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On a class of non-local phase-field models for tumor growth with possibly singular potentials, chemotaxis, and active transport

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Abstract

This paper provides a unified mathematical analysis of a family of non-local diffuse interface models for tumor growth describing evolutions driven by long-range interactions. These integro-partial differential equations model cell-to-cell adhesion by a non-local term and may be seen as non-local variants of the corresponding local model proposed by Garcke *et al* (2016). The model in consideration couples a non-local Cahn–Hilliard equation for the tumor phase variable with a reaction–diffusion equation for the nutrient concentration, and takes into account also significant mechanisms such as chemotaxis and active transport. The system depends on two relaxation parameters: a viscosity coefficient and parabolic-regularization coefficient on the chemical potential. The first part of the paper is devoted to the analysis of the system with both regularizations. Here, a rich spectrum of results is presented. Weak well-posedness is first addressed, also including singular potentials. Then, under suitable conditions, existence of strong solutions enjoying the separation property is proved. This allows also to obtain a refined stability estimate with respect to the data, including both chemotaxis and active transport. The second part of the paper is devoted to the study of the asymptotic behavior of the system as the relaxation parameters vanish. The asymptotics are analyzed when the parameters approach

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zero both separately and jointly, and exact error estimates are obtained. As a by-product, well-posedness of the corresponding limit systems is established.

1. Introduction

In the last decades, a vivid interest has been devoted to the challenging project of modeling tumor growth. The main responsible of deaths due to cancer is often the formation of metastases in the late stages of the pathology, when tumor cells spread also to separate parts of the host tissue and give rise to secondary tumor masses. Several clinical studies have confirmed that the primary mechanism leading to this process is identified in the ability of cells to invade adjacent tissues [81]. Invasion and metastasis have then deserved the unfortunate denomination of 'hallmarks of cancer' in [61]. Mathematical modeling has then become a fundamental tool in order to describe and possibly predict these underlying processes: validation, analysis, and simulation are crucial steps in the direction of designation of anti-tumoral therapies. Many mathematical models have been proposed to capture the complexity of the underlying biological and chemical phenomena: in this direction we refer to the seminal works [3, 6, 8, 9, 25, 37] and references therein.

From the continuum diffuse-interface approach to tumor growth, the cellular adhesion is introduced by embodying surface tension force at the tumor surface and this procedure has been successfully employed in many instances. In these models, the tumor evolution is described by introducing an order parameter φ, taking values between -1 and 1, and representing the local concentration of tumoral cells. The regions $\{\varphi = 1\}$ and $\{\varphi = -1\}$ represent the pure tumorous and healthy phases, respectively, whereas the diffuse interface $\{-1 < \varphi < 1\}$ models the narrow transition layer separating them. One of the major advantages of this modeling approach is that, unlike free boundary models, it takes into account also possible delicate behaviors such as topological changes in the tumorous regions, occurring for example during breakup and coalescence phenomena. The second main variable employed in the diffuse-interface description of tumor dynamics is the local concentration σ of a certain nutrient (e.g. oxygen, glucose), in which the tissue in consideration is embedded. The tumor is supposed to proliferate by absorption of such nutrient, and reversely the evolution of the nutrient is influenced by the consumption by the tumor cells. The key idea behind the diffuse-interface modeling consists then of a non-trivial coupling of a phase-field-type equation for φ, usually Cahn–Hilliard equation accounting for the phase segregation, with a reaction–diffusion equation for σ. The proliferation and coupling terms appearing in the system vary from model to model, and may take into account also further biological mechanisms exhibited by the tumor such as apoptosis, cell-to-cell adhesion, proliferation, chemotaxis, and active transport.

The classical local Cahn–Hilliard equation can be obtained as the conserved dynamics in the $(H^1)^*$-metric generated by the variational derivative of the Ginzburg–Landau free energy $\mathcal{E}_{\mathrm{loc}}$ with respect to the order parameter φ, where

$$\mathcal{E}_{\mathrm{loc}}(\varphi) := \int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi) \right).$$

Here, F is a so-called double-well potential, possessing two global minima, with typical choices being, in the order, the regular potential, the logarithmic potential and the double obstacle

potential defined as:

$$F_{\mathrm{pol}}(r) := \frac{1}{4}(r^2 - 1)^2, \quad r \in \mathbb{R}, \tag{1.1}$$

$$F_{\mathrm{log}}(r) := \frac{\theta}{2}\left[(1+r)\ln(1+r) + (1-r)\ln(1-r)\right] - \frac{\theta_0}{2}r^2, \quad r \in (-1,1), \quad 0 < \theta < \theta_0, \tag{1.2}$$

$$F_{\mathrm{dob}}(r) := \begin{cases} c(1 - r^2) & \text{if } r \in [-1,1], \\ +\infty & \text{otherwise,} \end{cases} \quad c > 0. \tag{1.3}$$

In the context of tumor growth, the energy $\mathcal{E}_{\mathrm{loc}}$ accounts for cell-to-cell adhesion. More specifically, the term in F models the fact that tumor cells prefer to adhere to each other rather than to non-tumor cells (hence the pure phases tend to concentrate), while the gradient term penalizes too scattered tumor patterns (hence high oscillations of φ). Despite the fact that phase segregation described by means of the local Cahn–Hilliard equation is widely accepted in literature, the local model is not effective in capturing cell-to-cell and cell-to-matrix adhesion phenomena driven by long-range competitions. In the context of tumor growth models, neglecting long-range interaction is an enormous drawback. Indeed, as we have pointed out above, the crucial biological process responsible of the evolution of the cancer diseases are tumor-cell invasion and the formation of metastases. The spread of secondary distant tumor masses is typically a long-range interaction process, and cannot be captured by means of the local modelling approach. One of the possible ways to include long-range competitions and make the model more accurate in describing cell-invasion and metastases-formation is to switch to a *non-local* model instead. This fact has been widely confirmed in the applied literature on biological engineering and applied analysis, for which we refer to the numerous contributions [2, 11–13, 36, 55, 74] and the references therein. In particular, the mentioned results and their subsequent developments agree that cell-adhesion is typically a non-local-in-space phenomenon, and represent then the crucial milestone for the validation and simulation of non-local models in the context of tumor growth dynamics.

In the framework of diffuse-interface modeling of tumor growth, long-range interactions can be incorporated by modifying the local energy $\mathcal{E}_{\mathrm{loc}}$ with a non-local one. By following the ground-breaking work done by Giacomin and Lebowitz on non-local Cahn–Hilliard equations [56–58] (see also [14, 38–40, 46]), we substitute the classical local Ginzburg–Landau free energy functional with the corresponding non-local Helmholtz free energy given by

$$\mathcal{E}_{\mathrm{nonloc}}(\varphi) := \frac{1}{4}\int_{\Omega \times \Omega} J(x-y)|\varphi(x) - \varphi(y)|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_{\Omega} F(\varphi).$$

Here J stands for a sufficiently fast decaying kernel, such as the classical Bessel or Newtonian potentials. Let us emphasize again that a non-local free energy as $\mathcal{E}_{\mathrm{nonloc}}$ is not motivated here by merely mathematical interests, but it has a fundamental meaning in the modeling of tumor growth: it previously appeared in [83] and is indeed crucial in describing the long-range interaction processes involved in cell-invasion that would otherwise be left out in a local model. For more details on the non-local Cahn–Hilliard equation we refer to the introduction of [46], where a rich description concerning the state of the art on the equation is performed. Besides, let us mention [24, 30, 59, 60] for mathematical results related to variations of the classical Cahn–Hilliard equations with possibly singular potentials, and to the recent works [27–29, 71] dealing with the asymptotic convergence of non-local Cahn–Hilliard equations to the respective local ones when the kernel suitably peaks around zero.

The goal of this paper is to introduce and investigate a class of non-local phase-field models for tumor growth inspired by the work by Garcke *et al* [54]. Let $\varepsilon, \tau \geqslant 0$, $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, and $T > 0$ a fixed final time horizon. We consider a two-parameter class of non-local models in the following form:

$$\varepsilon \partial_t \mu + \partial_t \varphi - \Delta \mu = (P\sigma - A)h(\varphi) \qquad \text{in } (0, T) \times \Omega, \qquad (1.4)$$

$$\mu = \tau \partial_t \varphi + a\varphi - J * \varphi + F'(\varphi) - \chi\sigma \qquad \text{in } (0, T) \times \Omega, \qquad (1.5)$$

$$\partial_t \sigma - \Delta\sigma + B(\sigma - \sigma_S) + C\sigma h(\varphi) = -\eta\Delta\varphi \quad \text{in } (0, T) \times \Omega, \qquad (1.6)$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}(\sigma - \eta\varphi) = 0 \qquad \text{on } (0, T) \times \partial\Omega, \qquad (1.7)$$

$$\varepsilon\mu(0) = \varepsilon\mu_0, \qquad \varphi(0) = \varphi_0, \qquad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \qquad (1.8)$$

Let us briefly review the role of the occurring symbols. The variable φ represents the difference in volume fractions between tumoral and healthy cells, with $\{\varphi = 1\}$ being the pure tumoral phase, and $\{\varphi = -1\}$ being the pure healthy phase. The variable μ is the chemical potential associated to φ, and σ represents the concentration of the unknown surrounding nutrient, with the following convention: $\sigma \simeq 1$ represents a rich nutrient concentration, whereas $\sigma \simeq 0$ a poor one. Furthermore, we indicate with $\mathbf{n}$ and $\partial_{\mathbf{n}}$ the normal vector and the corresponding directional derivative, J is a spatial convolution kernel, with $a := J * 1$, while F' represents the derivative of a double-well potential F. Precise assumptions are given in section 2 below.

The parameter $\tau \geqslant 0$ represents the viscosity coefficient associated to the Cahn–Hilliard equation, while $\varepsilon \geqslant 0$ is a relaxation coefficient providing a parabolic regularization on the chemical potential. The constants P, A, B, and C are fixed positive real numbers, taking into account the proliferation rate of tumoral cells by consumption of nutrient, the apoptosis rate, the consumption rate of the nutrient with respect to a pre-existing concentration σ_S, and the nutrient consumption rate, respectively. Moreover, χ and η are fixed non-negative constants, modeling the chemotaxis and active transport effects, respectively. For further insights concerning the modeling aspects, let us refer to [54] (see also [48, 49]), where the authors, after deriving some models from thermodynamic principles, underline how it is possible to decouple the chemotaxis by the active transport mechanism. It is worth mentioning that, at least formally, by setting $\varepsilon = \tau = 0$ and by substituting the non-locality $a\varphi - J*\varphi$ with the 'corresponding local term' $-\Delta\varphi$, we obtain exactly a particular case of the setting analyzed in [54]. Here, we highlight that Sherratt *et al* point out in [74] that cell adhesion is intrinsically a non-local in space phenomenon, whereas the chemotaxis mechanism is on the other hand of local nature (see [65]). This motivates the medical and modeling relevance of system (1.4)–(1.8) in which a non-local term is considered in equation (1.5), capturing long-term interaction processes occurring in cell-invasion, but still keeping the terms related to chemotaxis of local nature in (1.5) and (1.6). For a situation in which non-local chemotaxis is addressed, we refer to [7].

Let us comment further on the structure of the parameters in system (1.4)–(1.8). As far as the coefficients are concerned, one can identify two main classes. The parameters P, A, B, C, χ, η are structural coefficients of the model itself: they arise directly from the practical description of the tumor dynamics, and each one is linked to an exact undergoing biological process. For example, P and A take into account proliferation and death of tumor cells, B and C calibrate diffusion of nutrient with respect to a pre-existing concentration σ_S, and, more importantly, χ and η render the tendency of tumor cells to attract nutrient and to move towards regions with high levels of nutrient, respectively. The second group of parameters, namely ε and τ, are connected on the other hand to specific mathematical regularizations of the model, and have

to be considered as small perturbations acting on the original limit system (i.e., (1.4)–(1.8) with $\varepsilon = \tau = 0$). In this perspective, it is of upmost importance to stress that the introduction of ε and τ *is not* aimed at a mere mathematical technical virtuosity, but is finalized instead to the inclusion in the model of specific biological/physical mechanisms that are relevant to the tumor growth description. For instance, the coefficient $\varepsilon > 0$ is necessary in order to deal with possible singular potentials F due to the presence of a mass source in the Cahn–Hilliard equation. As such, bearing in mind that singular potentials are actually more relevant in phase-segregation, the choice of analyzing the case $\varepsilon > 0$ has then to be interpreted as an additional possibility to include thermodynamically–relevant potentials in the analysis, and not as a blunt mathematical exercise. In the same spirit, the introduction of the parameter τ in the model is not end in itself, but is aimed at keeping the relevant cross-diffusion mechanisms of chemotaxis and active transport, which otherwise could not be covered by the model. This being stressed, the regularized system (1.4)–(1.8) that we propose is purposely very general, in order to provide a larger variety of frameworks that could be covered by the model and that could adapt to different practical scenarios.

Up to the author's knowledge, there are still few contributions devoted to the mathematical analysis of non-local tumor growth models: we recognize [42, 44, 45]. By contrast, the local situation has been the subject of intensive studies. At first, let us point out some models which neglect velocity contribution which are somehow variations of the model introduced by Hawkins-Daarud *et al* in [63] (see also [62, 64]). In this direction, we mention [41], where the well-posedness of the system is shown under general polynomial growth type assumptions for the involved potentials. In [15] (along with the related works [17, 18]), the authors consider some regularized version compared to [41], by adding the same regularization that we have introduced here on the viscosity and the dissipation of the chemical potential. Owing to these terms the authors were able to extend the setting of some analytic results including in the investigation also singular and possibly non-regular potentials like the logarithmic (1.2) or the double obstacle one (1.3). Moreover, the authors established in which sense it is possible to let these regularizations parameters to zero, recovering some of the results already proved in [41]: in this sense, our work is somehow inspired by these contributions. Let us also refer to [21, 22], where a similar investigation was performed for fractional models. Furthermore, in order to better emulate *in-vivo* tumor-growth, other authors have proposed to include fluid motion by further coupling previous systems with a velocity law of Darcy's or Brinkmann's-type; we refer in particular to [1, 31, 32, 35, 45, 47–50, 54, 66, 83]. We point out the recent work [53] (see also [69, 70]) wrote by the second-named author in collaboration with Garcke and Lam, where elasticity effects are taken into account as physical evidence have shown that the presence of the extracellular matrix or rigid bone can assert significant influences on tumor proliferation. For multi-species tumor growth models, we point out [26, 43, 51].

Moreover, a wide number of results concerning further analysis on these models have been performed. In this direction, we mention the optimal control problems addressed by [16, 19, 20, 33, 34, 52, 67, 82]. In particular, we mention [23] which deals with the optimal control problem for the corresponding local version of system (1.4)–(1.8), and we also point out [20, 75–79], where similar relaxed models have been investigated from the optimal control viewpoint. Let us also point out [10, 15, 72], where some long-time behavior for similar models is addressed. To conclude the overview, let us mention the work [73], where a phase-field model for tumor growth has been analyzed also taking into account possible stochastic perturbations of the system. The paper, written by the first-named author in collaboration with Orrieri and Rocca, focuses on well-posedness and optimal control of treatment when two Wiener-type noises act on the proliferation of tumor cells and evolution of nutrient.

Let us present now the main results of the present paper.

The first part of the work is devoted to the analysis of the system (1.4)–(1.8) when both regularizations are present, i.e. with $\varepsilon, \tau > 0$. In this setting, we first investigate existence of weak solutions, even when singular potentials as (1.2) or (1.3) are present, also including chemotaxis and active transport. Secondly, we show that without active transport (i.e. $\eta = 0$) continuous dependence on the data (hence uniqueness) holds for weak solutions. Furthermore, we investigate regularity properties of the solutions, and prove existence of strong solutions as well as separation results from the potential barriers. For strong solutions, we are finally able to refine the stability estimate with respect to the data, also including the case of chemotaxis and active transport.

The second part of the work is focused on the study of the asymptotic behavior of the system as $\varepsilon \searrow 0$ and/or $\tau \searrow 0$. These are performed both separately (i.e. $\varepsilon \searrow 0$ with $\tau > 0$, and $\tau \searrow 0$ with $\varepsilon > 0$) and jointly (i.e. $\varepsilon, \tau \searrow 0$). In each of these cases, under suitable conditions we are able to show convergence of the system to the respective limit problem, hence also the corresponding well-posedness. Also, we give the exact rates of convergence through precise error estimates.

Let us briefly mention here the mathematical challenges that we have to overcome in these asymptotics.

Passage to the limit as $\varepsilon \searrow 0$. In this first asymptotic study the parabolic regularization on μ is 'removed', resulting in lack of regularity on the chemical potential. As a consequence, due to the presence of proliferation terms in the Cahn–Hilliard equation, a very natural growth condition on the potential has to be required (cf (2.22)), allowing for any polynomial or first-order exponential potentials. The passage to the limit, hence the existence for the limit problem with $\varepsilon = 0$, is proved in the setting of no active transport term (i.e. $\eta = 0$), due to the need of a maximum principle argument for σ. As for the error estimate (and therefore the uniqueness for the limit system), a rate of convergence of order $\varepsilon^{1/4}$ is obtained by showing refined estimates on the solutions and exploiting a locally-Lipschitz assumption on the potential (still including the classical case (1.1) for example).

Passage to the limit as $\tau \searrow 0$. In the second passage to the limit, the viscosity of the Cahn–Hilliard equation vanishes, and this results is a loss of regularity on the phase-variable. The presence of $\varepsilon > 0$ still allows passing to the limit in very general settings, such as singular potentials, chemotaxis, and active transport, only requiring some compatibility conditions (smallness-type assumptions) on the constants. The separation from the potential barriers is not preserved though, as it is naturally expectable. Moreover, a corresponding error estimate showing a convergence rate of order $\tau^{1/2}$ is obtained (and therefore the uniqueness for the limit system).

Passage to the limit as $\varepsilon, \tau \searrow 0$. In the last passage to the limit, the parameters ε and τ vanish simultaneously. Here, the convergence is proved by proving some refined estimates on the solutions, depending on both parameters, and combining the assumptions above on the potential and the coefficients. Moreover, the error estimate (and the resulting well-posedness of the limit problem) is obtained with a rate of convergence of $\varepsilon^{1/4} + \tau^{1/2}$, under a suitable scaling on the two parameters.

The plan of the rest of the paper is as follows. In section 2 we set our notation and present the obtained results. The weak and strong well-posedness of (1.4)–(1.8) for $\varepsilon, \tau > 0$ is addressed in section 3. Then, sections 4–6 are completely devoted to the asymptotic analysis of the system as ε and τ approach zero, first separately and then jointly.

2. Setting, assumptions, and main results

Throughout the paper, $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain and $T > 0$ is a fixed final time. We set for convenience the spatiotemporal cylinders

$$Q := (0, T) \times \Omega, \qquad \Sigma := (0, T) \times \partial\Omega, \qquad Q_t := (0, t) \times \Omega, \quad t \in (0, T),$$

and we introduce the functional spaces

$$H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad W := \left\{ y \in H^2(\Omega) : \partial_{\mathbf{n}} y = 0 \text{ a.e. on } \partial\Omega \right\}$$

endowed with their natural norms $\|\cdot\|_H$, $\|\cdot\|_V$, and $\|\cdot\|_W$, respectively. Likewise, we use $\|\cdot\|_p$ to indicate the standard norm of the space $L^p(\Omega)$, for all $p \in [1, \infty]$. As usual, H is identified with its dual H^* through its Riesz isomorphism, so that

$$W \hookrightarrow V \hookrightarrow H \simeq H^* \hookrightarrow V^* \hookrightarrow W^*,$$

where all inclusions are dense, continuous, and compact. The duality pairing between V^* and V, and the scalar product in H will be denoted by the symbols $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$, respectively.

Moreover, for every $v \in V^*$ we set $v_\Omega := \frac{1}{|\Omega|} \langle v, 1 \rangle$ for the generalised mean value of v. Let us also recall the Poincaré–Wirtinger inequality

$$\|v\|_V^2 \leqslant C_\Omega \left(\|\nabla v\|_H^2 + |v_\Omega|^2 \right), \quad \forall\, v \in V, \tag{2.1}$$

where the constant $C_\Omega > 0$ depends only on Ω. Let us recall that the Laplace operator with homogeneous Neumann conditions may be seen as a variational operator

$$-\Delta : V \to V^*, \qquad \langle -\Delta v, \zeta \rangle := \int_\Omega \nabla v \cdot \nabla \zeta, \quad \forall\, v, \zeta \in V.$$

It is well know, as a consequence of the Poincaré–Wirtinger inequality (2.1), that the restriction of $-\Delta$ to the subspace of null-mean elements of V is injective, and that it possesses a well defined inverse

$$\mathcal{N} : \{v^* \in V^* : v^*_\Omega = 0\} \to \{v \in V : v_\Omega = 0\}.$$

Lastly, let $\mathcal{R} = I - \Delta : V \to V^*$ be the Riesz isomorphism of V, i.e. the map

$$\langle \mathcal{R}u, v \rangle := \int_\Omega (\nabla u \cdot \nabla v + uv), \quad \forall\, u, v \in V.$$

It is well-known that $\mathcal{R}_{|W}$ yields an isomorphism from W to H with well-defined inverse $\mathcal{R}^{-1} : H \to W$. In addition, for all $v \in V$, and $v^*, w^* \in V^*$, the following properties hold

$$\langle \mathcal{R}v, \mathcal{R}^{-1}v^* \rangle = \langle v^*, v \rangle, \qquad \langle v^*, \mathcal{R}^{-1}w^* \rangle = (v^*, w^*)_*,$$

where the symbol $(\cdot, \cdot)_*$ denotes the inner product of V^*. Furthermore, for every $f \in V$ it holds that

$$\|f\|_H^2 = \langle f, f \rangle = \langle \mathcal{R}f, \mathcal{R}^{-1}f \rangle \leqslant \|f\|_V \|\mathcal{R}^{-1}f\|_V \leqslant \|f\|_V \|f\|_{V^*}.$$

Besides, if $v^* \in H^1(0, T; V^*)$, we have for a.e. $t \in (0, T)$ that

$$\langle \partial_t v^*(t), \mathcal{R}^{-1}v^*(t) \rangle = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|v^*(t)\|_*^2.$$

The following structural assumptions on the data will be in order in the paper.

A1 P, A, B, C, χ, η are non-negative constants.

A2 $h : \mathbb{R} \to [0, +\infty)$ is bounded and Lipschitz continuous.

A3 $\sigma_S \in L^\infty(Q)$ and

$$0 \leqslant \sigma_S(t, x) \leqslant 1 \quad \text{for a.e.} (t, x) \in Q.$$

A4 $F := F_1 + F_2 \geqslant 0$, where

$$F_1 : \mathbb{R} \to [0, +\infty] \quad \text{is proper, convex, and lower semicontinuous,}$$

and

$$F_2 \in C^1(\mathbb{R}), \qquad F_2' : \mathbb{R} \to \mathbb{R} \quad \text{is Lipschitz continuous,} \qquad F_2'(0) = 0.$$

In particular, the subdifferential $\partial F_1 : \mathbb{R} \to 2^{\mathbb{R}}$ is well defined in the sense of convex analysis, and we assume that $0 \in \partial F_1(0)$. The Moreau regularization of F_1 and the Yosida approximation of ∂F_1 are defined, respectively, as

$$F_{1,\lambda} : \mathbb{R} \to [0, +\infty), \qquad F_{1,\lambda}(r) := F_1(0) + \int_0^r F_{1,\lambda}'(s) \, ds, \quad r \in \mathbb{R},$$

and

$$F_{1,\lambda}' : \mathbb{R} \to \mathbb{R}, \qquad F_{1,\lambda}' := \frac{I - (I + \lambda \partial F_1)^{-1}}{\lambda}, \quad \lambda > 0,$$

where I stands for the identity operator. We recall that $F_{1,\lambda}'$ is $\frac{1}{\lambda}$-Lipschitz continuous and we set

$$F_\lambda := F_{1,\lambda} + F_2.$$

A5 The kernel $J \in W_{\text{loc}}^{1,1}(\mathbb{R}^3)$ is such that $J(x) = J(-x)$ for a.e. $x \in \mathbb{R}^3$. For any measurable $v : \Omega \to \mathbb{R}$ we use the notation

$$(J * v)(x) := \int_\Omega J(x - y) v(y) \, dy, \quad x \in \Omega,$$

and set $a := J * 1$. Moreover, we assume that

$$a_* := \inf_{x \in \Omega} \int_\Omega J(x - y) \, dy = \inf_{x \in \Omega} a(x) \geqslant 0,$$

$$a^* := \sup_{x \in \Omega} \int_\Omega |J(x - y)| \, dy < +\infty, \qquad b^* := \sup_{x \in \Omega} \int_\Omega |\nabla J(x - y)| \, dy < +\infty,$$

and we set $c_a := \max\{a^* - a_*, 1\}$. Finally, we suppose that there exists a positive constant C_0 such that

$$a_* + \frac{w_1 - w_2}{r_1 - r_2} \geqslant C_0, \quad \forall\, r_i \in D(\partial F_1),\ \forall\, w_i \in \partial F_1(r_i) + F_2'(r_i),$$

$$i = 1, 2,\ r_1 \neq r_2.$$

Note that if F is of class C^2, the last condition is equivalent to the classical one

$$a_* + F''(r) \geqslant C_0 \quad \forall\, r \in D(F'),$$

where $D(F')$ denotes the domain of F'.

For convenience, we introduce the following upper bounds for the coefficients ε and τ

$$\varepsilon_0 := \min\left\{ \frac{1}{4c_a},\ \frac{1}{\max\{1, a^* - \min\{a^*, C_0\}\}},\ \frac{2C_0}{3(a^* + b^*)^2 K_0^2} \right\}, \quad \tau_0 := 1,$$

where K_0 denotes the norm of the continuous inclusion $H \hookrightarrow V^*$. This is only a technical requirement on the coefficients, which is clearly not restrictive as ε and τ have to be considered as small perturbations.

The first main result deals with existence of global weak solutions to the system (1.4)–(1.7) under very general assumptions on the data. In particular, any type of potential as in (1.1)–(1.3) is included in this first result.

Theorem 2.1 (Existence of weak solutions: $\varepsilon, \tau > 0$). *Assume* **A1–A5***, and let* $\varepsilon \in (0, \varepsilon_0)$ *and* $\tau \in (0, \tau_0)$. *Moreover, let the triple of initial data* $(\varphi_0, \mu_0, \sigma_0)$ *satisfy*

$$\varphi_0 \in V, \qquad F(\varphi_0) \in L^1(\Omega), \qquad \mu_0, \sigma_0 \in H. \tag{2.2}$$

Then, there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$ *such that*

$$\varphi \in H^1(0, T; H) \cap L^\infty(0, T; V), \tag{2.3}$$

$$\mu, \sigma \in H^1(0, T; V^*) \cap L^2(0, T; V), \tag{2.4}$$

$$\xi \in L^2(0, T; H), \tag{2.5}$$

where

$$\mu = \tau \partial_t \varphi + a\varphi - J * \varphi + \xi + F_2'(\varphi) - \chi\sigma, \quad \xi \in \partial F_1(\varphi) \quad a.e.\ in\ Q, \tag{2.6}$$

with $\varphi(0) = \varphi_0$, $\mu(0) = \mu_0$, $\sigma(0) = \sigma_0$ *in H, and such that*

$$\langle \partial_t(\varepsilon\mu + \varphi), \zeta \rangle + \int_\Omega \nabla\mu \cdot \nabla\zeta = \int_\Omega (P\sigma - A)h(\varphi)\zeta, \tag{2.7}$$

$$\langle \partial_t\sigma, \zeta \rangle + \int_\Omega \nabla\sigma \cdot \nabla\zeta + \int_\Omega (B(\sigma - \sigma_S) + Ch(\varphi))\,\zeta = \eta \int_\Omega \nabla\varphi \cdot \nabla\zeta, \tag{2.8}$$

for every $\zeta \in V$, *almost everywhere in* $(0, T)$. *Furthermore, if* $\eta = 0$ *and*

$$0 \leqslant \sigma_0(x) \leqslant 1 \quad for\ a.e.\,x \in \Omega, \tag{2.9}$$

then $\sigma(t) \in L^\infty(\Omega)$ *for all* $t \in [0, T]$ *and*

$$0 \leqslant \sigma(t, x) \leqslant 1 \quad for\ a.e.\,x \in \Omega, \quad \forall\, t \in [0, T]. \tag{2.10}$$

It is worth mentioning that, in the case of singular potentials such as (1.2) and (1.3), the assumption $F(\varphi_0) \in L^1(\Omega)$ entails that $\varphi_0 \in L^\infty(\Omega)$ and that $|\varphi_0(x)| \leqslant 1$ for almost every $x \in \Omega$.

The second result concerns continuous dependence of the data for weak solutions. This result applies again to any choice of the potential F, but we are forced (so far) to restrict ourselves to the case without active transport (i.e. $\eta = 0$).

Theorem 2.2 (Continuous dependence: $\varepsilon, \tau > 0$). *Assume* **A1**–**A5***, and let $\eta = 0$, $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in (0, \tau_0)$. Then there exists a constant $K > 0$ independent of τ such that, for any pair of initial data $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_i$, $i = 1, 2$, satisfying (2.2) and (2.9), and for any respective solutions $\{(\varphi_i, \mu_i, \sigma_i, \xi_i)\}_i$, $i = 1, 2$, satisfying (2.3)–(2.10), it holds that*

$$\|(\varepsilon\mu_1 + \varphi_1) - (\varepsilon\mu_2 + \varphi_2)\|_{L^\infty(0,T;V^*)} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)}$$

$$+ \tau^{1/2}\|\varphi_1 - \varphi_2\|_{C^0([0,T];H)} + \|\varphi_1 - \varphi_2\|_{L^2(0,T;H)}$$

$$+ \|\sigma_1 - \sigma_2\|_{C^0([0,T];H) \cap L^2(0,T;V)}$$

$$\leqslant K\left(\left\|(\varepsilon\mu_0^1 + \varphi_0^1) - (\varepsilon\mu_0^2 + \varphi_0^2)\right\|_{V^*} + \tau^{1/2}\left\|\varphi_0^1 - \varphi_0^2\right\|_H + \left\|\sigma_0^1 - \sigma_0^2\right\|_H\right). \quad (2.11)$$

As a consequence of the above result, we infer the uniqueness of the weak solution obtained in theorem 2.1 under the only additional requirement that $\eta = 0$. The next result deals with the regularity of weak solutions with respect to the data.

Theorem 2.3 (Regularity: $\varepsilon, \tau > 0$). *Assume* **A1**–**A5***, $\varepsilon \in (0, \varepsilon_0)$, and $\tau \in (0, \tau_0)$. Moreover, let the triple of initial data $(\varphi_0, \mu_0, \sigma_0)$ satisfy (2.2) and also*

$$\exists\, \xi_0 \in H : \xi_0 \in \partial F_1(\varphi_0) \quad a.e.\ in\ \Omega, \qquad \mu_0, \sigma_0 \in V, \quad (2.12)$$

and suppose that $t = 0$ is a Lebesgue point for σ_S with

$$\sigma_S(0) \in H. \quad (2.13)$$

Then, the solution $(\varphi, \mu, \sigma, \xi)$ to (2.3)–(2.8) given by theorem 2.1 satisfies

$$\varphi \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \quad (2.14)$$

$$\mu, \sigma - \eta\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad (2.15)$$

$$\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V). \quad (2.16)$$

Our next result is concerned with the separation property, magnitude regularity, and existence of strong solutions. In this direction, we postulate the following assumptions for F and J.

A6 Setting $(-\ell, \ell) := \operatorname{Int} D(\partial F_1)$, with $\ell \in [0, +\infty]$, we assume that

$$F \in C^4(-\ell, \ell), \qquad \lim_{r \to (\pm\ell)^\mp} \left[F'(r) - \chi\eta r\right] = \pm\infty.$$

It is worth pointing out that **A6** excludes potentials F of double-obstacle type as in (1.3). Nevertheless, the logarithmic potential (1.2) and any polynomial super-quadratic potential as (1.1) is allowed. Let us also remark that assuming the effective domain of ∂F_1 to be symmetric with respect to zero is mainly a matter of convenience, so to allow (1.1) and (1.2) to be included. In general, the symmetry condition for the domain of ∂F_1 it is not necessary from the analysis viewpoint, and one can always reconstruct this situation by renormalization of F.

As for the kernel, a natural requirement from the analytical point of view is to require

$$J \in W^{2,1}(\mathcal{B}_R), \quad \text{where } \mathcal{B}_R := \{x \in \mathbb{R}^3 : |x| < R := \operatorname{diam}(\Omega)\}, \quad R > 0. \quad (2.17)$$

However, this condition prevents some relevant cases of kernels such as the Newtonian or the Bessel potential from being considered. Following the ideas of [38, 46] (see also [5, definition 1]), it is possible to cover also these situations by replacing the above condition by assuming that J is *admissible* in the following sense.

Definition 2.4. A convolution kernel $J \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)$ is admissible if it satisfies:

- $J \in C^3(\mathbb{R}^3 \setminus \{0\})$.
- J is radially symmetric, i.e. $J(\cdot) = \widetilde{J}(|\cdot|)$ for a non-increasing $\widetilde{J} : \mathbb{R}_+ \to \mathbb{R}$.
- There exists $R_0 > 0$ such that $r \mapsto \widetilde{J}''(r)$ and $r \mapsto \widetilde{J}'(r)/r$ are monotone on $(0, R_0)$.
- There exists $C_d > 0$ such that $|D^3 J(x)| \leqslant C_d |x|^{-4}$ for every $x \in \mathbb{R}^3 \setminus \{0\}$.

Thus, we require

A7 J satisfies (2.17) or it is admissible in the sense of definition 2.4.

Theorem 2.5 (Existence of strong solutions, separation property: $\varepsilon, \tau > 0$).
Assume conditions **A1**–**A7**, *and let* $\varepsilon \in (0, \varepsilon_0)$ *and* $\tau \in (0, \tau_0)$. *Let the initial data* $(\varphi_0, \mu_0, \sigma_0)$ *satisfy* (2.2), (2.12), *and also*

$$\varphi_0 \in H^2(\Omega), \qquad \mu_0, \sigma_0 \in L^\infty(\Omega), \quad \exists\, r_0 \in (0, \ell): \ \|\varphi_0\|_{L^\infty(\Omega)} \leqslant r_0. \tag{2.18}$$

Then, the solution $(\varphi, \mu, \sigma, \xi)$ *to* (2.3)–(2.8) *given by theorems* 2.1 *and* 2.3 *satisfies*

$$\varphi \in W^{1,\infty}(0, T; V) \cap H^1(0, T; H^2(\Omega)), \qquad \partial_t \varphi \in L^\infty(Q),$$

$$\eta \varphi \in L^2(0, T; W), \tag{2.19}$$

$$\exists\, r^* \in (r_0, \ell): \quad \sup_{t \in [0,T]} \|\varphi(t)\|_{L^\infty(\Omega)} \leqslant r^*, \tag{2.20}$$

$$\mu, \sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty(Q). \tag{2.21}$$

In particular, equations (1.4)–(1.6) *hold almost everywhere in Q.*

Remark 2.6. (i) Note that the equation (1.5) at time 0 reads

$$\mu_0 = \tau \varphi'_0 + a\varphi_0 - J * \varphi_0 + F'(\varphi_0) - \chi\sigma_0,$$

where φ'_0 'represents' the initial value of the time-derivative of φ. Under the assumptions (2.2), (2.12), and (2.18) we have that $\varphi'_0 \in V \cap L^\infty(\Omega)$, hence the improved regularities $\partial_t \varphi \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$ and $\partial_t \varphi \in L^\infty(Q)$ obtained in theorem 2.5 are naturally expectable.

(ii) Let us point out that (2.18), in the case (1.2), prevents the initial tumor distribution φ_0 to possess any region occupied by solely tumorous cells as $r_0 < \ell$.

Relying on the extra-regularity and the separation property, we are able to show a refined continuous dependence result for strong solutions, where the stability estimates are verified in stronger topologies. Let us stress that in this case we are able to cover also the scenarios of chemotaxis and active transport, complementing thus the previous theorem 2.2.

Theorem 2.7 (Refined continuous dependence: $\varepsilon, \tau > 0$). *Assume* **A1**–**A7**, *let* $\varepsilon \in (0, \varepsilon_0)$, *and* $\tau \in (0, \tau_0)$. *Then for any pair of initial data* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_i$, $i = 1, 2$, *satisfying* (2.2), (2.12), *and* (2.18), *there exists a constant* $K > 0$ *such that, for any respective solutions* $\{(\varphi_i, \mu_i, \sigma_i, \xi_i)\}_i$, $i = 1, 2$, *satisfying* (2.3)–(2.8) *and* (2.19)–(2.21), *it holds that*

$$\|\mu_1 - \mu_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\varphi_1 - \varphi_2\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))}$$

$$+ \|\sigma_1 - \sigma_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)}$$

$$\leqslant K \left(\|\mu_0^1 - \mu_0^2\|_V + \|\varphi_0^1 - \varphi_0^2\|_{H^2(\Omega)} + \|\sigma_0^1 - \sigma_0^2\|_V \right),$$

where K *only depends on* $\Omega, T, \varepsilon, \tau, P, A, B, C, C_0, a_*, a^*, b^*, r^*, \|F\|_{C^4([-r^*,r^*])}$ *and* $\{(\varphi_0^i, \mu_0^i, \sigma_0^i)\}_{i=1,2}$.

In particular, under the assumptions (2.2), (2.12), and (2.18) on the data, we deduce that the uniqueness of strong solutions in the sense of theorem 2.5 holds.

We now will present the results concerning the asymptotic analysis of (1.4)–(1.8) with respect to the parameters ε and τ. To begin with, we consider the case $\varepsilon \searrow 0$, assuming $\tau > 0$ to be fixed. In this direction, we need to enforce the conditions on the potential F. In fact, proceeding with classical estimates, just a bound of $\nabla\mu$ in $L^2(0,T;H)$ can be proved, having no information on the behavior of μ in $L^2(0,T;H)$. This gap is usually bridged via the application of a Poincaré-type inequality, which yields the control of μ in $L^2(0,T;V)$. To this end, some control on the spatial mean of μ is necessary: if $\varepsilon > 0$ is fixed, this follows automatically from the estimates, whereas in the limit $\varepsilon \searrow 0$ it has to be obtained from a suitable prescription on the potential. Namely, the assumption

$$D(\partial F_1) = \mathbb{R}, \quad \exists\, C_F > 0 : |\partial F_1^0(r)| \leqslant C_F(F_1(r) + 1) \quad \forall\, r \in \mathbb{R}, \tag{2.22}$$

have to be prescribed for F, where $\partial F_1^0(r)$ stands for the element of $\partial F_1(r)$ having minimum modulus. This implies that for every $z \in H$ and $w \in \partial F_1(z)$ it holds

$$\int_\Omega |w| \leqslant C_F \int_\Omega (F_1(z) + 1).$$

Let us point out that the above requirement is met by all the regular potentials, everywhere defined on the real line, with polynomial or first-order exponential growth-rate. The next two results deal with the asymptotic behavior as $\varepsilon \searrow 0$ and the respective error estimate: as a by-product, these yield existence and uniqueness of solutions, as well as continuous dependence on the data, for the system (1.4)–(1.8) with $\varepsilon = 0$.

Theorem 2.8 (Asymptotics: $\varepsilon \searrow 0$). *Assume* **A1**–**A5**, (2.22), *and let* $\tau \in (0, \tau_0)$, *and* $\eta = 0$. *Suppose also that*

$$\varphi_{0,\tau} \in V, \qquad F(\varphi_{0,\tau}) \in L^1(\Omega), \qquad \sigma_{0,\tau} \in H. \tag{2.23}$$

For every $\varepsilon \in (0, \varepsilon_0)$, *let the initial data* $(\varphi_{0,\varepsilon\tau}, \mu_{0,\varepsilon\tau}, \sigma_{0,\varepsilon\tau})$ *satisfy assumptions* (2.2) *and* (2.9), *and denote by* $(\varphi_{\varepsilon\tau}, \mu_{\varepsilon\tau}, \sigma_{\varepsilon\tau}, \xi_{\varepsilon\tau})$ *the respective unique weak solution to the system* (1.4)–(1.8) *obtained from theorem* 2.1. *In addition, we assume that, as* $\varepsilon \searrow 0$,

$$\varphi_{0,\varepsilon\tau} \rightharpoonup \varphi_{0,\tau} \quad \text{in } V, \qquad \sigma_{0,\varepsilon\tau} \to \sigma_{0,\tau} \quad \text{in } H, \tag{2.24}$$

and

$$\exists\, M_0 > 0 : \varepsilon^{1/2}\|\mu_{0,\varepsilon\tau}\|_H + \|F(\varphi_{0,\varepsilon\tau})\|_{L^1(\Omega)} \leqslant M_0 \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{2.25}$$

Then, there exists a quadruplet $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$, *with*

$$\varphi_\tau \in H^1(0,T;H) \cap L^\infty(0,T;V), \qquad \mu_\tau \in L^2(0,T;V),$$

$$\sigma_\tau \in H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q), \quad 0 \leqslant \sigma_\tau(t,x) \leqslant 1$$

$$\text{for a.e.} x \in \Omega, \quad \forall\, t \in [0,T],$$

$$\xi_\tau \in L^2(0,T;H),$$

such that

$$\langle \partial_t \varphi_\tau, \zeta \rangle + \int_\Omega \nabla \mu_\tau \cdot \nabla \zeta = \int_\Omega (P\sigma_\tau - A) h(\varphi_\tau) \zeta,$$

$$\langle \partial_t \sigma_\tau, \zeta \rangle + \int_\Omega \nabla \sigma_\tau \cdot \nabla \zeta + B \int_\Omega (\sigma_\tau - \sigma_S) \zeta + C \int_\Omega \sigma_\tau h(\varphi_\tau) \zeta = 0,$$

for every $\zeta \in V$, almost everywhere in $(0,T)$, and

$$\mu_\tau = \tau \partial_t \varphi_\tau + a\varphi_\tau - J * \varphi_\tau + \xi_\tau + F_2'(\varphi_\tau) - \chi \sigma_\tau, \quad \xi_\tau \in \partial F_1(\varphi_\tau) \quad \text{a.e. in } Q,$$

$$\varphi_\tau(0) = \varphi_{0,\tau}, \qquad \sigma_\tau(0) = \sigma_{0,\tau} \quad \text{a.e. in } \Omega.$$

Moreover, as $\varepsilon \searrow 0$, along a non-relabelled subsequence it holds that

$$\varphi_{\varepsilon\tau} \overset{*}{\rightharpoonup} \varphi_\tau \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{2.26}$$

$$\mu_{\varepsilon\tau} \rightharpoonup \mu_\tau \quad \text{in } L^2(0,T;V), \tag{2.27}$$

$$\sigma_{\varepsilon\tau} \overset{*}{\rightharpoonup} \sigma_\tau \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q), \tag{2.28}$$

$$\xi_{\varepsilon\tau} \rightharpoonup \xi_\tau \quad \text{in } L^2(0,T;H), \tag{2.29}$$

$$\varepsilon\mu_{\varepsilon\tau} \to 0 \quad \text{in } C^0([0,T];H) \cap L^2(0,T;V), \tag{2.30}$$

hence in particular that

$$\varphi_{\varepsilon\tau} \to \varphi_\tau \quad \text{in } C^0([0,T];H), \qquad \sigma_{\varepsilon\tau} \to \sigma_\tau \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H). \tag{2.31}$$

Theorem 2.9 (Error estimate: $\varepsilon \searrow 0$). *In the setting of theorem* 2.8*, if also*

$$F \in C^1(\mathbb{R}), \qquad |F'(r) - F'(s)| \leqslant C_F(1 + |r|^2 + |s|^2)|r-s| \quad \forall\, r,s \in \mathbb{R}, \tag{2.32}$$

then the solution $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$ to the system (1.4)–(1.8) *with $\varepsilon = 0$ is unique. Moreover, suppose that there exists $M_0 > 0$ such that*

$$\varepsilon^{1/4} \left(\|\mu_{0,\varepsilon\tau}\|_V + \|\sigma_{0,\varepsilon\tau}\|_V + \|F'(\varphi_{0,\varepsilon\tau})\|_H \right) \leqslant M_0 \quad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{2.33}$$

Then, the convergences obtained in theorem 2.8 *hold along the entire sequence $\varepsilon \searrow 0$, and there exists $K_\tau > 0$, independent of ε, such that the following error estimate holds:*

$$\|\varphi_{\varepsilon\tau} - \varphi_\tau\|_{L^\infty(0,T;H)} + \|\mu_{\varepsilon\tau} - \mu_\tau\|_{L^2(0,T;V)} + \|\sigma_{\varepsilon\tau} - \sigma_\tau\|_{L^\infty(0,T;H) \cap L^2(0,T;V)}$$

$$\leqslant K_\tau \left(\varepsilon^{1/4} + \|\varphi_{0,\varepsilon\tau} - \varphi_{0,\tau}\|_H + \|\sigma_{0,\varepsilon\tau} - \sigma_{0,\tau}\|_H \right).$$

Remark 2.10. Note that given $(\varphi_{0,\tau}, \sigma_{0,\tau})$ satisfying (2.23), a natural choice for the approximating sequence of initial data $(\varphi_{0,\varepsilon\tau}, \sigma_{0,\varepsilon\tau})$ satisfying (2.24), (2.25) and (2.33) is given by the solutions to the elliptic problems

$$\varphi_{0,\varepsilon\tau} + \varepsilon^{1/2}\mathcal{R}\varphi_{0,\varepsilon\tau} = \varphi_{0,\tau}, \qquad \sigma_{0,\varepsilon\tau} + \varepsilon^{1/2}\mathcal{R}\sigma_{0,\varepsilon\tau} = \sigma_{0,\tau}.$$

In this case, if for example $\sigma_{0,\tau} \in V$, it is immediate to check that

$$\|\varphi_{0,\varepsilon\tau} - \varphi_{0,\tau}\|_H + \|\sigma_{0,\varepsilon\tau} - \sigma_{0,\tau}\|_H \leqslant M_0\varepsilon^{1/4}$$

for a certain $M_0 > 0$, so that the rate of convergence given by theorem 2.9 is exactly $1/4$.

The second asymptotic study that we are going to address is the one as $\tau \searrow 0$, when $\varepsilon > 0$ is fixed. In this case, the presence of the parabolic regularization on μ provided by $\varepsilon > 0$ allows considering also very general potentials and to avoid assumptions as (2.22). The limit as $\tau \searrow 0$ corresponds instead to a vanishing viscosity argument on the system in consideration. We expect then to lose, at the limit $\tau = 0$, time regularity on the solutions, as well as the separation principle. The next two results deal with the asymptotic behavior as $\tau \searrow 0$ and the respective error estimate: again, as a by-product, these yield existence and uniqueness of solutions for the system (1.4)–(1.8) with $\tau = 0$.

Theorem 2.11 (Asymptotics: $\tau \searrow 0$). *Assume* **A1**–**A5**, *$\varepsilon \in (0, \varepsilon_0)$, and*

$$0 \leqslant \chi < \sqrt{c_a}, \qquad (\chi + \eta + 4c_a\chi)^2 < 8c_aC_0 + 4\chi\eta. \tag{2.34}$$

Moreover, let us suppose that

$$\varphi_{0,\varepsilon}, \mu_{0,\varepsilon}, \sigma_{0,\varepsilon} \in H, \qquad F(\varphi_{0,\varepsilon}) \in L^1(\Omega). \tag{2.35}$$

For every $\tau \in (0, \tau_0)$, let the initial data $(\varphi_{0,\varepsilon\tau}, \mu_{0,\varepsilon\tau}, \sigma_{0,\varepsilon\tau})$ satisfy (2.2), and denote by $(\varphi_{\varepsilon\tau}, \mu_{\varepsilon\tau}, \sigma_{\varepsilon\tau}, \xi_{\varepsilon\tau})$ the corresponding weak solution to (1.4)–(1.8) obtained from theorem 2.1. Suppose also that, as $\tau \searrow 0$,

$$\varphi_{0,\varepsilon\tau} \to \varphi_{0,\varepsilon} \quad \text{in } H, \qquad \mu_{0,\varepsilon\tau} \to \mu_{0,\varepsilon} \quad \text{in } H, \qquad \sigma_{0,\varepsilon\tau} \to \sigma_{0,\varepsilon} \quad \text{in } H, \tag{2.36}$$

and

$$\exists\, M_0 > 0 : \tau^{1/2}\|\varphi_{0,\varepsilon\tau}\|_V + \|F(\varphi_{0,\varepsilon\tau})\|_{L^1(\Omega)} \leqslant M_0 \quad \forall\, \tau \in (0, \tau_0). \tag{2.37}$$

Then, there exists a quadruplet $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon, \xi_\varepsilon)$, with

$$\varphi_\varepsilon, \mu_\varepsilon \in L^\infty(0, T; H) \cap L^2(0, T; V), \qquad \varepsilon\mu_\varepsilon + \varphi_\varepsilon \in H^1(0, T; V^*) \cap L^2(0, T; V),$$

$$\sigma_\varepsilon \in H^1(0, T; V^*) \cap L^2(0, T; V), \qquad \xi_\varepsilon \in L^2(0, T; V),$$

such that

$$\langle \partial_t(\varepsilon\mu_\varepsilon + \varphi_\varepsilon), \zeta \rangle + \int_\Omega \nabla\mu_\varepsilon \cdot \nabla\zeta = \int_\Omega (P\sigma_\varepsilon - A)h(\varphi_\varepsilon)\zeta,$$

$$\langle \partial_t\sigma_\varepsilon, \zeta \rangle + \int_\Omega \nabla\sigma_\varepsilon \cdot \nabla\zeta + B\int_\Omega (\sigma_\varepsilon - \sigma_S)\zeta + C\int_\Omega \sigma_\varepsilon h(\varphi_\varepsilon)\zeta = \eta\int_\Omega \nabla\varphi_\varepsilon \cdot \nabla\zeta,$$

for every $\zeta \in V$, almost everywhere in $(0, T)$, and

$$\mu_\varepsilon = a\varphi_\varepsilon - J * \varphi_\varepsilon + \xi_\varepsilon + F_2'(\varphi_\varepsilon) - \chi\sigma_\varepsilon, \qquad \xi_\varepsilon \in \partial F_1(\varphi_\varepsilon) \quad a.e.\ in\ Q,$$

$$\varphi_\varepsilon(0) = \varphi_{0,\varepsilon}, \qquad \sigma_\varepsilon(0) = \sigma_{0,\varepsilon} \quad a.e.\ in\ \Omega.$$

Moreover, as $\tau \searrow 0$, along a non-relabelled subsequence it holds that

$$\varphi_{\varepsilon\tau} \xrightarrow{*} \varphi_\varepsilon \quad in\ L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.38}$$

$$\mu_{\varepsilon\tau} \xrightarrow{*} \mu_\varepsilon \quad in\ L^\infty(0,T;H) \cap L^2(0,T;V), \tag{2.39}$$

$$\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau} \rightharpoonup \varepsilon\mu_\varepsilon + \varphi_\varepsilon \quad in\ H^1(0,T;V^*) \cap L^2(0,T;V), \tag{2.40}$$

$$\sigma_{\varepsilon\tau} \rightharpoonup \sigma_\varepsilon \quad in\ H^1(0,T;V^*) \cap L^2(0,T;V), \tag{2.41}$$

$$\xi_{\varepsilon\tau} \rightharpoonup \xi_\varepsilon \quad in\ L^2(0,T;H), \tag{2.42}$$

$$\tau\varphi_{\varepsilon\tau} \to 0 \quad in\ H^1(0,T;H) \cap L^\infty(0,T;V), \tag{2.43}$$

and also that

$$\varphi_{\varepsilon\tau} \to \varphi_\varepsilon \quad in\ L^2(0,T;H), \qquad \mu_{\varepsilon\tau} \to \mu_\varepsilon \quad in\ L^2(0,T;H), \tag{2.44}$$

$$\sigma_{\varepsilon\tau} \to \sigma_\varepsilon \quad in\ C^0([0,T];V^*) \cap L^2(0,T;H). \tag{2.45}$$

Furthermore, if $\eta = 0$ and $\sigma_{0,\varepsilon\tau}$ satisfies (2.9) for all $\tau > 0$, then the limit σ_ε satisfies (2.10) as well, and

$$\sigma_{\varepsilon\tau} \xrightarrow{*} \sigma_\varepsilon \quad in\ L^\infty(Q).$$

Theorem 2.12 (Error estimate: $\tau \searrow 0$). *In the setting of theorem 2.11, suppose that $\eta = 0$. Then the solution $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon, \xi_\varepsilon)$ to the system (1.4)–(1.8) with $\tau = 0$ is unique, the convergences obtained in theorem 2.11 hold along the entire sequence $\tau \searrow 0$, and there exists $K_\varepsilon > 0$, independent of τ, such that the following error estimate holds:*

$$\|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}) - (\varepsilon\mu_\varepsilon + \varphi_\varepsilon)\|_{L^\infty(0,T;V^*)} + \|\varphi_{\varepsilon\tau} - \varphi_\varepsilon\|_{L^2(0,T;H)}$$

$$+ \|\mu_{\varepsilon\tau} - \mu_\varepsilon\|_{L^2(0,T;H)} + \|\sigma_{\varepsilon\tau} - \sigma_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$

$$\leqslant K_\varepsilon \left(\tau^{1/2} + \|(\varepsilon\mu_{0,\varepsilon\tau} + \varphi_{0,\varepsilon\tau}) - (\varepsilon\mu_{0,\varepsilon} + \varphi_{0,\varepsilon})\|_{V^*} + \|\sigma_{0,\varepsilon\tau} - \sigma_{0,\varepsilon}\|_H\right).$$

Remark 2.13. Note that given $(\varphi_{0,\varepsilon}, \mu_{0,\varepsilon}, \sigma_{0,\varepsilon})$ satisfying (2.35), a natural choice for the approximating sequence $(\varphi_{0,\varepsilon\tau}, \mu_{0,\varepsilon\tau}, \sigma_{0,\varepsilon\tau})$ is given by the solutions to the elliptic problems

$$\varphi_{0,\varepsilon\tau} + \tau\mathcal{R}\varphi_{0,\varepsilon\tau} = \varphi_{0,\varepsilon}, \qquad \mu_{0,\varepsilon\tau} + \tau\mathcal{R}\mu_{0,\varepsilon\tau} = \mu_{0,\varepsilon}, \qquad \sigma_{0,\varepsilon\tau} + \tau\mathcal{R}\sigma_{0,\varepsilon\tau} = \sigma_{0,\varepsilon}.$$

In such a case, hypotheses (2.36) and (2.37) are readily satisfied. Moreover, if for example $\varphi_{0,\varepsilon}, \mu_{0,\varepsilon}, \tau_{0,\varepsilon} \in V$, it is immediate to check that, there is $M_0 > 0$, independent of τ, such that

$$\|\varphi_{0,\varepsilon\tau} - \varphi_{0,\varepsilon}\|_H + \|\mu_{0,\varepsilon\tau} - \mu_{0,\varepsilon}\|_H + \|\sigma_{0,\varepsilon\tau} - \sigma_{0,\varepsilon}\|_H \leqslant M_0\tau^{1/2},$$

so that the rate of convergence given by theorem 2.12 is exactly $1/2$.

The last two results we present deal with the asymptotic study of the system (1.4)–(1.8) as the parameters ε and τ go to 0 simultaneously. Again, as a by-product, these yield existence and uniqueness of solutions for the limit system (1.4)–(1.8) with $\varepsilon = \tau = 0$.

Theorem 2.14 (Asymptotics: $\varepsilon, \tau \searrow 0$). *Assume* **A1**–**A5**, (2.22), (2.34), $\eta = 0$, *and suppose that*

$$\varphi_0, \sigma_0 \in H, \qquad F(\varphi_0) \in L^1(\Omega). \tag{2.46}$$

For every $\varepsilon \in (0, \varepsilon_0)$ and $\tau \in (0, \tau_0)$, let the initial data $(\varphi_{0,\varepsilon\tau}, \mu_{0,\varepsilon\tau}, \sigma_{0,\varepsilon\tau})$ satisfy (2.2) and (2.9), and denote by $(\varphi_{\varepsilon\tau}, \mu_{\varepsilon\tau}, \sigma_{\varepsilon\tau}, \xi_{\varepsilon\tau})$ the respective unique weak solution to the system (1.4)–(1.8) obtained from theorem 2.1. Suppose also that, as $(\varepsilon, \tau) \to (0, 0)$,

$$\varphi_{0,\varepsilon\tau} \to \varphi_0 \quad \text{in } H, \qquad \sigma_{0,\varepsilon\tau} \to \sigma_0 \quad \text{in } H, \tag{2.47}$$

and that there exists $M_0 > 0$ such that

$$\tau^{1/2}\|\varphi_{0,\varepsilon\tau}\|_V + \varepsilon^{1/2}\|\mu_{0,\varepsilon\tau}\|_H + \|F(\varphi_{0,\varepsilon\tau})\|_{L^1(\Omega)} \leqslant M_0 \quad \forall\, (\varepsilon, \tau) \in (0, \varepsilon_0) \times (0, \tau_0). \tag{2.48}$$

Then, there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$, with

$$\varphi \in H^1(0, T; V^*) \cap L^2(0, T; V),$$

$$\mu = a\varphi - J * \varphi + \xi + F_2'(\varphi) - \chi\sigma \in L^2(0, T; V),$$

$$\sigma \in H^1(0, T; V^*) \cap L^2(0, T; V) \cap L^\infty(Q),$$

$$0 \leqslant \sigma(t, x) \leqslant 1 \quad \text{for a.e. } x \in \Omega, \quad \forall\, t \in [0, T],$$

$$\xi \in L^2(0, T; V), \qquad \xi \in \partial F_1(\varphi) \quad \text{a.e. in } Q,$$

$$\varphi(0) = \varphi_0, \qquad \sigma(0) = \sigma_0 \quad \text{a.e. in } \Omega,$$

such that, for every $\zeta \in V$, almost everywhere in $(0, T)$, it holds

$$\langle \partial_t \varphi, \zeta \rangle + \int_\Omega \nabla\mu \cdot \nabla\zeta = \int_\Omega (P\sigma - A)h(\varphi)\zeta,$$

$$\langle \partial_t \sigma, \zeta \rangle + \int_\Omega \nabla\sigma \cdot \nabla\zeta + B\int_\Omega (\sigma - \sigma_S)\zeta + C\int_\Omega \sigma h(\varphi)\zeta = 0.$$

Moreover, as $(\varepsilon, \tau) \to (0, 0)$, along a non-relabelled subsequence it holds that

$$\varphi_{\varepsilon\tau} \overset{*}{\rightharpoonup} \varphi \quad \text{in } L^\infty(0, T; H) \cap L^2(0, T; V), \tag{2.49}$$

$$\mu_{\varepsilon\tau} \rightharpoonup \mu \quad \text{in } L^2(0, T; V), \tag{2.50}$$

$$\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau} \rightharpoonup \varphi \quad \text{in } H^1(0, T; V^*) \cap L^2(0, T; V), \tag{2.51}$$

$$\sigma_{\varepsilon\tau} \overset{*}{\rightharpoonup} \sigma \quad \text{in } H^1(0, T; V^*) \cap L^2(0, T; V) \cap L^\infty(Q), \tag{2.52}$$

$$\varepsilon\mu_{\varepsilon\tau} \to 0 \quad \text{in } C^0([0, T]; H) \cap L^2(0, T; V), \tag{2.53}$$

$$\tau\varphi_{\varepsilon\tau} \to 0 \quad \text{in } H^1(0, T; H) \cap L^\infty(0, T; V), \tag{2.54}$$

hence in particular that

$$\varphi_{\varepsilon\tau} \to \varphi \quad \text{in } L^2(0,T;H), \qquad \sigma_{\varepsilon\tau} \to \sigma \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H). \quad (2.55)$$

Theorem 2.15 (Error estimate: $\varepsilon, \tau \searrow 0$). *In the setting of theorem* 2.14*, assume* (2.32)*, and suppose also that there exist constants $c_F, M_0 > 0$ such that*

$$F(r) \geqslant c_F |r|^4 - c_F^{-1} \quad \forall\, r \in \mathbb{R}. \quad (2.56)$$

Then, the solution $(\varphi, \mu, \sigma, \xi)$ to the system (1.4)–(1.8) *with $\varepsilon = \tau = 0$ is unique. Moreover, let the approximating initial data verify* (2.48) *and, for every $(\varepsilon, \tau) \in (0, \varepsilon_0) \times (0, \tau_0)$,*

$$\frac{\varepsilon^{1/4}}{\tau^{1/2}} \left(\|\mu_{0,\varepsilon\tau}\|_H + \|F'(\varphi_{0,\varepsilon\tau})\|_H \right) + \varepsilon^{1/4} \left(\|\mu_{0,\varepsilon\tau}\|_V + \|\sigma_{0,\varepsilon\tau}\|_V \right) \leqslant M_0. \quad (2.57)$$

Then, the convergences obtained in theorem 2.14 *hold along every subsequence $(\varepsilon_k, \tau_k)_k$ satisfying*

$$\limsup_{k \to \infty} \frac{\varepsilon_k^{1/2}}{\tau_k} < +\infty, \quad (2.58)$$

and in this case, there exists $K > 0$, independent of k, such that the following error estimate holds:

$$\|\varphi_{\varepsilon_k\tau_k} - \varphi\|_{C^0([0,T];V^*) \cap L^2(0,T;H)} + \|\sigma_{\varepsilon_k\tau_k} - \sigma\|_{C^0([0,T];H) \cap L^2(0,T;V)}$$
$$\leqslant K \left(\varepsilon_k^{1/4} + \tau_k^{1/2} + \|\varphi_{0,\varepsilon_k\tau_k} - \varphi_0\|_{V^*} + \|\sigma_{0,\varepsilon_k\tau_k} - \sigma_0\|_H \right).$$

Throughout the paper we convey to use the symbol M to indicate constants depending only on structural data. So, its meaning may change from line to line without further comments. Moreover, we will sometimes add a self-explanatory subscript to stress its possible dependence.

3. Analysis of the system with $\varepsilon, \tau > 0$

This section is devoted to the proof of the results concerning the behavior of the system with $\varepsilon, \tau > 0$, namely the existence of weak solutions contained in theorem 2.1, the continuous dependence result contained in theorem 2.2, the regularity property of theorem 2.3, the existence of strong solution and separation in theorem 2.5, and the refined continuous dependence result in theorem 2.7. Let us recall that throughout this section $\varepsilon, \tau > 0$ are fixed.

3.1. The approximation

To prove the existence of solutions we rely on an approximation procedure based on the two parameters $n \in \mathbb{N}$ and $\lambda > 0$, involving a Faedo–Galerkin approximation on the functional spaces and the Yosida approximation on the potential (cf **A4**), respectively.

Let $(e_j)_{j \in \mathbb{N}}$ and $(l_j)_{j \in \mathbb{N}}$ be the sequences of eigenfunctions and eigenvalues of the operator $-\Delta$ with homogeneous Neumann conditions, renormalized in such a way that $\|e_j\|_H = 1$ for all $j \in \mathbb{N}$. Then it is well known that $(e_j)_j$ is a complete orthonormal system in H, and orthogonal in V. For every $n \in \mathbb{N}$, let $\mathcal{W}_n := \text{span}\{e_1, \ldots, e_n\}$, and define $\Pi_n : H \to \mathcal{W}_n$ as the orthogonal projection on $\mathcal{W}_n$ with respect to the scalar product of H. Then, as $n \to \infty$, it holds that $\Pi_n v \to v$ in H (resp. V or W) for every $v \in H$ (resp. V or W). We consider the following approximated

problem: we then consider the following approximated system: find a triplet $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ such that

$$\varepsilon\partial_t\mu_{\lambda,n} + \partial_t\varphi_{\lambda,n} - \Delta\mu_{\lambda,n} = \Pi_n[(P\sigma_{\lambda,n} - A)h(\varphi_{\lambda,n})] \qquad \text{in } Q, \qquad (3.1)$$

$$\mu_{\lambda,n} = \tau\partial_t\varphi_{\lambda,n} + a\varphi_{\lambda,n} - J * \varphi_{\lambda,n} + \Pi_n F'_\lambda(\varphi_{\lambda,n}) - \chi\sigma_{\lambda,n} \qquad \text{in } Q, \qquad (3.2)$$

$$\partial_t\sigma_{\lambda,n} - \Delta\sigma_{\lambda,n} + B(\sigma_{\lambda,n} - \sigma_{S,n}) + \Pi_n[C\sigma_{\lambda,n}h(\varphi_{\lambda,n})] = -\eta\Delta\varphi_{\lambda,n} \quad \text{in } Q, \qquad (3.3)$$

$$\partial_{\mathbf{n}}\mu_{\lambda,n} = \partial_{\mathbf{n}}(\sigma_{\lambda,n} - \eta\varphi_{\lambda,n}) = 0 \qquad \text{on } \Sigma, \qquad (3.4)$$

$$\mu_{\lambda,n}(0) = \Pi_n\mu_0, \qquad \varphi_{\lambda,n}(0) = \Pi_n\varphi_0, \qquad \sigma_{\lambda,n}(0) = \Pi_n\sigma_0 \qquad \text{in } \Omega, \qquad (3.5)$$

where $\sigma_{S,n} := \Pi_n\sigma_S$, in the form

$$\varphi_{\lambda,n}(t,x) := \sum_{j=1}^n \alpha_j^{\lambda,n}(t)e_j(x), \qquad \mu_{\lambda,n}(t,x) := \sum_{j=1}^n \beta_j^{\lambda,n}(t)e_j(x),$$

$$\sigma_{\lambda,n}(t,x) := \sum_{j=1}^n \gamma_j^{\lambda,n}(t)e_j(x),$$

for $t \in [0,T]$, $x \in \Omega$, and $j \in \{1,\ldots,n\}$. Moreover, let us introduce the vectors

$$\boldsymbol{\alpha}^{\lambda,n}, \boldsymbol{\beta}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n} : [0,T] \to \mathbb{R}^n,$$

by

$$\boldsymbol{\alpha}^{\lambda,n} := (\alpha_1^{\lambda,n},\ldots,\alpha_n^{\lambda,n})^{\mathrm{T}}, \qquad \boldsymbol{\beta}^{\lambda,n} := (\beta_1^{\lambda,n},\ldots,\beta_n^{\lambda,n})^{\mathrm{T}}, \qquad \boldsymbol{\gamma}^{\lambda,n} := (\gamma_1^{\lambda,n},\ldots,\gamma_n^{\lambda,n})^{\mathrm{T}}.$$

Plugging these expression in (3.1)–(3.5) and taking arbitrary $e_i \in \mathcal{W}_n$ as test functions, for $i = 1,\ldots,n$, we deduce that $(\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n})$ solves the approximated system if and only if $(\boldsymbol{\alpha}^{\lambda,n}, \boldsymbol{\beta}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n})$ solves the following system of ODEs, for $i = 1,\ldots,n$:

$$\varepsilon\partial_t\beta_i^{\lambda,n} + \partial_t\alpha_i^{\lambda,n} + l_i\beta_i^{\lambda,n} = \int_\Omega \left(P\sum_{j=1}^n \gamma_j^{\lambda,n}e_j - A\right) h\left(\sum_{j=1}^n \alpha_j^{\lambda,n}e_j\right) e_i,$$

$$\beta_i^{\lambda,n} = \tau\partial_t\alpha_i^{\lambda,n} + \sum_{j=1}^n \alpha_j^{\lambda,n}\int_\Omega ae_je_i - \sum_{j=1}^n \alpha_j^{\lambda,n}\int_\Omega (J * e_j)e_i$$

$$+ \int_\Omega F'_\lambda\left(\sum_{j=1}^n \alpha_j^{\lambda,n}e_j\right) e_i - \chi\gamma_i^{\lambda,n}, \quad \partial_t\gamma_i^{\lambda,n} + l_i\gamma_i^{\lambda,n} + B\left(\gamma_i^{\lambda,n} - \int_\Omega \sigma_{S,n}e_i\right)$$

$$+ C\sum_{j=1}^n \gamma_j^{\lambda,n}\int_\Omega h\left(\sum_{m=1}^n \alpha_m^{\lambda,n}e_m\right) e_je_i = \eta l_i\alpha_i^{\lambda,n},$$

$$\alpha_i^{\lambda,n}(0) = (\varphi_0, e_i)_H, \quad \beta_i^{\lambda,n}(0) = (\mu_0, e_i)_H, \quad \gamma_i^{\lambda,n}(0) = (\sigma_0, e_i)_H.$$

Since $h, F'_\lambda : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and h is bounded, such initial value system can be written in the form

$$\begin{cases} \partial_t(\boldsymbol{\alpha}^{\lambda,n}, \boldsymbol{\beta}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n}) &= g_{\lambda,n}(\boldsymbol{\alpha}^{\lambda,n}, \boldsymbol{\beta}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n}), \\ (\boldsymbol{\alpha}^{\lambda,n}, \boldsymbol{\beta}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n})(0) &= ((\varphi_0, e_i)_H, (\mu_0, e_i)_H, (\sigma_0, e_i)_H), \end{cases}$$

where $g_{\lambda,n} : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ is locally Lipschitz continuous and linearly bounded. Hence, by the Cauchy–Peano theorem, the system above admits a unique global solution $\boldsymbol{\alpha}^{\lambda,n}, \boldsymbol{\beta}^{\lambda,n}, \boldsymbol{\gamma}^{\lambda,n} \in C^1([0, T]; \mathbb{R}^n)$, implying that

$$\varphi_{\lambda,n}, \mu_{\lambda,n}, \sigma_{\lambda,n} \in C^1([0, T]; \mathcal{W}_n)$$

are the unique solutions to the approximated problem (3.1)–(3.5).

3.2. Uniform estimates

We prove uniform estimates independent of λ and n, still keeping $\varepsilon, \tau > 0$ fixed.

Testing (3.1) by $\mu_{\lambda,n}$, (3.2) by $-\partial_t\varphi_{\lambda,n}$, (3.3) by $\sigma_{\lambda,n}$, taking the sum and integrating over $(0, t)$, yields by symmetry of the kernel J, for every $t \in [0, T]$,

$$\frac{\varepsilon}{2}\|\mu_{\lambda,n}(t)\|_H^2 + \int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + \tau\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2 + \frac{1}{4}\int_{\Omega\times\Omega}J(x-y)|\varphi_{\lambda,n}(t,x) - \varphi_{\lambda,n}(y,t)|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_\Omega F_\lambda(\varphi_{\lambda,n}(t)) + \frac{1}{2}\|\sigma_{\lambda,n}(t)\|_H^2$$

$$+ \int_{Q_t}|\nabla\sigma_{\lambda,n}|^2 + \int_{Q_t}\left(B + Ch(\varphi_{\lambda,n})\right)|\sigma_{\lambda,n}|^2$$

$$= \frac{\varepsilon}{2}\|\Pi_n\mu_0\|_H^2 + \frac{1}{4}\int_{\Omega\times\Omega}J(x-y)|\Pi_n\varphi_0(x) - \Pi_n\varphi_0(y)|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \int_\Omega F_\lambda(\Pi_n\varphi_0) + \frac{1}{2}\|\Pi_n\sigma_0\|_H^2 + \int_{Q_t}(P\sigma_{\lambda,n} - A)h(\varphi_{\lambda,n})\mu_{\lambda,n}$$

$$+ \chi\int_{Q_t}\sigma_{\lambda,n}\partial_t\varphi_{\lambda,n} + B\int_{Q_t}\sigma_{S,n}\sigma_{\lambda,n} + \eta\int_{Q_t}\nabla\varphi_{\lambda,n}\cdot\nabla\sigma_{\lambda,n}.$$

Now, note that by assumption **A5** we have

$$\frac{1}{4}\int_{\Omega\times\Omega}J(x-y)|\varphi_{\lambda,n}(t,x) - \varphi_{\lambda,n}(t,y)|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\int_\Omega\left[a(x)|\varphi_{\lambda,n}|^2 - (J*\varphi_{\lambda,n})\varphi_{\lambda,n}\right](t,x)\,\mathrm{d}x$$

$$\geqslant \frac{a_*}{2}\|\varphi_{\lambda,n}(t)\|_H^2 - \frac{1}{2}\|J*\varphi_{\lambda,n}(t)\|_H\|\varphi_{\lambda,n}(t)\|_H \geqslant \frac{a_* - a^*}{2}\|\varphi_{\lambda,n}(t)\|_H^2, \quad (3.6)$$

and similarly

$$\frac{1}{4}\int_{\Omega\times\Omega}J(x-y)|\Pi_n\varphi_0(x) - \Pi_n\varphi_0(y)|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\int_\Omega\left[a|\Pi_n\varphi_0|^2 - (J*\Pi_n\varphi_0)\Pi_n\varphi_0\right](x)\,\mathrm{d}x$$

$$\leqslant \frac{a^* + a^*}{2}\|\Pi_n\varphi_0\|_H^2 \leqslant a^*\|\varphi_0\|_H^2.$$

Using that $F_\lambda \geqslant 0$, (3.6) along with the definition of c_a, recalling also that h is non-negative and bounded and that Π_n is a contraction on H, owing to the Young inequality we infer that

$$\frac{\varepsilon}{2}\|\mu_{\lambda,n}(t)\|_H^2 + \int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + \tau\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2 + \frac{1}{2}\|\sigma_{\lambda,n}(t)\|_H^2 + \int_{Q_t}|\nabla\sigma_{\lambda,n}|^2$$

$$\leqslant \frac{\varepsilon}{2}\|\mu_0\|_H^2 + a^*\|\varphi_0\|_H^2 + \|F_\lambda(\Pi_n\varphi_0)\|_{L^1(\Omega)} + \frac{1}{2}\|\sigma_0\|_H^2 + \frac{c_a}{2}\|\varphi_{\lambda,n}(t)\|_H^2$$

$$+ \int_{Q_t}(P\sigma_{\lambda,n} - A)h(\varphi_{\lambda,n})\mu_{\lambda,n} + \frac{1}{4}\int_{Q_t}|\sigma_{\lambda,n}|^2 + |Q|B^2\|\sigma_{S,n}\|_{L^\infty(Q)}^2$$

$$+ \chi\int_{Q_t}\sigma_{\lambda,n}\partial_t\varphi_{\lambda,n} + \eta\int_{Q_t}\nabla\varphi_{\lambda,n}\cdot\nabla\sigma_{\lambda,n}. \tag{3.7}$$

Here, we recall that $a^* - a_* \geqslant 0$ which entails that $c_a = \max\{a^* - a_*, 1\} > 0$. Then, we test equation (3.1) by $4c_a(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})$ and (3.2) by $-4c_a\Delta\varphi_{\lambda,n}$, add the resulting equalities and integrate over $(0,t)$ and by parts, getting, thanks to assumption **A5**,

$$2c_a\|(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})(t)\|_H^2 + 4c_a\varepsilon\int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + 2c_a\tau\|\nabla\varphi_{\lambda,n}(t)\|_H^2$$

$$+ 4c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2 \leqslant 2c_a\|\Pi_n(\varepsilon\mu_0 + \varphi_0)\|_H^2 + 2c_a\tau\|\nabla\Pi_n\varphi_0\|_H^2$$

$$+ 4c_a\int_{Q_t}(P\sigma_{\lambda,n} - A)h(\varphi_{\lambda,n})(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})$$

$$+ 4c_a\chi\int_{Q_t}\nabla\sigma_{\lambda,n}\cdot\nabla\varphi_{\lambda,n} + 8c_ab^*\|\varphi_{\lambda,n}\|_{L^2(Q_t)}\|\nabla\varphi_{\lambda,n}\|_{L^2(Q_t)},$$

from which we infer, thanks to the Young inequality and the boundedness of h, that

$$2c_a\|(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})(t)\|_H^2 + 4c_a\varepsilon\int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + 2c_a\tau\|\nabla\varphi_{\lambda,n}(t)\|_H^2$$

$$+ 2c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2 \leqslant 4c_a\varepsilon^2\|\mu_0\|_H^2 + 4c_a\|\varphi_0\|_H^2 + 2c_a\tau\|\nabla\varphi_0\|_H^2 + 4c_a\chi\int_{Q_t}\nabla\sigma_{\lambda,n}\cdot\nabla\varphi_{\lambda,n}$$

$$+ M\left(1 + \int_{Q_t}|\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n}|^2 + \int_{Q_t}|\varphi_{\lambda,n}|^2 + \int_{Q_t}|\sigma_{\lambda,n}|^2\right). \tag{3.8}$$

for a constant $M > 0$, independent of λ, n, ε, and τ. Summing (3.7) and (3.8), we infer that, possibly updating M,

$$\frac{\varepsilon}{2}\|\mu_{\lambda,n}(t)\|_H^2 + (1 + 4c_a\varepsilon)\int_{Q_t}|\nabla\mu_{\lambda,n}|^2 + \tau\int_{Q_t}|\partial_t\varphi_{\lambda,n}|^2 + \frac{1}{2}\|\sigma_{\lambda,n}(t)\|_H^2$$

$$+ \int_{Q_t}|\nabla\sigma_{\lambda,n}|^2 + 2c_a\|(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})(t)\|_H^2 + 2c_a\tau\|\nabla\varphi_{\lambda,n}(t)\|_H^2$$

$$+ 2c_aC_0\int_{Q_t}|\nabla\varphi_{\lambda,n}|^2 \leqslant \left(\frac{\varepsilon}{2} + 4c_a\varepsilon^2\right)\|\mu_0\|_H^2 + (a^* + 4c_a)\|\varphi_0\|_H^2 + 2c_a\tau\|\nabla\varphi_0\|_H^2$$

$$+ \|F_\lambda(\Pi_n\varphi_0)\|_{L^1(\Omega)} + \frac{1}{2}\|\sigma_0\|_H^2 + \frac{c_a}{2}\|\varphi_{\lambda,n}(t)\|_H^2 + \chi\int_{Q_t}\sigma_{\lambda,n}\partial_t\varphi_{\lambda,n}$$

$$+ (\eta + 4c_a\chi) \int_{Q_t} \nabla\sigma_{\lambda,n} \cdot \nabla\varphi_{\lambda,n} + M\left(1 + \int_{Q_t} |\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n}|^2\right.$$

$$\left.+ \int_{Q_t} |\varphi_{\lambda,n}|^2 + \int_{Q_t} |\sigma_{\lambda,n}|^2\right) + \int_{Q_t} (P\sigma_{\lambda,n} - A)h(\varphi_{\lambda,n})\mu_{\lambda,n}. \tag{3.9}$$

Note that

$$\frac{c_a}{2}\|\varphi_{\lambda,n}(t)\|_H^2 \leqslant c_a\|(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})(t)\|_H^2 + c_a\varepsilon^2\|\mu_{\lambda,n}(t)\|_H^2,$$

where the two terms on the right-hand side can be incorporated in the left-hand side of (3.9) as $2c_a - c_a = c_a > 0$ and $\frac{\varepsilon}{2} - c_a\varepsilon^2 \geqslant \frac{\varepsilon}{4}$ (since $\varepsilon \in (0, \frac{1}{4c_a})$). Furthermore, using the Young inequality we have

$$\chi \int_{Q_t} \sigma_{\lambda,n}\partial_t\varphi_{\lambda,n} + (\eta + 4c_a\chi)\int_{Q_t} \nabla\sigma_{\lambda,n} \cdot \nabla\varphi_{\lambda,n}$$

$$\leqslant \frac{\tau}{2}\int_{Q_t} |\partial_t\varphi_{\lambda,n}|^2 + \frac{\chi^2}{2\tau}\int_{Q_t} |\sigma_{\lambda,n}|^2 + \frac{1}{2}\int_{Q_t} |\nabla\sigma_{\lambda,n}|^2 + \frac{(\eta + 4c_a\chi)^2}{2}\int_{Q_t} |\nabla\varphi_{\lambda,n}|^2.$$

Collecting the above estimates, we infer that

$$\frac{\varepsilon}{4}\|\mu_{\lambda,n}(t)\|_H^2 + (1 + 4c_a\varepsilon)\int_{Q_t} |\nabla\mu_{\lambda,n}|^2 + \frac{\tau}{2}\int_{Q_t} |\partial_t\varphi_{\lambda,n}|^2$$

$$+ \frac{1}{2}\|\sigma_{\lambda,n}(t)\|_H^2 + \int_{Q_t} |\nabla\sigma_{\lambda,n}|^2 + c_a\|(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})(t)\|_H^2$$

$$+ 2c_a\tau\|\nabla\varphi_{\lambda,n}(t)\|_H^2 + 2c_a C_0\int_{Q_t} |\nabla\varphi_{\lambda,n}|^2$$

$$\leqslant \frac{3}{2}\varepsilon\|\mu_0\|_H^2 + (a^* + 4c_a)\|\varphi_0\|_H^2 + 2c_a\tau\|\nabla\varphi_0\|_H^2 + \|F_\lambda(\Pi_n\varphi_0)\|_{L^1(\Omega)}$$

$$+ \frac{1}{2}\|\sigma_0\|_H^2 + M\left(1 + \int_{Q_t} |\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n}|^2 + \int_{Q_t} |\varphi_{\lambda,n}|^2 + \int_{Q_t} |\sigma_{\lambda,n}|^2\right)$$

$$+ \frac{\chi^2}{2\tau}\int_{Q_t} |\sigma_{\lambda,n}|^2 + \frac{1}{2}\int_{Q_t} |\nabla\sigma_{\lambda,n}|^2 + \frac{(\eta + 4c_a\chi)^2}{2}\int_{Q_t} |\nabla\varphi_{\lambda,n}|^2$$

$$+ \int_{Q_t} (P\sigma_{\lambda,n} - A)h(\varphi_{\lambda,n})\mu_{\lambda,n}. \tag{3.10}$$

Moreover, the last term on the right-hand side can be easily bounded owing to Young's inequality.

Then, we fix $\lambda > 0$, and since F_λ has at most quadratic growth (depending on λ) and $\varphi_0 \in H$, we have that $\|F_\lambda(\Pi_n\varphi_0)\|_{L^1(\Omega)} \leqslant M_\lambda$ uniformly in $n \in \mathbb{N}$, for a certain $M_\lambda > 0$ independent of n. Therefore, Gronwall's lemma yields that

$$\|\mu_{\lambda,n}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}^2 + \|\varphi_{\lambda,n}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}^2 + \|\sigma_{\lambda,n}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}^2 \leqslant M_\lambda,$$

$$\tag{3.11}$$

where the constant M_λ is independent of n (but not of τ and ε). Furthermore, by comparison in equations (3.1) and (3.3), we deduce that

$$\|\partial_t(\varepsilon\mu_{\lambda,n} + \varphi_{\lambda,n})\|^2_{L^2(0,T;V^*)} + \|\partial_t\mu_{\lambda,n}\|^2_{L^2(0,T;V^*)} + \|\partial_t\sigma_{\lambda,n}\|^2_{L^2(0,T;V^*)} \leqslant M_\lambda. \qquad (3.12)$$

3.3. Passage to the limit

We pass now to the limit, keeping $\varepsilon, \tau > 0$ fixed, first as $n \to \infty$ and then as $\lambda \searrow 0$. From the estimates (3.11)–(3.12) and the Aubin–Lions compactness theorems (see, e.g., [80, corollary 4]), we deduce that the exists a triplet $(\varphi_\lambda, \mu_\lambda, \sigma_\lambda)$, with

$$\varphi_\lambda \in H^1(0,T;H) \cap L^\infty(0,T;V), \qquad \mu_\lambda, \sigma_\lambda \in H^1(0,T;V^*) \cap L^2(0,T;V),$$

such that, as $n \to \infty$,

$$\varphi_{\lambda,n} \overset{*}{\rightharpoonup} \varphi_\lambda \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V), \qquad \varphi_{\lambda,n} \to \varphi_\lambda \quad \text{in } C^0([0,T];H),$$

$$\mu_{\lambda,n} \rightharpoonup \mu_\lambda \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V), \qquad \mu_{\lambda,n} \to \mu_\lambda \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H),$$

$$\sigma_{\lambda,n} \rightharpoonup \sigma_\lambda \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V), \qquad \sigma_{\lambda,n} \to \sigma_\lambda \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H).$$

Since F'_λ is Lipschitz continuous and h is Lipschitz continuous and bounded, it is a standard matter to pass the limit in the approximated problem (3.1)–(3.5) as $n \to \infty$ to obtain, for every test function $\zeta \in V$,

$$\langle \partial_t(\varepsilon\mu_\lambda + \varphi_\lambda), \zeta \rangle + \int_\Omega \nabla\mu_\lambda \cdot \nabla\zeta = \int_\Omega (P\sigma_\lambda - A)h(\varphi_\lambda)\zeta, \qquad (3.13)$$

$$\mu_\lambda = \tau\partial_t\varphi_\lambda + a\varphi_\lambda - J * \varphi_\lambda + F'_\lambda(\varphi_\lambda) - \chi\sigma_\lambda, \qquad (3.14)$$

$$\langle \partial_t\sigma_\lambda, \zeta \rangle + \int_\Omega \nabla\sigma_\lambda \cdot \nabla\zeta + \int_\Omega [B(\sigma_\lambda - \sigma_S) + C\sigma_\lambda h(\varphi_\lambda)]\zeta = \eta \int_\Omega \nabla\varphi_\lambda \cdot \nabla\zeta, \quad (3.15)$$

almost everywhere in $(0,T)$, and

$$\mu_\lambda(0) = \mu_0, \qquad \varphi_\lambda(0) = \varphi_0, \qquad \sigma_\lambda(0) = \sigma_0 \quad \text{a.e. in } \Omega \qquad (3.16)$$

meaning that $(\varphi_\lambda, \mu_\lambda, \sigma_\lambda)$ satisfy the analogous of conditions (2.6)–(2.8) at level λ.

Clearly, by weak lower semicontinuity of the norms and the convex integrands, passing to the lim inf as $n \to \infty$ in the estimates (3.11) and (3.12), and recalling that $F_\lambda \leqslant F$, we infer that there exists $M > 0$, independent of λ (but not of ε and τ), such that

$$\|\mu_\lambda\|^2_{H^1(0,T;V^*)\cap L^2(0,T;V)} + \|\varphi_\lambda\|^2_{H^1(0,T;H)\cap L^\infty(0,T;V)}$$

$$+ \|\sigma_\lambda\|^2_{H^1(0,T;V^*)\cap L^2(0,T;V)} \leqslant M. \qquad (3.17)$$

Furthermore, the estimate (3.17) readily implies, by comparison in (3.14), that

$$\left\|F'_{1,\lambda}(\varphi_\lambda)\right\|^2_{L^2(0,T;H)} \leqslant M. \qquad (3.18)$$

Hence, there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$, with

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

$$\mu, \sigma \in H^1(0,T;V^*) \cap L^2(0,T;V), \quad \xi \in L^2(0,T;H),$$

such that, as $\lambda \to 0$,

$$\varphi_\lambda \overset{*}{\rightharpoonup} \varphi \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V), \quad \varphi_\lambda \to \varphi \quad \text{in } C^0([0,T];H),$$

$$\mu_\lambda \rightharpoonup \mu \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V), \quad \mu_\lambda \to \mu \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H),$$

$$\sigma_\lambda \rightharpoonup \sigma \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V), \quad \sigma_\lambda \to \sigma \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H),$$

$$F'_{1,\lambda}(\varphi_\lambda) \rightharpoonup \xi \quad \text{in } L^2(0,T;H).$$

The graph convergence of $F'_{1,\lambda}$ to ∂F_1, as $\lambda \to 0$, implies that $\xi \in \partial F_1(\varphi)$ almost everywhere in Q. Moreover, by the Lipschitz continuity of F'_2 and h, and the boundedness of h, we have that

$$h(\varphi_\lambda) \to h(\varphi) \quad \text{in } L^p(Q) \quad \forall\, p \geqslant 1, \qquad F'_2(\varphi_\lambda) \to F'_2(\varphi) \quad \text{in } L^2(0,T;H).$$

Consequently, letting $\lambda \to 0$ in the variational formulation of (3.13)–(3.16), we obtain exactly (2.6)–(2.8) completing the proof concerning the existence of weak solutions in theorem 2.1.

3.4. Maximum principle for σ

We prove here the last assertion of theorem 2.1, concerning a maximum principle for σ under the additional requirement that $\eta = 0$. Testing equation (2.8) by $f_+(\sigma) := (\sigma - 1)_+$, we have

$$\frac{1}{2}\|f_+(\sigma(t))\|_H^2 + \int_{Q_t} f'_+(\sigma)|\nabla\sigma|^2 + B\int_{Q_t} f_+(\sigma)(\sigma - \sigma_S) + C\int_{Q_t} f_+(\sigma)\sigma h(\varphi) = 0,$$

where we have used the fact that $f_+(\sigma_0) = 0$. Since f_+ is non-decreasing and h is non-negative, we infer that the second and fourth terms on the left-hand side are non-negative so that

$$\frac{1}{2}\|f_+(\sigma(t))\|_H^2 + B\int_{Q_t} f_+(\sigma)(\sigma - \sigma_S) \leqslant 0. \tag{3.19}$$

Moreover, since $\sigma_S \leqslant 1$ by assumption **A3**, we have that

$$B\int_{Q_t} f_+(\sigma)(\sigma - \sigma_S) = B\int_{Q_t \cap \{\sigma > 1\}} (\sigma - 1)(\sigma - \sigma_S) \geqslant 0.$$

Therefore, coming back to (3.19), we realize that $f_+(\sigma(t)) = 0$ which gives us the upper bound $\sigma(t) \leqslant 1$ a.e. in Ω, for every $t \in [0,T]$, as desired. The lower inequality follows by a similar argument testing by $f_-(\sigma) := -\sigma_-$.

3.5. Continuous dependence

Let us prove here the continuous dependence of theorem 2.2. To begin with, bearing in mind the notation introduced in theorem 2.2, we set $\varphi := \varphi_1 - \varphi_2, \mu := \mu_1 - \mu_2, \sigma := \sigma_1 - \sigma_2, \xi := \xi_1 - \xi_2, \varphi_0 := \varphi_0^1 - \varphi_0^2, \mu_0 := \mu_0^1 - \mu_0^2, \sigma_0 := \sigma_0^1 - \sigma_0^2$. Then, we consider the difference of system (1.4)–(1.8) written for the two solutions to obtain

$$\varepsilon\partial_t\mu + \partial_t\varphi - \Delta\mu = P\sigma h(\varphi_1) + (P\sigma_2 - A)(h(\varphi_1) - h(\varphi_2)) \quad \text{in } Q, \tag{3.20}$$

$$\mu = \tau\partial_t\varphi + a\varphi - J * \varphi + \xi + F'_2(\varphi_1) - F'_2(\varphi_2) - \chi\sigma \quad \text{in } Q, \tag{3.21}$$

$$\partial_t\sigma - \Delta\sigma + B\sigma + C\sigma h(\varphi_1) = C\sigma_2(h(\varphi_2) - h(\varphi_1)) - \eta\Delta\varphi \quad \text{in } Q, \tag{3.22}$$

$$\partial_\mathbf{n}\mu = \partial_\mathbf{n}(\sigma - \eta\varphi) = 0 \quad \text{on } \Sigma, \tag{3.23}$$

$$\mu(0) = \mu_0, \qquad \varphi(0) = \varphi_0, \qquad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{3.24}$$

Next, we test the equation (3.20) by $\mathcal{R}^{-1}(\varepsilon\mu + \varphi)$, (3.21) by $-\varphi$, (3.22) by σ, and take the sum to get, after integration on $[0, t]$,

$$
\frac{1}{2}\|(\varepsilon\mu + \varphi)(t)\|_{V^*}^2 + \varepsilon\int_{Q_t}|\mu|^2 + \frac{\tau}{2}\|\varphi(t)\|_H^2
$$

$$
+ \int_{Q_t}\left[a|\varphi|^2 + \xi\varphi + (F_2'(\varphi_1) - F_2'(\varphi_2))\varphi\right] + \frac{1}{2}\|\sigma(t)\|_H^2 + \int_{Q_t}|\nabla\sigma|^2
$$

$$
+ \int_{Q_t}(B + Ch(\varphi_1))|\sigma|^2
$$

$$
= \frac{1}{2}\|\varepsilon\mu_0 + \varphi_0\|_{V^*}^2 + \frac{\tau}{2}\|\varphi_0\|_H^2 + \frac{1}{2}\|\sigma_0\|_H^2 + \int_{Q_t}(\chi\sigma + J * \varphi)\varphi
$$

$$
+ \int_{Q_t}[C\sigma_2(h(\varphi_2) - h(\varphi_1))]\sigma + \int_{Q_t}[\mu + P\sigma h(\varphi_1)
$$

$$
+ (P\sigma_2 - A)(h(\varphi_1) - h(\varphi_2))]\mathcal{R}^{-1}(\varepsilon\mu + \varphi) + \eta\int_{Q_t}\nabla\varphi \cdot \nabla\sigma. \tag{3.25}
$$

Note that the last term on the left-hand side is non-negative due to the positivity of h. Hence, using the monotonicity of ∂F_1 and recalling assumption **A5**, we have

$$
\int_{Q_t}\left[a|\varphi|^2 + \xi\varphi + (F_2'(\varphi_1) - F_2'(\varphi_2))\varphi\right] + \int_{Q_t}(B + Ch(\varphi_1))|\sigma|^2 \geqslant C_0\int_{Q_t}|\varphi|^2.
$$

Moreover, under the assumption $\eta = 0$, we have, owing to (2.10) that $\sigma_2 \in L^\infty(Q)$ with $\|\sigma_2\|_{L^\infty(Q)} \leqslant 1$ and that the last term on the right-hand side of (3.25) disappears. Let us estimate the remaining terms on the right-hand side. First of all, recalling that K_0 denotes the norm of the inclusion $H \hookrightarrow V^*$, by the Young inequality we have that, for every $\delta_1, \delta_2 > 0$,

$$
\int_{Q_t}[\mu + P\sigma h(\varphi_1) + (P\sigma_2 - A)(h(\varphi_1) - h(\varphi_2))]\mathcal{R}^{-1}(\varepsilon\mu + \varphi)
$$

$$
\leqslant \delta_1\varepsilon\int_{Q_t}|\mu|^2 + \delta_2\int_{Q_t}|\varphi|^2 + \frac{P^2\|h\|_{L^\infty(\mathbb{R})}^2}{2}\int_{Q_t}|\sigma|^2
$$

$$
+ K_0^2\left(\frac{1}{4\delta_1\varepsilon} + \frac{1}{2} + \frac{(P + A)^2\|h'\|_{L^\infty(\mathbb{R})}^2}{4\delta_2}\right)\int_0^t\|(\varepsilon\mu + \varphi)(s)\|_{V^*}^2\,\mathrm{d}s.
$$

Secondly, analogous computations yield

$$
\chi\int_{Q_t}\sigma\varphi + \int_{Q_t}[C\sigma_2(h(\varphi_2) - h(\varphi_1))]\sigma \leqslant \delta_2\int_{Q_t}|\varphi|^2 + \frac{\chi^2 + C^2\|h'\|_{L^\infty(\mathbb{R})}^2}{2\delta_2}\int_{Q_t}|\sigma|^2.
$$

Finally, we have that

$$
\int_{Q_t}(J * \varphi)\varphi \leqslant \int_0^t\|J * \varphi(s)\|_V\|\varphi(s)\|_{V^*}\,\mathrm{d}s \leqslant (a^* + b^*)\int_0^t\|\varphi(s)\|_H\|\varphi(s)\|_{V^*}\,\mathrm{d}s
$$

$$
\leqslant \delta_2\int_{Q_t}|\varphi|^2 + \frac{(a^* + b^*)^2}{4\delta_2}\int_0^t\|\varphi(s)\|_{V^*}^2\,\mathrm{d}s
$$

$$\leqslant \delta_2 \int_{Q_t} |\varphi|^2 + \frac{(a^* + b^*)^2}{2\delta_2} \int_0^t \|(\varepsilon\mu + \varphi)(s)\|_{V^*}^2 \, \mathrm{d}s$$

$$+ \frac{\varepsilon(a^* + b^*)^2 K_0^2}{2\delta_2} \left(\varepsilon \int_{Q_t} |\mu|^2 \right).$$

Rearranging the terms we deduce that

$$\frac{1}{2} \|(\varepsilon\mu + \varphi)(t)\|_{V^*}^2 + \varepsilon \int_{Q_t} |\mu|^2 + \frac{\tau}{2} \|\varphi(t)\|_H^2$$

$$+ C_0 \int_{Q_t} |\varphi|^2 + \frac{1}{2} \|\sigma(t)\|_H^2 + \int_{Q_t} |\nabla\sigma|^2$$

$$\leqslant \frac{1}{2} \|\varepsilon\mu_0 + \varphi_0\|_{V^*}^2 + \frac{\tau}{2} \|\varphi_0\|_H^2 + \frac{1}{2} \|\sigma_0\|_H^2$$

$$+ M_{\delta_1,\delta_2,\varepsilon} \int_0^t \left(\|\sigma(s)\|_H^2 + \|(\varepsilon\mu + \varphi)(s)\|_{V^*}^2 \right) \, \mathrm{d}s$$

$$+ \left(\delta_1 + \frac{\varepsilon(a^* + b^*)^2 K_0^2}{2\delta_2} \right) \varepsilon \int_{Q_t} |\mu|^2 + 3\delta_2 \int_{Q_t} |\varphi|^2$$

for some positive constant $M_{\delta_1,\delta_2,\varepsilon}$ depending on the data of the problem and ε, but independent of τ. Now, it clear that the last two terms on the right-hand side can be incorporated in the corresponding ones on the left provided to choose and fix $\delta_1, \delta_2 > 0$ such that

$$\delta_1 + \frac{\varepsilon(a^* + b^*)^2 K_0^2}{2\delta_2} < 1, \quad 3\delta_2 < C_0.$$

An elementary computation shows that this is possible if and only if

$$\frac{\varepsilon(a^* + b^*)^2 K_0^2}{2} < \frac{C_0}{3},$$

which is indeed guaranteed since $\varepsilon < \varepsilon_0$ and by the smallness assumption on ε_0. The thesis follows then by the Gronwall lemma.

3.6. Further regularity

This section is devoted to the proof of theorem 2.3, concerning regularity of weak solutions, when $\varepsilon, \tau > 0$. To begin with, we improve the regularity of φ and σ by showing that the approximate solutions $(\varphi_\lambda, \mu_\lambda, \sigma_\lambda)$ to the system (3.13)–(3.16) satisfy further estimates uniformly in λ. We proceed formally, to avoid a further regularization on the system based on time discretizations. First, we analyse the system (3.13)–(3.16) at the initial time $t = 0$ and let us claim that there exists a unique pair $(\varphi'_{0,\lambda}, \mu'_{0,\lambda}, \sigma'_{0,\lambda}) \in H \times V^* \times V^*$ such that, in Ω,

$$\begin{cases} \varepsilon\mu'_{0,\lambda} + \varphi'_{0,\lambda} - \Delta\mu_0 = (P\sigma_0 - A)h(\varphi_0), \\ \mu_0 = \tau\varphi'_{0,\lambda} + a\varphi_0 - J * \varphi_0 + F'_\lambda(\varphi_0) - \chi\sigma_0, \\ \sigma'_{0,\lambda} - \Delta\sigma_0 + B(\sigma_0 - \sigma_S(0)) + C\sigma_0 h(\varphi_0) = -\eta\Delta\varphi_0. \end{cases}$$

Indeed, the existence and uniqueness of $\sigma'_{0,\lambda}$ is given by the third equation and the assumptions (2.2), (2.12) and (2.13). It follows directly then from the second equation the unique definition

for $\varphi'_{0,\lambda}$, and finally from the first equation the one of $\mu'_{0,\lambda}$. Furthermore, from the second equation and assumption (2.12) it follows that $(\varphi'_{0,\lambda})_\lambda$ is uniformly bounded in H, which in turn yields that $(\mu'_{0,\lambda})_\lambda$ is uniformly bounded in V^*.

Bearing this in mind, we test (3.13) by $\partial_t\mu_\lambda$, the time-derivative of (3.14) by $-\partial_t\varphi_\lambda$, (3.15) by $\partial_t(\sigma_\lambda - \eta\varphi_\lambda)$, and take the sum: after integrating in time we obtain

$$
\begin{aligned}
&\varepsilon\int_{Q_t}|\partial_t\mu_\lambda|^2 + \frac{1}{2}\|\nabla\mu_\lambda(t)\|_H^2 + \frac{\tau}{2}\|\partial_t\varphi_\lambda(t)\|_H^2 + \int_{Q_t}(a + F''_\lambda(\varphi_\lambda))|\partial_t\varphi_\lambda|^2 \\
&\quad + \int_{Q_t}|\partial_t\sigma_\lambda|^2 + \frac{1}{2}\|\nabla(\sigma_\lambda - \eta\varphi_\lambda)(t)\|_H^2 \\
&= \frac{1}{2}\|\nabla\mu_0\|_H^2 + \frac{\tau}{2}\|\varphi'_{0,\lambda}\|_H^2 + \frac{1}{2}\|\nabla(\sigma_0 - \eta\varphi_0)\|_H^2 \\
&\quad + \int_{Q_t}(P\sigma_\lambda - A)h(\varphi_\lambda)\partial_t\mu_\lambda + \int_{Q_t}(J * (\partial_t\varphi_\lambda) + (\eta + \chi)\partial_t\sigma_\lambda)\,\partial_t\varphi_\lambda \\
&\quad + \int_{Q_t}(B(\sigma_S - \sigma_\lambda) - Ch(\varphi_\lambda)\sigma_\lambda)(\partial_t\sigma_\lambda - \eta\partial_t\varphi_\lambda).
\end{aligned}
\tag{3.26}
$$

Now, the second term on the right-hand side is uniformly bounded in λ thanks to the remarks above, and so is the first one by assumption. Hence, recalling again **A5** we infer that

$$
\begin{aligned}
&\varepsilon\int_{Q_t}|\partial_t\mu_\lambda|^2 + \frac{1}{2}\|\nabla\mu_\lambda(t)\|_H^2 + \frac{\tau}{2}\|\partial_t\varphi_\lambda(t)\|_H^2 + C_0\int_{Q_t}|\partial_t\varphi|^2 \\
&\quad + \int_{Q_t}|\partial_t\sigma_\lambda|^2 + \frac{1}{2}\|\nabla(\sigma_\lambda - \eta\varphi_\lambda)(t)\|_H^2 \\
&\leqslant M + \frac{\varepsilon}{2}\int_{Q_t}|\partial_t\mu_\lambda|^2 + \frac{1}{2\varepsilon}\int_{Q_t}|(P\sigma_\lambda - A)h(\varphi)|^2 + \frac{1}{2}\int_{Q_t}|\partial_t\sigma_\lambda|^2 \\
&\quad + \left(a^* + (\eta + \chi)^2 + \frac{\eta^2}{2}\right)\int_{Q_t}|\partial_t\varphi_\lambda|^2 + \frac{3}{2}\int_{Q_t}|B(\sigma_S - \sigma_\lambda) - Ch(\varphi_\lambda)\sigma_\lambda|^2.
\end{aligned}
$$

Taking the estimate (3.17) into account and using the boundedness of h and σ_S we infer that

$$
\|\varphi_\lambda\|_{W^{1,\infty}(0,T;H)}^2 + \|\mu_\lambda\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}^2 + \|\sigma_\lambda\|_{H^1(0,T;H)}^2 + \|\sigma_\lambda - \eta\varphi_\lambda\|_{L^\infty(0,T;V)}^2 \leqslant M
$$

for some $M > 0$ independent of λ. As we already know that $(\varphi_\lambda)_\lambda$ is uniformly bounded in $L^\infty(0,T;V)$ by (3.17), it is now a standard matter to pass to the limit as $\lambda \to 0$: recalling (2.3) and (2.4) and using a comparison argument for the linear combination $\sigma - \eta\varphi$, we have

$$
\varphi \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \qquad \mu, \sigma - \eta\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V),
$$
$$
\sigma \in H^1(0,T;H) \cap L^2(0,T;V).
$$

Moreover, note that (1.4) and (1.6) can be rewritten as

$$
\varepsilon\partial_t\mu - \Delta\mu = f_\mu := (P\sigma - A)h(\varphi) - \partial_t\varphi \quad \text{in } Q,
\tag{3.27}
$$
$$
\partial_t(\sigma - \eta\varphi) - \Delta(\sigma - \eta\varphi) = f_\sigma := -B(\sigma - \sigma_S) - Ch(\varphi) - \eta\partial_t\varphi \quad \text{in } Q,
\tag{3.28}
$$

endowed with homogeneous Neumann boundary conditions and initial data $\mu_0, \sigma_0 - \eta\varphi_0 \in V$. Since the forcing terms and the initial data satisfy $f_\mu, f_\sigma \in L^2(0,T;H)$, the classical parabolic

regularity theory yields

$$\mu, \sigma - \eta\varphi \in L^2(0, T; W),$$

completing the proof of theorem 2.3.

3.7. Strong solutions and separation principle

We focus here on the proof of theorem 2.5 concerning existence of strong solutions, separation property, and magnitude regularity, still in the case $\varepsilon, \tau > 0$. Let us stress that the separation result will allow us to exploit the regularity of the linear combination $\sigma - \eta\varphi$ to derive further regularity for φ and σ.

First of all, by virtue of theorem 2.3 we realize that (3.27) consists of a parabolic equation in the variable μ with source term $f_\mu \in L^\infty(0, T; H)$, and with initial datum $\mu_0 \in L^\infty(\Omega)$ by (2.18). Therefore, an application of [68, theorem 7.1, p 181] yields that

$$\mu \in L^\infty(Q).$$

In a similar fashion, we notice that in (3.28) we have initial datum $\sigma_0 - \eta\varphi_0 \in V \cap L^\infty(\Omega)$ and forcing term $f_\sigma \in L^\infty(0, T; H)$ by virtue of theorem 2.3. Hence, an application of [68, theorem 7.1, p 181] yields again

$$\sigma - \eta\varphi \in L^\infty(Q).$$

Furthermore, we claim that from assumption **A7** we can deduce further regularity also for the term $J * \varphi$. Indeed, every kernel verifying definition 2.4 satisfy the following result, whose proof can be found, e.g., in [5, lemma 2].

Lemma 3.1. *Assume that the kernel J is admissible in the sense of the definition* 2.4. *Then, for every* $p \in (1, \infty)$, *there exists a positive constant* C_p *such that*

$$\|\nabla(\nabla J * \psi)\|_{L^p(\Omega)^{3\times3}} \leqslant C_p \|\psi\|_{L^p(\Omega)} \quad \forall \, \psi \in L^p(\Omega). \tag{3.29}$$

As a consequence, by taking $p = 2$ in (3.29), we deduce that

$$\|J * \varphi\|_{L^\infty(0,T;H^2(\Omega))} \leqslant C_2 \|\varphi\|_{L^\infty(0,T;H)},$$

which readily implies, thanks to the continuous inclusion $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, that

$$J * \varphi \in L^\infty(0, T; H^2(\Omega)) \cap L^\infty(Q).$$

We are now ready to prove the separation property. To this end, note that, taking these remarks into account, under the assumption **A6** on F, we can rewrite equation (2.6) as

$$\tau\partial_t\varphi + a\varphi + F'(\varphi) - \chi\eta\varphi = f_\varphi := \mu + \chi(\sigma - \eta\varphi) + J * \varphi. \tag{3.30}$$

Besides, we have already proved that $f_\varphi \in L^2(0, T; H^2(\Omega)) \cap L^\infty(Q)$, so that there exists a constant $\overline{M} > 0$ such that

$$\|f_\varphi\|_{L^\infty(Q)} \leqslant \overline{M}.$$

Next, by **A6** and (2.18) we infer the existence of $r^* \in (r_0, \ell)$ such that

$$F'(r) - \chi\eta r \geqslant \overline{M} \quad \forall \, r \in (r^*, \ell), \qquad F'(r) - \chi\eta r \leqslant -\overline{M} \quad \forall \, r \in (-\ell, -r^*).$$

We claim that this choice entails $\varphi(t) \leqslant r^*$ almost everywhere in Ω, for all $t \in [0, T]$. In fact, by testing (3.30) by $(\varphi - r^*)_+$ and integrating on $[0, t]$, we immediately infer that

$$\frac{\tau}{2} \|(\varphi(t) - r^*)_+\|_H^2 + \int_{Q_t} a\varphi(\varphi - r^*)_+ = \frac{\tau}{2} \|(\varphi_0 - r^*)_+\|_H^2$$

$$+ \int_{Q_t} \left[f_\varphi - (F'(\varphi) - \chi\eta\varphi) \right] (\varphi - r^*)_+.$$

Now, since $r^* \in (r_0, \ell)$ and $\|\varphi_0\|_{L^\infty(\Omega)} \leqslant r_0$, the first term on the right-hand side vanishes. Moreover, by definition of $\overline{M}$ and r^* we have that

$$\int_{Q_t} \left[f_\varphi - (F'(\varphi) - \chi\eta\varphi) \right] (\varphi - r^*)_+$$

$$= \int_{Q_t \cap \{\varphi > r^*\}} \left[f_\varphi - (F'(\varphi) - \chi\eta\varphi) \right] (\varphi - r^*) \leqslant 0.$$

Recalling also **A5**, we infer that, for every $t \in [0, T]$,

$$\frac{\tau}{2} \|(\varphi(t) - r^*)_+\|_H^2 + a_* \int_{Q_t \cap \{\varphi > r^*\}} \varphi(\varphi - r^*) \leqslant 0.$$

Hence, since the second term on the left-hand side is non-negative, we deduce that

$$(\varphi(t) - r^*)_+ = 0 \quad \forall\, t \in [0, T], \quad \text{i.e.}$$

$$\varphi(t, x) \leqslant r^* \quad \text{for a.e.} x \in \Omega \quad \forall\, t \in [0, T],$$

as required. The other inequality $\varphi \geqslant -r^*$ can be deduced analogously by testing by $-(\varphi + r^*)_-$ instead. Thus, we have shown that

$$\sup_{t \in [0,T]} \|\varphi(t)\|_{L^\infty(\Omega)} \leqslant r^*, \quad \text{with } r^* \in (r_0, \ell).$$

Let us now show the $L^2(0, T; W)$-regularity for σ and $\eta\varphi$. To this end, for an exponent $p > 1$ yet to be chosen, we test the gradient of (3.30) by $|\nabla\varphi|^{p-2}\nabla\varphi$ and integrate over Q_t to obtain, by assumption **A5** and the Hölder and generalized Young inequalities, that

$$\frac{\tau}{p} \sup_{s \in [0,t]} \|\nabla\varphi(s)\|_{L^p(\Omega)}^p + C_0 \int_{Q_t} |\nabla\varphi|^p$$

$$= \frac{\tau}{p} \|\nabla\varphi_0\|_{L^p(\Omega)}^p + \chi\eta \int_{Q_t} |\nabla\varphi|^p - \int_{Q_t} (\nabla a)\varphi |\nabla\varphi|^{p-2}\nabla\varphi$$

$$+ \int_{Q_t} \nabla f_\varphi \cdot |\nabla\varphi|^{p-2}\nabla\varphi \leqslant \frac{\tau}{p} \|\nabla\varphi_0\|_{L^p(\Omega)}^p + \chi\eta \int_{Q_t} |\nabla\varphi|^p + \frac{\tau}{2p} \sup_{s \in [0,t]} \|\nabla\varphi(s)\|_{L^p(\Omega)}^p$$

$$+ \frac{[4(p-1)]^{p-1}(b^*)^p}{p\tau^{p-1}} \|\varphi\|_{L^1(0,T;L^p(\Omega))}^p + \frac{[4(p-1)]^{p-1}}{p\tau^{p-1}} \|\nabla f_\varphi\|_{L^1(0,T;L^p(\Omega))}^p.$$

Owing to the already proved regularities $f_\varphi \in L^2(0, T; H^2(\Omega))$ and $\varphi \in L^2(0, T; V)$, we deduce in particular that $\nabla f_\varphi \in L^2(0, T; V)$ so that, using the embedding $V \hookrightarrow L^6(\Omega)$, also $\nabla f_\varphi, \varphi \in$

$L^2(0, T; L^6(\Omega))$. Moreover, $\varphi_0 \in H^2(\Omega)$ also entails that $\nabla\varphi_0 \in L^6(\Omega)$. Choosing then $p = 6$ and using the Gronwall lemma yields

$$\varphi \in L^\infty(0, T; W^{1,6}(\Omega)). \tag{3.31}$$

Now, for brevity we proceed formally: a rigorous argument can be reproduced on suitable approximations. Applying the second-order differential operator $\partial_{x_i x_j}$ $(i, j = 1, 2, 3)$ to equation (3.30), testing it by $\partial_{x_i x_j}\varphi$, and integrating on $[0, t]$ lead to

$$\frac{\tau}{2}\left\|\partial_{x_i x_j}\varphi(t)\right\|_H^2 + \int_{Q_t} \left(a + F''(\varphi)\right)|\partial_{x_i x_j}\varphi|^2 = \frac{\tau}{2}\left\|\partial_{x_i x_j}\varphi_0\right\|_H^2$$

$$+ \int_{Q_t} \partial_{x_i x_j} f_\varphi \partial_{x_i x_j}\varphi + \chi\eta \int_{Q_t} |\partial_{x_i x_j}\varphi|^2 - \int_{Q_t} \left[\partial_{x_i} a \partial_{x_j}\varphi + \partial_{x_j} a \partial_{x_i}\varphi\right.$$

$$\left. + (\partial_{x_i x_j} a)\varphi + F'''(\varphi)\partial_{x_i}\varphi\partial_{x_j}\varphi\right] \partial_{x_i x_j}\varphi.$$

Now, due to the already proved separation property $\|\varphi\|_{L^\infty(Q)} \leqslant r^* < \ell$, and recalling that $F \in C^3(-\ell, \ell)$ by **A6**, we have that $F'''(\varphi) \in L^\infty(Q)$. Hence, exploiting **A5**, using the Young inequality, and summing on $i, j = 1, 2, 3$ we deduce, recalling that $\varphi \in L^2(0, T; V)$, that

$$\frac{\tau}{2}\|\varphi(t)\|_{H^2(\Omega)}^2 + C_0 \int_0^t \|\varphi(s)\|_{H^2(\Omega)}^2 \, ds$$

$$\leqslant \frac{\tau}{2}\|\varphi_0\|_{H^2(\Omega)}^2 + (2 + \chi\eta) \int_0^t \|\varphi(s)\|_{H^2(\Omega)}^2 \, ds$$

$$+ \frac{1}{2}\|f_\varphi\|_{L^2(0,T;H^2(\Omega))}^2 + 2\int_{Q_t} |\nabla a|^2 |\nabla\varphi|^2 + \frac{1}{2}\int_{Q_t} \sum_{i,j=1}^3 |\partial_{x_i x_j} a|^2 |\varphi|^2$$

$$+ \frac{1}{2}\|F'''(\varphi)\|_{L^\infty(Q)}^2 \int_{Q_t} |\nabla\varphi|^4.$$

Moreover, $\|\nabla a\|_{L^\infty(\Omega)} \leqslant b^*$ by **A5**, $\|a\|_{W^{2,p}(\Omega)} \leqslant C_p$ for all $p \in (1, +\infty)$ by (3.29) and $\varphi \in L^\infty(0, T; V)$, so that the Hölder inequality yields

$$\int_{Q_t} |\nabla a|^2 |\nabla\varphi|^2 \leqslant (b^*)^2 \|\varphi\|_{L^2(0,T;V)}^2 \leqslant M$$

and, by the continuous embedding $V \hookrightarrow L^4(\Omega)$, also that

$$\int_{Q_t} \sum_{i,j=1}^3 |\partial_{x_i x_j} a|^2 |\varphi|^2 \leqslant \|a\|_{W^{2,4}(\Omega)}^2 \|\varphi\|_{L^4(0,T;L^4(\Omega))}^2 \leqslant M'\|\varphi\|_{L^4(0,T;V)}^2 \leqslant M$$

for certain constants $M, M' > 0$. Using then (3.31), we are left with

$$\frac{\tau}{2}\|\varphi(t)\|_{H^2(\Omega)}^2 + C_0 \int_0^t \|\varphi(s)\|_{H^2(\Omega)}^2 \, ds$$

$$\leqslant \frac{\tau}{2}\|\varphi_0\|_{H^2(\Omega)}^2 + M\left(1 + \int_0^t \|\varphi(s)\|_{H^2(\Omega)}^2 \, ds\right)$$

so that a Gronwall argument produces

$$\varphi \in L^\infty(0, T; H^2(\Omega)).$$

At this point, the equation for σ can be written also as

$$\partial_t \sigma - \Delta \sigma = \tilde{f}_\sigma := -B(\sigma - \sigma_S) - C\sigma h(\varphi) - \eta \Delta \varphi \in L^\infty(0, T; H),$$

with initial datum $\sigma_0 \in V \cap L^\infty(\Omega)$. Hence, by parabolic regularity theory and again [68, theorem 7.1], we deduce that

$$\sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty(Q).$$

Since we already know that $\sigma - \eta\varphi \in L^2(0, T; W)$, by comparison we also infer

$$\eta\varphi \in L^2(0, T; W).$$

To conclude, we go back to equation (3.30) and note that, by difference, we have also the regularity

$$\partial_t \varphi \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(Q)$$

which completes the proof of theorem 2.5.

3.8. Refined continuous dependence

We prove here the refined stability estimates contained in theorem 2.7 which is now possible in light of the strong regularity result established by theorem 2.5. It is worth pointing out that both the chemotaxis and active transport mechanisms are now included in the analysis. Employing the same notation of subsection 3.5, we consider the system (3.20)–(3.24) and test (3.20) by $\partial_t \mu$, the time-derivative of (3.21) by $-\partial_t \varphi$, (3.22) by $\partial_t(\sigma - \eta\varphi)$, and integrate over $[0, t]$, to obtain

$$\varepsilon \int_{Q_t} |\partial_t \mu|^2 + \frac{1}{2}\|\nabla\mu(t)\|_H^2 + \frac{\tau}{2}\|\partial_t\varphi(t)\|_H^2 + \int_{Q_t} (a + F''(\varphi_1))|\partial_t\varphi|^2$$

$$+ \int_{Q_t} |\partial_t\sigma|^2 + \frac{1}{2}\|\nabla(\sigma - \eta\varphi)(t)\|_H^2$$

$$= \frac{1}{2}\|\nabla\mu_0\|_H^2 + \frac{\tau}{2}\|\varphi_0'\|_H^2 + \frac{1}{2}\|\nabla(\sigma_0 - \eta\varphi_0)\|_H^2$$

$$+ \int_{Q_t} [P\sigma h(\varphi_1) + (P\sigma_2 - A)(h(\varphi_1) - h(\varphi_2))]\,\partial_t\mu$$

$$+ \int_{Q_t} [(F''(\varphi_2) - F''(\varphi_1))\partial_t\varphi_2 + \chi\partial_t\sigma + J * \partial_t\varphi]\,\partial_t\varphi + \eta\int_{Q_t}\partial_t\sigma\partial_t\varphi$$

$$+ \int_{Q_t} [C\sigma_2(h(\varphi_2) - h(\varphi_1)) - C\sigma h(\varphi_1) - B\sigma]\,(\partial_t\sigma - \eta\partial_t\varphi).$$

First of all, notice that φ_0' is such that

$$\mu_0 = \tau\varphi_0' + a\varphi_0 - J * \varphi_0 + F'(\varphi_0^1) - F'(\varphi_0^2) - \chi\sigma_0.$$

Since the initial data satisfy (2.2), (2.12), and (2.18), for $i = 1, 2$ we have that $\varphi_0' \in V \cap L^\infty(\Omega)$. Now, recalling that $F \in C^3([-r_0, r_0])$, we have

$$\|\varphi_0'\|_H \leqslant \frac{1}{\tau} \left(\|\mu_0\|_H + 2a^*\|\varphi_0\|_H + \|F''\|_{C^0([-r_0, r_0])} \|\varphi_0\|_H + \chi\|\sigma_0\|_H \right).$$

Secondly, by the separation property for φ_1 and φ_2, we have $\|\varphi_i\|_{L^\infty(Q)} \leqslant r^* < \ell$ for $i = 1, 2$ and combined with $F \in C^3([-r^*, r^*])$ we have $F'' \in W^{1,\infty}(-r^*, r^*)$, so that

$$|F''(\varphi_1) - F''(\varphi_2)| \leqslant \|F'''\|_{C^0([-r^*, r^*])} |\varphi_1 - \varphi_2| \quad \text{a.e. in } Q.$$

Taking this information into account, using **A5**, and exploiting the regularities $h \in W^{1,\infty}(\mathbb{R})$, $\sigma_2 \in L^\infty(Q)$, and $\partial_t \varphi_2 \in L^\infty(Q)$, we invoke the Young inequality to infer

$$\int_{Q_t} |\partial_t \mu|^2 + \|\nabla\mu(t)\|_H^2 + \|\partial_t\varphi(t)\|_H^2 + \int_{Q_t} |\partial_t\varphi|^2 + \int_{Q_t} |\partial_t\sigma|^2 + \|\nabla(\sigma - \eta\varphi)(t)\|_H^2$$

$$\leqslant M \left(\|\mu_0\|_V^2 + \|\varphi_0\|_H^2 + \|\sigma_0\|_H^2 + \|\nabla(\sigma_0 - \eta\varphi_0)\|_H^2 + \int_{Q_t} (|\sigma|^2 + |\varphi|^2 + |\partial_t\varphi|^2) \right), \quad (3.32)$$

where the constant $M > 0$ may depend on ε, τ and on structural data. Now, we take the gradient of (3.21) and test it by $\nabla\varphi$, getting

$$\frac{\tau}{2}\|\nabla\varphi(t)\|_H^2 + \int_{Q_t} (a + F''(\varphi_1))|\nabla\varphi|^2$$

$$= \frac{\tau}{2}\|\nabla\varphi_0\|_H^2 + \int_{Q_t} \left(F''(\varphi_2) - F''(\varphi_1) \right) \nabla\varphi_2 \cdot \nabla\varphi$$

$$+ \int_{Q_t} (\nabla\mu + \chi\nabla\sigma + (\nabla J) * \varphi - (\nabla a)\varphi) \cdot \nabla\varphi.$$

Using **A5**, along with the Lipschitz continuity of F'' on $[-r^*, r^*]$, and the identity

$$\chi\nabla\sigma \cdot \nabla\varphi = \chi(\nabla(\sigma - \eta\varphi) + \eta\nabla\varphi) \cdot \nabla\varphi,$$

and the Young inequality lead to

$$\frac{\tau}{2}\|\nabla\varphi(t)\|_H^2 + C_0 \int_{Q_t} |\nabla\varphi|^2 \leqslant \frac{\tau}{2}\|\nabla\varphi_0\|_H^2 + \|F'''\|_{C^0([-r^*, r^*])} \int_{Q_t} |\varphi|\|\nabla\varphi_2\|\nabla\varphi|$$

$$+ \int_{Q_t} |\nabla\mu|^2 + \chi^2 \int_{Q_t} |\nabla(\sigma - \eta\varphi)|^2$$

$$+ (1 + \chi\eta) \int_{Q_t} |\nabla\varphi|^2 + 2(b^*)^2 \int_{Q_t} |\varphi|^2.$$

From the embedding $V \hookrightarrow L^4(\Omega)$, Hölder's inequality and the regularity $\varphi_2 \in L^\infty(0, T; H^2(\Omega))$, we find

$$\int_{Q_t} |\varphi|\|\nabla\varphi_2\|\nabla\varphi| \leqslant M' \int_0^t \|\varphi(s)\|_V \|\varphi_2(s)\|_{H^2(\Omega)} \|\nabla\varphi(s)\|_H \, ds$$

$$\leqslant M \int_0^t \|\varphi(s)\|_V^2 \, ds$$

for some constants $M, M' > 0$. We deduce then that, possibly updating M, for every $t \in [0, T]$,

$$\|\nabla\varphi(t)\|_H^2 \leqslant M\left(\|\nabla\varphi_0\|_H^2 + \int_{Q_t}|\nabla\mu|^2 + \int_{Q_t}|\nabla(\sigma - \eta\varphi)|^2 + \int_0^t\|\varphi(s)\|_V^2\,\mathrm{d}s\right). \quad (3.33)$$

We now combine the estimates (3.32) and (3.33) to infer that, for all $t \in [0, T]$,

$$\int_{Q_t}|\partial_t\mu|^2 + \|\nabla\mu(t)\|_H^2 + \|\partial_t\varphi(t)\|_H^2 + \|\nabla\varphi(t)\|_H^2 + \int_{Q_t}|\partial_t\sigma|^2 + \|\nabla\sigma(t)\|_H^2$$

$$\leqslant M\left(\|\mu_0\|_V^2 + \|\varphi_0\|_V^2 + \|\sigma_0\|_V^2 + \int_0^t\left(\|\nabla\mu(s)\|_H^2 + \|\sigma(s)\|_V^2\right.\right.$$

$$\left.\left.+ \|\varphi(s)\|_V^2 + \|\partial_t\varphi(s)\|_H^2\right)\,\mathrm{d}s\right).$$

Since the quantities $\|\sigma_2\|_{L^\infty(Q)}$, $\|\partial_t\varphi_2\|_{L^\infty(Q)}$, and $\|\varphi_2\|_{L^\infty(0,T;H^2(\Omega))}$ appearing implicitly in the constant M can be in turn handled in terms on the norms of the initial data appearing in (2.2), (2.12), and (2.18), we can close the estimate by the Gronwall lemma. Moreover, comparison in equation (3.20) produces

$$\|\Delta\mu\|_{L^2(0,T;H)} \leqslant M\left(\|\varphi\|_{H^1(0,T;H)} + \|\partial_t\mu\|_{L^2(0,T;H)} + \|\sigma\|_{L^2(0,T;H)}\right),$$

where all the terms on the right-hand side have already been estimated. Similarly, from (3.21) we get

$$\|\partial_t\varphi\|_{L^\infty(0,T;V)} \leqslant M\left(\|\mu\|_{L^\infty(0,T;V)} + \|\varphi\|_{L^\infty(0,T;V)} + \|\sigma\|_{L^\infty(0,T;V)}\right),$$

while from (3.22) we get

$$\|\Delta(\sigma - \eta\varphi)\|_{L^2(0,T;H)} \leqslant M\left(\|\sigma\|_{H^1(0,T;H)} + \|\varphi\|_{L^2(0,T;H)}\right).$$

Collecting the above estimates, along with elliptic regularity theory, we deduce that

$$\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)}^2 + \|\varphi\|_{W^{1,\infty}(0,T;V)}^2$$

$$+ \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}^2 + \|\sigma - \eta\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)}^2$$

$$\leqslant M\left(\|\mu_0\|_V^2 + \|\varphi_0\|_V^2 + \|\sigma_0\|_V^2\right). \quad (3.34)$$

To complete the proof, we need to show a stability estimate for $\partial_t\varphi$ and σ also in $L^2(0,T;H^2(\Omega))$ and $L^2(0,T;W)$, respectively. In this direction, for any $i, j = 1, 2, 3$, we apply the differential operator $\partial_{x_i x_j}$ to (3.21) and test the obtained equation by $\partial_{x_i x_j}\varphi$, getting

$$\frac{\tau}{2}\left\|\partial_{x_i x_j}\varphi(t)\right\|_H^2 + \int_{Q_t}(a + F''(\varphi_1))|\partial_{x_i x_j}\varphi|^2$$

$$= \frac{\tau}{2}\left\|\partial_{x_i x_j}\varphi_0\right\|_H^2 + \int_{Q_t}\partial_{x_i x_j}(\mu + \chi(\sigma - \eta\varphi) + J * \varphi)\partial_{x_i x_j}\varphi$$

$$+ \chi\eta\int_{Q_t}|\partial_{x_i x_j}\varphi|^2 - \int_{Q_t}(\partial_{x_i}a\partial_{x_j}\varphi + \partial_{x_j}a\partial_{x_i}\varphi + (\partial_{x_i x_j}a)\varphi)\partial_{x_i x_j}\varphi$$

$$+ \int_{Q_t}\left[(F''(\varphi_2) - F''(\varphi_1))\partial_{x_i x_j}\varphi_2 + (F'''(\varphi_2) - F'''(\varphi_1))\partial_{x_i}\varphi_1\partial_{x_j}\varphi_2\right]\partial_{x_i x_j}\varphi$$

$$- \int_{Q_t}\left[F'''(\varphi_1)\partial_{x_i}\varphi_1\partial_{x_j}\varphi + F'''(\varphi_2)\partial_{x_i}\varphi\partial_{x_j}\varphi_2\right]\partial_{x_i x_j}\varphi.$$

We recall that, due to **A6**, $F \in C^4([-r^*, r^*])$, so that F''' is Lipschitz continuous on $[-r^*, r^*]$, and as a consequence of the separation result, also $F'''(\varphi_i) \in L^\infty(Q)$, for $i = 1, 2$. Now, we use the Hölder and Young inequalities and sum on $i, j = 1, 2, 3$: proceeding as in subsection 3.7 and exploiting assumptions **A5** and **A7**, we get

$$\frac{\tau}{2}\|\varphi(t)\|^2_{H^2(\Omega)} + C_0 \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)} \, ds$$

$$\leqslant \frac{\tau}{2}\|\varphi_0\|^2_{H^2(\Omega)} + M \left(\|\mu\|^2_{L^2(0,T;W)} + \|\sigma - \eta\varphi\|^2_{L^2(0,T;W)} \right.$$

$$+ \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)} \, ds \right) + M \sum_{i,j=1}^3 \left(\int_{Q_t} |\varphi|^2 \left(|\partial_{x_i x_j}\varphi_2|^2 + |\partial_{x_i}\varphi_1|^2 |\partial_{x_j}\varphi_2|^2 \right) \right.$$

$$+ \int_{Q_t} \left(|\partial_{x_i}\varphi_1|^2 |\partial_{x_j}\varphi|^2 + |\partial_{x_i}\varphi|^2 |\partial_{x_j}\varphi_2|^2 \right) \bigg) .$$

The first bracket on the right-hand side can be controlled using (3.34) and the Gronwall lemma, while the sum-term can be estimated using the Hölder inequality and the continuous inclusions $V \hookrightarrow L^4(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ by

$$\int_0^t \|\varphi(s)\|^2_{L^\infty(\Omega)} \left(\|\varphi_2(s)\|^2_{H^2(\Omega)} + \|\nabla\varphi_1(s)\|^2_V \|\nabla\varphi_2(s)\|^2_V \right) \, ds$$

$$+ \int_0^t \|\nabla\varphi(s)\|^2_{L^4(\Omega)} \left(\|\nabla\varphi_1(s)\|^2_{L^4(\Omega)} + \|\nabla\varphi_2(s)\|^2_{L^4(\Omega)} \right) \, ds$$

$$\leqslant M' \left(\|\varphi_2\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\varphi_1\|^2_{L^\infty(0,T;H^2(\Omega))} \|\varphi_2\|^2_{L^\infty(0,T;H^2(\Omega))} \right)$$

$$\times \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)} \, ds + M' \left(\|\varphi_1\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\varphi_2\|^2_{L^\infty(0,T;H^2(\Omega))} \right)$$

$$\times \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)} \, ds.$$

Taking these estimates into account and recalling the regularity $\varphi_1, \varphi_2 \in L^\infty(0, T; H^2(\Omega))$, we conclude that

$$\|\varphi(t)\|^2_{H^2(\Omega)} \leqslant \|\varphi_0\|^2_{H^2(\Omega)} + M \left(\|\mu\|^2_{L^2(0,T;W)} + \|\sigma - \eta\varphi\|^2_{L^2(0,T;W)} + \int_0^t \|\varphi(s)\|^2_{H^2(\Omega)} \, ds \right)$$

so that Gronwall's lemma along with the above estimates produces

$$\|\varphi\|^2_{L^\infty(0,T;H^2(\Omega))} \leqslant M \left(\|\mu_0\|^2_V + \|\varphi_0\|^2_{H^2(\Omega)} + \|\sigma_0\|^2_V \right) .$$

The stability estimate for σ in $L^2(0, T; W)$ follows by comparison in (3.22) and elliptic regularity theory. Finally, by comparison in equation (3.21) we also infer the stability estimate for $\partial_t\varphi$ in $L^2(0, T; H^2(\Omega))$, concluding the proof of theorem 2.7.

4. Asymptotics as $\varepsilon \searrow 0$

This section is completely devoted to discuss the asymptotic behavior of system (1.4)–(1.8) as $\varepsilon \searrow 0$, when $\tau > 0$ is fixed. Namely, we aim at proving theorems 2.8 and 2.9. Henceforth, let us assume τ to be positive and fixed. Moreover, using the notation introduced by theorem 2.8, we indicate with $(\varphi_{\varepsilon\tau}, \mu_{\varepsilon\tau}, \sigma_{\varepsilon\tau}, \xi_{\varepsilon\tau})$ the unique weak solution to (1.4)–(1.8) with $\varepsilon, \tau > 0$.

4.1. Uniform estimates

Proceeding as in subsection 3.2, we perform the analogous estimates that we used to deduce (3.10). In particular, since the implicit constant M in (3.10) is independent of ε and τ, recalling that we are assuming $\eta = 0$, we realize that

$$
\frac{\varepsilon}{4}\|\mu_{\varepsilon\tau}(t)\|_H^2 + (1 + 4c_a\varepsilon)\int_{Q_t}|\nabla\mu_{\varepsilon\tau}|^2 + \frac{\tau}{2}\int_{Q_t}|\partial_t\varphi_{\varepsilon\tau}|^2 + \int_\Omega F(\varphi_{\varepsilon\tau}(t))
$$

$$
+ \frac{1}{2}\|\sigma_{\varepsilon\tau}(t)\|_H^2 + \int_{Q_t}|\nabla\sigma_{\varepsilon\tau}|^2 + c_a\|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau})(t)\|_H^2
$$

$$
+ 2c_a\tau\|\nabla\varphi_{\varepsilon\tau}(t)\|_H^2 + 2c_aC_0\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2
$$

$$
\leqslant \frac{3}{2}\varepsilon\|\mu_{0,\varepsilon\tau}\|_H^2 + (a^* + 4c_a)\|\varphi_{0,\varepsilon\tau}\|_H^2 + 2c_a\tau\|\nabla\varphi_{0,\varepsilon\tau}\|_H^2
$$

$$
+ \|F(\varphi_{0,\varepsilon\tau})\|_{L^1(\Omega)} + \frac{1}{2}\|\sigma_{0,\varepsilon\tau}\|_H^2
$$

$$
+ M\left(1 + \int_{Q_t}|\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}|^2 + \int_{Q_t}|\varphi_{\varepsilon\tau}|^2 + \int_{Q_t}|\sigma_{\varepsilon\tau}|^2\right) + \frac{\chi^2}{2\tau}\int_{Q_t}|\sigma_{\varepsilon\tau}|^2
$$

$$
+ \frac{1}{2}\int_{Q_t}|\nabla\sigma_{\varepsilon\tau}|^2 + 8c_a^2\chi^2\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2 + \int_{Q_t}(P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})\mu_{\varepsilon\tau}. \qquad (4.1)
$$

All the terms referring to the initial data on the right-hand side are uniformly bounded in ε by virtue of assumptions (2.24) and (2.25). Moreover, all the remaining terms can be handled using the Gronwall lemma, except for the last one. To this end, note that by the Poincaré–Wirtinger inequality (2.1), using the fact that h is bounded, and the uniform bound $\|\sigma_{\varepsilon\tau}\|_{L^\infty(Q)} \leqslant 1$, we have

$$
\int_{Q_t}(P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})\mu_{\varepsilon\tau}
$$

$$
\leqslant \int_{Q_t}(P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})(\mu_{\varepsilon\tau} - (\mu_{\varepsilon\tau})_\Omega) + \int_{Q_t}(P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})(\mu_{\varepsilon\tau})_\Omega
$$

$$
\leqslant \frac{1}{2}\int_{Q_t}|\nabla\mu_{\varepsilon\tau}|^2 + M + (P + A)\|h\|_{L^\infty(\mathbb{R})}t^{1/2}\|(\mu_{\varepsilon\tau})_\Omega\|_{L^2(0,t)}.
$$

Furthermore, noting that $(a\varphi_{\varepsilon\tau} - J * \varphi_{\varepsilon\tau})_\Omega = 0$, by comparison in equation (1.5) we get

$$
(\mu_{\varepsilon\tau})_\Omega = \tau(\partial_t\varphi_{\varepsilon\tau})_\Omega + (\xi_{\varepsilon\tau} + F_2'(\varphi_{\varepsilon\tau}))_\Omega - \chi(\sigma_{\varepsilon\tau})_\Omega,
$$

so that thanks to assumption (2.22) implies that

$$\|(\mu_{\varepsilon\tau})_\Omega\|_{L^2(0,t)}$$

$$\leqslant \tau\|\partial_t\varphi_{\varepsilon\tau}\|_{L^2(Q_t)} + \|\xi_{\varepsilon\tau} + F_2'(\varphi_{\varepsilon\tau})\|_{L^2(0,t;L^1(\Omega))} + \chi\|\sigma_{\varepsilon\tau}\|_{L^2(0,t;H)}$$

$$\leqslant M\left(1 + \tau^2\int_{Q_t}|\partial_t\varphi_{\varepsilon\tau}|^2 + \sup_{s\in[0,t]}\int_\Omega F(\varphi_{\varepsilon\tau}(s)) + \sup_{s\in[0,t]}\|\sigma_{\varepsilon\tau}(s)\|_H^2\right),$$

for a certain constant $M > 0$, independent of ε. Putting this information together, we first choose $t \in [0, T_0]$, where $T_0 \in (0, T]$ is fixed sufficiently small so that the term corresponding to $t^{1/2}$ can be incorporated on the left-hand side, for example by picking a T_0 such that

$$(P + A)\|h\|_{L^\infty(\mathbb{R})}T_0^{1/2} < \frac{1}{2M}.$$

We then take supremum in $t \in [0, T_0]$ on the left-hand side of the inequality (4.1) and rearrange the terms: the estimate can be closed on the time interval $[0, T_0]$ using the Gronwall lemma. As the choice of T_0 is independent of ε, τ, and of the initial data (it only depends on A, P, C_F, h, and χ), repeating the same argument we can close the estimate also on $[T_0, 2T_0]$, and so on, so that a classical patching argument guarantees the existence of a constant $M > 0$, independent of ε, such that

$$\|\varphi_{\varepsilon\tau}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\sigma_{\varepsilon\tau}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leqslant M, \tag{4.2}$$

$$\|(\mu_{\varepsilon\tau})_\Omega\|_{L^2(0,T)} + \|\nabla\mu_{\varepsilon\tau}\|_{L^2(0,T;H)} + \varepsilon^{1/2}\|\mu_{\varepsilon\tau}\|_{L^\infty(0,T;H)} \leqslant M. \tag{4.3}$$

From estimate (4.3), the Poincaré–Wirtinger inequality yields

$$\|\mu_{\varepsilon\tau}\|_{L^2(0,T;V)} \leqslant M. \tag{4.4}$$

Lastly, by comparison in (1.6), we also deduce that

$$\|\sigma_{\varepsilon\tau}\|_{H^1(0,T;V^*)} \leqslant M, \tag{4.5}$$

while by comparison in (1.5) we have that

$$\|\xi_{\varepsilon\tau}\|_{L^2(0,T;H)} \leqslant M. \tag{4.6}$$

4.2. Passage to the limit

From the estimates (4.2)–(4.6) and classical compactness arguments, we infer the existence of a quadruplet $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$ with

$$\varphi_\tau \in H^1(0,T;H)\cap L^\infty(0,T;V), \qquad \mu_\tau \in L^2(0,T;V),$$

$$\sigma_\tau \in H^1(0,T;V^*)\cap L^2(0,T;V), \qquad \xi_\tau \in L^2(0,T;H),$$

such that, as $\varepsilon \searrow 0$, along a non-relabelled subsequence, it holds that the weak, weak* and strong convergences (2.26)–(2.31) are fulfilled. We are then left to show that $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$ yields a solution to (1.4)–(1.8) with $\varepsilon = 0$ in the sense of theorem 2.8. In this direction, let us exploit the strong convergence of the phase variable (2.31) along with the continuity and boundedness of h, and Lebesgue convergence theorem, to deduce that, as $\varepsilon \searrow 0$,

$$h(\varphi_{\varepsilon\tau}) \to h(\varphi_\tau) \quad \text{in } L^p(Q) \quad \forall\, p \geqslant 1, \qquad F_2'(\varphi_{\varepsilon\tau}) \to F_2'(\varphi_\tau) \quad \text{in } C^0([0,T];H).$$

Moreover, the strong–weak closure of ∂F_1 (see, e.g., [4, corollary 2.4, p 41]) entails that $\xi_\tau \in \partial F_1(\varphi_\tau)$ almost everywhere in Q. Lastly, it is not difficult to pass to the limit in the weak formulation of (1.4)–(1.8) to conclude that $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$ solves (1.4)–(1.8) with $\varepsilon = 0$, as we claimed. The maximum principle for σ_τ can be then obtained repeating the argument of subsection 3.4 leading to $\sigma_\tau \in L^\infty(Q)$. This concludes the proof of theorem 2.8.

4.3. Error estimate

We focus here on the error estimate as $\varepsilon \searrow 0$ presented by theorem 2.9 under the additional assumptions (2.32) and (2.33).

First of all, we need to deduce an additional estimate on $\partial_t \mu_{\varepsilon\tau}$. Arguing as in subsection 3.6, by considering (3.26) and multiplying it by $\varepsilon^{1/2}$ (recall that $\eta = 0$), we obtain

$$\varepsilon^{3/2} \int_{Q_t} |\partial_t \mu_{\varepsilon\tau}|^2 + \frac{\varepsilon^{1/2}}{2} \|\nabla \mu_{\varepsilon\tau}(t)\|_H^2 + \frac{\tau \varepsilon^{1/2}}{2} \|\partial_t \varphi_{\varepsilon\tau}(t)\|_H^2$$

$$+ C_0 \varepsilon^{1/2} \int_{Q_t} |\partial_t \varphi_{\varepsilon\tau}|^2 + \varepsilon^{1/2} \int_{Q_t} |\partial_t \sigma_{\varepsilon\tau}|^2 + \frac{\varepsilon^{1/2}}{2} \|\nabla \sigma_{\varepsilon\tau}(t)\|_H^2$$

$$\leqslant \frac{\varepsilon^{1/2}}{2} \|\nabla \mu_{0,\varepsilon\tau}\|_H^2 + \frac{\tau \varepsilon^{1/2}}{2} \|\varphi'_{0,\varepsilon\tau}\|_H^2 + \frac{\varepsilon^{1/2}}{2} \|\nabla \sigma_{0,\varepsilon\tau}\|_H^2$$

$$+ \varepsilon^{1/2} \int_{Q_t} (P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})\partial_t \mu_{\varepsilon\tau} + \varepsilon^{1/2} \int_{Q_t} (J * (\partial_t \varphi_{\varepsilon\tau}) + \chi \partial_t \sigma_{\varepsilon\tau})\partial_t \varphi_{\varepsilon\tau}$$

$$+ \varepsilon^{1/2} \int_{Q_t} (B(\sigma_S - \sigma_{\varepsilon\tau}) - Ch(\varphi_{\varepsilon\tau})\sigma_{\varepsilon\tau}) \partial_t \sigma_{\varepsilon\tau}.$$

The last two terms on the right-hand side can be easily handled as in subsection 3.6, using the averaged Young inequality. Moreover, since $\varphi'_{0,\varepsilon\tau}$ satisfies

$$\mu_{0,\varepsilon\tau} = \tau \varphi'_{0,\varepsilon\tau} + a\varphi_{0,\varepsilon\tau} - J * \varphi_{0,\varepsilon\tau} + F'(\varphi_{0,\varepsilon\tau}) - \chi \sigma_{0,\varepsilon\tau},$$

the first three terms on the right-hand side of the inequality above are uniformly bounded in ε thanks to the assumptions (2.24), (2.25) and (2.33). As for the fourth term, this can be treated using integration by parts in time and the boundedness of $\sigma_{\varepsilon\tau}$ in (2.10) as

$$- \varepsilon^{1/2} P \int_{Q_t} \partial_t \sigma_{\varepsilon\tau} h(\varphi_{\varepsilon\tau})\mu_{\varepsilon\tau} - \varepsilon^{1/2} \int_{Q_t} (P\sigma_{\varepsilon\tau} - A)h'(\varphi_{\varepsilon\tau})\partial_t \varphi_{\varepsilon\tau}\mu_{\varepsilon\tau}$$

$$+ \varepsilon^{1/2} \int_\Omega (P\sigma_{\varepsilon\tau}(t) - A)h(\varphi_{\varepsilon\tau}(t))\mu_{\varepsilon\tau}(t) - \varepsilon^{1/2} \int_\Omega (P\sigma_{0,\varepsilon\tau} - A)h(\varphi_{0,\varepsilon\tau})\mu_{0,\varepsilon\tau}$$

$$\leqslant \frac{\varepsilon^{1/2}}{4} \int_{Q_t} |\partial_t \sigma_{\varepsilon\tau}|^2 + \varepsilon^{1/2} P^2 \|h\|_{L^\infty(\mathbb{R})}^2 \|\mu_{\varepsilon\tau}\|_{L^2(0,T;H)}^2$$

$$+ \varepsilon^{1/2}(P + A)\|h\|_{W^{1,\infty}(\mathbb{R})} \left(\|\partial_t \varphi_{\varepsilon\tau}\|_{L^2(0,T;H)} \|\mu_{\varepsilon\tau}\|_{L^2(0,T;H)} + 2\|\mu_{\varepsilon\tau}\|_{C^0([0,T];H)} \right),$$

where the right-hand side is uniformly bounded in ε thanks to (4.2)–(4.6). Putting this information together, we deduce that

$$\varepsilon^{3/4} \|\mu_{\varepsilon\tau}\|_{H^1(0,T;H)} + \varepsilon^{1/4} \|\mu_{\varepsilon\tau}\|_{L^\infty(0,T;V)} \leqslant M, \tag{4.7}$$

$$\varepsilon^{1/4} \|\varphi_{\varepsilon\tau}\|_{W^{1,\infty}(0,T;H)} + \varepsilon^{1/4} \|\sigma_{\varepsilon\tau}\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leqslant M. \tag{4.8}$$

We are now ready to show the error estimate. Taking the difference between the unique solution $(\varphi_{\varepsilon\tau}, \mu_{\varepsilon\tau}, \sigma_{\varepsilon\tau}, \xi_{\varepsilon\tau})$ to (1.4)–(1.8) with $\varepsilon, \tau > 0$ and $\eta = 0$ and the solution $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$ to (1.4)–(1.8) with $\varepsilon = \eta = 0$ obtained in subsection 4.2 leads us to

$$\varepsilon\partial_t\mu_{\varepsilon\tau} + \partial_t\varphi - \Delta\mu = P\sigma h(\varphi_{\varepsilon\tau}) + (P\sigma_\tau - A)(h(\varphi_{\varepsilon\tau}) - h(\varphi_\tau)) \quad \text{in } Q, \qquad (4.9)$$

$$\mu = \tau\partial_t\varphi + a\varphi - J * \varphi + F'(\varphi_{\varepsilon\tau}) - F'(\varphi_\tau) - \chi\sigma \qquad\qquad \text{in } Q, \quad (4.10)$$

$$\partial_t\sigma - \Delta\sigma + B\sigma + C\sigma h(\varphi_{\varepsilon\tau}) = C\sigma_\tau(h(\varphi_\tau) - h(\varphi_{\varepsilon\tau})) \qquad\qquad \text{in } Q, \quad (4.11)$$

$$\partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\sigma = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Sigma, \quad (4.12)$$

$$\varphi(0) = \varphi_0, \qquad \sigma(0) = \sigma_0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \quad (4.13)$$

where the equations are intended in the usual variational setting, and where we have set $\varphi := \varphi_{\varepsilon\tau} - \varphi_\tau, \mu := \mu_{\varepsilon\tau} - \mu_\tau, \sigma := \sigma_{\varepsilon\tau} - \sigma_\tau, \varphi_0 := \varphi_{0,\varepsilon\tau} - \varphi_{0,\tau}, \sigma_0 := \sigma_{0,\varepsilon\tau} - \sigma_{0,\tau}$. Next, we multiply (4.9) by $\tau\mu$, (4.10) by $\mu - \varphi$, (4.11) by σ, add the resulting equality and integrate over Q_t to obtain, thanks to assumption **A5**,

$$\int_{Q_t} |\mu|^2 + \tau\int_{Q_t} |\nabla\mu|^2 + \frac{\tau}{2}\|\varphi(t)\|_H^2 + C_0\int_{Q_t} |\varphi|^2 + \frac{1}{2}\|\sigma(t)\|_H^2$$

$$+ \int_{Q_t} |\nabla\sigma|^2 + \int_{Q_t} (B + Ch(\varphi_{\varepsilon\tau}))|\sigma|^2$$

$$\leqslant \frac{\tau}{2}\|\varphi_0\|_H^2 + \frac{1}{2}\|\sigma_0\|_H^2 - \int_{Q_t} \varepsilon\partial_t\mu_{\varepsilon\tau}\tau\mu + \int_{Q_t} (\mu + \chi\sigma)\varphi$$

$$+ \int_{Q_t} C\sigma_\tau(h(\varphi_\tau) - h(\varphi_{\varepsilon\tau}))\sigma + \tau\int_{Q_t} [P\sigma h(\varphi_{\varepsilon\tau}) + (P\sigma_\tau - A)$$

$$\times (h(\varphi_{\varepsilon\tau}) - h(\varphi_\tau))]\mu + \int_{Q_t} \left(a\varphi - J * \varphi + F'(\varphi_{\varepsilon\tau}) - F'(\varphi_\tau) - \chi\sigma\right)\mu.$$

Let us estimate the terms on the right-hand side separately. The third and fourth ones yield, thanks to the Young inequality and the refined estimate (4.7),

$$-\int_{Q_t} \varepsilon\partial_t\mu_{\varepsilon\tau}\tau\mu + \int_{Q_t} (\mu + \chi\sigma)\varphi$$

$$\leqslant \frac{1}{2}\int_{Q_t} |\mu|^2 + \tau^2\varepsilon^2\|\partial_t\mu_{\varepsilon\tau}\|_{L^2(0,T;H)}^2 + 2\int_{Q_t} |\varphi|^2 + \frac{\chi^2}{4}\int_{Q_t} |\sigma|^2$$

$$\leqslant M\varepsilon^{1/2} + \frac{1}{2}\int_{Q_t} |\mu|^2 + 2\int_{Q_t} |\varphi|^2 + \frac{\chi^2}{4}\int_{Q_t} |\sigma|^2,$$

for a certain constant M independent of ε. The fifth and sixth terms can be easily handled using the Young inequality, the Lipschitz continuity and boundedness of h, and the uniform bound $\|\sigma_\tau\|_{L^\infty(Q)} \leqslant 1$, as

$$\tau\int_{Q_t} (P\sigma h(\varphi_{\varepsilon\tau}) + (P\sigma_\tau - A)(h(\varphi_{\varepsilon\tau}) - h(\varphi_\tau)))\mu + \int_{Q_t} C\sigma_\tau(h(\varphi_\tau) - h(\varphi_{\varepsilon\tau}))\sigma$$

$$\leqslant \frac{1}{4}\int_{Q_t} |\mu|^2 + \|h\|_{W^{1,\infty}(\mathbb{R})}^2 \left((2\tau^2 P^2 + C^2)\int_{Q_t} |\sigma|^2 + \left(\frac{1}{4} + 2\tau^2(P + A)^2\right)\int_{Q_t} |\varphi|^2\right).$$

Moreover, the last term satisfies, thanks to the Young inequality and the growth assumption (2.32),

$$
\int_{Q_t} (a\varphi - J * \varphi + F'(\varphi_{\varepsilon\tau}) - F'(\varphi_\tau) - \chi\sigma)\mu
$$

$$
\leqslant \frac{1}{8} \int_{Q_t} |\mu|^2 + 12(a^*)^2 \int_{Q_t} |\varphi|^2 + 6\chi^2 \int_{Q_t} |\sigma|^2 + C_F \int_{Q_t} (1 + |\varphi_{\varepsilon\tau}|^2 + |\varphi_\tau|^2)|\varphi||\mu|,
$$

where, thanks to the inclusion $V \hookrightarrow L^6(\Omega)$ and the Hölder inequality,

$$
\int_{Q_t} (1 + |\varphi_{\varepsilon\tau}|^2 + |\varphi_\tau|^2)|\varphi||\mu|
$$

$$
\leqslant \int_0^t \left(|\Omega|^{1/3} + \|\varphi_{\varepsilon\tau}\|_6^2 + \|\varphi_\tau\|_6^2 \right) \|\varphi(s)\|_H \|\mu(s)\|_6 \, \mathrm{d}s
$$

$$
\leqslant M \left(1 + \|\varphi_{\varepsilon\tau}\|_{L^\infty(0,T;V)}^2 + \|\varphi_\tau\|_{L^\infty(0,T;V)}^2 \right) \int_0^t \|\varphi(s)\|_H \|\mu(s)\|_V \, \mathrm{d}s,
$$

which yields, thanks to the estimate (4.2) and again the Young inequality, that

$$
\int_{Q_t} (1 + |\varphi_{\varepsilon\tau}|^2 + |\varphi_\tau|^2)|\varphi||\mu| \leqslant \min\{1/16, \tau/2\}\|\mu\|_{L^2(0,t;V)}^2 + M_\tau \int_{Q_t} |\varphi|^2
$$

for a certain constant $M_\tau > 0$ independent of ε. Hence, collecting the above estimates we obtain

$$
\min\{1/16, \tau/2\}\|\mu\|_{L^2(0,t;V)}^2 + \frac{\tau}{2}\|\varphi(t)\|_H^2 + \frac{1}{2}\|\sigma(t)\|_H^2 + \int_{Q_t} |\nabla\sigma|^2
$$

$$
\leqslant M \left(\varepsilon^{1/2} + \frac{\tau}{2}\|\varphi_0\|_H^2 + \frac{1}{2}\|\sigma_0\|_H^2 + \int_{Q_t} |\varphi|^2 + \int_{Q_t} |\sigma|^2 \right),
$$

where the updated constant M depends on τ, and the initial data $(\varphi_{0,\tau}, \sigma_{0,\tau})$. The error estimate follows then by the Gronwall lemma.

Finally, it is not difficult to check that exactly the same argument performed here yields uniqueness of the solution $(\varphi_\tau, \mu_\tau, \sigma_\tau, \xi_\tau)$ for the system (1.4)–(1.8) at $\varepsilon = 0$, even without assumption (2.33). This reality implies then that the convergences as $\varepsilon \searrow 0$ hold along the entire sequence ε which completes the proof of theorem 2.9.

5. Asymptotics as $\tau \searrow 0$

Let us now investigate the behavior of system (1.4)–(1.8) as $\tau \searrow 0$ by proving theorems 2.11 and 2.12. Proceeding as before, notice that throughout this section we assume $\varepsilon \in (0, \varepsilon_0)$ to be fixed.

5.1. Uniform estimates

Performing the same estimates as in subsection 3.2, and noting that the constant M in (3.9) is independent of τ, ε, λ, and n, we infer that

$$\frac{\varepsilon}{2}\|\mu_{\varepsilon\tau}(t)\|_H^2 + (1 + 4c_a\varepsilon)\int_{Q_t}|\nabla\mu_{\varepsilon\tau}|^2 + \tau\int_{Q_t}|\partial_t\varphi_{\varepsilon\tau}|^2$$

$$+ \frac{1}{2}\|\sigma_{\varepsilon\tau}(t)\|_H^2 + \int_{Q_t}|\nabla\sigma_{\varepsilon\tau}|^2 + 2c_a\|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau})(t)\|_H^2$$

$$+ 2c_a\tau\|\nabla\varphi_{\varepsilon\tau}(t)\|_H^2 + 2c_aC_0\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2$$

$$\leqslant \frac{3}{2}\varepsilon\|\mu_{0,\varepsilon\tau}\|_H^2 + (a^* + 4c_a)\|\varphi_{0,\varepsilon\tau}\|_H^2 + 2c_a\tau\|\nabla\varphi_{0,\varepsilon\tau}\|_H^2$$

$$+ \|F(\varphi_{0,\varepsilon\tau})\|_{L^1(\Omega)} + \frac{1}{2}\|\sigma_{0,\varepsilon\tau}\|_H^2 + \frac{c_a}{2}\|\varphi_{\varepsilon\tau}(t)\|_H^2$$

$$+ \chi\int_{Q_t}\sigma_{\varepsilon\tau}\partial_t\varphi_{\varepsilon\tau} + (\eta + 4c_a\chi)\int_{Q_t}\nabla\sigma_{\varepsilon\tau}\cdot\nabla\varphi_{\varepsilon\tau}$$

$$+ M\left(1 + \int_{Q_t}|\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}|^2 + \int_{Q_t}|\varphi_{\varepsilon\tau}|^2 + \int_{Q_t}|\sigma_{\varepsilon\tau}|^2\right)$$

$$+ \int_{Q_t}(P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})\mu_{\varepsilon\tau}. \tag{5.1}$$

First of all, note that all the terms on the right-hand side referring to the initial data are uniformly bounded in τ due to assumptions (2.36) and (2.37). Moreover, since $\varepsilon \in (0, \frac{1}{4c_a})$ we have a bound from below on the left-hand side in the form

$$2c_a\|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau})(t)\|_H^2 + \frac{\varepsilon}{2}\|\mu_{\varepsilon\tau}(t)\|_H^2$$

$$\geqslant 2c_a\|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau})(t)\|_H^2 + 2c_a\varepsilon^2\|\mu_{\varepsilon\tau}(t)\|_H^2$$

$$\geqslant (c_a - \rho)\|\varphi_{\varepsilon\tau}(t)\|_H^2 + 2\rho\varepsilon^2\|\mu_{\varepsilon\tau}(t)\|_H^2 \tag{5.2}$$

for every $\rho \in (0, c_a)$. Hence the corresponding term $\frac{c_a}{2}\|\varphi_{\varepsilon\tau}(t)\|_H^2$ on the right-hand side can be incorporated on the left-hand side of (5.1), provided we choose $\rho < c_a/2$. Furthermore, from the boundedness of h the last term in (5.1) can be easily handled using the Young inequality and the Gronwall lemma. Hence, we only need to estimate the terms involving χ and η. To this end, we first use integration by parts and the equation (2.8) to deduce, thanks to the Young inequality and the boundedness of h, that

$$\chi\int_{Q_t}\sigma_{\varepsilon\tau}\partial_t\varphi_{\varepsilon\tau} = -\chi\int_0^t\langle\partial_t\sigma_{\varepsilon\tau}(s), \varphi_{\varepsilon\tau}(s)\rangle\,\mathrm{d}s + \chi\int_\Omega\sigma_{\varepsilon\tau}(t)\varphi_{\varepsilon\tau}(t) - \chi\int_\Omega\sigma_{0,\varepsilon\tau}\varphi_{0,\varepsilon\tau}$$

$$= \chi\int_{Q_t}\nabla\sigma_{\varepsilon\tau}\cdot\nabla\varphi_{\varepsilon\tau} + \chi\int_{Q_t}(B(\sigma_{\varepsilon\tau} - \sigma_S) + C\sigma_{\varepsilon\tau}h(\varphi_{\varepsilon\tau}))\varphi_{\varepsilon\tau}$$

$$- \chi\eta\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2 + \chi\int_\Omega\sigma_{\varepsilon\tau}(t)\varphi_{\varepsilon\tau}(t) - \chi\int_\Omega\sigma_{0,\varepsilon\tau}\varphi_{0,\varepsilon\tau}$$

$$\leqslant \chi\int_{Q_t}\nabla\sigma_{\varepsilon\tau}\cdot\nabla\varphi_{\varepsilon\tau} - \chi\eta\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2 + M\left(1 + \int_{Q_t}|\varphi_{\varepsilon\tau}|^2 + \int_{Q_t}|\sigma_{\varepsilon\tau}|^2\right)$$

$$+ \delta\chi^2\|\varphi_{\varepsilon\tau}(t)\|_H^2 + \frac{1}{4\delta}\|\sigma_{\varepsilon\tau}(t)\|_H^2, \tag{5.3}$$

for every $\delta > 0$. Now, it is immediate to check that assumption (2.34) yields $\frac{1}{2} < \frac{c_a}{2\chi^2}$ (with the convention that $\frac{1}{\chi} = +\infty$ if $\chi = 0$): hence

$$\exists\, \bar{\delta} \in \left(\frac{1}{2}, \frac{c_a}{2\chi^2}\right) \quad \text{such that} \quad \bar{\delta}\chi^2 < \frac{c_a}{2}, \qquad \frac{1}{4\bar{\delta}} < \frac{1}{2}, \tag{5.4}$$

so that we can incorporate the last two terms on the right-hand side of (5.3) on the left-hand side of (5.1). Taking these remarks into account, we are left with

$$2\rho\varepsilon^2\|\mu_{\varepsilon\tau}(t)\|_H^2 + \left(\frac{c_a}{2} - \rho - \bar{\delta}\chi^2\right)\|\varphi_{\varepsilon\tau}(t)\|_H^2 + (1 + 4c_a\varepsilon)\int_{Q_t}|\nabla\mu_{\varepsilon\tau}|^2$$

$$+ \tau\int_{Q_t}|\partial_t\varphi_{\varepsilon\tau}|^2 + \left(\frac{1}{2} - \frac{1}{4\bar{\delta}}\right)\|\sigma_{\varepsilon\tau}(t)\|_H^2 + \int_{Q_t}|\nabla\sigma_{\varepsilon\tau}|^2$$

$$+ 2c_a\tau\|\nabla\varphi_{\varepsilon\tau}(t)\|_H^2 + (2c_aC_0 + \chi\eta)\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2$$

$$\leqslant M\left(1 + \int_{Q_t}|\mu_{\varepsilon\tau}|^2 + \int_{Q_t}|\varphi_{\varepsilon\tau}|^2 + \int_{Q_t}|\sigma_{\varepsilon\tau}|^2\right)$$

$$+ (\chi + \eta + 4c_a\chi)\int_{Q_t}\nabla\sigma_{\varepsilon\tau}\cdot\nabla\varphi_{\varepsilon\tau}, \tag{5.5}$$

which holds for every $\rho \in (0, c_a/2)$. By choosing $\bar{\delta}$ such that (5.4) are fulfilled, it is also possible to choose and fix $\bar{\rho} \in (0, c_a/2)$ such that

$$\frac{c_a}{2} - \bar{\rho} - \bar{\delta}\chi^2 > 0.$$

Next, we use again the averaged Young inequality to obtain, for every $\kappa > 0$,

$$(\chi + \eta + 4c_a\chi)\int_{Q_t}\nabla\sigma_{\varepsilon\tau}\cdot\nabla\varphi_{\varepsilon\tau} \leqslant \kappa\int_{Q_t}|\nabla\sigma_{\varepsilon\tau}|^2 + \frac{(\chi + \eta + 4c_a\chi)^2}{4\kappa}\int_{Q_t}|\nabla\varphi_{\varepsilon\tau}|^2$$

where the two terms on the right-hand side can be incorporated on the left-hand side of (5.5) provided to choose κ such that

$$\kappa < 1, \quad \frac{(\chi + \eta + 4c_a\chi)^2}{4\kappa} < 2c_aC_0 + \chi\eta.$$

Easy computations show that this is possible if and only if

$$\frac{(\chi + \eta + 4c_a\chi)^2}{4(2c_aC_0 + \chi\eta)} < 1$$

which is verified owing to (2.34).

Therefore, after rearranging the terms and using the Gronwall lemma, we infer that there exists a constant $M > 0$, which may depend on ε, but it is independent of τ, such that

$$\|\varphi_{\varepsilon\tau}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mu_{\varepsilon\tau}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}$$

$$+ \|\sigma_{\varepsilon\tau}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leqslant M, \tag{5.6}$$

$$\tau^{1/2}\|\varphi_{\varepsilon\tau}\|_{H^1(0,T;H)} + \tau^{1/2}\|\varphi_{\varepsilon\tau}\|_{L^\infty(0,T;V)} \leqslant M, \tag{5.7}$$

yielding in turn, by comparison in equations (1.4) and (1.6),

$$\|\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}\|_{H^1(0,T;V^*)} + \|\sigma_{\varepsilon\tau}\|_{H^1(0,T;V^*)} \leqslant M. \tag{5.8}$$

Testing equation (1.5) by $\xi_{\varepsilon\tau}$ and using the estimate (5.6), its is a standard matter to deduce also that

$$\|\xi_{\varepsilon\tau}\|_{L^2(0,T;H)} \leqslant M. \tag{5.9}$$

5.2. Passage to the limit

The estimates (5.6)–(5.9) and classical compactness results (see, e.g., [80, section 8, corollary 4]) ensure that there exists a quadruplet $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon, \xi_\varepsilon)$ with

$$\varphi_\varepsilon, \mu_\varepsilon \in L^\infty(0,T;H) \cap L^2(0,T;V), \qquad \lambda_\varepsilon := \varepsilon\mu_\varepsilon + \varphi_\varepsilon \in H^1(0,T;V^*) \cap L^2(0,T;V),$$

$$\sigma_\varepsilon \in H^1(0,T;V^*) \cap L^2(0,T;V), \qquad \xi_\varepsilon \in L^2(0,T;H),$$

such that, as $\tau \searrow 0$ (on a subsequence) it holds that (2.38)–(2.43) and (2.44)–(2.45) are satisfied, and also that

$$\lambda_{\varepsilon\tau} \rightharpoonup \lambda_\varepsilon \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V),$$

$$\lambda_{\varepsilon\tau} \to \lambda_\varepsilon \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H).$$

Moreover, let us claim that the above strong convergences imply the strong convergences

$$\mu_{\varepsilon\tau} \to \mu_\varepsilon \quad \text{in } L^2(0,T;H), \qquad \varphi_{\varepsilon\tau} \to \varphi_\varepsilon \quad \text{in } L^2(0,T;H). \tag{5.10}$$

To this end, we argue as in [18, section 3], checking that the sequence $\{\lambda_{\varepsilon\tau}\}_\tau$ is a Cauchy sequence in $L^2(0,T;H)$. Let us pick two arbitrary $\tau, \tau' > 0$ and take the difference of the corresponding equation (1.5) for τ and τ'. Next, we multiply the resulting equation by ε, add to both sides $\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}$, test the resulting equation by $\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}$, and integrate over Q_t to obtain

$$\int_{Q_t} |\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}|^2 + \varepsilon \int_{Q_t} \left(a(\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}) + \xi_{\varepsilon\tau} - \xi_{\varepsilon\tau'} + F_2'(\varphi_{\varepsilon\tau}) - F_2'(\varphi_{\varepsilon\tau'}) \right)$$

$$\times (\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}) \leqslant \int_{Q_t} \left((\lambda_{\varepsilon\tau} - \lambda_{\varepsilon\tau'}) - \varepsilon(\tau\partial_t\varphi_{\varepsilon\tau} - \tau'\partial_t\varphi_{\varepsilon\tau'}) + \varepsilon\chi(\sigma_{\varepsilon\tau} - \sigma_{\varepsilon\tau'}) \right)$$

$$\times (\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}) + \varepsilon \int_{Q_t} J * (\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'})(\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}).$$

Owing to (2.38)–(2.43) and (2.44)–(2.45) we easily infer that the first term on the right-hand side goes to zero as $\tau, \tau' \to 0$. Moreover, on the left-hand side we have, thanks to assumption **A5**,

$$\int_{Q_t} \left(a(\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}) + \xi_{\varepsilon\tau} - \xi_{\varepsilon\tau'} + F_2'(\varphi_{\varepsilon\tau}) - F_2'(\varphi_{\varepsilon\tau'}) \right)(\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'})$$

$$\geqslant C_0 \int_{Q_t} |\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}|^2,$$

while the last term on the right-hand side satisfies

$$\int_{Q_t} J * (\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'})(\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}) \leqslant a^* \int_{Q_t} |\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}|^2.$$

Rearranging the terms leads us to

$$(1 + (C_0 - a^*)\varepsilon) \int_{Q_t} |\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'}|^2$$

$$\leqslant \int_{Q_t} \left((\lambda_{\varepsilon\tau} - \lambda_{\varepsilon\tau'}) - \varepsilon(\tau\partial_t\varphi_{\varepsilon\tau} - \tau'\partial_t\varphi_{\varepsilon\tau'}) + \chi(\sigma_{\varepsilon\tau} - \sigma_{\varepsilon\tau'}) \right) (\varphi_{\varepsilon\tau} - \varphi_{\varepsilon\tau'})$$

where the right-hand side converges to 0 as $\tau \searrow 0$. Since $\varepsilon a^* < \varepsilon C_0 + 1$ as a consequence of the smallness assumption on ε_0, this yields the second of (5.10) and by comparison also the first one follows, as we claimed.

With the strong convergence of the phase variable at disposal it is now straightforward to infer by combining the boundedness of h and the Lebesgue convergence theorem that, as $\tau \searrow 0$,

$$h(\varphi_{\varepsilon\tau}) \to h(\varphi_\varepsilon) \quad \text{in } L^p(Q) \quad \forall\, p \geqslant 1, \qquad F_2'(\varphi_{\varepsilon\tau}) \to F_2'(\varphi_\varepsilon) \quad \text{in } L^2(0, T; H).$$

Hence, since $\xi_\varepsilon \in \partial F_1(\varphi_\varepsilon)$ by the strong–weak closure of ∂F_1, it is a standard matter to pass to the limit as $\tau \searrow 0$ in the weak formulation of (1.4)–(1.8) and deduce that the limit $(\mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon, \xi_\varepsilon)$ yields a solution to (1.4)–(1.8) with $\tau = 0$. Notice in particular that by difference in the limit equation (1.5) we deduce the further regularity $\xi_\varepsilon \in L^2(0, T; V)$, while the last assertion of theorem 2.11 follows as before by repeating the computations of subsection 3.4 completing the proof of theorem 2.11.

5.3. Error estimate

The last result of this section follows with few changes from the proof of the continuous dependence estimate (2.11) established in theorem 2.2.

Indeed, we can repeat almost the same computations performed in subsection 3.5 with the choices

$$(\varphi_1, \mu_1, \sigma_1, \xi_1) := (\varphi_{\varepsilon\tau}, \mu_{\varepsilon\tau}, \sigma_{\varepsilon\tau}, \xi_{\varepsilon\tau}), \qquad (\varphi_2, \mu_2, \sigma_2, \xi_2) := (\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon, \xi_\varepsilon).$$

Moreover, by setting $\varphi := \varphi_{\varepsilon\tau} - \varphi_\varepsilon$, $\mu := \mu_{\varepsilon\tau} - \mu_\varepsilon$, $\sigma := \sigma_{\varepsilon\tau} - \sigma_\varepsilon$, $\varphi_0 := \varphi_{0,\varepsilon\tau} - \varphi_{0,\varepsilon}$, $\mu_0 := \mu_{0,\varepsilon\tau} - \mu_{0,\varepsilon}$, and $\sigma_0 := \sigma_{0,\varepsilon\tau} - \sigma_{0,\varepsilon}$, recalling that we are assuming $\eta = 0$ we infer from (3.25) that

$$\frac{1}{2}\|(\varepsilon\mu + \varphi)(t)\|_{V^*}^2 + \varepsilon \int_{Q_t} |\mu|^2 + C_0 \int_{Q_t} |\varphi|^2 + \frac{1}{2}\|\sigma(t)\|_H^2 + \int_{Q_t} |\nabla\sigma|^2$$

$$\leqslant -\tau \int_{Q_t} \partial_t\varphi_{\varepsilon\tau}\varphi + \frac{1}{2}\|\varepsilon\mu_0 + \varphi_0\|_{V^*}^2 + \frac{1}{2}\|\sigma_0\|_H^2$$

$$+ \int_{Q_t} (\chi\sigma + J * \varphi)\varphi + \int_{Q_t} [C\sigma_\varepsilon(h(\varphi_\varepsilon) - h(\varphi_{\varepsilon\tau}))]\sigma$$

$$+ \int_{Q_t} [\mu + P\sigma h(\varphi_{\varepsilon\tau}) + (P\sigma_\varepsilon - A)(h(\varphi_{\varepsilon\tau}) - h(\varphi_\varepsilon))]\, \mathcal{R}^{-1}(\varepsilon\mu + \varphi).$$

All the terms on the right-hand side, except the first one, can be handled in exactly the same way as in subsection 3.5. As for the first one, we use the Young inequality and estimate (5.7) to infer, for every $\delta > 0$,

$$-\tau \int_{Q_t} \partial_t \varphi_{\varepsilon\tau} \varphi \leqslant \delta \int_{Q_t} |\varphi|^2 + \frac{\tau^2}{4\delta} \int_{Q_t} |\partial_t \varphi_{\varepsilon\tau}|^2 \leqslant \delta \int_{Q_t} |\varphi|^2 + M_\delta \tau,$$

so that the first term on the right-hand side can be absorbed on the left provided to choose again δ small enough, which is indeed possible as we noted in subsection 3.5. We can now argue as before and conclude using Gronwall's lemma. Moreover, the same argument on the limit problem yields uniqueness of solution for the system with $\tau = 0$, hence also that the convergences hold along the entire sequence and the proof of theorem 2.12 is concluded.

6. Asymptotics as both $\varepsilon, \tau \searrow 0$

The last issue we are going to address here concerns the joint asymptotic limit as both $\varepsilon, \tau \searrow 0$. Let us recall that in this section we are supposing that $\eta = 0$.

6.1. Uniform estimates

Let us come back to estimate (3.9) with $\eta = 0$. We have

$$\frac{\varepsilon}{2} \|\mu_{\varepsilon\tau}(t)\|_H^2 + (1 + 4c_a\varepsilon) \int_{Q_t} |\nabla \mu_{\varepsilon\tau}|^2 + \tau \int_{Q_t} |\partial_t \varphi_{\varepsilon\tau}|^2 + \int_\Omega F(\varphi_{\varepsilon\tau}(t))$$

$$+ \frac{1}{2} \|\sigma_{\varepsilon\tau}(t)\|_H^2 + \int_{Q_t} |\nabla \sigma_{\varepsilon\tau}|^2 + 2c_a \|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau})(t)\|_H^2$$

$$+ 2c_a\tau \|\nabla \varphi_{\varepsilon\tau}(t)\|_H^2 + 2c_a C_0 \int_{Q_t} |\nabla \varphi_{\varepsilon\tau}|^2$$

$$\leqslant \frac{3}{2} \varepsilon \|\mu_{0,\varepsilon\tau}\|_H^2 + (a^* + 4c_a) \|\varphi_{0,\varepsilon\tau}\|_H^2 + 2c_a\tau \|\nabla \varphi_{0,\varepsilon\tau}\|_H^2 + \|F(\varphi_{0,\varepsilon\tau})\|_{L^1(\Omega)}$$

$$+ \frac{1}{2} \|\sigma_{0,\varepsilon\tau}\|_H^2 + \frac{c_a}{2} \|\varphi_{\varepsilon\tau}(t)\|_H^2 + \chi \int_{Q_t} \sigma_{\varepsilon\tau} \partial_t \varphi_{\varepsilon\tau} + 4c_a\chi \int_{Q_t} \nabla \sigma_{\varepsilon\tau} \cdot \nabla \varphi_{\varepsilon\tau}$$

$$+ M \left(1 + \int_{Q_t} |\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}|^2 + \int_{Q_t} |\varphi_{\varepsilon\tau}|^2 + \int_{Q_t} |\sigma_{\varepsilon\tau}|^2 \right)$$

$$+ \int_{Q_t} (P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})\mu_{\varepsilon\tau}, \tag{6.1}$$

where the constant $M > 0$ is independent of both ε and τ. Now, all the terms on the right-hand side referring to the initial data are uniformly bounded in both ε and τ thanks to assumptions (2.47) and (2.48). Moreover, as done in (5.2), on the left-hand side we have

$$2c_a \|(\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau})(t)\|_H^2 + \frac{\varepsilon}{2} \|\mu_{\varepsilon\tau}(t)\|_H^2 \geqslant (c_a - \rho)\|\varphi_{\varepsilon\tau}(t)\|_H^2 + 2\rho\varepsilon^2 \|\mu_{\varepsilon\tau}(t)\|_H^2$$

for every $\rho \in (0, c_a/2)$, so that the term on the right-hand side of the above inequality can be absorbed on the left-hand side of (6.1). Furthermore, proceeding again as in subsection 5.1 and

recalling that here $\eta = 0$, we have

$$\chi \int_{Q_t} \sigma_{\varepsilon\tau} \partial_t \varphi_{\varepsilon\tau} = -\chi \int_0^t \langle \partial_t \sigma_{\varepsilon\tau}(s), \varphi_{\varepsilon\tau}(s) \rangle \, \mathrm{d}s$$

$$+ \chi \int_\Omega \sigma_{\varepsilon\tau}(t) \varphi_{\varepsilon\tau}(t) - \chi \int_\Omega \sigma_{0,\varepsilon\tau} \varphi_{0,\varepsilon\tau}$$

$$\leqslant \chi \int_{Q_t} \nabla \sigma_{\varepsilon\tau} \cdot \nabla \varphi_{\varepsilon\tau} + M \left(1 + \int_{Q_t} |\varphi_{\varepsilon\tau}|^2 + \int_{Q_t} |\sigma_{\varepsilon\tau}|^2 \right)$$

$$+ \delta \chi^2 \|\varphi_{\varepsilon\tau}(t)\|_H^2 + \frac{1}{4\delta} \|\sigma_{\varepsilon\tau}(t)\|_H^2,$$

for every $\delta > 0$. Moreover, we can choose $\bar\delta$ such that (5.4) are satisfied, so that the corresponding two terms on the right-hand side can be incorporated on the left. The remaining terms on the right-hand side of (6.1) containing χ can be handled as, for every $\kappa > 0$,

$$(\chi + 4c_a\chi) \int_{Q_t} \nabla \sigma_{\varepsilon\tau} \cdot \nabla \varphi_{\varepsilon\tau} \leqslant \kappa \int_{Q_t} |\nabla \sigma_{\varepsilon\tau}|^2 + \frac{(\chi + 4c_a\chi)^2}{4\kappa} \int_{Q_t} |\nabla \varphi_{\varepsilon\tau}|^2.$$

Again, the two terms on the right can be incorporated on the left-hand side of (6.1) provided that we choose κ such that

$$\kappa < 1, \quad \frac{(\chi + 4c_a\chi)^2}{4\kappa} < 2c_aC_0,$$

which is indeed possible since (2.34) and the fact that $\eta = 0$ yield $\frac{(\chi + 4c_a\chi)^2}{8c_aC_0} < 1$. To close the estimate, we only need to handle the last term on the right-hand side of (6.1): this can be done exactly in the same way as in subsection 4.1. Indeed, on the right-hand side we have, thanks to the boundedness of h and the fact that $\|\sigma_{\varepsilon\tau}\|_{L^\infty(Q)} \leqslant 1$,

$$\int_{Q_t} (P\sigma_{\varepsilon\tau} - A) h(\varphi_{\varepsilon\tau}) \mu_{\varepsilon\tau} \leqslant \frac{1}{2} \int_{Q_t} |\nabla \mu_{\varepsilon\tau}|^2 + M' \left(1 + T_0^{1/2} \|(\mu_{\varepsilon\tau})_\Omega\|_{L^2(0,t)} \right)$$

for every $t \in [0, T_0]$ and $T_0 < T$, where M' only depends on P, A, and h. Furthermore, by comparison in equation (1.5) and thanks to (2.22), since $\tau \in (0, 1)$, we have

$$|(\mu_{\varepsilon\tau}(t))_\Omega| \leqslant M'' \left(1 + \tau \int_{Q_t} |\partial_t \varphi_{\varepsilon\tau}(t)|^2 + \sup_{s \in [0,t]} \int_\Omega F(\varphi_{\varepsilon\tau}(s)) \right.$$

$$\left. + \sup_{s \in [0,t]} \|\sigma_{\varepsilon\tau}(s)\|_H^2 \right),$$

where $M'' > 0$ only depends on C_F and χ. Hence, using a patching argument as in subsection 4.1, we deduce the following uniform estimates

$$\|\varphi_{\varepsilon\tau}\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\mu_{\varepsilon\tau}\|_{L^2(0,T;V)}$$

$$+ \|\sigma_{\varepsilon\tau}\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leqslant M, \tag{6.2}$$

$$\|F(\varphi_{\varepsilon\tau})\|_{L^\infty(0,T;L^1(\Omega))} \leqslant M, \tag{6.3}$$

$$\varepsilon^{1/2} \|\mu_{\varepsilon\tau}\|_{L^\infty(0,T;H)} + \tau^{1/2} \|\varphi_{\varepsilon\tau}\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leqslant M. \tag{6.4}$$

Comparison then in the system gives us in particular that

$$\|\xi_{\varepsilon\tau}\|_{L^2(0,T;H)} + \|\sigma_{\varepsilon\tau}\|_{H^1(0,T;V^*)} + \|\varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}\|_{H^1(0,T;V^*)} \leqslant M, \tag{6.5}$$

as well as

$$\varepsilon\tau^{1/2}\|\mu_{\varepsilon\tau}\|_{H^1(0,T;V^*)} \leqslant M. \tag{6.6}$$

The uniform bound for $\sigma_{\varepsilon\tau}$ in $L^\infty(Q)$ can be obtained as before using subsection 3.4.

6.2. Passage to the limit

The estimates (6.2)–(6.6) ensure, thanks to the classical compactness results, that there exists a quadruplet $(\varphi, \mu, \sigma, \xi)$, with

$$\varphi \in H^1(0,T;V^*) \cap L^2(0,T;V), \qquad \mu \in L^2(0,T;V),$$

$$\sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^\infty(Q),$$

$$0 \leqslant \sigma(t,x) \leqslant 1 \quad \text{for a.e. } x \in \Omega, \quad \forall\, t \in [0,T],$$

$$\xi \in L^2(0,T;H),$$

such that, as $(\varepsilon,\tau) \searrow 0$ it holds that, along a non-relabelled subsequence, (2.49)–(2.55) are fulfilled. In addition, setting $\lambda_{\varepsilon\tau} := \varepsilon\mu_{\varepsilon\tau} + \varphi_{\varepsilon\tau}$, we have

$$\lambda_{\varepsilon\tau} \rightharpoonup \varphi \quad \text{in } H^1(0,T;V^*) \cap L^2(0,T;V),$$

$$\lambda_{\varepsilon\tau} \to \varphi \quad \text{in } C^0([0,T];V^*) \cap L^2(0,T;H),$$

$$\xi_{\varepsilon\tau} \rightharpoonup \xi \quad \text{in } L^2(0,T;H).$$

In particular, by difference we deduce that

$$\varphi_{\varepsilon\tau} = \lambda_{\varepsilon\tau} - \varepsilon\mu_{\varepsilon\tau} \to \varphi \quad \text{in } L^2(0,T;H)$$

which readily implies that $\xi \in \partial F_1(\varphi)$ almost everywhere in Q, and that

$$h(\varphi_{\varepsilon\tau}) \to h(\varphi) \quad \text{in } L^p(Q) \quad \forall\, p \geqslant 1, \qquad F_2'(\varphi_{\varepsilon\tau}) \to F_2'(\varphi) \quad \text{in } L^2(0,T;H).$$

It is then a standard matter to let $(\varepsilon,\tau) \searrow 0$ in the weak formulation of (1.4)–(1.8) to conclude. Note in particular that by difference in the limit equation (1.5) we deduce the further regularity $\xi \in L^2(0,T;V)$, which concludes the proof of theorem 2.14.

6.3. Error estimate

In this last subsection we prove the error estimate as both ε and τ go to zero.

   The idea is to adapt the argument presented in subsection 4.3. First of all, we need to prove a refined estimate: proceeding as in subsection 4.3, we know that

$$\varepsilon^{3/2}\int_{Q_t} |\partial_t\mu_{\varepsilon\tau}|^2 + \frac{\varepsilon^{1/2}}{2}\|\nabla\mu_{\varepsilon\tau}(t)\|_H^2 + \frac{\tau\varepsilon^{1/2}}{2}\|\partial_t\varphi_{\varepsilon\tau}(t)\|_H^2$$

$$+ C_0\varepsilon^{1/2}\int_{Q_t} |\partial_t\varphi_{\varepsilon\tau}|^2 + \varepsilon^{1/2}\int_{Q_t} |\partial_t\sigma_{\varepsilon\tau}|^2 + \frac{\varepsilon^{1/2}}{2}\|\nabla\sigma_{\varepsilon\tau}(t)\|_H^2$$

$$\leqslant \frac{\varepsilon^{1/2}}{2}\|\nabla\mu_{0,\varepsilon\tau}\|_H^2 + \frac{\tau\varepsilon^{1/2}}{2}\|\varphi'_{0,\varepsilon\tau}\|_H^2 + \frac{\varepsilon^{1/2}}{2}\|\nabla\sigma_{0,\varepsilon\tau}\|_H^2$$

$$+ \varepsilon^{1/2}\int_{Q_t}(P\sigma_{\varepsilon\tau} - A)h(\varphi_{\varepsilon\tau})\partial_t\mu_{\varepsilon\tau} + \varepsilon^{1/2}\int_{Q_t}(J * (\partial_t\varphi_{\varepsilon\tau}) + \chi\partial_t\sigma_{\varepsilon\tau})\partial_t\varphi_{\varepsilon\tau}$$

$$+ \varepsilon^{1/2}\int_{Q_t}(B(\sigma_S - \sigma_{\varepsilon\tau}) - Ch(\varphi_{\varepsilon\tau})\sigma_{\varepsilon\tau})\partial_t\sigma_{\varepsilon\tau}. \tag{6.7}$$

The first and third terms on the right-hand side are uniformly bounded in ε and τ due to assumptions (2.48) and (2.57). As for the second term on the right-hand side, using (1.5) we realize that

$$\mu_{0,\varepsilon\tau} = \tau\varphi'_{0,\varepsilon\tau} + a\varphi_{0,\varepsilon\tau} - J * \varphi_{0,\varepsilon\tau} + F'(\varphi_{0,\varepsilon\tau}) - \chi\sigma_{0,\varepsilon\tau},$$

so that, multiplying both sides by $\varepsilon^{1/4}/\tau^{1/2}$ and squaring,

$$\tau\varepsilon^{1/2}\|\varphi'_{0,\varepsilon\tau}\|_H^2 \leqslant 5\frac{\varepsilon^{1/2}}{\tau}\left(\|\mu_{0,\varepsilon\tau}\|_H^2 + 2(a^*)^2\|\varphi_{0,\varepsilon\tau}\|_H^2\right.$$

$$\left. + \|F'(\varphi_{0,\varepsilon\tau})\|_H^2 + \chi^2\|\sigma_{0,\varepsilon\tau}\|_H^2\right),$$

from which we deduce by (2.57) that the second term on the right-hand side of (6.7) is uniformly bounded in ε and τ. Let us focus on the fourth term on the right-hand side: proceeding as in subsection 4.3, this can be bounded using integration by parts and the Young inequality by the quantity

$$\frac{\varepsilon^{1/2}}{4}\int_{Q_t}|\partial_t\sigma_{\varepsilon\tau}|^2 + M\varepsilon^{1/2}\left(\|\mu_{\varepsilon\tau}\|_{L^2(0,T;H)}^2 + \|\partial_t\varphi_{\varepsilon\tau}\|_{L^2(0,T;H)}^2 + \varepsilon^{1/2}\|\mu_{\varepsilon\tau}\|_{C^0([0,T];H)}\right)$$

for a positive constant M independent of ε and τ. The first term can be then incorporated on the left-hand side, and the remaining others are uniformly bounded in ε and τ thanks to the estimates (6.2), (6.4), and condition (2.58) on (ε, τ). Finally, noting that

$$\varepsilon^{1/2}\int_{Q_t}(J * \partial_t\varphi_{\varepsilon\tau})\partial_t\varphi_{\varepsilon\tau} \leqslant (a^* + b^*)\varepsilon^{1/2}\int_0^t\|\partial_t\varphi_{\varepsilon\tau}(s)\|_H\|\partial_t\varphi_{\varepsilon\tau}(s)\|_{V^*}\,\mathrm{d}s$$

$$\leqslant M\varepsilon^{1/2}\|\partial_t\varphi_{\varepsilon\tau}\|_{L^2(0,T;H)}^2,$$

the remaining terms on the right-hand side of (6.7) can be handled similarly, using the averaged Young inequality, estimate (6.2)–(6.4), and condition (2.58). Thus, there exists $M > 0$, independent of both ε and τ, such that

$$\varepsilon^{3/4}\|\mu_{\varepsilon\tau}\|_{H^1(0,T;H)} + \varepsilon^{1/4}\|\mu_{\varepsilon\tau}\|_{L^\infty(0,T;V)} \leqslant M, \tag{6.8}$$

$$\tau^{1/2}\varepsilon^{1/4}\|\varphi_{\varepsilon\tau}\|_{W^{1,\infty}(0,T;H)} + \varepsilon^{1/4}\|\sigma_{\varepsilon\tau}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leqslant M. \tag{6.9}$$

We are now ready to show the error estimate. Setting $\overline{\varphi} := \varphi_{\varepsilon\tau} - \varphi$, $\overline{\mu} := \mu_{\varepsilon\tau} - \mu$, $\overline{\sigma} := \sigma_{\varepsilon\tau} - \sigma$, $\overline{\varphi}_0 := \varphi_{0,\varepsilon\tau} - \varphi_0$, and $\overline{\sigma}_0 := \sigma_{0,\varepsilon\tau} - \sigma_0$, we write the difference of the system (1.4)–(1.8) with $\eta = 0$ at $\varepsilon, \tau > 0$ and $\varepsilon = \tau = 0$ to find that

$$\varepsilon\partial_t\mu_{\varepsilon\tau} + \partial_t\overline{\varphi} - \Delta\overline{\mu} = P\overline{\sigma}h(\varphi_{\varepsilon\tau}) + (P\sigma - A)(h(\varphi_{\varepsilon\tau}) - h(\varphi)) \quad \text{in } Q, \tag{6.10}$$

$$\overline{\mu} = \tau\partial_t\varphi_{\varepsilon\tau} + a\overline{\varphi} - J * \overline{\varphi} + F'(\varphi_{\varepsilon\tau}) - F'(\varphi) - \chi\overline{\sigma} \quad \text{in } Q, \tag{6.11}$$

$$\partial_t \overline{\sigma} - \Delta \overline{\sigma} + B\overline{\sigma} + C\overline{\sigma}h(\varphi_{\varepsilon\tau}) = C\sigma(h(\varphi) - h(\varphi_{\varepsilon\tau})) \quad \text{in } Q, \tag{6.12}$$

$$\partial_{\mathbf{n}}\overline{\mu} = \partial_{\mathbf{n}}\overline{\sigma} = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Sigma, \tag{6.13}$$

$$\overline{\varphi}(0) = \overline{\varphi}_0, \qquad \overline{\sigma}(0) = \overline{\sigma}_0 \qquad\qquad\qquad\qquad \text{in } \Omega, \tag{6.14}$$

where the equations have to be intended in the usual variational framework. We test (6.10) by $\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega)$, (6.11) by $\overline{\varphi} - (\overline{\varphi})_\Omega$, (6.12) by $\overline{\sigma}$, integrate over Q_t, add the resulting equalities and use **A5** to get

$$\frac{1}{2}\|(\overline{\varphi} - (\overline{\varphi})_\Omega)(t)\|_{V^*}^2 + C_0 \int_{Q_t} |\overline{\varphi}|^2 + \frac{1}{2}\|\overline{\sigma}(t)\|_H^2 + \int_{Q_t} |\nabla\overline{\sigma}|^2 + \int_{Q_t} (B + Ch(\varphi_{\varepsilon\tau}))|\overline{\sigma}|^2$$

$$= \frac{1}{2}\|\overline{\varphi}_0 - (\overline{\varphi}_0)_\Omega\|_{V^*}^2 + \frac{1}{2}\|\overline{\sigma}_0\|_H^2 - \varepsilon\int_{Q_t} \partial_t \mu_{\varepsilon\tau} \mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega)$$

$$+ \int_{Q_t} \overline{\varphi}(\chi\overline{\sigma} - \tau\partial_t\varphi_{\varepsilon\tau}) + \int_{Q_t} \overline{\mu}(\overline{\varphi})_\Omega + \int_{Q_t} (J * \overline{\varphi})\overline{\varphi} + C\int_{Q_t} \sigma(h(\varphi) - h(\varphi_{\varepsilon\tau}))\overline{\sigma}$$

$$+ \int_{Q_t} (P\overline{\sigma}h(\varphi_{\varepsilon\tau}) + (P\sigma - A)(h(\varphi_{\varepsilon\tau}) - h(\varphi)))\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega). \tag{6.15}$$

Now, note that the Young inequality and the estimates (6.4) and (6.8) yield

$$-\varepsilon\int_{Q_t} \partial_t \mu_{\varepsilon\tau}\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega) + \int_{Q_t} \overline{\varphi}(\chi\overline{\sigma} - \tau\partial_t\varphi_{\varepsilon\tau})$$

$$\leqslant \varepsilon^2\|\partial_t\mu_{\varepsilon\tau}\|_{L^2(0,T;H)}^2 + \frac{1}{4}\int_0^t \|\mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_H^2 \, \mathrm{d}s + \frac{C_0}{4}\int_{Q_t} |\overline{\varphi}|^2$$

$$+ \frac{2}{C_0}\left(\tau^2\|\partial_t\varphi_{\varepsilon\tau}\|_{L^2(0,T;H)}^2 + \chi^2\int_{Q_t} |\overline{\sigma}|^2\right)$$

$$\leqslant \frac{C_0}{4}\int_{Q_t} |\overline{\varphi}|^2 + M\left(\varepsilon^{1/2} + \tau + \int_0^t \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_{V^*}^2 \, \mathrm{d}s + \int_{Q_t} |\overline{\sigma}|^2\right),$$

for a certain constant $M > 0$ independent of ε and τ. Furthermore, using the boundedness and Lipschitz continuity of h, and the fact that $\|\sigma\|_{L^\infty(Q)} \leqslant 1$, the last two terms in (6.15) can be handled again by the Young inequality as

$$C\int_{Q_t} \sigma(h(\varphi) - h(\varphi_{\varepsilon\tau}))\overline{\sigma} + \int_{Q_t} (P\overline{\sigma}h(\varphi_{\varepsilon\tau}) + (P\sigma - A)(h(\varphi_{\varepsilon\tau}) - h(\varphi)))$$

$$\times \mathcal{N}(\overline{\varphi} - (\overline{\varphi})_\Omega) \leqslant \frac{C_0}{4}\int_{Q_t} |\overline{\varphi}|^2 + M\left(\int_{Q_t} |\overline{\sigma}|^2 + \int_0^t \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_{V^*}^2 \, \mathrm{d}s\right)$$

for a certain $M > 0$ independent of τ and ε, and similarly we have the estimate

$$\int_{Q_t} (J * \overline{\varphi})\overline{\varphi} \leqslant (a^* + b^*)\int_0^t \|\overline{\varphi}(s)\|_H \|\overline{\varphi}(s)\|_{V^*} \, \mathrm{d}s$$

$$\leqslant \frac{C_0}{8}\int_{Q_t} |\overline{\varphi}|^2 + M\int_0^t \|\overline{\varphi}(s)\|_{V^*}^2 \, \mathrm{d}s.$$

Finally, as for the fifth term on the right-hand side of (6.15) we have, for a positive δ yet to be chosen,

$$\int_{Q_t} \overline{\mu}(\overline{\varphi})_\Omega = |\Omega| \int_0^t (\overline{\varphi}(s))_\Omega (\overline{\mu}(s))_\Omega \, ds$$

$$\leqslant \delta \int_0^t |(\overline{\mu}(s))_\Omega|^2 \, ds + \frac{|\Omega|^2}{4\delta} \int_0^t |(\overline{\varphi}(s))_\Omega|^2 \, ds,$$

where, by comparison in equation (6.11) and by using the estimate (6.4),

$$\int_0^t |(\overline{\mu}(s))_\Omega|^2 \, ds \leqslant M \left(\tau + \int_0^t \|F'(\varphi_{\varepsilon\tau}(s)) - F'(\varphi(s))\|_{L^1(\Omega)}^2 \, ds + \chi^2 \int_{Q_t} |\overline{\sigma}|^2 \right).$$

Next, owing to (2.32) and the Hölder inequality, we infer that

$$\int_0^t \|F'(\varphi_{\varepsilon\tau}(s)) - F'(\varphi(s))\|_{L^1(\Omega)}^2 \, ds$$

$$\leqslant C_F^2 \int_0^t \left\| (1 + |\varphi_{\varepsilon\tau}|^2 + |\varphi|^2)\overline{\varphi} \right\|_{L^1(\Omega)}^2 \, ds$$

$$\leqslant M' \left(1 + \|\varphi_{\varepsilon\tau}\|_{L^\infty(0,T;L^4(\Omega))}^4 + \|\varphi\|_{L^\infty(0,T;L^4(\Omega))}^4 \right) \int_0^t \|\overline{\varphi}(s)\|_H^2 \, ds$$

for some $M' > 0$ independent of ε and τ, which yields in turn, due to assumption (2.56) and to the previous estimates,

$$\int_0^t \|F'(\varphi_{\varepsilon\tau}(s)) - F'(\varphi(s))\|_{L^1(\Omega)}^2 \, ds \leqslant M^* \int_{Q_t} |\overline{\varphi}|^2$$

for a constant $M^* > 0$ independent of ε and τ. Thus, collecting the above estimates and rearranging the terms, we see that choosing $\delta > 0$ sufficiently small, for example $\delta = \frac{C_0}{4MM^*}$, we are left with

$$\frac{1}{2} \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(t)\|_{V^*}^2 + \frac{C_0}{8} \int_{Q_t} |\overline{\varphi}|^2 + \frac{1}{2} \|\overline{\sigma}(t)\|_H^2 + \int_{Q_t} |\nabla\overline{\sigma}|^2$$

$$\leqslant \frac{1}{2} \|\overline{\varphi}_0\|_{V^*}^2 + \frac{1}{2} \|\overline{\sigma}_0\|_H^2 + M \left(\varepsilon^{1/2} + \tau + \int_0^t \|(\overline{\varphi} - (\overline{\varphi})_\Omega)(s)\|_{V^*}^2 \, ds \right.$$

$$\left. + \int_{Q_t} |\overline{\sigma}|^2 + \int_0^t \|\overline{\varphi}(s)\|_{V^*}^2 \, ds \right). \tag{6.16}$$

In order to conclude, we only need to handle the last terms on the right-hand side of (6.16). To this end, note that integrating equation (6.10) on Ω and testing by $(\overline{\varphi})_\Omega$ yields, using the estimate (6.8), the Young inequality, the boundedness of σ and h, and the Lipschitz continuity of h,

$$\frac{1}{2} |(\overline{\varphi}(t))_\Omega|^2 = \frac{1}{2} |(\overline{\varphi}_0)_\Omega|^2 - \varepsilon \int_0^t (\overline{\varphi}(s))_\Omega (\partial_t \mu_{\varepsilon\tau}(s))_\Omega \, ds$$

$$+ \int_0^t (P\overline{\sigma}(s)h(\varphi_{\varepsilon\tau}(s)) + (P\sigma(s) - A)(h(\varphi_{\varepsilon\tau}(s)) - h(\varphi(s))))_\Omega (\overline{\varphi}(s))_\Omega \, ds$$

$$\leqslant \frac{1}{2} \|\overline{\varphi}_0\|_{V^*}^2 + M \left(\varepsilon^{1/2} + \int_0^t |(\overline{\varphi}(s))_\Omega|^2 \, ds + \int_{Q_t} |\overline{\sigma}|^2 \right) + \frac{C_0}{16} \int_{Q_t} |\overline{\varphi}|^2. \tag{6.17}$$

Summing then (6.16) and (6.17), we infer that

$$\|\overline{\varphi}(t)\|_{V^*}^2 + \int_{Q_t}|\overline{\varphi}|^2 + \|\overline{\sigma}(t)\|_H^2 + \int_{Q_t}|\nabla\overline{\sigma}|^2$$

$$\leqslant M\left(\|\overline{\varphi}_0\|_{V^*}^2 + \|\overline{\sigma}_0\|_H^2 + \varepsilon^{1/2} + \tau + \int_0^t\|\overline{\varphi}(s)\|_{V^*}^2\,\mathrm{d}s + \int_{Q_t}|\overline{\sigma}|^2\right)$$

for a certain constant M, independent of ε and τ. Therefore, we invoke the Gronwall lemma to conclude. It is not difficult to check that the same argument performed here yields uniqueness of solutions for the limit problem, even without assuming (2.48) and (2.57). This concludes the proof of theorem 2.15.

7. Conclusions

Large part of the applied literature on tumor growth modeling agrees that relevant biological mechanisms such as cell-to-cell adhesion are typically a non-local process. In this spirit, in this paper we introduce and investigate from a mathematical perspective a wide class non-local models of tumor growth capturing long-range interactions in cell-invasion. The analyzed model contains two regularization coefficients ε and τ, which allow the investigation in very broad scenarios such as the thermodynamically–relevant potentials and crucial mechanisms of chemotaxis and active transport. Then, we perform a complete asymptotic analysis showing how the parameters ε and τ may approach zero, both separately and jointly, allowing us to establish well-posedness of the limiting systems obtained by formally setting to zero those coefficients.

Acknowledgments

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