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Calculus of Variations

Effects of corners in surface superconductivity

Michele Correggi[1] · Emanuela L. Giacomelli[2]

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Abstract
We study the Ginzburg–Landau functional describing an extreme type-II superconductor wire with cross section with finitely many corners at the boundary. We derive the ground state energy asymptotics up to $o(1)$ errors in the surface superconductivity regime, i.e., between the second and third critical fields. We show that, compared to the case of smooth domains, each corner provides an additional contribution of order $\mathcal{O}(1)$ depending on the corner opening angle. The corner energy is in turn obtained from an implicit model problem in an infinite wedge-like domain with fixed magnetic field. We also prove that such an auxiliary problem is well-posed and its ground state energy bounded and, finally, state a conjecture about its explicit dependence on the opening angle of the sector.

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Contents

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✉ Michele Correggi
michele.correggi@gmail.com

Emanuela L. Giacomelli
emanuela.giacomelli.elg@gmail.com

[1] Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy

[2] Department of Mathematics, Universität Tübingen, Auf der Morgenstelle, 10, 72076 Tübingen, Germany

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1 Introduction

The phenomenon of conventional superconductivity (see, e.g., [55] for a review of the physics of superconductors) is nowadays very well understood at the microscopic level thanks to the Bardeen–Cooper–Schrieffer (BCS) theory [8]: a collective behavior of the current carriers in the material is responsible for a sudden drop of the resistivity below a certain critical temperature. It is however astonishing how a phenomenological model as the Ginzburg–Landau (GL) theory [42] is capable of predicting most of the key equilibrium features of the phenomenon, in particular concerning the response of the superconducting material to an external field. When it was introduced in the '50s, indeed, the GL model was motivated only from purely phenomenological considerations. Only later it was shown that the GL theory emerges as an effective macroscopic model from the BCS theory suitably close to the critical temperature [37,40,43].

The interplay between superconductivity and strong magnetic fields is known to generate a very rich variety of physical phenomena since the pioneering works of Abrikosov [2] and St. James and De Gennes [54] in the late '50s/early '60s, who predicted the occurrence of the famous vortex lattice and of surface superconductivity, respectively, working only in the framework of the GL theory. In extreme synthesis, the response of a type-II superconducting material to the external magnetic field can vary from a perfect repulsion of the field (*Meissner effect*), for small fields, to a complete loss of superconductivity, for sufficiently strong ones. In between, several different phases of the material can be observed, ranging from various kinds of vortex states to configurations where the superconduction gets restricted to boundary regions. Each of these phase transitions can be associated with a *critical magnetic field* marking the threshold for the transition: the three major critical fields are

- the *first critical field*, which separate the Meissner behavior, i.e., when the magnetic field inside the material is zero and superconductivity is unaffected, from states where the penetration of the field has occurred at least at isolated points (*vortices*), where superconductivity is lost;
- the *second critical field*, above which the superconducting behavior gets confined at the surface of the sample (*surface superconductivity*);
- the *third critical field*, which marks the complete loss of superconductivity.

Let us now introduce in more detail the GL theory: the free energy of the material is given by a nonlinear functional, which in the case of a superconducting infinite wire of cross

section $\Omega \subset \mathbb{R}^2$ reads in suitable units

$$\mathcal{E}_\varepsilon^{\mathrm{GL}}[\psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \frac{1}{2b\varepsilon^2} (2|\psi|^2 - |\psi|^4) \right\} + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \, |\mathrm{curl}\mathbf{A} - 1|^2, \tag{1.1}$$

where $\varepsilon, b > 0$ are two parameters depending on the *London penetration depth* and the intensity of the applied magnetic field, which is assumed to be parallel to the wire. The function ψ, a.k.a. *wave function* or *order parameter*, is complex, while $\mathbf{A}$ is the *induced magnetic potential*, whose curl yields the intensity of the magnetic field outside and inside the sample (measured in units ε^{-2}). The physical meaning of the order parameter is twofold: $|\psi|^2$ yields the relative density of Cooper pairs and, at the same time, the phase of ψ contains the information about the stationary *current* flowing in the superconductor, i.e.,

$$\mathbf{j}[\psi] := \tfrac{i}{2} (\psi \nabla \psi^* - \psi^* \nabla \psi) = \mathrm{Im} (\psi^* \nabla \psi) . \tag{1.2}$$

Hence, one typically speaks of a *normal state*, if $\psi = 0$ and $\mathbf{A}$ is such that $\mathrm{curl}\mathbf{A} = 1$, while the *perfect superconducting state* is identified by $|\psi| = 1$, $\mathbf{A} = 0$. Whenever $|\psi|$ is non-vanishing everywhere but not identically 1, the superconductor is said to be in a *mixed state*. Any equilibrium state of the sample minimizes the free energy (1.1) and thus we set

$$E_\varepsilon^{\mathrm{GL}} := \min_{(\psi, \mathbf{A}) \in \mathscr{D}^{\mathrm{GL}}} \mathcal{E}_\varepsilon^{\mathrm{GL}}[\psi, \mathbf{A}], \tag{1.3}$$

and denote by $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ any minimizing configuration, where

$$\mathscr{D}^{\mathrm{GL}} = \left\{ (\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2) \,\middle|\, \mathrm{curl}\mathbf{A} - 1 \in L^2(\mathbb{R}^2) \right\}. \tag{1.4}$$

We provide some details about the above minimization and the properties of any minimizing configuration $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ in "Appendix B.1". We also use the following convention: if we need to specify the dependence on the domain Ω, we write $\mathcal{E}_\varepsilon^{\mathrm{GL}}[\psi, \mathbf{A}; \Omega]$ for the functional and $E_\varepsilon^{\mathrm{GL}}(\Omega)$ for the corresponding ground state energy.

In the rest of the paper we are going to study the minimization (1.3) in the asymptotic regime

$$\varepsilon \ll 1, \tag{1.5}$$

corresponding to an *extreme type-II superconductor*. Under this idealization, one can identify the mathematical counterparts of the critical values of the external magnetic field described above in terms of properties of the minimizing configuration $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ and it is also possible to precisely identify the behavior of such thresholds (see, e.g., [53] for an extensive discussion of the first phase transition). In particular, assuming that Ω is a simply connected domain with *smooth boundary* $\partial\Omega$, the *second critical field* associated with the transition from *bulk* to *surface superconductivity* is identified with $b = 1$ [33, Chpt. 10.6] and thus with a field of intensity

$$H_{c2} = \frac{1}{\varepsilon^2}, \tag{1.6}$$

based on sharp estimates (*Agmon estimates*) of the decay of ψ^{GL} in the distance from the boundary (see "Appendix B.3"); the *third critical field* marking the transition to the normal

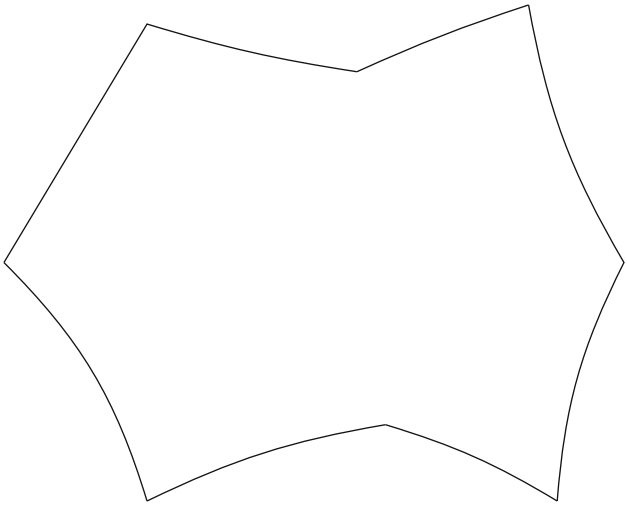

Fig. 1 A domain Ω with Lipschitz boundary and finitely many corners on $\partial\Omega$

state on the other hand corresponds to $b = \Theta_0^{-1} > 1$, where $\Theta_0 \simeq 0.59$ is a universal constant, i.e., more precisely [33, Chpt. 13],

$$H_{c3} = \frac{1}{\Theta_0 \varepsilon^2} + \mathcal{O}(1). \tag{1.7}$$

1.1 Setting: domains with corners

In this paper we are exclusively concerned with the behavior of the superconductor for very strong magnetic fields *above the second critical one*, i.e., we always assume that $h_{ex} > H_{c2}$, or, more concretely,

$$b > 1. \tag{1.8}$$

The main novelty of this paper compared to other works on the GL functional above the second critical field is that we assume that Ω is a bounded domain with a *Lipschitz boundary*, i.e., we allow for the presence of *corners* on $\partial\Omega$ (see Fig. 1). Indeed, apart from few physics papers (see [14,31,52]), the GL theory on domains with corners has already been studied only in [15,45,46,50], with the focus on the third critical field though, and in [19], whose results are improved in this work.

The main reason why it is interesting to study the behavior of the GL functional in domains with corners for large magnetic fields is that for smaller fields one expects that the presence of corners does not affect the salient features of superconductivity. Indeed, the occurrence of vortices but also their uniform distribution and arrangement in regular lattices, which occur for magnetic fields below H_{c2}, are bulk phenomena and, as such, little influenced by the boundary regularity. On the opposite, the surface superconductivity regime, where the density of Cooper pairs is non-vanishing only at and close to the boundary, might clearly depend on the presence of singularities along $\partial\Omega$. It is then important to know if and to what extent corners can modify the boundary behavior, even more so, considered that in physics

experiments it is hardly possible to distinguish between a sample with smooth boundary and another which has corners there (see, e.g., [48, Fig. 1]).

We now specify in more detail our assumptions on the domain Ω. First, we require that it is simply connected and its boundary $\partial\Omega$ is Lipschitz (see, e.g., [44, Def. 1.4.5.1]) and, more concretely, it is a curvilinear polygon of class C^∞, given by smooth pieces glued together at finitely many points, where however the curvature remains finite (no cusps). These assumptions are the same made, e.g., in [15,19,45].

Assumption 1 *(Piecewise smooth boundary)* Let Ω be a bounded open subset of $\mathbb{R}^2$. We assume that $\partial\Omega$ is a smooth curvilinear polygon, i.e., for every $\mathbf{r} \in \partial\Omega$ there exists a neighborhood U of $\mathbf{r}$ and a map $\mathbf{\Phi} : U \to \mathbb{R}^2$, such that

(1) $\mathbf{\Phi}$ is injective;
(2) $\mathbf{\Phi}$ together with $\mathbf{\Phi}^{-1}$ (defined from $\mathbf{\Phi}(U)$) are smooth;
(3) the region $\Omega \cap U$ coincides with either $\{\mathbf{r} \in \Omega \mid (\mathbf{\Phi}(\mathbf{r}))_1 < 0\}$ or $\{\mathbf{r} \in \Omega \mid (\mathbf{\Phi}(\mathbf{r}))_2 < 0\}$ or $\{\mathbf{r} \in \Omega \mid (\mathbf{\Phi}(\mathbf{r}))_1 < 0, (\mathbf{\Phi}(\mathbf{r}))_2 < 0\}$, where $(\mathbf{\Phi})_j$ stands for the $j-$th component of $\mathbf{\Phi}$.

Assumption 2 *(Boundary with corners)* We assume that the set $\Sigma := \{\mathbf{r}_1, \dots, \mathbf{r}_N\}$ of corners of $\partial\Omega$, i.e., the points where the normal $\boldsymbol{\nu}$ does not exist, is non empty but finite and given by N points. We denote by β_j the angle of the $j-$th corner (measured towards the interior) and by s_j its boundary coordinate.

The inward normal $\boldsymbol{\nu}$ to $\partial\Omega$ is thus defined almost everywhere and jumps at the corners. More precisely, we can find a counterclockwise parametrization $\boldsymbol{\gamma}(\mathsf{s}) : [0, |\partial\Omega|) \to \partial\Omega$ of $\partial\Omega$ which is smooth a.e., i.e., for $s \neq s_j$, $j = 1, \dots, N$, s_j being the curvilinear coordinate of the j-th corner, and such that $|\boldsymbol{\gamma}'(\mathsf{s})| = 1$. The mean curvature $\mathfrak{K}(\mathsf{s})$ is then defined a.e. through the identity

$$\boldsymbol{\gamma}''(\mathsf{s}) = \mathfrak{K}(\mathsf{s})\boldsymbol{\nu}(\mathsf{s}). \tag{1.9}$$

We can then introduce a convenient system of tubular coordinates in a neighborhood of the boundary (see also [33, Appendix F]) far from the corners: for any $(\mathsf{s}, \mathsf{t}) \in \mathcal{A}_{\mathsf{t}_0}$, where

$$\mathcal{A}_{\mathsf{t}_0} := \left\{ (\mathsf{s}, \mathsf{t}) \in [0, |\partial\Omega|) \times [0, \mathsf{t}_0) \;\middle|\; \min_{j=1,\dots,N} |s - s_j| > c\mathsf{t}_0 \right\},$$

with $\mathsf{t}_0 \ll 1$ small enough and c depending only on the corner opening angles, we can set

$$\mathbf{r}(\mathsf{s}, \mathsf{t}) =: \boldsymbol{\gamma}'(\mathsf{s}) + \mathsf{t}\boldsymbol{\nu}(\mathsf{s}), \tag{1.10}$$

which identifies a diffeomorphism from $\mathcal{A}_{t_0}$ to the region

$$\Omega_{\mathsf{t}_0} := \{\mathbf{r} \in \Omega \mid \mathrm{dist}(\mathbf{r}, \partial\Omega) < \mathsf{t}_0, \mathrm{dist}(\mathbf{r}, \Sigma) > c\mathsf{t}_0\},$$

satisfying

$$\mathrm{dist}\,(\mathbf{r}(\mathsf{s}, \mathsf{t}), \partial\Omega) = \mathsf{t}. \tag{1.11}$$

Therefore, such a map is invertible in the same region and identifies a system of local coordinates (s, t) there. The parameters s, t (with the latter defined through (1.11)) are well-posed also closer to the corners but there they do not identify a diffeomorphism and thus a set of coordinates.

1.2 Heuristics

Before entering the discussion of what is mathematically known on the phenomenon of surface superconductivity, we resume here its key features for smooth domains, neglecting errors and remainders: if $b > 1$, as $\varepsilon \to 0$,

- the order parameter ψ^{GL} is non-vanishing only close to $\partial\Omega$; more precisely it is exponentially small in ε at distances from the boundary much larger than ε;
- the induced magnetic field $\mathrm{curl}\mathbf{A}^{\mathrm{GL}}$ is suitably close to the applied one, i.e., a uniform magnetic field of unit strength, and, consequently, one can find a local gauge close to $\partial\Omega$ in which $\mathbf{A}^{\mathrm{GL}}$ is purely tangential and $|\mathbf{A}^{\mathrm{GL}}| = \mathrm{dist}\{\mathbf{r}, \partial\Omega\}$;
- the modulus of ψ^{GL} is essentially independent of the tangential coordinate s and therefore optimizes an effective one-dimensional problem where the only variable is the distance from the boundary;
- the phase of ψ^{GL} is on the other hand constant in t and linear in s, with rapid oscillations, or, more precisely, the current (1.2) is constant along s.

Summing up, we expect that $\mathbf{A}^{\mathrm{GL}}$ can be locally replaced by $-\mathsf{t}\mathbf{e}_{\mathsf{s}}$ close to $\partial\Omega$ and

$$\left|\psi^{\mathrm{GL}}(\mathbf{r})\right| \simeq f\left(\tfrac{\mathsf{t}}{\varepsilon}\right), \qquad \mathbf{j}[\psi^{\mathrm{GL}}] \simeq \frac{|\Omega|}{\varepsilon^2} - \frac{\alpha}{\varepsilon}, \tag{1.12}$$

for some f positive and $\alpha \in \mathbb{R}$, which leads to

$$\psi^{\mathrm{GL}}(\mathbf{r}) \simeq f(\mathsf{t}/\varepsilon) e^{-\frac{i\alpha\mathsf{s}}{\varepsilon}} e^{i\phi_\varepsilon(\mathbf{r})}, \tag{1.13}$$

ϕ_ε standing for the gauge transformation mentioned above. Note the scaling factors $1/\varepsilon$ we have extracted for later convenience, so that f and α are quantities of order $\mathcal{O}(1)$.

If we plug the ansatz (1.13) into the GL energy (1.1), we get

$$\frac{|\partial\Omega|}{\varepsilon} \int_0^{\ell_\varepsilon} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}(2f^2 - f^4) \right\}, \tag{1.14}$$

i.e., up to the prefactor $|\partial\Omega|/\varepsilon$, a one-dimensional (1D) energy functional evaluated on f and depending on the real parameter α. The value $\ell_\varepsilon > 0$ is to some extent arbitrary and is chosen much larger than 1 in order to cover all the superconducting layer: we make the following explicit choice

$$\ell_\varepsilon := c_1 |\log \varepsilon|, \tag{1.15}$$

for a large constant c_1. The minimization of the 1D functional above and some variants of it w.r.t. both f and α is discussed in "Appendix A". This identifies the leading term contribution in the GL energy asymptotics $E_\star^{\mathrm{1D}}/\varepsilon$, the optimal 1D profile $f_\star(t)$ and the optimal phase $\alpha_\star$.

The next-to-leading order term in the GL energy asymptotics is of order $\mathcal{O}(1)$ and depends on the mean curvature of the boundary: one can indeed refine the 1D model problem (1.14) keeping track of $\mathcal{O}(\varepsilon)$ contributions coming from the curvature-dependent terms due to the change of coordinates $\mathbf{r} \to (\mathsf{s}, \mathsf{t})$. Indeed, if we define the rescaled tubular coordinates as

$$\begin{cases} t := \mathsf{t}/\varepsilon \in [0, \ell_\varepsilon], \\ s := \mathsf{s}/\varepsilon \in \left[0, \frac{|\partial\Omega|}{\varepsilon}\right]. \end{cases} \tag{1.16}$$

we get $\mathrm{d}\mathbf{r} = \mathrm{d}\mathsf{t}\mathrm{d}\mathsf{s}\,(1 - \mathfrak{K}(\mathsf{s})\mathsf{t})$, or, equivalently,

$$\mathrm{d}\mathbf{r} = \varepsilon^2 \mathrm{d}t\mathrm{d}s\,(1 - \varepsilon k(s)t)\,, \tag{1.17}$$

where we have set

$$k(s) := \mathfrak{K}(\varepsilon s). \tag{1.18}$$

1.3 Summary

In this paper we continue the analysis started with [19]. The expansion (2.10) provides indeed only the leading order term in the energy asymptotics and does not capture the corner effects, that we are going to investigate. More precisely, we prove that the presence of corners modifies the $\mathcal{O}(1)$ term in the expansion (2.2). We also identify the model problem which yields such a new contribution in terms of a genuine 2D model. Finally, we prove that the pointwise estimate of $|\psi^{\mathrm{GL}}|$ in terms of $f_\star$ still holds far from the corners, precisely as in the smooth case.

After having introduced some notation in § 1.4, we define in § 2 the effective model in the corner region and state our main results about the GL energy asymptotics and the pointwise estimate of the order parameter far from corners. Further comments about the effective model and a conjecture about the possible explicit expression of the effective energy are contained in § 2.3. The well-posedness of the effective model problem is proven in detail in § 3, whereas in § 4 and § 5 we provide the energy lower bound and the rest of the arguments needed to complete the proof of our main results, respectively. The Appendix is divided in three parts: in "Appendix A" we discuss the effective 1D problems and their related properties; the GL minimization and some useful technical estimates are treated in "Appendix B"; finally, "Appendix C" recalls the salient steps of the proof of the energy asymptotics in domains with smooth boundaries, which are used to complete the proof of energy expansion.

1.4 Notation

Given their key role in the rest of the paper, we recall the definitions (1.10) and (1.16) of tubular coordinates (s, t) and their rescaled counterparts (s, t). We stress that (s, t) or, equivalently, (s, t) provide a smooth diffeomorphism, e.g., in

$$\left\{\mathbf{r} \in \mathcal{A}_\varepsilon \,\middle|\, \operatorname{dist}(\mathbf{r}, \Sigma) \geqslant \varepsilon |\log \varepsilon|\right\},$$

where Σ is the set of corner positions and

$$\mathcal{A}_\varepsilon := \left\{\mathbf{r} \in \Omega \,\middle|\, \operatorname{dist}(\mathbf{r}, \partial\Omega) \leqslant \varepsilon \ell_\varepsilon\right\}, \tag{1.19}$$

for $\varepsilon \ll 1$, where (see (1.15))

$$\ell_\varepsilon := c_1 |\log \varepsilon|$$

and c_1 is large enough constant, which is set once for all (see next (2.24)). Given a differentiable function $\psi(\mathbf{r})$ and a vector $\mathbf{A}(\mathbf{r})$, the transformations induced by the change of coordinates $\mathbf{r} \to (\mathsf{s}, \mathsf{t})$ are

$$(\nabla\psi)(\mathbf{r}(\mathsf{s}, \mathsf{t})) = (1 - \mathfrak{K}(\mathsf{s})\mathsf{t})^{-1} \left(\partial_{\mathsf{s}}\widetilde{\psi}\right)\mathbf{e}_{\mathsf{s}} + \left(\partial_{\mathsf{t}}\widetilde{\psi}\right)\mathbf{e}_{\mathsf{t}}, \tag{1.20}$$

where we have set $\widetilde{\psi}(\mathsf{s}, \mathsf{t}) := \psi(\mathbf{r}(\mathsf{s}, \mathsf{t}))$ and

$$\mathbf{e}_{\mathsf{s}} := \boldsymbol{\gamma}'(\mathsf{s}), \qquad \mathbf{e}_{\mathsf{t}} := \boldsymbol{\nu}(\mathsf{s}), \tag{1.21}$$

for short. As a consequence, for any vector $\mathbf{A}$,

$$(\operatorname{curl}\mathbf{A})(\mathbf{r}(s,t)) = -\partial_t\left(\mathbf{A}(\mathbf{r}(s,t))\cdot\mathbf{e}_s\right) + (1-\mathfrak{K}(s)t)^{-1}\left[\partial_s\left(\mathbf{A}(\mathbf{r}(s,t))\cdot\mathbf{e}_t\right) + \mathfrak{K}(s)\mathbf{A}(\mathbf{r}(s,t))\cdot\mathbf{e}_s\right]. \tag{1.22}$$

We are going to make use of Landau symbols, with the following convention: given two functions $f(x)$, $g(x)$, with $g > 0$,

- $f = \mathcal{O}(g)$, if $\lim_{x\to 0^+} |f|/g \leqslant C$;
- $f = o(g)$, if $\lim_{x\to 0^+} |f|/g = 0$;
- $f \sim g$, if $f = \mathcal{O}(g)$ and $\lim_{x\to 0^+} |f|/g > 0$;
- for $f > 0$, $f \ll g$ or $f \gg g$, if $f = o(g)$ or $g = o(f)$, respectively;
- for $f > 0$, $f \lesssim g$ or $f \gtrsim g$, if $f = \mathcal{O}(g)$ or $g = \mathcal{O}(f)$, respectively.

We also commit a little abuse of the notation by using the symbols $\mathcal{O}(\,)$ and $o(\,)$ inside an inequality to mean a precise direction of the estimate. As usual, $\mathcal{O}(\,)$ and $o(\,)$ stand for quantities whose sign is not known. In case of functions of two or more variables, we point out the parameter, whose asymptotics we are considering, by adding a label, e.g., $o_x(\,)$ or $\mathcal{O}_x(\,)$ is meant to stress that we are estimating the behavior of the function as $x \to 0^+$. Finally, we say that a quantity is $\mathcal{O}(\varepsilon^\infty)$, as $\varepsilon \to 0^+$, if it is smaller than any power of ε, i.e., it is exponentially small in ε. We will also use the following convention: $\mathcal{O}(\varepsilon^a|\log\varepsilon|^\infty)$, $a > 0$, stands for a quantity which is bounded by $\varepsilon^a|\log\varepsilon|^b$ for some large but finite power $b > 0$, which is however not relevant since the $|\log\varepsilon|$-factor is always dominated by ε-powers.

2 Main results

2.1 State of the art

We briefly review here the most recent and relevant results on surface superconductivity, which are related to the analysis carried on in this paper (see [16] for a more detailed review). After the series of works [17,23–25], following [49], where the problem was first investigated, and [4,34], the phenomenon of 2D surface superconductivity in domains with smooth boundaries is well understood: combining [23, Thm. 1] with [24, Lemma 2.1] (see also [17]), one gets that, whenever

$$1 < b < \Theta_0^{-1}, \tag{2.1}$$

the GL energy asymptotics is given by

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega|E_\star^{\mathrm{1D}}}{\varepsilon} - 2\pi\, E_{\mathrm{corr}} + o(1), \tag{2.2}$$

where

$$E_\star^{\mathrm{1D}} := \inf_{\alpha\in\mathbb{R}} \inf_{f\in\mathscr{D}^{\mathrm{1D}}} \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}(2f^2 - f^4) \right\}, \tag{2.3}$$

and

$$E_{\mathrm{corr}} := \int_0^{+\infty} \mathrm{d}t\, t \left\{ |\partial_t f_\star|^2 + f_0^2 \left(-\alpha_\star(t+\alpha_\star) - \frac{1}{b} + \frac{1}{2b} f_\star^2 \right) \right\} = \frac{1}{3} f_\star^2(0)\alpha_\star - E_\star^{\mathrm{1D}}, \tag{2.4}$$

$\alpha_\star$, $f_\star$ being a pair of minimizers of (2.3) (see "Appendix A.1"). Note that (2.2) can also be rewritten as

$$E_\varepsilon^{\mathrm{GL}} = \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s \; E^{\mathrm{1D}}_{k(s)} + \mathcal{O}(\varepsilon|\log\varepsilon|^\infty),$$

with a more precise remainder term and where E_k^{1D} is defined in (A.6) in "Appendix A.2". Expanding further E_k^{1D}, the next-to-leading order correction in (2.2) can be shown to be

$$-E_{\mathrm{corr}} \int_{\partial\Omega} \mathrm{d}\mathsf{s}\, \mathfrak{K}(\mathsf{s}) + o(1) = -2\pi E_{\mathrm{corr}} + o(1), \tag{2.5}$$

by the Gauss–Bonnet theorem, because Ω is flat and the Euler characteristic is equal to 1. Moreover, in [17] the quantity E_{corr} is numerically evaluated and it is shown that it is positive, which has some important consequences on the distribution of superconductivity near the boundary: regions with larger curvature attract Cooper pairs, which concentrate more there (to first order), although to leading order superconductivity is uniform at the boundary.

Indeed, a consequence of (2.3) is that [23, Thm. 1] the density $|\psi^{\mathrm{GL}}|^2$ is L^2-close to the reference density $f_\star$. Such an estimate can in fact be strengthened in two directions:

- in [23, Thm. 2] it is proven that there exists a boundary layer $\mathcal{A}_{\mathrm{bl}} \subset \{\mathbf{r}|\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant \varepsilon|\log\varepsilon|\}$, containing the bulk of superconductivity, where Pan's conjecture holds true, i.e.,
$$\left\| \left|\psi^{\mathrm{GL}}(\,\cdot\,)\right| - f_\star(\mathrm{dist}(\,\cdot\,,\partial\Omega)/\varepsilon)\right\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} = o(1); \tag{2.6}$$
- the approximation of $|\psi^{\mathrm{GL}}|$ in terms of $f_\star$ holds also locally [24, Thm 1.1] and one can explicitly derive the asymptotics of the density of superconductivity (in fact, the L^4 norm of ψ^{GL}) in any reasonable subdomain contained in Ω.

It is expected that a regime of surface superconductivity with similar features occurs also for genuine 3D samples but so far only partial results are available [38,39]. In particular, in [38, Thm 1.1] (see also [36]) it is shown that such a regime does exist and the leading order term in the energy asymptotics can be identified, although in terms of a rather implicit effective problem. In [39, Thm. 1.5] it is then proven that, when the magnetic field is parallel to the 3D boundary, the effective model is still given by the 1D functional (2.3) above.

One of the major differences for samples with non-smooth boundary is that one expects [15,45–47,50] a shift of the third critical field, provided there is at least one corner with acute opening angle $0 < \beta < \pi$: more precisely, the transition to the normal state should occur [15, Thm. 1.4] for applied fields larger than

$$H_{\mathrm{c3}} = \frac{1}{\mu(\beta)\varepsilon^2} \tag{2.7}$$

where

$$\mu(\beta) := \inf \mathrm{spec}_{L^2(W_\beta)} \left(-\left(\nabla + \tfrac{1}{2} i \mathbf{x}^\perp\right)^2 \right), \tag{2.8}$$

is the ground state energy of a Schrödinger operator with uniform magnetic field in an infinite sector W_β of angle β. The above result is however conditioned to the inequality $\mu(\beta) < \Theta_0$ (see also [51, Chpt. 8.2] and references therein), which is known to be true for $0 < \beta < \pi/2 + \epsilon$ [11,30,46] but is expected to hold in the whole interval $\beta \in (0, \pi)$, based on numerical simulations (see, e.g., [11,12,30]).

As the applied field gets closer to (2.7) from below, the order parameter concentrates around the corner of smallest opening angle and becomes smaller and smaller everywhere else. Hence, one can speak of a *corner superconductivity* regime occurring before the transition to the normal state. On the other hand, in [19], we proved that, if $1 < b < \Theta_0$, superconductivity is still uniform along the boundary (although only in L^2 sense), leading to the conjectured existence of another critical field

$$H_{\text{corner}} = \frac{1}{\Theta_0 \varepsilon^2}, \tag{2.9}$$

which marks the transition from surface to corner concentration. Indeed, if $1 < b < \Theta_0$, then [19, Thm 1.1]

$$E_\varepsilon^{\text{GL}} = \frac{|\partial\Omega| E_\star^{\text{1D}}}{\varepsilon} + \mathcal{O}(|\log\varepsilon|^2), \tag{2.10}$$

and, more importantly,

$$\left\| |\psi^{\text{GL}}(\,\cdot\,)|^2 - f_\star^2\,(\text{dist}(\,\cdot\,,\partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon|\log\varepsilon|), \tag{2.11}$$

which implies, to leading order, uniform distribution of superconductivity along the boundary layer.

The result of [15] has also been recently improved in [45], where the presence of several corners is taken into account and shown that, under the same unproven assumption, one can identify several critical fields associated to the concentration of the order parameter close to the respective corner. We also stress that, as noted in [7, Rmk. 1.9] (see also [5,6]), the behavior of superconductivity in presence of corners is expected to be recovered in the case of magnetic steps, i.e., for applied magnetic fields with a jump singularity along a curve.

2.2 GL energy and density asymptotics

Before stating our main results, we have to define the effective problem near the corners. Here we only provide a sketchy definition and in the next § 2.3 we comment further about its well-posedness and heuristic meaning. The model problem is given by first minimizing the GL functional with given magnetic potential and unit magnetic field in large wedge-like domain (see Fig. 2), and then subtracting the surface energy of the outer boundary of the wedge. The wedge domain is supposed to describe the rectified and rescaled area close to each corner, where the only relevant parameter is the opening angle β_j.

We thus define the corner energy as

$$\boxed{E_{\text{corner},\beta} := \lim_{\ell\to+\infty}\lim_{L\to+\infty}\left(-2LE_0^{\text{1D}}(\ell) + \inf_{\psi\in\mathscr{D}_\star(\Gamma_\beta(L,\ell))}\mathcal{E}_1^{\text{GL}}\left[\psi,\mathbf{F};\Gamma_\beta(L,\ell)\right]\right),} \tag{2.12}$$

where $E_0^{\text{1D}}(\ell)$ is a 1D effective energy analogous to (2.3), which is explicitly given by (see "Appendix A.3" for further details)

$$E_0^{\text{1D}}(\ell) := \inf_{\alpha\in\mathbb{R}}\inf_{f\in H^1([0,\ell])}\mathcal{E}_{0,\alpha}^{\text{1D}}[f], \tag{2.13}$$

with

$$\mathcal{E}_{0,\alpha}^{\text{1D}}[f] := \int_0^\ell \text{d}t\left\{|\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}(2f^2 - f^4)\right\}, \tag{2.14}$$

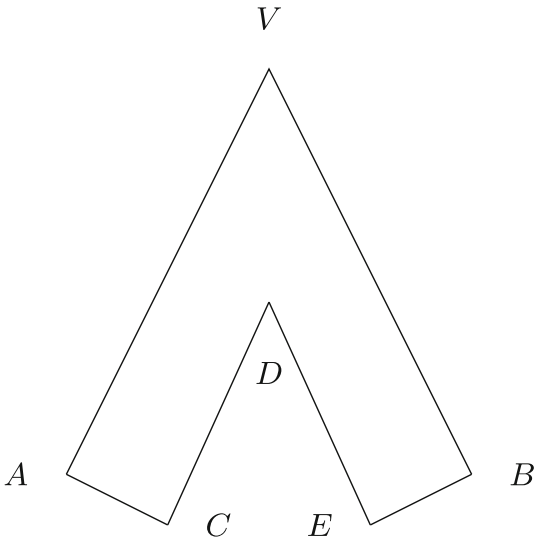

Fig. 2 The region $\Gamma_\beta(L,\ell)$, where β is the opening angle $\widehat{AVB}$, $L = |\overline{AV}| = |\overline{VB}|$ and $\ell = |\overline{AC}| = |\overline{EB}|$

and we denote by $\alpha_0 \in \mathbb{R}$, $f_0 \in H^1([0,\ell])$ a corresponding minimizing pair, i.e., $E_0^{\mathrm{1D}}(\ell) = \mathcal{E}_{0,\alpha_0}^{\mathrm{1D}}[f_0]$. The minimization domain is

$$\mathscr{D}_\star(\Gamma_\beta(L,\ell)) := \left\{\psi \in H^1(\Gamma_\beta(L,\ell)) \;\middle|\; \psi|_{\partial\Gamma_{\mathrm{bd}}\cup\partial\Gamma_{\mathrm{in}}} = \psi_\star\right\}. \tag{2.15}$$

where the boundaries are identified in Fig. 2 by the segments $\partial\Gamma_{\mathrm{bd}} = \overline{AC}\cup\overline{EB}$ and $\partial\Gamma_{\mathrm{in}} = \overline{CD}\cup\overline{DE}$ and

$$\psi_\star(\mathbf{r}(s,t)) := f_0(t)\exp\left\{-i\alpha_0 s - \tfrac{1}{2}ist\right\}, \qquad \text{for } |s| \geqslant \tfrac{\ell}{\tan(\beta/2)}. \tag{2.16}$$

Here, we have denoted by (s,t) a set of tubular coordinates similar to (1.10), yielding a smooth parametrization of $\Gamma_\beta(L,\ell)\cap\{\mathrm{dist}(\mathbf{r},V)\geqslant c\ell\}$. Any function in $\mathscr{D}_\star(\Gamma_\beta(L,\ell))$ has thus to satisfy Dirichlet non-zero boundary conditions on $\partial\Gamma_{\mathrm{in}}$ and $\partial\Gamma_{\mathrm{bd}}$ in trace sense, whose role is going to be explained in next § 2.3. Finally, the magnetic potential $\mathbf{F}$ is fixed and equals

$$\mathbf{F}(\mathbf{r}) := \tfrac{1}{2}(-y,x) =: \tfrac{1}{2}\mathbf{r}^\perp, \tag{2.17}$$

in a coordinate system chosen[1] as in Fig. 3. We also point out that the existence of the limit in (2.12) is not trivial at all and, in fact, it will be the main content of Proposition 2.2. Furthermore, the GL functional in the second term on the r.h.s. of (2.12) is independent of ε, but still contains the parameter $b \in (1,\Theta_0^{-1})$.

The main result we prove in this paper is about the GL energy asymptotics as $\varepsilon\to 0$, i.e., we derive the expansion of $E_\varepsilon^{\mathrm{GL}}$ up to correction of order $o(1)$. Compared with the case of domains with smooth boundary, some new terms of order $\mathcal{O}(1)$ appear: each corner indeed contributes to the energy by $E_{\mathrm{corner},\beta_j}$, β_j being the corresponding opening angle.

Theorem 2.1 (GL energy asymptotics) *Let $\Omega\subset\mathbb{R}^2$ be any bounded simply connected domain satisfying Assumptions 1 and 2. Then, for any fixed*

$$1 < b < \Theta_0^{-1}, \tag{2.18}$$

[1] In fact, any choice of the coordinate system would lead to the same energy because of rotational invariance of the GL functional and its gauge symmetry, which allows to incorporate any translation of the origin.

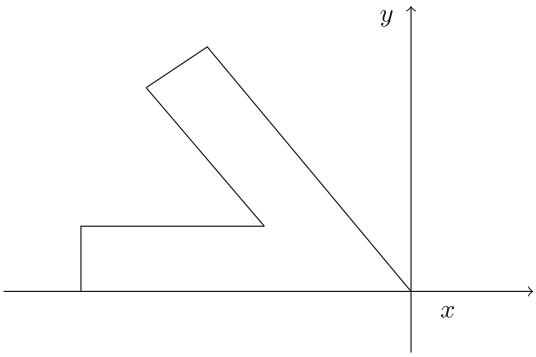

Fig. 3 Cartesian coordinates for the corner domain

as $\varepsilon \to 0$, *it holds*

$$\boxed{E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_\star^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}s\, \mathfrak{K}(s) + \sum_{j=1}^{N} E_{\mathrm{corner},\beta_j} + o(1).} \tag{2.19}$$

Remark 2.1 **(Critical field H_{c2})** In smooth domains the regime of surface superconductivity corresponds to the parameter interval $1 < b < \Theta_0^{-1}$, namely the second critical field is $b = 1$, while the third one is precisely $b = \Theta_0^{-1}$. This is motivated by the results in [24,25] in combination with [32,35], where it is proven that, for $b < 1$, there is still superconductivity in the bulk, while, for $b > \Theta_0^{-1}$, the normal state is a global minimizer of the GL functional, respectively. The condition $b > 1$ is expected to be sharp also for domains with corners and, consequently, we expect that the second critical field is given by

$$H_{\mathrm{c2}} = \frac{1}{\varepsilon^2}, \tag{2.20}$$

The value $1/\varepsilon^2$ can actually be taken as a definition of the second critical field, but, as for smooth domains, it would be necessary to show that, for $b \leqslant 1$, there is still superconductivity in the bulk. This has not yet been proven in case of samples with corners, but, based on the results proven in [35], it is highly expected.

Remark 2.2 **(Critical field H_{corner})** The result proven in Theorem 2.1 substantiates even more than [19] the conjecture about the appearance of an additional critical field

$$H_{\mathrm{corner}} = \frac{1}{\Theta_0 \varepsilon^2}, \tag{2.21}$$

when corners are present along the sample boundary. Indeed, combining (2.19) and, more importantly, next Proposition 2.1, with [15, Thm. 1.6] (see also [45, Thm. 1.2]), which states the exponential decay of ψ^{GL} in the distance from Σ (still, based on the unproved conjecture on the linear model), one concludes that superconductivity is uniform along the boundary layer until the threshold $b = \Theta_0^{-1}$ is crossed and, then, concentrates close to the corners with smallest opening angles. More precisely, assuming that all the angles β_j are acute and different, one can identify [45, Rmk. 1.4] a sequence of N critical fields

$$H_{\mathrm{corner}} = H_{\mathrm{corner},0} \leqslant H_{\mathrm{corner},1} \leqslant \ldots \leqslant H_{\mathrm{corner},N-1} \leqslant H_{\mathrm{corner},N} = H_{\mathrm{c3}}, \tag{2.22}$$

with

$$H_{\text{corner},j} = \frac{1}{\mu(\beta_j)\varepsilon^2}, \qquad \text{for } 1 \leqslant j \leqslant N, \tag{2.23}$$

so that, in between $H_{\text{corner},j-1}$ and $H_{\text{corner},j}$, the material is superconducting only close to the $j-$th corner $\mathbf{r}_j$. Let us stress that all these results are conditioned by the request $\mu(\beta_j) < \Theta_0$ for all the corners, which is expected to hold true (but not proven) for any acute angle $0 < \beta_j < \pi$.

Once the energy asymptotics is obtained, it is natural to ask whether one can extract information about the behavior of the order parameter, which would then give access to the physically relevant quantities, as the density of Cooper pairs. As already proven in [19, Thm. 1.1], the distribution of superconductivity along the boundary layer is uniform to leading order (see (2.11)). Note that such an estimate goes along with the exponential decay proven in (B.16), which implies that $\psi^{\text{GL}} = o(1)$ at distance much larger than ε from the boundary $\partial\Omega$: we can indeed restrict out attention to the boundary layer $\mathcal{A}_\varepsilon$ defined in (1.19), since

$$\psi^{\text{GL}}(\mathbf{r}) = \mathcal{O}\left(\varepsilon^{c_1 \cdot c(b)+1}\right), \qquad \text{in } \Omega \setminus \mathcal{A}_\varepsilon, \tag{2.24}$$

and by taking c_1 large, the above quantity can be made arbitrarily small. We thus denote it as $\mathcal{O}(\varepsilon^\infty)$, to stress that it is an arbitrarily large power of ε.

Remark 2.3 **(Refined L^2 estimate)**
An almost direct consequence of the energy asymptotics (2.19) is an improvement of the bound (2.11): setting

$$\Omega_{\text{smooth}} := \{\mathbf{r} \in \Omega | \text{dist}(\mathbf{r}, \Sigma) \geqslant c_2 \varepsilon |\log \varepsilon|\}, \tag{2.25}$$

for some large enough constant $c_2 > 0$, one has

$$\left\| |\psi^{\text{GL}}(\,\cdot\,)|^2 - {f_\star}^2 \left(\text{dist}(\,\cdot\,, \partial\Omega)/\varepsilon\right) \right\|_{L^2(\Omega_{\text{smooth}})} = o(\varepsilon |\log \varepsilon|). \tag{2.26}$$

The estimates (2.11) or (2.26) do not exclude however the presence of vortices or region with very little superconductivity left close to the boundary and, therefore, one would like to prove a bound in a stronger norm, e.g., in L^∞, which is stated in the next Proposition 2.1.

Proposition 2.1 (GL order parameter) *Under the same assumptions of Theorem 2.1,*

$$\boxed{\left\| |\psi^{\text{GL}}(\boldsymbol{r})| - f_\star(0) \right\|_{L^\infty(\partial\Omega_{\text{smooth}} \cap \partial\Omega)} = o(1).} \tag{2.27}$$

Remark 2.4 **(Uniform distribution of superconductivity)** The estimate (2.27) can in fact be extended to the boundary layer of points $\mathbf{r}$ such that $\text{dist}(\mathbf{r}, \partial\Omega_{\text{smooth}}) \leqslant \varepsilon\sqrt{|\log \varepsilon|}$, in the very same way as the analogous result in [23, Thm. 2]. An important consequence is the uniformity of superconductivity in $\mathcal{A}_\varepsilon$, where one has

$$\left|\psi^{\text{GL}}(\,\cdot\,)\right| \sim f_\star\left(\text{dist}(\,\cdot\,, \partial\Omega)/\varepsilon\right), \tag{2.28}$$

not only in weak sense, as proven in [19], but also pointwise. Strictly speaking, the corner regions are excluded, but, on the one hand, their overall area is $\mathcal{O}(N\varepsilon^2|\log\varepsilon|^2)$, i.e., much smaller than $|\mathcal{A}_\varepsilon|$, and, on the other, we do expect the minimizer of the corner problem to be close to $f_\star$ almost everywhere but very close to the corner. An interested reader might wonder whether it is possible to show that ψ^{GL} is close to such an effective minimizer in the corner region, but this presumably requires to get some more information about the effective problem (2.12) as well as extract a more precise estimate of the remainders in (2.19).

Remark 2.5 **(Current along $\partial\Omega$)** An important consequence of (2.6) in smooth domains is the non-vanishing of ψ^{GL} close to the boundary, because of the strict positivity of $f_\star$, and thus surface superconductivity is robust w.r.t. the inclusion of the applied magnetic field. In addition, this allows to estimate the current (1.2) along the boundary or, equivalently, the total winding number of ψ^{GL} on $\partial\Omega$ [23, Thm. 3]:

$$\deg\left(\psi^{\mathrm{GL}}, \partial\Omega\right) = \frac{|\Omega|}{\varepsilon^2} - \frac{\alpha_\star}{\varepsilon}(1 + o(1)). \tag{2.29}$$

Such a behavior is similar (although physically different) to the ultrafast rotation regime for angular velocities larger than the third critical one of rotating Bose-Einstein condensates, when vortices are expelled from the boundary region [18,27] (see also [21,22,28] for further results on rotating condensates). In presence of corners, (2.27) guarantees the non-vanishing of ψ^{GL} only far from the corners and prevents us to estimate the current on $\partial\Omega$. Indeed, the pointwise estimate of the gradient (B.13) allows a variation of order 1 of ψ^{GL} on a scale ε, which is much smaller than the tangential length of the corner region, thus implying that ψ^{GL} may a priori vanish there.

2.3 Corner effective energy

We now give more details about the corner effective problem. Let us start by identifying more precisely the corner region depicted in Fig. 2. It is meant as a suitable stretching and rescaling (on a scale ε) of a local area around any corner of Ω of tangential and normal lengths both of order $\varepsilon|\log\varepsilon|$, as $\varepsilon \to 0$. For later convenience, however, we consider a region where the tangential length L along the angle is different from the normal length ℓ. Let then $\Gamma_\beta(L, \ell)$ be a triangle-like region as in Fig. 2, where β is the opening angle at the vertex V and side lengths $L, \ell > 0$. In order to reproduce the shape of Fig. 2, we always assume that

$$\ell \leqslant \tan\left(\tfrac{\beta}{2}\right) L. \tag{2.30}$$

We recall the definition of the boundaries $\partial\Gamma_{\mathrm{in}}, \partial\Gamma_{\mathrm{bd}}$ provided in § 2 and denote by $\partial\Gamma_{\mathrm{out}}$ the outer boundary $\overline{AVB}$, so that $\partial\Gamma_\beta(L, \ell) = \partial\Gamma_{\mathrm{out}} \cup \partial\Gamma_{\mathrm{in}} \cup \partial\Gamma_{\mathrm{bd}}$.

The effective energy in the corner region is given by a suitably rescaled GL energy with fixed magnetic potential (2.17). The effective variational model is then

$$E_{\mathrm{corner},\beta}(L, \ell) := -2LE_0^{\mathrm{1D}}(\ell) + \inf_{\psi\in\mathscr{D}_\star(\Gamma_\beta(L,\ell))} \mathcal{E}_1^{\mathrm{GL}}\left[\psi, \mathbf{F}; \Gamma_\beta(L, \ell)\right], \tag{2.31}$$

where $\mathscr{D}_\star(\Gamma_\beta(L, \ell))$ is defined in (2.15). The heuristics behind the choice (2.31) is that in the surface superconductivity regime each portion of the boundary of the sample yields a (leading order) energy contribution proportional to $E_\star^{\mathrm{1D}}$ times its length, which equals $E_\star^{\mathrm{1D}}|\partial\Gamma_{\mathrm{out}}| = 2LE_\star^{\mathrm{1D}}$ in the case of $\Gamma_\beta(L, \ell)$. Indeed, the boundaries $\partial\Gamma_{\mathrm{in}}$ and $\partial\Gamma_{\mathrm{bd}}$ are not expected to give any energy contribution. More precisely, $\partial\Gamma_{\mathrm{in}}$ is immersed in the bulk, where the order parameter is exponentially small in ℓ and it could have been removed from the outset by consider a solid wedge; similarly, $\partial\Gamma_{\mathrm{bd}}$ is a fictitious boundary, whose role is to separate the corner region from the rest. Mathematically, the non-zero Dirichlet conditions on $\partial\Gamma_{\mathrm{in}}$ and $\partial\Gamma_{\mathrm{bd}}$ in the minimization domain $\mathscr{D}_\star$ guarantee that those portions of the boundary do not contribute to surface superconductivity.

Once the boundary energy $2LE_\star^{\mathrm{1D}}$ has been subtracted, what remains is precisely the additional energy due to the presence of the corner. Such an energy is indeed of purely geometric nature and is generated by the constraint on the boundary $\partial\Gamma_{\mathrm{out}}$: in order to

reproduce the correct energy along $\partial\Gamma_{\text{out}}$, the minimizer must behave like the model order parameter $f_\star(t)e^{-i\alpha_\star s}$ in a layer of width $\mathcal{O}(1)$ around $\partial\Gamma_{\text{out}}$, but the coordinate s has a jump on the bisectrix of the domain and thus such a behavior is allowed only close far from the corner. The modulus of the minimizer $f_\star(t)$ is in fact well defined and continuous everywhere, since it depends on the normal coordinate which is continuous as well. Hence, in order to glue together the two model profiles, any minimizer must accommodate a non-trivial phase factor, which must be genuinely 2D, because no 1D function can adjust the jump of $-i\alpha_\star s$ along the bisectrix. Unfortunately, the explicit expression of such a phase remains unknown, expect in certain specific cases (for almost flat angles, see [20]).

The GL energy functional appearing in (2.31) is gauge invariant but we have chosen to work in a prescribed local gauge, i.e., we have made an explicit choice of the vector potential $\mathbf{F}$, generating a unit magnetic field. In this respect the GL energy in (2.31) is similar to the effective functional studied in [15, Eq. (1.11)], although both the parameter regime and the domain are slightly different. Such a difference reflects indeed the different behavior of the minimizer: in the present setting it decays in the distance from the outer boundary, whereas in [15], the decay is in the distance from the corner.

Recalling that L and ℓ are obtained via a rescaling from the tangential and normal length of the corner region and thus, in the original problem in Ω, are actually of order $|\log\varepsilon| \gg 1$, we have to study the limit $L, \ell \to +\infty$ of (2.31).

Proposition 2.2 (Corner energy) *Let $\{\ell_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ be two monotone sequences with $\ell_n, L_n \to +\infty$, as $n \to +\infty$, and $\beta \in (0, 2\pi)$, such that $1 \ll \ell_n \leqslant \tan(\beta/2)\ L_n \leqslant C\ell_n^a$ for some $a > 0$. Then, for any $1 < b < \Theta_0^{-1}$, the limit*

$$\lim_{n\to+\infty} E_{\text{corner},\beta}(L_n, \ell_n) =: E_{\text{corner},\beta} \tag{2.32}$$

exists, it is finite and independent of the chosen sequences.

As stated in Proposition 2.2 (see also Proposition 3.4), the corner energy $E_{\text{corner},\beta}$ is bounded for any $\beta \in (0, 2\pi)$, although we have no information on its sign. In fact, it might as well be zero. In a companion paper [20] however we prove that, when β is close to π, this is not the case (see also below).

Once the well-posedness of the model problem has been proven, it is then natural to ask whether one can derive the explicit dependence of $E_{\text{corner},\beta}$ on the angle β. So far we have not found such an expression but, based on some heuristic arguments, we formulate below an unproven conjecture, which is inspired again by the Gauss–Bonnet theorem. Indeed, the first order correction to the GL energy asymptotics in smooth domains reads equivalently

$$-E_{\text{corn}} \int_0^{|\partial\Omega|} \mathrm{d}s\ \mathfrak{K}(s) = -2\pi E_{\text{corn}}. \tag{2.33}$$

In presence of corner singularities on $\partial\Omega$, the Gauss–Bonnet theorem has to be modified to take into account the corners: the only correction is that the integral of the curvature must now be performed over the smooth part of $\partial\Omega$ and each corner yields a contribution proportional to its opening angle

$$\int_{\partial\Omega_{\text{smooth}}} \mathrm{d}s\ \mathfrak{K}(s) + \sum_{j=1}^{N} (\pi - \beta_j) = 2\pi.$$

Therefore, one can think of the above identity as if each corner contributes to the mean curvature with a Dirac mass multiplied by $\pi - \beta_j$ and the integral is meant in distributional

sense, i.e., formally replacing the curvature $\mathfrak{K}(\mathsf{s})$ with

$$\mathfrak{K}(\mathsf{s}) + \sum_{j=1}^{N} (\pi - \beta_j)\delta(\mathsf{s}_j),$$

which, if substituted on the r.h.s. of (2.33), yields

$$-E_{\mathrm{corr}} \int_{\partial\Omega_{\mathrm{smooth}}} \mathrm{d}\mathsf{s}\, \mathfrak{K}(\mathsf{s}) - E_{\mathrm{corr}} \sum_{j=1}^{N} (\pi - \beta_j).$$

After a direct comparison with the asymptotics proven in Theorem 2.1, i.e.,

$$-E_{\mathrm{corr}} \int_{\partial\Omega_{\mathrm{smooth}}} \mathrm{d}\mathsf{s}\, \mathfrak{K}(\mathsf{s}) + \sum_{j=1}^{N} E_{\mathrm{corner},\beta_j}$$

it is then very natural to state the conjecture below. Note that, if true, the conjecture would imply that the next-to-leading order term in the GL energy expansion would always be given by $-2\pi E_{\mathrm{corr}}$, irrespective of the presence of corners.

Conjecture 1 *(Corner energy)* For any $1 < b < \Theta_0^{-1}$ and $\beta \in (0, 2\pi)$, one has

$$E_{\mathrm{corner},\beta} = -(\pi - \beta) E_{\mathrm{corr}}. \tag{2.34}$$

Remark 2.6 **(Acute/obtuse angles)** In the linear case, i.e., for a magnetic Schrödinger operator with uniform magnetic field in an infinite wedge, it is expected [15, Rmk. 1.1] and numerically verified [1,13] that the ground state energy changes for acute or obtuse angles: for the former it is a strictly increasing function of the angle, which equals Θ_0 for flat angles, while it is believed to remain constantly equal to Θ_0 for any obtuse angle. On the opposite, in the nonlinear case, the above Conjecture would provide the same expression for acute and obtuse angles.

As already anticipated, we prove in [20] that in a wedge with opening angle $\pi - \delta$, $0 < \delta \ll 1$, the corner energy is given by

$$E_{\mathrm{corner},\beta} = -\delta E_{\mathrm{corr}} + \mathcal{O}(\delta^{4/3} |\log \delta|) + \mathcal{O}(\ell^{-\infty}), \tag{2.35}$$

i.e., it coincides to leading order in δ with the conjectured expression. Furthermore, this also shows that the corner energy $E_{\mathrm{corner},\beta}$ is non-trivial, at least for angles close to the flat one.

3 Corner effective problems

This section is mainly devoted to the proof of Proposition 2.2, i.e., the existence of the limit defining the effective energy contribution of each corner, and the discussion of the properties of such a limit. For later convenience, we also study another minimization problem in Γ_β with different boundary conditions and show that it asymptotically provides the same effective energy (Proposition 3.5).

3.1 Surface superconductivity in a finite strip

We start by studying a simple minimization problem in a finite strip. Similar problems have already been studied in [4,23,49], taking into account the limit of an infinite strip. Here, instead, the focus is more on boundary conditions and their effect on the ground state energy. We are going to apply the corresponding obtained results to the minimization in (2.12) to derive Proposition 3.5.

After a local gauge transformation and blow-up on a scale ε, the leading contribution to the GL energy of a portion of the boundary layer of Ω of tangential length εL and normal length $\varepsilon\ell$, suitably far from any corner, is (see, e.g., [23, Lemmas 2 & 4])

$$\mathcal{G}[\psi; R(L,\ell)] := \int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t \left\{ |\partial_t \psi|^2 + |(\partial_s - it)\,\psi|^2 - \frac{1}{2b}\left(2|\psi|^2 - |\psi|^4\right) \right\}, \tag{3.1}$$

where $L, \ell > 0$, $b \in (1, \Theta_0^{-1})$ and $R(L,\ell)$ stands for the rectangle

$$R(L,\ell) := [0,L] \times [0,\ell], \qquad \text{with } \ell \gg 1. \tag{3.2}$$

We study two simple minimization problems associated to the energy (3.1). First, we set

$$E_{\mathrm{D}}(R(L,\ell)) := \inf_{\psi \in \mathscr{D}_{\mathrm{D}}(R(L,\ell))} \mathcal{G}[\psi; R(L,\ell)], \tag{3.3}$$

and denote by ψ_{D} any corresponding minimizer. The minimization domain is given by

$$\mathscr{D}_{\mathrm{D}}(R(L,\ell)) := \Big\{ \psi \in H^1(R(L,\ell)) \;\Big|\; \psi(0,t) = f_0(t),\, \psi(L,t) = f_0(t)e^{-i\alpha_0 L}, \\ \psi(s,\ell) = f_0(\ell)e^{-i\alpha_0 s} \Big\}, \tag{3.4}$$

where the boundary conditions are meant in trace $H^{1/2}$-sense and we recall that f_0, α_0 is a minimizing pair (see also "Appendix A.2") of (2.13). The label D stands for the Dirichlet-type conditions at $s = 0$ and $s = L$. The heuristic meaning of such conditions is the following:

- on the boundary between the surface and the bulk region, i.e., for $t = \ell$, the order parameter is exponentially small and the same holds true for $f_0(\ell)$, so the contribution of the boundary conditions there is expected to be negligible; for this reason we could as well have set $\psi = 0$ at $t = \ell$, but this would make the analysis more complicated;
- at the normal boundaries $s = 0$ or $s = L$, the order parameter is set equal to the ideal minimizer (see § 1.2);
- no condition is set on the boundary $t = 0$, which is meant to coincide with a blow-up of a portion of $\partial\Omega$: this is crucial to capture the key features of surface superconductivity and leads to Neumann conditions along the line $t = 0$.

By setting $\psi := \chi + f_0(t)e^{-i\alpha_0 s}$, one can reduce the variational problem (3.3) to the minimization of a functional of χ with zero Dirichlet conditions on the boundaries $s = 0, L$ and $t = \ell$. This easily allows to deduce (see, e.g., [41, Chapt. 4]) the existence of a minimizer, its smoothness and the fact that any minimizer solves

$$-(\nabla - it\mathbf{e}_s)^2\,\psi = \frac{1}{b}\left(1 - |\psi|^2\right)\psi. \tag{3.5}$$

The alternative version of (3.4) is provided by a modification of the energy: we define

$$\widetilde{\mathcal{G}}[\psi; R(L,\ell)] := \mathcal{G}[\psi; R(L,\ell)] - \int_0^\ell \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)}\, j_t[\psi]\Big|_{s=0}^{s=L}, \tag{3.6}$$

where F_0 is the potential function (see also "Appendix A.3")

$$F_0(t) := 2\int_0^t \mathrm{d}\eta\ (\eta+\alpha_0)\, f_0^2(\eta), \tag{3.7}$$

and j_t is the normal component of the current $\mathbf{j}[\psi]$ given in (1.2), i.e., $j_t[\psi] = \frac{i}{2}(\psi\,\partial_t\psi^* - \psi^*\partial_t\psi)$. The boundary terms appearing in (3.6) are non-trivial only if the phase of ψ varies along the normal to the boundary, which is obviously not the case for the reference function $f_0(t)e^{-i\alpha_0 s}$. The reason why such terms have been added to the energy will become clear later (see the proof of Proposition 3.1 and in particular (3.22)). The minimization of (3.6) is performed on a domain without constraints on the boundaries $s=0$ and $s=L$, i.e., we set

$$E_{\mathrm{N}}(R(L,\ell)) := \inf_{\psi\in\mathscr{D}_{\mathrm{N}}(R(L,\ell))} \widetilde{\mathcal{G}}\,[\psi;\, R(L,\ell)]\,, \tag{3.8}$$

where

$$\mathscr{D}_{\mathrm{N}}(R(L,\ell)) := \left\{\psi\in H^1(R(L,\ell)) \;\middle|\; \psi(s,\ell) = f_0(\ell)e^{-i\alpha_0 s}\right\} \tag{3.9}$$

and we denote by ψ_{N} any corresponding minimizer, which enjoys the same properties as ψ_{D}, except for conditions of magnetic Neumann-type at $s=0$ and $s=L$, i.e.,

$$\left[(\partial_s + i\alpha_0)\,\psi_{\mathrm{N}} - i\frac{F_0(t)}{f_0^2(t)}\,(\partial_t\psi_{\mathrm{N}})\right]\Bigg|_{s=0,L} = 0. \tag{3.10}$$

The surface superconductivity behavior occurs for $1<b<\Theta_0^{-1}$ and is characterized by the emergence of the 1D effective model (2.13) or, equivalently, (2.3).

Proposition 3.1 (GL energies on a finite strip) *For any* $1<b<\Theta_0^{-1}$ *and* $L>0$, *as* $\ell\to\infty$,

$$E_{\mathrm{D/N}}(R(L,\ell)) = L\left(E_0^{\mathrm{1D}}(\ell) + \mathcal{O}(\ell^{-\infty})\right) = L\left(E_\star^{\mathrm{1D}} + \mathcal{O}(\ell^{-\infty}))\right). \tag{3.11}$$

Remark 3.1 **(Boundary conditions)** The boundary condition $\psi_{\mathrm{N}}(s,\ell) = f_0(\ell)e^{-i\alpha_0 s}$ is needed for the asymptotics (3.11) to hold true. The reason is that otherwise one would get an additional energy contribution from the boundary $t=\ell$, i.e., the energy would be twice the value appearing in (3.11). Indeed, without the condition at $t=\ell$, exploiting the gauge invariance of (3.1) and replacing $\psi, -t\mathbf{e}_s$ with $\psi^* e^{i\ell s}, -(\ell-t)\mathbf{e}_s$, one can exchange the boundaries $t=0$ and $t=\ell$, leaving the energy unaffected.

Proof We first observe that the last estimate is in fact stated in Lemma A.2 in "Appendix A.3". The rest of the statement is actually proven by showing separately that the first estimate holds true for both E_{D} and E_{N}.

Let us first consider $E_{\mathrm{D}}(R(L,\ell))$. For the upper bound, we test $\mathcal{G}$ on the trial state $f_0(t)e^{-i\alpha_0 s}$, which immediately yields $E_{\mathrm{D}}(R(L,\ell)) \leqslant LE_0^{\mathrm{1D}}(\ell)$. For the corresponding lower bound we use the same energy splitting used, e.g., in [24], i.e., we set

$$\psi_{\mathrm{D}}(s,t) =: f_0(t)e^{-i\alpha_0 s}u(s,t), \tag{3.12}$$

which, via an integration by parts and the variational equation (A.7), leads to

$$E_{\mathrm{D}}(R(L,\ell)) = LE_0^{\mathrm{1D}} + \mathcal{E}_0\,[u;\, R(L,\ell)]\,, \tag{3.13}$$

where

$$\mathcal{E}_0\,[u;\,R(L,\,\ell)] := \int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\; f_0^2 \left\{ \left|\nabla_{s,t} u\right|^2 - 2(t+\alpha_0)\, j_s[u] + \frac{f_0^2}{2b}(1-|u|^2)^2 \right\}, \tag{3.14}$$

and $j_s[\psi]$ is the tangential component of (1.2), i.e., explicitly $j_s[\psi] = \frac{i}{2}\,(\psi\,\partial_s\psi^* - \psi^*\partial_s\psi)$. We stress that the decoupling does not generate any boundary term because f_0' vanishes both at $t = 0$ and $t = \ell$ by (A.17): the only non-trivial computation is the following integration by parts

$$\int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\, \left[|u|^2 f_0'^2 + f_0 f_0' \partial_t\, |u|^2 \right] = -\int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\, |u|^2 f_0 f_0'',$$

where f_0'' can then be replaced via the variational Eq. (A.7).

The key ingredient to bound from below $\mathcal{E}_0[u]$ is the pointwise positivity of the cost function (see (A.21) and (A.22) in "Appendix A.3")

$$K_0 := f_0^2 + F_0, \tag{3.15}$$

in $I_{\bar\ell} = [0,\,\bar\ell]$ given by (A.23) (recall that $\bar\ell = \ell + \mathcal{O}(1)$ by (A.24)). Indeed, we integrate by parts twice:

$$\begin{aligned} &-2\int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\; (t+\alpha_0) f_0^2(t)\, j_s[u] \\ &= -\int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\; F_0'(t)\, j_s[u] = \int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\; F_0(t) \partial_t\, j_s[u] \\ &= 2\int_0^L \mathrm{d}s \int_0^\ell \mathrm{d}t\; F_0(t)\, \mathrm{Im}\left(\partial_t u^* \partial_s u\right) + \int_0^\ell \mathrm{d}t\; F_0(t)\, j_t[u]\Big|_{s=0}^{s=L}, \end{aligned} \tag{3.16}$$

where the boundary terms of the first integration by parts vanish, because $F_0(0) = F_0(\ell) = 0$, and the last terms vanish as well, since, due to boundary conditions, $u(0,t) = u(L,t) = 1$ and thus $j_t[u] = 0$ there.

Using (A.25) and the simple bound $2|\mathrm{Im}(ab)| \leqslant |a|^2 + |b|^2$, one then obtains as in [23, Eq. (4.38)] (see also [23, Sect. 2.3 & Proof of Prop. 4.2])

$$\begin{aligned} \mathcal{E}_0\,[u;\,R(L,\,\ell)] &\geqslant \int_0^L \mathrm{d}s \int_0^{\bar\ell} \mathrm{d}t\; \left\{ K_0(t)\left(|\partial_s u|^2 + |\partial_t u|^2\right) + \frac{1}{2b} f_0^4 (1-|u|^2)^2 \right\} \\ &+ \int_0^L \mathrm{d}s \int_{\bar\ell}^\ell \mathrm{d}t\; \left\{ f_0^2\, |\nabla u|^2 + 2F_0(t)\, \mathrm{Im}\left(\partial_t u^* \partial_s u\right) \right\} \\ &\geqslant \int_0^L \mathrm{d}s \int_{\bar\ell}^\ell \mathrm{d}t\; \left\{ f_0^2\, |\nabla u|^2 + 2F_0(t)\, \mathrm{Im}\left(\partial_t u^* \partial_s u\right) \right\}, \end{aligned} \tag{3.17}$$

by (A.22) and the positivity of the last term on the r.h.s. of the first line. It thus remains to estimate the quantity on the r.h.s. of (3.17) above, which can be done by integrating by parts back:

$$\begin{aligned} &\int_0^L \mathrm{d}s \int_{\bar\ell}^\ell \mathrm{d}t\; \left\{ f_0^2\, |\nabla u|^2 + 2F_0(t)\, \mathrm{Im}\left(\partial_t u^* \partial_s u\right) \right\} \\ &= \int_0^L \mathrm{d}s \int_{\bar\ell}^\ell \mathrm{d}t\; \left\{ f_0^2\, |\nabla u|^2 - 2(t+\alpha_0)\, j_s[f_0 u] \right\} - 2F_0(\bar\ell) \int_0^L \mathrm{d}s\; j_s[u]\Big|_{t=\bar\ell}. \end{aligned} \tag{3.18}$$

Now, exploiting (A.11) and the fact that $\bar{\ell} = \ell + \mathcal{O}(1)$, we deduce that

$$f_0(t) = \mathcal{O}(\ell^{-\infty}), \qquad f_0'(t) = \mathcal{O}(\ell^{-\infty}), \qquad \text{for any } t \geqslant \bar{\ell}. \tag{3.19}$$

Hence, $|\nabla\psi_{\mathrm{D}}| = f_0\,|\nabla u| + \mathcal{O}(\ell^{-\infty})$ in $I_\ell \setminus I_{\bar{\ell}}$. Now, since $F_0(\ell) = 0$, $F_0(\bar{\ell}) \leqslant C\ell f_0^2(\bar{\ell})$, we can bound the boundary term (last term in (3.18)) by

$$CL \sup_{s\in[0,L]} \left|\psi_{\mathrm{D}}(s,\bar{\ell})\right| \left|\nabla\psi_{\mathrm{D}}(s,\bar{\ell})\right| = L\mathcal{O}(\ell^{-\infty}),$$

thanks to (B.38) and the bound $\|\nabla\psi_{\mathrm{D}}\|_\infty \leqslant C$ on the gradient of ψ_{D} (see (B.13)). For the same reason, the first term on the r.h.s. of (3.18) can be bounded from below via Cauchy inequality and (B.17) by

$$-C\int_0^L \mathrm{d}s \int_{\bar{\ell}}^{\ell} \mathrm{d}t\ (t+\alpha_0)^2\,|\psi_{\mathrm{D}}|^2 = \mathcal{O}(L\ell^{-\infty}),$$

which finally yields,

$$\int_0^L \mathrm{d}s \int_{\bar{\ell}}^{\ell} \mathrm{d}t\ \left\{ f_0^2\,|\nabla u|^2 + 2F_0(t)\,\mathrm{Im}\left(\partial_t u^* \partial_s u\right)\right\} = \mathcal{O}(L\ell^{-\infty}), \tag{3.20}$$

and thus the statement.

The proof for the modified functional (3.6) is very similar. The upper bound is obtained by evaluating the energy on the trial state $f_0(t)e^{-i\alpha_0 s}$: notice that the phase of such a function is independent of t, then the normal component j_t of its current is identically zero and therefore the boundary terms in $\widetilde{\mathcal{G}}$ do not yield any additional contribution. The final outcome is the very same bound $E_{\mathrm{N}}(R(L,\ell)) \leqslant LE_0^{\mathrm{1D}}(\ell)$ as before.

One can then apply the splitting technique, setting (for a different u than before)

$$\psi_{\mathrm{N}}(s,t) =: f_0(t)e^{-i\alpha_0 s}u(s,t), \tag{3.21}$$

to get the identity $E_{\mathrm{N}}(R(L,\ell)) = LE_0^{\mathrm{1D}} + \widetilde{\mathcal{E}}_0[u;\,R(L,\ell)]$, where

$$\widetilde{\mathcal{E}}_0[u;\,R(L,\ell)] := \mathcal{E}_0[u;\,R(L,\ell)] - \int_0^{\ell} \mathrm{d}t\ F_0(t)\,j_t[u]\Big|_{s=0}^{s=L}. \tag{3.22}$$

The proof of the lower bound is then completely analogous to the one above: the only non-trivial observation is that the first integration by parts in (3.16) generates the same outcome, because of the vanishing of F_0 at the boundaries, and the last terms in (3.16) are exactly compensated by the boundary terms in the functional (3.22), so that they sum up to zero. Actually, this was the main reason to add those terms to (3.6) in first place. The lower bound then follows from the positivity of K_0, exactly as above. □

A straightforward adaptation of the above arguments leads to the following result on a modified problem with twisted boundary conditions, which is going to play a role later.

Proposition 3.2 (GL energy with twisted boundary conditions) *Let* $\varkappa \in [0, 2\pi)$, $1 < b < \Theta_0^{-1}$ *and* $L > 0$. *Let also*

$$E_{\mathrm{D},\varkappa}(R(L,\ell)) := \inf_{\psi\in\mathscr{D}_{\mathrm{D},\varkappa}(R(L,\ell))} \mathcal{G}[\psi; R(L,\ell)], \tag{3.23}$$

$$\mathscr{D}_{\mathrm{D},\varkappa}(R(L,\ell)) := \left\{ \psi \in \mathscr{D}_{\mathrm{N}}(R(L,\ell)) \,\middle|\, \psi(0,t) = f_0(t)e^{i\varkappa}, \psi(L,t) = f_0(t)e^{-i\alpha_0 L} \right\}. \tag{3.24}$$

Then, as $\ell \to \infty$,

$$E_0^{\mathrm{1D}}(\ell)L + \mathcal{O}(L\ell^{-\infty}) \leqslant E_{\mathrm{D},\varkappa}(R(L,\ell)) \leqslant E_0^{\mathrm{1D}}(\ell)L + \frac{C}{L}. \tag{3.25}$$

Proof The lower bound is obtained via the splitting technique and the positivity of the cost function as discussed in the proof of Proposition 3.1. For the upper bound it is sufficient to test the functional on the trial state

$$f_0(t)e^{-i\alpha_0 s}e^{i\frac{\varkappa(L-s)}{L}},$$

and recall the optimality of the phase α_0 yielding (A.8). □

We conclude this section with a result which will be used later in the paper. In extreme synthesis it states that, if one has an a priori upper bound on $\mathcal{E}_0[u, R(L,\ell)]$, then it is possible to extract some useful information on the corresponding order parameter $\psi(s,t)$ and show for instance that it is pointwise close to $f_0(t)e^{-i\alpha_0 s}$ up to a smooth phase factor.

Proposition 3.3 (Order parameter estimates) *Let* ψ *be a solution of* (3.5) *in the strip* $R(L,\ell)$, *with* $\ell \geqslant t_0 > 0$ *and* $L > 0$, *satisfying the boundary conditions in* (3.9) *and* (3.10), *and let* u *be defined as in* (3.21). *Let also* $\widetilde{\mathcal{E}}_0[u; R(L,\ell)]$ *be the functional defined in* (3.22) *in the strip* $R(L,\ell)$ *and assume that*

$$\widetilde{\mathcal{E}}_0[u; R(L,\ell)] \leqslant \mathfrak{e} \leqslant 1, \tag{3.26}$$

for some $\mathfrak{e} > 0$. *Then, if* $1 < b < \Theta_0^{-1}$,

$$\left\| f_0^2 \nabla u \right\|^2_{L^2(R(L,\ell))} \leqslant C\mathfrak{e} + \mathcal{O}(L\ell^{-\infty}). \tag{3.27}$$

Moreover, for any $0 < T \leqslant \bar{\ell}$, *there exists a finite constant* $C > 0$, *such that*

$$||\psi(s,t)| - f_0(t)| \leqslant \frac{C\mathfrak{e}^{1/4} + \mathcal{O}(L\ell^{-\infty})}{\sqrt{\min_{[0,T]} f_0}}, \qquad \text{for any } (s,t) \in R_{L,T}, \tag{3.28}$$

$$\left| \partial_s \int_0^\ell \mathrm{d}t \ |\psi|^2 \right| \Big|_{s=L} \leqslant C \left\{ \mathfrak{e} + \sqrt{\mathfrak{e}L} + \frac{1}{L} \left[\frac{\mathfrak{e}^{1/4} + \mathcal{O}(L\ell^{-\infty})}{\sqrt{\min_{[0,T]} f_0}} + e^{-c(b)T} \right] + L \right\}. \tag{3.29}$$

Proof Applying elliptic regularity theory to the equation satisfied by ψ one can prove as in Lemma B.1 that

$$\|\nabla\psi\|_{L^\infty(R(L,\ell))} \leqslant C, \tag{3.30}$$

Furthermore, ψ satisfies the Agmon estimates (B.17) and (B.38).

The key estimate is then the positivity of the cost function K_0 in $I_{\bar{\ell}}$, as well as the lower bound given by (A.27), i.e.,

$$K_0(t) = f_0^2(t) + F_0(t) \geqslant c_b f_0^4(t), \qquad \text{for any } t \in I_{\bar{\ell}}. \tag{3.31}$$

Indeed, by acting as in the proof of Proposition 3.1, one immediately gets

$$\widetilde{\mathcal{E}}_0[u,\, R(L,\ell)] \geqslant \int_{R(L,\bar{\ell})} \mathrm{d}s\mathrm{d}t\; K_0(t)\,|\nabla u|^2 + \frac{1}{2b}\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t\; f_0^4(t)\left(1-|u|^2\right)^2 + \mathcal{O}(L\ell^{-\infty}). \tag{3.32}$$

Plugging in (3.31) above, one obtains (3.27) and

$$\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t\; f_0^4(t)\left(1-|u|^2\right)^2 \leqslant C\mathfrak{e} + \mathcal{O}(L\ell^{-\infty}). \tag{3.33}$$

We now address (3.28): the starting point is provided by (3.33), which essentially implies that $|u|$ is approximately constant and equal to 1. The idea of proof goes back to [10] and it has been used several times since then (see, e.g., [27]). Fix $0 < T \leqslant \bar{\ell}$ and assume by contradiction that there was a point $(s_0, t_0) \in R_{L,T}$, where

$$\big||\psi(s_0,t_0)| - f_0(t_0)\big| \geqslant \frac{c\,\bar{\mathfrak{e}}^{1/4}}{\sqrt{\min_{[0,T]} f_0}}, \tag{3.34}$$

for suitable $c > 0$ and $\bar{\mathfrak{e}} \geqslant \mathfrak{e}$ to be adjusted later. Then, by (3.30) and the analogous bound for $|f_0'(t)|$ (see (A.11)), we deduce that there would exist also a ball of radius $\varrho := c'\bar{\mathfrak{e}}^{1/4}/\sqrt{f_0(t_0)}$ centered in (s_0, t_0), with c' a constant proportional to c and depending only on the a priori bounds on the gradients, so that

$$\big||\psi(s,t)| - f_0(t)\big| \geqslant \frac{1}{2}\frac{c\,\bar{\mathfrak{e}}^{1/4}}{\sqrt{\min_{[0,T]} f_0}}, \qquad \text{in } \mathcal{B}_\varrho(s_0,t_0)\cap R_{L,T}.$$

Furthermore, we can also assume that at least one quarter of the ball is contained inside $R_{L,T}$. Hence,

$$\int_{R(\ell,T)} \mathrm{d}s\mathrm{d}t\; f_0^4(t)\left(1-|u|^2\right)^2 = \int_{R(\ell,T)} \mathrm{d}s\mathrm{d}t\; \left(f_0^2(t)-|\psi|^2\right)^2 \geqslant \frac{\pi c^2}{16}\varrho^2\bar{\mathfrak{e}}^{1/2}\min_{[0,T]} f_0 \geqslant Cc^4\bar{\mathfrak{e}}$$

where C is a positive constant independent of c. Therefore, by taking c large enough and $\bar{\mathfrak{e}} = \mathfrak{e} + \mathcal{O}(L\ell^{-\infty})$, we would get a contradiction with (3.33), which completes the proof.

In order to finally get (3.29), we can restrict the integration to the interval $t \in [0, \bar{\ell}]$, since the rest is exponentially small. We then compute

$$\partial_s \int_0^{\bar{\ell}} \mathrm{d}t\; |\psi(s,t)|^2\Big|_{s=L} = \int_0^L \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t\; \left[\chi(s)\partial_s^2\,|\psi(s,t)|^2 + \chi'(s)\partial_s\,|\psi(s,t)|^2\right], \tag{3.35}$$

for any smooth χ such that $\chi(L) = 1$. Taking $\chi(s) = s/L$, we get

$$\left|\int_0^\delta \mathrm{d}s \int_0^{\bar{\ell}} \mathrm{d}t\; \chi'(s)\partial_s\,|\psi(s,t)|^2\right| \leqslant \frac{1}{L}\int_0^{\bar{\ell}} \mathrm{d}t\; \left||\psi(\delta,t)|^2 - |\psi(0,t)|^2\right| \leqslant \frac{C}{L}\left[\frac{\mathfrak{e}^{1/4}+\mathcal{O}(L\ell^{-\infty})}{\sqrt{\min_{[0,T]} f_0}} + e^{-c(b)T}\right],$$

by (3.28) and (B.38). For the first term on the r.h.s. of (3.35), we extend the integration in t to ℓ: using Neumann boundary conditions at $t = \ell$, one gets

$$\int_{\bar{\ell}}^{\ell} \mathrm{d}t\ \partial_s^2 |\psi|^2 = \int_{\bar{\ell}}^{\ell} \mathrm{d}t\ \Delta |\psi|^2 + \partial_t |\psi|^2\big|_{t=\bar{\ell}} = \int_{\bar{\ell}}^{\ell} \mathrm{d}t\ \Delta |\psi|^2 + \mathcal{O}(\ell^{-\infty}),$$

by (B.38) and the pointwise bound on the gradient of ψ. Hence, exploiting the Neumann conditions also at $t = 0$ and (3.5), we obtain

$$\begin{aligned}\int_0^{\bar{\ell}} \mathrm{d}t\ \partial_s^2 |\psi|^2 &= \int_0^{\ell} \mathrm{d}t\ \Delta |\psi|^2 + \mathcal{O}(\ell^{-\infty}) = \int_0^{\ell} \mathrm{d}t\ \left[2\mathrm{Re}\left(\psi^* \Delta\psi\right) + 2|\nabla\psi|^2\right] + \mathcal{O}(\ell^{-\infty}) \\ &= 2\int_0^{\ell} \mathrm{d}t\ \left[|(\nabla - it\mathbf{e}_s)\psi|^2 - \tfrac{1}{b}\left(1 - |\psi|^2\right)|\psi|^2\right] + \mathcal{O}(\ell^{-\infty}),\end{aligned}$$

which yields, after integration in s,

$$\begin{aligned}&\int_0^L \mathrm{d}s \int_0^{\ell} \mathrm{d}t\ \chi(s)\partial_s^2 |\psi(s,t)|^2 \\ &\quad = 2\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t\ \chi(s)\left[|(\nabla - it\mathbf{e}_s)\psi|^2 - \tfrac{1}{b}\left(1 - |\psi|^2\right)|\psi|^2\right] + \mathcal{O}(L\ell^{-\infty}). \qquad (3.36)\end{aligned}$$

In order to estimate the quantity on the r.h.s. of the expression above we observe that

$$\begin{aligned}&\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t\ \chi(s)\left[|(\nabla - it\mathbf{e}_s)\psi|^2 - \tfrac{1}{b}\left(1 - |\psi|^2\right)|\psi|^2\right] \\ &\quad \leqslant \widetilde{\mathcal{E}}_0[u] + \frac{1}{2b}\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t\ \left|f_0^4 - |\psi|^4\right| + 2\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t\ (\chi(s) - 1)(t + \alpha_0) f_0^2 j_s[u] \\ &\quad \leqslant C\left[\mathfrak{e} + \sqrt{\mathfrak{e}L} + L\right],\end{aligned}$$

by (3.30) and (3.33). Altogether we get (3.29). □

3.2 Properties of $E_{\mathrm{corner},\beta}(L,\ell)$

In the present and following Sections, we study the effective model introduced in (2.12) and specifically prove the existence of the limit as well as its boundedness. The key properties we are going to use in the proof of Proposition 2.2 are:

- change of gauge to replace the magnetic potential $\mathbf{F}$ with $\mathbf{a} \simeq -t\mathbf{e}_s$ (Lemma 3.1);
- uniform boundedness of $E_{\mathrm{corner},\beta}(L,\ell)$ and existence of the limit $L, \ell \to +\infty$ over suitable subsequences (Proposition 3.4);
- further properties of the effective model and, in particular, its dependence on the boundary conditions (§ 3.4).

We recall the corner energy defined in (2.31) and set

$$\mathcal{G}_{\mathbf{F}}[\psi] := \mathcal{E}_1^{\mathrm{GL}}\left[\psi, \mathbf{F}; \Gamma_\beta(L,\ell)\right]; \qquad (3.37)$$

$$E_\beta(L,\ell) := \inf_{\psi \in \mathscr{D}_\star(\Gamma_\beta(L,\ell))} \mathcal{G}_{\mathbf{F}}[\psi], \qquad (3.38)$$

where both the energy $\mathcal{E}_1^{\mathrm{GL}}$, the minimization domain and the corner region are introduced in § 2. Any corresponding minimizer is denoted by ψ_β. Before proceeding further, we introduce

an auxiliary problem in $\Gamma_\beta(L,\ell)$, modified by the addition of analogous boundary terms as in (3.6). Such a problem will appear in the proof of the main theorem. We set

$$\widetilde{\mathcal{G}}_{\mathbf{F}}[\psi] := \mathcal{E}_1^{\mathrm{GL}}\left[\psi,\mathbf{F};\Gamma_\beta(L,\ell)\right] - \int_0^\ell \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t\left[\psi(\mathbf{r}(s,t))e^{\frac{i}{2}st}\right]\Big|_{s=-L}^{s=L}; \quad (3.39)$$

$$\widetilde{E}_\beta(L,\ell) := \inf_{\psi\in\widetilde{\mathscr{D}}_\star(\Gamma_\beta(L,\ell))} \widetilde{\mathcal{G}}_{\mathbf{F}}[\psi], \quad (3.40)$$

where

$$\widetilde{\mathscr{D}}_\star(\Gamma_\beta(L,\ell)) := \left\{\psi\in H^1(\Gamma_\beta(L,\ell)) \;\middle|\; \psi|_{\partial\Gamma_{\mathrm{in}}} = \psi_\star\right\}, \quad (3.41)$$

and $\psi_\star$ is defined in (2.16). Note that the boundary terms are slightly different than the ones considered in (3.6), which is due to the presence of an additional phase in $\psi_\star$ compared to $f_0 e^{-i\alpha_0 s}$, due to the different choice of the magnetic potential.

In the next Lemma 3.1, we show that the vector potential $\mathbf{F}$ can be replaced with $\mathbf{a}$, such that far from the corners

$$\mathbf{a}(\mathbf{r}(s,t)) \simeq -t\mathbf{e}_s, \quad (3.42)$$

in boundary coordinates (s,t). It is not difficult to figure out that there exists no smooth gauge transformation implementing the above change globally in $\Gamma_\beta(L,\ell)$, in particular close to the bisectrix. More precisely, we define the wedge-domain $\Gamma_\beta(L,\ell)\setminus\widetilde{\Gamma}_\beta(L,\ell)$ (as depicted in Fig. 4) through

$$\Gamma_\beta(L,\ell)\setminus\widetilde{\Gamma}_\beta(L,\ell) := \left\{\mathbf{r}\in\Gamma_\beta(L,\ell) \;\middle|\; \tfrac{1}{2}\beta - \tfrac{1}{\ell^3} \leqslant \vartheta \leqslant \tfrac{1}{2}\beta + \tfrac{1}{\ell^3}\right\}, \quad (3.43)$$

in polar coordinates $(\varrho,\vartheta)\in[0,\ell]\times[0,\beta]$. Hence, we obviously have

$$\left|\Gamma_\beta(L,\ell)\setminus\widetilde{\Gamma}_\beta(L,\ell)\right| = \mathcal{O}(\ell^{-1}). \quad (3.44)$$

The potential $\mathbf{a}$ is thus such that there exists a gauge phase $\phi_{\mathbf{F}}\in H^1(\Gamma_\beta(L,\ell))$ so that

$$\mathbf{a} = \mathbf{F} + \nabla\phi_{\mathbf{F}}, \quad (3.45)$$

and

$$\mathbf{a} = -t\mathbf{e}_s, \qquad \text{in } \widetilde{\Gamma}_\beta(L,\ell). \quad (3.46)$$

As already explained, because of the jump of $\mathbf{e}_s$ along the bisectrix of the sector, one can not set $\mathbf{a} = -t\mathbf{e}_s$ everywhere. However, we require that

$$\mathbf{a} = \mathcal{O}(\ell^4), \qquad \text{in } \Gamma_\beta(L,\ell), \quad (3.47)$$

which is in fact a constraint only in $\Gamma_\beta(L,\ell)\setminus\widetilde{\Gamma}_\beta(L,\ell)$. In next Lemma 3.1 we investigate the existence of such a phase $\phi_{\mathbf{F}}$. Note that

$$\operatorname{curl}\mathbf{a} = \operatorname{curl}(-t\mathbf{e}_s) = 1, \qquad \text{in } \Gamma_\beta(L,\ell), \quad (3.48)$$

thanks to (1.22) and the gauge invariance of the curl.

Lemma 3.1 (Gauge choice) *For any $L,\ell>0$ satisfying (2.30) and so that*

$$\frac{\left|\Gamma_\beta(L,\ell)\right|}{2\pi\left|\partial\Gamma_\beta(L,\ell)\right|}\in\mathbb{Z}, \quad (3.49)$$

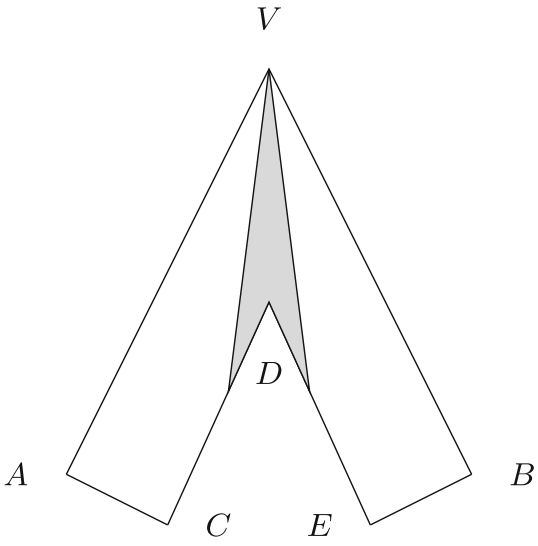

Fig. 4 The region $\Gamma_\beta(L,\ell)\setminus\widetilde{\Gamma}_\beta(L,\ell)$ (shaded area)

there exists a vector potential $\boldsymbol{a}\in L^\infty(\Gamma_\beta(L,\ell))$ *and a phase* $\phi_{\boldsymbol{F}}\in H^1(\Gamma_\beta(L,\ell))$ *satisfying* (3.45)*,* (3.46) *and* (3.47)*, such that*

$$\inf_{\psi\in\mathscr{D}_\star(\Gamma_\beta(L,\ell))}\mathcal{G}_{\boldsymbol{F}}[\psi]=\inf_{\psi\in\mathscr{D}_{\mathrm{D}}(\Gamma_\beta(L,\ell))}\mathcal{G}_{\boldsymbol{a}}[\psi], \tag{3.50}$$

$$\inf_{\psi\in\widetilde{\mathscr{D}}_\star(\Gamma_\beta(L,\ell))}\widetilde{\mathcal{G}}_{\boldsymbol{F}}[\psi]=\inf_{\psi\in\mathscr{D}_{\mathrm{N}}(\Gamma_\beta(L,\ell))}\widetilde{\mathcal{G}}_{\boldsymbol{a}}[\psi], \tag{3.51}$$

where

$$\mathscr{D}_{\mathrm{D}}(\Gamma_\beta(L,\ell)):=\left\{\psi\in H^1(\Gamma_\beta(L,\ell))\;\middle|\;\psi|_{\partial\Gamma_{\mathrm{bd}}\cup\partial\Gamma_{\mathrm{in}}}=\psi_0\right\}, \tag{3.52}$$

$$\mathscr{D}_{\mathrm{N}}(\Gamma_\beta(L,\ell)):=\left\{\psi\in H^1(\Gamma_\beta(L,\ell))\;\middle|\;\psi|_{\partial\Gamma_{\mathrm{in}}}=\psi_0\right\}, \tag{3.53}$$

$$\psi_0(s,t):=f_0(t)e^{-i\alpha_0 s}. \tag{3.54}$$

Remark 3.2 **(Constraint on L,ℓ)** The condition (3.49) reads $L-\frac{2\ell}{\tan\beta}=c(\beta,L\ell)\mathbb{Z}$ where $c(\beta,\ell)$ is uniformly bounded as $L,\ell\to+\infty$. More precisely

$$c(\beta,L,\ell)\xrightarrow[L\to+\infty]{}0,\qquad c(\beta,L,\ell)\xrightarrow[\ell\to+\infty]{}c(\beta),$$

uniformly in the other parameters. Hence, given generic $\ell, L\to+\infty$, it suffices to replace L with $L+\mathcal{O}(1)$ to enforce (3.49).

Proof The two different minimization problem can be treated in the same way. It suffices to prove the existence of the gauge phase $\phi_{\mathbf{F}}$ and, in order to recover (3.46), we set

$$\phi_{\mathbf{F}}(s,t):=-\tfrac{1}{2}st,\qquad\text{in }\widetilde{\Gamma}_\beta(L,\ell). \tag{3.55}$$

Note that such a phase is actually the same gauge phase used in [33, Appendix F] or in [24, Eq. (4.7)] with vector potential set equal to $\mathbf{F}$ and recovers the additional phase factor in the boundary terms in (3.39). Such a phase is in $H^1(\widetilde{\Gamma}_\beta(L,\ell))$ but its definition can not be extended to the whole $\Gamma_\beta(L,\ell)$. We can however continue $\phi_{\mathbf{F}}$ arbitrarily in $\Gamma_\beta(L,\ell)\setminus\widetilde{\Gamma}_\beta(L,\ell)$, just requiring continuity through the boundary of the region. There are infinitely

many ways of doing that and at least one such that the bound (3.47) is satisfied (e.g., a linear interpolation).

In order to complete the proof, we need to show that $\psi e^{i\phi_{\mathbf{F}}}$ is still a single-valued function. It is not difficult to see [33, Appendix F] that, to this purpose, one has to correct (3.55) by ϖs, where

$$\begin{aligned}\varpi &= \frac{1}{2\pi\left|\partial\Gamma_\beta(L,\ell)\right|}\int_{\Gamma_\beta(L,\ell)}\mathrm{d}\mathbf{r}\,\mathrm{curl}\mathbf{F} - \left\lfloor\frac{1}{2\pi\left|\partial\Gamma_\beta(L,\ell)\right|}\int_{\Gamma_\beta(L,\ell)}\mathrm{d}\mathbf{r}\,\mathrm{curl}\mathbf{F}\right\rfloor\\ &= \frac{\left|\Gamma_\beta(L,\ell)\right|}{2\pi\left|\partial\Gamma_\beta(L,\ell)\right|} - \left\lfloor\frac{\left|\Gamma_\beta(L,\ell)\right|}{2\pi\left|\partial\Gamma_\beta(L,\ell)\right|}\right\rfloor,\end{aligned}$$

where $\lfloor\cdot\rfloor$ stands for the integer part. However, by the assumption (3.49), $\varpi = 0$ and no additional phase is needed. □

From now on we are to going to study only the minimization on the r.h.s. of (3.50) in Lemma 3.1, with the vector potential $\mathbf{a}$ satisfying (3.45), (3.46) and (3.47). In order to guarantee that (3.49) is satisfied, however, we restrict the analysis to suitable monotone sequences $\{\ell_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$, such that

$$\ell_n, L_n \xrightarrow[n\to+\infty]{} +\infty, \qquad (2.30) \text{ and } (3.49) \text{ hold}, \tag{3.56}$$

and consider $E_{\mathrm{corner},\beta}(L_n,\ell_n)$ in the following. More precisely, we are going to study the quantity

$$E_\beta(L_n,\ell_n) := \inf_{\psi\in\mathscr{D}_{\mathrm{D}}(\Gamma_\beta(L_n,\ell_n))}\mathcal{G}_{\mathbf{a}}[\psi;\Gamma_\beta(L_n,\ell_n))]. \tag{3.57}$$

Any minimizer of (3.57) is denoted by ψ_n, i.e.,

$$E_\beta(L_n,\ell_n) := \mathcal{G}_{\mathbf{a}}\left[\psi_n;\Gamma_\beta(L_n,\ell_n))\right]. \tag{3.58}$$

The existence of such a minimizer follows by standard arguments as well as the fact that any ψ_n solves the variational equation

$$\begin{cases} -(\nabla + i\mathbf{a})^2\,\psi_n = \frac{1}{b}(1-|\psi_n|^2)\psi_n, & \text{in } \Gamma_\beta(L,\ell),\\ \psi_n = f_0(t(\mathbf{r}))e^{-i\alpha_0 s(\mathbf{r})}, & \text{on } \partial\Gamma_{\mathrm{bd}}\cup\partial\Gamma_{\mathrm{in}},\\ \mathbf{n}\cdot(\nabla+i\mathbf{a})\,\psi_n = 0, & \text{on } \partial\Gamma_{\mathrm{out}}.\end{cases} \tag{3.59}$$

Note that the equation above coincides with (3.5) far from the vertex, where boundary coordinates are well posed and $\mathbf{a} = -t\mathbf{e}_s$. We can thus apply to ψ_n the results in Lemmas B.1, B.3, B.5 and B.6.

3.3 Boundedness and existence of the limit

We start by proving the uniform boundedness of $E_{\mathrm{corner},\beta}(L,\ell)$ as a function of ℓ, L.

Proposition 3.4 (Boundedness of $E_{\mathrm{corner},\beta}(L_n,\ell_n)$) *Let $\{\ell_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ satisfy (3.56). Then, for any $1 < b < \Theta_0^{-1}$, there exists a finite constant $C < +\infty$ independent of n, such that*

$$\left|E_{\mathrm{corner},\beta}(L_n,\ell_n)\right| \leqslant C. \tag{3.60}$$

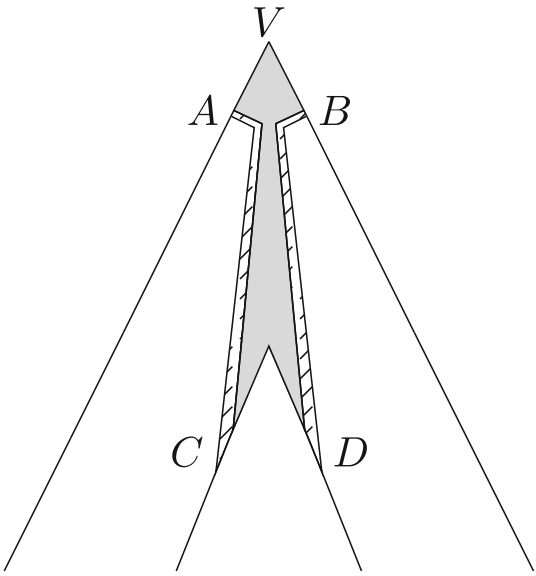

Fig. 5 The partition of unity χ, η

Proof We first discuss the boundedness from below, which is the most difficult property to prove, and show that

$$E_{\mathrm{corner},\beta}(L_n, \ell_n) \geqslant -C, \tag{3.61}$$

for some finite $0 < C < +\infty$. The key tool is a suitable partition of unity, which isolates the region where we want to retain the energy and allow us to discard the rest. We thus consider two smooth positive functions χ and η, such that $\chi^2 + \eta^2 = 1$ and whose supports are described, e.g., in Fig. 5: we assume that $\eta \equiv 1$ inside the shaded area, while $\chi \equiv 1$ in the white area.

The dashed regions is where the supports of the two functions overlap. We choose the angle $\widehat{CVD}$ equal to $\beta/2$ for concreteness but any angle of order 1 would work. The distance of the points A and B from the vertex V is also taken of order 1. Furthermore, the width of the transition regions can also be taken in such a way that

$$|\nabla\chi| = \mathcal{O}(1), \qquad |\nabla\eta| = \mathcal{O}(1). \tag{3.62}$$

The rationale behind the choice of the partition of unity is that the energy contribution coming from the support of χ reconstructs the leading term $2LE_0^{\mathrm{1D}}(\ell)$, up to an $\mathcal{O}(1)$ error, while the rest provides a correction of order $\mathcal{O}(1)$. Therefore, the support of χ must contain the outer boundary $\partial\Gamma_{\mathrm{out}}$ up to $\mathcal{O}(1)$ regions and the magnetic potential must be equal to $-t\mathbf{e}_s$ there. Hence, the area close to the bisectrix is included in the support η, because there the magnetic potential is unknown.

The key ingredient of the proof is then the IMS formula [29, Thm. 3.2], which yields

$$\begin{aligned} E_\beta(L_n, \ell_n) &= \mathcal{G}[\chi\psi_n] + \mathcal{G}[\eta\psi_n] - \int_{\Gamma_\beta(L,\ell)} \mathrm{d}\mathbf{r}\, |\nabla\chi|^2 |\psi_n|^2 - \int_{\Gamma_\beta(L,\ell)} \mathrm{d}\mathbf{r}\, |\nabla\eta|^2 |\psi_n|^2 \\ &= \mathcal{G}[\chi\psi_n] + \mathcal{G}[\eta\psi_n] + \mathcal{O}(1), \end{aligned} \tag{3.63}$$

where we have exploited the decay (B.35) to bound the contributions on the supports of $\nabla\chi$, $\nabla\eta$. We now claim that there exists a finite constant independent of n so that

$$\mathcal{G}_{\mathbf{a}}[\eta\psi_n] \geqslant -C, \tag{3.64}$$

$$\mathcal{G}_{\mathbf{a}}[\chi\psi_n] = 2L_n E_0^{1\mathrm{D}} + \mathcal{O}(1), \tag{3.65}$$

which combined with (3.63) yields (3.60).

Let us first consider the first estimate above: dropping from the energy all the positive terms, we get

$$\mathcal{G}_{\mathbf{a}}[\eta\psi_n] \geqslant -C\int_{\mathrm{supp}(\eta)} \mathrm{d}\mathbf{r}\ |\psi_n(\mathbf{r})|^2 \geqslant -C, \tag{3.66}$$

by the decay of ψ_n as above. To complete the proof it remains only to deal with (3.65): since $\mathrm{supp}(\chi)$ is actually composed of two disconnected sets, denoted by T_- (on the right of Fig. 5) and T_+, we can use boundary coordinates in both regions $T_\pm$. We can then apply the splitting technique described in the proof of Proposition 3.1 and set

$$\chi(s,t)\psi_n(\mathbf{r}(s,t)) =: \begin{cases} f_0(t)e^{-i\alpha_0 s}u_-(s,t), & \text{in } T_-, \\ f_0(t)e^{-i\alpha_0 s}u_+(s,t), & \text{in } T_+. \end{cases} \tag{3.67}$$

The same computation which leads to (3.13) yields now (recall (3.14))

$$\mathcal{G}_{\mathbf{a}}[\chi\psi_n] = -\frac{1}{b}\int_{T_-\cup T_+} \mathrm{d}s\mathrm{d}t\ f_0^4(t) + \mathcal{E}_0[u_-;T_-] + \mathcal{E}_0[u_+;T_+]. \tag{3.68}$$

Finally, as long as $1 < b < \Theta_0^{-1}$, one can prove that the energies $\mathcal{E}_0[u_-;T_-]$ and $\mathcal{E}_0[u_+;T_+]$ are both positive, exactly as in (3.17), leading to

$$\mathcal{G}_{\mathbf{a}}[\chi\psi_n] \geqslant -\frac{1}{b}\int_{T_-} \mathrm{d}s\mathrm{d}t\ f_0^4(t) - \frac{1}{b}\int_{T_+} \mathrm{d}s\mathrm{d}t\ f_0^4(t). \tag{3.69}$$

The last step is the estimate of the two integrals on the r.h.s. of (3.69) above: the identity (A.9) and the exponential decay (A.10) (both with $k = 0$) imply

$$-\frac{1}{b}\int_{T_\pm} \mathrm{d}s\mathrm{d}t\ f_0^4(t) \geqslant L_n E_0^{1\mathrm{D}} - C\int_0^\ell \mathrm{d}t\ te^{-2(t+\alpha_0)^2} + \mathcal{O}(1) \geqslant L_n E_0^{1\mathrm{D}} + \mathcal{O}(1),$$

which together with (3.69) completes the lower bound proof.

The opposite side of the inequality (3.61) can be proven by simply using $\chi f_0(t)e^{-i\alpha_0 s}$ as a trial state (more precisely, setting $u_\pm = 1$ in (3.67)). We omit the calculations, since they are totally analogous to the ones above. □

We are now in position to prove the first important result of this section.

Proof of Proposition 2.2 The first important observation is that $E_{\mathrm{corner},\beta}(L,\ell)$ is a monotone non-increasing function of L and as such it admits a limit. Indeed, for any $L_a < L_b$, one can easily construct a trial state for the energy in $\Gamma_\beta(L_b,\ell)$ by extending the minimizer in $\Gamma_\beta(L_a,\ell)$ and setting the trial state equal to $f_0(t)e^{-i\alpha_0 s}$ where the minimizer is not defined. The outcome of the trivial computation is the inequality $E_{\mathrm{corner},\beta}(L_a,\ell) \leqslant E_{\mathrm{corner},\beta}(L_b,\ell)$.

Let $\{\ell_n\}_{n\in\mathbb{N}}, \{L_n\}_{n\in\mathbb{N}}$ be two monotone subsequences such that $\lim_{n\to+\infty}\ell_n = \lim_{n\to+\infty}L_n = +\infty$ and (3.56) is satisfied (see Remark 3.2). By the monotonicity in L

of the energy and its boundedness, we know that for any $\varepsilon > 0$ and any given $\bar{n} \in \mathbb{N}$, there exists $\bar{n}_2(\bar{n}) \in \mathbb{N}$, such that

$$\left|E_{\text{corner},\beta}(L_n, \ell_{\bar{n}}) - E_{\text{corner},\beta}(L_m, \ell_{\bar{n}})\right| < \tfrac{1}{3}\varepsilon, \tag{3.70}$$

for any $n, m > \bar{n}_2$.

Furthermore, by the exponential decay of the minimizer and its derivatives (B.34), one gets

$$\left|E_{\text{corner},\beta}(L_n, \ell_n) - E_{\text{corner},\beta}(L_n, \ell_m)\right| \leqslant C L_n e^{-c \min\{\ell_n, \ell_m\}}.$$

Hence, if the sequences satisfy the condition

$$L_n \leqslant C \ell_n^a, \qquad \text{for some } a > 0, \tag{3.71}$$

we can conclude that there exists $\bar{n}_1 \in \mathbb{N}$, such that

$$\left|E_{\text{corner},\beta}(L_n, \ell_n) - E_{\text{corner},\beta}(L_n, \ell_m)\right| < \tfrac{1}{3}\varepsilon \tag{3.72}$$

for $n, m > \bar{n}_1$.

In conclusion, we can estimate

$$\begin{aligned}\left|E_{\text{corner},\beta}(L_n, \ell_n) - E_{\text{corner},\beta}(L_m, \ell_m)\right| &\leqslant \left|E_{\text{corner},\beta}(L_n, \ell_n) - E_{\text{corner},\beta}(L_n, \ell_{\bar{n}_1+1})\right| \\ &+ \left|E_{\text{corner},\beta}(L_n, \ell_{\bar{n}_1+1}) - E_{\text{corner},\beta}(L_m, \ell_{\bar{n}_1+1})\right| \\ &+ \left|E_{\text{corner},\beta}(L_m, \ell_{\bar{n}_1+1}) - E_{\text{corner},\beta}(L_m, \ell_m)\right| < \varepsilon \end{aligned} \tag{3.73}$$

for any $n, m > \max\{\bar{n}_1, \bar{n}_2(\bar{n}_1 + 1)\}$, so that the sequence is Cauchy and the limit exists. The independence of the chosen subsequences relies on the uniqueness of the limit, while the uniform boundedness has been proven in Proposition 3.4. □

3.4 Neumann and Dirichlet problems in $\Gamma_\beta(L, \ell)$

We are going to study the Neumann problem (3.40) on the monotone subsequences $\{\ell_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ introduced in the previous § 3.2, i.e., such that (3.56) holds. Our main goal here is to show that, as in the case of the strip, the Dirichlet and Neumann energies coincide asymptotically as $n \to +\infty$. This is going to play a key role in the proof of our main result, since it implies the identity

$$\begin{aligned} E_{\text{corner},\beta} &= \lim_{n\to+\infty} \left(-2LE_0^{\text{1D}}(\ell_n) + E_\beta(L_n, \ell_n)\right) \\ &= \lim_{n\to+\infty} \left(-2L_n E_0^{\text{1D}}(\ell_n) + \widetilde{E}_\beta(L_n, \ell_n)\right). \end{aligned} \tag{3.74}$$

Before proving the result we need however a technical lemma on a variational problem with twisted boundary conditions, whose proof is postponed at the end of the section. Let then $\varkappa \in [0, 2\pi)$ as in Proposition 3.2 and set

$$E_{\beta,\varkappa}(L, \ell) := \inf_{\psi\in\mathscr{D}_{\text{D},\varkappa}(\Gamma_\beta(L,\ell))} \widetilde{\mathcal{G}}_{\mathfrak{a}}[\psi], \tag{3.75}$$

$$\mathscr{D}_{\text{D},\varkappa}(\Gamma_\beta(L, \ell)) := \left\{\psi \in H^1(\Gamma_\beta(L, \ell)) \;\middle|\; \psi|_{\partial\Gamma_{\text{in}}\cup\{s=-L\}} = \psi_0,\ \psi|_{s=L} = \psi_0 e^{i\varkappa}\right\}. \tag{3.76}$$

Lemma 3.2 *Let $\{\ell_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ be two monotone subsequences such that (3.56) holds. Then,*

$$E_\beta(L_n,\ell_n) = E_{\beta,\varkappa}(L_n,\ell_n) + o_n(1). \tag{3.77}$$

Proposition 3.5 (Dirichlet and Neumann energies) *Let $\{\ell_n\}_{n\in\mathbb{N}}$, $\{L_n\}_{n\in\mathbb{N}}$ be two monotone subsequences such that (3.56) holds. Then, for any $1 < b < \Theta_0^{-1}$,*

$$E_\beta(L_n,\ell_n) - \widetilde{E}_\beta(L_n,\ell_n) = o_n(1). \tag{3.78}$$

Proof In view of the vanishing of the boundary terms in the functional $\widetilde{\mathcal{G}}_{\mathbf{a}}[\psi]$ on any ψ belonging to $\mathscr{D}_{\mathrm{D}}(\Gamma_\beta(L_n,\ell_n))$ (see also the proof of Proposition 3.1) and the trivial inclusion $\mathscr{D}_{\mathrm{D}}(\Gamma_\beta(L_n,\ell_n)) \subset \mathscr{D}_{\mathrm{N}}(\Gamma_\beta(L_n,\ell_n))$, we deduce the inequality

$$\widetilde{E}_\beta(L_n,\ell_n) \leqslant E_\beta(L_n,\ell_n). \tag{3.79}$$

Hence, we only have to prove the opposite inequality, i.e.,

$$E_\beta(L_n,\ell_n) \leqslant \widetilde{E}_\beta(L_n,\ell_n) + o_n(1). \tag{3.80}$$

Preliminarily, we observe that the quantity $\widetilde{E}_{\mathrm{corner},\beta}(L_n,\ell_n)$ admits a limit, which is independent of the chosen sequences, exactly as $E_{\mathrm{corner},\beta}(L_n,\ell_n)$. The argument to prove it is the same as in the proof of Proposition 2.2; therefore we spell in detail only the estimates showing that $\widetilde{E}_{\mathrm{corner},\beta}(L,\ell)$ is monotone in L for fixed ℓ, up to an exponentially small error term: let $L_a < L_b$, then we have

$$\widetilde{E}_\beta(L_b,\ell) = \widetilde{\mathcal{G}}_{\mathbf{a}}\left[\widetilde{\psi}_{L_b,\ell};\,\Gamma_\beta(L_a,\ell)\right] + \widetilde{\mathcal{G}}_{\mathbf{a}}\left[\widetilde{\psi}_{L_b,\ell};\,R_\pm\right] \geqslant \widetilde{E}_\beta(L_a,\ell) + \widetilde{\mathcal{G}}_{\mathbf{a}}\left[\widetilde{\psi}_{L_b,\ell};\,R_\pm\right], \tag{3.81}$$

where $R_\pm$ are the rectangular regions $[L_a,L_b]\times[0,\ell]$ and $[-L_b,-L_a]\times[0,\ell]$, respectively. Applying Proposition 3.1, we get

$$\widetilde{\mathcal{G}}_{\mathbf{a}}\left[\widetilde{\psi}_{L_b,\ell};\,R_\pm(L_b-L_a,\ell)\right] \geqslant (L_b-L_a)E_0^{\mathrm{1D}}(\ell) + \mathcal{O}\left((L_b-L_a)\ell^{-\infty}\right),$$

which, plugged into (3.81), yields

$$\widetilde{E}_{\mathrm{corner},\beta}(L_b,\ell) \geqslant \widetilde{E}_{\mathrm{corner},\beta}(L_a,\ell) + \mathcal{O}\left((L_b-L_a)\ell^{-\infty}\right). \tag{3.82}$$

Let $\{\delta_n\}_{n\in\mathbb{N}}$ be such that $0 \leqslant \delta_n \leqslant 1$ and the pair of sequences $\{L_n-\delta_n\}_{n\in\mathbb{N}}$, $\{\ell'_n\}_{n\in\mathbb{N}}$ satisfies the same conditions (3.56) as $\{L_n\}$, $\{\ell_n\}$. Note that we have also $\ell'_n = \ell_n + \mathcal{O}(\delta_n)$, because of (3.49) (see also Remark 3.2). We denote by $\widetilde{\psi}_n$ and $\widetilde{\psi}_{n,\delta_n}$ for short any energy minimizer in $\Gamma_\beta(L_n,\ell_n)$ and $\Gamma_\beta(L_n-\delta_n,\ell'_n)$, respectively. The splitting technique used to derive (3.13), yields (recall (3.12), (3.22) and (3.67))

$$\begin{aligned}
&\widetilde{E}_\beta(L_n,\ell_n)\\
&= \widetilde{\mathcal{G}}_{\mathbf{a}}\big[\widetilde{\psi}_n;\Gamma_\beta(L_n-\delta_n,\ell'_n)\big] + 2E_0^{\mathrm{1D}}(\ell'_n)\delta_n + \widetilde{\mathcal{E}}_0\left[u_-;\,R_-\right] + \widetilde{\mathcal{E}}_0\left[u_+;\,R_+\right] + \mathcal{O}(\ell_n^{-\infty})\\
&\geqslant \widetilde{E}_\beta(L_n-\delta_n,\ell'_n) + 2E_0^{\mathrm{1D}}(\ell'_n)\delta_n + \widetilde{\mathcal{E}}_0\left[u_-;\,R_-\right] + \widetilde{\mathcal{E}}_0\left[u_+;\,R_+\right] + \mathcal{O}(\ell_n^{-\infty}),
\end{aligned} \tag{3.83}$$

where $R_- = [-L_n,-L_n+\delta_n]\times[0,\ell'_n]$ and $R_+ = [L_n-\delta_n,L_n]\times[0,\ell'_n]$ and $u_\pm$ are defined as in (3.12). Hence, we get that (recall that $\widetilde{\mathcal{E}}_0[u] \geqslant \mathcal{O}(\ell^{-\infty})$ if $1 < b < \Theta_0^{-1}$)

$$\widetilde{\mathcal{E}}_0\left[u_\pm;\,R_\pm\right] \leqslant \widetilde{E}_{\mathrm{corner},\beta}(L_n,\ell_n) - \widetilde{E}_{\mathrm{corner},\beta}(L_n-\delta_n,\ell'_n) + \mathcal{O}(\ell_n^{-\infty}) =: \mathfrak{e}_n = o_n(1), \tag{3.84}$$

for any $\delta_n \leqslant 1$, since the two quantities $\widetilde{E}_{\mathrm{corner},\beta}(L_n, \ell_n)$, $\widetilde{E}_{\mathrm{corner},\beta}(L_n - \delta_n, \ell_n')$ admit the same limit, as proven above.

Now, we claim that (3.84) implies that, up to a phase, $\widetilde{\psi}_n$ is pointwise close to $f_0(t)e^{-i\alpha_0 s}$ in the region $R_- \cup R_+$ and, in particular, along the boundary $\partial\Gamma_{\mathrm{bd}}$. Indeed, applying Proposition 3.3 to the functionals $\widetilde{\mathcal{E}}_0\left[u_\pm; R_\pm\right]$ (with δ_n in place of L), we get that

$$\int_{-L_n}^{-L_n+\delta_n} \mathrm{d}s \int_0^{\bar{\ell}_n} \mathrm{d}t\ f_0^4\, |\nabla u_-|^2 + \int_{L_n-\delta_n}^{L_n} \mathrm{d}s \int_0^{\bar{\ell}_n} \mathrm{d}t\ f_0^4\, |\nabla u_+|^2 \leqslant C\mathfrak{e}_n + \mathcal{O}(\ell_n^{-\infty}). \tag{3.85}$$

Furthermore, fixing some $0 < T_n \leqslant \bar{\ell}_n$, then, for any $t \in [0, T_n]$ and any $s \in [L_n - \delta_n, L_n]$ or $s \in [-L_n, -L_n + \delta_n]$,

$$\left|\left|\widetilde{\psi}_n(\mathbf{r}(s,t))\right| - f_0(t)\right| \leqslant \frac{C\mathfrak{e}_n^{1/4} + \mathcal{O}(\ell_n^{-\infty})}{\sqrt{\min_{[0,T_n]} f_0}}, \tag{3.86}$$

$$\left|\partial_s \int_0^{\ell_n} \mathrm{d}t\ \left|\widetilde{\psi}_n\right|^2\right|\Big|_{s=\pm L_n} \leqslant C\left\{\sqrt{\mathfrak{e}_n} + \frac{1}{\delta_n}\left[\frac{\mathfrak{e}_n{}^{1/4} + \mathcal{O}(\delta_n \ell_n^{-\infty})}{\sqrt{\min_{[0,T_n]} f_0}} + e^{-c(b)T_n}\right] + \delta_n\right\}. \tag{3.87}$$

In order to simplify the discussion, let us assume that the errors $\mathcal{O}(\ell_n^{-\infty})$ appearing on the r.h.s. of (3.85) and (3.86) are much smaller than $\mathfrak{e}_n$, since, if this is not the case, i.e., $\mathfrak{e}_n$ is exponentially small in ℓ_n, then the argument is actually much simpler. Then, if we pick T_n in such a way that

$$f_0(T_n) = \mathfrak{e}_n^{1/12} = o_n(1), \tag{3.88}$$

if the r.h.s. is larger than $f_0(\bar{\ell}_n)$, or $T_n = \bar{\ell}_n$ otherwise, then

$$\|\nabla u_\pm\|^2_{L^2(\widetilde{R}_\pm)} \leqslant C\mathfrak{e}_n^{2/3} = o_n(1), \tag{3.89}$$

$$\|1 - |u_\pm|\|_{L^\infty(\widetilde{R}_\pm)} \leqslant C\mathfrak{e}_n^{1/8} = o_n(1), \qquad \left\|\widetilde{\psi}_n\right\|_{L^\infty(R_\pm \setminus \widetilde{R}_\pm)} \leqslant Ce^{-\frac{1}{2}c(b)T_n}, \tag{3.90}$$

where $\widetilde{R}_+ := [L_n - \delta_n, L_n] \times [0, T_n]$ and we used (B.35). Note that, by the pointwise lower bound on f_0 stated in (A.10), we find that

$$C\sqrt{|\log \mathfrak{e}_n|} \geqslant T_n \geqslant 2\sqrt{|\log \mathfrak{e}_n|}(1 + o_n(1)) \gg 1. \tag{3.91}$$

Now, we claim that (3.89) and (3.90) imply that $u_\pm$ is close in L^2 sense to a constant phase factor $e^{i\varkappa_\pm}$, $\varkappa_\pm \in \mathbb{R}$, or, equivalently, $\widetilde{\psi}_n \simeq f_0(t)e^{-i(\alpha_0 s - \varkappa_\pm)}$ in $\widetilde{R}^\pm$. By applying the Poincaré inequality

$$\int_{\widetilde{R}_+} \mathrm{d}s\mathrm{d}t\ |h - \langle h\rangle|^2 \leqslant C\int_{\widetilde{R}_+} \mathrm{d}s\mathrm{d}t\ \left\{T_n^2\,|\partial_t h|^2 + \delta_n^2\,|\partial_s h|^2\right\},$$

where $\langle h\rangle$ is the average of h over $\widetilde{R}_+$, to $h = u_+/|u_+|$, which is well posed since u_+ does not vanish in $\widetilde{R}_+$ by (3.90), we obtain that there exists $\varkappa_+ \in [0, 2\pi)$ such that

$$\left\|\frac{u_+}{|u_+|} - e^{i\varkappa_+}\right\|^2_{L^2(\widetilde{R}_+)} \leqslant CT_n^2\mathfrak{e}_n^{2/3} = o_n(1),$$

thanks to (3.91). This in turn yields the desired estimate via (3.90):

$$\left\|u_+ - e^{i\varkappa_+}\right\|^2_{L^2(\widetilde{R}_+)} \leqslant C\left[T_n^2\mathfrak{e}_n^{2/3} + \delta_n T_n \mathfrak{e}_n^{1/4}\right] = o_n(1). \tag{3.92}$$

The idea is now to exploit the information collected above to construct a trial state and prove an upper bound on $E_\beta(L_n,\ell_n)$ in terms of $\widetilde{E}_\beta(L_n-\delta_n,\ell_n)$ via Lemma 3.2: we set $\psi_{\rm trial}(\mathbf{r}):=\widetilde{\psi}_n(\mathbf{r})$ close to the corner, while sufficiently far from it,

$$\psi_{\rm trial}(\mathbf{r}):=\widetilde{\psi}_n(\mathbf{r})+\eta(s(\mathbf{r}))\left(f_0(t(\mathbf{r}))e^{-i\alpha_0 s(\mathbf{r})}e^{i\varkappa_\pm}-\widetilde{\psi}_n(\mathbf{r})\right), \tag{3.93}$$

where the phases $\varkappa_\pm$ are the constants appearing in (3.92). The function η is smooth and satisfies $\eta(\pm L_n)=1$, $\mathrm{supp}(\eta)\subset[-L_n,-L_n+\delta_n]\cup[L_n-\delta_n,L_n]\times[0,\ell_n]$ and $|\nabla\eta|=\mathcal{O}(\delta_n^{-1})$. Obviously, the trial state $\psi_{\rm trial}$ does not belong to $\mathscr{D}_{\rm D}$ but $e^{-i\varkappa_-}\psi_{\rm trial}\in\mathscr{D}_{{\rm D},\varkappa}$ (recall (3.76)) with $\varkappa=\varkappa_+-\varkappa_-$. Hence, using Lemma 3.2, we can estimate

$$\begin{aligned}E_\beta(L_n,\ell_n)&\leqslant E_{\beta,\varkappa_+-\varkappa_-}(L_n,\ell_n)+o_n(1)\leqslant\mathcal{G}_{\mathbf{a}}[\psi_{\rm trial};\Gamma_\beta(L_n,\ell_n)]+o_n(1)\\&\leqslant\widetilde{E}_\beta(L_n-\delta_n,\ell_n)+\int_0^{\ell_n}\mathrm{d}t\,\frac{F_0(t)}{f_0^2(t)}j_t\big[\widetilde{\psi}_n\big]\Big|_{s=-L_n-\delta_n}^{s=L_n-\delta_n}+\mathcal{G}_{\mathbf{a}}\left[\psi_{\rm trial};R_+\cup R_-\right]+o_n(1).\end{aligned} \tag{3.94}$$

Let us first consider the boundary terms at L_n, since the ones at $-L_n$ are perfectly equivalent: thanks to the boundary conditions (3.10) satisfied by $\widetilde{\psi}_n$, we get

$$\int_0^{\ell_n}\mathrm{d}t\,\frac{F_0(t)}{f_0^2(t)}j_t\big[\widetilde{\psi}_n\big]\Big|_{s=L_n-\delta_n}=-\frac{1}{2}\int_0^{\ell_n}\mathrm{d}t\,\partial_s\big|\widetilde{\psi}_n\big|^2\Big|_{s=L_n}+\int_{L_n-\delta_n}^{L_n}\int_0^{\ell_n}\mathrm{d}t\,\frac{F_0(t)}{f_0^2(t)}\partial_s j_t\big[\widetilde{\psi}_n\big].$$

Integrating by parts as in (3.16) and using the Agmon bound provided by Lemma B.5 as well as the inequalities (A.22) and (A.25) , one can show that the second term on the r.h.s. of the expression above is bounded by

$$\begin{aligned}&\left|\int_{L_n-\delta_n}^{L_n}\int_0^{\ell_n}\mathrm{d}t\,\frac{F_0(t)}{f_0^2(t)}\partial_s j_t\big[\widetilde{\psi}_n\big]\right|\\&\leqslant 2\int_{L_n-\delta_n}^{L_n}\int_0^{T_n}\mathrm{d}t\,\big|\partial_t\widetilde{\psi}_n\big|\big|\partial_s\widetilde{\psi}_n\big|+2e^{-c(b)T_n}\int_{R_+\setminus\widetilde{R}_+}\mathrm{d}s\mathrm{d}t\;e^{c(b)t}\big|\nabla\widetilde{\psi}_n\big|^2\\&\leqslant C\left[\|f_0\nabla u\|^2_{L^2(\widetilde{R}_+)}+e^{-c(b)T_n}\right]\leqslant C\left(\mathfrak{e}_n^{5/6}+e^{-c(b)T_n}\right),\end{aligned} \tag{3.95}$$

while the first one can be estimated via (3.87), so obtaining

$$\left|\int_0^{\ell_n}\mathrm{d}t\,\frac{F_0(t)}{f_0^2(t)}j_t\big[\widetilde{\psi}_n\big]\Big|_{s=L_n-\delta_n}\right|\leqslant C\left[\sqrt{\mathfrak{e}_n}+\frac{\mathfrak{e}_n{}^{5/24}+e^{-c(b)T_n}}{\delta_n}+\delta_n\right]. \tag{3.96}$$

We now focus on the energy contributions of the regions $R_\pm$ (third term on the r.h.s. of (3.94)): For simplicity, we are going to consider only the energy in the region R_+, since the corresponding one in R_- can be bounded in the very same way. We have

$$\begin{aligned}\mathcal{G}_{\mathbf{a}}\left[\psi_{\rm trial};R_+\right]&\leqslant 10\left\|(\nabla-it\mathbf{e}_s)\,\widetilde{\psi}_n\right\|^2_{L^2(R_+)}+4\left\|\nabla\eta\left(f_0e^{-i\alpha_0 s+i\varkappa}-\widetilde{\psi}_n\right)\right\|^2_{L^2(R_+)}\\&+8\left\|(\nabla-it\mathbf{e}_s)\,f_0e^{-i\alpha_0 s+i\varkappa}\right\|^2_{L^2(R_+)}+\frac{1}{b}\left(\|\widetilde{\psi}\|^4_{L^4(R_+)}+\|f_0\|^4_{L^4(R_+)}\right)\\&\leqslant\frac{C}{\delta_n^2}\left\|f_0e^{-i\alpha_0 s+i\varkappa}-\widetilde{\psi}_n\right\|^2_{L^2(R_+)}+C\left(\mathfrak{e}_n^{5/6}+e^{-c(b)T_n}\right)+\mathcal{O}(\delta_n)\end{aligned} \tag{3.97}$$

thanks to (A.10) and (B.34) and where the first term on the r.h.s. has been bounded by Cauchy inequality, exploiting (3.85), (B.34), (B.35) and the splitting technique:

$$\begin{aligned}\left\|(\nabla - it\mathbf{e}_s)\,\widetilde{\psi}_n\right\|^2_{L^2(R_+)} &\leqslant \int_{R_+} \mathrm{d}s\mathrm{d}t\ f_0^2\left\{|\nabla u_+|^2 - 2(t+\alpha_0)j_s[u_+] + \tfrac{1}{b}\left(1-f_0^2\right)|u_+|^2\right\}\\ &\leqslant C\left[\|f_0\nabla u\|^2_{L^2(\widetilde{R}_+)} + e^{-c(b)T_n}\right] + \mathcal{O}(\delta_n) \leqslant C\left(\mathfrak{e}_n^{5/6} + e^{-c(b)T_n}\right) + \mathcal{O}(\delta_n). \end{aligned} \tag{3.98}$$

We now exploit (3.90) and (3.92) to deduce that

$$\begin{aligned}\left\|f_0 e^{-i\alpha_0 s + i\varkappa} - \widetilde{\psi}_n\right\|^2_{L^2(R_+)} &\leqslant \int_{\widetilde{R}_+} \mathrm{d}s\mathrm{d}t\ f_0^2\left|u - e^{i\varkappa}\right|^2 + C\delta_n e^{-\frac{1}{2}c(b)T_n}\\ &\leqslant C\left[T_n^2\mathfrak{e}_n^{2/3} + \delta_n T_n\mathfrak{e}_n^{1/4} + e^{-\frac{1}{2}c(b)T_n}\right]. \end{aligned} \tag{3.99}$$

Putting together (3.94) with (3.96), (3.97) and (3.99), we finally get

$$\begin{aligned} E_\beta(L_n,\ell_n) &\leqslant \widetilde{E}_\beta(L_n-\delta_n,\ell_n)\\ &\quad + C\left\{\sqrt{\mathfrak{e}_n} + \frac{\mathfrak{e}_n{}^{5/24} + e^{-c(b)T_n}}{\delta_n} + \frac{T_n^2\mathfrak{e}_n^{2/3} + \delta_n T_n\mathfrak{e}_n^{1/4} + e^{-\frac{1}{2}c(b)T_n}}{\delta_n^2} + \delta_n\right\}\\ &\leqslant \widetilde{E}_\beta(L_n-\delta_n,\ell_n) + C\left[\frac{T_n^2\mathfrak{e}_n^{2/3} + \delta_n T_n\mathfrak{e}_n^{1/4} + e^{-\frac{1}{2}c(b)T_n}}{\delta_n^2} + \delta_n\right] + o_n(1)\\ &\leqslant \widetilde{E}_\beta(L_n-\delta_n,\ell_n) + C\left[\max\left\{T_n^{2/3}\mathfrak{e}_n^{2/9}, T_n^2\mathfrak{e}_n^{1/8}\right\} + e^{-\frac{1}{10}c(b)T_n}\right] + o_n(1)\\ &= \widetilde{E}_\beta(L_n,\ell_n) + o_n(1), \end{aligned} \tag{3.100}$$

where we have optimized over δ_n by taking $\delta_n = \max\{T_n^{2/3}\mathfrak{e}_n^{2/9}, T_n^2\mathfrak{e}_n^{1/8}\} + e^{-\frac{1}{5}c(b)T_n}$ and used that $T_n = \mathcal{O}(\sqrt{|\log\mathfrak{e}_n|})$. □

Proof of Lemma 3.2 We first observe that the existence of the limit as $n \to +\infty$ of $E_{\beta,\varkappa}(L_n,\ell_n) - 2E_0^{\mathrm{1D}}(\ell_n)L_n$ can be shown as in the proof of Proposition 2.2. Hence for any $1 \ll \delta_n \ll \min\{\ell_n, L_n\}$, we have

$$E_{\beta,\varkappa}(L_n,\ell_n) - E_{\beta,\varkappa}(L_n-\delta_n,\ell_n) + 2E_0^{\mathrm{1D}}(\ell_n)\delta_n = o_n(1).$$

By a trivial testing of the functional, exploiting the above estimate as well as Propositions 3.1 and 3.2, one gets

$$\begin{aligned} E_\beta(L_n,\ell_n) &\leqslant E_{\beta,\varkappa}(L_n-\delta_n,\ell_n) + E_{\mathrm{D},\varkappa}(R(\delta_n,\ell_n))\\ &\leqslant E_{\beta,\varkappa}(L_n-\delta_n,\ell_n) + 2E_0^{\mathrm{1D}}(\ell_n)\delta_n + \mathcal{O}(\delta_n^{-1}) + o_n(1) = E_{\beta,\varkappa}(L_n,\ell_n) + o_n(1). \end{aligned} \tag{3.101}$$

The proof of the opposite inequality is identical. □

4 Proof of the energy lower bound

In this Section we prove the lower bound to the GL energy which in combination with the upper bound proven in Proposition 5.1, stated in next Section, will provide the proof of Theorem 2.1.

Proposition 4.1 (GL energy lower bound) *Let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain satisfying Assumptions 1 and 2. Then, for any fixed $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$, it holds*

$$E^{\mathrm{GL}} \geqslant \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}s\, \mathfrak{K}(s) + \sum_{j=1}^{N} E_{\mathrm{corner},\beta_j} + o(1). \tag{4.1}$$

We recall the definition of the superconducting boundary layer

$$\mathcal{A}_\varepsilon := \left\{ \mathbf{r} \in \Omega \,\middle|\, \mathrm{dist}\,(\mathbf{r}, \partial\Omega) \leqslant \varepsilon \ell_\varepsilon \right\},$$

with (see (1.15)) $\ell_\varepsilon = c_1 |\log \varepsilon|$, for a large constant c_1. The smooth part of the boundary layer is defined as

$$\mathcal{A}_{\mathrm{cut},\varepsilon} := \left\{ \mathbf{r} \in \mathcal{A}_\varepsilon \,\middle|\, |s(\mathbf{r}) - \mathsf{s}_j| \geqslant \varepsilon L_\varepsilon \right\}, \tag{4.2}$$

where s_j is the coordinate along $\partial\Omega$ of the j-th corner and

$$L_\varepsilon = c_2(\varepsilon) |\log \varepsilon|, \tag{4.3}$$

for some

$$\tfrac{c_1}{\tan(\beta/2)} \leqslant c_2(\varepsilon) \leqslant C, \tag{4.4}$$

so that (3.56) holds. The corner regions are denoted by $\Gamma_{j,\varepsilon}$, $j \in \{1, \dots, N\}$, and coincide with the complement of $\mathcal{A}_{\mathrm{cut},\varepsilon}$:

$$\Gamma_{j,\varepsilon} := \left(\mathcal{A}_\varepsilon \setminus \mathcal{A}_{\mathrm{cut},\varepsilon}\right) \cap \left\{ \mathbf{r} \in \mathcal{A}_\varepsilon \mid \mathrm{dist}(\mathbf{r}, \mathbf{r}_j) \leqslant C\varepsilon L_\varepsilon \right\}. \tag{4.5}$$

In $\mathcal{A}_{\mathrm{cut},\varepsilon}$, one can use the tubular coordinates (s, t) defined in (1.10) as well as their rescaled counterparts given in (1.16). We denote by $\mathcal{A}$ the rescaling of the boundary layer $\mathcal{A}_\varepsilon$. Similarly, the set obtained via rescaling of the domain $\mathcal{A}_{\mathrm{cut},\varepsilon}$ is denoted by $\mathcal{A}_{\mathrm{cut}}$, i.e., with a little abuse of notation,

$$\mathcal{A}_{\mathrm{cut}} := \left([0, s_1 - L_\varepsilon] \cup [s_1 + L_\varepsilon, s_2 - L_\varepsilon] \cup \cdots \cup \left[s_N + L_\varepsilon, \tfrac{|\partial\Omega|}{\varepsilon} \right] \right) \times [0, c_1 |\log \varepsilon|], \tag{4.6}$$

while Γ_j stands for the rescaling of the domain $\Gamma_{j,\varepsilon}$, i.e., $\Gamma_j := \left\{ \mathbf{r}' \in \mathbb{R}^2 \,\middle|\, \mathbf{r}_j + \varepsilon \mathbf{r}' \in \Gamma_{j,\varepsilon} \right\}$.

Before proceeding further, we summarize the main steps of the proof of the lower bound. We are going to treat the smooth part of the layer and the corner regions differently. In order to extract the $\mathcal{O}(1)$ contributions to the energy, it is indeed necessary to retain in the smooth part of the layer the terms depending on the boundary curvature. The same precision is not needed close to the corners. There, however, the procedure is more involved, since we have to reconstruct the model problem discussed in § 2.3.

- The first step is the *replacement of the magnetic vector potential* (§ 4.1). The idea is to replace $\mathbf{A}^{\mathrm{GL}}$ with $-\mathbf{t}\mathsf{e}_\mathsf{s} + \frac{1}{2}\mathfrak{K}(\mathsf{s})\mathsf{t}^2 + \varepsilon\delta_\varepsilon$ far enough from the corners by means of a suitable gauge change. Close to the corners, on the other hand, we replace $\mathbf{A}^{\mathrm{GL}}$ with $\mathbf{F}$ (Lemma 4.3) by means of a priori bounds of the difference between $\mathbf{A}^{\mathrm{GL}}$ and $\mathbf{F}$ (see "Appendix B.2");
- The second step is the *rectification of the corner regions* (§ 4.2): via a suitable diffeomorphism, we map the corner region as in Fig. 6 onto a domain with the same shape as $\Gamma_\beta(L, \ell)$ in Fig. 2; this allows us to reduce the lower bound to the corner effective problem introduced in (2.12);

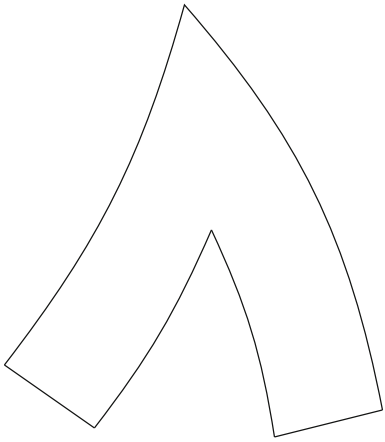

Fig. 6 A typical corner region $\Gamma_{j,\varepsilon}$ (or, after rescaling, Γ_j) before the rectification

- The third step is simply the *completion of the lower bound* (§ 4.3), where we just glue together the lower bounds near the corners with the one in the smooth part of the domain discussed in "Appendix C".

4.1 Replacement of the magnetic field

In $\mathcal{A}_{\mathrm{cut},\varepsilon}$ we aim at bounding from below the GL energy by the reduced energy functional $\mathcal{G}_\varepsilon[\psi;\mathcal{A}_{\mathrm{cut}}]$, where

$$\mathcal{G}_\varepsilon[\psi;\mathcal{A}_{\mathrm{cut}}] := \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k(s)t)\left\{|\partial_t\psi|^2 + \tfrac{1}{(1-\varepsilon k(s)t)^2}\,|(\partial_s + ia_\varepsilon(s,t))\,\psi|^2 - \tfrac{1}{2b}\left(2|\psi|^2 - |\psi|^4\right)\right\}. \tag{4.7}$$

$$\mathbf{a}_\varepsilon(s,t) := a_\varepsilon(s,t)\mathbf{e}_s, \qquad a_\varepsilon(s,t) = -t + \tfrac{1}{2}k(s)t^2 + \varepsilon\delta_\varepsilon, \tag{4.8}$$

and $\psi(s,t) = \psi^{\mathrm{GL}}(\mathbf{r}(s,t))e^{-i\phi_\varepsilon(\mathbf{r}(s,t))}$, with ϕ_ε a suitable gauge phase (see (4.10) below).

The replacement procedure by means of a local gauge choice is well described in [33, Appendix F] for smooth domains and, in more details, in [24, Sect. 5.1]. A similar discussion is extended in presence of corners at the boundary in [19, Sect. 2.4], where however the energy of the corner regions is dropped.

Lemma 4.1 (Replacement of the magnetic potential in $\mathcal{A}_{\mathrm{cut},\varepsilon}$) *Under the assumptions of Proposition 4.1, there exists $\phi_\varepsilon \in C^\infty(\Omega)$ such that, setting $\psi(s,t) := \psi^{\mathrm{GL}}(\boldsymbol{r}(s,t))e^{i\phi_\varepsilon(\boldsymbol{r}(s,t))}$, we get, as $\varepsilon \to 0$,*

$$\mathcal{E}^{\mathrm{GL}}_\varepsilon\left[\psi^{\mathrm{GL}}, \boldsymbol{A}^{\mathrm{GL}}; \mathcal{A}_{\mathrm{cut},\varepsilon}\right] \geqslant \mathcal{G}_\varepsilon[\psi;\mathcal{A}_{\mathrm{cut}}] + \mathcal{O}(\varepsilon^2|\log\varepsilon|^2). \tag{4.9}$$

Proof As described above there are three operations, which are performed simultaneously, to get (4.9):

- change to boundary tubular coordinates (s,t);
- extraction of a suitable gauge phase to replace $\mathbf{A}^{\mathrm{GL}}$ with $-\mathsf{t}\mathbf{e}_\mathsf{s} + \frac{1}{2}\mathfrak{K}(\mathsf{s})\mathsf{t}^2 + \varepsilon\delta_\varepsilon$;
- rescaling of all the lengths (e.g., via (1.16)).

As anticipated, the above procedures have been already discussed in the literature, therefore we omit the details for the sake of brevity. We only provide the expression of the gauge phase

for later convenience

$$\phi_\varepsilon(s,t) := -\frac{1}{\varepsilon}\int_0^t \mathrm{d}\eta\, \mathbf{A}^{\mathrm{GL}}(\mathbf{r}(\varepsilon s,\varepsilon\eta))\cdot \boldsymbol{v}(\varepsilon s) - \frac{1}{\varepsilon}\int_0^s \mathrm{d}\xi\, \mathbf{A}^{\mathrm{GL}}(\mathbf{r}(\varepsilon\xi,0))\cdot \boldsymbol{\gamma}'(\varepsilon\xi) + \mathcal{O}(\varepsilon)s. \tag{4.10}$$

□

In the corner regions, on the opposite, it suffices to use a priori bounds on the solutions of the GL equations to substitute $\mathbf{A}^{\mathrm{GL}}$ with $\mathbf{F}$ (recall (2.17)). Before doing that, we need however a preparatory lemma:

Lemma 4.2 *For any $j=1,\dots,N$, as $\varepsilon\to 0$,*

$$\left\|\left(\nabla + i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|_{L^2(\Gamma_{j,\varepsilon})} = \mathcal{O}(|\log\varepsilon|). \tag{4.11}$$

Proof The idea is to exploit the variational equation for ψ^{GL} in (B.6), to compute

$$\begin{aligned}\left\|\left(\nabla + i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|^2_{L^2(\Gamma_{j,\varepsilon})} &= \frac{1}{\varepsilon^2}\int_{\Gamma_{j,\varepsilon}} \mathrm{d}\mathbf{r}\,\left(1-\left|\psi^{\mathrm{GL}}\right|^2\right)\left|\psi^{\mathrm{GL}}\right|^2\\ &+\int_{\partial\Gamma_{j,\varepsilon}\setminus\partial\Omega}\mathrm{d}x\,\psi^{\mathrm{GL}*}\boldsymbol{\nu}\cdot\left(\nabla+i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}} = \mathcal{O}(|\log\varepsilon|^2),\end{aligned} \tag{4.12}$$

by the bounds (B.10), (B.13) and the boundary conditions on ψ^{GL} (recall (B.6)). □

We can now perform the vector potential replacement:

Lemma 4.3 (Replacement of the magnetic field in $\Gamma_{j,\varepsilon}$) *For any $j=1,\dots,N$, there exists $\psi_j\in H^1(\mathcal{A}_\varepsilon)$, so that, as $\varepsilon\to 0$,*

$$\mathcal{E}^{\mathrm{GL}}_\varepsilon\left[\psi^{\mathrm{GL}},\boldsymbol{A}^{\mathrm{GL}};\Gamma_{j,\varepsilon}\right] \geqslant \mathcal{E}^{\mathrm{GL}}_1\left[\psi_j,\boldsymbol{F};\Gamma_j\right] + \mathcal{O}(\varepsilon^{3/5}). \tag{4.13}$$

Remark 4.1 **(Kinetic energy in the corner regions)** Combining Lemma 4.2 with Lemma 4.3, one can easily deduce that (4.11) holds true with $(\psi_j,\mathbf{F})$ in place of $(\psi^{\mathrm{GL}},\mathbf{A}^{\mathrm{GL}})$. More precisely, let ψ_j as in (4.13), then

$$\left\|(\nabla+i\mathbf{F})\,\psi_j\right\|_{L^2(\Gamma_j)} = \mathcal{O}(|\log\varepsilon|). \tag{4.14}$$

Proof A straightforward computation yields

$$\begin{aligned}&\left\|\left(\nabla+i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|^2_{L^2(\Gamma_{j,\varepsilon})} - \left\|\left(\nabla+i\tfrac{\mathbf{F}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|^2_{L^2(\Gamma_{j,\varepsilon})}\\ &= -2\mathrm{Im}\int_{\Gamma_{j,\varepsilon}}\mathrm{d}\mathbf{r}\,\left[\left(\nabla+i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right]^*\left(\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}-\tfrac{\mathbf{F}}{\varepsilon^2}\right)\psi^{\mathrm{GL}} - \int_{\Gamma_{j,\varepsilon}}\mathrm{d}\mathbf{r}\,\left|\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}-\tfrac{\mathbf{F}}{\varepsilon^2}\right|^2\left|\psi^{\mathrm{GL}}\right|^2\\ &\geqslant -\delta\left\|\left(\nabla+i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|^2_{L^2(\Gamma_{j,\varepsilon})} - \tfrac{1}{\varepsilon^4}\left(\tfrac{1}{\delta}+1\right)\left\|\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right\|^2_{L^2(\Gamma_{j,\varepsilon})}\\ &\geqslant -C\delta|\log\varepsilon|^2 - \tfrac{1}{\varepsilon^4}\left(\tfrac{1}{\delta}+1\right)\left\|\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right\|^2_{L^{2p}(\Omega)}\left|\Gamma_{j,\varepsilon}\right|^{1-\frac{1}{p}},\end{aligned} \tag{4.15}$$

for any $p\in[2,\infty)$, where we have used Lemma 4.2. Plugging now (B.11), which reads for $p'\in[2,+\infty)$

$$\left\|\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right\|_{L^{p'}(\Omega)} = \mathcal{O}(\varepsilon^{7/4}), \tag{4.16}$$

thanks to the Agmon decay (B.16), we get

$$\left\|\left(\nabla + i\,\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|^2_{L^2(\Gamma_{j,\varepsilon})} - \left\|\left(\nabla + i\,\tfrac{\mathbf{F}}{\varepsilon^2}\right)\psi^{\mathrm{GL}}\right\|^2_{L^2(\Gamma_{j,\varepsilon})} \geqslant -C\left[\delta|\log\varepsilon|^2 - \tfrac{1}{\sqrt{\varepsilon}}\left(\tfrac{1}{\delta}+1\right)\left(\varepsilon^2|\log\varepsilon|^2\right)^{1-\frac{1}{p}}\right] \geqslant -C\varepsilon^{3/4-1/p}|\log\varepsilon|^2 \tag{4.17}$$

after an optimization over δ (i.e., taking $\delta = \varepsilon^{3/4-1/p}|\log\varepsilon|^{-1/p}$). The proof is then completed by exploiting the invariance of the energy under combined translations and global gauge change. □

4.2 Rectification of the corner regions

We aim at estimating from below the energy close to the corners (first term on the r.h.s. of (4.13)) by the minimal energy of the model problem introduced in (2.12) and discussed in § 3. To this purpose there are two difficulties to overcome. First, one needs to force the boundary conditions in the minimization of the corner energies appearing in (4.13), since an unconstrained minimization would lead to unwanted contribution from the normal boundaries at $s_j \pm L_\varepsilon$. Such a problem will however be solved by exploiting Proposition 3.5.

The second issue is less trivial: the model problem is indeed defined on a domain whose boundary is straight, while typically the boundaries of the domains Γ_j have a non-trivial curvature, as in Fig. 6. Of course, being the corner regions rather small and the curvature bounded, the corrections induced by this adjustment are of lower order (Lemma 4.4).

Let us introduce some notation: we are going to denote by $\Gamma_{j,\mathrm{rect}}$ the corner region of opening angle β_j with straight sides, longitudinal length L_ε and normal width $\ell_\varepsilon(1+o(1))$. We do not require the inner boundaries of $\Gamma_{j,\mathrm{rect}}$ to be straight since the exponential decay of any GL minimizer makes such a boundary irrelevant. We also choose the coordinates in such a way that the corner coincides with the origin. Hence, except for the inner boundaries, $\Gamma_{j,\mathrm{rect}}$ coincide with the region described in Fig. 2, up to a rotation:

$$\Gamma_{j,\mathrm{rect}} \simeq \mathcal{R}\Gamma_{\beta_j}(L_\varepsilon, \ell_\varepsilon), \tag{4.18}$$

with $\mathcal{R}$ a rotation around the axis perpendicular to the plane passing through the corner.

Lemma 4.4 (Rectification of the corner) *Let ψ_j be the H^1 function in (4.13). Then, there exists a diffeomorphism $\boldsymbol{R} : \Gamma_j \to \Gamma_{j,\mathrm{rect}}$, so that, setting $\widetilde{\psi}_j(\boldsymbol{R}) := \psi_j(\boldsymbol{r}(\boldsymbol{R}))$,*

$$\mathcal{E}_1^{\mathrm{GL}}\left[\psi_j, \boldsymbol{F}; \Gamma_j\right] = \mathcal{E}_1^{\mathrm{GL}}\left[\widetilde{\psi}_j, \boldsymbol{F}; \Gamma_{\beta_j}(L_\varepsilon, \ell_\varepsilon)\right] + \mathcal{O}(\varepsilon|\log\varepsilon|^\infty). \tag{4.19}$$

Proof We want to map the region Γ_j onto $\Gamma_{j,\mathrm{rect}}$ via a suitable diffeomorphism and exploit the fact that, thanks to the boundedness of curvature and the size of the region, such a map is suitably close to the identity. A similar a trick has already been used, e.g., in [15]. Indeed, there exists a smooth map $\mathbf{R}(\mathbf{r}) : \Gamma_j \to \Gamma_{j,\mathrm{rect}}$ which is one-to-one, such that $\mathbf{R}(0) = 0$ and

$$R_j(\mathbf{r}) = r_j\left(1 + \mathcal{O}(\varepsilon|\log\varepsilon|)\right), \qquad \partial_j R_k(\mathbf{r}) = \delta_{jk} + \mathcal{O}(\varepsilon|\log\varepsilon|^\infty). \tag{4.20}$$

Using such a map, we get

$$\mathcal{E}_1^{\mathrm{GL}}\left[\psi_j, \mathbf{F}; \Gamma_j\right] = \int_{\Gamma_{j,\mathrm{rect}}} \mathrm{d}\mathbf{R}\left\{\left|\left(J\nabla_{\mathbf{R}} + i\widetilde{\mathbf{F}}\right)\widetilde{\psi}_j\right|^2 - \tfrac{1}{2b}\left(2\left|\widetilde{\psi}_j\right|^2 - \left|\widetilde{\psi}_j\right|^4\right)\right\}, \tag{4.21}$$

where J is the jacobian matrix associated to the change of coordinates $\mathbf{r} \to \mathbf{R}$ and

$$\widetilde{\psi}_j(\mathbf{R}) := \psi_j(\mathbf{r}(\mathbf{R})), \qquad \widetilde{\mathbf{F}}(\mathbf{R}) := \mathbf{F}(\mathbf{r}(\mathbf{R})). \tag{4.22}$$

By (4.20),

$$\widetilde{\mathbf{F}}(\mathbf{R}) = \tfrac{1}{2}\mathbf{R}^{\perp}\left(1+\mathcal{O}(\varepsilon|\log\varepsilon|)\right). \tag{4.23}$$

Therefore, we can estimate from below the r.h.s. of (4.21) exactly as in (4.15), using (4.14) in Remark 4.1, to get

$$\begin{aligned}\mathcal{E}_1^{\mathrm{GL}}\left[\psi_j,\mathbf{F};\Gamma_j\right] &\geqslant \int_{\Gamma_{j,\mathrm{rect}}} \mathrm{d}\mathbf{r}\left\{\left|(\nabla+i\mathbf{F})\,\widetilde{\psi}_j\right|^2-\tfrac{1}{2b}\left(2\left|\widetilde{\psi}_j\right|^2-\left|\widetilde{\psi}_j\right|^4\right)\right\}\\ &-C\left[\delta|\log\varepsilon|^2+\left(1+\tfrac{1}{\delta}\right)\varepsilon^2|\log\varepsilon|^4\right]\geqslant \mathcal{E}_1^{\mathrm{GL}}\left[\widetilde{\psi}_j,\mathbf{F};\Gamma_{j,\mathrm{rect}}\right]+\mathcal{O}(\varepsilon|\log\varepsilon|^{\infty}).\end{aligned} \tag{4.24}$$

The last step is then the replacement of the region $\Gamma_{j,\mathrm{rect}}$ with $\Gamma_{\beta_j}(L_\varepsilon,\ell_\varepsilon)$, which can be done exploiting the rotational invariance of the GL functional, and the exponential decay of ψ^{GL} given by (B.16) (and thus of ψ). □

4.3 Completion of the lower bound

We are now in position to complete the proof of the lower bound.

Proof of Proposition 4.1 Combining the results proven in Lemma 4.1, Lemmas 4.3 and 4.4, we get

$$E^{\mathrm{GL}}\geqslant \mathcal{G}_\varepsilon[\psi;\mathcal{A}_{\mathrm{cut}}]+\sum_{j=1}^{N}\mathcal{E}_1^{\mathrm{GL}}\left[\widetilde{\psi}_j,\mathbf{F};\Gamma_{\beta_j}(L_\varepsilon,\ell_\varepsilon)\right]+\mathcal{O}(\varepsilon^{3/5}). \tag{4.25}$$

At this stage the energy contributions of the smooth part of the domain and its complement have been completely decoupled, so we can bound them from below separately. In fact, the lower bound to $\mathcal{G}_\varepsilon[\psi;\mathcal{A}_{\mathrm{cut}}]$ can be simply taken from [24, Prop. 4]:

$$\begin{aligned}E^{\mathrm{GL}}&\geqslant \frac{|\partial\Omega|E_0^{1\mathrm{D}}}{\varepsilon}-2L_\varepsilon E_0^{1\mathrm{D}}-E_{\mathrm{corr}}\int_0^{|\partial\Omega|}\mathrm{d}s\,\mathfrak{K}(s)\\ &+\sum_{j=1}^{N}\left[\mathcal{E}_1^{\mathrm{GL}}\left[\widetilde{\psi}_j,\mathbf{F};\Gamma_{\beta_j}(L_\varepsilon,\ell_\varepsilon)\right]-\int_0^{c_0|\log\varepsilon|}\mathrm{d}t\;\frac{F_0(t)}{f_0^2(t)}\,j_t\left[\psi(s,t)\right]\Big|_{s=s_j-L_\varepsilon}^{s=s_j+L_\varepsilon}\right]+\mathcal{O}(\varepsilon^{3/5}).\end{aligned} \tag{4.26}$$

Now, recalling the definitions of ψ in Lemma 4.1 and $\widetilde{\psi}_j$ in Lemma 4.4, we claim that

$$\mathcal{E}_1^{\mathrm{GL}}\left[\widetilde{\psi}_j,\mathbf{F};\Gamma_{\beta_j}(L_\varepsilon,\ell_\varepsilon)\right]-\int_0^{c_0|\log\varepsilon|}\mathrm{d}t\;\frac{F_0(t)}{f_0^2(t)}\,j_t\left[\psi(s,t)\right]\Big|_{s=s_j-L_\varepsilon}^{s=s_j+L_\varepsilon}=\widetilde{\mathcal{G}}_{\mathbf{F}}\left[\widetilde{\psi}_j\right]+\mathcal{O}(\varepsilon^{3/5}), \tag{4.27}$$

where the functional $\widetilde{\mathcal{G}}_{\mathbf{F}}$ is defined in (3.39).

We note that, because of the rigid translation and rotation, the boundaries $\partial\Gamma_{\beta_j,\mathrm{bd}}$ coincides with the portion of the lines $s=s_j\pm L_\varepsilon$ in $\mathcal{A}$. Therefore, in order to replace ψ with $\widetilde{\psi}_j$ in the boundary terms in (4.27), we need to estimate the contribution of the gauge phase as well as the effect of the rectification. Next, we observe that

$$j_t[\psi]|_{s=s_j\pm L_\varepsilon} = j_t\left[\psi_j((\mathbf{r}(\varepsilon s,\varepsilon t))-\mathbf{r}_j)/\varepsilon)e^{-i\phi_\varepsilon(s,t)}e^{-i\frac{\mathbf{F}(\mathbf{r}_j)\cdot\mathbf{r}'}{\varepsilon}}\right]\Big|_{s=s_j\pm L_\varepsilon}$$
$$= \left[j_t\left[\psi_j((\mathbf{r}(\varepsilon s,\varepsilon t))-\mathbf{r}_j)/\varepsilon)\right]-\partial_t\left(\phi_\varepsilon(s,t)+\tfrac{1}{\varepsilon}\mathbf{F}(\mathbf{r}_j)\cdot\mathbf{r}'\right)\right]\Big|_{s=s_j\pm L_\varepsilon}. \tag{4.28}$$

Recalling (4.22), we immediately see that the first term on the r.h.s. of the expression above in fact reconstructs the boundary terms in $\widetilde{\mathcal{G}}_{\mathbf{F}}$. Moreover, by direct computation

$$-\partial_t\left(\phi_\varepsilon+\tfrac{1}{\varepsilon}\mathbf{F}(\mathbf{r}_j)\cdot\mathbf{r}'\right)=\tfrac{1}{\varepsilon}\left(\mathbf{A}^{\mathrm{GL}}(\mathbf{r})-\mathbf{F}(\mathbf{r}_j)\right)\cdot\mathbf{e}_t=\tfrac{1}{\varepsilon}\left(\mathbf{A}^{\mathrm{GL}}(\mathbf{r})-\mathbf{F}(\mathbf{r})\right)\cdot\mathbf{e}_t+\tfrac{1}{2}\mathbf{r}'^{\perp}\cdot\mathbf{e}_t,$$

where we have used the change of coordinates $\mathbf{r}=\mathbf{r}_j+\varepsilon\mathbf{r}'$. Furthermore, the properties of the diffeomorphism discussed in the proof of Lemma 4.4 imply that

$$\tfrac{1}{2}\mathbf{r}'^{\perp}\cdot\mathbf{e}_t=\tfrac{1}{2}s+\mathcal{O}(\varepsilon|\log\varepsilon|^\infty).$$

Hence, each boundary term can be rewritten

$$\int_0^{c_0|\log\varepsilon|}\mathrm{d}t\ \frac{F_0(t)}{f_0^2(t)}j_t\left[\psi(s,t)\right]\Big|_{s=s_j-L_\varepsilon}^{s=s_j+L_\varepsilon}=\int_0^{c_0|\log\varepsilon|}\mathrm{d}t\ \frac{F_0(t)}{f_0^2(t)}\left[j_t\left[\widetilde{\psi}(\mathbf{r}(s,t))\right]+\tfrac{1}{2}s\right]\Big|_{s=-L_\varepsilon}^{s=+L_\varepsilon}$$
$$-\frac{1}{\varepsilon}\int_0^{c_0|\log\varepsilon|}\mathrm{d}t\ \frac{F_0(t)}{f_0^2(t)}\left(\mathbf{A}^{\mathrm{GL}}(\mathbf{r}(s,t))-\mathbf{F}(\mathbf{r}(s,t))\right)\cdot\mathbf{e}_t\Big|_{s=s_j-L_\varepsilon}^{s=s_j+L_\varepsilon}+\mathcal{O}(\varepsilon|\log\varepsilon|^\infty) \tag{4.29}$$

and it only remains to bound the last term. This can be done exploiting once more (B.11): setting $g(\mathbf{r}):=F_0(t(\mathbf{r}))/f_0^2(t(\mathbf{r}))$ for short and using the vanishing of F_0 at $t=0$, $c_1|\log\varepsilon|$, we get

$$\frac{1}{\varepsilon^3}\left|\int_{\Gamma_{j,\varepsilon}}\mathrm{d}\mathbf{r}\ \nabla\cdot\left[g\left(\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right)\right]\right|=\frac{1}{\varepsilon^3}\left|\int_{\Gamma_{j,\varepsilon}}\mathrm{d}\mathbf{r}\ (\nabla g)\cdot\left(\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right)\right|$$
$$\leqslant\frac{C|\log\varepsilon|^5}{\varepsilon^3}\left\|\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right\|_{L^1(\Gamma_{j,\varepsilon})}, \tag{4.30}$$

where we have used that both $\mathbf{A}^{\mathrm{GL}}$ and $\mathbf{F}$ are divergence free and the estimate

$$|\nabla g(\mathbf{r})|=\left|\partial_t\frac{F_0(t)}{f_0^2(t)}\right|\leqslant 2\,|t+\alpha_0|+\left|\frac{F_0(t)f_0'(t)}{f_0^3(t)}\right|\leqslant C|\log\varepsilon|^5, \tag{4.31}$$

by (A.3), (A.11) and the simple bound (A.25). On the other hand, by (4.16),

$$\left\|\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right\|_{L^1(\Gamma_{j,\varepsilon})}\leqslant\left\|\mathbf{A}^{\mathrm{GL}}-\mathbf{F}\right\|_{L^p(\Gamma_{j,\varepsilon})}\left|\Gamma_{j,\varepsilon}\right|^{1-\frac{1}{p}}\leqslant C\varepsilon^{7/4}\left(\varepsilon^2|\log\varepsilon|\right)^{1-\frac{1}{p}}, \tag{4.32}$$

for $p\in[2,+\infty)$, which implies (4.27) via (4.30).

Putting together (4.26) with (4.27) and observing that, by the Agmon decay, we can impose the boundary condition $\widetilde{\psi}=\psi_0$ along the interior boundary $\partial\Gamma_{\mathrm{in}}$ up to $\mathcal{O}(\varepsilon^\infty)$ errors, we thus get

$$E^{\mathrm{GL}}\geqslant\frac{|\partial\Omega|E_0^{1\mathrm{D}}}{\varepsilon}-2L_\varepsilon E_0^{1\mathrm{D}}-E_{\mathrm{corr}}\int_0^{|\partial\Omega|}\mathrm{d}s\ \mathfrak{K}(s)$$
$$+\sum_{j\in\Sigma}\inf_{\psi\in\widetilde{\mathscr{D}}_\star(\Gamma_{\beta_j}(L_\varepsilon,\ell_\varepsilon))}\widetilde{\mathcal{G}}_{\mathbf{F}}\left[\widetilde{\psi},\Gamma_{\beta_j}(L_\varepsilon,\ell_\varepsilon)\right]+\mathcal{O}(\varepsilon^{3/5})$$

$$= \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}s\, \mathfrak{K}(s) + \sum_{j\in\Sigma} \left(\widetilde{E}_{\beta_j}(L_\varepsilon, \ell_\varepsilon) - 2L_\varepsilon E_0^{\mathrm{1D}} \right) + \mathcal{O}(\varepsilon^{3/5}). \quad (4.33)$$

The final step of the proof is the application of Proposition 3.5, which yields (see (3.74))

$$\widetilde{E}_{\beta_j}(L_\varepsilon, \ell_\varepsilon) - 2L_\varepsilon E_0^{\mathrm{1D}} = E_{\mathrm{corner},\beta_j} + o(1), \quad (4.34)$$

and thus the result. □

5 Other proofs

We complete in this Section the proofs of the results proven in the paper, i.e., specifically, we prove the energy upper bound matching the lower bound proven in Proposition 4.1. Finally, we show how the energy asymptotics can be used to deduce a pointwise estimate of the order parameter.

5.1 Upper bound and energy asymptotics

We state the main result of this section in the following Proposition 5.1. Note that Propositions 4.1 and 5.1 together completes the proof of Theorem 2.1, with the simple exception of the replacement of E_0^{1D} with $E_\star^{\mathrm{1D}}$, which can be done up to remainders of order $\mathcal{O}(\varepsilon^\infty)$ by Lemma A.2.

Proposition 5.1 (GL energy upper bound) *Let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain satisfying Assumptions 1 and 2. Then, for any fixed $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$, it holds*

$$E^{\mathrm{GL}} \leqslant \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} - E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}s\, \mathfrak{K}(s) + \sum_{j=1}^{N} E_{\mathrm{corner},\beta_j} + o(1). \quad (5.1)$$

Proof As usual the upper bound is obtained by testing the GL functional on a suitable trial state. As a vector potential, we pick $\mathbf{F} = \frac{1}{2}(-y, x)$. The order parameter on the other hand is much more involved: the idea is to recover the trial state given in [24, Eq. (4.14)] far from the corners and glue it to the minimizers of the effective energies in every corner. To retain the curvature corrections, as in [23] (see also "Appendix C"), we decompose the smooth part of the layer into cells of tangential length of order ε. The order parameter is constructed in such a way that its modulus is close to $f_{k_n}(\varepsilon \mathsf{t})$ (see "Appendix A.2") in each cell, k_n being the average curvature, and to the modulus of the corner minimizer $\psi_{\beta_j}(\mathbf{r})$ in the $j-$th corner region, respectively. The phase of ψ_{trial} on the other hand is given by a gauge phase analogous to (4.10), but defined in terms of the vector potential $\mathbf{F}$, plus the optimal phase $\exp\{-i\alpha_{k_n}\mathsf{s}/\varepsilon\}$ in each cell. An additional phase is then added to patch together such factors. Explicitly, we set

$$\psi_{\mathrm{trial}}(\mathbf{r}) := \chi\left(t(\mathbf{r})\right) \cdot \begin{cases} g(s(\mathbf{r}), t(\mathbf{r})) e^{-iS(s(\mathbf{r}))} e^{i\phi_{\mathrm{trial}}(s(\mathbf{r}),t(\mathbf{r}))}, & \text{for } \mathbf{r} \in \widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon}, \\ \widetilde{\psi}_{\beta_j,\varkappa_j}\left(\mathcal{R}^{-1}(\mathbf{r} - \mathbf{r}_j)/\varepsilon\right), & \text{for } \mathbf{r} \in \Gamma_{j,\varepsilon}, \end{cases} \quad (5.2)$$

where $\mathcal{R}$ is the rotation defined in (4.18) and $\varkappa_j$ a suitable phase factor, given in (5.5) below. Moreover, for some $a > 0$,

$$\widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon} := \left\{ \mathbf{r} \in \mathcal{A}_{\mathrm{cut},\varepsilon} \;\middle|\; |s(\mathbf{r}) - s_j| \geqslant L_\varepsilon + \varepsilon^a, \forall j = 1, \dots, N \right\}, \quad (5.3)$$

is a subdomain of $\mathcal{A}_{\mathrm{cut},\varepsilon}$ where boundary coordinates are well defined. In $\mathcal{A}_{\mathrm{cut},\varepsilon} \setminus \widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon}$, we take care of the transition from the smooth part of the layer to the corner region: for any given $j = 1, \dots, N$, we concretely set

$$\psi_{\mathrm{trial}}(\mathbf{r}) := \zeta(s)\, f_0(t)\, e^{-iS(s_j \pm L_\varepsilon)} + (1 - \zeta(s))\, f_{k_\pm}(t)\, e^{-iS(s_j \pm L_\varepsilon \pm \varepsilon^a)}, \tag{5.4}$$

for any $|s(\mathbf{r}) - s_j \pm L_\varepsilon| \leqslant \varepsilon^a$ and where we have denoted $k_\pm$ the average curvature in the cells $\mathscr{C}_\pm$ adjacent to the j-th corner region. The smooth cut-off function ζ is chosen in such a way that it is positive and

$$\zeta(s_j \pm L_\varepsilon) = 1, \qquad \zeta(s_j \pm L_\varepsilon \pm \varepsilon^a) = 0, \qquad |\zeta'| \leqslant C\varepsilon^{-a}.$$

Furthermore,

- the smooth cut-off function $\chi(t)$ equals 1 for any $t \leqslant c_1 |\log\varepsilon|$ and vanishes for $t \geqslant |\log\varepsilon|^2$, so that its gradient is bounded by $\mathcal{O}(|\log\varepsilon|^{-2})$;
- the gauge phase ϕ_{trial} is given by (4.10) with $\mathbf{F}$ in place of $\mathbf{A}^{\mathrm{GL}}$; to leading order such a phase equals $-\frac{1}{2}st$, so recovering part of the phase of $\psi_\star$ (recall (2.16));
- the function $g(s,t)$ is taken directly from [24, Eq. (4.15)] (see also [24, Eq. (4.18)]): it equals f_{k_n} in $\mathscr{C}_n$, up to a smaller correction χ_n, which allows the continuous transition from f_{k_n} to $f_{k_{n+1}}$;
- similarly, the phase $S(s)$ is given by [24, Eqs. (4.20) & (4.21)]: to leading order $S(s) = -i\alpha_{k_n} s$ in $\mathscr{C}_n$, but, as for the density, one needs to add a higher order correction taking into account the jump from α_{k_n} to $\alpha_{k_{n+1}}$;
- $\widetilde{\psi}_{\beta_j,\varkappa_j}(\mathbf{r}')$ is close to the minimizer of the effective energy (3.75) in $\Gamma_{\beta_j}(L_\varepsilon, \ell_\varepsilon)$, where $\varkappa_j := \varkappa_{+,j} - \varkappa_{-,j}$ and

$$\varkappa_{\pm,j} := -S(s_j \pm L_\varepsilon) + \alpha_0(s_j \pm L_\varepsilon) = -\int_0^{s_j \pm L_\varepsilon} \mathrm{d}\xi \ \big(\alpha_{k(\xi)} - \alpha_0\big) + \mathcal{O}(|\log\varepsilon|^\infty). \tag{5.5}$$

  Applying the rectification procedure described in Lemma 4.4, we set

$$\widetilde{\psi}_{\beta_j,\varkappa_j}(\mathbf{r}') = \psi_{\beta_j,\varkappa_j}(\mathbf{R}(\mathbf{r})), \tag{5.6}$$

  where $\mathbf{R}$ is the diffeomorphism of Lemma 4.4 and $\psi_{\beta,\varkappa}$ any minimizer of (3.75).

We now sketch the main steps in the computation of the energy of $(\psi_{\mathrm{trial}}, \mathbf{F})$, which were already discussed elsewhere and focus afterwards on the new estimates:

- since $\mathrm{curl}\mathbf{F} = 1$ in Ω, the last term in the GL energy functional (1.1) vanishes;
- the integration can be restricted to $\mathcal{A}_\varepsilon$, where the cut-off function χ is 1 and all the rest of the energy can be discarded thanks to the exponential decay of the modulus of ψ_{trial} as well as its derivatives (inherited from f_0 and f_k, see (A.10));
- the gauge phase ϕ_{trial} allows to replace $\mathbf{F}$ with $\big(-t + \frac{1}{2}\varepsilon k(s) t^2\big)\mathbf{e}_s$ in $\widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon}$, as in Lemma 4.1 up to an error of order $\mathcal{O}(\varepsilon|\log\varepsilon|^\infty)$;
- the energy bound in $\widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon}$ is taken from [24] and stated in Proposition C.1:

$$\mathcal{E}^{\mathrm{GL}}\big[\psi_{\mathrm{trial}}, \mathbf{F}; \mathcal{A}_{\mathrm{cut},\varepsilon}\big] \leqslant \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} - 2L_\varepsilon N E_0^{\mathrm{1D}} - \varepsilon E_{\mathrm{corr}} \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s\, k(s) + o(1).$$

Given the discussion above, it remains to compute the energy of ψ_{trial} in the region $\mathcal{A}_{\mathrm{cut},\varepsilon} \setminus \widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon}$ as well as the energy contributions of all the corner regions $\Gamma_{j,\varepsilon}$. Let us start by considering the latter: getting rid of the diffeomorphism up to small errors, close to each

corner we recover $E_{\beta_j,\varkappa_j}(L_\varepsilon,\ell_\varepsilon) = E_{\beta_j}(L_\varepsilon,\ell_\varepsilon) + o(1)$ by Lemma 3.2. Summing up, we get

$$\sum_{j=1}^{N} E_{\beta_j}(L_\varepsilon,\ell_\varepsilon) + o(1) = \sum_{j=1}^{N} E_{\mathrm{corner},\beta_j} + 2N E_0^{\mathrm{1D}} L_\varepsilon + o(1), \tag{5.7}$$

where we have exploited the existence of the limit proven in Proposition 2.2.

Finally, let us consider the energy in $\mathcal{A}_{\mathrm{cut},\varepsilon} \setminus \widetilde{\mathcal{A}}_{\mathrm{cut},\varepsilon}$ and restrict ourselves to the interval $[s_j - L_\varepsilon - \varepsilon^a, s_j - L_\varepsilon]$: the area of the region is of order $\varepsilon^{2+a}|\log\varepsilon|$ and we can thus discard all the terms involving f_0, $f_{k_\pm}$ and their derivatives there up to errors of order $\mathcal{O}(\varepsilon^a|\log\varepsilon|^\infty)$ by (A.12) and (A.10). The only non-trivial term to estimate is thus the kinetic energy of the cut-off function ζ: by grouping together the terms in a convenient way, one has to bound at the boundary $s_j - L_\varepsilon$ the quantity

$$\begin{aligned} & C\int_{s_j-L_\varepsilon-\varepsilon^a}^{s_j-L_\varepsilon} \mathrm{d}s \int_0^{\ell_\varepsilon} \mathrm{d}t \ \left|\zeta'(s)\right|^2 \left| f_0(t)\, e^{-iS(s_j-L_\varepsilon)} - f_{k_-}(t)\, e^{-iS(s_j-L_\varepsilon-\varepsilon^a)}\right|^2 \\ & \leqslant C\varepsilon^{-2a}\int_{s_j-L_\varepsilon-\varepsilon^a}^{s_j-L_\varepsilon} \mathrm{d}s \int_0^{\ell_\varepsilon} \mathrm{d}t \left[\left| f_0(t) - f_{k_-}(t)\right|^2 + f_0^2(t) \left| e^{-iS(s_j-L_\varepsilon)} - e^{-iS(s_j-L_\varepsilon-\varepsilon^a)}\right|^2\right] \\ & \leqslant C\left[\varepsilon^{1-a} + \varepsilon^a|\log\varepsilon|^\infty\right], \end{aligned} \tag{5.8}$$

by the identity

$$S(s) = \alpha_0 s + \int_0^s \mathrm{d}\xi \ \left(\alpha_{k(\xi)} - \alpha_0\right) + \mathcal{O}(|\log\varepsilon|^\infty). \tag{5.9}$$

Putting together all the energy contributions, we get (5.1). □

5.2 Order parameter

As proven in [19], any minimizing ψ^{GL} is such that its modulus is suitably close in $L^2(\mathcal{A}_\varepsilon)$ to the 1D profile $f_0(\mathrm{dist}(\,\cdot\,,\partial\Omega)/\varepsilon)$. The presence of the corners affects the estimate only at the precision one can approximate $|\psi^{\mathrm{GL}}|$ with f_0, since the result in [19, Thm. 1.1] is proven by neglecting the corner regions. The improved energy asymptotics of Theorem 2.1 obviously suggests that such an estimate can in fact be strengthened. Indeed, we prove here that a pointwise estimate of the difference $|\psi^{\mathrm{GL}}| - f_0$ holds true in the smooth part of the boundary layer.

Proof of Proposition 2.1 The starting point is the combination of the energy upper bound (5.1), with the stronger lower bound which can be obtained by combining the arguments of the proof of Proposition 4.1 with Lemma C.1: in each cell contained in the smooth part of the boundary layer, we can retain the positive contribution appearing on the r.h.s. of (C.13). The final outcome is the estimate

$$\sum_{n=1}^{M_\varepsilon} \int_{\mathscr{C}_n} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k_n t)\, f_n^4 (1-|u_n|^2)^2 = o(1). \tag{5.10}$$

A direct consequence is the estimate stated in (2.26) in Remark 2.3. Furthermore, (5.10) is the key ingredient of a typical argument (first used in [10]) to deduce a pointwise estimate of $|u_n|$ and thus $|\psi^{\mathrm{GL}}| = f_0|u_n|$ (see [24, Proof of Thm. 2, Step 2] but also the proof of (3.28) in Proposition 3.3).

Instead of providing all the details, we comment only on the needed adaptations. First of all, one has to select a subdomain of $\mathscr{C}_n$, where a suitable lower bound on the density f_n holds true. In our case, we can restrict the analysis to the layer $\{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant c\varepsilon\}$, where f_n is bounded from below by a positive constant independent of ε. As a consequence, the argument of [24, Proof of Thm. 2, Step 1] leads to

$$|\nabla |u_n|| \leqslant C, \qquad \text{in } \{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant c\varepsilon\} \cap \mathscr{C}_n. \tag{5.11}$$

With such a bound at disposal, the aforementioned argument applies straightforwardly in the boundary region $\{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant c\varepsilon\}$, leading to the pointwise estimate

$$||u_n| - 1| = o(1), \qquad \text{in } \{\mathrm{dist}(\mathbf{r}, \partial\Omega) \leqslant c\varepsilon\} \cap \mathscr{C}_n, \tag{5.12}$$

which immediately yields (2.27). □

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Appendix A. One-dimensional effective energies

In this Appendix we recall some known results about the effective one-dimensional problems, which are known to play a role in surface superconductivity. More details can be found, e.g., in [17,23–25].

A.1. Effective model on the half-line

The model problem describing the behavior of the order parameter along the normal direction to the boundary $\partial\Omega$ in the surface superconductivity regime is given in first approximation by the energy

$$\mathcal{E}^{\mathrm{1D}}_{\star,\alpha}[f] := \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}(2f^2 - f^4) \right\}, \tag{A.1}$$

where t is the rescaled distance to the boundary and $\alpha \in \mathbb{R}$ is a parameter.

For any $\alpha \in \mathbb{R}$, the functional (A.1) admits (see, e.g., [23, Prop. 3.1] or [24, Prop. 5]) a unique minimizer in the domain $\mathscr{D}^{\mathrm{1D}} = \left\{ f \in H^1(\mathbb{R}^+;\mathbb{R}) \mid tf(t) \in L^2(\mathbb{R}^+) \right\}$ with ground state energy $E^{\mathrm{1D}}_{\star,\alpha}$. The optimal profile is obtained by optimizing over α, as in (2.3). The infimum can be easily shown to actually be a minimum, i.e., there exists an $\alpha_\star \in \mathbb{R}$, where

the minimum is achieved. We also denote by $f_\star$ the corresponding profile, i.e., the minimizer of $E^{\mathrm{1D}}_{\star,\alpha_\star}$, which satisfies

$$0 < f_\star \leqslant C \exp\left\{-\tfrac{1}{2}(t+\alpha_\star)^2\right\}. \tag{A.2}$$

A.2. Curvature-dependent one-dimensional models

It is convenient to introduce a generalization of (A.1), which takes into account the effects of the boundary curvature:

$$\mathcal{E}^{\mathrm{1D}}_{k,\alpha}[f] := \int_0^\ell \mathrm{d}t(1-\varepsilon k t)\left\{|\partial_t f|^2 + V_{k,\alpha}(t) f^2 - \frac{1}{2b}(2f^2 - f^4)\right\}, \tag{A.3}$$

where $k \in \mathbb{R}$ is the rescaled mean curvature, which is assumed to be constant here,

$$V_{k,\alpha}(t) = \frac{1}{(1-\varepsilon k t)^2}\left(t + \alpha - \tfrac{1}{2}\varepsilon k t^2\right)^2 \tag{A.4}$$

and $\ell = \ell(\varepsilon) \gg 1$ is an ε-dependent quantity satisfying

$$|\log\varepsilon| \lesssim \ell \ll \varepsilon^{-1}. \tag{A.5}$$

For any $\alpha \in \mathbb{R}$ we denote the ground state energy of (A.3) by $E^{\mathrm{1D}}_{k,\alpha}$. The corresponding optimal energy is

$$E^{\mathrm{1D}}_k := \inf_{\alpha\in\mathbb{R}} E^{\mathrm{1D}}_{k,\alpha} = \inf_{\alpha\in\mathbb{R}} \inf_{f\in\mathscr{D}^{\mathrm{1D}}_\ell} \mathcal{E}^{\mathrm{1D}}_{k,\alpha}[f], \tag{A.6}$$

where $\mathscr{D}^{\mathrm{1D}}_\ell = H^1(I_\ell;\mathbb{R})$, with $I_\ell := [0,\ell]$, and one can prove the existence of a minimizing $\alpha_k \in \mathbb{R}$, i.e., $E^{\mathrm{1D}}_k = E^{\mathrm{1D}}_{k,\alpha_k}$. The corresponding profile is then denoted by f_k, which is therefore the unique minimizer of $E^{\mathrm{1D}}_{k,\alpha_k}$. We also set $\mathcal{E}^{\mathrm{1D}}_k := \mathcal{E}^{\mathrm{1D}}_{k,\alpha_k}$, accordingly. Note that, unlike $\mathcal{E}^{\mathrm{1D}}_{\star,\alpha}$, the new energy functional $\mathcal{E}^{\mathrm{1D}}_{k,\alpha}$ depends on ε in the measure, in the potential $V_{k,\alpha}$ and possibly in the upper extreme of the integration domain ℓ.

The dependence on the curvature k of the model problem (A.6) is investigated in [24, Props. 1 & 2], where it is shown that all the relevant quantities are essentially continuous in k. We sum up here the main properties of the limiting functionals (A.3) and the corresponding minimizers (see [23, Sect. 3] and [24, Appendix A] for the proofs):

- f_k is a smooth non-negative function monotonically decreasing for $t \gg 1$, such that $\|f_k\|_\infty \leqslant 1$ and

$$-f_k'' + \tfrac{\varepsilon k}{1-\varepsilon k t} f_k' + V_{k,\alpha}(t) f_k = \tfrac{1}{b}\left(1 - f_k^2\right) f_k, \tag{A.7}$$

  in I_ℓ with Neumann boundary conditions $f_k'(0) = f_k'(\ell) = 0$;
- for any $1 \leqslant b < \Theta_0^{-1}$, f_k is strictly positive and satisfies the *optimality condition*

$$\int_0^\ell \mathrm{d}t\, \frac{1}{1-\varepsilon k t}\left(t + \alpha_k - \tfrac{1}{2}\varepsilon k t^2\right) f_k^2(t) = 0; \tag{A.8}$$

- for any $k \in \mathbb{R}$,

$$E^{\mathrm{1D}}_k = -\frac{1}{2b}\int_0^\ell \mathrm{d}t\,(1-\varepsilon k t) f_k^4(t). \tag{A.9}$$

- for any $1 \leqslant b < \Theta_0^{-1}$ and for ε small enough, there exist two positive and finite constants $c, c' > 0$, such that

$$c \exp\left\{-\tfrac{1}{2}\left(t+\tfrac{1}{2}\right)^2\right\} \leqslant f_k(t) \leqslant c' \exp\left\{-\tfrac{1}{2}\left(t+\alpha_k\right)^2\right\}. \tag{A.10}$$

We add to the above bounds the following estimate, which is quite similar to what is proven in [24, Lemma 9]:

Lemma A.1 *For any $1 \leqslant b < \Theta_0^{-1}$ and for $\varepsilon \ll 1$, there exists a finite constant $C > 0$, such that*

$$\left|f_k'(t)\right| \leqslant C\, e^{-\frac{1}{4}t^2}, \qquad \text{for any } t \in [0, \ell], \tag{A.11}$$

$$\left|f_k'(t)\right| \leqslant C t^3 f_k(t), \qquad \text{for any } t \in [1, \ell]. \tag{A.12}$$

Proof For the proof of (A.11) we simply notice that, by integrating the variational Eq. (A.7) multiplied by $f_k(t)$ in $[t, \ell]$ and using Neumann boundary conditions, we obtain

$$\left|f_k'(t)\right| \leqslant C \int_t^\ell \mathrm{d}\eta\, \eta^2\, f_k(\eta)$$

Then, the result is a consequence of the decay of f_k (A.10).

The proof of (A.12) follows along the same lines of [24, Proof of Lemma 9]. □

As first discussed in [23, Sect. 3], a key role in the study of the effective 1D models is played by the following *potential function*

$$\begin{aligned} F_k(t) &:= 2\int_0^t \mathrm{d}\eta\, \frac{1}{1-\varepsilon k\eta}\left(\eta+\alpha_k-\tfrac{1}{2}\varepsilon k\eta^2\right) f_k^2(\eta) \\ &= -f_k'^2(t) + (t+\alpha_k)^2 f_k^2(t) - \tfrac{1}{b}\left(1-\tfrac{1}{2}f_k^2(t)\right) f_k^2(t) + \mathcal{O}(\varepsilon k). \end{aligned} \tag{A.13}$$

which heuristically provides the energy gain of a single vortex at a distance εt from the boundary. Similarly, the overall energy cost of a vortex is given by the *cost function*

$$K_k(t) := f_k^2(t) + F_k(t), \tag{A.14}$$

The properties of both functions are summed up below (see [23, Sect. 3] and [24, Appendix A]): for any $1 < b < \Theta_0^{-1}$ and $k \in \mathbb{R}$,

- $F_k(t) \leqslant 0$, for any $t \in [0, \ell]$;
- $F_k(0) = F_k(\ell) = 0$;
- let $\bar{t}_{k,\ell} > 0$ be such that ($\bar{t}_{k,\ell}$ is uniquely defined by monotonicity of f_k for large t)

$$\bar{I}_{k,\ell} := \left\{t \in (0,\ell) \;\middle|\; f_k(t) \geqslant \ell^3 f_k(\ell)\right\} = \left[0, \bar{t}_{k,\ell}\right], \tag{A.15}$$

then,

$$K_k(t) \geqslant 0, \qquad \text{for any } t \in [0, \bar{t}_{k,\ell}]. \tag{A.16}$$

A.3. Effective model on an interval with $k = 0$

A special case of the 1D models discussed in the previous Section is the one obtained for $k = 0$. It is in fact an approximation of the 1D effective energy $E_\star^{1\mathrm{D}}$ obtained by minimizing the energy on a finite interval $[0, \ell]$, $\ell \gg 1$, rather than in the whole of $\mathbb{R}^+$ (see [23, Sect. 3]). There is indeed a unique minimizing pair f_0, α_0 of $\mathcal{E}_{0,\alpha}^{1\mathrm{D}}[f]$ over $f \in H^1([0,\ell])$ positive

and $\alpha \in \mathbb{R}$. Like f_k, f_0 solves the variational Eq. (A.7) in the interval $[0, \ell]$ with α_0 in place of α_k. In addition, f_0 satisfies Neumann boundary conditions

$$f_0'(0) = f_0'(\ell) = 0. \tag{A.17}$$

Furthermore, all the properties (A.8)–(A.12) (with $k = 0$) hold true for f_0 as well. In particular, f_0 is monotonically decreasing for $t \geqslant t_0$, where t_0 is the unique maximum of f_0, satisfying

$$0 < t_0 \leqslant |\alpha_0| + \tfrac{1}{\sqrt{b}}. \tag{A.18}$$

Lemma A.2 *For any $1 < b < \Theta_0^{-1}$, if $\ell \gg 1$, then*

$$E_0^{\mathrm{1D}} = E_\star^{\mathrm{1D}} + \mathcal{O}(\ell^{-\infty}). \tag{A.19}$$

Proof It suffices to prove that for any finite $\alpha \in \mathbb{R}$,

$$E_{\star,\alpha}^{\mathrm{1D}} - E_{0,\alpha}^{\mathrm{1D}} = \mathcal{O}(\ell^{-\infty}), \tag{A.20}$$

which immediately implies (A.19), since the minima of both functionals are achieved for bounded α (see, e.g., [23, Cor. 3.2 & Lemma 3.1]). However, (A.20) above is a trivial consequence of the exponential decays (A.2) and (A.10). □

A very important consequence of the properties of f_0 [23, Prop. 3.5] is that, as for f_k, if we set (recall (3.7))

$$K_0(t) := f_0^2(t) + F_0(t), \tag{A.21}$$

where

$$F_0(t) := 2\int_0^t \mathrm{d}\eta\ (\eta + \alpha_0)\, f_0^2(\eta) = -2\int_t^\ell \mathrm{d}\eta\ (\eta + \alpha_0)\, f_0^2(\eta),$$

then, for $1 < b < \Theta_0^{-1}$,

$$K_0(t) \geqslant 0, \qquad \text{for any } t \in I_{\bar\ell}, \tag{A.22}$$

with

$$I_{\bar\ell} := \left\{ t \in (0, \ell) \;\middle|\; f_0(t) \geqslant \ell^3 f_0(\ell) \right\} = \left[0, \bar\ell\right]; \tag{A.23}$$

$$\bar\ell = \ell + \mathcal{O}(1). \tag{A.24}$$

In the whole interval $[0, \ell]$, we can use (A.13) in combination with (A.12) to estimate

$$|F_0(t)| = -F_0(t) \leqslant C t^6 f_0^2(t), \qquad \forall t \in [1, \ell]. \tag{A.25}$$

Remark A.1 **(Positivity of the cost function)** An interested reader might wonder whether $K_0(t)$ is in fact positive in the whole of $[0, \ell]$. There is however a simple argument showing that this is not the case and $\exists t_m \in I_\ell$, where

$$K_0(t_m) < 0. \tag{A.26}$$

To prove this, one first note that K_0 is convex. Hence, since $K_0'(0) = 2\alpha_0 f_0^2(0) < 0$, $K_0'(\ell) = 2(\ell + \alpha_0)^2 f_0^2(\ell) > 0$, there must be a minimum point at $0 < t_m < \ell$. In fact, by

a close inspection of the condition $K_0'(t_m) = 0$, it is possible to prove that $t_m = \ell + \mathcal{O}(1)$. However, the analogue of (A.13) and the criticality condition $K'(t_m)$ imply that

$$K_0(t_m) = \left(1 - \tfrac{1}{b} + \tfrac{1}{2b} f_0^2(t_m)\right) f_0^2(t_m) - \ell^2 f_0^2(\ell)(1 + o(1)) = -\left(\ell^2 + o\left(\ell^2\right)\right) f_0^2(\ell) < 0.$$

We complete the section, by showing that the positivity in (A.22) can in fact be strengthen and promoted to a sort of coercivity of K_0.

Proposition A.1 (Coercivity of K_0) *For any $1 < b < \Theta_0^{-1}$, if $\ell \gg 1$, there exists a constant $c_b > 0$, such that*

$$K_0(t) \geqslant c_b f_0^4(t), \qquad \text{for any } t \in I_{\tilde{\ell}}. \tag{A.27}$$

Proof The proof idea is quite similar to the one used in the proof of [23, Prop. 3.5]. We provide the details for the sake of completeness. We set

$$\widetilde{K}(t) := K_0(t) - \tfrac{1}{\ell^4} f_0^2(t) - c_b f_0^4(t) + \gamma_\ell, \tag{A.28}$$

where $\gamma_\ell := (\ell + \alpha_0)^2 f_0^2(\ell) - \frac{1}{b}\left(1 - \frac{1}{2} f_0^2(\ell)\right) f_0^2(\ell)$, so that, by the identity (A.13), one gets

$$\widetilde{K}(t) = \left[1 - \tfrac{1}{\ell^4} - \tfrac{1}{b} + \left(\tfrac{1}{2b} - c_b\right) f_0^2(t)\right] f_0^2(t) - f_0'^2(t) + (t + \alpha_0)^2 f_0^2(t). \tag{A.29}$$

Now, if one can prove that $\widetilde{K} \geqslant 0$ in $[0, \ell]$, the result then easily follows because

$$\min_{I_{\tilde{\ell}}} \left(\tfrac{1}{\ell^4} f_0^2(\ell) - \gamma_\ell\right) \geqslant \min_{I_{\tilde{\ell}}} \left[\tfrac{1}{\ell^4} f_0^2(t) - \ell^2 f_0^2(\ell)\right] \geqslant 0,$$

for ℓ large enough.

Let us then address the positivity of (A.28): at the boundary of the interval we have

$$\widetilde{K}(0) \geqslant \left(1 - \tfrac{1}{\ell^4} - c_b f_0^2(0)\right) f_0^2(0) > 0, \qquad \widetilde{K}(\ell) \geqslant \left(1 - \tfrac{1}{\ell^4} - c_b f_0(\ell)^2\right) f_0^2(\ell) > 0, \tag{A.30}$$

if $c_b < 1/f_0^2(0)$ and $\ell \gg 1$. Hence, the function can become negative only in the interior of I_ℓ, so let us look for its minimum points t_m, which must satisfy $\widetilde{K}'(t_m) = 0$, yielding

$$\left(1 - \tfrac{1}{\ell^4} - 4c_b f_0^2(t_m)\right) f_0'(t_m) = -(t_m + \alpha_0)\, f_0(t_m).$$

If we now take $c_b < 1/(4\,\|f_0\|_\infty^2)$, we can solve the above identity w.r.t. to $f'(t_m)$ and plug it into (A.29), so obtaining

$$\begin{aligned}\widetilde{K}(t_m) = &\left[1 - \tfrac{1}{\ell^4} - \tfrac{1}{b} - \frac{2\left(\frac{1}{\ell^4} + 4c_b f_0^2(t_m)\right)\left(1 - \frac{1}{2\ell^4} - 2c_b f_0^2(t_m)\right)}{\left(1 - \frac{1}{\ell^4} - 4c_b f_0(t_m)\right)^2}(t_m + \alpha_0)^2\right] f_0^2(t_m) \\ &+ \left(\tfrac{1}{2b} - c_b\right) f_0^4(t_m) \geqslant \left[1 - \tfrac{C}{\ell^2} - \tfrac{1}{b} - 8c_b t_m^2 f_0^2(t_m)\right] f_0^2(t_m) \geqslant 0,\end{aligned} \tag{A.31}$$

if $\ell \gg 1$ and c_b is taken small enough so that $c_b \leqslant \frac{1}{2b}$ and $c_b < \left(1 - \frac{1}{b}\right)/(8\,\|t f_0\|_\infty^2)$ (recall (A.10)). Putting all the conditions together, we see that the result is proven if we take

$$c_b < \min\left\{\tfrac{b-1}{8b\|t f_0\|_\infty^2}, \tfrac{1}{2b}, \tfrac{1}{4\|f_0\|_\infty^2}, \tfrac{1}{f_0^2(0)}\right\}.$$

□

Appendix B. Technical estimates

In this Appendix we collect several technical estimates, which are used in the paper. Throughout this Appendix, Ω will denote a bounded and simply connected domain $\Omega \subset \mathbb{R}^2$ satisfying Assumptions 1 and 2. We recall that for any bounded domain $\Omega \subset \mathbb{R}^2$ with locally Lipschitz boundary, all the usual So*bolev embeddings hold true [3, Thm. 5.4]. In particular, in what follows, we often use that, given a domain Ω with the strong local Lipschitz property (see [3, Def. 4.5]), for all $p \in [2, \infty)$ and for all $\alpha \in [0, 1)$,

$$H^1(\Omega) \hookrightarrow L^p(\Omega), \qquad H^2(\Omega) \hookrightarrow W^{1,p}(\Omega), \qquad W^{2,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \tag{B.1}$$

where $C^{0,\alpha}$ stands for the space of Hölder continuous functions with exponent α. We also note that the diamagnetic inequality is verified in a piecewise smooth domain as well, i.e., for every $\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that $(\nabla + i\mathbf{A})\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, one has

$$|\nabla|\psi|| \leqslant |(\nabla + i\mathbf{A})\psi|, \qquad \text{for a.e. } \mathbf{r} \in \Omega. \tag{B.2}$$

B.1. Minimization of the GL energy

For the sake of completeness, we briefly discuss the minimization of the GL functional in domains with Lipschitz boundary. The material is mostly taken from [33] (see in particular [33, Chpt. 15 & Sect. D.2.3]).

As proven in [33, Thm. 15.3.1], there exists a minimizing pair $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ for $\mathcal{G}^{\mathrm{GL}}_\kappa[\psi, \mathbf{A}]$, such that $(\psi, \mathbf{A} - \mathbf{F}) \in H^1(\Omega) \times W^{1,2}_{0,0}(\mathbb{R}^2)$, where $W^{1,2}_{0,0}(\mathbb{R}^2)$ is a suitable Sobolev space properly defined in [33, Eq. (D.12)] and $\mathbf{F}$ given in (2.17)). In addition, we may fix the gauge in such a way that

$$\nabla \cdot \mathbf{A}^{\mathrm{GL}} = 0. \tag{B.3}$$

This determines the potential up to an additive constant, which can be chosen so that

$$\left\| \mathbf{A}^{\mathrm{GL}} - \mathbf{F} \right\|_{H^1(\Omega;\mathbb{R}^2)} \leqslant C \left\| \mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1 \right\|_{L^2(\mathbb{R}^2)}, \tag{B.4}$$

which in turn implies [33, Lemma 15.3.2] that $\mathrm{curl}\left(\mathbf{A}^{\mathrm{GL}} - \mathbf{F}\right) = 0$ or, equivalently,

$$\mathrm{curl}\mathbf{A}^{\mathrm{GL}} = 1, \qquad \text{in } \mathbb{R}^2 \setminus \Omega. \tag{B.5}$$

Hence, when we evaluate $\mathcal{E}^{\mathrm{GL}}_\varepsilon$ on the minimizing configuration, we may restrict the integration domain in the last term in (1.1) to Ω.

Finally, any critical point $(\psi, \mathbf{A})$ of $\mathcal{E}^{\mathrm{GL}}$ and in particular the minimizing pair $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ satisfies the GL variational equations

$$\begin{cases} -\left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2}\right)^2 \psi = \frac{1}{\varepsilon^2}\left(1 - |\psi|^2\right)\psi, & \text{in } \Omega, \\ -\frac{1}{\varepsilon^2}\nabla^\perp \mathrm{curl}\mathbf{A} = \mathbf{j}_{\mathbf{A}}[\psi]\mathbb{1}_\Omega, & \text{in } \mathbb{R}^2, \\ \boldsymbol{\nu} \cdot \left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2}\right)\psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{B.6}$$

where we have denoted by $\mathbf{j}_{\mathbf{A}}$ the current

$$\mathbf{j}_{\mathbf{A}}[\psi] := \frac{i}{2}\left[\psi\left(\nabla - i\frac{\mathbf{A}}{\varepsilon^2}\right)\psi^* - \psi^*\left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2}\right)\psi\right] = \mathrm{Im}\left(\psi^*\left(\nabla + i\frac{\mathbf{A}}{\varepsilon^2}\right)\psi\right). \tag{B.7}$$

Any minimizing pair is smooth in the interior of Ω and continuous at the boundary. More precisely, for any $\tilde{\Omega} \subset \Omega$ with $\partial\tilde{\Omega} \subset \Omega^\circ$ smooth, and for any $\alpha \in [0, 1)$

$$\psi^{\mathrm{GL}} \in C^\infty(\tilde{\Omega}), \qquad \mathbf{A}^{\mathrm{GL}} \in C^\infty(\tilde{\Omega}; \mathbb{R}^2); \tag{B.8}$$

$$\psi^{\mathrm{GL}} \in C^{0,\alpha}(\overline{\Omega}), \qquad \mathbf{A}^{\mathrm{GL}} \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^2), \tag{B.9}$$

as it can be seen by applying standard arguments in elliptic theory (see, e.g., [41]).

B.2. Elliptic estimates

We now state useful estimates valid for any critical point of $\mathcal{E}^{\mathrm{GL}}$. The following bounds are direct consequences of (B.6) [33, Chpts. 10, 11 & 15]:

$$\|\psi\|_{L^\infty(\Omega)} \leqslant 1. \tag{B.10}$$

$$\left\|\mathbf{A}^{\mathrm{GL}} - \mathbf{F}\right\|_{L^p(\Omega)} \leqslant C\varepsilon \, \|\psi\|_{L^2(\Omega)} \, \|\psi\|_{L^4(\Omega)}. \tag{B.11}$$

We also have a quantitative estimate of the magnetic gradient of ψ, which is however limited by the presence of corners at the boundary.

Lemma B.1 *Let $\psi, \boldsymbol{A}$ solve (B.6) and let*

$$\Omega_\varepsilon := \left\{ \boldsymbol{r} \in \Omega \,\middle|\, \mathrm{dist}(\boldsymbol{r}, \Sigma) \geqslant \varepsilon \right\}, \tag{B.12}$$

then

$$\left\| \left(\nabla + i \frac{\boldsymbol{A}}{\varepsilon^2} \right) \psi \right\|_{L^\infty(\Omega_\varepsilon)} \leqslant \frac{C}{\varepsilon}. \tag{B.13}$$

Proof The result can be deduced from the Eq. (B.6) and, in particular, the first one, applying in a suitable way, e.g., [9, Lemma A.1]. □

The counterpart of (B.13) for any minimizer ψ of the corner problems (3.38) and (3.40) reads

$$\|(\nabla + i\mathbf{a})\,\psi\|_{L^\infty(\{|s(\mathbf{r})|\geqslant 1\})} = \mathcal{O}(1), \tag{B.14}$$

and combining it with, e.g., (B.34) proven in next "Appendix B.3", we also get

$$\|\nabla\psi\|_{L^\infty(\{|s(\mathbf{r})|\geqslant 1\})} = \mathcal{O}(1). \tag{B.15}$$

B.3. Agmon estimates

Another typical key tool in the study of the GL theory is the estimate of the decay properties (*Agmon estimates*) of any solution $(\psi, \mathbf{A})$ of the GL variational Eq. (B.6) in the surface superconductivity regime, i.e., when the intensity of the applied magnetic field is such that $h_{\mathrm{ex}} > H_{\mathrm{c2}}$. The result is in fact inherited from the linear problem associated to the GL energy, i.e., a magnetic Schrödinger operator, and does not exploit the nonlinearity. The presence of corners does not influence the exponential decay of the order parameter away from the boundary [33, Sect. 15.3.1]. More precisely, for any $b > 1$ and for any $(\psi, \mathbf{A})$ solving (B.6) [15, Thm. 4.4],

$$\int_\Omega \mathrm{d}\mathbf{r} \, \exp\left\{ \tfrac{c(b)\,\mathrm{dist}(\mathbf{r},\partial\Omega)}{\varepsilon} \right\} \left\{ |\psi|^2 + \varepsilon^2 \left| \left(\nabla + i \tfrac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 \right\} = \mathcal{O}(\varepsilon), \tag{B.16}$$

where $c(b) > 0$, for $b > 1$, is independent of ε. When $b \to 1^+$, the above bound becomes non-optimal because of the vanishing of $c(b)$ and one can in fact prove other estimates showing a power law decay of ψ [35]. Similarly, in presence of corners, the result might not be optimal for $b > \Theta_0^{-1}$: assuming that there is at least one angle β along the boundary such that $\mu(\beta) < \Theta_0$, one can prove [15, Thm. 1.6] a stronger decay w.r.t. the distance from that corner. Here, $\mu(\beta)$ stands for the ground state energy of the magnetic Schrödinger operator in an infinite wedge of opening angle β with unit magnetic field.

The translation of (B.16) in the setting of § 3.1, i.e., a GL functional with fixed parameter $\varepsilon = 1$ in a finite strip $R(\ell, L)$ is as follows:

Lemma B.2 *Let ψ solve (3.5) and satisfy the boundary conditions alternatively in (3.4), (3.9) or (3.10) in $R(\ell, L)$ with $\ell, L > 0$. Then, for any $b > 1$, there exists a constant $c(b) > 0$, such that*

$$\int_{R(L,\ell)} \mathrm{d}s\mathrm{d}t \; e^{c(b)\, t} \left\{ |\psi|^2 + |(\nabla - i t \boldsymbol{e}_s)\, \psi|^2 \right\} = \mathcal{O}(L). \tag{B.17}$$

In the paper, we use Agmon estimates also for the corner effective problem. We discuss here such an extension to the setting of the effective problem formulated in (2.12) and discussed in § 2.3.

Lemma B.3 *Let $\Gamma_\beta(L, \ell)$ be the region given in Fig. 2, with $L, \ell \gg 1$ and $L \lesssim \ell^a$, for some $a > 1$. Let also ψ be a solution of (3.59), with $b > 1$. Then, there exists a constant $c(b) > 0$, such that*

$$\int_{\Gamma_\beta(L,\ell)} \mathrm{d}\boldsymbol{r} \; e^{c(b)\, \mathrm{dist}(\boldsymbol{r}, \partial\Gamma_{\mathrm{out}})} \left\{ |\psi|^2 + |(\nabla + i\boldsymbol{a})\, \psi|^2 \right\} = \mathcal{O}(L). \tag{B.18}$$

The above result is a simple adaptation of (B.16) to the effective problem in $\Gamma_\beta(L, \ell)$. The only difference is that the magnetic potential $\mathbf{a}$ is given and not a minimizer of the energy. Before discussing its proof, however, we first state a technical lemma, which follows from a standard inequality for the magnetic gradient and the equation solved by ψ.

Lemma B.4 *Let $\Gamma_\beta(L, \ell)$ be the region given in Fig. 2, with $L, \ell \gg 1$ and $L \lesssim \ell^a$, for some $a > 1$. Let also ψ be a solution of (3.59) and let ξ be a smooth real function. Then, for any set $S \subset \Gamma_\beta(L, \ell)$ with Lipschitz boundary,*

$$\begin{aligned} \int_S \mathrm{d}\boldsymbol{r} \left\{ |\psi|^2 \, (\nabla\xi)^2 + \tfrac{1}{b} \, |\xi\psi|^2 \left(1 - |\psi|^2\right) \right\} &\geqslant \int_S \mathrm{d}\boldsymbol{r} \, \mathrm{curl}(\boldsymbol{a}) \, |\xi\psi|^2 \\ - \int_{\partial S} \mathrm{d}x \left\{ \tfrac{1}{2} \xi^2 \boldsymbol{\nu} \cdot \nabla |\psi|^2 + \boldsymbol{\tau} \cdot \boldsymbol{j}_{\boldsymbol{a}}[\xi\psi] \right\}, \end{aligned} \tag{B.19}$$

where $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ stand for the tangential and normal unit vectors to ∂S, respectively.

Proof We start by integrating the following trivial bound (see, e.g., [26, Lemma 3.2]) for any u weakly differentiable and $\mathbf{a} \in L^\infty$ (we set $\mathbf{a} := (a_1, a_2)$)

$$\begin{aligned} |(\nabla + i\mathbf{a})\, u|^2 &= |(\partial_1 + i a_1 - i\,(\partial_2 + i a_2))\, u|^2 - \mathrm{curl}\, \mathbf{j}[u] - \mathbf{a} \cdot \nabla^\perp |u|^2 \\ &\geqslant -\mathrm{curl}\, \mathbf{j}[u] - \mathbf{a} \cdot \nabla^\perp |u|^2, \end{aligned}$$

which yields, taking $u = \xi\psi$,

$$\int_S \mathrm{d}\mathbf{r} \; |(\nabla + i\mathbf{a})\, \xi\psi|^2 \geqslant \int_S \mathrm{d}\mathbf{r} \, \mathrm{curl}(\mathbf{a}) \, |\xi\psi|^2 - \int_{\partial S} \mathrm{d}x \left\{ \boldsymbol{\tau} \cdot \mathbf{j}[\xi\psi] + \boldsymbol{\tau} \cdot \mathbf{a} |\xi\psi|^2 \right\},$$

(B.20)

after an integration by parts of the last term and the use of Stokes theorem. Note that the last two terms can be combined to reconstruct the magnetic current $\mathbf{j_a}$. To complete the proof it suffices to use the Eq. (3.59) to compute the term on the l.h.s. The additional boundary term in (B.19) is produced by the integration by parts of the cross product term $\xi\nabla\xi\cdot\nabla|\psi|^2$ to reconstruct the term $\psi^*\Delta\psi$ + h.c. of the variational equation. □

Proof of Lemma B.3 As anticipated the result is a simple adaptation of (B.16) (see [33, Proof of Thm. 12.2.1]). The key ingredient is the inequality (B.19), applied to $S = \Gamma_\beta(L,\ell)$, together with the following choice of the function ξ:

$$\xi(\mathbf{r}) = \xi(t(\mathbf{r})) = e^{at(\mathbf{r})} f(t), \tag{B.21}$$

with the function f such that $|f'| \leqslant C$ and

$$f = \begin{cases} 1, & \text{for } t \in [1, +\infty], \\ 0, & \text{for } t \in \left[0, \frac{1}{2}\right]. \end{cases}$$

We first estimate the boundary terms appearing in (B.19):

$$\int_{\partial\Gamma_\beta(L,\ell)} \mathrm{d}x\ \xi^2 \boldsymbol{\nu}\cdot\nabla|\psi|^2 = \mathcal{O}(1), \tag{B.22}$$

because $\xi = 0$ on $\partial\Gamma_{\text{out}}$,

$$\left|\xi^2\boldsymbol{\nu}\cdot\nabla|\psi|^2\right| \leqslant Ce^{2a\ell} f_0(\ell) = \mathcal{O}(\ell^{-\infty}), \qquad \text{on } \partial\Gamma_{\text{in}}, \tag{B.23}$$

and

$$\int_{\partial\Gamma_{\text{bd}}} \mathrm{d}x\ \left|\xi^2\boldsymbol{\nu}\cdot\nabla|\psi|^2\right| \leqslant C\int_0^\ell \mathrm{d}t\ e^{2at} f_0(t) \leqslant C, \tag{B.24}$$

where we have used the boundary conditions on ψ, the exponential decay of f_0 (A.10) and the estimate (B.13), which yields

$$|\nabla|\psi|| \leqslant |(\nabla + i\mathbf{a})\,\psi| \leqslant C, \qquad \text{for dist}\,(\mathbf{r}, \mathbf{r}_0) \geqslant 1, \tag{B.25}$$

$\mathbf{r}_0$ being the position of the corner. Similarly,

$$\int_{\partial\Gamma_\beta(L,\ell)} \mathrm{d}x\ \boldsymbol{\tau}\cdot\mathbf{j_a}[\xi\psi] = \mathcal{O}(1), \tag{B.26}$$

thanks to the vanishing at 0 of ξ and the bounds

$$\int_{\partial\Gamma_{\text{in}}} \mathrm{d}x\ \left|\boldsymbol{\tau}\cdot\mathbf{j_a}[\xi\psi]\right| \leqslant C\int_{\partial\Gamma_{\text{in}}} \mathrm{d}x\ e^{2a\ell} f_0^2(\ell) = \mathcal{O}(\ell^{-\infty}), \tag{B.27}$$

$$\int_{\partial\Gamma_{\text{bd}}} \mathrm{d}x\ \left|\boldsymbol{\tau}\cdot\mathbf{j_a}[\xi\psi]\right| \leqslant C\int_0^\ell \mathrm{d}t\ e^{2at} f_0(t) \leqslant C, \tag{B.28}$$

as in (B.23) and (B.24), respectively.

The rest of the proof is identical to [33, Proof of Thm. 12.2.1]: the estimates (B.22) and (B.26) above together with (B.19) imply

$$\left(1 - \tfrac{1}{b}\right)\|\xi\psi\|^2_{L^2(\Gamma_\beta(L,\ell))} \leqslant \int_{\Gamma_\beta(L,\ell)} \mathrm{d}\mathbf{r}\ (\nabla\xi)^2\,|\psi|^2 + \mathcal{O}(1). \tag{B.29}$$

Noticing now that

$$\left|\xi'\right|^2 \leqslant 2(1+\epsilon)a^2 f^2 e^{2at} + \left(1+\tfrac{1}{\epsilon}\right) f'^2 e^{2at} \leqslant 2(1+\epsilon)a^2 f^2 e^{2at} + C(\epsilon)e^{2at}, \tag{B.30}$$

we conclude that

$$\left(1-\tfrac{1}{b}-2(1+\epsilon)a^2\right)\int_{t(\mathbf{r})\geqslant\frac{1}{2}} \mathrm{d}\mathbf{r}\, e^{2at(\mathbf{r})}\,|\psi|^2 \leqslant C\int_{t(\mathbf{r})\leqslant 1} \mathrm{d}\mathbf{r}\,|\psi|^2 + \mathcal{O}(1), \tag{B.31}$$

and since we can always find $\epsilon > 0$ and $a(\epsilon) > 0$ so that the factor on the l.h.s. of the above expression is positive, we obtain the result for the order parameter. The estimate of the magnetic gradient however follows using (3.59) once more and the bound just proven. □

We complete the discussion of the decaying properties of the order parameter with a refined version of the estimate proven in Lemma B.3: we consider a solution of the differential Eq. (3.59) and show that, in a subdomain of tangential length of order $\mathcal{O}(1)$, the r.h.s. of (B.18) is $\mathcal{O}(1)$ as well. In order to state a more precise bound there, we identify two model domains, i.e., a rectangle S_{strip} of tangential side length $\mathcal{O}(1)$ far from the corner and the region close to it S_{corner}. More precisely, we set

$$S_{\mathrm{strip}} := \left\{\mathbf{r}\in\Gamma_\beta(L,\ell)\,\middle|\,\bar{s}_1\leqslant s(\mathbf{r})\leqslant\bar{s}_2\right\}, \qquad \bar{s}_2-\bar{s}_1\leqslant C, \tag{B.32}$$

and either $\bar{s}_1 \geqslant \ell/\tan(\beta/2)$ or $\bar{s}_2 \leqslant -\ell/\tan(\beta/2)$, which ensures that in S_{strip} we can use the coordinates (s,t) and it corresponds to $[\bar{s}_1,\bar{s}_2]\times[0,\ell]$. The other region S_{corner} is

$$S_{\mathrm{corner}} := \left\{\mathbf{r}\in\Gamma_\beta(L,\ell)\,\middle|\,\bar{s}_1\leqslant \mathrm{dist}(\mathbf{r},\mathbf{r}_0)\leqslant\bar{s}_2\right\}, \qquad \bar{s}_2-\bar{s}_1\leqslant C, \tag{B.33}$$

and $1\leqslant\bar{s}_1,\bar{s}_2\leqslant C\ell$, i.e., it is a wedge-like domain where boundary coordinates can not be used globally.

Lemma B.5 *Let $S_\sharp$ be one of the two domains defined in (B.32) and (B.33). Let also ψ be a solution of (3.59), with $b>1$. Then, there exists a constant $c(b)>0$, such that*

$$\int_{S_\sharp} \mathrm{d}\boldsymbol{r}\, e^{c(b)\,\mathrm{dist}(\boldsymbol{r},\partial\Gamma_{\mathrm{out}})}\left\{|\psi|^2+|(\nabla+i\boldsymbol{a})\,\psi|^2\right\} = \mathcal{O}(1). \tag{B.34}$$

Proof The proof is identical to the one of Lemma B.3, with the only difference due to the estimate of boundary terms. Exploiting (B.25) and the other properties of ψ and f_0, it is however easy to show that those terms provide contributions of order $\mathcal{O}(1)$, as well as the r.h.s. of (B.31), which leads to the result. A short comment is in order for regions close to the corner, where the pointwise bound (B.25) might fail: there one can always arrange the domain S in such a way that the boundary ∂S is far enough from $\partial\Gamma_{\mathrm{bd}}$ (still at a distance of order 1 from the corner) so that (B.25) applies, while on $\partial S\cap\partial\Gamma_{\mathrm{out}}$, the gradient estimate is not used. □

We finally provide a simple bound which is a direct consequence of (B.34).

Lemma B.6 *Let ψ be a solution of (3.59), with $b>1$. Then, there exists a finite constant C, such that*

$$|\psi(\boldsymbol{r})| \leqslant Ce^{-\frac{1}{2}c(b)\mathrm{dist}(\boldsymbol{r},\partial\Gamma_{\mathrm{out}})}, \tag{B.35}$$

where $c(b)$ is the constant appearing in (B.34).

Proof The result is proven by contradiction. Suppose that there was a point $\bar{\mathbf{r}} \in \Gamma_\beta(L, \ell)$, with $\mathrm{dist}(\bar{\mathbf{r}}, \partial\Gamma_{\mathrm{out}}) \geqslant 1$ and $\mathrm{dist}(\bar{\mathbf{r}}, \mathbf{r}_0) \geqslant 1$, so that

$$|\psi(\bar{\mathbf{r}})|\, e^{\frac{1}{2}c(b)\mathrm{dist}(\bar{\mathbf{r}},\partial\Gamma_{\mathrm{out}})} > C_0, \tag{B.36}$$

for some given $C_0 > 0$. Then, thanks to the pointwise bound (B.25), we can always construct a square Q of unit side length containing $\bar{\mathbf{r}}$, such that

$$|\psi(\mathbf{r})|\, e^{\frac{1}{2}c(b)\mathrm{dist}(\mathbf{r},\partial\Gamma_{\mathrm{out}})} \geqslant \tfrac{1}{2}C_0, \qquad \text{in } Q. \tag{B.37}$$

We are here assuming that C_0 is large enough, so that

$$\inf_{\mathbf{r}\in Q}\left(|\psi(\mathbf{r})|\, e^{\frac{1}{2}c(b)\mathrm{dist}(\mathbf{r},\partial\Gamma_{\mathrm{out}})}\right) \geqslant C_0 - \sqrt{2}\left(\|\nabla|\psi|\|_\infty + \tfrac{1}{2}c(b)\right) \geqslant \tfrac{1}{2}C_0.$$

Hence,

$$\int_Q \mathrm{d}\mathbf{r}\ |\psi(\mathbf{r})|^2\, e^{c(b)\mathrm{dist}(\mathbf{r},\partial\Gamma_{\mathrm{out}})} \geqslant \tfrac{1}{4}C_0^2,$$

which contradicts (B.34), if C_0 is large enough, since Q is fixed. □

Reformulating the above result for the variational problems in the strip considered in § 3.1 yields the pointwise estimates

$$|\psi(s,t)| \leqslant Ce^{-\frac{1}{2}c(b)t}, \tag{B.38}$$

for any ψ solving (3.5) and where $c(b)$ is the same constant appearing in Lemma B.2.

Appendix C. Local energy estimates

In this Section we sum up the salient points of the energy estimate in the smooth part of the boundary layer. Thanks to Agmon estimates (see "Appendix B.3"), we can restrict our analysis to the boundary layer (1.19), i.e.,

$$\mathcal{A}_\varepsilon = \left\{\mathbf{r}\in\Omega \,\middle|\, \mathrm{dist}\,(\mathbf{r},\partial\Omega) \leqslant \varepsilon\ell_\varepsilon\right\},$$

but here we will focus on its smooth component defined in (4.6):

$$\mathcal{A}_{\mathrm{cut}} = \left([0, s_1 - L_\varepsilon] \cup [s_1 + L_\varepsilon, s_2 - L_\varepsilon] \cup \cdots \cup \left[s_N + L_\varepsilon, \tfrac{|\partial\Omega|}{\varepsilon}\right]\right) \times [0, c_1|\log\varepsilon|],$$

where s_j, $j = 1, \ldots, N$ is the tangential coordinate of the $j-$th vertex. By Lemma 4.1, we can take as starting point of our analysis the effective functional introduced in (4.7):

$$\mathcal{G}_\varepsilon[\psi, \mathcal{A}_{\mathrm{cut}}] = \int_{\mathcal{A}_{\mathrm{cut}}} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k(s)t)\left\{|\partial_t\psi|^2 + \tfrac{1}{(1-\varepsilon k(s)t)^2}|(\partial_s - it)\psi|^2 - \tfrac{1}{2b}(2|\psi|^2 - |\psi|^4)\right\}$$

and its ground state energy

$$G_{\mathcal{A}_{\mathrm{cut}}} := \inf_{\psi\in H^1_{\mathrm{per}}(\mathcal{A}_{\mathrm{cut}})} \mathcal{G}_\varepsilon[\psi, \mathcal{A}_{\mathrm{cut}}], \tag{C.1}$$

where $H^1_{\mathrm{per}}(\mathcal{A}_{\mathrm{cut}}) := \{\psi \in H^1(\mathcal{A}_{\mathrm{cut}}) \,|\, \psi(0,t) = \psi(|\partial\Omega|/\varepsilon, t), \forall t \in [0,\ell]\}$. We also denote

$$\mathcal{I}_{\mathrm{smooth}} := \bigcup_{j=1}^{N}\left[s_j + L_\varepsilon, s_{j+1} - L_\varepsilon\right], \tag{C.2}$$

with the identification $s_{N+1} = s_1 + |\partial\Omega|/\varepsilon$. The material presented in this Section is essentially taken from [24] (see, in particular [24, Lemmas 3, 6 and 7]), but an important difference in the lower bound is given by the presence of holes in the boundary layer $\mathcal{A}_{\rm cut}$, where the corner regions have been removed. The key tool in the strategy is the decomposition of $\mathcal{A}_{\rm cut}$ into cells:

$$\mathcal{A}_{\rm cut} = \bigcup_{n=1}^{M_\varepsilon} \mathscr{C}_n, \qquad \mathscr{C}_n := [\sigma_n, \sigma_{n+1}] \times [0, c_1 |\log \varepsilon|], \tag{C.3}$$

with $|\sigma_{n+1} - \sigma_n| \propto 1$ and $M_\varepsilon \propto |\mathcal{I}_{\rm smooth}|/\varepsilon$. We then approximate the curvature $k(s)$ of the boundary in each cell by its mean value

$$k_n := \int_{\sigma_n}^{\sigma_{n+1}} \mathrm{d}s\, k(s) \tag{C.4}$$

and set for short $\alpha_n := \alpha_{k_n}$, $f_n(t) := f_{k_n}(t)$ (recall the notation of "Appendix A.2").

Proposition C.1 (Upper bound to $G_{\mathcal{A}_{\rm cut}}$) *For any fixed $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$, it holds*

$$G_{\mathcal{A}_{\rm cut}} \leqslant \frac{|\partial\Omega| E_0^{\rm 1D}}{\varepsilon} - 2L_\varepsilon N E_0^{\rm 1D} - \varepsilon E_{\rm corr} \int_0^{\frac{|\partial\Omega|}{\varepsilon}} \mathrm{d}s\, k(s) + o(1). \tag{C.5}$$

Proof See [24, Sect. 4.1]. □

We now complement (C.5) with a matching lower bound. As already pointed out, the proposition below is the analogue of [24, Lemma 6] but the effect of the holes in the smooth part of the domain now becomes apparent in the additional boundary terms appearing on the r.h.s. of (C.6). Those terms are matched in [24, Proof of Lemma 7, Step 2] by the corresponding boundary contributions coming from the cells which are missing in the present setting.

Proposition C.2 (Lower bound) *Let $\psi(s,t) \in H^1(\mathcal{A}_{\rm cut})$ be a function enjoying the same properties as $\psi^{\rm GL}(\boldsymbol{r}(s,t))$. Then, for any $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$, it holds*

$$\begin{aligned} \mathcal{G}_{\mathcal{A}_{\rm cut}}[\psi] \geqslant{}& \frac{|\partial\Omega| E_0^{\rm 1D}}{\varepsilon} - 2L_\varepsilon N E_0^{\rm 1D} - \varepsilon E_{\rm corr} \int_0^{|\partial\Omega|} \mathrm{d}s\, k(s) \\ &- \sum_{j=1}^{N} \int_0^{c_1|\log\varepsilon|} \mathrm{d}t\ \frac{F_0(t)}{f_0^2(t)}\, j_t\left[\psi(s,t)\right]\Big|_{s=s_j-L_\varepsilon}^{s=s_j+L_\varepsilon} + o(1). \end{aligned} \tag{C.6}$$

Proof The starting point is the very same splitting performed in [24, Lemma 6], which is analogous to what we did in the proof of Proposition 3.1: in each cell $\mathscr{C}_n$, we set

$$\psi(\mathbf{r}(s,t)) =: u_n(s,t) f_n(t) e^{-i\alpha_n s}, \tag{C.7}$$

where u_n plays the same role as u in the decoupling (3.12). Such a splitting procedure allows to extract from each cell the desired energy, i.e.,

$$E_{k_n}^{\rm 1D}(\sigma_{n+1} - \sigma_n) + \mathcal{E}_n[u_n], \tag{C.8}$$

where the reduced energies are

$$\begin{aligned} \mathcal{E}_n[u] := \int_{\sigma_n}^{\sigma_{n+1}} \mathrm{d}s \int_0^{\ell} \mathrm{d}t\, (1 - \varepsilon k_n t) f_n^2 \Big\{ |\partial_t u|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} |\partial_s u|^2 - 2b_n(t)\, j_s[u] \\ + \tfrac{1}{2b} f_n^2 (1 - |u|^2)^2 \Big\}, \end{aligned} \tag{C.9}$$

with $b_n(t) = \frac{1}{(1-\varepsilon k_n t)^2}(t + \alpha_n - \frac{1}{2}\varepsilon k_n t^2)$. By [25, Lemma 2.1], the first terms of (C.8) above sum up to

$$\frac{|\partial\Omega| E_0^{1\mathrm{D}}}{\varepsilon} - 2L_\varepsilon N E_0^{1\mathrm{D}} - \varepsilon E_{\mathrm{corr}} \int_{\mathcal{I}_{\mathrm{smooth}}} \mathrm{d}s\, k(s) + o(1). \tag{C.10}$$

If $1 < b < \Theta_0^{-1}$, the reduced functionals $\mathcal{E}_n[u_n]$ can be proven to be positive [24, Lemma 7] and can thus be dropped from the lower estimate, again up to small errors. Here, however, the major difference with [24] occurs: the positivity of $\mathcal{E}_n[u_n]$ is proven in [24, Lemma 7] via an integration by parts and exploits the pointwise positivity of the cost function K_k (see (A.14) and (A.16)), but the estimate of the boundary terms emerging from the integration has to be adjusted. Such terms have the form

$$-\int_0^{c_1|\log\varepsilon|} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t\left[\psi(s,t)\right]\Big|_{s=\sigma_n}^{s=\sigma_{n+1}}.$$

The sum of all the terms is shown in [24, Lemma 7] to be small, but this requires (see [24, Step 2 and eq. (5.33)]) to pair the term coming from one cell at σ_n with the one generated in the adjacent cell again at σ_n. In our setting, due to the absence of corner regions in $\mathcal{A}_{\mathrm{cut}}$, some boundary terms are missing. Such terms are precisely given by

$$\sum_{j\in\Sigma} \int_0^{c_1|\log\varepsilon|} \mathrm{d}t\, \frac{F_0(t)}{f_0^2(t)} j_t[\psi(s,t)]\Big|_{s=s_j-L_\varepsilon}^{s=s_j+L_\varepsilon}, \tag{C.11}$$

and have to be added and subtracted to apply [24, Lemma 7], leading to (C.6). □

Note that in both the upper and lower bounds (C.5) and (C.6), we can easily replace the integral over $\mathcal{I}_{\mathrm{smooth}}$ with the integral over the whole boundary, since

$$\varepsilon E_{\mathrm{corr}} \int_{\mathcal{I}_{\mathrm{smooth}}} \mathrm{d}s\, k(s) = E_{\mathrm{corr}} \int_0^{|\partial\Omega|} \mathrm{d}s\, \mathfrak{K}(\mathrm{s}) + \mathcal{O}(\varepsilon|\log\varepsilon|), \tag{C.12}$$

by the boundedness of the curvature.

We conclude the Section with an important corollary of the above lower bound, which will be used to prove a uniform estimate of $|\psi^{\mathrm{GL}}|$ in the smooth part of the layer.

Lemma C.1 (Lower bound on the reduced energies) *Let u_n be defined in (C.7) and $\mathcal{E}_n$ be given by (C.9). Then, if $1 < b < \Theta_0^{-1}$, as $\varepsilon \to 0$,*

$$\sum_{n=1}^{M_\varepsilon} \mathcal{E}_n[u_n] \geqslant |\log\varepsilon|^{-4} \sum_{n=1}^{M_\varepsilon} \int_{\mathscr{C}_n} \mathrm{d}s\mathrm{d}t\, (1-\varepsilon k_n t) f_n^2 \left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2}|\partial_s u_n|^2\right]$$
$$+ \frac{1}{2b} \sum_{n=1}^{M_\varepsilon} \int_{\mathscr{C}_n} \mathrm{d}s\mathrm{d}t\, (1-\varepsilon k_n t) f_n^4 (1-|u_n|^2)^2 + o(1). \tag{C.13}$$

Proof See [24, Proof of Lemma 7]. □

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