular convergence: see Musielak [58].

Definition 2.4 (Modular convergence). A sequence $\left(x_{n}\right)_{n} \subset L_{\varphi}$ is called modular convergent to $x \in L_{\varphi}$ if there exist an $\alpha>0$ such that $\varphi\left(\alpha\left(x_{n}-x\right)\right) \rightarrow 0$ as $n \rightarrow \infty$.

Thanks to Lemma 2.2, it is clear that modular convergence is weaker than the norm convergence. Actually, the former is strictly weaker than the latter, and they are equivalent if and only if $\varphi\left(x_{n}\right) \rightarrow 0$ implies $\varphi\left(2 x_{n}\right) \rightarrow 0$, for every sequence $\left(x_{n}\right)_{n} \subset L_{\varphi}$.

We conclude the preliminary section with an overview on the duality properties of $E_{\varphi}$ and $L_{\varphi}$. These are indeed crucial in order to build a suitable variational framework for PDEs in modular spaces.
Definition 2.5. The convex conjugate $\varphi^{*}: X^{*} \rightarrow[0, \infty]$ of $\varphi: X \rightarrow[0, \infty]$ is defined as

$$
\varphi^{*}(y):=\sup _{x \in X}\left\{\langle y, x\rangle_{X^{*}, X}-\varphi(x)\right\}, \quad y \in X .
$$

Lemma 2.6. If $\varphi$ is a lower semicontinuous convex semi-modular on $X$ and $L_{\varphi} \hookrightarrow X$ densely, then $\varphi^{*}: X^{*} \rightarrow[0,+\infty]$ is a lower semicontinuous convex semi-modular on $X^{*}$.

Proof. We know that $\varphi^{*}$ is lower semicontinuous, proper, and convex. It is also immediate to check using the definition that $\varphi^{*}(0)=0$ and $\varphi^{*}(-y)=\varphi^{*}(y)$ for all $y \in X^{*}$. Moreover, let $y \in X^{*}$ be such that $\varphi^{*}(\alpha y)=0$ for every $\alpha>0$. Take now an arbitrary $x \in L_{\varphi}$ and choose $\eta>0$ such that $\varphi(\eta x)<+\infty$. Then, by the Young inequality and the symmetry of $\varphi$, for all $\alpha>0$ we have

$$
\pm \eta \alpha\langle y, x\rangle_{X^{*}, X} \leq \varphi( \pm \eta x)+\varphi^{*}(\alpha y)=\varphi(\eta x),
$$

yielding

$$
\left|\langle y, x\rangle_{X^{*}, X}\right| \leq \frac{\varphi(\eta x)}{\eta \alpha} \quad \forall \alpha>0
$$

Since $\varphi(\eta x)<+\infty$, letting $\alpha \rightarrow+\infty$ we have

$$
\langle y, x\rangle_{X^{*}, X}=0 \quad \forall x \in L_{\varphi}
$$

from which $y=0$ in $X^{*}$ be density of $L_{\varphi}$ in $X$. Hence, $\varphi^{*}$ is a semi-modular on $X^{*}$.
This lemma ensures that $\varphi^{*}$ is always a (lower semicontinuous convex) semi-modular whenever so is $\varphi$ and $L_{\varphi}$ is dense in $X$. However, note that even if we additionally require that $\varphi$ is a modular, it is not necessarily true that $\varphi^{*}$ is a modular as well. Without additional assumptions on $\varphi$, this is actually false (as it happens for example for $\varphi(x)=\|x\|_{X}, x \in X$ ).
The last results that we present here concern the duality properties of the restriction of $\varphi$ to $E_{\varphi}$. These will be fundamental in the paper.
Lemma 2.7. Let $\varphi$ be a lower semicontinuous convex semi-modular on $X$ satisfying condition (2.4). Then, the restriction

$$
\bar{\varphi}: E_{\varphi} \rightarrow[0,+\infty), \quad \bar{\varphi}:=\varphi_{\mid E_{\varphi}},
$$

is a lower semicontinuous convex semi-modular on $E_{\varphi}$, and its convex conjugate

$$
\bar{\varphi}^{*}: E_{\varphi}^{*} \rightarrow[0,+\infty], \quad \bar{\varphi}^{*}(y):=\sup _{x \in E_{\varphi}}\left\{\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}-\varphi(x)\right\}, \quad y \in E_{\varphi}^{*},
$$

is a lower semicontinuous convex semi-modular on $E_{\varphi}^{*}$.

Proof. It is clear that $\bar{\varphi}$ is proper, convex and lower semicontinuous, since $E_{\varphi} \hookrightarrow X$ continuously by Proposition 2.3 and $\varphi$ is lower semicontinuous on $X$. It is also trivial that $\bar{\varphi}(0)=0$ and that $\bar{\varphi}(-x)=\bar{\varphi}(x)$ for all $x \in E_{\varphi}$. Moreover, if $x \in E_{\varphi}$ satisfies $\bar{\varphi}(\alpha x)=0$ for all $\alpha>0$, then clearly $\varphi(\alpha x)=0$ for all $\alpha>0$, hence $x=0$ since $\varphi$ is a semi-modular. This shows that $\bar{\varphi}$ is a lower semicontinuous convex semi-modular on $E_{\varphi}$. Consequently, since $E_{\bar{\varphi}}=L_{\bar{\varphi}}=E_{\varphi}$ (so in particular $L_{\bar{\varphi}}$ is trivially dense in $E_{\varphi}$ ), by Lemma 2.6 with the choice $X=E_{\varphi}$ we have that $\bar{\varphi}^{*}$ is a lower semicontinuous convex semi-modular on $E_{\varphi}^{*}$.

Lemma 2.7 ensures then that the modular spaces

$$
\begin{array}{ll}
L_{\bar{\varphi}^{*}}:=\left\{y \in E_{\varphi}^{*}: \exists \alpha>0:\right. & \left.\bar{\varphi}^{*}(\alpha y)<+\infty\right\}, \\
E_{\bar{\varphi}^{*}}:=\left\{y \in E_{\varphi}^{*}: \forall \alpha>0:\right. & \left.\bar{\varphi}^{*}(\alpha y)<+\infty\right\},
\end{array}
$$

endowed with the norm

$$
\|y\|_{\bar{\varphi}^{*}}:=\inf \left\{\lambda>0: \quad \bar{\varphi}^{*}(y / \lambda) \leq 1\right\}, \quad y \in L_{\bar{\varphi}^{*}},
$$

are well-defined normed spaces and satisfy $E_{\bar{\varphi}^{*}} \subset L_{\bar{\varphi}^{*}} \subset E_{\varphi}^{*}$. The following result gives a further characterization in terms of completeness.

Proposition 2.8. Let $\varphi$ be a lower semicontinuous convex semi-modular on $X$ satisfying condition (2.4). Then, it holds that

$$
\begin{equation*}
\|y\|_{\bar{\varphi}^{*}} \leq\|y\|_{E_{\varphi}^{*}} \leq 2\|y\|_{\bar{\varphi}^{*}} \quad \forall y \in L_{\bar{\varphi}^{*}} \tag{2.5}
\end{equation*}
$$

In particular, the modular spaces $\left(L_{\bar{\varphi}^{*}},\|\cdot\|_{\bar{\varphi}^{*}}\right)$ and $\left(E_{\bar{\varphi}^{*}},\|\cdot\|_{\bar{\varphi}^{*}}\right)$ are Banach spaces, and it holds

$$
E_{\bar{\varphi}^{*}} \hookrightarrow L_{\bar{\varphi}^{*}}=E_{\varphi}^{*} .
$$

Moreover, if $s>1$ in (2.4) and $E_{\varphi} \hookrightarrow X$ densely, then also $X^{*} \hookrightarrow E_{\bar{\varphi}^{*}}$ continuously.
Proof. Step 1. Let $y \in L_{\bar{\varphi}^{*}}$ : by definition of dual norm and by the Young inequality we have

$$
\|y\|_{E_{\varphi}^{*}}=\sup \left\{\langle y, x\rangle: x \in E_{\varphi},\|x\|_{\varphi} \leq 1\right\} \leq \bar{\varphi}^{*}(y)+\sup \left\{\varphi(x): x \in E_{\varphi},\|x\|_{\varphi} \leq 1\right\} .
$$

Now, let $x \in E_{\varphi}$ be such that $\|x\|_{\varphi} \leq 1$. If $\left(\delta_{k}\right)_{k} \subset(0,1)$ is such that $\delta_{k} \nearrow 1$ as $k \rightarrow \infty$, then we have that $\left\|\delta_{k} x\right\|_{\varphi}<1$ for every $k$, hence also, by Lemma 2.2 (2),

$$
\varphi\left(\delta_{k} x\right) \leq\left\|\delta_{k} x\right\|_{\varphi}=\delta_{k}\|x\|_{\varphi} \leq \delta_{k} .
$$

Since $\delta_{k} x \rightarrow x$ in $X$, letting $k \rightarrow \infty$ we get, by lower semicontinuity of $\varphi$,

$$
\varphi(x) \leq \liminf _{k \rightarrow \infty} \varphi\left(\delta_{k} x\right) \leq 1 \quad \forall x \in E_{\varphi}:\|x\|_{\varphi} \leq 1
$$

Putting this information together, we deduce that

$$
\|y\|_{E_{\varphi}^{*}} \leq 1+\bar{\varphi}^{*}(y) \quad \forall y \in L_{\bar{\varphi}^{*}} .
$$

Now, let $\lambda>0$ be such that $\bar{\varphi}^{*}(y / \lambda) \leq 1$. The inequality just proved implies, by arbitrariness of $y \in L_{\bar{\varphi}^{*}}$, that

$$
\frac{1}{\lambda}\|y\|_{E_{\varphi}^{*}}=\|y / \lambda\|_{E_{\varphi}^{*}} \leq 1+\bar{\varphi}^{*}(y / \lambda) \leq 2
$$

from which $\|y\|_{E_{\varphi}^{*}} \leq 2 \lambda$. By arbitrariness of $\lambda$ the right-inequality in (2.5) follows. As for the left-inequality, for $y \in E_{\varphi}^{*} \backslash\{0\}$ we have

$$
\bar{\varphi}^{*}\left(\frac{y}{\|y\|_{E_{\varphi}^{*}}}\right)=\sup _{x \in E_{\varphi}}\left\{\frac{\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}}{\|y\|_{E_{\varphi}^{*}}}-\varphi(x)\right\} \leq \sup _{x \in E_{\varphi}}\left\{\|x\|_{\varphi}-\varphi(x)\right\} .
$$

Now, if $x \in E_{\varphi}$ and $\|x\|_{\varphi}>1$, from Lemma 2.2 (3) we know that $\varphi(x) \geq\|x\|_{\varphi}$, hence also $\|x\|_{\varphi}-\varphi(x) \leq 0$. It follows then that

$$
\bar{\varphi}^{*}\left(\frac{y}{\|y\|_{E_{\varphi}^{*}}}\right) \leq \sup _{x \in E_{\varphi}}\left\{\|x\|_{\varphi}-\varphi(x)\right\} \leq \sup _{\|x\|_{\varphi} \leq 1}\left\{\|x\|_{\varphi}-\varphi(x)\right\} \leq 1 .
$$

This yields that $y \in L_{\bar{\varphi}^{*}}$ and $\|y\|_{\bar{\varphi}^{*}} \leq\|y\|_{E_{E^{*}}}$, as desired. Hence, the inequality (2.5) is proved. As a byproduct, this implies also that $L_{\bar{\varphi}^{*}}=E_{\varphi}^{*}$, and that the dual norm on $E_{\varphi}^{*}$ is equivalent to the $\|\cdot\|_{\bar{\varphi}^{*}}$-norm.
Let us show now that $E_{\bar{\varphi}^{*}}$ and $L_{\bar{\varphi}^{*}}$ are Banach spaces. To this end, let $\left(y_{n}\right)_{n} \subset E_{\bar{\varphi}^{*}}$ be a Cauchy sequence: then, by (2.5) it follows that it is also Cauchy in $E_{\varphi}^{*}$. By completeness of $E_{\varphi}^{*}$, there is $y \in E_{\varphi}^{*}$ such that $y_{n} \rightarrow y$ in $E_{\varphi}^{*}$. Proceeding now as in the proof of Proposition 2.3, by the lower semicontinuity of $\bar{\varphi}^{*}$ on $E_{\varphi}^{*}$ we deduce that $y \in E_{\bar{\varphi}^{*}}$, hence again by (2.5) that $y_{n} \rightarrow y$ in $E_{\bar{\varphi}^{*}}$. The case of $L_{\varphi^{*}}$ is entirely analogous.
Step 2. Let us suppose that $s>1$ and $E_{\varphi} \hookrightarrow X$ densely, and show that $X^{*} \hookrightarrow E_{\bar{\varphi}^{*}}$. First of all, note that we can identify $X^{*}$ with a closed subspace of $E_{\varphi}^{*}$ : indeed, denoting by $i: E_{\varphi} \rightarrow X$ the inclusion, it is easily seen that since $E_{\varphi} \hookrightarrow X$ densely the adjoint operator $i^{*}: X^{*} \rightarrow E_{\varphi}^{*}$ is linear, continuous, and injective. Hence, we can identify $X^{*} \cong i^{*}\left(X^{*}\right) \hookrightarrow E_{\varphi}^{*}$, getting

$$
E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow X, \quad X^{*} \hookrightarrow E_{\varphi}^{*}
$$

Secondly, let us show that $\left(\bar{\varphi}^{*}\right)_{\mid X^{*}} \leq \varphi^{*}$. Indeed, for every $y \in X^{*}$ we have

$$
\begin{aligned}
\bar{\varphi}^{*}(y) & =\sup _{x \in E_{\varphi}}\left\{\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}-\varphi(x)\right\} \\
& =\sup _{x \in E_{\varphi}}\left\{\langle y, x\rangle_{X^{*}, X}-\varphi(x)\right\} \\
& \leq \sup _{x \in X}\left\{\langle y, x\rangle_{X^{*}, X}-\varphi(x)\right\} \\
& =\varphi^{*}(y) .
\end{aligned}
$$

Finally, we are now ready to conclude. Indeed, recalling that $s>1$, taking conjugates in (2.4) yields, after a standard computation,

$$
\varphi^{*}(y) \leq \frac{s-1}{s}(c s)^{-\frac{1}{s-1}}\|y\|_{X^{*}}^{\frac{s}{s-1}} \quad \forall y \in X^{*},
$$

which implies that actually $\varphi^{*}: X^{*} \rightarrow[0,+\infty)$ is everywhere defined on $X^{*}$, and also that $\varphi^{*}(\alpha y)<$ $+\infty$ for every $y \in X^{*}$ and $\alpha>0$. Since $\left(\bar{\varphi}^{*}\right)_{\mid X^{*}} \leq \varphi^{*}$, this shows that $X^{*} \subset E_{\bar{\varphi}^{*}}$. Moreover, for every arbitrary $y \in X^{*}$ and for any $\lambda=\lambda(y)>0$ such that

$$
\lambda \geq\left(\frac{s-1}{s}(c s)^{-\frac{1}{s-1}}\right)^{\frac{s-1}{s}}\|y\|_{X^{*}},
$$

it clearly holds that

$$
\bar{\varphi}^{*}(y / \lambda) \leq \varphi^{*}(y / \lambda) \leq \frac{s-1}{s}(c s)^{-\frac{1}{s-1}}\|y\|_{X^{*}}^{\frac{s}{s-1}} \lambda^{-\frac{s}{s-1}} \leq 1
$$

Hence, by definition of $\|\cdot\|_{\bar{\varphi}^{*}}$ we have then that

$$
\|y\|_{\bar{\varphi}^{*}} \leq\left(\frac{s-1}{s}(c s)^{-\frac{1}{s-1}}\right)^{\frac{s-1}{s}}\|y\|_{X^{*}} \quad \forall y \in X^{*},
$$

so that the inclusion $X^{*} \hookrightarrow E_{\bar{\varphi}^{*}}$ is continuous, as required.

## 3. Extended variational setting and main result

In this section, we fix the assumptions and introduce the extended variational setting that will be used in the paper. After this, we present the main well posedness result.
3.1. Assumptions. Throughout the paper, we will work in the following framework.

H0: $H$ is a real separable Hilbert space, $\varphi: H \rightarrow[0, \infty]$ is a lower semicontinuous convex semi-modular on $H$, and there exist constants $c>0$ and $s>1$ such that

$$
\varphi(x) \geq c\|x\|_{H}^{s} \quad \forall x \in H
$$

By Proposition 2.3 applied with $\rho(z)=c z^{s}, z \geq 0$, this implies that the modular spaces $\left(E_{\varphi},\|\cdot\|_{\varphi}\right)$ and $\left(L_{\varphi},\|\cdot\|_{\varphi}\right)$ are well-defined Banach spaces with continuous inclusions $E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow H$. From now on, $H$ is identified to its dual space $H^{*}$ by the Riesz isomorphism, and norm and scalar product in $H$ are denoted by $\|\cdot\|_{H}$ and $(\cdot, \cdot)$, respectively. The convex conjugate of $\varphi$ is defined as

$$
\varphi^{*}: H \rightarrow[0,+\infty], \quad \varphi^{*}(y):=\sup _{x \in H}\{(y, x)-\varphi(x)\}, \quad x \in H .
$$

$\mathbf{H 1}: E_{\varphi}$ is dense in $H$, and there exists a separable reflexive Banach space $V_{0} \hookrightarrow E_{\varphi}$ continuously and densely, such that $\varphi$ is bounded on bounded subsets of $V_{0}$.

The existence of such $V_{0}$ is an assumption of technical nature, and can be seen a separability-type requirement for $E_{\varphi}$. This is satisfied in the majority of applications (see Section 6). The density of $E_{\varphi}$ ensures first that we can identify $H$ with a closed subspace of $E_{\varphi}^{*}$. More specifically, denoting by $i: E_{\varphi} \rightarrow H$ the (continuous) inclusion and recalling that $H \cong H^{*}$, we have that the adjoint operator $i^{*}: H \rightarrow E_{\varphi}^{*}$ is linear, continuous, and injective: indeed, linearity and continuity follow trivially by the continuous inclusion $E_{\varphi} \hookrightarrow H$, while the injectivity is an immediate consequence of the density. Hence, one can identify $H$ with the closed subspace $i^{*}(H) \subset E_{\varphi}^{*}$. We have then the following continuous inclusions

$$
E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow H \hookrightarrow E_{\varphi}^{*} .
$$

We introduce the restricted semi-modular

$$
\bar{\varphi}: E_{\varphi} \rightarrow[0,+\infty), \quad \bar{\varphi}:=\varphi_{\mid E_{\varphi}}
$$

and its convex conjugate

$$
\bar{\varphi}^{*}: E_{\varphi}^{*} \rightarrow[0, \infty], \quad \bar{\varphi}^{*}(y):=\sup _{x \in E_{\varphi}}\left\{\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}-\varphi(x)\right\}, \quad x \in E_{\varphi} .
$$

By Lemma 2.7 and Proposition 2.8, we know that $\bar{\varphi}^{*}$ is a lower semicontinuous convex semimodular on $E_{\varphi}^{*}$, with $\left(\bar{\varphi}^{*}\right)_{\mid H}=\varphi^{*}$, and that the modular spaces $\left(E_{\bar{\varphi}^{*}},\|\cdot\|_{\bar{\varphi}^{*}}\right)$ and $\left(L_{\bar{\varphi}^{*}},\|\cdot\|_{\bar{\varphi}^{*}}\right)$ are well-defined Banach spaces, with continuous inclusions

$$
H \hookrightarrow E_{\bar{\varphi}^{*}} \hookrightarrow L_{\bar{\varphi}^{*}}=E_{\varphi}^{*} .
$$

H2: either one of the following conditions holds:
H2i: $E_{\varphi} \hookrightarrow L_{\varphi}$ densely, or
H2ii: $H \hookrightarrow L_{\bar{\varphi}^{*}}$ densely.
The main consequence of this assumption is that it allows to properly define an extended concept of duality between the spaces $L_{\varphi}$ and $L_{\bar{\varphi}^{*}}$. Specifically, by $\mathbf{H 0}-\mathbf{H 1}$ we have the continuous inclusions

$$
V_{0} \hookrightarrow E_{\varphi} \hookrightarrow L_{\varphi} \hookrightarrow H \hookrightarrow E_{\bar{\varphi}^{*}} \hookrightarrow L_{\bar{\varphi}^{*}}=E_{\varphi}^{*} \hookrightarrow V_{0}^{*}
$$

However, a priori it is not true that there exists a duality pairing between $L_{\varphi}$ and $L_{\bar{\varphi}^{*}}$ generalizing the scalar product of $H$. This is because $L_{\varphi}$ may be strictly bigger than $E_{\varphi}$, and $L_{\bar{\varphi}^{*}}$ may be strictly bigger than $H$ (see Secction 6). Assumption H2 is fundamental as it allows to extend the scalar product of $H$ to a duality between $L_{\varphi}$ and $L_{\bar{\varphi}^{*}}$. Due to the importance of this result, we collect it in the following lemma.
Lemma 3.1. Assume $\mathbf{H 0} \mathbf{- H 2}$. Then, there exists a unique continuous bilinear form

$$
[\cdot, \cdot]: L_{\bar{\varphi}^{*}} \times L_{\varphi} \rightarrow \mathbb{R},
$$

extending the scalar product of $H$, in the sense that $[y, \cdot]: L_{\varphi} \rightarrow \mathbb{R}$ and $[\cdot, x]: L_{\bar{\varphi}^{*}} \rightarrow \mathbb{R}$ are linear and continuous for every $y \in L_{\bar{\varphi}^{*}}$ and $x \in L_{\varphi}$, respectively, and such that

$$
\begin{array}{cl}
{[y, x]=(y, x)} & \forall x \in L_{\varphi}, \forall y \in H, \\
{[y, x]=\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}} & \forall x \in E_{\varphi}, \forall y \in L_{\bar{\varphi}^{*}} .
\end{array}
$$

Furthermore, the following generalized Hölder inequality holds:

$$
\begin{equation*}
[y, x] \leq 2\|y\|_{\bar{\varphi}^{*}}\|x\|_{\varphi} \quad \forall(x, y) \in L_{\varphi} \times L_{\bar{\varphi}^{*}} . \tag{3.1}
\end{equation*}
$$

Proof. CASE 1. We prove here the lemma in the case of $\mathbf{H} 2$ i, i.e. if $E_{\varphi} \hookrightarrow L_{\varphi}$ densely. Let $x \in L_{\varphi}$ and $y \in L_{\bar{\varphi}^{*}}=E_{\varphi}^{*}$. If either $x=0$ or $y=0$, then we set $[y, x]:=0$. Let us suppose that $x \neq 0$ and $y \neq 0$. Take a sequence $\left(x_{n}\right)_{n \in \mathbb{N}} \subset E_{\varphi}$ such that $x_{n} \rightarrow x$ in $L_{\varphi}$, and define

$$
\left[y, x_{n}\right]:=\left\langle y, x_{n}\right\rangle_{E_{\varphi}^{*}, E_{\varphi}}, \quad n \in \mathbb{N} .
$$

Now, for every $n, k \in \mathbb{N}$, by the Young inequality we have

$$
\begin{aligned}
\left|\left\langle\frac{y}{\|y\|_{\bar{\varphi}^{*}}}, \frac{x_{n}-x_{k}}{\left\|x_{n}-x_{k}\right\|_{\varphi}}\right\rangle_{E_{\varphi}^{*}, E_{\varphi}}\right| & \leq \bar{\varphi}^{*}\left(\frac{y}{\|y\|_{\bar{\varphi}^{*}}}\right)+\bar{\varphi}\left(\frac{x_{n}-x_{k}}{\left\|x_{n}-x_{k}\right\|_{\varphi}}\right) \\
& =\bar{\varphi}^{*}\left(\frac{y}{\|y\|_{\bar{\varphi}^{*}}}\right)+\varphi\left(\frac{x_{n}-x_{k}}{\left\|x_{n}-x_{k}\right\|_{\varphi}}\right) \\
& \leq 2,
\end{aligned}
$$

yielding

$$
\left|\left[y, x_{n}-x_{k}\right]\right| \leq 2\|y\|_{\bar{\varphi}^{*}}\left\|x_{n}-x_{k}\right\|_{\varphi} \rightarrow 0 .
$$

Hence, $\left(\left[y, x_{n}\right]_{\varphi}\right)_{n}$ is a Cauchy sequence in $\mathbb{R}$, and we can define

$$
[y, x]:=\lim _{n \rightarrow \infty}\left[y, x_{n}\right] .
$$

A similar argument shows that this definition is independent of the choice of the approximating sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$. Moreover, it is straightforward to check that $[\cdot, \cdot]$ is a continuous bilinear form extending the scalar product of $H$. It is also the only continuous bilinear form on $L_{\bar{\varphi}^{*}} \times L_{\varphi}$ doing so: indeed, if $[\cdot, \cdot]_{1}$ and $[\cdot,]_{2}$ are continuous bilinear forms extending the scalar product of $H$, they would coincide on $L_{\bar{\varphi}^{*}} \times E_{\varphi}$, hence they coincide on the whole $L_{\bar{\varphi}^{*}} \times L_{\varphi}$ by density of $E_{\varphi}$ in $L_{\varphi}$.
CASE 2. In the case of $\mathbf{H} 2 \mathbf{i i}$, i.e. when $H \hookrightarrow L_{\bar{\varphi}^{*}}$ densely, the argument is very similar, so we omit the details. Let $x \in L_{\varphi} \subset H$ and $y \in L_{\bar{\varphi}^{*}}$, with $x \neq 0$ and $y \neq 0$ (otherwise the definition is trivial). Take a sequence $\left(y_{n}\right)_{n \in \mathbb{N}} \subset H$ such that $y_{n} \rightarrow y$ in $L_{\bar{\varphi}^{*}}$, and define

$$
\left[y_{n}, x\right]:=\left(y_{n}, x\right) .
$$

As before, for every $n, k \in \mathbb{N}$, by the Young inequality we have

$$
\begin{aligned}
\left|\left(\frac{y_{n}-y_{k}}{\left\|y_{n}-y_{k}\right\|_{\bar{\varphi}^{*}}}, \frac{x}{\|x\|_{\varphi}}\right)\right| & \leq \varphi^{*}\left(\frac{y_{n}-y_{k}}{\left\|y_{n}-y_{k}\right\|_{\bar{\varphi}^{*}}}\right)+\varphi\left(\frac{x}{\|x\|_{\varphi}}\right) \\
& =\bar{\varphi}^{*}\left(\frac{y_{n}-y_{k}}{\left\|y_{n}-y_{k}\right\|_{\bar{\varphi}^{*}}}\right)+\varphi\left(\frac{x}{\|x\|_{\varphi}}\right) \\
& \leq 2
\end{aligned}
$$

yielding $\left|\left[y_{n}-y_{m}, x\right]\right| \leq 2\left\|y_{n}-y_{k}\right\|_{\varphi^{*}}\|x\|_{\varphi} \rightarrow 0$. Therefore, we can define

$$
[y, x]:=\lim _{n \rightarrow \infty}\left[y_{n}, x\right]_{\varphi} .
$$

As before, this definition is independent of the choice of the approximating sequence $\left(y_{n}\right)_{n}$, and $[,, \cdot]$ is the unique continuous bilinear form on $L_{\bar{\varphi}^{*}} \times L_{\varphi}$ extending the scalar product of $H$.

Remark 3.2 (Assumption H0). Assumption H0 is very natural in applications to nonlinear evolution problems: for example, for a wide class of evolutionary PDEs of parabolic type a natural choice for the space $H$ is $L^{2}(\Omega)$, with $\Omega$ being a sufficiently smooth domain in $\mathbb{R}^{d}$, while $\varphi$ is the convex part of the energy driving the evolution. Several examples are given in Section 6.

Remark 3.3 (Assumption H1). As far as assumption H1 is concerned, one can easily check that this is verified by a huge class of potentials $\varphi$, not necessarily of polynomial growth. More importantly, H1 includes several interesting examples in which $E_{\varphi}$ is not reflexive, hence cannot be framed in classical variational structures: again, a spectrum of applications is given in Section 6. It excludes, however, some singular Orlicz-Sobolev spaces associated, for instance, to convex functions defined of bounded intervals and blowing up at the extreme points. In these extreme cases, the space $E_{\varphi}$ often reduces to the trivial space $\{0\}$.

Remark 3.4 (Assumption H2). Let us spend a few words on the idea behind H2. This hypothesis requires the density either of $E_{\varphi}$ in $L_{\varphi}$, or of $H$ in $L_{\bar{\varphi}^{*}}$. In the former case, we are supposing that the modular spaces $E_{\varphi}$ in $L_{\varphi}$ are "not too different": this is verified when the potential $\varphi$ satisfies a so-called $\Delta_{2}$-type condition (see Section 6), and in such a case it actually holds that $E_{\varphi}=L_{\varphi}$. In the latter case, by contrast, what we are supposing is that the modular space $L_{\bar{\varphi}^{*}}$ is not "too much bigger" than $H$ itself. Again, this happens when the conjugate function $\varphi^{*}$ satisfies a $\Delta_{2}$-type condition. The main advantage of assumption $\mathbf{H} \mathbf{2}$ is that in the majority of applications either $\varphi$ or $\varphi^{*}$ always satisfy a $\Delta_{2}$-condition: the rough idea is that whenever $\varphi$ is not $\Delta_{2}$ (for example if it grows super-polynomially at infinity) then by contrast its conjugate $\varphi^{*}$ behaves in a $\Delta_{2}$-fashion (it grows sub-polynomially), and viceversa. This allows to include in assumption $\mathbf{H} 2$ a wide variety of very singular problems, where $E_{\varphi}$ and $L_{\varphi}$ are not necessarily reflexive.
3.2. The extended variational setting. The main idea is to work on the triplet

$$
L_{\varphi} \hookrightarrow H \hookrightarrow L_{\bar{\varphi}^{*}},
$$

where both inclusions are continuous, the first one is also dense, and the second one is dense if H2ii holds. Let us point out that such variational setting is non-standard for the following main reasons. First, the space $L_{\varphi}$ is allowed to be non-reflexive, thus including several applications to singular PDEs of evolutionary type. Secondly, the space $L_{\bar{\varphi}^{*}}$ is not the dual of $L_{\varphi}$, and the duality pairing between them is not given by the classical duality, but by the generalized bilinear form $[,, \cdot]$. Finally, the space $L_{\varphi}$ and $L_{\bar{\varphi}^{*}}$ are not necessarily separable.

The lack of separability and reflexivity in evolution problems is a crucial issue that creates several difficulties. As for separability, the main issue concerns measurability for vector-valued functions: indeed, by the Pettis measurability theorem, a necessary condition for a Banach-space-valued function to be Bochner-measurable is that it is essentially separably-valued (see, for example, [27, Sect. II, Thm. 2]). This forces us to work on spaces of weakly-measurable functions instead. As for reflexivity, the main drawback that we need to face is the following. If $X$ is a reflexive Banach space, then it is well-known that Sobolev-Bochner spaces in the form $W^{1, p}(a, b ; X)$, where $p \geq 1$ and $[a, b] \subset \mathbb{R}$ is a bounded interval, can be characterized as spaces of absolutely continuous functions in $L^{p}(a, b ; X)$ with almost everywhere derivative in $L^{p}(a, b ; X)$. In particular, the reflexivity of $X$ implies that any absolutely continuous function $[a, b] \rightarrow X$ is almost everywhere differentiable: see for example [6, Thm. 1.16]. Nevertheless, if $X$ is not reflexive, these results are actually false, and such characterization of $W^{1, p}(a, b ; X)$ spaces is no longer valid. In this case, one has to additionally require the almost everywhere differentiability, as this is not granted by the absolute continuity itself (see [14] and [50, Thm. 8.57]). In particular, if $X$ is not reflexive, there exist absolutely continuous functions [a,b] $\rightarrow X$ that are nowhere differentiable (e.g. [50, Ex. 8.30 and 8.32]).

Let us introduce some notation for the spaces of vector-valued integrable functions that we will use. We refer the reader to [27, Sect. II] for the general theory of integration. From now on, $T>0$
is a fixed final time. We set

$$
\begin{aligned}
& L_{w}^{1}\left(0, T ; L_{\varphi}\right):=\left\{v:[0, T] \rightarrow L_{\varphi}: \quad[y, v] \in L^{1}(0, T) \quad \forall y \in L_{\bar{\varphi}^{*}}\right\}, \\
& L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right):=\left\{v:[0, T] \rightarrow L_{\bar{\varphi}^{*}}: \quad[v, x] \in L^{1}(0, T) \quad \forall x \in L_{\varphi}\right\}, \\
& L^{1}\left(0, T ; L_{\varphi}\right):=\left\{v:[0, T] \rightarrow L_{\varphi} \text { strongly measurable : }\|v\|_{\varphi} \in L^{1}(0, T)\right\}, \\
& L^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right):=\left\{v:[0, T] \rightarrow L_{\bar{\varphi}^{*}} \text { strongly measurable : }\|v\|_{\bar{\varphi}^{*}} \in L^{1}(0, T)\right\} .
\end{aligned}
$$

Let us point out that under assumption H2i we have that $L_{\varphi}$ is separable, since so is $E_{\varphi}$ by H1, hence elements in $L_{w}^{1}\left(0, T ; L_{\varphi}\right)$ are also strongly measurable in this case. Under assumption H2ii, we have instead that $L_{\bar{\varphi}^{*}}$ is separable, since so is $H$, and in this case elements of $L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$ are strongly measurable.

The following result holds also for non-reflexive spaces, hence are fundamental in our setting: the reader can refer to [ 6, Thm. 1.17]
Proposition 3.5. Let $X$ be a Banach space and $u \in L^{p}(a, b ; X)$, with $1 \leq p \leq \infty$ and $[a, b] \subset \mathbb{R}$. Then $u \in W^{1, p}(a, b ; X)$ if and only if there exists an absolutely continuous function $u^{0}:[a, b] \rightarrow X$, i.e. $u^{0} \in A C([0, T] ; X)$, which is almost everywhere differentiable on $[a, b]$ with $\frac{d u^{0}}{d t} \in L^{p}(0, T ; X)$, such that $u(t)=u^{0}(t)$ for almost every $t \in(a, b)$. In such case, $\frac{d u^{0}}{d t}$ coincides with the weak derivative $\partial_{t} u$ of $u$.

Inspired by Proposition 3.5, it is natural to define

$$
\begin{aligned}
W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right):=\{ & v:[0, T] \rightarrow L_{\bar{\varphi}^{*}}: \quad \exists v^{\prime} \in L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right): \\
& {\left.[v(t), x]=[v(0), x]+\int_{0}^{t}\left[v^{\prime}(s), x\right] d s \quad \forall x \in L_{\varphi}\right\} . }
\end{aligned}
$$

Note that $W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$ only implies weak* continuity in $E_{\varphi}^{*}=L_{\bar{\varphi}^{*}}$, and not absolute continuity as in the classical case.
3.3. Main results. We are ready to present here the main results of the paper. From now on, $T>0$ is a a fixed final time.

The first result that we present is a fundamental computational tool in order to handle evolution equations in singular modular spaces. It ensures that under assumptions $\mathbf{H} 0-\mathbf{H} 2$ the novel variational setting $\left(L_{\varphi}, H, L_{\varphi^{*}}\right)$ with duality given by $[\cdot, \cdot]$ is actually suited for dealing with evolution problems, even without the classical reflexivity/separabilty assumptions. This is an interesting generalization to the non-reflexive and non-separable case of a well-know "chain-rule" property for vector-valued functions.

Theorem 3.6 (Generalized chain rule). Assume $\mathbf{H 0} \mathbf{- H 2}$, and let

$$
u \in W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right) \cap L_{w}^{1}\left(0, T ; L_{\varphi}\right),
$$

be such that

$$
\partial_{t} u=u_{1}^{\prime}+u_{2}^{\prime}, \quad \text { with } \quad u_{1}^{\prime} \in L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right), \quad u_{2}^{\prime} \in L^{1}(0, T ; H) .
$$

If there exists $\alpha>0$ such that

$$
\varphi(\alpha u), \varphi^{*}\left(\alpha u_{1}^{\prime}\right) \in L^{1}(0, T),
$$

then $u \in C^{0}([0, T] ; H)$, the function $t \mapsto\|u(t)\|_{H}^{2}, t \in[0, T]$, is absolutely continuous, and it holds that

$$
\begin{equation*}
\left[\partial_{t} u, u\right]=\frac{d}{d t} \frac{1}{2}\|u\|_{H}^{2} \quad \text { a.e. in }(0, T) . \tag{3.2}
\end{equation*}
$$

The second result concerns existence of variational solutions for evolution equations in modular spaces.

Theorem 3.7 (Existence of solutions). Assume $\mathbf{H 0} \mathbf{- H 2}$ and let

$$
\begin{equation*}
u_{0} \in H, \quad f \in L^{1}(0, T ; H) \tag{3.3}
\end{equation*}
$$

Then, there exists a unique pair $(u, \xi)$, with

$$
\begin{align*}
& u \in W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right) \cap C^{0}([0, T] ; H) \cap L_{w}^{1}\left(0, T ; L_{\varphi}\right),  \tag{3.4}\\
& \xi \in L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)  \tag{3.5}\\
& \varphi(u), \bar{\varphi}^{*}(\xi) \in L^{1}(0, T), \tag{3.6}
\end{align*}
$$

such that

$$
\begin{equation*}
\partial_{t} u+\xi=f \quad \text { in } L_{\bar{\varphi}^{*}} \quad \text { a.e. in }(0, T), \quad u(0)=u_{0}, \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\varphi(u)+[\xi, x-u] \leq \varphi(x) \quad \forall x \in E_{\varphi}, \quad \text { a.e. in }(0, T) . \tag{3.8}
\end{equation*}
$$

Moreover, the following energy equality holds:

$$
\begin{equation*}
\frac{1}{2}\|u(t)\|_{H}^{2}+\int_{0}^{t}[\xi(s), u(s)] d s=\frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{t}(f(s), u(s)) d s \quad \forall t \in[0, T] . \tag{3.9}
\end{equation*}
$$

In particular, under H2i it holds that $u \in L^{1}\left(0, T ; L_{\varphi}\right)$, while under H2ii it holds that $u \in$ $W^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$ and $\xi \in L^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$.

Condition (3.8) is the natural generalization of the classical subdifferential inclusion

$$
\begin{equation*}
\xi \in \partial \varphi(u) \quad \text { a.e. in }(0, T) . \tag{3.10}
\end{equation*}
$$

In our setting, $\xi$ is less regular than $H$, as $\xi$ only belongs to $L_{\bar{\varphi}^{*}}$, hence the classical differential inclusion (3.10) makes no sense here. Similarly, the classical relaxation of the inclusion (3.10) given by

$$
\begin{equation*}
\xi \in \partial \bar{\varphi}(u) \quad \text { a.e. in }(0, T), \tag{3.11}
\end{equation*}
$$

is useless as well, as $u$ is not necessarily $E_{\varphi}$-valued. For these reasons, the introduction of the novel duality $[\cdot, \cdot]$ is crucial, as it allows to give sense to the subdifferential inclusion in the spaces $L_{\varphi}-L_{\bar{\varphi}^{*}}$ as done in (3.8). Clearly, this is a very natural extension of both (3.10) and (3.11): indeed, whenever $\xi$ is $H$-valued (or $u$ is $E_{\varphi}$-valued, respectively) then (3.8) is equivalent to (3.10) (or (3.11), respectively). An equivalent formulation of the relaxed condition (3.8) is given by

$$
\begin{equation*}
[\xi, u]=\varphi(u)+\bar{\varphi}^{*}(\xi) \quad \text { a.e. in }(0, T) . \tag{3.12}
\end{equation*}
$$

Finally, the last result that we present is a continuous dependence result, with ensures that the evolution problem is actually well-posed.
Theorem 3.8 (Continuous dependence on the data). Assume H0-H2, and let $\left(u_{0}^{1}, f_{1}\right)$ and $\left(u_{0}^{2}, f_{2}\right)$ satisfy (3.3). Then, for any respective solutions $\left(u_{1}, \xi_{1}\right)$ and $\left(u_{2}, \xi_{2}\right)$ to (3.4)-(3.8), it holds that

$$
\left\|u_{1}-u_{2}\right\|_{C^{0}([0, T] ; H)}^{2}+\left\|\left[\xi_{1}-\xi_{2}, u_{1}-u_{2}\right]\right\|_{L^{1}(0, T)} \leq 2\left(\left\|u_{0}^{1}-u_{0}^{2}\right\|_{H}^{2}+\left\|f_{1}-f_{2}\right\|_{L^{1}(0, T ; H)}^{2}\right) .
$$

## 4. Proof of the generalized "Chain rule"

This section is devoted to the proof of Theorem 3.6. Let us work then in the notation and setting of Theorem 3.6. The proof is organized in several steps.
4.1. Time-regularization. First of all, the idea is to regularize $u$ in time using convolutions. For every $n \in \mathbb{N}$, we introduce the convolution operator

$$
\begin{aligned}
& T_{n}: L^{1}(0, T) \rightarrow L^{1}(0, T), \\
& \left(T_{n} v\right)(t):=n \int_{0}^{T} v(s) \varrho((t-s) n) d s, \quad t \in[0, T], \quad v \in L^{1}(0, T),
\end{aligned}
$$

where $\varrho \in C_{c}^{\infty}(\mathbb{R})$ is nonnegative with $\int_{\mathbb{R}} \varrho(s) d s=1, \varrho(t)=\varrho(-t)$, and $\operatorname{supp} \varrho \subset[-1,1]$. It is wellknown that $T_{n}$ is linear, continuous, and sub-Markovian, in the sense that, for every $v \in L^{1}(0, T)$,

$$
0 \leq v \leq 1 \quad \text { a.e. in }(0, T) \quad \Rightarrow \quad 0 \leq T_{n} v \leq 1 \quad \text { a.e. in }(0, T)
$$

Also, it holds that

$$
\left\|T_{n} v\right\|_{L^{1}(0, T)} \leq\|v\|_{L^{1}(0, T)} \quad \forall v \in L^{1}(0, T)
$$

Furthermore, for any Banach space $X$, it is clear that $T_{n}$ can be extended to the vector-valued operator

$$
T_{n}^{X}: L^{1}(0, T ; X) \rightarrow L^{1}(0, T ; X)
$$

and it is well-known that for every $v \in L^{1}(0, T ; X)$ it holds that $T_{n}^{X} v \rightarrow v$ in $L^{1}(0, T ; X)$ as $n \rightarrow \infty$, see for instance [50, Thm 8.20-8.21].
4.2. Proof under assumption H2ii. Let us prove the result under assumption H2ii first. As we pointed out above, this implies that $L_{\bar{\varphi}^{*}}$ is separable, so that actually

$$
\begin{equation*}
u \in W^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right) \tag{4.1}
\end{equation*}
$$

In particular, $u:[0, T] \rightarrow L_{\bar{\varphi}^{*}}$ is Bochner-measurable, hence also Borel-measurable, and $\|u\|_{\bar{\varphi}^{*}}$ is a measurable function. We show now that actually also $\|u\|_{\varphi}$ and $\|u\|_{H}$ are measurable functions. To this end, we use the following lemma.

Lemma 4.1. Set

$$
\Psi_{*}: L_{\bar{\varphi}^{*}} \rightarrow[0,+\infty], \quad \Psi_{*}(y):= \begin{cases}\|y\|_{H} & \text { if } y \in H \\ +\infty & \text { otherwise }\end{cases}
$$

and

$$
\Psi_{H}: H \rightarrow[0,+\infty], \quad \Psi_{H}(y):= \begin{cases}\|y\|_{\varphi} & \text { if } y \in L_{\varphi} \\ +\infty & \text { otherwise }\end{cases}
$$

Then, $\Psi_{*}$ and $\Psi_{H}$ are convex, proper, and lower semicontinuous. In particular, $H$ is a Borel subset of $L_{\bar{\varphi}^{*}}$ and that $L_{\varphi}$ is a Borel subset of $H$.

Proof. It is clear that $\Psi_{*}$ and $\Psi_{H}$ are convex and proper. Moreover, the lower semicontinuity of $\Psi_{*}$ it follows directly from the reflexivity of $H$. As for $\Psi_{H}$, given $\left(y_{n}\right)_{n} \subset L_{\varphi}$ and $y \in H$ such that $y_{n} \rightarrow y$ in $H$ and $\left\|y_{n}\right\|_{\varphi} \leq C$ for some arbitrary constant $C>0$, we need to check that $y \in L_{\varphi}$ and $\|y\|_{\varphi} \leq C$. To this end, we note that the real sequence $\left(\left\|y_{n}\right\|_{\varphi}\right)_{n}$ is bounded, so that there exists $\lambda \geq 0$ and a subsequence $\left(y_{n_{k}}\right)_{k}$ such that $\lambda_{k}:=\left\|y_{n_{k}}\right\|_{\varphi} \rightarrow \lambda$ as $k \rightarrow \infty$. If $\lambda=0$, then trivially $y=0 \in L_{\varphi}$. Otherwise, if $\lambda>0$ then $\lambda_{k}>\lambda / 2$ for $k$ sufficiently large, so that

$$
\limsup _{k \rightarrow \infty}\left\|\frac{y_{n_{k}}}{\lambda_{k}}-\frac{y}{\lambda}\right\|_{H} \leq \frac{2}{\lambda} \limsup _{k \rightarrow \infty}\left\|y_{n_{k}}-y\right\|_{H}+\|y\|_{H} \limsup _{k \rightarrow \infty}\left|\frac{1}{\lambda_{k}}-\frac{1}{\lambda}\right|=0 .
$$

Hence $y_{n_{k}} / \lambda_{k} \rightarrow y / \lambda$ in $H$, and by lower semicontinuity of $\varphi$ and definition of $\lambda_{k}$ we have

$$
\varphi(y / \lambda) \leq \liminf _{k \rightarrow \infty} \varphi\left(y_{n_{k}} / \lambda_{k}\right) \leq 1,
$$

which implies that $y \in L_{\varphi}$ and $\|y\|_{\varphi} \leq \lambda \leq C$. Hence, also $\Psi_{H}$ is lower semicontinuous.
Eventually, the choice of $\Psi_{*}$ and $\Psi_{H}$ implies, by lower semicontinuity, that the sets $\left\{y \in L_{\bar{\varphi}^{*}}\right.$ :
$\left.\Psi_{*}(y) \leq K\right\}$ and $\left\{x \in H: \Psi_{H}(x) \leq K\right\}$ are closed in $L_{\bar{\varphi}^{*}}$ and $H$, respectively, for all $K>0$. Hence, since

$$
H=\bigcup_{k \in \mathbb{N}}\left\{y \in L_{\bar{\varphi}^{*}}: \Psi_{*}(y) \leq \frac{1}{k}\right\}, \quad L_{\varphi}=\bigcup_{k \in \mathbb{N}}\left\{x \in H: \Psi_{H}(x) \leq \frac{1}{k}\right\}
$$

we deduce that $H$ is a Borel subset of $L_{\bar{\varphi}^{*}}$ and that $L_{\varphi}$ is a Borel subset of $H$.
Consequently, by Lemma 4.1, the facts that $u$ is essentially $L_{\varphi^{-}}$-valued and $L_{\bar{\varphi}^{*}}$-Borel measurable implies that $u:[0, T] \rightarrow L_{\varphi}$ and $u:[0, T] \rightarrow H$ are Borel measurable as well. In particular, this implies that $\|u\|_{\varphi},\|u\|_{H}:[0, T] \rightarrow \mathbb{R}$ are measurable.

Thanks to H0 and the separability of $H$, the condition $\varphi(\alpha u) \in L^{1}(0, T)$ implies that

$$
u \in L^{s}(0, T ; H)
$$

Moreover, noting that from Lemma $2.2(2)-(3)$ we have $\alpha\|u\|_{\varphi} \leq 1+\varphi(\alpha u)$, we infer that

$$
\|u\|_{\varphi} \in L^{1}(0, T) .
$$

Now, for every $n \in \mathbb{N}$ we set $u_{n}:=T_{n}^{H} u$ and note that

$$
u_{n} \in C^{k}([0, T] ; H) \quad \forall k \in \mathbb{N}
$$

where, as $n \rightarrow \infty$, we have

$$
\begin{align*}
u_{n} \rightarrow u & \text { in } L^{s}(0, T ; H),  \tag{4.2}\\
\partial_{t} u_{n} \rightarrow \partial_{t} u & \text { in } L^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right),  \tag{4.3}\\
T_{n}^{H} u_{2}^{\prime} \rightarrow u_{2}^{\prime} & \text { in } L^{1}(0, T ; H) \tag{4.4}
\end{align*}
$$

Furthermore, note that $u_{n}$ is essentially $L_{\varphi}$-valued: indeed, thanks to the abstract Jensen inequality for sub-Markovian operators (see Haase [34, Thm. 3.4]), for almost every $t \in(0, T)$ we have that

$$
\varphi\left(\alpha u_{n}(t)\right)=\varphi\left(T_{n}^{H}(\alpha u)(t)\right) \leq T_{n}[\varphi(\alpha u)](t)
$$

Since $\varphi(\alpha u) \in L^{1}(0, T)$, by definition of $T_{n}$ one has also that $T_{n}(\varphi(\alpha u)) \in L^{1}(0, T)$ for all $n \in \mathbb{N}$, which yields in turn by comparison that $\varphi\left(\alpha u_{n}\right) \in L^{1}(0, T)$ for all $n \in \mathbb{N}$. This clearly implies that $u_{n}$ is essentially $L_{\varphi}$-valued and, after integration in time and by contraction of $T_{n}$ in $L^{1}(0, T)$, that

$$
u_{n}:(0, T) \rightarrow L_{\varphi}, \quad\left\|\varphi\left(\alpha u_{n}\right)\right\|_{L^{1}(0, T)} \leq\|\varphi(\alpha u)\|_{L^{1}(0, T)}
$$

Furthermore, for every $y \in H$ and almost every $t \in(0, T)$, by definition of $T_{n}$ and $T_{n}^{H}$ we have that

$$
\left(y, u_{n}(t)\right)=\left(y, T_{n}^{H} u(t)\right)=T_{n}(y, u(t)) \quad \forall y \in H
$$

As $H$ is dense in $L_{\bar{\varphi}^{*}}$ by $\mathbf{H 2 i i}$, this implies that

$$
\left[y, u_{n}(t)\right]=T_{n}[y, u(t)] \quad \forall y \in L_{\bar{\varphi}^{*}}, \quad \text { for a.e. } t \in(0, T)
$$

Since $[y, u] \in L^{1}(0, T)$ for every $y \in L_{\bar{\varphi}^{*}}$, letting $n \rightarrow \infty$ we deduce that

$$
\begin{equation*}
\left[y, u_{n}\right] \rightarrow[y, u] \quad \text { in } L^{1}(0, T), \quad \forall y \in L_{\bar{\varphi}^{*}} . \tag{4.5}
\end{equation*}
$$

Now, since in particular $u_{n} \in C^{1}([0, T] ; H)$, it is well-known (see [6, Thm. 1.9]) that

$$
\frac{d}{d t} \frac{1}{2}\left\|u_{n}\right\|_{H}^{2}=\left(\partial_{t} u_{n}, u_{n}\right)=\left[\partial_{t} u_{n}, u_{n}\right] \quad \text { in }[0, T]
$$

yielding, after integration in time

$$
\begin{equation*}
\frac{1}{2}\left\|u_{n}(t)\right\|_{H}^{2}-\frac{1}{2}\left\|u_{n}(s)\right\|_{H}^{2}=\int_{s}^{t}\left[\partial_{t} u_{n}(r), u_{n}(r)\right] d r, \quad \forall s, t \in[0, T] \tag{4.6}
\end{equation*}
$$

By the abstract Jensen inequality, since $T_{n}^{L_{\bar{\varphi}^{*}}}$ coincides with $T_{n}^{H}$ on $L^{1}(0, T ; H)$,

$$
\begin{aligned}
& \int_{s}^{t}\left[\partial_{t} u_{n}(r), u_{n}(r)\right] d r \leq \int_{0}^{T}\left|\left[\partial_{t} u_{n}(r), u_{n}(r)\right]\right| d r \\
& \leq \frac{1}{\alpha^{2}} \int_{0}^{T} \varphi\left(\alpha u_{n}\right)+\frac{1}{\alpha^{2}} \int_{0}^{T} \bar{\varphi}^{*}\left(\alpha T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}\right)+\int_{0}^{T}\left\|T_{n}^{H} u_{2}^{\prime}(s)\right\|_{H}\left\|u_{n}(s)\right\|_{H} d s \\
& \leq \frac{1}{\alpha^{2}} \int_{0}^{T} \varphi(\alpha u)+\frac{1}{\alpha^{2}} \int_{0}^{T} \bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right)+\int_{0}^{T}\left\|u_{2}^{\prime}(s)\right\|_{H}\left\|u_{n}(s)\right\|_{H} d s
\end{aligned}
$$

Moreover, using the Hölder inequality and the weighted Young inequality on the last term of the right-hand side in the form

$$
a b \leq \frac{1}{4} a^{2}+b^{2} \quad \forall a, b \geq 0
$$

we deduce that

$$
\begin{aligned}
\int_{s}^{t}\left[\partial_{t} u_{n}(r), u_{n}(r)\right] d r & \leq \frac{1}{\alpha^{2}} \int_{0}^{T} \varphi(\alpha u)+\frac{1}{\alpha^{2}} \int_{0}^{T} \bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right)+\left\|u_{n}\right\|_{L^{\infty}(0, T ; H)}\left\|u_{2}^{\prime}\right\|_{L^{1}(0, T ; H)} \\
& \leq \frac{1}{\alpha^{2}} \int_{0}^{T} \varphi(\alpha u)+\frac{1}{\alpha^{2}} \int_{0}^{T} \bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right)+\frac{1}{4}\left\|u_{n}\right\|_{L^{\infty}(0, T ; H)}^{2}+\left\|u_{2}^{\prime}\right\|_{L^{1}(0, T ; H)}^{2}
\end{aligned}
$$

Taking supremum in $t \in[0, T]$ in (4.6) and rearranging the terms, we obtain then

$$
\frac{1}{4}\left\|u_{n}\right\|_{L^{\infty}(0, T ; H)}^{2} \leq \frac{1}{2}\left\|u_{n}(s)\right\|_{H}^{2}+C \quad \forall s \in[0, T]
$$

where

$$
C:=\frac{1}{\alpha^{2}} \int_{0}^{T} \varphi(\alpha u)+\frac{1}{\alpha^{2}} \int_{0}^{T} \bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right)+\left\|u_{2}^{\prime}\right\|_{L^{1}(0, T ; H)}^{2}
$$

is independent of $n \in \mathbb{N}$. Taking square roots at both sides, and integrating with respect to $s$ on $(0, T)$ we get

$$
\frac{T}{2}\left\|u_{n}\right\|_{L^{\infty}(0, T ; H)} \leq \frac{1}{\sqrt{2}}\left\|u_{n}\right\|_{L^{1}(0, T ; H)}+\sqrt{C} T \leq \frac{1}{\sqrt{2}}\|u\|_{L^{1}(0, T ; H)}+\sqrt{C} T
$$

from which we infer that

$$
\left\|u_{n}\right\|_{L^{\infty}(0, T ; H)} \leq C
$$

so that $u \in L^{\infty}(0, T ; H)$ and

$$
\begin{equation*}
u_{n} \stackrel{*}{\rightharpoonup} u \quad \text { in } L^{\infty}(0, T ; H) \tag{4.7}
\end{equation*}
$$

Now, we have

$$
\left[\partial_{t} u_{n}, u_{n}\right]=\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}\right]+\left(T_{n}^{H} u_{2}^{\prime}, u_{n}\right)
$$

where thanks to (4.7) and (4.4) it holds

$$
\int_{s}^{t}\left(T_{n}^{H} u_{2}^{\prime}(r), u_{n}(r)\right) d r \rightarrow \int_{s}^{t}\left(u_{2}^{\prime}(r), u(r)\right) d r \quad \forall s, t \in[0, T]
$$

Moreover, thanks to the convergences (4.3) and (4.5) of $\left(u_{n}\right)_{n}$, we have that, possibly extracting a non-relabelled subsequence,

$$
\left[y, u_{n}\right] \rightarrow[y, u] \quad \forall y \in L_{\bar{\varphi}^{*}} \quad \text { a.e. in }(0, T), \quad T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime} \rightarrow u_{1}^{\prime} \quad \text { in } L_{\bar{\varphi}^{*}} \quad \text { a.e. in }(0, T)
$$

and similarly, since $\|u\|_{\varphi} \in L^{1}(0, T)$, that

$$
T_{n}\|u\|_{\varphi} \rightarrow\|u\|_{\varphi} \quad \text { a.e. in }(0, T)
$$

By the the Hölder inequality (3.1), using again the abstract Jensen inequality applied to the convex function $\|\cdot\|_{\varphi}$, we also have then almost everywhere on $(0, T)$

$$
\begin{aligned}
\left|\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}\right]-\left[u_{1}^{\prime}, u\right]\right| & \leq\left|\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}-u_{1}^{\prime}, u_{n}\right]\right|+\left|\left[u_{1}^{\prime}, u-u_{n}\right]\right| \\
& \leq 2\left\|T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}-u_{1}^{\prime}\right\|_{\bar{\varphi}^{*}}\left\|T_{n}^{H} u\right\|_{\varphi}+\left|\left[u_{1}^{\prime}, u-u_{n}\right]\right| \\
& \leq 2\left\|T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}-u_{1}^{\prime}\right\|_{\bar{\varphi}^{*}} T_{n}\|u\|_{\varphi}+\left|\left[u_{1}^{\prime}, u-u_{n}\right]\right| \rightarrow 0,
\end{aligned}
$$

yielding

$$
\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}\right] \rightarrow\left[u_{1}^{\prime}, u\right] \quad \text { a.e. in }(0, T) .
$$

Moreover, thanks to the Young inequality and the abstract Jensen inequality for sub-markovian operators (see again [34, Thm. 3.4]),

$$
\begin{aligned}
\pm \alpha^{2}\left[T_{n}^{L_{\varphi^{*}}} u_{1}^{\prime}, u_{n}\right] & \leq \varphi\left( \pm \alpha u_{n}\right)+\bar{\varphi}^{*}\left(\alpha T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}\right)=\varphi\left(\alpha T_{n}^{H} u\right)+\bar{\varphi}^{*}\left(\alpha T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}\right) \\
& \leq T_{n} \varphi(\alpha u)+T_{n} \bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right)
\end{aligned}
$$

This implies that

$$
\left|\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}\right]\right| \leq \frac{1}{\alpha^{2}} T_{n}\left(\varphi(\alpha u)+\bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right)\right) \quad \text { a.e. in }(0, T), \quad \forall n \in \mathbb{N}
$$

Since by assumption $\varphi(\alpha u)+\bar{\varphi}^{*}\left(\alpha u_{1}^{\prime}\right) \in L^{1}(0, T)$, the right-hand side of such inequality converges in $L^{1}(0, T)$, hence in particular is uniformly integrable on $(0, T)$. By comparison, we infer that the sequence $\left(\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}\right]\right)_{n}$ in uniformly integrable on $(0, T)$ as well. By Vitali's dominated convergence theorem we obtain

$$
\left[T_{n}^{L_{\varphi_{\varphi}}} u_{1}^{\prime}, u_{n}\right] \rightarrow\left[u_{1}^{\prime}, u\right] \quad \text { in } L^{1}(0, T) .
$$

taking these remarks into account and letting now $n \rightarrow \infty$ in (4.6), we obtain that

$$
\frac{1}{2}\|u(t)\|_{H}^{2}-\frac{1}{2}\|u(s)\|_{H}^{2}=\int_{s}^{t}\left[\partial_{t} u(r), u(r)\right] d r \quad \text { for a.e. } s, t \in(0, T) .
$$

Clearly, this implies that $t \mapsto\|u(t)\|_{H}^{2}$ is absolutely continuous on $[0, T]$. In particular, the equality holds for every $s, t \in[0, T]$. Moreover, since $u \in C^{0}\left([0, T] ; E_{\varphi}^{*}\right)$ by (4.1), we have that the function $t \mapsto(u(t), x)$ is continuous for every $x \in E_{\varphi}$. Now, for any $\bar{t} \in[0, T]$ and $\left(t_{k}\right)_{k} \subset[0, T]$ such that $t_{k} \rightarrow \bar{t}$ as $k \rightarrow \infty$, since $u \in L^{\infty}(0, T ; H)$ on a non-relabelled subsequence we have $u\left(t_{k}\right) \rightharpoonup y$ in $H$ for a certain $y \in H$. Moreover, for any $x \in E_{\varphi}$ it holds that $\left(u\left(t_{k}\right), x\right) \rightarrow(u(\bar{t}), x)$, from which $(y, x)=(u(\bar{t}), x)$. As $E_{\varphi}$ is dense in $H$, we deduce that $y=u(\bar{t})$. This shows that $u:[0, T] \rightarrow H$ is weakly continuous. As we have already proved that $t \mapsto\|u(t)\|_{H}$ is continuous, we infer that $u \in C^{0}([0, T] ; H)$.
4.3. Proof under assumption H2i. Let us consider now the case of assumption H2i. The proof is very similar to the case $\mathbf{H} 2 \mathbf{i i}$, the main difference being that the roles of $L_{\varphi}$ and $L_{\bar{\varphi}^{*}}$ are exchanged. Indeed, under H2i we have that $L_{\varphi}$ is separable, hence $u \in L^{1}\left(0, T ; L_{\varphi}\right)$ and we can set for any $n \in \mathbb{N} u_{n}:=T_{n}^{L_{\varphi}} u$, getting in particular

$$
u_{n} \rightarrow u \quad \text { in } L^{1}\left(0, T ; L_{\varphi}\right) .
$$

Proceeding as before one can also show that $u \in L^{\infty}(0, T ; H)$ and

$$
u_{n} \stackrel{*}{\rightharpoonup} u \quad \text { in } L^{\infty}(0, T ; H) .
$$

Moreover, $\partial_{t} u_{n}=T_{n}^{L_{\varphi^{*}}} u_{1}^{\prime}+T_{n}^{H} u_{2}^{\prime}$, where

$$
T_{n}^{H} u_{2}^{\prime} \rightarrow u_{2}^{\prime} \quad \text { in } L^{1}(0, T ; H)
$$

and

$$
\begin{aligned}
& {\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, x\right] \rightarrow\left[u_{1}^{\prime}, x\right] \quad \text { in } L^{1}(0, T), \quad \forall x \in L_{\varphi},} \\
& \left\|T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}\right\|_{\bar{\varphi}^{*}} \leq T_{n}\left\|u_{1}^{\prime}\right\|_{\bar{\varphi}^{*}} \quad \text { a.e. in }(0, T) .
\end{aligned}
$$

On a not relabelled subsequence, by the Hölder inequality we have the almost everywhere convergence

$$
\begin{aligned}
\left|\left[T_{n}^{L_{\varphi^{*}}} u_{1}^{\prime}, u_{n}\right]-\left[u_{1}^{\prime}, u\right]\right| & \leq\left|\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}-u\right]\right|+\left|\left[T_{n}^{L_{\varphi^{*}}} u_{1}^{\prime}-u_{1}^{\prime}, u\right]\right| \\
& \leq 2\left\|T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}\right\|_{\bar{\varphi}^{*}}\left\|u_{n}-u\right\|_{\varphi}+\left|\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}-u_{1}^{\prime}, u\right]\right| \rightarrow 0,
\end{aligned}
$$

Hence, writing $\left[\partial_{t} u_{n}, u_{n}\right]=\left[T_{n}^{L_{\varphi^{*}}} u_{1}^{\prime}, u_{n}\right]+\left(T_{n}^{H} u_{2}^{\prime}, u_{n}\right)$, on the one hand we have again

$$
\int_{s}^{t}\left(T_{n}^{H} u_{2}^{\prime}(r), u_{n}(r)\right) d r \rightarrow \int_{s}^{t}\left(u_{2}^{\prime}(r), u(r)\right) d r \quad \forall s, t \in[0, T],
$$

and on the other hand, proceeding as before using the abstract Jensen inequality and the Vitali convergence theorem, we infer that

$$
\left[T_{n}^{L_{\bar{\varphi}^{*}}} u_{1}^{\prime}, u_{n}\right] \rightarrow\left[u_{1}^{\prime}, u\right] \quad \text { in } L^{1}(0, T) .
$$

This allows to pass to the limit as $n \rightarrow \infty$ as in the CASE H2ii and obtain

$$
\frac{1}{2}\|u(t)\|_{H}^{2}-\frac{1}{2}\|u(s)\|_{H}^{2}=\int_{s}^{t}\left[\partial_{t} u(r), u(r)\right] d r \quad \text { for a.e. } s, t \in(0, T) .
$$

Hence, $t \mapsto\|u(t)\|_{H}^{2}$ is absolutely continuous on $[0, T]$, and $u \in L^{\infty}(0, T ; H)$. In particular, the equality holds for every $s, t \in[0, T]$. Moreover, since now $u \in W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$, we only have that $u$ is weakly continuous in $E_{\varphi}^{*}$. Still, this ensures that the function $t \mapsto(u(t), x)$ is continuous for every $x \in E_{\varphi}$. As $u \in L^{\infty}(0, T ; H)$ and $E_{\varphi}$ is dense in $H$, this implies that $u:[0, T] \rightarrow H$ is weakly continuous. As we have already proved that $t \mapsto\|u(t)\|_{H}$ is continuous, we infer that $u \in C^{0}([0, T] ; H)$. This concludes the proof of Theorem 3.6.

## 5. Proof of well-posedness

This section is devoted to the proof of well-posedness contained in Theorems 3.7-3.8.
5.1. The approximation. Let us denote by $A:=\partial \varphi: H \rightarrow 2^{H}$ the subdifferential of $\varphi$. We recall the Young inequality

$$
(y, x) \leq \varphi(x)+\varphi^{*}(y) \quad \forall x, y \in H
$$

and point out that the equality holds if and only if $y \in A(x)$.
For every $\lambda>0$, let $\varphi_{\lambda}: H \rightarrow[0,+\infty)$ be the Moreau-Yosida regularization of $\varphi$, defined as

$$
\varphi_{\lambda}(x):=\inf _{y \in H}\left\{\varphi(y)+\frac{1}{2 \lambda}\|x-y\|_{H}^{2}\right\}, \quad x \in H
$$

From classical results of convex and monotone analysis (see [6, Ch. 2]), we have that $\varphi_{\lambda} \in C^{1}(H)$, with $D \varphi_{\lambda}=A_{\lambda}$, where $A_{\lambda}: H \rightarrow H$ is the Yosida approximation of $A$. Let us recall that $A_{\lambda}$ is defined as

$$
A_{\lambda}(x):=\frac{x-J_{\lambda}(x)}{\lambda}, \quad x \in H
$$

where we have denoted by $J_{\lambda}: H \rightarrow H$ the resolvent of $A$, namely

$$
J_{\lambda}(x):=(I+\lambda A)^{-1}(x), \quad x \in H
$$

It is well known that $A_{\lambda}$ is $\frac{1}{\lambda}$-Lipschitz continuous, $J_{\lambda}$ is 1 -Lipschitz continuous, and that $A_{\lambda}(x) \in$ $A\left(J_{\lambda}(x)\right)$ for every $x \in H$. Moreover, $\varphi_{\lambda}$ satisfies

$$
\begin{aligned}
& \varphi\left(J_{\lambda}(x)\right) \leq \varphi_{\lambda}(x) \leq \varphi(x) \quad \forall x \in H \\
& \lim _{\lambda \searrow 0} \varphi_{\lambda}(x)=\varphi(x) \quad \forall x \in H
\end{aligned}
$$

We study the approximated problem

$$
\left\{\begin{array}{l}
\partial_{t} u_{\lambda}+A_{\lambda}\left(u_{\lambda}\right)=f \\
u_{\lambda}(0)=u_{0}
\end{array}\right.
$$

Since $A_{\lambda}$ is Lipschitz-continuous, from the classical theory of nonlinear evolution equations (see again [6] or [13, Prop. 3.4]) such approximated problem admits a unique solution

$$
u_{\lambda} \in W^{1,1}(0, T, H)
$$

5.2. Uniform estimates. Let us prove some uniform estimates on $\left(u_{\lambda}\right)_{\lambda}$, independent of $\lambda$. To this end, testing the approximated equation by $u_{\lambda}$ and integrating in time yields, for every $t \in[0, T]$,

$$
\frac{1}{2}\left\|u_{\lambda}(t)\right\|_{H}^{2}+\int_{0}^{t}\left(A_{\lambda}\left(u_{\lambda}(s)\right), u_{\lambda}(s)\right) d s=\frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{t}\left(f(s), u_{\lambda}(s)\right) d s
$$

Now, on the left-hand side, since $A_{\lambda}\left(u_{\lambda}\right) \in A\left(J_{\lambda}\left(u_{\lambda}\right)\right)$, by the Young inequality we have

$$
\begin{aligned}
\left(A_{\lambda}\left(u_{\lambda}\right), u_{\lambda}\right) & =\left(A_{\lambda}\left(u_{\lambda}\right), J_{\lambda}\left(u_{\lambda}\right)\right)+\lambda\left\|A_{\lambda}\left(u_{\lambda}\right)\right\|_{H}^{2} \\
& =\varphi\left(J_{\lambda}\left(u_{\lambda}\right)\right)+\varphi^{*}\left(A_{\lambda}\left(u_{\lambda}\right)\right)+\lambda\left\|A_{\lambda}\left(u_{\lambda}\right)\right\|_{H}^{2}
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\frac{1}{2}\left\|u_{\lambda}(t)\right\|_{H}^{2} & +\int_{0}^{t} \varphi\left(J_{\lambda}\left(u_{\lambda}(s)\right)\right) d s+\int_{0}^{t} \varphi^{*}\left(A_{\lambda}\left(u_{\lambda}(s)\right)\right) d s+\lambda \int_{0}^{t}\left\|A_{\lambda}\left(u_{\lambda}(s)\right)\right\|_{H}^{2} d s \\
& =\frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{t}\left(f(s), u_{\lambda}(s)\right) d s \\
& \leq \frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{t}\|f(s)\|_{H}\left\|u_{\lambda}(s)\right\|_{H} d s
\end{aligned}
$$

The Gronwall lemma implies then that there exists $M>0$, independent of $\lambda$, such that

$$
\begin{align*}
\left\|u_{\lambda}\right\|_{C^{0}([0, T] ; H)}^{2} & \leq M  \tag{5.1}\\
\left\|\varphi\left(J_{\lambda}\left(u_{\lambda}\right)\right)\right\|_{L^{1}(0, T)}+\left\|\varphi^{*}\left(A_{\lambda}\left(u_{\lambda}\right)\right)\right\|_{L^{1}(0, T)}+\lambda\left\|A_{\lambda}\left(u_{\lambda}\right)\right\|_{L^{2}(0, T ; H)}^{2} & \leq M \tag{5.2}
\end{align*}
$$

Recalling also assumption H0, this implies that

$$
\begin{equation*}
\left\|J_{\lambda}\left(u_{\lambda}\right)\right\|_{L^{s}(0, T ; H)}^{s} \leq M \tag{5.3}
\end{equation*}
$$

Now, by the estimates (5.1) and (5.3) there exist $u \in L^{\infty}(0, T ; H)$ and $\tilde{u} \in L^{s}(0, T ; H)$ such that, as $\lambda \searrow 0$,

$$
\begin{align*}
u_{\lambda} \stackrel{*}{\rightharpoonup} u \quad \text { in } L^{\infty}(0, T ; H),  \tag{5.4}\\
J_{\lambda}\left(u_{\lambda}\right) \rightharpoonup \tilde{u} \quad \text { in } L^{s}(0, T ; H) . \tag{5.5}
\end{align*}
$$

Moreover, note that by (5.2) and the definition of $A_{\lambda}$ we have

$$
\left\|J_{\lambda}\left(u_{\lambda}\right)-u_{\lambda}\right\|_{L^{2}(0, T ; H)}=\lambda\left\|A_{\lambda}\left(u_{\lambda}\right)\right\|_{L^{2}(0, T ; H)} \leq M \lambda^{1 / 2} \rightarrow 0
$$

which implies that $\tilde{u}=u$ and

$$
\begin{equation*}
J_{\lambda}\left(u_{\lambda}\right) \rightharpoonup u \quad \text { in } L^{s}(0, T ; H) \tag{5.6}
\end{equation*}
$$

By the weak lower semicontinuity of convex integrands, convergence (5.6), and estimate (5.2), we deduce then that

$$
\int_{0}^{T} \varphi(u(s)) d s \leq \liminf _{\lambda \searrow 0} \int_{0}^{T} \varphi\left(J_{\lambda}\left(u_{\lambda}(s)\right)\right) d s \leq M
$$

It follows that $\varphi(u) \in L^{1}(0, T)$, so that $u$ is essentially $L_{\varphi}$-valued. Since by Lemma $4.1 L_{\varphi}$ is a Borel subset of $H$ and $u$ is strongly measurable in $H$, we infer that $u$ is Borel-measurable in $L_{\varphi}$ : hence, by definition of Borel-measurability we have that $\|u\|_{\varphi}$ is measurable. Furthermore, as a consequence of Lemma 2.2 we have that

$$
\|u\|_{\varphi} \leq 1+\varphi(u) \quad \text { a.e. in }(0, T)
$$

so that by comparison $\|u\|_{\varphi} \in L^{1}(0, T)$. In order to prove that $u \in L_{w}^{1}\left(0, T ; L_{\varphi}\right)$, we need to show that $[y, u] \in L^{1}(0, T)$ for every $y \in L_{\bar{\varphi}^{*}}$. To this end, since $u:(0, T) \rightarrow L_{\varphi}$ is Borel-measurable, by definition of weak topology on $L_{\varphi}$ it follows that $u:(0, T) \rightarrow L_{\varphi}$ is weakly measurable in the classical sense, i.e. $\langle y, u\rangle_{L_{\varphi}^{*}, L_{\varphi}}$ is measurable for every $y \in L_{\varphi}^{*}$. Now, under assumption H2i we have that $L_{\varphi}$ is separable (because so is $E_{\varphi}$ ), hence by the Pettis theorem [27, Thm. 2, Ch. II] we obtain that $u:(0, T) \rightarrow L_{\varphi}$ is strongly measurable: it follows that in this case we have actually that $u \in L^{1}\left(0, T ; L_{\varphi}\right)$. In particular, since $[y, \cdot]: L_{\varphi} \rightarrow \mathbb{R}$ is linear continuous by Lemma 3.1, by composition we infer that also $[y, u]:(0, T) \rightarrow \mathbb{R}$ is measurable, hence $u \in L_{w}^{1}\left(0, T ; L_{\varphi}\right)$. Alternatively, under assumption H2ii we observe that $[y, u] \in L^{1}(0, T)$ for every $y \in H$ since $u \in L^{1}(0, T ; H)$ : hence, the density of $H$ in $L_{\bar{\varphi}^{*}}$ readily implies that $[y, u] \in L^{1}(0, T)$ also for every $y \in L_{\bar{\varphi}^{*}}$, and this shows indeed that $u \in L_{w}^{1}\left(0, T ; L_{\varphi}\right)$.

In order to deduce some compactness for $\left(A_{\lambda}\left(u_{\lambda}\right)\right)_{\lambda}$, we need the following lemma.
Lemma 5.1. For any reflexive Banach space $V_{0}$ such that $V_{0} \hookrightarrow E_{\varphi}$ continuously and densely, it holds that $E_{\varphi}^{*} \hookrightarrow V_{0}^{*}$ continuously and densely. Furthermore, the convex conjugate of $\varphi_{0}:=$ $\varphi_{\mid V_{0}}: V_{0} \rightarrow[0,+\infty)$ is given by

$$
\varphi_{0}^{*}: V_{0}^{*} \rightarrow[0,+\infty], \quad \varphi_{0}^{*}(y)= \begin{cases}\bar{\varphi}^{*}(y) & \text { if } y \in E_{\varphi}^{*} \\ +\infty & \text { if } y \in V_{0}^{*} \backslash E_{\varphi}^{*}\end{cases}
$$

If also $\varphi$ is bounded on bounded sets of $V_{0}$, then

$$
\begin{equation*}
\lim _{\|y\|_{V_{0}^{*}} \rightarrow+\infty} \frac{\varphi_{0}^{*}(y)}{\|y\|_{V_{0}^{*}}}=+\infty \tag{5.7}
\end{equation*}
$$

Proof. The fact that $E_{\varphi}^{*} \hookrightarrow V_{0}^{*}$ continuously is an immediate consequence of the density of $V_{0}$ in $E_{\varphi}$, while the fact that $E_{\varphi}^{*} \hookrightarrow V_{0}^{*}$ densely follows from the reflexivity of $V_{0}$ by a classical argument. Let us compute the convex conjugate of $\varphi_{0}$. If $V_{0}=E_{\varphi}$, then the conclusion is trivial, so let us suppose then that $V_{0} \subset E_{\varphi}$ strictly. First of all, we show $\varphi_{0}^{*}=+\infty$ on $V_{0}^{*} \backslash E_{\varphi}^{*}$. Let $y \in V_{0}^{*} \backslash E_{\varphi}^{*}$ : this means that $y: V_{0} \rightarrow \mathbb{R}$ cannot be extended to a continuous linear functional on $E_{\varphi}$, i.e. there is no constant $C>0$ such that $\langle y, x\rangle_{V_{0}^{*}, V_{0}} \leq C\|x\|_{\varphi}$ for all $x \in V_{0}$. Moreover, for all $x \in E_{\varphi} \backslash V_{0}$, by density of $V_{0}$ in $E_{\varphi}$ there is a sequence $\left(x_{n}\right)_{n} \subset V_{0}$ such that $x_{n} \rightarrow x$ in $E_{\varphi}$. If for all $x \in E_{\varphi} \backslash V_{0}$ there exists $C_{x}>0$ such that $\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}} \leq C_{x}\left\|x_{n}\right\|_{\varphi}$ for all $n \in \mathbb{N}$, then, due to the continous embedding $V_{0} \hookrightarrow E_{\varphi}$, one could extend by density $y$ to a continuous linear functional on $E_{\varphi}$. However, this is not possible since $y \in V_{0}^{*} \backslash E_{\varphi}^{*}$ : consequently, there exists $\bar{x} \in E_{\varphi} \backslash V_{0}$ and a sequence $\left(x_{n}\right)_{n} \subset V_{0}$ such that

$$
x_{n} \rightarrow \bar{x} \quad \text { in } E_{\varphi}, \quad\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}}>n\left\|x_{n}\right\|_{\varphi} \quad \forall n \in \mathbb{N} .
$$

In particular, $\varphi\left(x_{n}\right) \rightarrow \varphi(\bar{x})$ and, since $\bar{x} \neq 0,\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}} \rightarrow+\infty$ : hence

$$
\begin{aligned}
\varphi_{0}^{*}(y) & =\sup _{x \in V_{0}}\left\{\langle y, x\rangle_{V_{0}^{*}, V_{0}}-\varphi_{0}(x)\right\} \geq \sup _{n \in \mathbb{N}}\left\{\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}}-\varphi\left(x_{n}\right)\right\} \\
& \geq \limsup _{n \rightarrow \infty}\left(\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}}-\varphi\left(x_{n}\right)\right) \\
& =\limsup _{n \rightarrow \infty}\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}}-\varphi(\bar{x})=+\infty .
\end{aligned}
$$

This shows that $\varphi_{0}^{*}=+\infty$ on $V_{0}^{*} \backslash E_{\varphi}^{*}$. Let us prove now that $\left(\varphi_{0}^{*}\right)_{\mid E_{\varphi}^{*}}=\bar{\varphi}^{*}$. Let $y \in E_{\varphi}^{*}$ arbitrary: we have

$$
\varphi_{0}^{*}(y)=\sup _{x \in V_{0}}\left\{\langle y, x\rangle_{V_{0}^{*}, V_{0}}-\varphi(x)\right\} \leq \sup _{x \in E_{\varphi}}\left\{\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}-\varphi(x)\right\}=\bar{\varphi}^{*}(y) .
$$

On the other hand, for all $x \in E_{\varphi}$ there is $\left(x_{n}\right)_{n} \subset V_{0}$ such that $x_{n} \rightarrow x$ in $E_{\varphi}$, so that

$$
\left\langle y, x_{n}\right\rangle_{E_{\varphi}^{*}, E_{\varphi}}=\left\langle y, x_{n}\right\rangle_{V_{0}^{*}, V_{0}} \leq \varphi_{0}^{*}(y)+\varphi\left(x_{n}\right) \quad \forall n \in \mathbb{N},
$$

hence also, noting that $\varphi\left(x_{n}\right) \rightarrow \varphi(x)$,

$$
\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}} \leq \varphi_{0}^{*}(y)+\varphi(x) \quad \forall x \in E_{\varphi} .
$$

It follows that

$$
\varphi_{0}^{*}(y) \geq \sup _{x \in E_{\varphi}}\left\{\langle y, x\rangle_{E_{\varphi}^{*}, E_{\varphi}}-\varphi(x)\right\}=\bar{\varphi}^{*}(y) .
$$

The shows that $\left(\varphi_{0}^{*}\right)_{\mid E_{\varphi}^{*}}=\bar{\varphi}^{*}$, as required.
Finally, let us show that $\varphi_{0}^{*}$ is superlinear at $\infty$. To this end, by the Young inequality we have that

$$
\varphi_{0}^{*}(y) \geq\langle y, x\rangle_{V_{0}^{*}, V_{0}}-\varphi(x) \quad \forall x \in V_{0}, \quad \forall y \in V_{0}^{*} .
$$

Since $V_{0}$ is reflexive, for any $y \in V_{0}^{*} \backslash\{0\}$, there is $x_{y} \in V_{0}$ such that $\left\langle y, x_{y}\right\rangle_{V_{0}^{*}, V_{0}}=\|y\|_{V_{0}^{*}}^{2}=\left\|x_{y}\right\|_{V_{0}}^{2}$. Choosing $x=L x_{y}\|y\|_{V_{0}^{*}}^{-1}$ for arbitrary $L>0$ in the last inequality yields

$$
\varphi_{0}^{*}(y) \geq L\|y\|_{V_{0}^{*}}-\varphi\left(L x_{y}\|y\|_{V_{0}^{*}}^{-1}\right) \quad \forall y \in V_{0}^{*} \backslash\{0\}, \quad \forall L>0,
$$

where

$$
\left\|L x_{y}\right\| y\left\|_{V_{0}^{*}}^{-1}\right\|_{V_{0}}=L \quad \forall y \in V_{0}^{*} \backslash\{0\}, \quad \forall L>0 .
$$

Since $\varphi$ is bounded on bounded subsets of $V_{0}$, there exists $C_{L}>0$ such that

$$
\varphi_{0}^{*}(y) \geq L\|y\|_{V_{0}^{*}}-C_{L} \quad \forall y \in V_{0}^{*} \backslash\{0\}, \quad \forall L>0
$$

Hence, for all arbitrary $K>0$, choosing $L=2 K$, for all $y \in V_{0}^{*}$ with $\|y\|_{V_{0}^{*}} \geq C_{2 K} / K$ we have

$$
\frac{\varphi_{0}^{*}(y)}{\|y\|_{V_{0}^{*}}} \geq 2 K-\frac{C_{2 K}}{\|y\|_{V_{0}^{*}}} \geq K
$$

As $K>0$ is arbitrary, we can conclude.
By assumption H1, Lemma 5.1, and estimate (5.2), we deduce that

$$
\int_{0}^{T} \varphi_{0}^{*}\left(A_{\lambda}\left(u_{\lambda}(s)\right)\right) d s \leq M
$$

where

$$
\lim _{\|y\|_{V_{0}^{*} \rightarrow+\infty}} \frac{\varphi_{0}^{*}(y)}{\|y\|_{V_{0}^{*}}^{*}}=+\infty .
$$

In particular, there exists an increasing sequence $\left(r_{n}\right)_{n}$ of positive numbers such that

$$
\varphi_{0}^{*}(y) \geq n\|y\|_{V_{0}^{*}} \quad \forall y \in V_{0}^{*}, \quad\|y\|_{V_{0}^{*}} \geq r_{n}, \quad \forall n \in \mathbb{N} .
$$

This readily implies that $\left(A_{\lambda}\left(u_{\lambda}\right)\right)_{\lambda}$ is bounded in $L^{1}\left(0, T ; V_{0}^{*}\right)$. Moreover, for any measurable $I \subset[0, T]$ we have

$$
\begin{aligned}
\left.\int_{I} \| A_{\lambda}\left(u_{\lambda}\right)\right) \|_{V_{0}^{*}} & \left.\int_{\left.I \cap\left\{\| A_{\lambda}\left(u_{\lambda}\right)\right) \|_{V_{0}^{*}}<r_{n}\right\}} \| A_{\lambda}\left(u_{\lambda}\right)\right) \|_{V_{0}^{*}}+\int_{\left.I \cap\left\{\| A_{\lambda}\left(u_{\lambda}\right)\right)\left\|_{\left.V_{0}^{*} \geq r_{n}\right\}}\right\| A_{\lambda}\left(u_{\lambda}\right)\right) \|_{V_{0}^{*}}} \\
& \leq|I| r_{n}+\frac{1}{n} \int_{I} \varphi_{0}^{*}\left(A_{\lambda}\left(u_{\lambda}\right)\right) \leq|I| r_{n}+\frac{M}{n} .
\end{aligned}
$$

Hence, for any arbitrary $\varepsilon>0$, choosing $\bar{n}=\bar{n}(\varepsilon)$ sufficiently large such that $M / \bar{n} \leq \varepsilon / 2$, and setting $\delta=\delta(\varepsilon):=\varepsilon r_{\bar{n}}^{-1} / 2$, we have that

$$
\left.\sup _{\lambda>0} \int_{I} \| A_{\lambda}\left(u_{\lambda}\right)\right) \|_{V_{0}^{*}} \leq \varepsilon \quad \forall I \subset[0, T], \quad|I| \leq \delta .
$$

This implies that the family $\left(A_{\lambda}\left(u_{\lambda}\right)\right)_{\lambda}$ is uniformly integrable in $L^{1}\left(0, T ; V_{0}^{*}\right)$, hence also by the Dunford-Pettis theorem that

$$
\left(A_{\lambda}\left(u_{\lambda}\right)\right)_{\lambda} \text { is sequentially weakly compact in } L^{1}\left(0, T ; V_{0}^{*}\right) .
$$

We deduce that there exists $\xi \in L^{1}\left(0, T ; V_{0}^{*}\right)$ such that, on a not relabelled subsequence,

$$
\begin{equation*}
A_{\lambda}\left(u_{\lambda}\right) \rightharpoonup \xi \quad \text { in } L^{1}\left(0, T ; V_{0}^{*}\right) \tag{5.8}
\end{equation*}
$$

Furthermore, by the weak lower semicontinuity of the convex integrand $\int_{0}^{T} \varphi_{0}^{*}(\cdot) d s$ and the estimate (5.2), we have

$$
\int_{0}^{T} \varphi_{0}^{*}(\xi(s)) d s \leq \liminf _{\lambda \searrow 0} \int_{0}^{T} \varphi_{0}^{*}\left(A_{\lambda}\left(u_{\lambda}(s)\right)\right) d s \leq M
$$

Thanks to Lemma 5.1, this implies that actually $\xi(t) \in E_{\varphi}^{*}$ for almost every $t \in(0, T)$ and that $\bar{\varphi}^{*}(\xi) \in L^{1}(0, T)$, hence in particular that $\xi$ is essentially $L_{\bar{\varphi}^{*}}$-valued. Moreover, proceeding as in the proof of Theorem 3.6 we have that $L_{\bar{\varphi}^{*}}$ is a Borel subset of $V_{0}^{*}$, hence $\xi:[0, T] \rightarrow L_{\bar{\varphi}^{*}}$ is Borel measurable and $\|\xi\|_{\bar{\varphi}^{*}}$ is measurable. Since $\|y\|_{\bar{\varphi}^{*}} \leq 1+\bar{\varphi}^{*}(y)$ for all $y \in L_{\bar{\varphi}^{*}}$, we have that $\|\xi\|_{\bar{\varphi}^{*}} \in L^{1}(0, T)$. Furthermore, it also holds that $\xi \in L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$. Indeed, under H2ii this is immediate since $L_{\bar{\varphi}^{*}}$ is separable and $\xi \in L^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$, while under $\mathbf{H} 2 \mathbf{i}$ the weak measurability follows directly from the density of $V_{0}$ in $L_{\varphi}$ and the strong measurability of $\xi$ in $V_{0}^{*}$.
5.3. Passage to the limit. The approximated problem can be written as

$$
u_{\lambda}(t)+\int_{0}^{t} A_{\lambda}\left(u_{\lambda}(s)\right) d s=u_{0}+\int_{0}^{t} f(s) d s \quad \forall t \in[0, T]
$$

Fix now $t \in[0, T]$ arbitrary. By the convergence (5.8), it follows that as $\lambda \searrow 0$

$$
\int_{0}^{t} A_{\lambda}\left(u_{\lambda}(s)\right) d s \rightharpoonup \int_{0}^{t} \xi(s) d s \text { in } V_{0}^{*}
$$

By comparison, we deduce that $u_{\lambda}(t)$ converges weakly in $V_{0}^{*}$, yielding thanks to (5.4) that

$$
u_{\lambda}(t) \rightharpoonup u(t) \quad \text { in } H
$$

Hence, we have that

$$
u(t)+\int_{0}^{t} \xi(s) d s=u_{0}+\int_{0}^{t} f(s) d s \quad \forall t \in[0, T]
$$

This implies in particular also that $u \in W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right)$, with $\partial_{t} u=-\xi+f$, hence also $u \in$ $C^{0}([0, T] ; H)$ by Theorem 3.6, and $(u, \xi)$ solves (3.4)-(3.7).

We only need to show that $\xi$ is the weak realization of $\partial \varphi(u)$ in $L_{\bar{\varphi}^{*}}$, namely condition (3.8). To this end, note that we already proved that for every $t \in[0, T]$

$$
\frac{1}{2}\left\|u_{\lambda}(t)\right\|_{H}^{2}+\int_{0}^{t}\left(A_{\lambda}\left(u_{\lambda}(s)\right), u_{\lambda}(s)\right) d s=\frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{t}\left(f(s), u_{\lambda}(s)\right) d s
$$

which yields, by the weak lower semicontinuity of the $H$-norm and the convergence (5.4),

$$
\begin{aligned}
\limsup _{\lambda \searrow 0} \int_{0}^{T}\left(A_{\lambda}\left(u_{\lambda}(s)\right), u_{\lambda}(s)\right) d s & =\frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{T}(f(s), u(s)) d s-\frac{1}{2} \liminf _{\lambda \searrow 0}\left\|u_{\lambda}(T)\right\|_{H}^{2} \\
& \leq \frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{T}(f(s), u(s)) d s-\frac{1}{2}\|u(T)\|_{H}^{2} .
\end{aligned}
$$

Furthermore, since we have proved that

$$
\partial_{t} u+\xi=f \quad \text { in } L_{\bar{\varphi}^{*}}, \quad u(0)=u_{0}
$$

taking the $[\cdot, \cdot]$ duality with $u$ and integrating on $(0, t)$, using Theorem 3.6 we get exactly that

$$
\frac{1}{2}\|u(t)\|_{H}^{2}+\int_{0}^{t}[\xi(s), u(s)] d s=\frac{1}{2}\left\|u_{0}\right\|_{H}^{2}+\int_{0}^{t}(f(s), u(s)) d s \quad \forall t \in[0, T]
$$

which in particular proves the energy equality (3.9). Putting this information together we obtain

$$
\begin{equation*}
\underset{\lambda \searrow 0}{\limsup } \int_{0}^{T}\left(A_{\lambda}\left(u_{\lambda}(s)\right), u_{\lambda}(s)\right) d s \leq \int_{0}^{T}[\xi(s), u(s)] d s \tag{5.9}
\end{equation*}
$$

Now, recalling that $A_{\lambda} \in A\left(J_{\lambda}(\cdot)\right)$, we have that

$$
\varphi\left(J_{\lambda}\left(u_{\lambda}\right)\right)+\left(A_{\lambda}\left(u_{\lambda}\right), z-J_{\lambda}\left(u_{\lambda}\right)\right) \leq \varphi(z) \quad \text { a.e. in }(0, T), \quad \forall z \in L^{2}(0, T ; H)
$$

so in particular it holds that

$$
\int_{0}^{T} \varphi\left(J_{\lambda}\left(u_{\lambda}(s)\right)\right) d s+\int_{0}^{T}\left(A_{\lambda}\left(u_{\lambda}(s)\right), z(s)-J_{\lambda}\left(u_{\lambda}(s)\right)\right) d s \leq \int_{0}^{T} \varphi(z(s)) d s
$$

Now, we want to let $\lambda \searrow 0$ in the inequality. To this end, note first that the convergence (5.6) and the weak lower semicontinuity of the convex integrands yields

$$
\int_{0}^{T} \varphi(u(s)) d s \leq \liminf _{\lambda \searrow 0} \int_{0}^{T} \varphi\left(J_{\lambda}\left(u_{\lambda}(s)\right)\right) d s
$$

Secondly, the weak convergence (5.8) readily implies that, for all $z \in L^{\infty}\left(0, T ; V_{0}\right)$,

$$
\left.\lim _{\lambda \geq 0} \int_{0}^{T}\left(A_{\lambda}\left(u_{\lambda}(s)\right), z(s)\right)\right) d s=\int_{0}^{T}\langle\xi(s), z(s)\rangle_{V_{0}^{*}, V_{0}} d s=\int_{0}^{T}[\xi(s), z(s)] d s
$$

Finally, the limsup inequality (5.9) yields

$$
\begin{aligned}
& \underset{\lambda \searrow 0}{\limsup } \int_{0}^{T}\left(A_{\lambda}\left(u_{\lambda}(s)\right), J_{\lambda}\left(u_{\lambda}(s)\right)\right) d s \\
& =\underset{\lambda \searrow 0}{\limsup } \int_{0}^{T}\left[\left(A_{\lambda}\left(u_{\lambda}(s)\right), u_{\lambda}(s)\right)-\lambda\left\|A_{\lambda}\left(u_{\lambda}(s)\right)\right\|_{H}^{2}\right] d s \\
& \leq \underset{\lambda \searrow 0}{\limsup } \int_{0}^{T}\left(A_{\lambda}\left(u_{\lambda}(s)\right), u_{\lambda}(s)\right) d s \leq \int_{0}^{T}[\xi(s), u(s)] d s
\end{aligned}
$$

Hence, letting $\lambda \searrow 0$ we infer that, for all

$$
\int_{0}^{T} \varphi(u(s)) d s+\int_{0}^{T}[\xi(s), z(s)-u(s)] d s \leq \int_{0}^{T} \varphi(z(s)) d s \quad \forall z \in L^{\infty}\left(0, T ; V_{0}\right)
$$

By a standard localization procedure and by the density of $V_{0}$ in $E_{\varphi}$ we have

$$
\varphi(u)+[\xi, x-u] \leq \varphi(x) \quad \forall x \in E_{\varphi}, \quad \text { a.e. in }(0, T) .
$$

This complete the proof of condition (3.8) and of existence of solutions in Theorem 3.7.
5.4. Continuous dependence. We prove here the continuous dependence in Theorem 3.8, which in particular implies uniqueness of solutions.

In the setting and notations of Theorem 3.8 we have that

$$
\partial_{t}\left(u_{1}-u_{2}\right)+\xi_{1}-\xi_{2}=f_{1}-f_{2} \quad \text { a.e. in }(0, T), \quad\left(u_{1}-u_{2}\right)(0)=u_{0}^{1}-u_{0}^{2}
$$

Moreover, note that by convexity and symmetry of $\varphi$ we have

$$
\varphi\left(\frac{u_{1}-u_{2}}{2}\right) \leq \frac{1}{2} \varphi\left(u_{1}\right)+\frac{1}{2} \varphi\left(u_{2}\right) \in L^{1}(0, T)
$$

and similarly

$$
\bar{\varphi}^{*}\left(\frac{\partial_{t} u_{1}-\partial_{t} u_{2}}{2}\right) \leq \frac{1}{2} \bar{\varphi}^{*}\left(\partial_{t} u_{1}\right)+\frac{1}{2} \bar{\varphi}^{*}\left(\partial_{t} u_{2}\right) \in L^{1}(0, T)
$$

Consequently, taking the $[\cdot, \cdot]$ duality with $u_{1}-u_{2}$, integrating on $(0, t)$ and using Theorem 3.6 we have, for every $t \in[0, T]$,

$$
\begin{aligned}
& \frac{1}{2}\left\|\left(u_{1}-u_{2}\right)(t)\right\|_{H}^{2}+\int_{0}^{t}\left[\left(\xi_{1}-\xi_{2}\right)(s),\left(u_{1}-u_{2}\right)(s)\right] d s \\
& \quad=\frac{1}{2}\left\|u_{0}^{1}-u_{0}^{2}\right\|_{H}^{2}+\int_{0}^{t}\left(\left(f_{1}-f_{2}\right)(s),\left(u_{1}-u_{2}\right)(s)\right) d s
\end{aligned}
$$

Now, from (3.8) we know that

$$
\varphi\left(u_{1}\right)+\bar{\varphi}^{*}\left(\xi_{1}\right)=\left[\xi_{1}, u_{1}\right], \quad \varphi\left(u_{2}\right)+\bar{\varphi}^{*}\left(\xi_{2}\right)=\left[\xi_{2}, u_{2}\right]
$$

from which

$$
\left[\xi_{1}-\xi_{2}, u_{1}-u_{2}\right]=\varphi\left(u_{1}\right)+\bar{\varphi}^{*}\left(\xi_{1}\right)+\varphi\left(u_{2}\right)+\bar{\varphi}^{*}\left(\xi_{2}\right)-\left[\xi_{1}, u_{2}\right]-\left[\xi_{2}, u_{1}\right]
$$

By the Young inequality we also deduce that

$$
\left[\xi_{1}, u_{2}\right] \leq \bar{\varphi}^{*}\left(\xi_{1}\right)+\varphi\left(u_{2}\right), \quad\left[\xi_{2}, u_{1}\right] \leq \bar{\varphi}^{*}\left(\xi_{2}\right)+\varphi\left(u_{1}\right)
$$

so that putting everything together we infer that

$$
\left[\xi_{1}-\xi_{2}, u_{1}-u_{2}\right] \geq 0 \quad \text { a.e. in }(0, T)
$$

We deduce that

$$
\frac{1}{2}\left\|\left(u_{1}-u_{2}\right)(t)\right\|_{H}^{2} \leq \frac{1}{2}\left\|u_{0}^{1}-u_{0}^{2}\right\|_{H}^{2}+\int_{0}^{t}\left\|\left(f_{1}-f_{2}\right)(s)\right\|_{H}\left\|\left(u_{1}-u_{2}\right)(s)\right\|_{H} d s
$$

for every $t \in[0, T]$, and the thesis follows then by the Gronwall lemma.

## 6. Applications

In this section, we thoroughly discuss a wide spectrum of applications. First of all, we show that the classical variational theory in reflexive and separable spaces is covered as a special case of our results. Then, we show the applicability of our theory to much more general examples, such as evolution equations in singular Orlicz spaces, in Muselak-Orlicz spaces, and Musielak-OrliczSobolev spaces. These cover, among many other examples, PDEs in variable-exponent Sobolev spaces, in double-phase spaces, and PDE with dynamic boundary conditions driven by singular potentials.
6.1. The classical variational theory. We show here that the classical variational theory for evolution equations is covered by our results: in this direction we refer the reader to the main contributions [6, Thm. 4.10] and [2,12].

Let $H$ be a Hilbert space and $V$ a separable reflexive Banach space such that $V \hookrightarrow H$ continuously and densely. Let $\varphi: V \rightarrow[0,+\infty)$ be convex, lower semicontinuous, with $\varphi(0)=0$, such that $V=D(\partial \varphi)$, where $D(\partial \varphi)$ denotes a domain of $\partial \varphi$, and there exist constants $c_{1}, c_{2}>0$ and $p \geq 2$ such that

$$
\langle y, x\rangle \geq c_{1}\|x\|_{V}^{p}, \quad\|y\|_{V^{*}}^{p^{\prime}} \leq c_{2}\left(1+\|x\|_{V}^{p}\right), \quad \forall x \in V, \quad \forall y \in \partial \varphi(x)
$$

where $p^{\prime}:=\frac{p}{p-1}$. Let also $u_{0} \in H$ and $f \in L^{2}(0, T ; H)$. Then the classical variational theory ensures that there exists a unique $(u, \xi)$ with

$$
u \in W^{1, p^{\prime}}\left(0, T ; V^{*}\right) \cap C^{0}([0, T] ; H) \cap L^{p}(0, T ; V), \quad \xi \in L^{p^{\prime}}\left(0, T ; V^{*}\right)
$$

such that

$$
\partial_{t} u+\xi=f, \quad \xi \in \partial \varphi(u), \quad u(0)=u_{0}
$$

Let us compare this with our results. In this setting, it is not difficult to check that $\varphi$ (suitably extended to $+\infty$ on $H \backslash V$ ) is actually a modular, and that assumption $\mathbf{H 0}$ holds with the choice $s=p$. Furthermore, the growth conditions on $\partial \varphi$ imply that

$$
c_{1}^{\prime}\|x\|_{V}^{p} \leq \varphi(x) \leq c_{2}^{\prime}\left(1+\|x\|_{V}^{p}\right) \quad \forall x \in V
$$

for some constants $c_{1}^{\prime}, c_{2}^{\prime}>0$. Consequently, we easily deduce that in this case

$$
E_{\varphi}=L_{\varphi}=V
$$

so that H1 holds with the trivial choice $V_{0}=V$. Clearly, one has then $\bar{\varphi}=\varphi, E_{\varphi}^{*}=V^{*}$, and $\bar{\varphi}^{*}=\varphi^{*}$. The growth conditions on $\varphi$ yields

$$
c_{3}\|y\|_{V^{*}}^{p^{\prime}}-1 / c_{3}^{\prime} \leq \varphi^{*}(y) \leq c_{4}^{\prime}\left(1+\|y\|_{V^{*}}^{p^{\prime}}\right) \quad y \in V^{*}
$$

for some $c_{3}^{\prime}, c_{4}^{\prime}>0$, so that $E_{\bar{\varphi}^{*}}=L_{\bar{\varphi}^{*}}=E_{\varphi}^{*}=V^{*}$. Finally, since $V$ is both reflexive and dense in $H$, it is a standard matter to check that both H2i and H2ii are satisfied.

Our main result Theorem 3.7 implies that for all $u_{0} \in H$ and $f \in L^{2}(0, T ; H)$, there is a unique pair $(u, \xi)$ with

$$
\begin{aligned}
& u \in W_{w}^{1,1}\left(0, T ; V^{*}\right) \cap C^{0}([0, T] ; H) \cap L_{w}^{1}(0, T ; V), \quad \xi \in L_{w}^{1}\left(0, T ; V^{*}\right) \\
& \varphi(u), \bar{\varphi}^{*}(\xi) \in L^{1}(0, T)
\end{aligned}
$$

such that

$$
\partial_{t} u+\xi=f, \quad \xi \in \partial \varphi(u), \quad u(0)=u_{0}
$$

Since $V$ and $V^{*}$ are separable, one actually has

$$
u \in W^{1,1}\left(0, T ; V^{*}\right) \cap L^{1}(0, T ; V), \quad \xi \in L^{1}\left(0, T ; V^{*}\right)
$$

Moreover, thanks to the growth conditions on $\varphi$, we immediately see that the regularity $\varphi(u)$, $\bar{\varphi}^{*}(\xi) \in L^{1}(0, T)$ yields

$$
u \in L^{p}(0, T ; V), \quad \xi \in L^{p^{\prime}}\left(0, T ; V^{*}\right)
$$

Finally, by comparison in the equation we have $\partial_{t} u \in L^{p^{\prime}}\left(0, T ; V^{*}\right)$ as well.
Hence, in this simplified setting, our Theorem 3.7 actually coincides with the classical existence result from the variational theory.
6.2. Reaction-diffusion equations. We present here a first simple example of a class of PDEs that falls out of the classical variational setting presented in Subsection 6.1, but that can nonetheless be covered by our main existence result.

Let us consider partial differential equations in the form

$$
\begin{cases}\partial_{t} u-\operatorname{div}(\partial M(\nabla u))+\partial N(u) \ni f & \text { in }(0, T) \times \Omega  \tag{6.1}\\ u=0 & \text { on }(0, T) \times \partial \Omega \\ u(0)=0 & \text { in } \Omega\end{cases}
$$

where $\Omega \subset \mathbb{R}^{d}$ is a bounded Lipschitz domain, $T>0$ is a fixed final time, the data are chosen as $u_{0} \in L^{2}(\Omega)$ and $f \in L^{2}((0, T) \times \Omega)$, and $M: \mathbb{R}^{d} \rightarrow[0,+\infty)$ and $N: \mathbb{R} \rightarrow \mathbb{R}$ are convex, even, lower semicontinuous and polynomially bounded by above and below. More specifically, we suppose that there exist $c_{1}, c_{2}>0$ and $p, q \geq 2$ such that

$$
\begin{aligned}
c_{1}|x|^{p} \leq y \cdot x, & |y|^{p^{\prime}} \leq c_{2}\left(1+|x|^{p}\right)
\end{aligned} \quad \forall x \in \mathbb{R}^{d}, \quad \forall y \in \partial M(x),\left.~ 子\right|^{q^{\prime}} \leq c_{2}\left(1+|r|^{q}\right) \quad \forall r \in \mathbb{R}, \quad \forall w \in \partial N(r), ~ \$
$$

where $p^{\prime}$ and $q^{\prime}$ are the conjugate exponents of $p$ and $q$, respectively. Here the case $M(\nabla u)=|\nabla u|^{p}$ leads to thoroughly studied $p$-Laplace operator: see, for example, [48, 49].

Since $p$ and $q$ may be different in general, this setting does not fall directly in the classical framework presented in Subsection 6.1, as the coercivity condition is not satisfied if $p \neq q$. However, let us show that it can be treated by using our results.

One can consider the modular $\varphi: L^{2}(\Omega) \rightarrow[0,+\infty]$ defined as

$$
\varphi(u):= \begin{cases}\int_{\Omega} M(\nabla u)+\int_{\Omega} N(u) & \text { if } u \in W_{0}^{1, p}(\Omega) \cap L^{q}(\Omega) \\ +\infty & \text { otherwise }\end{cases}
$$

With this notation, the $\operatorname{PDE}$ (6.1) can be written in the abstract form

$$
\partial_{t} u+\partial \varphi(u) \ni f, \quad u(0)=u_{0}
$$

by choosing

$$
H:=L^{2}(\Omega), \quad V:=W_{0}^{1, p}(\Omega) \cap L^{q}(\Omega), \quad V^{*}=W^{-1, p^{\prime}}(\Omega)+L^{q^{\prime}}(\Omega)
$$

Thanks to the growth conditions on $M$ and $N$ we deduce that

$$
c_{1}^{\prime}\left(\|\nabla u\|_{L^{p}(\Omega)}^{p}+\|u\|_{L^{q}(\Omega)}^{q}\right) \leq \varphi(u) \leq c_{2}^{\prime}\left(1+\|\nabla u\|_{L^{p}(\Omega)}^{p}+\|u\|_{L^{q}(\Omega)}^{q}\right) \quad \forall u \in V
$$

Consequently, assumption $\mathbf{H 0}$ holds with $s=\min \{p, q\}$. Furthermore, the growth condition readily implies that $V=E_{\varphi}=L_{\varphi}$ and $E_{\bar{\varphi}^{*}}=L_{\bar{\varphi}^{*}}=E_{\varphi}^{*}=V^{*}$, so that also $\mathbf{H 1} \mathbf{- H} \mathbf{H} \mathbf{i}-\mathbf{H} 2 \mathbf{i i}$ are satisfied, with the choice $V_{0}=V$. Moreover, the conjugate $\bar{\varphi}^{*}: V^{*} \rightarrow[0,+\infty)$ satisfies

$$
\bar{\varphi}^{*}(v)=\langle v, u\rangle-\varphi(u) \quad \forall u \in V, \quad \forall v \in \partial \bar{\varphi}(u)
$$

where by [ 6, Thm. 2.10] we have that

$$
\partial \bar{\varphi}(u)=\left\{-\operatorname{div} v_{1}+v_{2}: \quad v_{1} \in \partial M(\nabla u), v_{2} \in \partial N(u) \quad \text { a.e. in } \Omega\right\} \quad \forall u \in V
$$

In this expression, by assumption on $M$ and $N$ we have that $v_{1} \in L^{p^{\prime}}(\Omega)^{d}$ and $v_{2} \in L^{q^{\prime}}(\Omega)$, and the divergence is intended in the sense of distributions on $\Omega$.

The existence Theorem 3.7 ensures then that for all $u_{0} \in L^{2}(\Omega)$ and $f \in L^{2}((0, T) \times \Omega)$ there exists a unique pair $(u, \xi)$ with

$$
\begin{aligned}
& u \in W_{w}^{1,1}\left(0, T ; V^{*}\right) \cap C^{0}([0, T] ; H) \cap L_{w}^{1}(0, T ; V), \quad \xi \in L_{w}^{1}\left(0, T ; V^{*}\right) \\
& \varphi(u), \bar{\varphi}^{*}(\xi) \in L^{1}(0, T)
\end{aligned}
$$

such that

$$
\partial_{t} u+\xi=f, \quad \xi \in \partial \bar{\varphi}(u), \quad u(0)=u_{0} .
$$

Since $V$ and $V^{*}$ are separable, these conditions imply the existence and uniqueness of a solution of (6.1) with

$$
\begin{aligned}
& u \in L^{p}\left(0, T ; W_{0}^{1, p}(\Omega)\right) \cap L^{q}\left(0, T ; L^{q}(\Omega)\right), \\
& \partial_{t} u \in L^{p^{\prime}}\left(0, T ; W^{-1, p^{\prime}}(\Omega)\right)+L^{q^{\prime^{\prime}}}\left(0, T ; L^{q^{\prime}}(\Omega)\right), \\
& \xi=-\operatorname{div} \xi_{1}+\xi_{2}, \quad \xi_{1} \in L^{p^{\prime}}\left(0, T ; L^{p^{\prime}}(\Omega)^{d}\right), \quad \xi_{2} \in L^{q^{\prime}}\left(0, T ; L^{q^{\prime}}(\Omega)\right),
\end{aligned}
$$

and

$$
\partial_{t} u-\operatorname{div} \xi_{1}+\xi_{2} \ni f \quad \text { in } V^{*} \quad \text { a.e. in }(0, T), \quad u(0)=u_{0} .
$$

6.3. Singular PDEs in Musielak-Orlicz spaces. In this subsection, we examine our approach in the setting of Muselak-Orlicz spaces. For all the abstract theory and general properties we refer to the classical monograph $[58, \S 7]$. We will show that our results cover several interesting cases of singular PDEs, including evolution equations in both reflexive and non-reflexive spaces, such as Lebesgue spaces with variable exponents, double-phase spaces, Orlicz spaces, and weighted Lebesgue spaces.

Given a bounded domain $\Omega \subset \mathbb{R}^{d}$ regular enough, we consider $\varphi_{M}: L^{2}(\Omega) \rightarrow[0,+\infty]$ of the form

$$
\varphi_{M}(v)= \begin{cases}\int_{\Omega} M(x, v(x)) d x & \text { if } M(\cdot, v) \in L^{1}(\Omega) \\ +\infty & \text { otherwise }\end{cases}
$$

where $M: \Omega \times \mathbb{R} \rightarrow[0, \infty)$ is a generalized strong $\Phi$-function $[35]$ in the sense that
(1) $M(\cdot, z)$ is measurable for every $z \in \mathbb{R}$;
(2) $M(x, \cdot)$ is convex and continuous for almost every $x \in \Omega$;
(3) $M(x, 0)=\lim _{z \rightarrow 0} M(x, z)=0$ and $\lim _{z \rightarrow \infty} M(x, z)=+\infty$ for almost all $x \in \Omega$;
(4) there is $\varepsilon \in(0,1]$ such that $M(x, \varepsilon) \leq 1$ and $M(x, 1 / \varepsilon) \geq 1$ for almost all $x \in \Omega$;
(5) if $M(x, \alpha z)=0$ for all $\alpha>0$ and almost all $x \in \Omega$, then $z=0$;
(6) $M(x, z)=M(x,-z)$ for all $z \in \mathbb{R}$ and almost all $x \in \Omega$.

In this setting, $\varphi_{M}$ is a lower semicontinuous convex semi-modular on $L^{2}(\Omega)$. Let us use the classical notation $L^{M}(\Omega):=L_{\varphi_{M}}$ and $E^{M}(\Omega):=E_{\varphi_{M}}$ for the respective Muselak-Orlicz spaces. Moreover, if there exists $c>0$ such that

$$
M(x, z) \geq c z^{2} \quad \text { for a.e. } x \in \Omega, \quad \forall z \in \mathbb{R}
$$

then we have $L^{M}(\Omega) \subset L^{2}(\Omega)$. Hence, we can choose $H:=L^{2}(\Omega)$, and assumption H0 holds with $s=2$.

We denote by $M^{*}: \Omega \times \mathbb{R} \rightarrow[0,+\infty]$ the convex conjugate of $M$ with respect to its second variable, namely

$$
M^{*}(x, z):=\sup _{y \in \mathbb{R}}\{z y-M(x, y)\}, \quad(x, z) \in \Omega \times \mathbb{R}
$$

Definition 6.1. A function $M: \Omega \times \mathbb{R} \rightarrow[0,+\infty]$ satisfies the weak doubling condition $\Delta_{2}^{w}$ if there exist a constant $k \geq 2$ and $h \in L^{1}(\Omega)$ such that

$$
\begin{equation*}
M(x, 2 z) \leq k M(x, z)+h(x) \tag{6.2}
\end{equation*}
$$

for almost all $x \in \Omega$ and all $z \in \mathbb{R}$. Similarly, $M$ fulfils the condition $\nabla_{2}^{w}$ if $M^{*}$ satisfies $\Delta_{2}^{w}$. Whenever $h \equiv 0$, the conditions are referred to as strong $\Delta_{2}$ and strong $\nabla_{2}$.

Definition 6.2. Let $M, N: \Omega \times \mathbb{R} \rightarrow[0,+\infty]$. We say that $M \preceq N$ if there exist two constants $c_{1}$, $c_{2}>0$ and $h \in L^{1}(\Omega)$ such that

$$
M(x, z) \leq c_{1} N\left(x, c_{2} z\right)+h(x) \quad \text { for almost every } x \in \Omega \text { and for all } z \in \mathbb{R}
$$

We say that $M$ and $N$ are equivalent if $M \preceq N \preceq M$.
It is well-known that every function $M$ meeting the $\Delta_{2}^{w}\left(\nabla_{2}^{w}\right)$ condition has an equivalent function $N$ which satisfies the strong $\Delta_{2}\left(\nabla_{2}\right)$ condition.
Definition 6.3. A generalized strong $\Phi$-function $M$ is said locally integrable if, for every measurable compact subset $K \subset \Omega$ and for every $z \in \mathbb{R}$, it holds that

$$
\int_{K} M(x, z) d x<+\infty
$$

We recall the main properties of the Musielak-Orlicz spaces in the following proposition: for detailed proofs, the reader can refer to $[17,37,58,71]$.

Proposition 6.4. Let $M$ be a generalized strong $\Phi$-function. Then, the following holds:
(i) $E^{M}(\Omega)$ and $L^{M}(\Omega)$ are Banach spaces w.r.t. $\|\cdot\|_{\varphi_{M}}$, and $L^{\infty}(\Omega)$ is continuously embedded in $E^{M}(\Omega)$;
(ii) if $M$ is locally integrable, then the simple functions on $\Omega$ and the smooth functions with compact support $C_{c}^{\infty}(\Omega)$ are dense in $E^{M}(\Omega)$ w.r.t. the norm $\|\cdot\|_{\varphi_{M}}$. In particular, $E^{M}(\Omega)$ is separable. Moreover, we have the characterization of the dual $E^{M}(\Omega)^{*} \cong L^{M^{*}}(\Omega)$;
(iii) if $M$ satisfies the $\Delta_{2}$ condition, then $L^{M}(\Omega)=E^{M}(\Omega)$ and there exists $p>1$ such that $\frac{M(x, z)}{|z|^{p}} \rightarrow 0$ as $|z| \rightarrow+\infty$ for almost all $x \in \Omega$;
(iv) $L^{M}(\Omega)$ is reflexive if and only if $M$ satisfies both $\Delta_{2}^{w}$ and $\nabla_{2}^{w}$ conditions.

The evolution equation associated to this choice of $\varphi_{M}$ reads

$$
\begin{cases}\partial_{t} u(t, x)+\partial M(x, u(t, x)) \ni f(t, x) & (t, x) \in(0, T) \times \Omega  \tag{6.3}\\ u(0, x)=u_{0}(x) & x \in \Omega\end{cases}
$$

where, as in the previous subsection, $T>0$ is a fixed final time, and the data are chosen as $u_{0} \in L^{2}(\Omega)$ and $f \in L^{2}((0, T) \times \Omega)$.

We have already pointed out that in the setting above, assumption H0 is satisfied with the choices $H:=L^{2}(\Omega)$ and $s=2$. We analyze now in detail the validity of the hypotheses $\mathbf{H 1}$, H2i-H2ii in connection to the $\Delta_{2}$ and $\nabla_{2}$ conditions for $M$.

## $M$ satisfies both $\Delta_{2}$ and $\nabla_{2}$.

If $M$ satisfies both the $\Delta_{2}$ and $\nabla_{2}$ conditions, then $E^{M}(\Omega)=L^{M}(\Omega)$ is reflexive. Moreover, if also $M$ and $M^{*}$ are locally integrable on $\Omega$, by property (ii) of Proposition $6.4, E^{M}(\Omega)$ is separable and $E^{M}(\Omega)^{*}=L^{M^{*}}(\Omega)$. In particular, it follows that $E^{M}(\Omega)^{*}=L^{M^{*}}(\Omega)=E^{M^{*}}(\Omega)$. Hence, assumption H1 is satisfied by the trivial choice $V_{0}=E^{M}(\Omega)$, and assumptions H2i-H2ii hold since $E^{M}(\Omega)=L^{M}(\Omega)$ and $H$ is dense in $L^{M^{*}}(\Omega)=E^{M^{*}}(\Omega)$.

In this setting, our Theorem 3.7 ensures then that the equation (6.3) has a unique solution

$$
u \in W^{1,1}\left(0, T ; L^{M^{*}}(\Omega)\right) \cap C^{0}\left([0, T] ; L^{2}(\Omega)\right) \cap L^{1}\left(0, T ; L^{M}(\Omega)\right), \quad \xi \in L^{1}\left(0, T ; L^{M^{*}}(\Omega)\right)
$$

such that

$$
M(\cdot, u), M^{*}(\cdot, \xi) \in L^{1}(0, T)
$$

Let us stress that the separability properties of $L^{M}(\Omega)$ and $L^{M^{*}}(\Omega)$ ensure actually that the measurability in time of such solutions is intended in the usual strong sense. Moreover, thanks to
the characterisation of $\partial M$ in [6, Prop. 2.7] the subdifferential relation (3.8) in this case can be written pointwise and reads

$$
\xi \in \partial M(\cdot, u) \quad \text { a.e. in }(0, T) \times \Omega
$$

This framework where $M$ is both $\Delta_{2}$ and $\nabla_{2}$ allows to cover several interesting cases, coming mainly from the modelling of anisotropic/non-homogenous phenomena. We refer the interested reader to the surveys [38] and [17] for more details. For instance, the following well-known examples are included in this setting:

- Variable exponents spaces: $M(x, z)=|z|^{p(x)}$, where $p: \Omega \rightarrow(1,+\infty)$ is measurable and such that

$$
1<p^{-}:=\underset{x \in \Omega}{\operatorname{ess} \inf } p(x) \leq \underset{x \in \Omega}{\operatorname{ess} \sup } p(x)=: p^{+}<+\infty
$$

- Double phase spaces: $M(x, z)=|z|^{p(x)}+a(x)|z|^{q(x)}$, where $a: \Omega \rightarrow[0,+\infty)$ is measurable and bounded, and

$$
1<p^{-} \leq p<q \leq q^{+}<+\infty
$$

- Orlicz spaces: $M(x, z)=\Phi(z)$, independently of $x \in \Omega$, where $\Phi$ is $\Delta_{2}$ and $\nabla_{2}$.
- Weighted Lebesgue spaces.


## $M$ satisfies either $\Delta_{2}$ or $\nabla_{2}$.

Let us start with $M$ fulfilling the $\Delta_{2}$-condition (but not necessarily the $\nabla_{2}$ condition). In this case, $E^{M}(\Omega)$ coincides with $L^{M}(\Omega)$. Moreover, if $M$ is locally integrable on $\Omega$, by property (ii) of Proposition 6.4 the space $E^{M}(\Omega)$ is separable and dense in $L^{2}(\Omega)$. Furthermore, by property (iii) there exists $p>1$ such that $\lim _{|z| \rightarrow+\infty} M(x, z) /|z|^{p}=0$ for a.e. $x \in \Omega$ : hence, if we choose $V_{0}$ as the space $L^{p}(\Omega)$, then $L^{p}(\Omega)$ is dense $E^{M}(\Omega)$, separable and reflexive, and $\varphi_{M}$ is bounded on bounded subsets of $L^{p}(\Omega)$. This shows that H1 is satisfied. Furthermore, assumption $\mathbf{H} 2$ is trivially satisfied since H2i holds as $E^{M}(\Omega)=L^{M}(\Omega)$.

Let us consider now $M$ fulfilling the $\nabla_{2}$-condition (but not necessarily the $\Delta_{2}$ condition). In this case, $E^{M^{*}}(\Omega)=L^{M^{*}}(\Omega)$. Moreover, if $M$ is locally integrable, then as before we have that $E^{M}(\Omega)$ is separable and dense in $L^{2}(\Omega)$. Using also property (ii) of Proposition 6.4, we have that $C_{c}^{\infty}(\Omega)$ is dense in $E^{M}(\Omega)$ and $L^{\infty}(\Omega) \hookrightarrow E^{M}(\Omega)$ continuously. Hence, supposing that $\Omega$ is regular enough (for example, bounded with Lipschitz-boundary), by the classical Sobolev embeddings one can choose $V_{0}:=W^{m, p}(\Omega)$ with $m>\frac{d}{p}$, so that $C_{c}^{\infty}(\Omega) \subset W^{m, p}(\Omega) \hookrightarrow L^{\infty}(\Omega)$. This shows that assumption H1 holds. Furthermore, assumption H2 holds due to to the following argument: $M^{*}$ is $\Delta_{2}$, hence $L^{M^{*}}(\Omega)=E^{M^{*}}(\Omega)$ is separable, and the collection of simple functions is a countable and dense subset of $L^{M^{*}}(\Omega)$. Consequently, $L^{2}(\Omega)$ is dense in $L^{M^{*}}(\Omega)$, so that in particular H2ii is satisfied.

In this setting, Theorem 3.7 ensures then that the equation (6.3) has a unique solution

$$
u \in W_{w}^{1,1}\left(0, T ; L^{M^{*}}(\Omega)\right) \cap C^{0}\left([0, T] ; L^{2}(\Omega)\right) \cap L_{w}^{1}\left(0, T ; L^{M}(\Omega)\right), \quad \xi \in L_{w}^{1}\left(0, T ; L^{M^{*}}(\Omega)\right)
$$

such that

$$
M(\cdot, u), M^{*}(\cdot, \xi) \in L^{1}(0, T)
$$

Let us point out that the measurability in time is not intended in the strong sense. Nonetheless, if $M$ satisfies the $\Delta_{2}$ condition, then $L^{M}(\Omega)$ is separable and actually it holds that also $u \in$ $L^{1}\left(0, T ; L^{M}(\Omega)\right)$. Similarly, if $M$ satisfies the $\nabla_{2}$ conditions, then $L^{M^{*}}(\Omega)$ is separable and it holds also that $u \in W^{1,1}\left(0, T ; L^{M^{*}}(\Omega)\right)$ and $\xi \in L^{1}\left(0, T ; L^{M^{*}}(\Omega)\right)$. As before, the differential inclusion (3.8) can be written pointwise in $\Omega$.

This more general framework where $M$ is allowed to satisfy either the $\Delta_{2}$ or the $\nabla_{2}$ condition (but not necessarily both of them) allows to cover almost all relevant cases of PDEs in MusielakOrlicz spaces. For these reasons, the variational theory presented here is widely applicable to most interesting examples. For instance, let us mention the following [17]:

- Orlicz spaces: $M(x, z)=\Phi(z)$, independently of $x \in \Omega$, where $\Phi$ is either $\Delta_{2}$ or $\nabla_{2}$. For instance, $\Phi(z)=\exp |z|^{p}-1, p \geq 1$, or $\Phi(z)=(|z|+1) \ln (|z|+1)-|z|$.
6.4. Singular PDEs in Musielak-Orlicz-Sobolev spaces. In this subsection, we show that our results conver also evolution problems in Muselak-Orlicz-Sobolev spaces. Again, for the abstract theory we refer to the classical monograph [58, § 10]. This framework includes interesting cases such as singular or degenerate evolution equations in Sobolev spaces with variable exponents, double-phase spaces, Orlicz-Sobolev spaces, and weighted Sobolev spaces. For simplicity, we will only focus on homogeneous Dirichlet boundary conditions: other classical choices can be easily covered with natural adaptation of this approach.

Here, we assume that $N$ and $M$ are generalized strong $\Phi$-functions as in Subsection 6.3. Given a bounded domain $\Omega \subset \mathbb{R}^{d}$ regular enough, we recall the definitions of the following Muselak-OrliczSobolev spaces (see [17]):

$$
\begin{aligned}
W_{0}^{1} L^{M}(\Omega) & :=\left\{v \in W_{0}^{1,1}(\Omega): v,|\nabla v| \in L^{M}(\Omega)\right\}, \\
W_{0}^{1} E^{M}(\Omega) & :=\left\{v \in W_{0}^{1,1}(\Omega): v,|\nabla v| \in E^{M}(\Omega)\right\}, \\
V_{0}^{1} L^{M}(\Omega) & :=\left\{v \in W_{0}^{1,1}(\Omega):|\nabla v| \in L^{M}(\Omega)\right\}, \\
V_{0}^{1} E^{M}(\Omega) & :=\left\{v \in W_{0}^{1,1}(\Omega):|\nabla v| \in E^{M}(\Omega)\right\} .
\end{aligned}
$$

If $M$ is also locally integrable in the sense of Definition 6.3 then all the above spaces are actually Banach spaces. Moreover, the space of compactly supported smooth functions $C_{c}^{\infty}(\Omega)$ is dense in $W_{0}^{1} E^{M}(\Omega)$ and in $V_{0}^{1} E^{M}(\Omega)$ : a possible proof can be readily adapted to the arguments of [28, § 2]. Also, if $M$ is $\Delta_{2}$ and $\nabla_{2}$ then $W_{0}^{1} L^{M}(\Omega)=W_{0}^{1} E^{M}(\Omega)$ is reflexive (see [37, Thm. 6.3]).

We introduce the lower semicontinuous convex semi-modular $\varphi_{M, N}: L^{2}(\Omega) \rightarrow[0,+\infty]$ as

$$
\varphi_{M, N}(v)=\left\{\begin{array}{l}
\int_{\Omega} N(x, v(x)) d x+\int_{\Omega} M(x,|\nabla v(x)|) d x \\
\quad \text { if } N(\cdot, v) \in L^{1}(\Omega), \quad v \in W_{0}^{1,1}(\Omega), \quad M(\cdot,|\nabla v|) \in L^{1}(\Omega), \\
+\infty \\
\quad \text { otherwise } .
\end{array}\right.
$$

Supposing again that, for some $c>0$,

$$
M(x, z), N(x, z) \geq c z^{2} \quad \text { for a.e. } x \in \Omega, \quad \forall z \in \mathbb{R}
$$

we have that $L^{M}(\Omega), L^{N}(\Omega) \subset L^{2}(\Omega)$ and that $W_{0}^{1} L^{M}(\Omega), V_{0}^{1} L^{M}(\Omega) \subset H_{0}^{1}(\Omega)$. Hence, one can characterize the modular spaces associated to $\varphi_{M, N}$ in terms of the Musielak-Orlicz-Sobolev spaces related to $N$ and $M$ as

$$
\begin{aligned}
& L_{\varphi_{M, N}}=\left\{v \in L^{N}(\Omega) \cap H_{0}^{1}(\Omega):|\nabla v| \in L^{M}(\Omega)\right\}=L^{N}(\Omega) \cap V_{0}^{1} L^{M}(\Omega), \\
& E_{\varphi_{M, N}}=\left\{v \in E^{N}(\Omega) \cap H_{0}^{1}(\Omega):|\nabla v| \in E^{M}(\Omega)\right\}=E^{N}(\Omega) \cap V_{0}^{1} E^{M}(\Omega) .
\end{aligned}
$$

As before, we can choose then $H:=L^{2}(\Omega)$ and assumption $\mathbf{H 0}$ holds with $s=2$.
Now, we set $M: \Omega \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ as $M(x, z):=M(x,|z|)$, for $(x, z) \in \Omega \times \mathbb{R}^{d}$, so in particular $M$ is symmetric in the second argument. With this notation, the evolution equation associated to this choice of $\varphi_{M, N}$ reads

$$
\begin{cases}\partial_{t} u(t, x)-\operatorname{div} \partial \boldsymbol{M}(x, \nabla u(t, x))+\partial N(x, u(t, x)) \ni f(t, x) & (t, x) \in(0, T) \times \Omega  \tag{6.4}\\ u(t, y)=0 & (t, y) \in(0, T) \times \partial \Omega \\ u(0, x)=u_{0}(x) & x \in \Omega\end{cases}
$$

where again $T>0$ is a fixed final time, $u_{0} \in L^{2}(\Omega)$, and $f \in L^{2}((0, T) \times \Omega)$.

Let us now discuss the validity of the hypotheses H1, H2i-H2ii. These strongly depend on whether $M$ and/or $N$ satisfy the conditions $\Delta_{2}$ and/or $\nabla_{2}$ : for sake of brevity, we only focus on two cases, the other ones being analogous.

## $M$ and $N$ satisfy both $\Delta_{2}$ and $\nabla_{2}$.

By the properties above, we have that the space $E^{N}(\Omega) \cap V_{0}^{1} E^{M}(\Omega)=L^{N}(\Omega) \cap V_{0}^{1} L^{M}(\Omega)$ is reflexive, and separable if also $M$ and $N$ are locally integrable. In particular, it is immediate to check that $\mathbf{H} 1$ holds with the trivial choice $V_{0}:=E^{N}(\Omega) \cap V_{0}^{1} E^{M}(\Omega)$, as well as both assumptions H2i-H2ii.

Theorem 3.7 ensures then that the equation (6.4) has a unique solution

$$
u \in W^{1,1}\left(0, T ; E_{\varphi_{M, N}}^{*}\right) \cap C^{0}\left([0, T] ; L^{2}(\Omega)\right) \cap L^{1}\left(0, T ; E_{\varphi_{M, N}}\right), \quad \xi \in L^{1}\left(0, T ; E_{\varphi_{M, N}}^{*}\right),
$$

such that

$$
\varphi_{M, N}(u), \varphi_{M, N}^{*}(\xi) \in L^{1}(0, T)
$$

Let us comment now on the subdifferential relation (3.8). Let us notice that since $E_{\varphi_{M, N}}=$ $E^{N}(\Omega) \cap V_{0}^{1} E^{M}(\Omega)$ and $E_{\varphi_{M, N}}$ is dense in both $E^{N}(\Omega)$ and $V_{0}^{1} E^{M}(\Omega)$, we have the representation of the dual as

$$
E_{\varphi_{M, N}}^{*} \cong L^{N^{*}}(\Omega)+V_{0}^{1} E^{M}(\Omega)^{*} .
$$

Let us show now that the subdifferential relation (3.8) in this case can be written as

$$
\xi=\xi_{1}-\operatorname{div} \xi_{2}, \quad \xi_{1} \in \partial N(\cdot, u), \quad \xi_{2} \in \partial \boldsymbol{M}(\cdot, \nabla u) \quad \text { a.e. in }(0, T) \times \Omega .
$$

To this end, we notice that we have the representation $\varphi_{M, N}=\varphi_{N}+\psi_{M}$, where $\varphi_{N}$ is defined as in Subsection 6.3 with respect to $N$, and $\psi_{M}: L^{2}(\Omega) \rightarrow[0,+\infty]$ is given by

$$
\psi_{M}(v):= \begin{cases}\int_{\Omega} M(x,|\nabla v(x)|) d x & \text { if } v \in H_{0}^{1}(\Omega), \quad M(\cdot,|\nabla v|) \in L^{1}(\Omega) \\ +\infty & \text { otherwise } .\end{cases}
$$

Lemma 6.5. In this setting, if $M$ satisfies the $\Delta_{2}$ and $\nabla_{2}$ conditions, the subdifferential of the restriction $\psi: V_{0}^{1} E^{M}(\Omega) \rightarrow \mathbb{R}$ is the operator

$$
\begin{aligned}
& A_{M}: V_{0}^{1} E^{M}(\Omega) \rightarrow 2^{V_{0}^{1} E^{M}(\Omega)^{*}}, \\
& A_{M}(v):=\left\{-\operatorname{div} \eta: \quad \eta \in L^{M^{*}}(\Omega)^{d}, \quad \eta \in \partial \boldsymbol{M}(\cdot, \nabla v) \text { a.e. in } \Omega\right\} .
\end{aligned}
$$

Proof. The proof can be directly adapted to the arguments of [6, Thm. 2.17-2.18], by taking into account that under these assumptions the space $V_{0}^{1} E^{M}(\Omega)$ is separable, reflexive, and dense in $L^{2}(\Omega)$.
Lemma 6.6. In this setting, if $M$ and $N$ satisfy the $\Delta_{2}$ and $\nabla_{2}$ conditions, the subdifferential of the restriction $\varphi_{M, N}: E^{N}(\Omega) \cap V_{0}^{1} E^{M}(\Omega) \rightarrow \mathbb{R}$ is the operator

$$
\begin{aligned}
& A_{M, N}: E^{N}(\Omega) \cap V_{0}^{1} E^{M}(\Omega) \rightarrow 2^{L^{N^{*}}(\Omega)+V_{0}^{1} E^{M}(\Omega)^{*}} \\
& A_{M, N}(v):=\{-\operatorname{div} \eta+\xi: \eta \in L^{M^{*}}(\Omega)^{d}, \quad \xi \in L^{N^{*}}(\Omega), \\
&\eta \in \partial M(\cdot, \nabla v), \quad \xi \in \partial N(\cdot, v) \quad \text { a.e. in } \Omega\} .
\end{aligned}
$$

Proof. It follows from the classical result [6, Thm. 2.11, Rmk. 2.1].
$M$ and $N$ satisfy $\Delta_{2}$.
If $M$ and $N$ fulfil the $\Delta_{2}$-condition (but not necessarily the $\nabla_{2}$ condition), then we have that $\left.V_{0}^{1} E^{M}(\Omega)=V_{0}^{1} L^{M} \Omega\right)$ and $E^{N}(\Omega)=L^{N}(\Omega)$. Moreover, these spaces are separable and dense in $L^{2}(\Omega)$ if also $M$ and $N$ are locally integrable. As far as the choice of $V_{0}$ is concerned, combining the considerations made in Subsection 6.3, we can choose for example $V_{0}:=W_{0}^{1, p}(\Omega)$, where $p$
realises condition (iii) of Proposition 6.4 for $M$ and $N$. Clearly, $V_{0}$ is separable reflexive and dense in $E_{\varphi_{M, N}}$. Furthermore, assumption $\mathbf{H} 2$ is trivially satisfied since $\mathbf{H 2 i}$ holds as $E_{\varphi_{M, N}}=L_{\varphi_{M, N}}$.

Theorem 3.7 ensures then that the equation (6.4) has a unique solution

$$
u \in W_{w}^{1,1}\left(0, T ; E_{\varphi_{M, N}}^{*}\right) \cap C^{0}\left([0, T] ; L^{2}(\Omega)\right) \cap L^{1}\left(0, T ; L_{\varphi_{M, N}}\right), \quad \xi \in L_{w}^{1}\left(0, T ; L^{M^{*}}(\Omega)\right)
$$

such that

$$
\varphi_{M, N}(u), \varphi_{M, N}^{*}(\xi) \in L^{1}(0, T)
$$

and the differential inclusion (3.8) is satisfied.
6.5. Singular PDEs with dynamic boundary conditions. In this subsection we show that the variational theory presented in this papers also allows to consider PDEs with possibly singular dynamic boundary conditions. As a motivating example, let us focus now on problems in the following form:

$$
\begin{cases}\partial_{t} u-\Delta u+\partial M(\cdot, u) \ni f & \text { in }(0, T) \times \Omega  \tag{6.5}\\ u=u_{\Gamma} & \text { in }(0, T) \times \Gamma \\ \partial_{t} u_{\Gamma}+\partial_{\mathbf{n}} u+\partial M_{\Gamma}\left(\cdot, u_{\Gamma}\right) \ni f_{\Gamma} & \text { in }(0, T) \times \Gamma \\ \left(u, u_{\Gamma}\right)(0)=\left(u_{0}, u_{0, \Gamma}\right) & \text { in } \Omega \times \Gamma\end{cases}
$$

where $\Omega \subset \mathbb{R}^{d}(d \geq 2)$ is a bounded domain with sufficiently regular boundary $\Gamma, \Delta$ denotes the Laplace operator, and $\mathbf{n}$ is the outward unit normal vector on $\Gamma$. Here, $f \in L^{2}\left(0, T ; L^{2}(\Omega)\right)$ and $f_{\Gamma} \in L^{2}\left(0, T ; L^{2}(\Gamma)\right)$ represent two given forcing terms in the bulk and on the boundary, respectively, while $u_{0} \in L^{2}(\Omega)$ and $u_{0, \Gamma} \in L^{2}(\Gamma)$ are the given initial data. Moreover, $M$ and $M_{\Gamma}$ are taken as strong $\Phi$-fucntions on $\Omega$ and $\Gamma$, respectively, in the sense of conditions (1)-(5) of Subsection 6.3.

In order to frame the evolution problem (6.5) in the context of modular spaces, it is natural to consider the Hilbert space $H:=L^{2}(\Omega) \times L^{2}(\Gamma)$ and define $\varphi: H \rightarrow[0,+\infty]$ as

$$
\varphi\left(v, v_{\Gamma}\right):=\left\{\begin{array}{l}
\int_{\Omega}\left(\frac{1}{2}|\nabla v(x)|^{2}+M(x, v(x))\right) d x+\int_{\Gamma} M_{\Gamma}\left(y, v_{\Gamma}(y)\right) d y \\
\quad \text { if } v \in H^{1}(\Omega), \quad M(\cdot, v) \in L^{1}(\Omega) \\
v_{\left.\right|_{\Gamma}}=v_{\Gamma}, \quad M_{\Gamma}\left(\cdot, v_{\Gamma}\right) \in L^{1}(\Gamma) \\
+\infty \\
\quad \text { otherwise } .
\end{array}\right.
$$

It is not difficult to check that problem (6.5) can be formulated in an abstract way as

$$
\partial_{t} \mathbf{u}+\partial \varphi(\mathbf{u}) \ni \mathbf{f}, \quad \mathbf{u}(0)=\mathbf{u}_{0}
$$

where we have used the bold notation to denote a general element of $H$, namely $\mathbf{u}:=\left(u, u_{\Gamma}\right)$, $\mathbf{f}:=\left(f, f_{\Gamma}\right) \in L^{2}(0, T ; H)$ and $\mathbf{u}_{0}:=\left(u_{0}, u_{0, \Gamma}\right) \in H$. Let us point out that for the general element $\mathbf{v}=\left(v, v_{\Gamma}\right) \in H$ it is not necessary true that $v_{\Gamma}$ is the trace of $v$ on $\Gamma$. For this to be ensured, one needs on $v$ more regularity than just $L^{2}$ : for example, if $\mathbf{v}=\left(v, v_{\Gamma}\right) \in D(\varphi)$, then $v \in H^{1}(\Omega)$ and $v_{\Gamma}$ is its trace (which is indeed well-defined in $H^{1 / 2}(\Gamma)$ ).

Let us check that this problem can be framed in our variational setting. First of all, using the notation of Subsection 6.3 for the Musielak-Orlicz spaces on $\Omega$ and $\Gamma$, we have that

$$
\begin{aligned}
& L_{\varphi}=\left\{\mathbf{v} \in H^{1}(\Omega) \times H^{1 / 2}(\Gamma): \quad v_{\left.\right|_{\Gamma}}=v_{\Gamma}, v \in L^{M}(\Omega), v_{\Gamma} \in L^{M_{\Gamma}}(\Gamma)\right\} \\
& E_{\varphi}=\left\{\mathbf{v} \in H^{1}(\Omega) \times H^{1 / 2}(\Gamma):\right. \\
& \left.v_{\left.\right|_{\Gamma}}=v_{\Gamma}, v \in E^{M}(\Omega), v_{\Gamma} \in E^{M_{\Gamma}}(\Gamma)\right\}
\end{aligned}
$$

and the $\|\cdot\|_{\varphi}$-norm is equivalent to the norm of the intersection of the spaces appearing on the right-hand side, namely

$$
\frac{1}{4}\|\mathbf{v}\|_{\varphi} \leq\|v\|_{H^{1}(\Omega)}+\|v\|_{\varphi_{M}}+\left\|v_{\Gamma}\right\|_{\varphi_{M_{\Gamma}}} \leq 4\|\mathbf{v}\|_{\varphi} \quad \forall \mathbf{v} \in L_{\varphi}
$$

It is immediate that $\varphi$ is a lower semicontinuous convex semi-modular on $H$, which is indeed a separable Hilbert space. Besides, the $s$-coercivity of $\varphi$ in the sense of assumption H0 follows from the respective hypothesis on $M$ and $M_{\Gamma}$ as in Subsection 6.3 with $s=2$ (actually, in this case it is enough to require the above-mentioned coercivity only on $M_{\Gamma}$, thanks to a suitable Poincaré-type inequality and the gradient contribution in $\varphi$ ).

Secondly, let us check assumption H1. Using property (ii) of Proposition 6.4, we have that $L^{\infty}(\Omega) \times L^{\infty}(\Gamma) \hookrightarrow E^{M}(\Omega) \times E^{M_{\Gamma}}(\Gamma)$ continuously, and $C^{\infty}(\bar{\Omega}) \times C^{\infty}(\Gamma)$ is dense in $E^{M}(\Omega) \times$ $E^{M_{\Gamma}}(\Gamma)$. Consequently, a natural candidate for the space $V_{0}$ is

$$
V_{0}:=\left\{\mathbf{v} \in H^{m}(\Omega) \times H^{m-1 / 2}(\Omega): \quad v_{\left.\right|_{\Gamma}}=v_{\Gamma}\right\}, \quad m>\frac{d}{2}
$$

Indeed, clearly $V_{0}$ is separable, reflexive, and by the Sobolev embeddings we have the continuous inclusions

$$
H^{m}(\Omega) \hookrightarrow L^{\infty}(\Omega) \cap H^{1}(\Gamma), \quad H^{m-1 / 2}(\Gamma) \hookrightarrow L^{\infty}(\Gamma) \cap H^{1 / 2}(\Gamma)
$$

Recalling the equivalence of the $\|\cdot\|_{\varphi}$-norm above, this ensures that $V_{0} \hookrightarrow E_{\varphi}$ continuously. As far as the density of $V_{0}$ in $E_{\varphi}$ is concerned, given $\mathbf{v} \in E_{\varphi}$, since $H^{m}(\Omega)$ is dense in $E^{M}(\Omega) \cap H^{1}(\Omega)$, there is a sequence $\left(v_{n}\right)_{n} \subset H^{m}(\Omega)$ such that $v_{n} \rightarrow v_{\Gamma}$ in $E^{M}(\Omega) \cap H^{1}(\Omega)$. Clearly, the traces $\left(v_{\Gamma, n}\right)_{n} \subset H^{m-1 / 2}(\Gamma)$ satisfy $v_{\Gamma, n} \rightarrow v_{\Gamma}$ in $H^{1 / 2}(\Gamma)$. Furthermore, if $d=2$, then $\Gamma$ has dimension 1 and by the Sobolev embeddings we have $H^{1 / 2}(\Gamma) \hookrightarrow L^{\infty}(\Gamma) \hookrightarrow E^{M_{\Gamma}}(\Gamma)$ continuously, so that also $v_{\Gamma, n} \rightarrow v_{\Gamma}$ in $E^{M_{\Gamma}}(\Gamma)$, hence $\mathbf{v}_{n} \rightarrow \mathbf{v}$ in $E_{\varphi}$. If $d=3$, then $\Gamma$ has dimension 2 and by the Sobolev embeddings we have $H^{1 / 2}(\Gamma) \hookrightarrow L^{4}(\Gamma)$ : in this case, if we suppose that $M_{\Gamma}$ is controlled by a 4 -th power, then $L^{4}(\Gamma) \hookrightarrow E^{M_{\Gamma}}(\Gamma)$ and we can conclude as above.

Lastly, let us discuss assumption H2. Following the same line of Subsection 6.3 and without going into the details, the main idea is that we can either assume $\Delta_{2}$-type conditions in order to get H2i or $\nabla_{2}$-type conditions in order to get H2ii. For example, if both $M$ and $M_{\Gamma}$ satisfy the $\Delta_{2}$ condition, then we have $E^{M}(\Omega)=L^{M}(\Omega)$ and $E^{M_{\Gamma}}(\Gamma)=L^{M_{\Gamma}}(\Gamma)$, so that condition H2i is trivially satisfied.

Theorem 3.7 ensures then that the problem (6.5) admits a unique variational solution

$$
\mathbf{u} \in W_{w}^{1,1}\left(0, T ; L_{\bar{\varphi}^{*}}\right) \cap C^{0}([0, T] ; H) \cap L_{w}^{1}\left(0, T ; L_{\varphi}\right), \quad \xi \in L_{w}^{1}\left(0, T ; L_{\bar{\varphi}^{*}}\right),
$$

such that

$$
\varphi(\mathbf{u}), \varphi^{*}(\boldsymbol{\xi}) \in L^{1}(0, T),
$$

in the sense that

$$
\begin{aligned}
\int_{\Omega} u(t) \zeta & +\int_{\Gamma} u_{\Gamma}(t) \zeta_{\Gamma}+\int_{0}^{t} \int_{\Omega} \nabla u \cdot \nabla \zeta+\int_{0}^{t}[\xi, \zeta]_{\varphi_{M}}+\int_{0}^{t}\left[\xi_{\Gamma}, \zeta_{\Gamma}\right]_{\varphi_{M_{\Gamma}}} \\
& =\int_{\Omega} u_{0} \zeta+\int_{\Gamma} u_{0, \Gamma} \zeta_{\Gamma}+\int_{0}^{t} \int_{\Omega} f \zeta+\int_{0}^{t} \int_{\Gamma} f_{\Gamma} \zeta_{\Gamma} \quad \forall t \in[0, T], \quad \forall \zeta \in E_{\varphi} .
\end{aligned}
$$

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