

THE BORDIGA SURFACE AS CRITICAL LOCUS FOR 3-VIEW RECONSTRUCTIONS

MARINA BERTOLINI, ROBERTO NOTARI, AND CRISTINA TURRINI

ABSTRACT. In Computer Vision, images of dynamic or segmented scenes are modeled as linear projections from \mathbb{P}^k to \mathbb{P}^2 . The reconstruction problem consists in recovering the position of the projected objects and the projections themselves from their images, after identifying many enough correspondences between the images. A critical locus for the reconstruction problem is a variety in \mathbb{P}^k containing the objects for which the reconstruction fails. In this paper, we deal with projections both of points from $\mathbb{P}^4 \rightarrow \mathbb{P}^2$ and of lines from $\mathbb{P}^3 \rightarrow \mathbb{P}^2$. In both cases, we consider 3 projections, minimal number for a uniquely determined reconstruction. In the case of projections of points, we declinate the Grassmann tensors introduced in [21] in our context, and we use them to compute the equations of the critical locus. Then, given the ideal that defines this locus, we prove that, in the general case, it defines a Bordiga surface, or a scheme in the same irreducible component of the associated Hilbert scheme. Furthermore, we prove that every Bordiga surface is actually the critical locus for the reconstruction for suitable projections. In the case of projections of lines, we compute the defining ideal of the critical locus, that is the union of 3 α -planes and a line congruence of bi-degree $(3, 6)$ and sectional genus 5 in the Grassmannian $\mathbb{G}(1, 3) \subset \mathbb{P}^5$. This last surface is biregular to a Bordiga surface [40]. We use this fact to link the two reconstruction problems by showing how to compute the projections of one of the two settings, given the projections of the other one. The link is effective, in the sense that we describe an algorithm to compute the projection matrices.

1. INTRODUCTION

Linear projections from \mathbb{P}^3 to \mathbb{P}^2 are the natural geometric model for pictures of static three-dimensional scenes taken from pinhole cameras. Similarly linear projections from \mathbb{P}^k to \mathbb{P}^2 , with $k \geq 3$, arise when images of particular dynamic and segmented scenes are considered ([39, 21, 28, 38, 22]).

Given multiple images of an unknown scene, taken from unknown cameras, the *reconstruction* of the positions of cameras and scene points is a classical problem in Computer Vision, which has been generalized as well in the setting of higher dimensional projective spaces.

Sufficiently many images and sufficiently many sets of corresponding points in the given images should in principle allow for a successful projective reconstruction. Anyway, there exist sets of points, in the ambient space \mathbb{P}^k , for which the projective reconstruction fails. These configurations of points are called *critical*, which means that there exist other non projectively equivalent sets of points and cameras that give the same images in the view planes.

Critical loci for projections from \mathbb{P}^3 to \mathbb{P}^2 have been studied by many authors. Among the many papers on the subject, we recall [13, 27, 29, 19, 24, 20, 35, 2]. In the case of projections from higher dimensional \mathbb{P}^k to the projective plane \mathbb{P}^2 , when $k \geq 4$, critical loci were described in [11] in the case of one view, and in [7, 3, 6] in the case of multiple views.

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In this paper we focus on the case of three projections from \mathbb{P}^4 to \mathbb{P}^2 , that is the first non-classical case, since three views is the minimum number which allows us to reconstruct the scene, when the scene points are general (in a sense which will be clear later). Our purpose is to get a scheme-theoretical description of the critical locus. The critical locus comes out to be a classical surface in \mathbb{P}^4 , the so-called Bordiga surface ([12]). The approach used here to obtain the polynomials that generate the ideal of the critical locus is different from the one followed in [7]: we use the Grassmann tensor introduced in [21]. In [2], a first seminal case of this idea has been applied to two projections from \mathbb{P}^2 to \mathbb{P}^1 , while in [10] this approach is used to study the critical locus when it is a hypersurface. The construction of the critical locus given in [7] allowed the set-theoretical description of it, while the one given here through the Grassmann tensor allows us to compute the generators of the ideal of the critical locus and its first syzygy module, so giving a scheme-theoretical description of the critical locus itself. In more details, the ideal is minimally generated by 4 degree 3 forms that are the maximal minors of a 4×3 matrix with linear entries. From the Hilbert–Burch Theorem, it follows that the critical locus is a determinantal variety of codimension 2 and degree 6 in \mathbb{P}^4 , and so it belongs to the irreducible component of the Hilbert scheme containing Bordiga surfaces.

A very natural question arising as a consequence of the above results is whether every Bordiga surface in \mathbb{P}^4 is the critical locus of suitable projections. To give a positive answer to this question, we heavily use the geometry of Bordiga surfaces.

We recall that a Bordiga surface S is the blow-up of \mathbb{P}^2 at 10 general points, embedded in \mathbb{P}^4 via the complete linear system of the quartics through the 10 points. S contains exactly 10 lines, corresponding to the base points of the linear system. The ideal of the 10 points is determinantal, too. The 5×4 matrix of linear forms whose maximal minors are the generators of the ideal of the 10 points in \mathbb{P}^2 is strongly related to the matrix that presents the surface S in \mathbb{P}^4 . We use such a relation to prove that the generators of the first syzygy module of the ideal of S can be chosen in a very particular form. Finally, we use this particular presentation to prove that every Bordiga surface is the critical locus for suitable projections, that we explicitly construct.

Quite surprisingly, the Bordiga surface is linked to another classical problem in Computer Vision: reconstruction using lines. Given an unknown scene in \mathbb{P}^3 , consisting of a set of lines (i.e. a subset of the Grassmannian $\mathbb{G}(1, 3)$), and taken multiple images of this scene, in which corresponding lines in the different views are identified, the goal is the reconstruction of the scene. The reconstruction using lines is particularly significant in the real world, since images of lines can be more easily and accurately detected and tracked than points [30]. A classical result of Buchanan and Maybank shows that the critical locus for reconstruction in \mathbb{P}^3 using lines is essentially a line congruence of bidegree $(3, 6)$ and sectional genus 5 in $\mathbb{G}(1, 3)$. This congruence admits a biregular model in \mathbb{P}^4 which is a Bordiga surface [40]. Furthermore, Maybank proves that every Bordiga surface can be obtained in this way, i.e. as model in \mathbb{P}^4 of a congruence in $\mathbb{G}(1, 3)$ which arises as the critical locus for the reconstruction using lines. As a consequence of this, we have more deeply investigated, also from an algebraic point of view, the relations between the two critical loci. To do this we have computed the generators of the ideal of the critical congruence, obtaining its scheme-theoretical description, and proved that the whole critical locus in $\mathbb{G}(1, 3)$ is the union of the quoted congruence with three α -planes corresponding to the lines through the centers of projection.

Finally we have explicitly described how to link the projections from \mathbb{P}^3 to \mathbb{P}^2 used in the reconstruction using lines, with the ones from \mathbb{P}^4 to \mathbb{P}^2 used in the reconstruction using points, which give rise to biregular critical loci, and conversely. This provides a connection between the two reconstructions, a priori independent each other.

The plan of the paper is as follows. In sections 2 and 3 we recall the geometric constructions of the Bordiga surface and of the $(3, 6)$ line congruence respectively and, for

both of them, we give the algebraic translation of their constructions. Moreover in section 3 we present some results on the $(3, 6)$ line congruence we are not able to quote. Section 4 is devoted to recall the main facts on multiview geometry in higher dimension, in particular the description of the Grassmann tensor. In section 5 we prove that the generators of the ideal of the critical locus for 3 projections from \mathbb{P}^4 to \mathbb{P}^2 can be deduced via the appropriate Grassmann tensor and that this locus is actually a Bordiga surface. Moreover we prove also that the converse holds, i.e. every Bordiga surface S is the critical locus of suitable linear projections. In section 6 we provide the definition of critical locus for reconstruction using lines, we compute the defining ideal of such a critical locus and finally we show that the $(3, 6)$ congruence is the main component of this locus. Section 7 gives a bridge between the two above problems of criticality, by explicitly computing the equations of the maps between the two critical loci. From the maps we are able to compute the projections of one of the two settings given the projections of the other one. In section 8 we provide a Singular session in which we compute a degree 10 arithmetically Gorenstein curve contained in a given $(3, 6)$ line congruence. We need this curve to make effective the algorithm in section 7.

2. ON THE BORDIGA SURFACE

In this section, we recall the construction of the Bordiga surface and some properties of this surface we'll use later on in the paper. For more information on this surface, see [12],[16].

Let $\mathbb{P}^2 = \text{Proj}(S = \mathbb{C}[z_0, z_1, z_2])$ be the projective space of dimension 2 over the complex ground field \mathbb{C} , and let $p_1, \dots, p_{10} \in \mathbb{P}^2$ be ten general points. Let $\tilde{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 in p_1, \dots, p_{10} . The linear system $|4\pi^*L - E_1 - \dots - E_{10}|$, where L is the line divisor in \mathbb{P}^2 and E_i is the exceptional divisor associated to p_i , $i = 1, \dots, 10$, embeds $\tilde{\mathbb{P}}^2$ in $\mathbb{P}^4 = \text{Proj}(R = \mathbb{C}[y_0, \dots, y_4])$, and becomes the hyperplane divisor H of the image. If $\varphi : \tilde{\mathbb{P}}^2 \hookrightarrow \mathbb{P}^4$ is the embedding, $B = \varphi(\tilde{\mathbb{P}}^2)$ is named Bordiga surface. B is a smooth surface of degree 6 and sectional genus 3. φ maps each E_i onto a line, and moreover B contains exactly these 10 lines. It is possible to prove that these lines are pairwise skew and that no three of them are contained in a 3-dimensional linear space. The canonical divisor of B is $K_B = |-3\pi^*L + E_1 + \dots + E_{10}|$. By adjunction, it is possible to prove that the image of a plane cubic through 9 among p_1, \dots, p_{10} is a plane cubic in B , as well. Moreover, the plane containing it and the exceptional line associated to the 10th point are skew linear spaces. Finally, the image of a line containing no point among p_1, \dots, p_{10} is a rational normal quartic curve Γ contained in B .

Now, we give the algebraic translation of the previous geometric constructions.

Because of the generality assumption, $Z = \{p_1, \dots, p_{10}\}$ has Hilbert function $h_Z(j) = \min\{\binom{j+2}{2}, 10\}$ for $j \geq 0$, and so I_Z , defining ideal of Z in S , is generated by 5 forms g_0, \dots, g_4 of degree 4. As Z is a codimension 2 arithmetically Cohen–Macaulay (ACM, for short) closed subscheme of \mathbb{P}^2 , the ideal I_Z is determinantal and its minimal free resolution is

$$(1) \quad 0 \rightarrow S^4(-5) \xrightarrow{N_Z} S^5(-4) \rightarrow I_Z \rightarrow 0.$$

The forms g_0, \dots, g_4 are a basis of the complete linear system of quartic curves through p_1, \dots, p_{10} . If we define $\varphi : R \rightarrow S$ as $\varphi(y_i) = g_i, i = 0, \dots, 4$, then $I_B = \ker(\varphi)$ is the defining ideal of the Bordiga surface $B = \varphi(\tilde{\mathbb{P}}^2)$. I_B is determinantal too, and its minimal free resolution is

$$(2) \quad 0 \rightarrow R^3(-4) \xrightarrow{N_B} R^4(-3) \rightarrow I_B \rightarrow 0.$$

This proves that B is a codimension 2, ACM closed scheme in \mathbb{P}^4 with Hilbert polynomial $p_B(t) = (6t^2 + 2t + 2)/2$. From [18], it follows that

Proposition 2.1. *The Bordiga surface is the general element of the irreducible component of the Hilbert scheme $\mathcal{Hilb}_{p_B(t)}(\mathbb{P}^4)$ containing the codimension 2, ACM closed subschemes of \mathbb{P}^4 with Hilbert polynomial $p_B(t)$.*

From the point of view of liaison theory, the Bordiga surface is linked to a degree 3 rational normal scroll. In fact, let S be the surface complete intersection two general cubic hypersurfaces containing B . Then, by standard argument from liaison theory, the defining ideal of the residual surface B' has the following minimal free resolution

$$0 \rightarrow R^2(-3) \rightarrow R^3(-2) \rightarrow I_{B'} \rightarrow 0$$

and so B' is a degree 3 rational normal scroll, as claimed. Moreover, the intersection between B and B' is an arithmetically Gorenstein curve of degree 8 and socle degree 3, as its minimal free resolution is

$$0 \rightarrow R(-6) \rightarrow \begin{array}{c} R^2(-3) \\ \oplus \\ R^3(-4) \end{array} \rightarrow \begin{array}{c} R^3(-2) \\ \oplus \\ R^2(-3) \end{array} \rightarrow I_{B \cap B'} \rightarrow 0,$$

computed by mapping cone from the short exact sequence

$$0 \rightarrow I_S \rightarrow I_B \oplus I_{B'} \rightarrow I_{B \cap B'} \rightarrow 0.$$

Such a curve is the image in B of a curve in the linear system $|7\pi^*L - 2E_1 - \dots - 2E_{10}| = |-K_B + H|$ and so the curve is a twisted anticanonical divisor on B (see, [25] for generalities on such divisors). Such a linear system will play a role in next sections.

By adjunction ([32], pp.48–49), one can prove that the matrices N_Z and N_B are related each other by

$$(3) \quad N_B \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = N_Z^T \begin{pmatrix} y_0 \\ \vdots \\ y_4 \end{pmatrix}.$$

If $p \in \mathbb{P}^2 \setminus Z$, then $\text{rank}(N_Z \otimes \mathbb{C}(p)) = 4$ as (1) remains exact after tensorization by $\mathbb{C}(p)$, residue field of p . So p is mapped to the point in \mathbb{P}^4 whose homogeneous coordinates solve the linear system

$$(4) \quad (N_Z \otimes \mathbb{C}(p))^T \begin{pmatrix} y_0 \\ \vdots \\ y_4 \end{pmatrix} = \mathbf{0}.$$

If $p \in Z$, then $\text{rank}(N_Z \otimes \mathbb{C}(p)) = 3$, and so the solutions of (4) are the points of the exceptional line E_p . Moreover, if $p \in Z$, then $I_Z : I_p$ has a unique generator in degree 3, the defining form f_C of the plane cubic curve C through $Z \setminus \{p\}$. For every line $L \ni p$, we have that $lf_C \in \langle g_0, \dots, g_4 \rangle$, with $l \in \mathbb{C}[z_0, z_1, z_2]_1$ defining L , and so $lf_c = a_0(l)g_0 + \dots + a_4(l)g_4$ for suitable $a_i(l) \in \mathbb{C}$. Then, the hyperplane $a_0(l)y_0 + \dots + a_4(l)y_4 = 0$ contains $\varphi(C)$. As there are two linearly independent lines in \mathbb{P}^2 containing p , we have that $\varphi(C)$ is contained in a dimension 2 linear space, as well. Hence, $\varphi(C)$ is a plane cubic, because I_B is generated in degree 3. Now we want to explain why the above plane containing $\varphi(C)$ is skew with E_p . In fact, the solutions of (4) define the sub-linear system of plane quartics through p_1, \dots, p_{10} , singular at p . As the quartics defined by lf_C are smooth at p , then they are not in the sub-linear system above, and so the intersection of the two linear spaces is empty.

Now, if $L \subset \mathbb{P}^2$ is a general line, we can assume that $L \cap Z = \emptyset$. Let $p, q \in L$ be two distinct points and let $M_L = \begin{pmatrix} z_{0p} & z_{0q} \\ z_{1p} & z_{1q} \\ z_{2p} & z_{2q} \end{pmatrix}$ be the rank 2 matrix whose columns are homogeneous coordinates of the points p, q . Then $N_B M_L$ is a 4×2 matrix of generic rank

2 of linear forms in R , and so its maximal minors define a rational normal curve Γ of degree 4 contained in B .

3. ON THE LINE CONGRUENCE OF BI-DEGREE (3, 6) AND SECTIONAL GENUS 5

In this section, we recall the construction and some properties of the line congruence K of bi-degree (3, 6) and sectional genus 5 in the Grassmannian $\mathbb{G}(1, 3)$ of lines in \mathbb{P}^3 , following [40], [1]. Furthermore, we present three properties of K for which we are not able to quote a reference.

As in section 2, let $\widetilde{\mathbb{P}}^2$ be the blow-up of \mathbb{P}^2 at 10 general points p_1, \dots, p_{10} . The linear system $|7\pi^*L - 2E_1 - \dots - 2E_{10}|$ on $\widetilde{\mathbb{P}}^2$ embeds $\widetilde{\mathbb{P}}^2$ in $\mathbb{G}(1, 3) \subset \mathbb{P}^5 = \text{Proj}(T = \mathbb{C}[x_0, \dots, x_5])$. For a suitable choice of the coordinates, $\mathbb{G}(1, 3) = V(x_0x_5 - x_1x_4 + x_2x_3)$. Let $\psi : \widetilde{\mathbb{P}}^2 \rightarrow \mathbb{G}(1, 3)$ be such an embedding, and let $K = \psi(\widetilde{\mathbb{P}}^2)$ be its image. Then, K is a line congruence of bi-degree (3, 6) and sectional genus 5. The canonical divisor is $K_K = |-3\pi^*L + E_1 + \dots + E_{10}|$ while the hyperplane divisor is $H = |7\pi^*L - 2E_1 - \dots - 2E_{10}|$. Moreover, the converse holds true, that is to say, every such a surface is the embedding of $\widetilde{\mathbb{P}}^2$ with $|7\pi^*L - 2E_1 - \dots - 2E_{10}|$. In $\mathbb{G}(1, 3)$, the ideal sheaf $\mathcal{I}_{K|\mathbb{G}(1,3)}$ is generated in degree 3 and it holds

$$(5) \quad 0 \rightarrow \mathcal{O}_{\mathbb{G}(1,3)}^5 \rightarrow \mathcal{E}_2(1)^3 \rightarrow \mathcal{I}_{K|\mathbb{G}(1,3)}(3) \rightarrow 0$$

where \mathcal{E}_2 is the rank 2 vector bundle on $\mathbb{G}(1, 3)$ coming from the universal exact sequence

$$0 \rightarrow \mathcal{E}_1 \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{\mathbb{G}(1,3)} \rightarrow \mathcal{E}_2(1) \rightarrow 0.$$

The first result we are not able to quote concerns the shape of the minimal free resolution of the defining ideal I_K of K in T .

Proposition 3.1. *The minimal free resolution of I_K has the shape*

$$(6) \quad 0 \rightarrow T^5(-5) \rightarrow T^{12}(-4) \rightarrow \begin{array}{c} T(-2) \\ \oplus \\ T^7(-3) \end{array} \rightarrow I_K \rightarrow 0.$$

Proof. From (5), we get the resolution of $I_{K|\mathbb{G}(1,3)}$ over $T_{\mathbb{G}} = T/\langle x_0x_5 - x_1x_4 + x_2x_3 \rangle$, that is periodic, and it is

$$\dots \rightarrow T_{\mathbb{G}}^{12}(-5) \rightarrow T_{\mathbb{G}}^{12}(-4) \rightarrow T_{\mathbb{G}}^7(-3) \rightarrow I_{K|\mathbb{G}(1,3)} \rightarrow 0$$

as $H^0(\mathcal{E}_2(1)) = H^0(\mathcal{O}_{\mathbb{P}^3}(1))$. By [34], the minimal free resolution of I_K has the shape

$$0 \rightarrow T^5(-5) \rightarrow T^{12}(-4) \rightarrow \begin{array}{c} T(-2) \\ \oplus \\ T^7(-3) \end{array} \rightarrow I_K \rightarrow 0$$

as claimed, where the degree 2 generator of I_K is the defining form of $\mathbb{G}(1, 3)$. \square

In particular, K is ACM with Hilbert polynomial $p_K(t) = (9t^2 + t + 2)/2$. A result similar to Proposition 2.1 holds.

Proposition 3.2. *The line congruence of bi-degree (3, 6) and sectional genus 5 is the general element of the irreducible component of the Hilbert scheme $\text{Hilb}_{p_K(t)}(\mathbb{P}^5)$ containing the codimension 3, ACM closed subschemes of \mathbb{P}^5 with Hilbert polynomial $p_K(t)$ and resolution (6).*

Now, we go back to the geometric description of K .

As $\widetilde{\mathbb{P}}^2$ is embedded both in \mathbb{P}^4 as a Bordiga surface B , and in $\mathbb{G}(1, 3)$ as a suitable line congruence K , it is natural to look for maps from \mathbb{P}^4 to $\mathbb{G}(1, 3)$ that transform B into K .

In [40], it is proved that the map θ that makes commutative the diagram

$$(7) \quad \begin{array}{ccc} \tilde{\mathbb{P}}^2 & \xrightarrow{\varphi} & B \subset \mathbb{P}^4 \\ \downarrow \psi & \swarrow \theta & \\ K \subset \mathbb{G}(1, 3) & & \end{array}$$

is associated to the linear system $|\mathcal{O}_{\mathbb{P}^4}(2) \setminus \Gamma|$ where $\Gamma \subset B$ is a degree 4 rational normal curve in \mathbb{P}^4 , and so Γ is the image in B of a line in \mathbb{P}^2 . The map θ contracts lines that are bi-secant to Γ and so θ is an embedding of $\mathbb{P}^4 \setminus F$ where F is the secant variety to Γ , a degree 3 hypersurface. Moreover, θ maps F to a Veronese surface V .

Now, we can prove the second result we are not able to quote.

Theorem 3.1. *The line congruence K of bi-degree $(3, 6)$ and sectional genus 5 in $\mathbb{G}(1, 3)$ is Gorenstein linked to a Veronese surface V by an arithmetically Gorenstein surface S of degree 13 and socle degree 4.*

Proof. The map θ^{-1} is defined on $K \setminus C$ where $C = K \cap V$ and V is the contraction of the secant variety to Γ where θ is not an embedding. The curve C is in the linear system $|10\pi^*L - 3E_1 - \dots - 3E_{10}| = |-K_K + H|$ and so C is a twisted anticanonical divisor. By ([25], Lemma 5.4), C is arithmetically Gorenstein in \mathbb{P}^5 , and its minimal free resolution can be easily constructed by self-duality, because the minimal free resolution of I_V is a subcomplex of the resolution of I_C . Then, we have

$$0 \rightarrow T(-7) \rightarrow \begin{array}{ccc} T^3(-4) & T^8(-3) & T^6(-2) \\ \oplus & \rightarrow \oplus & \rightarrow \oplus \\ T^6(-5) & T^8(-4) & T^3(-3) \end{array} \rightarrow I_C \rightarrow 0.$$

Let $S = K \cup V$. Then, we have the short exact sequence

$$0 \rightarrow I_S \rightarrow I_K \oplus I_V \rightarrow I_C \rightarrow 0$$

from which we get the shape of the minimal free resolution of I_S :

$$(8) \quad 0 \rightarrow T(-7) \rightarrow \begin{array}{ccc} T^4(-4) & T(-2) \\ \oplus & \rightarrow \oplus \\ T(-5) & T^4(-3) \end{array} \rightarrow I_S \rightarrow 0$$

and so we get that S is arithmetically Gorenstein of degree 13 and socle degree 4, as claimed. Moreover, K is algebraically linked to V by S and the linkage is geometric, because both K and V are irreducible varieties, and so they have no irreducible component in common.

Conversely, let $V \subset \mathbb{G}(1, 3)$ be a Veronese surface, and let S be an arithmetically Gorenstein surface of degree 13 and socle degree 4 containing V and contained in the Grassmannian $\mathbb{G}(1, 3)$. Then, the Hilbert function of the Artinian reduction of S is $(1, 3, 5, 3, 1)$, and so its minimal free resolution is (8). Hence, the residual variety K has minimal free resolution (6) and so K is a line congruence of bi-degree $(3, 6)$ and sectional genus 5, as claimed. \square

Remark 3.1. As the shape of the resolution of I_C allows cancellations, the general element of $|-K_K + H|$ is not the intersection of K with a Veronese surface.

Now, we describe an algorithm, based on [31], to compute such a surface S , given a general line congruence K . It can be easily adapted to start from V . In last section 8, we implement it in Singular ([15]), as this explicit computation is needed in section 7, which is the algorithmic part of the paper.

Given a reduced Gröbner basis of I_K with respect to the degree reverse lexicographic ordering (degrevlex, for short) in T , the initial ideal $\text{in}(I_K)$ is generated by $\langle x_2x_3, x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0^2x_2, x_0x_1x_2, x_1^2x_2 \rangle$. With “general” line congruence, we mean that the generators of $\text{in}(I_K)$ are as large as possible w.r.t. the degrevlex ordering and the Hilbert function

of I_K . Then, we write polynomials p_1, \dots, p_5 whose coefficients are indeterminates, and whose initial terms are $x_0^3, x_0^2x_1, x_0x_1^2, x_0^2x_2, x_1^4$, respectively. Also in this case, we choose the largest monomial ideal compatible with the Hilbert function of S . We want to compute the indeterminate coefficients in such a way that $\{x_0x_5 - x_1x_4 + x_2x_3, p_1, p_2, p_3, p_4, p_5\}$ is a reduced Gröbner basis of I_S . At first, we reduce p_1, \dots, p_5 w.r.t. the Gröbner basis of I_K and we set equal to 0 their normal forms. This gives a linear system in the indeterminate coefficients. We compute the solutions of this system, and we substitute them in the polynomials p_1, \dots, p_5 . Then, we run the Buchberger algorithm on the input $\{x_0x_5 - x_1x_4 + x_2x_3, p_1, p_2, p_3, p_4, p_5\}$ and we set equal to zero their S-polynomials. We get a new set of equations, no more linear. It is possible to check, however, that the equations allow us to express all but two of the remaining variables as rational functions of the last two ones. More precisely, the two special variables are the coefficients of $x_1^2x_2, x_0x_1x_2$ in p_4 . For a random choice of them, we compute the polynomials p_1, \dots, p_5 and we get the arithmetically Gorenstein surface S we are looking for. As a by-product, we have that such surfaces form a dimension 2 family, whose parameter space contains an open subset of an affine plane \mathbb{A}^2 . This confirms that for every line in \mathbb{P}^2 , containing no point among p_1, \dots, p_{10} , we can construct a rational normal curve Γ in the Bordiga surface $B \subset \mathbb{P}^4$ and a curve C contained in $K \subset \mathbb{G}(1, 3)$ where the map θ is not an embedding.

As last result, we describe a geometric property of K .

We proved in section 2 that a plane cubic curve containing 9 of the 10 points in Z is mapped onto a plane cubic curve in B , as well. Now we prove that these plane curves are mapped by θ to plane curves again. Take, for example, an element in $|3\pi^*L - E_2 - \dots - E_{10}|$. The degree of its image in K is 3 and its genus, computed via adjunction, turns out to be 1. So it is a plane curve in K , too. The plane containing this curve is an α -plane in $\mathbb{G}(1, 3)$. To compute the equations of the α -plane, one can proceed as follows. Let $C_i \subset \mathbb{P}^2$ be the cubic curve not containing p_i , $i = 1, \dots, 10$, and let ℓ_{ij} be the line through p_i, p_j , with $i \neq j$. Then, $\ell_{ij}C_iC_j$ is a degree 7 plane curve singular at p_1, \dots, p_{10} , and so it is a linear combination of a basis of the linear system. These linear combinations give the equations of the hyperplanes in $\mathbb{G}(1, 3) \subset \mathbb{P}^5$ containing the image of C_i . If we select p_1, p_2, p_3 as distinguished points among p_1, \dots, p_{10} , and we set $\alpha_1, \alpha_2, \alpha_3$ the planes containing the images of C_1, C_2, C_3 , respectively, then each one of them intersects the other two ones, as the intersection point is image of the further intersection point of C_i and C_j other than the base points.

Now, we can prove the last result we are not able to quote.

Proposition 3.3. *With the previous notations, let Y be the union of the line congruence K and the three planes $\alpha_1, \alpha_2, \alpha_3$. Then, Y is an ACM surface of bi-degree $(6, 6)$ in $\mathbb{G}(1, 3)$ and the minimal free resolution of its defining ideal is*

$$(9) \quad 0 \rightarrow T^3(-6) \rightarrow \begin{array}{ccc} T^3(-4) & & T(-2) \\ & \oplus & \\ T^4(-5) & & T^4(-3) \end{array} \rightarrow I_Y \rightarrow 0.$$

Proof. To prove the result, we use mainly mapping cone procedure. The starting point is the following short exact sequence

$$0 \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2)} \rightarrow I_{K \cap \alpha_1} \oplus I_{K \cap \alpha_2} \rightarrow I_{K \cap \alpha_1 \cap \alpha_2} \rightarrow 0.$$

As $K \cap \alpha_i$ is a degree 3 plane curve, the minimal free resolution of $I_{K \cap \alpha_i}$ is Koszul, with 3 generators in degree 1 and a further generator in degree 3. On the other hand, $K \cap \alpha_1 \cap \alpha_2$ is a point, and so also the minimal free resolution of $I_{K \cap \alpha_1 \cap \alpha_2}$ is Koszul with 5 generators in degree 1. Furthermore, $K \cap (\alpha_1 \cup \alpha_2)$ is the union of two plane cubic curves meeting at a point, and so there are two minimal generators in degree 3.

Then, by mapping cone on the previous sequence, we get

$$\begin{array}{ccccccc}
& & & & T(-1) & & \\
& & & & \oplus & & \\
0 \rightarrow & T(-5) & \rightarrow & T^5(-4) & \rightarrow & T^8(-3) & \rightarrow T^4(-2) \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2)} \rightarrow 0. \\
& \oplus & & \oplus & & \oplus & \\
& T^2(-6) & & T^6(-5) & & T^6(-4) & \\
& & & & & \oplus & \\
& & & & & T^2(-3) &
\end{array}$$

Before going on, we construct the minimal free resolution of $I_{\alpha_1 \cup \alpha_2 \cup \alpha_3}$. We choose coordinates in \mathbb{P}^5 in such a way that $I_{\alpha_1} = \langle x_0, x_1, x_2 \rangle$, $I_{\alpha_2} = \langle x_0, x_3, x_4 \rangle$, $I_{\alpha_3} = \langle x_1, x_3, x_5 \rangle$, so to let them intersect pairwise. Hence, $I_{\alpha_1 \cup \alpha_2 \cup \alpha_3} = \langle x_0 x_1, x_0 x_3, x_0 x_5, x_1 x_3, x_1, x_4, x_2 x_3, x_2 x_4 x_5 \rangle$, and its minimal free resolution has the shape

$$\begin{array}{ccccccc}
& & & T^3(-4) & \rightarrow & T^8(-3) & \rightarrow & T^6(-2) \\
0 \rightarrow & T(-6) & \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow & \oplus \rightarrow I_{\alpha_1 \cup \alpha_2 \cup \alpha_3} \rightarrow 0. \\
& & & T^3(-5) & & T^3(-4) & & T(-3)
\end{array}$$

Now, we consider the short exact sequence

$$0 \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3)} \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2)} \oplus I_{K \cap \alpha_3} \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2) \cap \alpha_3} \rightarrow 0.$$

When we apply the mapping cone procedure, we consider that, by degree argument, the minimal free resolution of $I_{\alpha_1 \cup \alpha_2 \cup \alpha_3}$ is a sub-complex of the resolution of $I_{K \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3)}$, and that $K \cap (\alpha_1 \cup \alpha_2) \cap \alpha_3$ is a set of 2 points, and so its minimal free resolution is Koszul, too, with 4 generators in degree 1 and a generator in degree 2. Then, we get

$$\begin{array}{ccccccc}
& & & T^3(-4) & \rightarrow & T^8(-3) & \rightarrow & T^6(-2) \\
0 \rightarrow & T^4(-6) & \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow & \oplus \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3)} \rightarrow 0. \\
& & & T^{12}(-5) & & T^{12}(-4) & & T^4(-3)
\end{array}$$

Finally, from the short exact sequence

$$0 \rightarrow I_Y \rightarrow I_K \oplus I_{\alpha_1 \cup \alpha_2 \cup \alpha_3} \rightarrow I_{K \cap (\alpha_1 \cup \alpha_2 \cup \alpha_3)} \rightarrow 0$$

we get the result on the minimal free resolution of I_Y . From the minimal free resolution, it follows that Y is ACM. \square

4. ON MULTIVIEW GEOMETRY FOR PROJECTIONS FROM \mathbb{P}^4 TO \mathbb{P}^2

In this section we fix notation and terminology and give a short overview of classical facts in Computer Vision related to the problem of projective reconstruction of scenes and cameras from multiple views.

Even if all the definitions can be given in full generality, we restrict ourselves to the case of projections from \mathbb{P}^4 to \mathbb{P}^2 . In this context, a *camera* P is a linear projection from \mathbb{P}^4 onto \mathbb{P}^2 , from a line C_P , called *center of projection*. The target space \mathbb{P}^2 is called *view*. A *scene* is a set of points $\mathbf{X}_i \in \mathbb{P}^4$.

Using homogeneous coordinates in \mathbb{P}^4 and \mathbb{P}^2 , we identify P with a 3×5 matrix of maximal rank, defined up to a multiplicative constant. Hence C_P comes out to be the right annihilator of P .

Let us consider a set of cameras $P_j : \mathbb{P}^4 \setminus C_{P_j} \rightarrow \mathbb{P}^2$ projecting the same scene in \mathbb{P}^4 and the corresponding set of images in the different target planes. In this setting, proper linear subspaces (points or lines), $L_i, i = 1 \dots m$, of different views are said to be *corresponding* if there exists at least a point $\mathbf{X} \in \mathbb{P}^4$ such that $P_i(\mathbf{X}) \in L_i$ for all $i = 1 \dots m$.

In the context of multiple view geometry, the problem of *projective reconstruction* of a scene, given multiple images of it, is the following: given many enough scene points in \mathbb{P}^4 and identified a suitable number of *corresponding subspaces* on each image, one wants to get the projection matrices (up to projective transformations), i.e. the cameras, and the coordinates in \mathbb{P}^4 of the scene points.

Hartley and Schaffalitzky, [21], have constructed a set of multiview tensors, called *Grassmann tensors*, encoding the relations between sets of corresponding subspaces. We recall

here the basic elements of their construction in the case of three projections from \mathbb{P}^4 to \mathbb{P}^2 , which turns out to be the minimal number of views allowing projective reconstruction.

Consider three projections $P_j : \mathbb{P}^4 \setminus C_j \rightarrow \mathbb{P}^2$, $j = 1, 2, 3$, with centers C_1, C_2, C_3 in general position.

Let $\{L_1, L_2, L_3\}$ be three general linear subspaces (points or lines) of \mathbb{P}^2 of codimension $\alpha_1, \alpha_2, \alpha_3$, respectively. (L_1, L_2, L_3) is a triple of corresponding subspaces if and only if $(P_1)^{-1}(L_1) \cap (P_2)^{-1}(L_2) \cap (P_3)^{-1}(L_3)$ is not empty. From the Grassmann formula, if $\sum \alpha_j = 5$, $j = 1, 2, 3$, the existence of points in the previous intersection gives a constrain which allows us to construct the Grassmann tensor. Hartley and Schaffalitzky call the triple $(\alpha_1, \alpha_2, \alpha_3)$ a *profile* for the reconstruction problem. In our situation the only possible profiles are: $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1), (2, 1, 2)$ or $(1, 2, 2)$.

Let $\{L_1, L_2, L_3\}$ be three general linear subspaces of \mathbb{P}^2 as above and let S_j be the maximal rank matrix of type $3 \times (3 - \alpha_j)$ whose columns are a basis for L_j , $j = 1, 2, 3$. By definition, if the L_j 's are corresponding subspaces, there exists a point $\mathbf{X} \in \mathbb{P}^k$ such that $P_j(\mathbf{X}) \in L_j$ for $j = 1, 2, 3$. In other words there exist three vectors $\mathbf{v}_j \in \mathbb{C}^{3-\alpha_j}$ $j = 1, 2, 3$ such that:

$$(10) \quad \begin{pmatrix} S_1 & 0 & 0 & P_1 \\ 0 & S_2 & 0 & P_2 \\ 0 & 0 & S_3 & P_3 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{X} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The existence of a non-trivial solution $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{X})$ of system (10) implies that the coefficient matrix has determinant zero, as it is square of order 9 in our case. This determinant can be thought of as a tri-linear form (tensor) in the Plücker coordinates of the spaces L_j . This tensor is the *Grassmann tensor*.

In the following we explicitly construct such a tensor for the profile $(2, 2, 1)$, others being similar. Now L_1, L_2 are points and L_3 is a line. We denote by $(x_1, x_2, x_3), (y_1, y_2, y_3)$ the homogeneous coordinates of L_1 and L_2 , respectively, and by (z_1, z_2, z_3) and (w_1, w_2, w_3) the homogeneous coordinates of two points of L_3 . In this case the matrix of the coefficients of the linear system (10) becomes:

$$T_{L_1, L_2, L_3}^{P_1, P_2, P_3} = \begin{pmatrix} x_1 & 0 & 0 & 0 & P_1[1, 1] & P_1[1, 2] & P_1[1, 3] & P_1[1, 4] & P_1[1, 5] \\ x_2 & 0 & 0 & 0 & P_1[2, 1] & P_1[2, 2] & P_1[2, 3] & P_1[2, 4] & P_1[2, 5] \\ x_3 & 0 & 0 & 0 & P_1[3, 1] & P_1[3, 2] & P_1[3, 3] & P_1[3, 4] & P_1[3, 5] \\ 0 & y_1 & 0 & 0 & P_2[1, 1] & P_2[1, 2] & P_2[1, 3] & P_2[1, 4] & P_2[1, 5] \\ 0 & y_2 & 0 & 0 & P_2[2, 1] & P_2[2, 2] & P_2[2, 3] & P_2[2, 4] & P_2[2, 5] \\ 0 & y_3 & 0 & 0 & P_2[3, 1] & P_2[3, 2] & P_2[3, 3] & P_2[3, 4] & P_2[3, 5] \\ 0 & 0 & z_1 & w_1 & P_3[1, 1] & P_3[1, 2] & P_3[1, 3] & P_3[1, 4] & P_3[1, 5] \\ 0 & 0 & z_2 & w_2 & P_3[2, 1] & P_3[2, 2] & P_3[2, 3] & P_3[2, 4] & P_3[2, 5] \\ 0 & 0 & z_3 & w_3 & P_3[3, 1] & P_3[3, 2] & P_3[3, 3] & P_3[3, 4] & P_3[3, 5] \end{pmatrix}$$

where $P_i[j, k]$ denotes the element in position (j, k) of the matrix P_i .

If L_1, L_2 and L_3 are corresponding spaces then the linear system

$$(11) \quad T_{L_1, L_2, L_3}^{P_1, P_2, P_3} \begin{pmatrix} \lambda \\ \mu \\ \alpha \\ \beta \\ \mathbf{X} \end{pmatrix} = \mathbf{0}$$

has a non trivial solution, and so $\det(T_{L_1, L_2, L_3}^{P_1, P_2, P_3}) = 0$.

The converse is true for general L_1, L_2 and L_3 since we are looking for a non trivial solution of (11) in which \mathbf{X} is a point of \mathbb{P}^4 (i.e. $\mathbf{X} \neq \mathbf{0}$) and $\mathbf{X} \notin C_1 \cup C_2 \cup C_3$.

In particular this happens if $L_1 \notin P_1(C_2 \cup C_3)$ and $L_2 \notin P_2(C_1 \cup C_3)$. Under this hypothesis, if $\det(T_{L_1, L_2, L_3}^{P_1, P_2, P_3}) = 0$, then L_1, L_2 and L_3 are corresponding as the linear system (11) has a non trivial solution with \mathbf{X} as required. Indeed the case $\mathbf{X} = \mathbf{0}$ doesn't occur, since, in the opposite case, either L_1 or L_2 are not points, or L_3 is not a line. Moreover $\mathbf{X} \notin C_1 \cup C_2 \cup C_3$, for the assumption.

In conclusion, for the chosen profile $(2, 2, 1)$, one sees that $\det(T_{L_1, L_2, L_3}^{P_1, P_2, P_3}) = 0$ is indeed the tri-linear constraint between the coordinates \mathbf{x} and \mathbf{y} of the first and second view and the Plücker (i.e.dual) coordinates of the line $\langle \mathbf{z}, \mathbf{w} \rangle$ of the third view so to let them be corresponding.

5. CRITICAL LOCI FOR PROJECTIVE RECONSTRUCTION OF POINTS AND THEIR GEOMETRY

As discussed in the previous section, folklore on the reconstruction problem says that sufficiently many views and sufficiently many sets of corresponding points in the given views should allow one a successful projective reconstruction. This is generally true, but even in the classical set-up of two projections from \mathbb{P}^3 to \mathbb{P}^2 one can have non projectively equivalent pairs of sets of scene points and of cameras that produce the same images in the view planes, thus preventing reconstruction. Such configurations and the loci they describe are referred to as *critical*. In [7], critical loci for projective reconstruction of camera centers and scene points from multiple views for projections from \mathbb{P}^k to \mathbb{P}^2 have been introduced and studied. Here we shortly recall the basic definitions in the case in which we are interested in, i.e. three views from \mathbb{P}^4 to \mathbb{P}^2 . Moreover we study the critical locus \mathcal{X} for general projections. The non general case is under investigation, and some preliminary results are summarized in [9].

Definition 5.1. A set of points $\{\mathbf{X}_j\}$, $j = 1, \dots, N$, $N \gg 0$, in \mathbb{P}^4 is said to be a *critical configuration for projective reconstruction from three views* if there exist two collections of 3×5 full-rank projection matrices P_i and Q_i , $i = 1, 2, 3$, and a set of N points $\{\mathbf{Y}_j\} \subset \mathbb{P}^4$, non-projectively equivalent to $\{\mathbf{X}_j\}$, such that, for all i and j , $P_i(\mathbf{X}_j) = Q_i(\mathbf{Y}_j)$, up to homography in the image planes. The two sets $\{\mathbf{X}_j\}$ and $\{\mathbf{Y}_j\}$ are called *conjugate critical configurations*, with *associated conjugate matrices* $\{P_i\}$ and $\{Q_i\}$.

Remark 5.1. It can be proved that in the above definition $N \geq 7$ is enough.

The generators of the ideal of the critical locus \mathcal{X} can be obtained by making use of the Grassmann tensor introduced in the previous section.

Indeed, the Grassmann tensor $\mathcal{T}^{P_1, P_2, P_3}$ encodes the algebraic relations between corresponding subspaces in the different views of the projections P_1, P_2, P_3 . Hence by definition of critical set, if $\{\mathbf{X}_j, \mathbf{Y}_j\}$ are conjugate critical configurations, then, for each j , the projections $P_1(\mathbf{X}_j), P_2(\mathbf{X}_j)$ and $P_3(\mathbf{X}_j)$ are corresponding points not only for the projections P_1, P_2, P_3 , but for the projections Q_1, Q_2, Q_3 , too.

Following the construction of the previous section, we first choose the profile $(2, 2, 1)$.

If L'_1, L'_2, L'_3 is a triple of corresponding spaces for the projections Q_1, Q_2, Q_3 , where L'_1, L'_2 are points with homogeneous coordinates $L'_1 = (x'_1, x'_2, x'_3)^T$, $L'_2 = (y'_1, y'_2, y'_3)^T$ in the first two views respectively, and L'_3 is the line spanned by $\mathbf{z} = (z'_1, z'_2, z'_3)$ and $\mathbf{w} = (w'_1, w'_2, w'_3)$ in the third view, the trilinear relation between L'_1, L'_2, L'_3 is given by the vanishing of $\det(T_{L'_1, L'_2, L'_3}^{Q_1, Q_2, Q_3})$, where

$$T_{L'_1, L'_2, L'_3}^{Q_1, Q_2, Q_3} = \begin{pmatrix} L'_1 & 0 & 0 & 0 & Q_1 \\ 0 & L'_2 & 0 & 0 & Q_2 \\ 0 & 0 & \mathbf{z} & \mathbf{w} & Q_3 \end{pmatrix}.$$

Considering as corresponding spaces $L'_1 = P_1(\mathbf{X})$, $L'_2 = P_2(\mathbf{X})$ and any line L'_3 passing through $P_3(\mathbf{X})$, with \mathbf{X} any point in the critical locus, one gets that the determinant of

the matrix

$$M' = \begin{pmatrix} P_1(\mathbf{X}) & 0 & 0 & 0 & Q_1 \\ 0 & P_2(\mathbf{X}) & 0 & 0 & Q_2 \\ 0 & 0 & P_3(\mathbf{X}) & \mathbf{v} & Q_3 \end{pmatrix}$$

must vanish for every choice of $\mathbf{v} = (a, b, c)^T$.

This implies that all the cofactors of the elements a, b, c in M' must vanish.

Since the above reasoning holds for all the possible profiles $(2, 2, 1)$, $(2, 1, 2)$ or $(1, 2, 2)$, one sees that \mathbf{X} is in the critical locus if and only if one has the vanishing of all the maximal minors of

$$M = \begin{pmatrix} P_1(\mathbf{X}) & 0 & 0 & Q_1 \\ 0 & P_2(\mathbf{X}) & 0 & Q_2 \\ 0 & 0 & P_3(\mathbf{X}) & Q_3 \end{pmatrix}.$$

From the above discussion it follows that the maximal minors of M generate the ideal of the critical locus \mathcal{X} , as \mathbf{X} has to satisfy no other constraint.

Remark 5.2. In [7], the critical locus for a suitable number n of projections from \mathbb{P}^k to \mathbb{P}^2 has been introduced and studied in a more general context. If k is even, the critical locus is a determinantal variety, defined by $3n$ polynomials of degree n . In the particular case of three projections from \mathbb{P}^4 to \mathbb{P}^2 , up to a constant, these polynomials are exactly the maximal minors of M defined above.

Now we study the geometry of the critical locus \mathcal{X} in a general situation, where general means that the greatest common divisor of the generators of the ideal is trivial.

Since the generators have a trivial greatest common divisor, by Hilbert–Burch Theorem ([17], Theorem 20.15), a free resolution of the ideal $I_{\mathcal{X}}$ they generate is

$$(12) \quad 0 \rightarrow \begin{matrix} R^3(-4) \\ \oplus \\ R^5(-3) \end{matrix} \xrightarrow{M} R^9(-3) \rightarrow I_{\mathcal{X}} \rightarrow 0$$

where M is the above matrix.

By generality assumption, we can assume that $\text{rank} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = 5$. Hence, we can cancel the summand $R^5(-3)$ from the resolution, and we get the minimal free resolution

$$(13) \quad 0 \rightarrow R^3(-4) \xrightarrow{N_{\mathcal{X}}} R^4(-3) \rightarrow I_{\mathcal{X}} \rightarrow 0.$$

Let us assume that a non-vanishing minor is given by the last 5 rows and columns in M . Then if we write M as the following block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A of type 4×3 , and D invertible, we can reduce M with elementary operations on rows and columns to the block matrix

$$\begin{pmatrix} N_{\mathcal{X}} & 0 \\ 0 & I_5 \end{pmatrix}$$

where I_5 is the 5×5 identity matrix, 0 are null matrices of suitable type, and

$$N_{\mathcal{X}} = A - BD^{-1}C.$$

Because of the way it is obtained, $N_{\mathcal{X}}$ has linear entries. As the elementary operations on the columns correspond to base changes in the free module $R^3(-4) \oplus R^5(-3)$ while the ones on the rows correspond to base changes in the free module $R^9(-3)$, the global effect is to construct a new set of 4 generators of the ideal $I_{\mathcal{X}}$ again as maximal minors of $N_{\mathcal{X}}$. Then, the critical locus \mathcal{X} is a codimension 2 arithmetically Cohen–Macaulay scheme in

\mathbb{P}^4 with Hilbert polynomial $p(t) = (6t^2 + 2t + 2)/2 = p_B(t)$. From Proposition 2.1, we have the following

Proposition 5.1. *The general critical locus for projective reconstruction for three views from \mathbb{P}^4 to \mathbb{P}^2 is in the irreducible component of $\mathcal{Hilb}_{p_B(t)}(\mathbb{P}^4)$ containing the Bordiga surfaces.*

It is then natural to characterize the locus filled by critical loci inside this irreducible component. Quite surprisingly, this locus coincides with the whole irreducible component. In fact, we have

Theorem 5.1. *Let B be a Bordiga surface. Then B is a critical locus for the projective reconstruction from three views from \mathbb{P}^4 to \mathbb{P}^2 , that is to say, there exist two couples of three projections P_1, P_2, P_3 and Q_1, Q_2, Q_3 from \mathbb{P}^4 to \mathbb{P}^2 such that the associated critical locus is B .*

Proof. Let $Z \subset \mathbb{P}^2$ be a set of 10 points in uniform position and let B be the associated Bordiga surface. B is irreducible, of course. We choose a reference frame in \mathbb{P}^2 in such a way that the fundamental points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ are in Z . For this choice, from equation (3), we have that the columns of N_B in resolution (2) define three lines in \mathbb{P}^4 that lie on the Bordiga surface B . We take these lines as centers of three projections P_1, P_2, P_3 .

By performing elementary operations on the rows of $N_B = (n_{ij})$, we can assume that $n_{41} = 0$ and n_{11}, n_{21}, n_{31} are linearly independent.

Now, we want to prove that $n_{42} \neq 0$. Assume by contradiction that $n_{42} = 0$. Then, with easy computations, we have that I_B is generated by

$$n_{43} \det(N_{\widehat{14;3}}), n_{43} \det(N_{\widehat{24;3}}), n_{43} \det(N_{\widehat{34;3}}), \det(N_{\widehat{4}}).$$

So,

$$I_B = \langle n_{43}, \det(N_{\widehat{4}}) \rangle \cap \langle \det(N_{\widehat{14;3}}), \det(N_{\widehat{24;3}}), \det(N_{\widehat{34;3}}) \rangle$$

where $\widehat{ij; k}$ denotes the cancellation of the rows i, j and of the column k from N_B , while \widehat{i} denotes the cancellation of the i -th row, and hence B is not irreducible. In more details, $\langle n_{43}, \det(N_{\widehat{4}}) \rangle$ defines a cubic surface in a \mathbb{P}^3 while $\langle \det(N_{\widehat{14;3}}), \det(N_{\widehat{24;3}}), \det(N_{\widehat{34;3}}) \rangle$ defines a cone in \mathbb{P}^4 over a twisted cubic curve in a suitable \mathbb{P}^3 . Finally, the two surfaces meet along a twisted cubic curve. Then, $n_{42} \neq 0$.

Analogously we get $n_{43} \neq 0$.

Performing other elementary operations on the rows of N_B , we can get $n_{32} = 0$. In analogy to the previous case, $n_{33} \neq 0$ otherwise B would be non irreducible, as only an element in the third row of N_B is non-zero. Let $\mathbf{p}_3^1 \mathbf{X}, \mathbf{p}_3^2 \mathbf{X}, \mathbf{p}_3^3 \mathbf{X}$ be a basis of the ideal generated by $n_{i3}, i = 1, \dots, 4$. Moreover, let E be the 4×3 matrix such that

$$\begin{pmatrix} n_{13} \\ n_{23} \\ n_{33} \\ n_{43} \end{pmatrix} = E \begin{pmatrix} \mathbf{p}_3^1 \mathbf{X} \\ \mathbf{p}_3^2 \mathbf{X} \\ \mathbf{p}_3^3 \mathbf{X} \end{pmatrix}.$$

As $n_{33} \neq 0$ and $n_{43} \neq 0$, the last two rows of E are non-zero, as well.

Then, if we set

$$A = \begin{pmatrix} n_{11} & 0 & 0 \\ n_{21} & 0 & 0 \\ n_{31} & 0 & 0 \\ 0 & n_{42} & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & n_{12} & 0 \\ 0 & n_{22} & 0 \\ 0 & 0 & \mathbf{p}_3^1 \mathbf{X} \\ 0 & 0 & \mathbf{p}_3^2 \mathbf{X} \\ 0 & 0 & \mathbf{p}_3^3 \mathbf{X} \end{pmatrix}$$

we have that $N_B = A - FC$ for

$$F = \left(\begin{array}{cc|c} -1 & 0 & \\ 0 & -1 & \\ 0 & 0 & -E \\ 0 & 0 & \end{array} \right).$$

Let D be any invertible matrix of order 5 and take $B = FD$. Then, the maximal minor of the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

generate the same ideal I_B that defines the Bordiga surface B from which we started.

The matrices $P_1\mathbf{X}, P_2\mathbf{X}, P_3\mathbf{X}$ are the three matrices deduced by taking the rows of $\begin{pmatrix} A \\ C \end{pmatrix}$ three by three.

Let Q_1, Q_2, Q_3 the three matrices obtained by taking the rows of $\begin{pmatrix} B \\ D \end{pmatrix}$ three by three.

We want to check that Q_i has maximal rank 3. This is obvious for Q_3 because the rows of D are linearly independent. Q_1 has rank three too, because the first three rows of F are linearly independent and D is invertible. The last row of B is the last row of F times D , and so it is a linear combination of the last three rows of D with non-zero coefficients. So, it is not a linear combination of the first two rows of D , again because D has rank 5. Hence, Q_2 has maximal rank 3, too, and the proof is complete. \square

6. CRITICAL LOCI FOR PROJECTIVE RECONSTRUCTION OF LINES AND THEIR GEOMETRY

In the previous sections, we have dealt with the reconstruction problem for sets of points from their images via three projections from \mathbb{P}^4 to \mathbb{P}^2 . Moreover we have studied the critical loci for this problem.

A different reconstruction problem arises when one considers projections of lines in \mathbb{P}^3 instead of projections of points. This set-up has been considered and studied by various authors, in particular T.Buchanan [14] and S.J.Maybank [30]. Given a set of lines in \mathbb{P}^3 and n projections of these lines to \mathbb{P}^2 , T.Buchanan [14] shows that $n = 3$ is the minimum n such that it is possible to reconstruct the set from their images, up to projective transformations in \mathbb{P}^3 . Of course, also in this context, there is a natural notion of critical locus, consisting of lines in \mathbb{P}^3 .

Definition 6.1. A set of lines $\{\lambda_j\}$, $j = 1, \dots, N$, $N \gg 0$, in \mathbb{P}^3 is said to be a *critical configuration for projective reconstruction of lines from three views* if there exist two collections of 3×4 full-rank projection matrices φ_i and ψ_i , $i = 1, 2, 3$, and a set of N lines $\{\mu_j\}$ in \mathbb{P}^3 , non-projectively equivalent to $\{\lambda_j\}$, such that, for all i and j , $\varphi_i(\lambda_j) = \psi_i(\mu_j)$, up to homography in the (dual) image planes. The two sets $\{\lambda_j\}$ and $\{\mu_j\}$ are called *conjugate critical configurations*, with *associated conjugate matrices* $\{\varphi_i\}$ and $\{\psi_i\}$.

In [14], the author, using geometrical arguments, shows that given two triples of projections from \mathbb{P}^3 to \mathbb{P}^2 , the associated critical set for reconstruction via lines is a congruence K of bi-degree $(3, 6)$ and sectional genus 5 in the Grassmannian $\mathbb{G}(1, 3)$ (see section 3).

In this section, via an algebraic approach, we compute the defining ideal of the critical locus for this reconstruction problem. We prove that the critical locus is the union of the above line congruence and of the three α -planes in $\mathbb{G}(1, 3)$ corresponding to the three centers of projection in \mathbb{P}^3 .

To start, we denote by $\varphi_i : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ and $\psi_i : \mathbb{P}^3 \rightarrow \mathbb{P}^2$, $i = 1, 2, 3$, two triples of projections and we consider the associated critical locus for the reconstruction problem through lines considered by T.Buchanan [14]. Let $O_i \in \mathbb{P}^3$ (respectively, $O'_i \in \mathbb{P}^3$) be the

center of the projection φ_i (resp. ψ_i). As in the following we are interested in the rows of the matrices ψ_1, ψ_2, ψ_3 , we set

$$(14) \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} R_1 \\ \vdots \\ R_9 \end{pmatrix}$$

where R_i is a 1×4 matrix. As we consider general projections, we assume that the above matrix has rank 4.

As proved in [23], if we call l a line in \mathbb{P}^3 and l' its image in \mathbb{P}^2 via a projection $\varphi = (\varphi_{ij})$, the Plücker coordinates of l are transformed into the Plücker coordinates of l' via

$$\varphi \wedge \varphi = \begin{pmatrix} \left| \begin{array}{cc} \varphi_{21} & \varphi_{22} \\ \varphi_{31} & \varphi_{32} \end{array} \right| & \left| \begin{array}{cc} \varphi_{21} & \varphi_{23} \\ \varphi_{31} & \varphi_{33} \end{array} \right| & \left| \begin{array}{cc} \varphi_{22} & \varphi_{23} \\ \varphi_{32} & \varphi_{33} \end{array} \right| & \left| \begin{array}{cc} \varphi_{21} & \varphi_{24} \\ \varphi_{31} & \varphi_{34} \end{array} \right| & \left| \begin{array}{cc} \varphi_{22} & \varphi_{24} \\ \varphi_{32} & \varphi_{34} \end{array} \right| & \left| \begin{array}{cc} \varphi_{23} & \varphi_{24} \\ \varphi_{33} & \varphi_{34} \end{array} \right| \\ \left| \begin{array}{cc} \varphi_{31} & \varphi_{32} \\ \varphi_{11} & \varphi_{12} \end{array} \right| & \left| \begin{array}{cc} \varphi_{31} & \varphi_{33} \\ \varphi_{11} & \varphi_{13} \end{array} \right| & \left| \begin{array}{cc} \varphi_{32} & \varphi_{33} \\ \varphi_{12} & \varphi_{13} \end{array} \right| & \left| \begin{array}{cc} \varphi_{31} & \varphi_{34} \\ \varphi_{11} & \varphi_{14} \end{array} \right| & \left| \begin{array}{cc} \varphi_{32} & \varphi_{34} \\ \varphi_{12} & \varphi_{14} \end{array} \right| & \left| \begin{array}{cc} \varphi_{33} & \varphi_{34} \\ \varphi_{13} & \varphi_{14} \end{array} \right| \\ \left| \begin{array}{cc} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{array} \right| & \left| \begin{array}{cc} \varphi_{11} & \varphi_{13} \\ \varphi_{21} & \varphi_{23} \end{array} \right| & \left| \begin{array}{cc} \varphi_{12} & \varphi_{13} \\ \varphi_{22} & \varphi_{23} \end{array} \right| & \left| \begin{array}{cc} \varphi_{11} & \varphi_{14} \\ \varphi_{21} & \varphi_{24} \end{array} \right| & \left| \begin{array}{cc} \varphi_{12} & \varphi_{14} \\ \varphi_{22} & \varphi_{24} \end{array} \right| & \left| \begin{array}{cc} \varphi_{13} & \varphi_{14} \\ \varphi_{23} & \varphi_{24} \end{array} \right| \end{pmatrix}$$

Hence, we denote $\Phi_i = \varphi_i \wedge \varphi_i : \mathbb{P}^5 \rightarrow \mathbb{P}^2$ and similarly $\Psi_i = \psi_i \wedge \psi_i$ both for $i = 1, 2, 3$.

Remark 6.1. The center of the projection Φ_i is the 2-plane in \mathbb{P}^5 defined by $\Phi_i(\mathbf{X}) = \mathbf{0}$. They are contained in the Grassmannian $\mathbb{G}(1, 3)$ and parameterize the lines of \mathbb{P}^3 through O_i , for every i , i.e. they are α -planes.

In a natural way, then, we are induced to consider the reconstruction problem for points in \mathbb{P}^5 from three views, which turns out to be the minimum number needed for the reconstruction of a scene. Let $\mathbf{X} = (x_0, \dots, x_5)^T$ be a point in \mathbb{P}^5 . As done in section 5 while computing the critical locus for two triples of projections from \mathbb{P}^4 to \mathbb{P}^2 , one has to consider the 9×9 matrix

$$M = \begin{pmatrix} \Phi_1(\mathbf{X}) & \mathbf{0} & \mathbf{0} & \Psi_1 \\ \mathbf{0} & \Phi_2(\mathbf{X}) & \mathbf{0} & \Psi_2 \\ \mathbf{0} & \mathbf{0} & \Phi_3(\mathbf{X}) & \Psi_3 \end{pmatrix}$$

where $\mathbf{0}$ is the 3×1 null matrix.

Remark 6.2. In the case of two triples of projections from \mathbb{P}^5 to \mathbb{P}^2 , $\det(M) = 0$ is the defining equation of the hypersurface that is critical for the reconstruction for points in this setting. Indeed, $\det(M) = 0$ is equivalent to require that the linear system

$$M \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mathbf{Y} \end{pmatrix} = \mathbf{0}$$

has a non-trivial solution such that $\mathbf{Y} \neq \mathbf{0}$.

The condition $\det(M) = 0$ is not enough in our case to describe the critical locus since both \mathbf{X} and \mathbf{Y} are lines and so their coordinates must satisfy the equation of Klein hyperquadric. For \mathbf{X} we have $x_0x_5 - x_1x_4 + x_2x_3 = 0$. The situation for \mathbf{Y} is more involved. Indeed we have first to compute the coordinates of \mathbf{Y} by solving suitable homogeneous linear systems, then to impose that such coordinates satisfy the Klein equation.

If we denote by M_i the matrix obtained from M by erasing the i -th row, the homogeneous linear system becomes

$$M_i \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mathbf{Y} \end{pmatrix} = \mathbf{0} \quad 1 \leq i \leq 9.$$

It follows from Cramer's rule that

$$y_j = (-1)^{1+j} \det(M_{i, \widehat{j+4}}), \quad j = 0 \dots 5,$$

where $M_{\widehat{h,k}}$ is the matrix obtained from M by erasing the h -th row and the k -th column. Now we have to impose that the computed $y_0 \dots y_5$ satisfy $y_0 y_5 - y_1 y_4 + y_2 y_3 = 0$.

To simplify notations, we set

$$(15) \quad \Phi_i(\mathbf{X}) = \begin{pmatrix} \Phi_{i1} \\ \Phi_{i2} \\ \Phi_{i3} \end{pmatrix}$$

that are the natural variables to construct the generators of the ideal of the critical locus \mathcal{Y} . In fact, while $\det(M)$ is a degree 3 form in the Φ_{ij} 's, irreducible in the general case, the form $y_0 y_5 - y_1 y_4 + y_2 y_3$ has degree 6 and it is the product of three linear forms and a cubic form in the Φ_{ij} 's. To fix notations we denote by g_i the cubic arising from the linear system associated to M_i . For example, let us consider the linear system associated to M_1 . In this case, the linear forms are

$$\begin{aligned} \Phi_{21} \det \begin{pmatrix} R_7 \\ R_8 \\ R_9 \\ R_4 \end{pmatrix} + \Phi_{22} \det \begin{pmatrix} R_7 \\ R_8 \\ R_9 \\ R_5 \end{pmatrix} + \Phi_{23} \det \begin{pmatrix} R_7 \\ R_8 \\ R_9 \\ R_6 \end{pmatrix} &= 0, \\ \Phi_{31} \det \begin{pmatrix} R_4 \\ R_5 \\ R_6 \\ R_7 \end{pmatrix} + \Phi_{32} \det \begin{pmatrix} R_4 \\ R_5 \\ R_6 \\ R_8 \end{pmatrix} + \Phi_{33} \det \begin{pmatrix} R_4 \\ R_5 \\ R_6 \\ R_9 \end{pmatrix} &= 0, \\ \Phi_{12} b - \Phi_{13} a &= 0, \end{aligned}$$

where

$$a = (\Psi_1)_{21} b_{12} - (\Psi_1)_{22} b_{13} + (\Psi_1)_{23} b_{23} + (\Psi_1)_{24} b_{14} - (\Psi_1)_{25} b_{24} + (\Psi_1)_{26} b_{34},$$

$$b = (\Psi_1)_{31} b_{12} - (\Psi_1)_{32} b_{13} + (\Psi_1)_{33} b_{23} + (\Psi_1)_{34} b_{14} - (\Psi_1)_{35} b_{24} + (\Psi_1)_{36} b_{34}$$

and (b_{12}, \dots, b_{34}) are the Plücker coordinates of the line through the points

$$(\det((\psi_2)_{\hat{1}}), \det((\psi_2)_{\hat{2}}), \det((\psi_2)_{\hat{3}}), \det((\psi_2)_{\hat{4}}))$$

and

$$(\det((\psi_3)_{\hat{1}}), \det((\psi_3)_{\hat{2}}), \det((\psi_3)_{\hat{3}}), \det((\psi_3)_{\hat{4}})),$$

where, this time, \hat{h} means that we erase the h -th column.

We have then proved the following:

Theorem 6.1. *The defining ideal $I_{\mathcal{Y}}$ of the critical locus is generated by the Klein hyperquadric, $\det(M)$ and $g_1 \dots, g_9$.*

Now we compute the minimal generators and free resolution of the ideal $I_{\mathcal{Y}}$. It is straightforward to check that

$$g_i = \sum_{j,h,k}^{1,2,3} \Phi_{1j} \Phi_{2h} \Phi_{3k} \det \begin{pmatrix} R_i \\ R_j \\ R_{3+h} \\ R_{6+k} \end{pmatrix} \quad i = 1, \dots, 9$$

while

$$\det(M) = - \sum_{i,j,h}^{1,2,3} \Phi_{1i} \Phi_{2j} \Phi_{3h} c_{i,3+j,6+h}$$

where

$$\Phi_{ijh} = \begin{pmatrix} R_i \\ R_j \\ R_h \end{pmatrix} \text{ and}$$

$$c_{i,j,h} = \det \begin{pmatrix} \det((\Phi_1)_{\hat{1}}) & \det((\Phi_1)_{\hat{2}}) & \det((\Phi_1)_{\hat{3}}) & \det((\Phi_1)_{\hat{4}}) \\ \det((\Phi_2)_{\hat{1}}) & \det((\Phi_2)_{\hat{2}}) & \det((\Phi_2)_{\hat{3}}) & \det((\Phi_2)_{\hat{4}}) \\ \det((\Phi_3)_{\hat{1}}) & \det((\Phi_3)_{\hat{2}}) & \det((\Phi_3)_{\hat{3}}) & \det((\Phi_3)_{\hat{4}}) \\ \det((\Phi_{ijh})_{\hat{1}}) & \det((\Phi_{ijh})_{\hat{2}}) & \det((\Phi_{ijh})_{\hat{3}}) & \det((\Phi_{ijh})_{\hat{4}}) \end{pmatrix}.$$

Let J be the ideal generated by $g_1, \dots, g_9, \det(M)$. Then,

Proposition 6.1. *The minimal free resolution of J is*

$$0 \rightarrow T^3(-4) \rightarrow T^4(-3) \rightarrow J \rightarrow 0$$

and so the scheme defined by J is ACM of codimension 2 in \mathbb{P}^5 .

Proof. Without loss of generality, we assume that R_1, R_2, R_4, R_7 are linearly independent, and so they are a basis of \mathbb{C}^4 . Moreover, we set

$$R_i = \alpha_{i1}R_1 + \alpha_{i2}R_2 + \alpha_{i3}R_4 + \alpha_{i4}R_7 \quad i = 3, 5, 6, 8, 9$$

and

$$D = \det \begin{pmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \end{pmatrix}.$$

We remark that D is the coefficient of $\Phi_{12}\Phi_{21}\Phi_{31}$ in g_1 .

First, we verify that J is generated by g_1, \dots, g_4 . In fact, it holds

$$\begin{aligned} g_5 &= -\frac{\alpha_{31}\alpha_{54} - \alpha_{34}\alpha_{51}}{\alpha_{34}}g_1 - \frac{\alpha_{32}\alpha_{54} - \alpha_{34}\alpha_{52}}{\alpha_{34}}g_2 + \frac{\alpha_{54}}{\alpha_{34}}g_3 - \frac{\alpha_{33}\alpha_{54} - \alpha_{34}\alpha_{53}}{\alpha_{34}}g_4 \\ g_6 &= -\frac{\alpha_{31}\alpha_{64} - \alpha_{34}\alpha_{61}}{\alpha_{34}}g_1 - \frac{\alpha_{32}\alpha_{64} - \alpha_{34}\alpha_{62}}{\alpha_{34}}g_2 + \frac{\alpha_{64}}{\alpha_{34}}g_3 - \frac{\alpha_{33}\alpha_{64} - \alpha_{34}\alpha_{63}}{\alpha_{34}}g_4 \\ g_7 &= -\frac{\alpha_{31}}{\alpha_{34}}g_1 - \frac{\alpha_{32}}{\alpha_{34}}g_2 + \frac{1}{\alpha_{34}}g_3 - \frac{\alpha_{33}}{\alpha_{34}}g_4 \\ g_8 &= -\frac{\alpha_{31}\alpha_{84} - \alpha_{34}\alpha_{81}}{\alpha_{34}}g_1 - \frac{\alpha_{32}\alpha_{84} - \alpha_{34}\alpha_{82}}{\alpha_{34}}g_2 + \frac{\alpha_{84}}{\alpha_{34}}g_3 - \frac{\alpha_{33}\alpha_{84} - \alpha_{34}\alpha_{83}}{\alpha_{34}}g_4 \\ g_9 &= -\frac{\alpha_{31}\alpha_{94} - \alpha_{34}\alpha_{91}}{\alpha_{34}}g_1 - \frac{\alpha_{32}\alpha_{94} - \alpha_{34}\alpha_{92}}{\alpha_{34}}g_2 + \frac{\alpha_{94}}{\alpha_{34}}g_3 - \frac{\alpha_{33}\alpha_{94} - \alpha_{34}\alpha_{93}}{\alpha_{34}}g_4 \\ \det(M) &= \frac{1}{\alpha_{34}D}c_{2,3,4}g_1 - \frac{1}{\alpha_{34}D}c_{1,3,4}g_2 + \frac{1}{\alpha_{34}D}c_{1,2,4}g_3. \end{aligned}$$

The equalities have to be checked by verifying that the coefficients of $\Phi_{1i}\Phi_{2j}\Phi_{3h}$ are the same on both sides. It is very long, but straightforward. As example, we check some of them.

The first term we consider is $\Phi_{13}\Phi_{23}\Phi_{31}$ in the g_5 relation. The coefficient on the left side is equal to

$$\det \begin{pmatrix} R_5 \\ R_3 \\ R_6 \\ R_7 \end{pmatrix} = \det \begin{pmatrix} \alpha_{51}R_1 + \alpha_{52}R_2 + \alpha_{53}R_4 \\ \alpha_{31}R_1 + \alpha_{32}R_2 + \alpha_{33}R_4 \\ \alpha_{61}R_1 + \alpha_{62}R_2 + \alpha_{63}R_4 \\ R_7 \end{pmatrix} = D \det \begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{61} & \alpha_{62} & \alpha_{63} \end{pmatrix}.$$

On the right, we have

$$\begin{aligned}
& -\frac{\alpha_{31}\alpha_{54} - \alpha_{34}\alpha_{51}}{\alpha_{34}} \det \begin{pmatrix} R_1 \\ R_3 \\ R_6 \\ R_7 \end{pmatrix} - \frac{\alpha_{32}\alpha_{54} - \alpha_{34}\alpha_{52}}{\alpha_{34}} \det \begin{pmatrix} R_2 \\ R_3 \\ R_6 \\ R_7 \end{pmatrix} + \frac{\alpha_{54}}{\alpha_{34}} \det \begin{pmatrix} R_3 \\ R_6 \\ R_7 \end{pmatrix} - \\
& -\frac{\alpha_{33}\alpha_{54} - \alpha_{34}\alpha_{53}}{\alpha_{34}} \det \begin{pmatrix} R_4 \\ R_3 \\ R_6 \\ R_7 \end{pmatrix} = -\frac{\alpha_{31}\alpha_{54} - \alpha_{34}\alpha_{51}}{\alpha_{34}} \det \begin{pmatrix} R_1 \\ \alpha_{32}R_2 + \alpha_{33}R_4 \\ \alpha_{62}R_2 + \alpha_{63}R_4 \\ R_7 \end{pmatrix} - \\
& -\frac{\alpha_{32}\alpha_{54} - \alpha_{34}\alpha_{52}}{\alpha_{34}} \det \begin{pmatrix} R_2 \\ \alpha_{31}R_1 + \alpha_{33}R_4 \\ \alpha_{61}R_1 + \alpha_{63}R_4 \\ R_7 \end{pmatrix} - \frac{\alpha_{33}\alpha_{54} - \alpha_{34}\alpha_{53}}{\alpha_{34}} \det \begin{pmatrix} R_4 \\ \alpha_{31}R_1 + \alpha_{32}R_2 \\ \alpha_{61}R_1 + \alpha_{62}R_2 \\ R_7 \end{pmatrix} = \\
& = D \left\{ -\frac{\alpha_{54}}{\alpha_{34}} \left(\alpha_{31} \det \begin{pmatrix} \alpha_{32} & \alpha_{33} \\ \alpha_{62} & \alpha_{63} \end{pmatrix} - \alpha_{32} \det \begin{pmatrix} \alpha_{31} & \alpha_{33} \\ \alpha_{61} & \alpha_{63} \end{pmatrix} + \alpha_{33} \det \begin{pmatrix} \alpha_{31} & \alpha_{32} \\ \alpha_{61} & \alpha_{62} \end{pmatrix} \right) + \right. \\
& \left. + \left(\alpha_{51} \det \begin{pmatrix} \alpha_{32} & \alpha_{33} \\ \alpha_{62} & \alpha_{63} \end{pmatrix} - \alpha_{52} \det \begin{pmatrix} \alpha_{31} & \alpha_{33} \\ \alpha_{61} & \alpha_{63} \end{pmatrix} + \alpha_{53} \det \begin{pmatrix} \alpha_{31} & \alpha_{32} \\ \alpha_{61} & \alpha_{62} \end{pmatrix} \right) \right\} = \\
& = D \left\{ -\frac{\alpha_{54}}{\alpha_{34}} \det \begin{pmatrix} \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{61} & \alpha_{62} & \alpha_{63} \end{pmatrix} + \det \begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{61} & \alpha_{62} & \alpha_{63} \end{pmatrix} \right\}
\end{aligned}$$

and so the equality holds.

Now, we check the equality of the coefficients of $\Phi_{11}\Phi_{21}\Phi_{31}$ in the $\det(M)$ relation. On the left side, we have $-c_{147}$. On the right side, we have

$$\begin{aligned}
& + \frac{1}{\alpha_{34}D} c_{234} \det \begin{pmatrix} R_1 \\ R_1 \\ R_4 \\ R_7 \end{pmatrix} - \frac{1}{\alpha_{34}D} c_{134} \det \begin{pmatrix} R_2 \\ R_1 \\ R_4 \\ R_7 \end{pmatrix} + \frac{1}{\alpha_{34}D} \det \begin{pmatrix} R_3 \\ R_1 \\ R_4 \\ R_7 \end{pmatrix} = \\
& = \frac{1}{\alpha_{34}} c_{134} + \frac{1}{\alpha_{34}D} \det \begin{pmatrix} \alpha_{32}R_2 \\ R_1 \\ R_4 \\ R_7 \end{pmatrix} = \frac{1}{\alpha_{34}} \det \begin{pmatrix} \det((\Phi_1)_{\hat{1}}) & \dots \\ \det((\Phi_2)_{\hat{1}}) & \dots \\ \det((\Phi_3)_{\hat{1}}) & \dots \\ \det((\Phi_{134} - \alpha_{32}\Phi_{124})_{\hat{1}}) & \dots \end{pmatrix} = \\
& = \frac{1}{\alpha_{34}} \det \begin{pmatrix} \det((\Phi_1)_{\hat{1}}) & \dots \\ \det((\Phi_2)_{\hat{1}}) & \dots \\ \det((\Phi_3)_{\hat{1}}) & \dots \\ \det((\alpha_{34}\Phi_{174})_{\hat{1}}) & \dots \end{pmatrix} = -c_{147}
\end{aligned}$$

and so the equality holds also in this case.

As second step, we check that the minors with sign of the following matrix N are equal to g_1, \dots, g_4 .

$$\begin{pmatrix} \frac{D}{\alpha_{34}}\Phi_{11} & - \begin{vmatrix} \alpha_{31} & \alpha_{34} \\ \alpha_{51} & \alpha_{54} \end{vmatrix} \Phi_{22} - \begin{vmatrix} \alpha_{31} & \alpha_{34} \\ \alpha_{61} & \alpha_{64} \end{vmatrix} \Phi_{23} & -\alpha_{31}\Phi_{31} - \begin{vmatrix} \alpha_{31} & \alpha_{34} \\ \alpha_{81} & \alpha_{84} \end{vmatrix} \Phi_{32} - \begin{vmatrix} \alpha_{31} & \alpha_{34} \\ \alpha_{91} & \alpha_{94} \end{vmatrix} \Phi_{33} \\ \frac{D}{\alpha_{34}}\Phi_{12} & - \begin{vmatrix} \alpha_{32} & \alpha_{34} \\ \alpha_{52} & \alpha_{54} \end{vmatrix} \Phi_{22} - \begin{vmatrix} \alpha_{32} & \alpha_{34} \\ \alpha_{62} & \alpha_{64} \end{vmatrix} \Phi_{23} & -\alpha_{32}\Phi_{31} - \begin{vmatrix} \alpha_{32} & \alpha_{34} \\ \alpha_{82} & \alpha_{84} \end{vmatrix} \Phi_{32} - \begin{vmatrix} \alpha_{32} & \alpha_{34} \\ \alpha_{92} & \alpha_{94} \end{vmatrix} \Phi_{33} \\ \frac{D}{\alpha_{34}}\Phi_{13} & \alpha_{54}\Phi_{22} + \alpha_{64}\Phi_{23} & \Phi_{31} + \alpha_{84}\Phi_{32} + \alpha_{94}\Phi_{33} \\ 0 & \alpha_{34}\Phi_{21} - \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{53} & \alpha_{54} \end{vmatrix} \Phi_{22} - \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{63} & \alpha_{64} \end{vmatrix} \Phi_{23} & -\alpha_{33}\Phi_{31} - \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{83} & \alpha_{84} \end{vmatrix} \Phi_{32} - \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{93} & \alpha_{94} \end{vmatrix} \Phi_{33} \end{pmatrix}.$$

Once again, the check is straightforward. For example, let us consider the minor obtained by erasing the last row of N and let us compute the coefficient of $\Phi_{11}\Phi_{22}\Phi_{33}$ in the minor. It holds

$$\frac{D}{\alpha_{34}} \left\{ -\alpha_{94} \begin{vmatrix} \alpha_{32} & \alpha_{34} \\ \alpha_{52} & \alpha_{54} \end{vmatrix} + \alpha_{54} \begin{vmatrix} \alpha_{32} & \alpha_{34} \\ \alpha_{92} & \alpha_{94} \end{vmatrix} \right\} = \frac{D}{\alpha_{34}} \alpha_{34} \begin{vmatrix} \alpha_{52} & \alpha_{54} \\ \alpha_{92} & \alpha_{94} \end{vmatrix} = \det \begin{pmatrix} R_4 \\ R_1 \\ R_5 \\ R_9 \end{pmatrix}$$

and so we get $\det(N_4) = g_4$. With analogous computations, we get $\det(N_1) = -g_1$, $\det(N_2) = g_2$, and $\det(N_3) = -g_3$. As g_1, \dots, g_4 are irreducible for general choices, we get the claim by Hilbert–Burch Theorem. Moreover, N represents the map $T^3(-4) \rightarrow T^4(-3)$ in the resolution of J . \square

Remark 6.3. The scheme defined by J in \mathbb{P}^5 is a lifting of a Bordiga surface in \mathbb{P}^4 in the following sense. Let ℓ be a general linear form in T . Hence there is an isomorphism between R and T/ℓ . As J is ACM the minimal free resolution of J remains exact over T/ℓ and so $(J+\ell)/\ell$ defines a Bordiga surface in $\mathbb{P}^4 = \text{Proj}(R)$. Moreover, the matrices N and $N_{\mathcal{X}}$ of the two minimal free resolutions have the same properties: the three columns span linear spaces of codimension 3, in position $(4, 1)$ there is a zero, and the element in position $(3, 2)$ is a linear combination of the ones in positions $(1, 2)$ and $(2, 2)$. The main difference is that the linear spaces spanned by the columns are pairwise skew in the case of the Bordiga surface, while every couple of them has a point in common in the case of J , as the codimension 2 linear spaces are α -planes in this case, and so the intersection of two of them is the point corresponding to the line through the centers of the two nets of lines.

Now we can describe the geometry of the critical locus, starting from its defining ideal, and Buchanan’s result.

Theorem 6.2. *The critical locus for the reconstruction problem for a pair of three projections φ_i, ψ_j , $i, j = 1, 2, 3$, of lines from \mathbb{P}^3 to \mathbb{P}^2 is the union of a line congruence of bi-degree $(3, 6)$ and sectional genus 5 and the three α -planes associated to the projection centers of the φ_i ’s. Moreover, each α -plane intersects the line congruence along a degree 3 plane curve.*

Proof. The scheme defined by J is irreducible for general choices, and so the Klein hyperquadric is regular with respect to it. As the defining ideal of the critical locus \mathcal{Y} is

$I_{\mathcal{Y}} = J + (x_0x_5 - x_1x_4 + x_2x_3)T$, then its minimal free resolution is

$$0 \rightarrow T^3(-6) \rightarrow \begin{array}{c} T^4(-5) \\ \oplus \\ T^3(-4) \end{array} \rightarrow \begin{array}{c} T(-2) \\ \oplus \\ T^4(-3) \end{array} \rightarrow I_{\mathcal{Y}} \rightarrow 0$$

and so the Hilbert polynomial of \mathcal{Y} is $p_{\mathcal{Y}}(t) = (12t^2 - 8t + 8)/2$. Hence, \mathcal{Y} is a surface in \mathbb{P}^5 of degree 12. It is an easy check to verify that the centers $\alpha_1, \alpha_2, \alpha_3$ of the projections Φ_1, Φ_2, Φ_3 are the α -planes associated to the centers of $\varphi_1, \varphi_2, \varphi_3$ respectively and are contained in \mathcal{Y} , as every maximal minor of N is a combination of $\Phi_{i1}, \dots, \Phi_{i3}$, for $i = 1, 2, 3$. Moreover, we know that a line congruence K of bi-degree $(3, 6)$ and sectional genus 5 is contained in the critical locus. Then, by degree argument, $\mathcal{Y} = K \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$. By using the same argument as in Proposition 3.3, we get that each α_i intersects K along a degree 3 plane curve, as claimed. \square

7. A BRIDGE BETWEEN THE TWO RECONSTRUCTION PROBLEMS

In this section, we want to show that the two reconstruction problems considered in the previous sections are related each other, in the following sense. Given two triples of projections from \mathbb{P}^4 to \mathbb{P}^2 , and the corresponding critical locus $\mathcal{X} \subset \mathbb{P}^4$, it is possible to determine two triples of projections from \mathbb{P}^3 to \mathbb{P}^2 in such a way that the critical locus for the reconstruction problem for lines is the union of three suitable α -planes and the image of \mathcal{X} in $\mathbb{G}(1, 3)$ via the rational map $\theta : \mathbb{P}^4 \rightarrow \mathbb{G}(1, 3)$ quoted in (7). Furthermore, also the converse holds. Now we describe various steps to get the φ_i and ψ_j from the P_i and Q_j .

- (1) Let $P_i, Q_j : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ be projections, with $i, j = 1, 2, 3$. The critical locus $\mathcal{X} \subset \mathbb{P}^4$ for the reconstruction problem for points has been studied in section 5. In particular, we proved that \mathcal{X} is a Bordiga surface, and we computed the 4×3 matrix $N_{\mathcal{X}}$ whose columns generate the first syzygy module of $I_{\mathcal{X}}$ from the 9×8 matrix M whose maximal minors generate the same ideal $I_{\mathcal{X}}$.
- (2) As explained at the end of section 2, a degree 4 rational normal curve $\Gamma \subset \mathcal{X}$ can be obtained by taking the maximal minors q_0, \dots, q_5 of $N_{\mathcal{X}}M_L$ where M_L is a general 3×2 matrix of maximal rank. To fix the order, $q_0 = \det(N_{\mathcal{X}}M_L)_{\widehat{34}}$, $q_1 = \det(N_{\mathcal{X}}M_L)_{\widehat{24}}$, $q_2 = \det(N_{\mathcal{X}}M_L)_{\widehat{14}}$, $q_3 = \det(N_{\mathcal{X}}M_L)_{\widehat{23}}$, $q_4 = \det(N_{\mathcal{X}}M_L)_{\widehat{13}}$, $q_5 = \det(N_{\mathcal{X}}M_L)_{\widehat{12}}$, where, as usual, $\widehat{\cdot}$ means that the corresponding rows are omitted.
- (3) Following [40], we define the map $\theta : T = H_*^0(\mathcal{O}_{\mathbb{P}^5}) \rightarrow R = H_*^0(\mathcal{O}_{\mathbb{P}^4})$ by setting $\theta(x_i) = q_i, i = 0, \dots, 5$. Then, $\theta^{-1}(I_{\mathcal{X}}) = I_K$ where K is a line congruence of bi-degree $(3, 6)$ and sectional genus 5 as the ones studied in section 3.
- (4) From equation (3), we get N_Z from $N_{\mathcal{X}}$ where the columns of N_Z generate the first syzygy module of I_Z in $S = H_*^0(\mathcal{O}_{\mathbb{P}^2})$, Z being a set of 10 points in general position, among which there are $p_1(1 : 0 : 0), p_2(0 : 1 : 0), p_3(0 : 0 : 1)$. We remark that \mathcal{X} is the image of the embedding of $\widetilde{\mathbb{P}^2}$, blow-up of \mathbb{P}^2 in Z , via the linear system $|4\pi^*L - E_1 - \dots - E_{10}|$.
- (5) Let $C_i \subset \mathbb{P}^2$ be the plane cubic curve containing $Z \setminus \{p_i\}$, with $i = 1, 2, 3$. The equation of C_i is the only degree 3 generator of $I_Z : I_{p_i}$. Let $C'_i \subset \mathcal{X}$ be the image of C_i and $C''_i \subset K$ be the image of C'_i . The saturated ideal that defines C''_i is $\theta^{-1}(I_{C'_i})$ and so we can compute the plane α_i spanned by C''_i . The critical locus we are looking for is $\mathcal{Y} = K \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$.
- (6) The last map in the minimal free resolution of $I_{\mathcal{Y}}$ is $T^3(-6) \rightarrow T^4(-5) \oplus T^3(-4)$. Let $N_{\mathcal{Y}}$ be the matrix that represents $T^3(-6) \rightarrow T^4(-5)$: the maximal minors of this matrix define the ideal J and so we can recover the projection matrices Φ_i, Ψ_j from it, $i, j = 1, 2, 3$, as explained in Proposition 6.1. Hence, it is possible to compute the projections $\varphi_i, \psi_j : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ we want to construct.

Now, we give an algorithm that summarizes the previous discussion, and that can be easily implemented in Singular [15], for example.

Input: projection matrices $P_1, P_2, P_3, Q_1, Q_2, Q_3$.

Output: projection matrices $\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3$.

- (1.1) Given the 5×1 matrix $VY = [y_0, \dots, y_4]$, compute $P_1 \cdot VY, P_2 \cdot VY, P_3 \cdot VY$ and the 9×8 matrix M .
- (1.2) Compute the matrix $N_{\mathcal{X}} = A - BD^{-1}C$ as in section 5, and the ideal $I_{\mathcal{X}}$ of the maximal minors of $N_{\mathcal{X}}$.
 - (2) Choose a general 3×2 matrix M_L of numbers, compute $M_{\Gamma} = N_{\mathcal{X}}M_L$, and its maximal minors q_0, \dots, q_5 in the above order.
 - (3) Define the map θ by setting $\theta(x_i) = q_i, i = 0, \dots, 5$, and compute $I_K = \theta^{-1}(I_{\mathcal{X}})$.
- (4.1) Given the 3×1 matrix $VZ = [z_0, z_1, z_2]$, compute the matrix N_Z from $N_{\mathcal{X}}$ according to equation (3).
- (4.2) Compute g_0, \dots, g_4 , maximal minors of N_Z with alternating signs, and set I_Z the ideal they generate.
- (4.3) Define the map π by setting $\pi(y_i) = g_i, i = 0, \dots, 4$.
- (5.1) Compute $F_1 = (I_Z : \langle z_1, z_2 \rangle)_3, F_2 = (I_Z : \langle z_0, z_2 \rangle)_3, F_3 = (I_Z : \langle z_0, z_1 \rangle)_3$ and $I_{\alpha_i} = (\theta^{-1} \circ \pi^{-1}(F_i))_1, i = 1, 2, 3$, where $(\dots)_j$ means the degree j part of the ideal in parenthesis.
- (5.2) Compute $I_{\mathcal{Y}} = I_K \cap \left(\bigcap_{i=1,2,3} I_{\alpha_i} \right)$.
- (6.1) Compute the third matrix in the minimal free resolution of $I_{\mathcal{Y}}$ and take the submatrix with linear entries.
- (6.2) Put this matrix in the form of the proof of Proposition 6.1.
- (6.3) Compute matrices $\Phi_i, \Psi_j, 1 \leq i, j \leq 3$, from the last matrix, and the associated projection matrices φ_i, ψ_j .

Now, we describe the converse construction.

- (1) Given the projections $\varphi_i, \psi_j : \mathbb{P}^3 \rightarrow \mathbb{P}^2$, we can compute the maps Φ_i, Ψ_j and the critical locus $\mathcal{Y} \subset \mathbb{G}(1, 3)$ for the reconstruction problem for projections of lines, as explained in section 6. Moreover, we compute the line congruence K and the three α -planes that are the irreducible components of \mathcal{Y} .
- (2) As explained in section 3, we compute a Veronese surface $V \subset \mathbb{G}(1, 3)$ such that $K \cup V$ is an arithmetically Gorenstein surface of degree 13 and $K \cap V$ is a degree 10 arithmetically Gorenstein curve.
- (3) Let $\theta^{-1} : R \rightarrow T_{\mathbb{G}}$ be the map associated to $K \cap V$, where $T_{\mathbb{G}} = T/\langle x_0x_5 - x_1x_4 + x_2x_3 \rangle$ is the coordinate ring of the Grassmannian $\mathbb{G}(1, 3)$. Then, $\theta^{-1}(I_K) = I_{\mathcal{X}}$ where \mathcal{X} is a Bordiga surface.
- (4) From Theorem 5.1, we get projections $P_i, Q_j : \mathbb{P}^4 \rightarrow \mathbb{P}^2, 1 \leq i, j \leq 3$, such that \mathcal{X} is the associated critical locus for the reconstruction problem.

As before, we give the corresponding algorithm. It can be implemented in Singular [15].

Input: projection matrices $\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3$.

Output: projection matrices $P_1, P_2, P_3, Q_1, Q_2, Q_3$.

- (1.1) Compute $\Phi_i, \Psi_j, 1 \leq i, j \leq 3$, as in section 6.
- (1.2) Compute the critical locus \mathcal{Y} for the reconstruction problem for lines, and its irreducible components.
 - (2) In the line congruence K contained in \mathcal{Y} , compute a degree 10 arithmetically Gorenstein curve $C = K \cap V$ whose ideal is generated by 5 quadrics q_0, \dots, q_4 in addition to the Klein hyperquadric $x_0x_5 - x_1x_4 + x_2x_3$.
 - (3) Define $\theta^{-1} : R \rightarrow T_{\mathbb{G}}$ as $\theta^{-1}(y_i) = q_i, i = 0, \dots, 4$, and compute $\theta^{-1}(I_K) = I_{\mathcal{X}}$.

- (4) Compute the generators of the first syzygy module $N_{\mathcal{X}}$ of $I_{\mathcal{X}}$ and compute a basis from which it is possible to reconstruct the projections $P_i, Q_j : \mathbb{P}^4 \rightarrow \mathbb{P}^2, 1 \leq i, j \leq 3$, we are looking for.

8. A COMPUTER SESSION CONCERNING THE LINE CONGRUENCE

In this section, we present a computer session in Singular in which, starting from the ideal of a general line congruence K of bi-degree $(3, 6)$ and sectional genus 5, we compute a degree 13 arithmetically Gorenstein surface S such that $S = K \cup V$ where V is a Veronese surface, and $K \cap V$ is a degree 10 arithmetically Gorenstein curve that allows us to compute the map $\theta^{-1} : \mathbb{G}(1, 3) \rightarrow \mathbb{P}^4$.

Input: ideal I_K of a general line congruence.

Output: ideals I_S and I_C where $S = K \cup V, C = K \cap V$.

```

ring r = 0, (x(0..5), a(1..268)), dp;
ideal ik;    \ \ This ideal has to be given as input. The first generator is the Klein
hyperquadric.
option(redSB);
ideal k1 = std(ik);    \ \ reduced standard basis of ik
\ \ In the following lines, we compute general polynomials that will become a reduced
standard basis of is
matrix mon3[1][46] = x(5)3, x(4) * x(5)2, x(3) * x(5)2, x(2) * x(5)2, x(1) * x(5)2, x(0) *
x(5)2, x(4)2 * x(5), x(3) * x(4) * x(5), x(2) * x(4) * x(5), x(1) * x(4) * x(5), x(0) * x(4) *
x(5), x(3)2 * x(5), x(1) * x(3) * x(5), x(0) * x(3) * x(5), x(2)2 * x(5), x(1) * x(2) * x(5), x(0) *
x(2) * x(5), x(1)2 * x(5), x(0) * x(1) * x(5), x(0)2 * x(5), x(4)3, x(3) * x(4)2, x(2) * x(4)2, x(1) *
x(4)2, x(0) * x(4)2, x(3)2 * x(4), x(1) * x(3) * x(4), x(0) * x(3) * x(4), x(2)2 * x(4), x(1) * x(2) *
x(4), x(0) * x(2) * x(4), x(1)2 * x(4), x(0) * x(1) * x(4), x(0)2 * x(4), x(3)3, x(1) * x(3)2, x(0) *
x(3)2, x(1)2 * x(3), x(0) * x(1) * x(3), x(0)2 * x(3), x(2)3, x(1) * x(2)2, x(0) * x(2)2, x(1)2 *
x(2), x(0) * x(1) * x(2), x(1)3;
matrix cp1[46][1] = a(1..46);
poly p1 = (mon3 * cp1)[1, 1] + x(0)3;
matrix cp2[46][1] = a(47..92);
poly p2 = (mon3 * cp2)[1, 1] + x(0)2 * x(1);
matrix cp3[46][1] = a(93..138);
poly p3 = (mon3 * cp3)[1, 1] + x(0) * x(1)2;
matrix cp4[45][1] = a(139..183);
poly p4 = (submat(mon3, intvec(1), intvec(1..45)) * cp4)[1, 1] + x(0)2 * x(2);
matrix mon4[1][85] = x(5)4, x(4) * x(5)3, x(3) * x(5)3, x(2) * x(5)3, x(1) * x(5)3, x(0) *
x(5)3, x(4)2 * x(5)2, x(3) * x(4) * x(5)2, x(2) * x(4) * x(5)2, x(1) * x(4) * x(5)2, x(0) * x(4) *
x(5)2, x(3)2 * x(5)2, x(1) * x(3) * x(5)2, x(0) * x(3) * x(5)2, x(2)2 * x(5)2, x(1) * x(2) * x(5)2, x(0) *
x(2) * x(5)2, x(1)2 * x(5)2, x(0) * x(1) * x(5)2, x(0)2 * x(5)2, x(4)3 * x(5), x(3) * x(4)2 * x(5), x(2) *
x(4)2 * x(5), x(1) * x(4)2 * x(5), x(0) * x(4)2 * x(5), x(3)2 * x(4) * x(5), x(1) * x(3) * x(4) *
x(5), x(0) * x(3) * x(4) * x(5), x(2)2 * x(4) * x(5), x(1) * x(2) * x(4) * x(5), x(0) * x(2) * x(4) *
x(5), x(1)2 * x(4) * x(5), x(0) * x(1) * x(4) * x(5), x(0)2 * x(4) * x(5), x(3)3 * x(5), x(1) * x(3)2 *
x(5), x(0) * x(3)2 * x(5), x(1)2 * x(3) * x(5), x(0) * x(1) * x(3) * x(5), x(0)2 * x(3) * x(5), x(2)3 *
x(5), x(1) * x(2)2 * x(5), x(0) * x(2)2 * x(5), x(1)2 * x(2) * x(5), x(0) * x(1) * x(2) * x(5), x(1)3 *
x(5), x(4)4 + a(231) * x(3) * x(4)3, x(2) * x(4)3, x(1) * x(4)3, x(0) * x(4)3, x(3)2 * x(4)2, x(1) *
x(3) * x(4)2, x(0) * x(3) * x(4)2, x(2)2 * x(4)2, x(1) * x(2) * x(4)2, x(0) * x(2) * x(4)2, x(1)2 *
x(4)2, x(0) * x(1) * x(4)2, x(0)2 * x(4)2, x(3)3 * x(4), x(1) * x(3)2 * x(4), x(0) * x(3)2 * x(4), x(1)2 *
x(3) * x(4), x(0) * x(1) * x(3) * x(4), x(0)2 * x(3) * x(4), x(2)3 * x(4), x(1) * x(2)2 * x(4), x(0) *
x(2)2 * x(4), x(1)2 * x(2) * x(4), x(0) * x(1) * x(2) * x(4), x(1)3 * x(4), x(3)4, x(1) * x(3)3, x(0) *
x(3)3, x(1)2 * x(3)2, x(0) * x(1) * x(3)2, x(0)2 * x(3)2, x(1)3 * x(3), x(2)4, x(1) * x(2)3, x(0) *
x(2)3, x(1)2 * x(2)2, x(0) * x(1) * x(2)2, x(1)3 * x(2);

```

```

matrix cp5[85][1] = a(184..268);
poly p5 = (mon4 * cp5)[1, 1] + x(1)^4;
ideal rel;    \ \ This ideal will contain the relations among the a(j)'s
poly sp;    \ \ the following are local variables
int i;
int j;
int k;
int fl;
ideal mx = x(0..5);
matrix mx3[1][56] = std(mx^3);    \ \ monomials of degree 3
matrix mx4[1][126] = std(mx^4);    \ \ monomials of degree 4
matrix lp1[1][8] = std(lead(k1));    \ \ leading monomials of k1
for(i = 1; i <= 8; i = i + 1){k1[i] = k1[i]/leadcoef(k1[i]);};    \ \ normalized reduced
standard basis of k1;
\ \ we compute the normal form of p1 with respect to k1 and we collect the coefficients
of the monomials in the ideal rel
sp = p1;
for(j = 56; j >= 1; j = j - 1){
fl = 0;
k = 0;
while((k < 8)and(fl == 0)){
k = k + 1;
if(gcd(lp1[1, k], mx3[1, j]) == lp1[1, k]){fl = 1; };
if(fl == 0){
if(sp <> reduce(sp, std(mx3[1, j]))){
rel = rel, (sp - reduce(sp, std(mx3[1, j])))/mx3[1, j]; };
sp = reduce(sp, std(mx3[1, j])); };
if(fl == 1){
sp = sp - ((sp - reduce(sp, std(mx3[1, j])))/lp1[1, k]) * k1[k]; }; };
\ \ repeat verbatim the above computation for sp = p2, p3, p4 and we collect the coeffi-
cients of the monomials in the same ideal rel
sp = p2;
:
\ \ we compute the normal form of p5 with respect to k1 and we collect the coefficients
of the monomials in the same ideal rel
sp = p5;
for(j = 126; j >= 1; j = j - 1){
fl = 0;
k = 0;
while((k < 8)and(fl == 0)){
k = k + 1;
if(gcd(lp1[1, k], mx4[1, j]) == lp1[1, k]){fl = 1; };
if(fl == 0){
if(sp <> reduce(sp, std(mx4[1, j]))){
rel = rel, (sp - reduce(sp, std(mx4[1, j])))/mx4[1, j]; };
sp = reduce(sp, std(mx4[1, j])); };
if(fl == 1){
sp = sp - ((sp - reduce(sp, std(mx4[1, j])))/lp1[1, k]) * k1[k]; }; };
\ \ we compute a standard basis of rel, and we reduce p1, ..., p5 modulo it. After this
step, they belong to i_k
rel = std(rel);
poly p11 = reduce(p1, rel);

```

```

poly p21 = reduce(p2, rel);
poly p31 = reduce(p3, rel);
poly p41 = reduce(p4, rel);
poly p51 = reduce(p5, rel);
\\ we run Buchberger's algorithm of the set {i_k[1], p11, ..., p51} and we compute the
coefficient in such a way that they form a Gröbner basis. The relations are stored in rel
kill rel;
ideal rel;
matrix lp[1][6] = x(2) * x(3), x(0)^3, x(0)^2 * x(1), x(0) * x(1)^2, x(0)^2 * x(2), x(1)^4;
matrix pp[1][6] = x(2) * x(3) - x(1) * x(4) + x(0) * x(5), p11, p21, p31, p41, p51;
intmat b[9][2] = 1, 5, 2, 3, 2, 5, 3, 4, 3, 5, 2, 4, 4, 5, 4, 6, 3, 6;
for(i = 1; i <= 5; i = i + 1){
  sp = (lp[1, b[i, 2]] * pp[1, b[i, 1]] - lp[1, b[i, 1]] * pp[1, b[i, 2]]);
  sp = sp/gcd(lp[1, b[i, 2]], lp[1, b[i, 1]]);
  for(j = 126; j >= 1; j = j - 1){
    fl = 0;
    k = 0;
    while((k < 6)and(fl == 0)){
      k = k + 1;
      if(gcd(lp[1, k], mx4[1, j]) == lp[1, k]){fl = 1; };
      if(fl == 0){
        if(sp <> reduce(sp, std(mx4[1, j]))){
          rel = rel, (sp - reduce(sp, std(mx4[1, j])))/mx4[1, j]; };
          sp = reduce(sp, std(mx4[1, j])); };
        if(fl == 1){
          sp = sp - ((sp - reduce(sp, std(mx4[1, j])))/lp[1, k]) * pp[1, k]; }; }; };
      \\ we choose randomly the values of a(182), a(183) and compute a standard basis of rel:
in such a way, every coefficient has a numerical value that we substitute in p11, ..., p51
ideal rel1 = a(182) - random(-5, 5), a(183) - random(-5, 5), rel;
rel1 = std(rel1);
poly p12 = reduce(p11, rel1);
poly p22 = reduce(p21, rel1);
poly p32 = reduce(p31, rel1);
poly p42 = reduce(p41, rel1);
poly p52 = reduce(p51, rel1);
ideal i_s = i_k[1], p12, p22, p32, p42, p52;
ideal i_v = quotient(i_s, i_k);
ideal i_c = i_v, i_k;

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E-mail address: marina.bertolini@unimi.it

E-mail address: roberto.notari@polimi.it

E-mail address: cristina.turrini@unimi.it

DIPARTIMENTO DI MATEMATICA “F. ENRIQUES”, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, 20133 MILANO, ITALY

DIPARTIMENTO DI MATEMATICA “F.BRIOSCHI”, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI 32, 20133 MILANO