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Generalization of Heron's and Brahmagupta's equalities to any cyclic polygon

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Abstract. It is well known that Heron's equality provides an explicit formula for the area of a triangle, as a symmetric function of the lengths of its edges. It has been extended by Brahmagupta to quadrilaterals inscribed in a circle (cyclic quadrilaterals). A natural problem is trying to further generalize the result to cyclic polygons with a larger number of edges. Surprisingly, this has proved to be far from simple, and no explicit solutions exist for cyclic polygons having $n > 4$ edges. In this paper we investigate such a problem by following a new and elementary approach, based on the idea that the simple geometry underlying Heron's and Brahmagupta's equalities hides the real players of the game. In details, we propose to focus on the dissection of the edges determined by the incircles of a suitable triangulation of the cyclic polygon, showing that this approach leads to an explicit formula for the area as a symmetric function of the lengths of these segments. We also show that such a symmetry can be rediscovered in Heron's and Brahmagupta's results, which consequently represent special cases of the provided general equality.

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Keywords. Area, Cyclic polygon, Incircle, Inradius, Symmetric function.

1. Introduction

A natural and largely considered question in convex geometry is the determination of the area A of a convex polygon as a symmetric function of the lengths of its edges. The problem goes back to Heron of Alexandria, who was able to solve the problem in the case of a triangle. If a, b, c are the lengths of the edges, and p denotes the semiperimeter, then

$$A^2 = p(p-a)(p-b)(p-c). \tag{1}$$

Later, in the seventh century, Brahmagupta extended the result to cyclic quadrilaterals, namely to quadrilaterals inscribed in a circle (see for instance

[1]), proving that

$$A^2 = (p-a)(p-b)(p-c)(p-d), \tag{2}$$

where a, b, c, d are the lenghts of the edges of the quadrilateral.

Several results concerning the geometry of cyclic polygons have been obtained in different areas of research (see [2–4,6,11]), which points to a general interest in such geometric objects. It is therefore natural to try to further extend to cyclic polygons with a larger number of edges the nice and ancient formulae by Heron and Brahmagupta. Surprisingly, this has proved to be far from simple. In [8,9] an algebraic formulation of the problem led D.P. Robbins to find symmetric formulae for cyclic pentagons and cyclic hexagons. It was observed that Heron and Brahmagupta's formulae can be restated in a form where $16A^2$ represents a monic polynomial whose coefficients are symmetric polynomials in the squares of the edges. This generalizes to cyclic pentagons and hexagons, where polynomials of degree 7 and 38 respectively appear, but the formulae, even if they hold in the non convex case as well, do not provide explicit forms for the area (see also [7] for interesting comments and remarks). Symmetric formulae of the same kind have been conjectured [8], and later proved [5], even for heptagons and octagons, also illuminating some mysterious features of Robbin's results for the areas of cyclic pentagons and hexagons (see also [10] for further details on Robbin's conjectures). The resulting formulae are interesting, but very complex and only hold for cyclic polygons having very few edges, so these do not seem to provide a general picture that could be easily generalized to polygons with an arbitrarily large number of edges.

Moving from the above remarks, we propose here a different approach, based on the idea that the role of the edges in the simple geometric cases of Heron and Brahmagupta's equalities in fact hides the real players of the game. After giving in Sect. 2 the main notations and preliminaries, in Sects. 3 and 4 we show that the classical results for triangles and cyclic quadrilaterals can be rediscovered by using the lengths of the segments determined on the edges by the tangent points of suitable incircles, and arise from a simple symmetric formula concerning the area of a right triangle. As a consequence, the leitmotif of our paper is that instead of looking for the area as a symmetric function of the edges, the original problem should be tackled by means of segments cut on the edges by the incircles of the triangles of a triangulation of the cyclic polygon. Indeed, in Sect. 5, we prove a symmetric coordinate free equality that holds true for any cyclic polygon, and includes Heron's and Brahmagupta's results as special cases.

2. Notations and preliminaries

A convex cyclic polygon is a convex polygon inscribed in a circle. We denote by P_n a convex cyclic polygon having $n+2$ edges, and by $A(n)$ its area. In

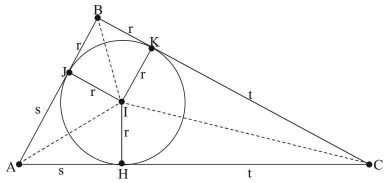

FIGURE 1. A right triangle ABC

case when the role of the vertices must be emphasized, we also denote the area by writing the vertices between vertical bars. For instance, $|ABC|$, means the area of a triangle ABC. In case several triangles $T_1, \ldots, T_m$ must be considered simultaneously, then we denote by A_j the area of T_j, $j \in \{1, \ldots, m\}$.

In view of a clearer presentation of the new proposed approach, we recall Pytagoras' theorem, as well as Heron and Brahmagupta's results concerning the area as a symmetric function of the lengths of the edges.

Theorem 1. (Pythagoras) *In a right triangle the area of the square whose edge is the hypotenuse is equal to the sum of the areas of the squares whose edges are the two legs.*

Theorem 2. (Heron's equality for triangles) *The area of a triangle is equal to* $\sqrt{p(p-a)(p-b)(p-c)}$*, where* a*,* b*,* c *are the lengths of its edges (taken in any order) and* $p = (a+b+c)/2$ *is its half perimeter.*

Theorem 3. (Brahmagupta's equality for cyclic quadrilaterals) *The area of a cyclic quadrilateral is equal to* $\sqrt{(p-a)(p-b)(p-c)(p-d)}$*, where* a*,* b*,* c*,* d *are the lengths of its edges (taken in any order) and* $p = (a+b+c+d)/2$ *is its half perimeter.*

Our main result is Theorem 8, where we prove a general symmetric formula for the area of a cyclic polygon having $n \geq 3$ edges, which includes Theorem 2 and Theorem 3 as particular cases.

3. Heron's equality

Let ABC be a right triangle and let I be its incenter (see Fig. 1). Since $\hat{B}$ is a right angle and since the incircle is tangent perpendicularly to the three edges of ABC, we have $r = IJ = IK = IH = BJ = BK$, where r is the inradius. The internal bisectors of ABC are concurrent in I and this implies $AJ = AH = s$ and $CH = CK = t$.

We have $|ABC| = |AIB| + |BIC| + |CIA|$. The half-perimeter of ABC is $p = r + s + t$ while the three triangles on the r.h.s. of the above equality have altitude r with respect to their edges AB, BC, and AC. Therefore, it results

$$|ABC| = r(r+s+t). \tag{3}$$

Remark 4. In case ABC is not a right triangle, Formula (3) generalizes to

$$|ABC| = R(r+s+t), \tag{4}$$

where R is the incircle of ABC, meaning that $|ABC|$ is a symmetric function in r, s, t.

Lemma 5. *The area of a right triangle ABC is equal to the area of the rectangle of edges $s = AH$ and $t = CH$, where H is the point where the incircle is tangent to the hypotenuse AC.*

Proof. Clearly $|ABC| = \dfrac{1}{2}(s+r)(t+r)$, and by (3) we get

$$st + rs + rt + r^2 = 2(r^2 + rs + rt)$$

which we simplify into $st = r^2 + rs + rt = r(r+s+t) = |ABC|$. $\square$

A new proof of Theorem 2 for right triangles. Using the same notation (as in Fig. 1) we need to show that $|ABC|^2 = str(r+s+t)$, but this is immediate, since $|ABC| = st$ by Lemma 5, and also $|ABC| = r(r+s+t)$ by (3). $\square$

The above proof shows that, in any right triangles, Heron's equality can be rediscovered by starting from the symmetric formula provided by Lemma 5. We wish now to extend such a result to any triangle.

A new proof of Theorem 1. We have $r(r+s+t) = st$. Multiplying by 2 and adding $s^2 + t^2$ to both sides we get $s^2 + t^2 + 2r^2 + 2rs + 2rt = s^2 + t^2 + 2st$, and consequently $(s+r)^2 + (r+t)^2 = (s+t)^2$. $\square$

The above proof shows that Pythagoras' Theorem can be rediscovered as a consequence of Heron's equality in right triangles. Now, thanks to Pythagoras' Theorem, we can easily extend Heron's equality to any triangle, which consequently follows from Heron's equality for right triangles.

For this, let ABC be a generic triangle, and let CH be the altitude on its edge AB (see Fig. 2), where we assume H between A and B (in any triangle there surely exists an altitude with this property).

Let $p = r+s+t$ be the semiperimeter of ABC, and let

$$\varphi = \sqrt{rst(r+s+t)} = \sqrt{p(p-a)(p-b)(p-c)}.$$

By Pythagoras' Theorem in AHC and CHB, we have

$$\begin{aligned} c^2 = (AH + HB)^2 &= AH^2 + BH^2 + 2(AH)(HB) \\ &= a^2 + b^2 - 2CH^2 + 2(AH)(HB) \\ &= a^2 + b^2 - 2CH^2 + 2\sqrt{(a^2 - CH^2)(b^2 - CH^2)} \end{aligned}$$

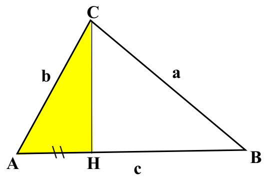

FIGURE 2. A generic triangle ABC

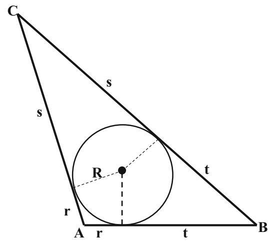

FIGURE 3. Incircle of a generic triangle ABC

and, solving for CH we get

$$CH = \frac{\sqrt{4a^2b^2 - (c^2 - a^2 - b^2)^2}}{2c} = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{c} = \frac{2}{c}\varphi.$$

Therefore, from $2|ABC| = cCH$, Heron's equality for ABC follows.

Remark 6. By (4) and Heron's equality written in the form $|ABC| = \sqrt{rst(r+s+t)}$ we can obtain the incircle R of any triangle as a symmetric function of r, s, t as follows (see Fig. 3).

$$R = \sqrt{\frac{rst}{p}} \tag{5}$$

since $p = r + s + t$ is the half perimeter of ABC.

4. Brahmagupta's equality

We wish now to show how Brahmagupta's equality can be rediscovered by exploiting the same idea of symmetry considered in the previous section. First of all, we prove the following result.

Theorem 7. *Let ABC and ADC be two triangles inscribed in the same circle. If s_1, t_1 and s_2, t_2 are the lengths of the two segments split on the common edge AC by the respective incircles, then*

$$|ABC||ACD| = s_1 s_2 t_1 t_2.$$

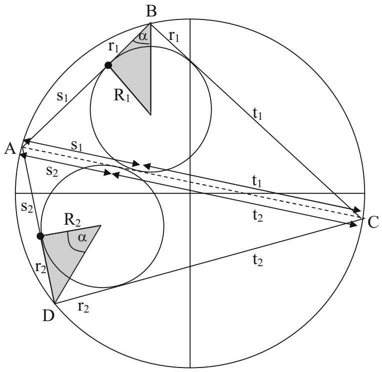

Figure 4. A general cyclic quadrilateral $ABCD$

Proof. In the cyclic quadrilateral $ABCD$ the halves of $A\hat{B}C$ and $A\hat{D}C$ are complementary angles. Therefore the shaded right triangles in Fig. 4 are similar, and consequently $\frac{R_1}{r_1} = \frac{r_2}{R_2}$, that is $R_1R_2 = r_1r_2$.

By (5) we have $p_1R_1^2 = r_1s_1t_1$ and $p_2R_2^2 = r_2s_2t_2$, since p_1, p_2 is the half perimeter of ABC and ADC respectively, so that

$$(p_1R_1p_2R_2)^2 = p_1(r_1s_1t_1)p_2(r_2s_2t_2) = p_1p_2(R_1R_2)(s_1t_1s_2t_2),$$

and consequently $|ABC||ACD| = p_1R_1p_2R_2 = s_1t_1s_2t_2$.

Remark 8. Since $R_1R_2 = r_1r_2$ we have also $|ABC||ACD| = p_1R_1p_2R_2 = p_1p_2r_1r_2$.

A new proof of Theorem 3. Let the cyclic quadrilateral of vertices A, B, C, D be split into ABC and ACD, as in Fig. 4, and assume $a = s_1 + r_1 = AB, b = t_1 + r_1 = BC, c = t_2 + r_2 = CD, d = s_2 + r_2 = AD$, so that $p = r_1 + r_2 + (s_1 + t_1) = r_1 + r_2 + (s_2 + t_2)$. Starting from $|ABCD|^2 = (|ABC| + |ACD|)^2 = |ABC|^2 + |ACD|^2 + 2|ABC||ACD|$, we use the previous theorem, and the Remark, to write $2|ABC||ACD| = s_1t_1s_2t_2 + p_1p_2r_1r_2$, where p_1, p_2 are the half perimeters of ABC and ACD, respectively. Moreover, by Heron's formula, $|ABC|^2 = (r_1 + s_1 + t_1)r_1s_1t_1$, and $|ACD|^2 = (r_2 + s_2 + t_2)r_2s_2t_2$. Then, also using $s_1 + t_1 = s_2 + t_2$, we get

$$\begin{aligned}
|ABCD|^2 &= (r_1 + s_1 + t_1)r_1s_1t_1 + (r_2 + s_2 + t_2)r_2s_2t_2 \\
&\quad + s_1t_1s_2t_2 + r_1r_2(r_1 + s_1 + t_1)(r_2 + s_2 + t_2) \\
&= r_1(r_1 + s_1 + t_1)(r_2(r_2 + s_2 + t_2) + s_1t_1) \\
&\quad + s_2t_2(r_2(r_2 + s_2 + t_2) + s_1t_1) \\
&= (r_2(r_2 + s_2 + t_2) + s_1t_1)(r_1(r_1 + s_1 + t_1) + s_2t_2) \\
&= (r_2(r_2 + s_1 + t_1) + s_1t_1)(r_1(r_1 + s_2 + t_2) + s_2t_2)
\end{aligned}$$

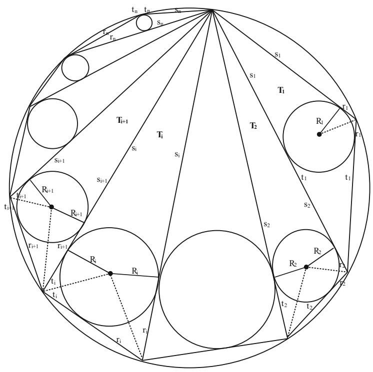

FIGURE 5. Triangulation of a cyclic polygon

$$
\begin{aligned}
&= (r_2 + t_1)(r_2 + s_1)(r_1 + s_2)(r_1 + t_2) \\
&= (p - a)(p - b)(p - c)(p - d).
\end{aligned}
$$

□

5. The area of a circular polygon having an arbitrary number of edges

In this section we generalize the previous results to a cyclic polygon P_n, having $n + 2$ edges for any $n \geq 1$. Let us observe that Heron's equality has been extended to Brahmagupta's equality by considering a cyclic quadrilateral Q as the union of two triangles, $Q = T_1 \cup T_2$, and then focusing on the segments r_1, s_1, t_1 and r_2, s_2, t_2 determined, respectively, on the edges of T_1 and T_2 by the tangent points of the corresponding incircles. This provides the square of the area of Q as a polynomial function, symmetric under the exchange of r_1, s_1, t_1 with r_2, s_2, t_2.

As a consequence we are inspired to investigate the square of the area $A(n)$ of a generic cyclic polygon P_n by looking at the partitions of the edges of P_n determined by the tangent points of the incircles of some triangulation. For this, let us first observe that P_n can always be assumed as the union of n consecutive triangles $T_1, T_2, \ldots, T_n$, all having a common vertex. For $i = 1, \ldots, n-1$, denote by $L_{i,i+1}$ the common edge between the two consecutive triangles T_i, T_{i+1}. Let p_j, A_j, R_j be, respectively, the semiperimeter, the area and the radius of the incircle of T_j, $j = 1, \ldots, n$. Also, let r_j, s_j, t_j be the segments cut on the edges of T_j by its incircle, where $L_{i,i+1} = s_i + t_i = s_{i+1} + r_{i+1}$, $i = 1, \ldots, n-1$ (see Fig. 5).

For the sake of brevity, and in order to avoid heavy notations, in the following theorems we assume all the meaningless products to be equal to 1.

Theorem 9. *Let P_n be a cyclic polygon consisting of $n+2$ edges, $n \geq 1$. Then*

$$A_h A_k = s_h t_h s_k r_k \prod_{i=h+1}^{k-1} \frac{s_i}{p_i} = p_h r_h p_k t_k \prod_{i=h+1}^{k-1} \frac{p_i}{s_i}, \text{ for } 1 \leq h < k \leq n. \tag{6}$$

Proof. In order to prove the first equality in (6) we apply Theorem 7 iteratively, so that

$$\begin{aligned} &A_h A_{h+1} = s_h t_h s_{h+1} r_{h+1} \\ &A_{h+1} A_{h+2} = s_{h+1} t_{h+1} s_{h+2} r_{h+2} \\ &\dots \\ &A_{k-1} A_k = s_{k-1} t_{k-1} s_k r_k. \end{aligned}$$

By multiplying on both sides we get

$$A_h (A_{h+1} A_{h+2} \dots A_{k-1})^2 A_k = s_h t_h \left(\prod_{i=h+1}^{k-1} r_i s_i^2 t_i \right) s_k r_k.$$

By Heron's equality applied to $T_{h+1}, T_{h+2}, \dots, T_{k-1}$ we have

$$A_h A_k = \frac{s_h t_h \left(\prod_{i=h+1}^{k-1} r_i s_i^2 t_i \right) s_k r_k}{\prod_{i=h+1}^{k-1} r_i s_i t_i p_i} = s_h t_h s_k r_k \prod_{i=h+1}^{k-1} \frac{s_i}{p_i},$$

and the first equality in (6) is obtained. For the proof of the second equality, let us observe that, by similitude, we have $\dfrac{R_i}{r_i} = \dfrac{t_{i+1}}{R_{i+1}}$, for all $i = 1, \dots, n-1$. Therefore we get

$$\begin{aligned} &R_h R_{h+1} = r_h t_{h+1} \\ &R_{h+1} R_{h+2} = r_{h+1} t_{h+2} \\ &\dots \\ &R_{k-1} R_k = r_{k-1} t_k. \end{aligned}$$

By multiplying on both sides, we obtain

$$R_h (R_{h+1} R_{h+2} \dots R_{k-1})^2 R_k = R_h R_k \prod_{i=h+1}^{k-1} R_i^2 = r_h t_k \prod_{i=h+1}^{k-1} r_i t_i,$$

and applying (5) to all R_i^2 we have

$$R_h R_k = \frac{r_h t_k \prod_{i=h+1}^{k-1} r_i t_i}{\prod_{i=h+1}^{k-1} \frac{r_i s_i t_i}{p_i}},$$

and consequently

$$R_h R_k = r_h t_k \prod_{i=h+1}^{k-1} \frac{p_i}{s_i} \text{ for } 1 \leq h < k \leq n. \tag{7}$$

The second equality in (6) follows immediately from (7), since $A_hA_k = p_hR_hp_kR_k$. $\square$

Assuming $P_n = T_1 \cup T_2 \cup \cdots \cup T_n$ as in Fig. 5, and using the same notations as above we can now prove a general formula for the area of a cyclic polygon with any number of edges.

Theorem 10. *Let P_n be a cyclic polygon with $n+2$ edges, $n \geq 1$, and let $A(n)$ be its area. Then*

$$A(n)^2 = \left(p_1r_1 + \sum_{q=2}^{n} r_qs_q \prod_{m=2}^{q-1} \frac{s_m}{p_m}\right)\left(s_1t_1 + \sum_{q=2}^{n} p_qt_q \prod_{m=2}^{q-1} \frac{p_m}{s_m}\right). \tag{8}$$

Proof. Since P_n is the union of $T_1, \dots, T_n$, by Heron's equality, and using both equalities in (6), we have

$$\begin{aligned} A^2 &= (A_1 + A_2 + \cdots + A_n)^2 \\ &= \sum_{j=1}^{n} A_j^2 + 2 \sum_{1 \leq h < k \leq n} A_hA_k \\ &= \sum_{j=1}^{n} p_jr_js_jt_j + \sum_{1\leq h<k\leq n} s_ht_hs_kr_k \prod_{i=h+1}^{k-1} \frac{s_i}{p_i} + \sum_{1\leq h<k\leq n} p_hr_hp_kt_k \prod_{i=h+1}^{k-1} \frac{p_i}{s_i}. \end{aligned}$$

Let's rearrange as follows (where each one of the three terms appearing in each bracket comes from the corresponding sum)

$$\begin{aligned} A^2 = p_1r_1 &\left(s_1t_1 + 0 + \sum_{k>1} p_kt_k \prod_{i=2}^{k-1} \frac{p_i}{s_i}\right) \\ &+ r_2s_2 \left(p_2t_2 + s_1t_1 + \frac{p_2}{s_2} \sum_{k>2} p_kt_k \prod_{i=3}^{k-1} \frac{p_i}{s_i}\right) \\ &+ r_3s_3\frac{s_2}{p_2}\left(\frac{p_2}{s_2}p_3t_3 + (s_1t_1 + p_2t_2) + \frac{p_2}{s_2}\frac{p_3}{s_3} \sum_{k>3} p_kt_k \prod_{i=4}^{k-1} \frac{p_i}{s_i}\right) \\ &+ r_4s_4\frac{s_2}{p_2}\frac{s_3}{p_3}\left(\frac{p_2}{s_2}\frac{p_3}{s_3}p_4t_4 + (s_1t_1 + p_2t_2 + \frac{p_2}{s_2}p_3t_3) + \frac{p_2}{s_2}\frac{p_3}{s_3}\frac{p_4}{s_4} \sum_{k>4} p_kt_k \prod_{i=5}^{k-1} \frac{p_i}{s_i}\right) \\ &+ \cdots + \\ &+ r_ns_n\frac{s_2}{p_2}\frac{s_3}{p_3}\cdots\frac{s_{n-1}}{p_{n-1}}\left(\frac{p_2}{s_2}\frac{p_3}{s_3}\cdots\frac{p_{n-1}}{s_{n-1}}p_nt_n\right. \\ &+ \left.\left(s_1t_1 + p_2t_2 + p_3t_3\frac{p_2}{s_2} + \cdots + p_{n-1}t_{n-1}\frac{p_2}{s_2}\frac{p_3}{s_3}\cdots\frac{p_{n-2}}{s_{n-2}}\right) + 0\right) \\ = &\left(p_1r_1 + \sum_{q=2}^{n} r_qs_q \prod_{m=2}^{q-1} \frac{s_m}{p_m}\right)\left(s_1t_1 + \sum_{q=2}^{n} p_qt_q \prod_{m=2}^{q-1} \frac{p_m}{s_m}\right). \end{aligned}$$

$\square$

Remark 11. We emphasize that the formula obtained for $A(n)^2$ is symmetric under mutually exchanging r_j with t_j, and s_j with p_j, for all $j \in \{1,\dots,n\}$. We also note that only the terms concerning the partitions of the edges of the polygon P_n are in fact necessary. Indeed, $p_1 = r_1 + s_1 + t_1$ can be immediately computed once we know the terms r_1, s_1, t_1 determined on the edges of P_n by the incircle of T_1. For $1 < q < n-1$, due to consecutiveness, we have $s_q = s_{q-1} + t_{q-1} - r_q$, so that, by recursion, we get

$$s_q = s_1 + \sum_{h=1}^{q-1} t_h - \sum_{k=2}^{q} r_k, \tag{9}$$

$$p_q = r_q + s_q + t_q = \begin{cases} s_1 + t_1 + t_2 & \text{if } q = 2 \\ s_1 + \sum_{h=1}^{q} t_h - \sum_{k=2}^{q-1} r_k & \text{if } 2 < q < n-1. \end{cases} \tag{10}$$

Consequently, all terms appearing in $A(n)^2$ can be computed once s_1, s_n, r_j, t_j are known.

Examples We can easily rediscover Heron's and Brahmagupta's results from the provided symmetric function.

• $n = 1$

$$A(1)^2 = (p_1 r_1 + 0)(s_1 t_1 + 0) = p_1 r_1 s_1 t_1$$

Heron's equality.

• $n = 2$

$$\begin{aligned} A(2)^2 &= (p_1 r_1 + r_2 s_2)(s_1 t_1 + p_2 t_2) \\ &= ((r_1 + s_1 + t_1) r_1 + r_2 s_2)(s_1 t_1 + (r_2 + s_2 + t_2) t_2) \\ &= ((r_1 + s_2 + r_2) r_1 + r_2 s_2)(s_1 t_1 + (s_1 + t_1 + t_2) t_2) \\ &= (r_1 + s_2)(r_1 + r_2)(t_2 + s_1)(t_2 + t_1). \end{aligned}$$

It is $2p = s_1 + t_1 + 2r_1 + r_2 + s_2 + 2t_2$, so, from $s_1 + t_1 = s_2 + r_2$ we get $p = r_1 + r_2 + s_2 + t_2 = r_1 + s_1 + t_1 + t_2$. From Fig. 5, since $n = 2$, we can assume the edges to be $a = s_2 + t_2$, $b = r_2 + t_2$, $c = r_1 + t_1$ and $d = r_1 + s_1$, then

$$A(2)^2 = (r_1 + r_2)(r_1 + s_2)(t_2 + s_1)(t_1 + t_2) = (p-a)(p-b)(p-c)(p-d)$$

Brahmagupta's equality.

• $n = 3$. **Generalization of Brahmagupta's equality to cyclic pentagons**

$$A(3)^2 = \left(p_1 r_1 + r_2 s_2 + r_3 s_3 \frac{s_2}{p_2}\right)\left(s_1 t_1 + p_2 t_2 + p_3 t_3 \frac{p_2}{s_2}\right).$$

• $n = 4$. **Generalization of Brahmagupta's equality to cyclic hexagons**

$$A(4)^2 = \left(p_1 r_1 + r_2 s_2 + r_3 s_3 \frac{s_2}{p_2} + r_4 s_4 \frac{s_2}{p_2}\frac{s_3}{p_3}\right)$$

$$\left(s_1 t_1 + p_2 t_2 + p_3 t_3 \frac{p_2}{s_2} + p_4 t_4 \frac{p_2}{s_2}\frac{p_3}{s_3}\right).$$

We can even extend the formula to $n = 0$ by assuming $A(0) = 0$, where P_0 is a polygon degenerated in a segment, which can be obtained by progressively removing an edge from a starting polygon P_n having $n + 2$ edges.

6. Conclusion and remarks

We have shown that Heron's and Brahmagupta's equalities can be extended to a formula that provides the square of the area of any convex cyclic polygon as a symmetric polynomial of the lengths of the segments determined on the edges by the incircles of a suitable triangulation. We remark that the formula is coordinate-free as one should expect from the intrinsic geometric nature of the problem. Otherwise, using for instance Green's theorem, it would be quite easy to provide a coordinate dependent result.

In our opinion the obtained formula is the natural generalization of what happens for triangles and cyclic quadrilaterals, where the lengths of the edges explicitly appear in the computation of the area. This is just because the number of involved edges is small, so that the segments determined by the edge partitions induced by the incircles can be easily related to the original lengths of the edges of the polygon. We also remark that the incircles can be constructed in an elementary way, so that the provided formula also determines an elementary computation of the square of the area of any convex cyclic polygon.

Some related open problems also arise, which would be worth considering in later possible works. For instance, it would be interesting to investigate the functional dependence of the assumed parameters r_q, s_q, t_q on the original edge lengths. Presumably, such functions are not simple, which could explain why an explicit formula for the area as a symmetric function of the edges has not been obtained in the literature. Also, the extension of the result to non-convex cyclic polygons seems to be an appealing and challenging task. A further analysis could be devoted to equality (8) in case it is obtained by using different triangulations of the same polygon, or to its possible simplification in suitable subclasses of cyclic polygons

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