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Variable Elasticity of Substitution in the Diamond Model: Dynamics and Comparisons

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Abstract. We study the dynamics shown by the discrete time Diamond overlapping-generations model with the VES production function in the form given by Revankar[10] and compare our results with those obtained by Brianzoni *et al.*[2] in the Solow model. We prove that, as in Brianzoni *et al.*[2], unbounded endogenous growth can emerge if the elasticity of substitution is greater than one; moreover, differently from Brianzoni *et al.*[2], the Diamond model can admit two positive steady states. We also prove that complex dynamics occur if the elasticity of substitution between production factors is less than one, confirming the results obtained by Brianzoni *et al.*[2]. Numerical simulations support the analysis.
Keywords: Variable Elasticity of Substitution, Diamond Growth Model, Fluctuations and Chaos, Bifurcation in Piecewise Smooth Dynamical Systems.

1 Introduction

The elasticity of substitution between production factors plays a crucial role in the theory of economic growth, it being one of the determinants of the economic growth level (see Klump and de La Grandville[6]).

Within the Solow model (see Solow[11], and Swan[12]) it was found that a country exhibiting a higher elasticity of substitution experiences greater capital (and output) per capita levels in the equilibrium state (see Klump and de La Grandville[6], Klump and Preissler[7], and Masanjala and Papageorgiou[8]). More recently, the role of the elasticity of substitution between production factors in the long run dynamics of the Solow model was investigated both considering the Constant Elasticity of Substitution production function (CES) (see Brianzoni *et al.*[1]) and the Variable Elasticity of Substitution production function (VES) (see Brianzoni *et al.*[2]). The results obtained demonstrate

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that fluctuations may arise if the elasticity of substitution between production factors falls below one.

Miyagiwa and Papageorgiou[9] moved the attention to the Diamond overlapping-generations model (Diamond[4]) while proving that, differently from the Solow setup, "if capital and labor are relatively substitutable, a country with a greater elasticity of substitution exhibits lower per capita output growth in both transient and steady state". To reach this conclusion they considered the normalized CES production function.

In the present work we consider the Diamond overlapping-generations model with the VES production function in the form given by Revankar[10] (see also Karagiannis *et al.*[5]). Our main goal is to study the local and global dynamics of the model to verify if the main result obtained by Brianzoni *et al.*[2] in the Solow model, i.e. cycles and complex dynamics may emerge if the elasticity of substitution between production factors is sufficiently low, still holds in the Diamond framework.

To summarize, the qualitative and quantitative long run dynamics of the Diamond growth model with VES production function are studied, to show that complex features can be observed and to compare the results obtained with the ones reached while considering the CES technology or the Solow framework.

2 The economic setup

Consider a discrete time setup, $t \in \mathbb{N}$, and let $y_t = f(k_t)$ be the production function in intensive form, mapping capital per worker k_t into output per worker y_t. Following Karagiannis *et al.*[5] we consider the Variable Elasticity of Substitution (VES) production function in intensive form with constant return to scale, as given by Revankar[10]:

$$y_t = f(k_t) = Ak_t^a[1 + bak_t]^{(1-a)}, \quad k_t \geq 0 \tag{1}$$

where $A > 0, \quad 0 < a < 1, \quad b \geq -1$; furthermore $1/k_t \geq -b$, in order to assure that $f(k_t) > 0$, $f'(k_t) > 0$ and $f''(k_t) < 0$, $\forall k_t > 0$, where

$$f'(k_t) = Aak_t^a(1 + abk_t)^{1-a}[k_t^{-1} + (1 - a)b(1 + abk_t)^{-1}]$$

and

$$f''(k_t) = A\frac{a(a-1)(1 + abk_t)^{-a-1}}{k_t^{2-a}}.$$

The elasticity of substitution between production factors is then given by

$$\sigma(k_t) = 1 + bk_t$$

hence $\sigma \geq (<)1$ iff $b \geq (<)0$. Thus the elasticity of substitution varies with the level of capital per capita, representing an index of economic development. Observe that, while the elasticity of substitution for the CES is constant along an isoquant, in the case of the VES it is constant only along a ray through the origin.

In the Diamond[4] overlapping-generations model a new generation is born at the beginning of every period. Agents are identical and live for two periods. In the first period each agent supplies a unit of labor inelastically and receives a competitive wage:

$$w_t = f(k_t) - k_t f'(k_t),$$

thus, taking into account the specification of the production function in (1), we obtain

$$w_t = Ak_t^a \frac{(1+2abk_t)(1-a)}{(1+abk_t)^a}. \tag{2}$$

As in Miyagiwa and Papageorgiou[9] we assume that agents save a fixed proportion $s \in (0,1)$ of the wage income to finance consumption in the second period of their lives. All savings are invested as capital to be used in the next period's production, so that the evolution of capital per capita is described by the following map

$$k_{t+1} = \phi(k_t) = \frac{s}{1+n} w_t = \frac{sA}{1+n} k_t^a \frac{(1+2abk_t)(1-a)}{(1+abk_t)^a}, \tag{3}$$

where $n > 0$ is the exogenous labor growth rate and capital depreciates fully.

As in Brianzoni *et al.*[2] we distinguish between the following two cases.

(a) If $b > 0$ the elasticity of substitution between production factors is greater than one and the standard properties of the production function are verified $\forall k_t > 0$; in this case k_t evolves according to (3).We do not consider the case $b = 0$ as $\sigma(k_t)$ becomes constant and equal to one, $\forall k_t \geq 0$, thus obtaining a particular case of the CES production function.

(b) If $b \in [-1, 0)$ the elasticity of substitution between production factors is less than one and the standard properties of the production function are verified for all $0 < k_t < -\frac{1}{b}$; in this case k_t evolves according to (3) iff $k_t \in [0, -1/b]$ while, following Karagiannis *et al.*[5] and Brianzoni *et al.*[2], if $k_t > -1/b$ then $k_t = \phi(-1/b)$.

3 Local and Global Dynamics

3.1 Elasticity of Substitution Greater than One

Let $b > 0$. Then the discrete time evolution of the capital per capita k_t is described by the continuous and differentiable map (3).

The establishment of the number of steady states is not trivial to solve, considering the high variety of parameters. As a generale result, the map ϕ always admits one fixed point characterized by zero capital per capita, i.e. $k = 0$ is a fixed point for any choice of parameter values. Anyway steady states which are economically interesting are those characterized by positive capital per worker. In order to determine the positive fixed points of ϕ, let us define the following function:

$$G(k) = \frac{1-a}{k^{1-a}} \frac{1+2abk}{(1+abk)^a}, k > 0 \tag{4}$$

where

$$G'(k) = \frac{(1-a)}{k^{2-a}(1+abk)^{1+a}}[(a-1)+(2a-1)abk], \tag{5}$$

then solutions of $G(k) = \frac{1+n}{sA}$ are positive fixed points of ϕ.

The following proposition establishes the number of fixed points of map ϕ.

Proposition 1 *Let ϕ given by (3).*

(i) Assume $b > 0$ and $a \leq \frac{1}{2}$. Then:
  (a) if $\frac{1+n}{sA} > (ab)^{1-a}2(1-a)$, ϕ has two fixed points given by $k_t = 0$ and $k_t = k^* > 0$;
  (b) if $0 < \frac{1+n}{sA} \leq (ab)^{1-a}2(1-a)$, ϕ has a unique fixed point given by $k_t = 0$.

(ii) Assume $b > 0$ and $a > \frac{1}{2}$ and let $k_m = \frac{1-a}{ab(2a-1)}$. Then:
  (a) if $\frac{1+n}{sA} < (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$, ϕ has a unique fixed point given by $k_t = 0$;
  (b) if $\frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$, ϕ has two fixed points given by $k_t = 0$ and $k^* = k_m$;
  (c) if $(\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) < \frac{1+n}{sA} < (ab)^{1-a}2(1-a)$, ϕ has three fixed points given by $k_t = 0$, $k_t = k_1$ and $k_t = k_2$, where $0 < k_1 < k_m < k_2$;
  (d) if $\frac{1+n}{sA} \geq (ab)^{1-a}2(1-a)$, ϕ has two fixed points given by $k_t = 0$ and $k^* > 0$, where $0 < k^* < k_m$.

Proof. $k_t = 0$ is a solution of equation $k_t = \phi(k_t)$ for all parameter values hence it is a fixed point. Function (4) is such that $G(k_t) > 0$ for all $k_t > 0$, furthermore $\lim_{k_t \to 0^+} G(k_t) = +\infty$ while $\lim_{k_t \to +\infty} G(k_t) = (ab)^{1-a}2(1-a)$.

(i) Observe that if $b > 0$ and $a \leq \frac{1}{2}$, $G(k)$ is strictly decreasing $\forall k_t > 0$ since $G'(k) < 0$. Hence $G(k_t)$ intersects the positive and constant function $g = \frac{1+n}{sA}$ in a unique positive value $k_t = k^*$ iff $\frac{1+n}{sA} > (ab)^{1-a}2(1-a)$.

(ii) Assume $a > \frac{1}{2}$ and $b > 0$ then G has a unique minimum point $k_m = \frac{1-a}{ab(2a-1)}$ where $G(k_m) = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$. Hence, if $(\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) < \frac{1+n}{sA} < (ab)^{1-a}2(1-a)$, then equation $G(k_t) = \frac{1+n}{sA}$ admits two positive solutions. Similarly, it can be observed that if $\frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$ or $\frac{1+n}{sA} \geq (ab)^{1-a}2(1-a)$ then $\phi(k_t)$ admits a unique positive fixed point. Trivially, for the other parameter values, equation $G(k_t) = \frac{1+n}{sA}$ has no positive solutions.

For what it concerns the local stability of the steady states the following proposition holds.

Proposition 2 *Let ϕ be as given in (3).*

(i) The equilibrium $k_t = 0$ is locally unstable for all parameter values.

(ii) If ϕ admits two fixed points then the equilibrium $k_t = k^* > 0$ is locally stable.

(iii) If ϕ admits three fixed points, then the equilibrium $k_t = k_1$ is locally stable while the equilibrium $k_t = k_2$ is locally unstable.

Proof. Firstly notice that function ϕ may be written in terms of function G being:

$$\phi(k) = \frac{sA}{1+n} kG(k) \tag{6}$$

hence $\phi'(k) = \frac{sA}{1+n}[G(k) + kG'(k)]$.

(i) Since $\lim_{k_t \to 0^+} G(k_t) = +\infty$ and $\lim_{k_t \to 0^+} kG'(k_t) = +\infty$, then $\phi'(0) = +\infty$ and consequently the origin is a locally unstable fixed point for map ϕ.

(ii) Assume that ϕ admits two fixed points. After some algebra it can be noticed that

$$\phi'(k) = \frac{a(1+a)sA}{1+n} \frac{(1+abk)^{-1-a}}{k^{1-a}} [2ab^2k^2 + 2b(1+a)k + 1] > 0 \qquad \forall k > 0 \tag{7}$$

hence $\phi(k)$ is strictly increasing and consequently k^* is locally stable. In the particular case in which $\frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$ then $k^* = k_m$ is a non hyperbolic fixed point: a tangent bifurcation occurs at which k^* is locally stable.

(iii) Assume that ϕ has three equilibria. Since $\phi'(k) > 0 \; \forall k > 0$ then point *(iii)* is easily proved.

The results concernig the existence and number of fixed points and their local stability when the elasticity of substitution between production factors is greater than one, are resumed in Fig. 1. We fix all the parameters but s and we show that, as s is increased, we pass from two to three and, finally, to one fixed point. Hence it can be observed that unbounded growth can emerge if the propensity to save in sufficiently high.

As in Brianzoni *et al.*[2], if the elasticity of substitution between the two factors is greater than one ($b > 0$), then unbounded endogenous growth can be observed but only simple dynamics can be produced. Anyway, differently from Brianzoni *et al.*[2], the growth model can exhibit two positive steady states so that the final outcome of the economy depends on the initial condition (in fact if $k_0 \in (0, k_2)$ then the convergence toward k_1 is observed while if $k_0 > k_2$ then unbounded endogenous growth is exhibited).

3.2 Elasticity of Substitution Less than One

Let $b \in [-1, 0)$. Then the discrete time evolution of the capital per capita k_t is described by the following continuous and piecewise smooth map:

$$k_{t+1} = F(k_t) = \begin{cases} \phi(k_t) & \forall k_t \in \left[0, -\frac{1}{b}\right] \\ \phi\left(-\frac{1}{b}\right) & \forall k_t > -\frac{1}{b} \end{cases}. \tag{8}$$

As it is easy to verify, F is non-differentiable in the point $k_t = -\frac{1}{b}$, which separates the state space into two regions $R_1 = \{(k) : 0 \leq k < -\frac{1}{b}\}$ and $R_2 = \{(k) : k > -\frac{1}{b}\}$. Furthermore, the map F is constant for $k_t > -\frac{1}{b}$ and non-linear for $0 \leq k_t \leq -\frac{1}{b}$. The following proposition describes the number of fixed points when $b \in [-1, 0)$.

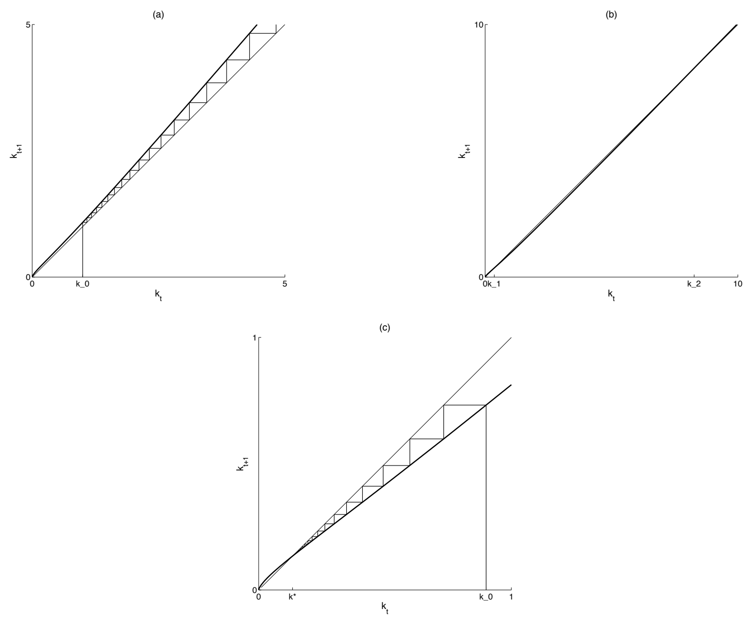

Fig. 1. Map ϕ, its fixed points and their stability for $b = 1$, $a = 0.7$, $A = 3$ and $n = 0.1$. (a) $s = 0.8$, (b) $s = 0.7$ and (c) $s = 0.6$.

Proposition 3 *Let F be as given in (8) and $b \in [-1, 0)$.*

(i) Assume $a > \frac{1}{2}$. Then F has two fixed points given by $k = 0$ and $k^* \in (0, -\frac{1}{2ab})$.

(ii) Assume $a \le \frac{1}{2}$ and $M = \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}}$. Then:

(a) if $\frac{1+n}{sA} \ge M$ there exist two fixed points given by $k = 0$ and $k^* \in (0, -\frac{1}{b}]$;

(b) if $\frac{1+n}{sA} < M$ there exist two fixed points given by $k = 0$ and $k^* = F(-\frac{1}{b})$.

Proof. It is easy to see that $k = 0$ is a fixed point for any choice of the parameter values.

(i) Firstly notice that $\phi \ge 0$ iff $k \in [0, -\frac{1}{2ab}]$ and $\phi(0) = \phi(-\frac{1}{2ab}) = 0$, so values of $k > -\frac{1}{2ab}$ are not economically significant. Moreover ϕ has a unique maximum point given by $k_M = \frac{-1-a+\sqrt{1+a^2}}{2ab}$ with $\phi(k_M) = \frac{sA}{1+n}\left(\frac{\sqrt{1+a^2}-1-a}{ab\sqrt{1+a^2}+1-a}\right)^a (1-a)(\sqrt{1+a^2}-a)$. Finally $\lim_{k\to 0^+} \phi'(k) = \infty$. Hence equation $\phi(k) = k$ has always a unique positive solution given by $k^* \in (0, -\frac{1}{2ab})$.

(ii) The positive fixed points of F such that $k \le -\frac{1}{b}$ are given by the solutions of equation $G(k) = \frac{1+n}{sA}$ with $G(k)$ as given in (4) and $G > 0$ defined in $(0, -\frac{1}{b}]$. Being $G'(k) = \frac{(1-a)}{k^{2-a}(1+abk)^{1+a}}[(a-1) + (2a-1)abk]$, G is strictly decreasing $\forall k \in (0, -\frac{1}{b}]$ with minimum point in $k_m = -\frac{1}{b}$ and $G(k_m) = G(-\frac{1}{b}) = \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} = M$. Hence $G(k) = \frac{1+n}{sA}$ has a unique positive solution $k^* \in (0, -\frac{1}{b}]$ iff $\frac{1+n}{sA} \geq \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}}$. Differently, the unique fixed point of F such that $k > -\frac{1}{b}$ is defined by $k^* = F(-\frac{1}{b}) = \phi(-\frac{1}{b})$ and it exists iff $F(-\frac{1}{b}) > -\frac{1}{b}$, which is equivalent to require $\frac{1+n}{sA} < \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}}$.

Let us move to the study of the local stability of the fixed points. Since

$$\phi'(k) = \frac{a(1+a)sA}{1+n}\frac{(1+abk)^{-1-a}}{k^{1-a}}[2ab^2k^2 + 2b(1+a)k + 1]$$

then $\lim_{k\to 0^+} \phi'(k) = +\infty$, so that the equilibrium characterized by zero capital-per capita is always locally unstable.

We firstly focus on the case with $a > \frac{1}{2}$. As it has been discussed, the related one dimensional map is continuous and differentiable in its domain $[0, -\frac{1}{2ab}]$. Furthermore, $\phi(0) = \phi(-\frac{1}{2ab}) = 0$ and $\phi''(k) < 0$ $\forall k \in (0, -\frac{1}{2ab})$, i.e. it is strictly concave. As a consequence map ϕ behaves as the logistic map, that is it exhibits the standard period doubling bifurcation cascade as one parameter is moved (see Devaney[3]).

The period doubling bifurcation cascade is observed, for instance, if A is increased. In fact it can be easily observed that $\phi(k_M)$ increases as A increases so that $\exists \bar{A}$ such that $\phi(k_M) > -\frac{1}{2ab}$ $\forall A > \bar{A}$, i.e. almost all trajectories are unfeasible. At $A = \bar{A}$ a final bifurcation occurs (the origin is a pre-periodic fixed point and ϕ is chaotic in a Cantor set) while $\forall A \in (0, \bar{A})$ the period doubling bifurcation cascade is observed (see Fig. 2 (a),(b) and (e)). Notice also that the situation presented in panel (b) becomes simpler if a greater value of b is considered (see Fig. 2 (c)), proving that in order to have complex dynamics b must be sufficiently low (as also showed in panel (d)).

In order to study the local stability of the positive fixed point when $a \leq \frac{1}{2}$ and $b \in [-1, 0)$ we observe that function F has a non differentiable point given by

$$P = \left(-\frac{1}{b}, F(-\frac{1}{b})\right), \tag{9}$$

where $F(-\frac{1}{b}) = \frac{sA}{1+n}(-b)^{-a}(1-a)^{1-a}(1-2a)$.

Notice that if P is above the main diagonal, the fixed point k^* is superstable being $F'(k^*) = 0$ and no complex dynamics can be exhibited. This case occurs, for instance, if A is great enough and the related situation is presented in Fig. 3 (a). If $\frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} = \frac{1+n}{sA}$ we get that $k^* = -\frac{1}{b}$, then a border collision of the superstable fixed point occurs.

If P is below the main diagonal then k^* may be locally stable or unstable and complex dynamics may arise.

The following Proposition states a sufficient condition for the existence of a stable 2-period cycle $\{k_1, k_2\}$ such that $k_i \in R_i, (i = 1, 2)$.

Proposition 4 *Let $b \in [-1, 0)$. For all b in the region defined as*

$$\Omega = \left\{ b : F^2(-\frac{1}{b}) > -\frac{1}{b} \cap \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} < \frac{1+n}{sA} \right\} \tag{10}$$

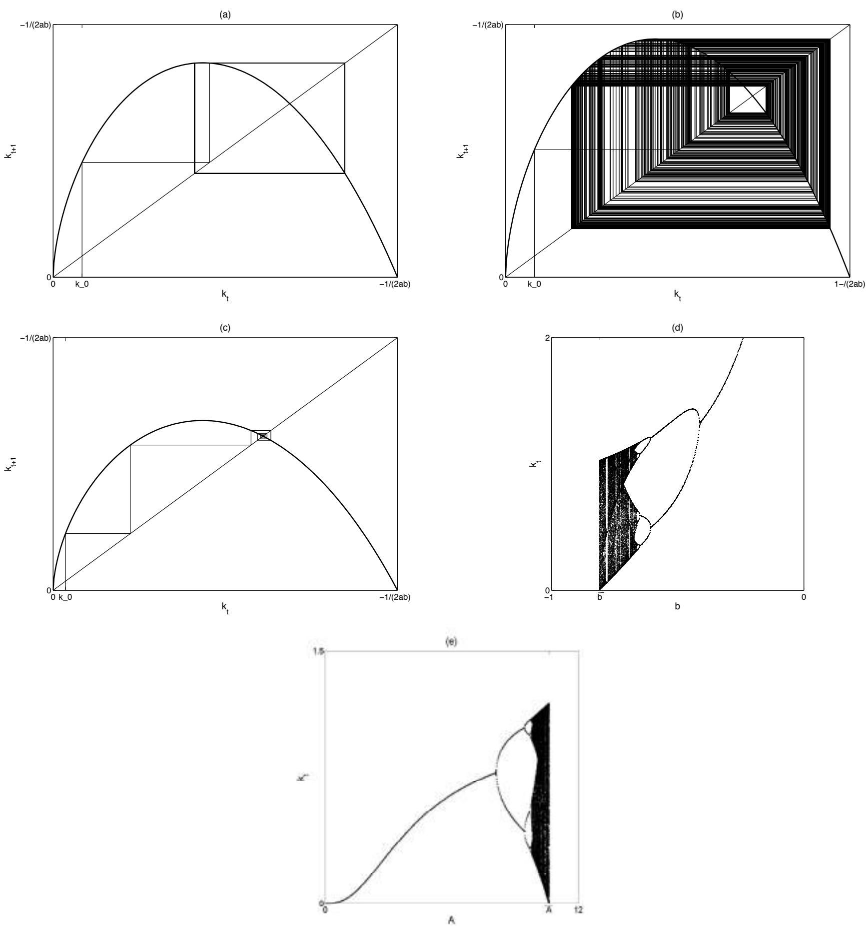

Fig. 2. $a = 0.6$, $n = 0.1$, $s = 0.7$. (a) If $b = -0.7$ and $A = 9$ a stable two period cycle is presented, while (b) if $A = 10$ complexity emerges. (c) Locally stable fixed point for $A = 10$ and $b = -0.3$. (d) Bifurcation diagram w.r.t. b. (e) Bifurcation diagram w.r.t. A.

map F admits a superstable 2-period cycle defined as $C_2 = \{F(-\frac{1}{b}), F^2(-\frac{1}{b})\}$.

Proof. A 2-cycle for map F is given by $\{k_1, k_2\}$ with $F(k_1) = k_2$ and $F(k_2) = k_1$. Let $k_0 > -\frac{1}{b}$ with $k_0 \in R_2$, then $k_1 = F(-\frac{1}{b})$ belongs to R_1 (being the point P below the main diagonal) and $k_2 = F(k_1) = F(F(-\frac{1}{b})) = F^2(-\frac{1}{b})$. If $F^2(-\frac{1}{b}) > -\frac{1}{b}$, then $F^2(-\frac{1}{b}) \in R_2$ and consequently $F(F^2(-\frac{1}{b})) = F(k_2) = F(-\frac{1}{b}) = k_1$. This proves the existence of a two period cycle. Moreover, the eigenvalue of such cycle is zero, since $F'(k_2) = 0$, therefore it is a superstable two period cycle.

Notice that in $F^2(-\frac{1}{b}) = -\frac{1}{b}$ a border collision bifurcation of the superstable 2-period cycle occurs. The superstable two period cycle is depicted in Fig. 3 (b).

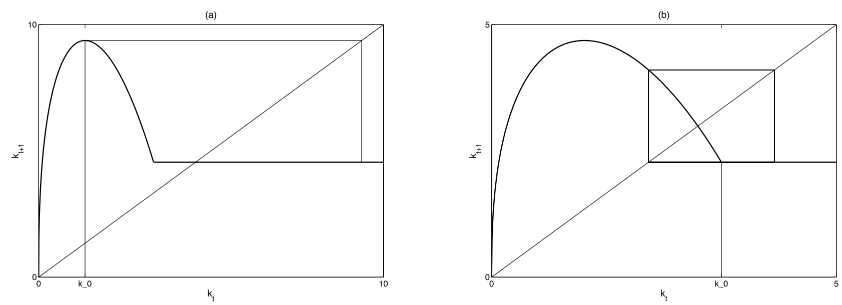

Fig. 3. $a = 0.4$, $n = 0.1$, $s = 0.7$. (a) If $b = -0.3$ and $A = 30$ the positive steady state is superstable. (b) The superstable two period cycle for $b = -0.3$ and $A = 15$.

In order to describe how complex dynamics may emerge if $a \leq \frac{1}{2}$, we recall that F is unimodal and $k_M = \frac{-1-a+\sqrt{1+a^2}}{2ab}$ is its maximum point.

If $k^* \in (0, k_M)$ (i.e. point $(k_M, F(k_M))$ is below the main diagonal), then k^* is globally stable $\forall k_0 \neq 0$. On the contrary, if $F(k_M) > k_M$ (i.e. point $(k_M, F(k_M))$ is above the main diagonal), then its eigenvalue is negative and k^* may lose stability only via a period-doubling bifurcation. Therefore, a necessary condition for a flip bifurcation is that that point $(k_M, F(k_M))$ is above the main diagonal.

To recap, as in Brianzoni *et al.*[2], our model can exhibit cycles or more complex dynamics iff P is below the main diagonal while the maximum point k_M is above the main diagonal. In this case all positive initial conditions produce trajectories converging to an attractor belonging to a trapping interval J defined in the following proposition.

Proposition 5 *Let* $\frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} < \frac{1+n}{sA}$ *and* $F(k_M) > k_M$. *Then the one-dimensional map* F *admits a trapping interval* J*, where* J *is defined as follows:*

1. $J = [F(-\frac{1}{b}), F(k_M)]$ *if* $F(k_M) \geq -\frac{1}{b}$,
2. $J = [F^2(k_M), F(k_M)]$ *if* $F(k_M) < -\frac{1}{b}$.

Proof. If the one-dimensional map F has a maximum point k_M above the main diagonal and point P is below the main diagonal, then through the graphical analysis it is possible to see that when the image of k_M belongs to $R_2 \cup \{-\frac{1}{b}\}$, then $J = [F(-\frac{1}{b}), F(k_M)]$ is mapped into itself; otherwise $J = [F^2(k_M), F(k_M)]$ is mapped into itself by F.

Since every initial condition $k_0 \neq 0$ creates bounded trajectories converging to an attractor included into the trapping interval J, it can be noticed that if $F(k_M) \geq -\frac{1}{b}$, the flat branch of map F is involved. Moreover, since all the points mapped in R_2 have the same trajectory of point $F(-\frac{1}{b})$, then the attractor will be a cycle. The transition from $F(k_M) \geq -\frac{1}{b}$ to $F(k_M) < -\frac{1}{b}$ corresponds to a border collision bifurcation.

In order to describe the qualitative dynamics occurring on set J, we consider the situation in which $k^* \in R_1$ is locally stable (as in Fig. 4 (a)), for instance b is close to zero. Then, as b decreases, k^* becomes unstable via flip bifurcation and a period doubling route to chaos occurs till a border collision bifurcation emerges at $F(k_M) = -\frac{1}{b}$. This bifurcation occurs at $b = b_c$ and a point of the attractor of F collides with point P. In Fig. 4 (b) and (c) the situations immediately before and after the border collision bifurcation occurring at $b_c \simeq -0.315$ are presented. Notice that after this bifurcation the qualitative dynamics drastically changes, passing from a complex attractor to a locally stable 5-period cycle. The related bifurcation diagram is presented in Fig. 4 (d).

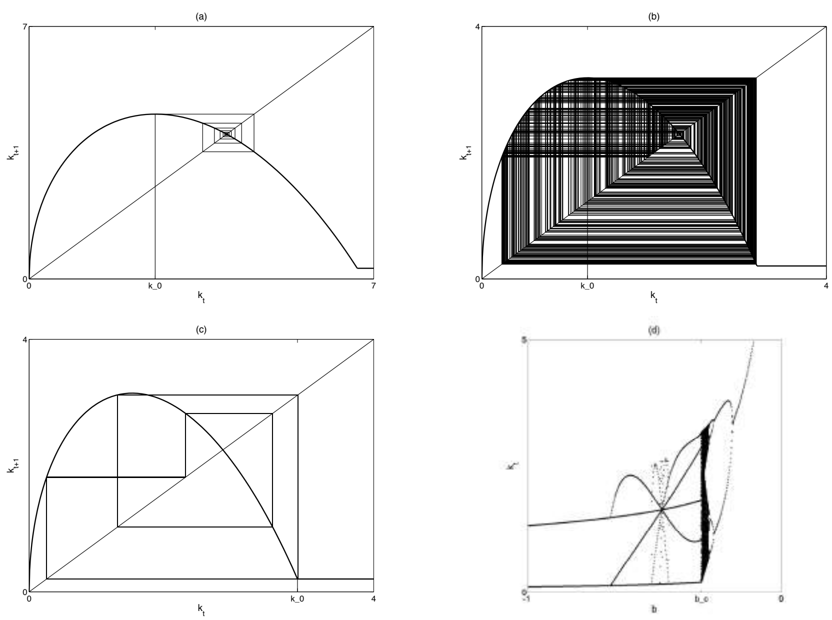

Fig. 4. $A = 10$, $a = 0.49$, $n = 0.1$, $s = 0.9$. (a) If $b = -0.15$ the fixed point is locally stable. (b) Situation before the border collision bifurcation, i.e. $b = -0.314$. (c) Situation immediately after the border collision bifurcation, i.e. $b = -0.316$. (d) Bifurcation diagram w.r.t. b.

As in Brianzoni *et al.*[2] if elasticity of substitution between production factors in less then one, then the system becomes more complex as b decreases since cycles or more complex features may be exhibited.

4 Conclusions

In this paper we considered the Diamond overlapping-generations model with the VES production function in the form given by Revankar[10]. We examined existence and stability conditions for steady state and the results of our analysis show that fluctuation or even chaotic patterns can be exhibited. As in Brianzoni *et al.*[2], cycles or complex dynamics can emerge if the elasticity of substitution between production factors is low enough. Moreover, unbounded endogenous growth can be observed. A new feature is due to the fact that, if elasticity of substitution is greater then one, then up to three fixed point can be exhibited.

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