Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 6831508, 9 pages
https://doi.org/10.1155/2018/6831508

Research Article

On the Effect of Labour Productivity on Growth: Endogenous Fluctuations and Complex Dynamics

F. Grassetti, C. Mammana, and E. Michetti

Department of Economics and Law, University of Macerata, Macerata, Italy

Correspondence should be addressed to C. Mammana; cristiana.mammana@unimc.it

Received 3 April 2018; Accepted 5 September 2018; Published 9 October 2018

Academic Editor: Josef Diblík

This paper introduces a sigmoidal production function that considers production possible even when the only input is labour. The long-run behaviour of an economy described by the neoclassical Solow-type growth model with differential savings is investigated considering the technology presented. It is found that labour productivity influences the existence of boom and bust periods as well as the level of capital per capita in equilibrium.

1. Introduction

Economic growth models explain a country's capital accumulation over time. One of the most used neoclassical models is the model developed by Solow and Swan (see Solow [1] and Swan [2]). The Solow-Swan model considers the growth rate of labour, the depreciation rate of capital, the saving rate, and the technology of production as determinants of the long-run behaviour of an economy. An extension of the Solow model considers two different saving propensities for workers and shareholders (see Böhm and Kaas [3]).

In literature, different technologies have been studied in the Solow framework to analyse the evolution of capital per capita over time. Production technologies with Constant Elasticity of Substitution between production factors (see Brianzoni [4, 5], Masanjala and Papageorgiou [6], and Papageorgiou and Saam [7]) and Variable Elasticity of Substitution (see Brianzoni et al. [8], Cheban et al. [9], and Karagiannis et al. [10]) have been considered in order to verify which parameters influence the generation of cycles and of more complex dynamics for an economy. It has been found that nonsimple dynamics arise when the elasticity of substitution is sufficiently low and shareholders save more than workers. Moreover, attention has been paid to sigmoidal production functions (see Brianzoni et al. [11, 12]), since they are able to describe growth dynamics of nondeveloped and developing countries.

In all the above-mentioned works the technology considered, $f(k)$, satisfies $f(0) = 0$, where k is the capital per capita. Therefore, all those works provide a production rule for which no capital can be generated without capital. Thus, they do not take into consideration a possibility of deriving from one of the assumptions of the Solow model: in his work, Solow [1] defined the capital stock as the amount of already-produced output. This implies that an initial production of capital should be possible when the only production factor is labour. Moreover, it is known that nondeveloped countries may base their early stage development only on labour (agriculture and handicraft manufacture), as Azariadis and Stachurski [13]) discussed.

In this paper we present a sigmoidal technology that satisfies $f(0) > 0$, providing the condition for an initial amount of production possible without capital. We then investigate the Solow-type model with differential savings considering the technology introduced, in order to verify how the long-run behaviour of an economy is influenced by labour productivity (a measure of the quantity of output that can be produced from labour).

The rest of the paper is organised as follows. In Section 2 we describe the production function, in Section 3 we study the existence and stability of the steady states, while in Section 4 we investigate the influence of labour productivity on long-run dynamics. Section 5 concludes the paper.

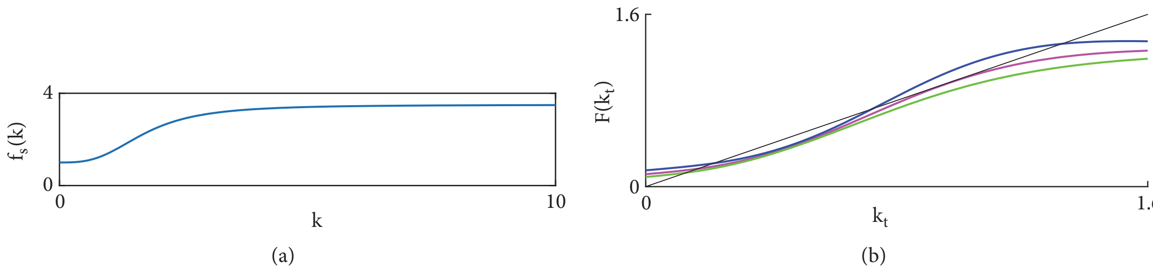


Figure 1: (a) Production function $f_s(k_t)$ for $\alpha = .5$, $\beta = .2$, $\rho = 3$, and $c = 1$. (b) Map $F(k_t)$ and its fixed points for $n = s_w = .2$, $\delta = s_r = .7$, $\alpha = 1$, $\beta = .5$, and $c = \alpha(\rho - 1)^2/4\beta\rho$; one fixed point for $\rho = 2.5$ (green map), two fixed points for $\rho = 2.8$ (purple map), and three fixed points for $\rho = 3.2$ (blue map).

2. Model Setup

Nonconcave production functions are used in literature to describe countries with a low level of economic development, since they present increasing returns to scale at an early stage of development and diminishing ones at a later stage (see, among all, Skiba [14] who introduced them). Capasso et al. [15] presented the sigmoidal production function mapping capital per worker into output per worker, as given by

$$f(k_t) = \frac{\alpha k_t^\rho}{1 + \beta k_t^\rho} \tag{1}$$

where α and β are nonnegative and $\rho \geq 2$ is the output elasticity of capital. Notice that an inflection point k_f exists such that $f(k_t)$ is convex for $k_t \in (0, k_f)$ and it is concave for $k \in (k_f, +\infty)$ with $k_f = ((\rho - 1)/\beta(\rho + 1))^{1/\rho}$, so it is a convex-concave production function. Such a function can be considered an extension of the Cobb–Douglas (CD) production function, since it approximates the CD case when $\alpha = 1$ and $\beta = 0$. Notice that $f(k)$ does not fulfil the Inada Conditions (see Barro and Sala-i Martin [16]), being $\lim_{k \longrightarrow 0} f'(k) = 0$. The S-shape behaviour of the function allows us to consider less-developed countries, but it does not take into account those economies driven by craftsmanship, i.e., in which low levels of output are possible even without capital, that is, with $f(0) > 0$. In order to consider these economies, we add a positive constant c to the sigmoidal production function. The resulting shifted sigmoidal (SS) production function is given by

$$f_s(k_t) = \frac{\alpha k_t^\rho}{1 + \beta k_t^\rho} + c \tag{2}$$

and its graph is in Figure 1(a).

Notice that when the SS production function is considered, production is possible without capital, being $f_s(0) = c > 0$. This kind of technology has been already considered in literature when studying the evolution of capital per capita over time (see, among all, the Nobel Prize Solow [1]).

We now consider the wage of a worker. When the per capita wage of a worker equals the marginal product of labour, that is, $w(k_t) = f(k_t) - k_t f'(k_t)$, and the SS production function is considered, then the wage is given by

$$w(k_t) = \frac{\alpha k_t^\rho (\beta k_t^\rho - \rho + 1)}{(\beta k_t^\rho + 1)^2} + c \tag{3}$$

being

$$f_s'(k_t) = \frac{\alpha \rho k_t^{\rho - 1}}{(1 + \beta k_t^\rho)^2}. \tag{4}$$

In order for the wage to be nonnegative, we assume $c \geq (\alpha/\beta)((\rho - 1)^2/4\rho) = c_{\min}$, so that the minimum point of $w(k)$ is either positive or equal to zero. This consideration establishes an economically meaningful lower bound of c for the setup, as stated in the following remark.

Remark 1. The shifted sigmoidal production function (SS) is given by

$$f_s(k_t) = \frac{\alpha k_t^\rho}{1 + \beta k_t^\rho} + c \tag{5}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\rho \geq 2$, and $c \geq (\alpha/\beta)((\rho - 1)^2/4\rho)$ for the wage being nonnegative.

As it can be observed in Figure 1(a), the SS production function describes economies in which output is generated even without capital. Recall that the elasticity of substitution (ES) between production factors measures the ease with which capital and labour can be substituted for one another and, for twice differentiable production functions, it is mathematically defined by (see Sato and Hoffman [17]):

$$\sigma(k) = -\frac{f'(k)\left(f(k) - kf'(k)\right)}{kf(k)f''(k)}. \tag{6}$$

Being

$$f_s''(k_t) = -\frac{\alpha \rho k_t^{\rho - 2}\left(\beta(\rho + 1)k_t^\rho - \rho + 1\right)}{(\beta k_t^\rho + 1)^3} \tag{7}$$

and the SS production function as given by (2), the resulting ES is

$$\sigma(k_t) = \frac{c(\beta k_t^\rho + 1)^2 + \alpha k_t^\rho(\beta k_t^\rho - \rho + 1)}{(\beta(\rho + 1)k_t^\rho - \rho + 1)(k_t^\rho(\alpha + \beta c) + c)} \tag{8}$$

Therefore, this production function belongs to the class of VES (Variable Elasticity of Substitution) production functions, since σ depends on the capital per capita level. Notice that the ES may be negative. Production functions with negative ES between inputs can be found in literature (see, among all, Prywas [18], Thompson and Taylor [19], Jurgen [20], and Grassetti et al. [21]). The work of Paterson [22] suggests that such eventuality occurs when production inputs are complementary.

Consider now the Solow-Swan-type growth model in which workers and shareholders have different but constant saving rates (respectively, $s_w \in (0,1)$ and $s_r \in (0,1)$) and assume that shareholders receive the marginal product of capital $\pi(k)$, while the wage rate $w(k)$ equals the marginal product of labour. Following Böhm and Kaas [3], the map describing the capital accumulation over time $t \in \mathbb{N}$ is then given by

$$k_{t+1} = F(k_t) = \frac{1}{1+n}\left[(1-\delta)k_t + s_w w(k_t) + s_r \pi(k_t)\right], \quad k_t \geq 0 \tag{9}$$

where the labour force grows at a rate $n \geq 0$ and $\delta \in (0,1]$ is the depreciation rate of capital.

In order to consider less-developed economies, in which production is possible using only labour, we assume that the technology is described by the SS production function as defined in (5). Consequently, the per capita wage of workers is given in (3), while the income per worker of a shareholder is given by

$$\pi(k_t) = \frac{\alpha\rho k_t^{\rho}}{\left(1+\beta k_t^{\rho}\right)^2}. \tag{10}$$

Taking into account (9), (3), and (10), the final continuous and smooth one-dimensional map describing the capital per capita evolution is given by

$$k_{t+1} = F(k_t) = \frac{1}{1+n}\left[(1-\delta)k_t + \frac{s_w \alpha k_t^{\rho}}{1+\beta k_t^{\rho}} + \frac{(s_r - s_w)\alpha\rho k_t^{\rho}}{\left(1+\beta k_t^{\rho}\right)^2} + s_w c\right], \tag{11}$$

where $F(k_t) \geq 0$ $\forall k_t \geq 0$ for all combinations of parameter values. Notice that in Brianzoni et al. [11], where a sigmoidal production function is considered, the condition $s_r \geq s_w$ is required in order to assure that $F(k_t)$ is nonnegative for all k_t. Differently, if the SS production function (as defined in Remark 1) is considered, then the final map $F(k_t)$ has always economic meaning, thus allowing us to consider both cases $s_r \geq s_w$ and $s_r < s_w$ and, therefore, this can be considered a generalisation of the previous work. This is possible because capital is not an essential input for production and hence even an economy where shareholders are not saving can be considered.

3. Equilibria and Local Dynamics

In this section we consider the question of the existence and stability of steady states for an economy described by the Solow-Swan growth model with differential savings as proposed in Böhm and Kaas [3], while considering the SS production function as defined in Remark 1.

3.1. Existence of Steady States. We now analyse the number of steady states map F may own, considering different conditions on parameters values.

The first difference from most of the models that consider the same initial framework (see, among all, Brianzoni et al. [12] and Klump and de La Grandville [23]) concerns the existence of a steady state characterised by zero capital per capita. While previous works established the existence of this equilibrium, when the SS production function is considered, the origin is not a fixed point, being $F(0) > 0$. This happens when production is possible without capital (i.e., only labour is necessary), as the works of Solow [1] and Miyagiwa and Papageorgiou [24] demonstrate. Notice that this model better describes the behaviour of nondeveloped economies: even in those countries in which only labour is available (no infrastructures, machineries, or investments are given), a minimum level of production is possible, i.e., agriculture and handicraft manufacture (with regard to the early stages of economic development see Azariadis and Stachurski [13]). While Brianzoni et al. [11] proved that, when the sigmoidal production function given by (5) is considered, a poverty trap exists (the origin is a stable fixed point); this work found that, if a technology in which the only necessary input factor is labour is considered, the existence of the 'vicious circle of poverty' (see Azariadis and Stachurski [13]) can be avoided. Notice that, even though an economy in which only labour is given will not converge to a zero growth rate, attention should be paid to the existence of multiple positive steady states: different equilibrium levels of capital per capita would explain the coexistence of nondeveloped, developing, and developed countries.

To this aim we now consider the following functions:

$$G(k_t) = \frac{k_t^{\rho-1}}{1+\beta k_t^{\rho}}\left(s_w + \rho\frac{\Delta_s}{1+\beta k_t^{\rho}}\right) \tag{12}$$

and

$$H(k_t) = \frac{n+\delta}{\alpha} - \frac{s_w c}{\alpha k_t}, \tag{13}$$

with $\Delta_s = s_r - s_w \in (-s_w, 1-s_w)$. Then the fixed points of (11) are the solutions of equation $G(k_t) = H(k_t)$, $k_t \geq 0$.

As far as the properties of function $G(k_t)$ are concerned, the following proposition holds.

Proposition 2. *Function $G(k_t)$ as given by (12) is defined $\forall k_t \geq 0$, it is continuous and differentiable such that $G(0) = \lim_{k_t \to +\infty} G(k_t) = 0$. Furthermore, there exist $0 < k_m < k_M$ such that*

(i) *if $((\rho-1)/\rho)s_w \leq s_r < 1$, then $G(k_t)$ admits a unique maximum point k_M;*

(ii) *if* $0 < s_r < ((\rho-1)/\rho)s_w$, *then* $G(k_t)$ *admits a minimum point* k_m *and a maximum point* k_M, $0 < k_m < k_M$.

Proof. The first part is trivial. As far as the number of critical points of $G(k_t)$ is concerned, it is possible to write

$$G'(k_t) = \frac{f(k_t)}{\alpha(1+M)^2 k_t^2}\left(a_1 M^2 + a_2 M + a_3\right) \tag{14}$$

where $f(k_t)$ is given by (1), $M = \beta k_t^{\rho}$, $a_1 = -s_w$, $a_2 = s_w(\rho - 2) - (\rho^2+\rho)\Delta_s$, and $a_3 = s_w(\rho-1) + (\rho^2+\rho)\Delta_s$.

(i) Consider $\Delta_s > 0$, then $G(k_t)$ admits a unique maximum point given by $k_M = [(-a_2 - \sqrt{a_2^2 - 4a_1a_3})/(2a_1\beta)]^{1/\rho}$ as it was proved in Brianzoni et al. [11].

(ii) Consider $\Delta_s < 0$, then $a_1 < 0$, $a_2 > 0$ while a_3 can be positive or negative. Define $Z(M) = a_1M^2 + a_2M + a_3$ and notice that $Z'(0) > 0$. Then

  (a) if $a_3 \geq 0$, i.e., $s_r \geq ((\rho-1)/\rho)s_w$, the equation $Z(M) = 0$ admits a unique positive solution; hence $G(k_t)$ admits a unique maximum point k_M;

  (b) if $a_3 < 0$, $Z(M) = 0$ may admit two, one, or zero solutions, depending on the sign of $Z(-a_2/2a_1) = -(a_2^2 - 4a_1a_3)/4a_1$. Notice that $Z(-a_2/2a_1) > 0$ iff $a_2^2 > 4a_1a_3$. After some computations, it can be observed that this inequality is always verified; hence, if $s_r < ((\rho-1)/\rho)s_w$, then $Z(M) = 0$ admits two solutions. The minimum and maximum points are given, respectively, by $k_m = [(-a_2 + \sqrt{a_2^2 - 4a_1a_3})/(2a_1\beta)]^{1/\rho}$ and k_M, where $0 < k_m < k_M$.

□

Taking into account Proposition 2 and considering that $H(k_t)$ given by (13) is a hyperbola with $H(k_t) < (n+\delta)/\alpha$ $\forall k_t > 0$, $\lim_{k_t\to 0^+} H(k_t) = -\infty$, $\lim_{k_t\to+\infty} H(k_t) = (n+\delta)/\alpha > 0$, $H' > 0$, and $H'' < 0$ $\forall k_t > 0$, then the following proposition holds.

Proposition 3. *Map* $F(k_t)$ *given by (11) admits at least one positive fixed point and at most three positive fixed points.*

Proof. Recall that $H(k_t)$ is concave, $\lim_{k_t\to+\infty} H(k_t) = (n+\delta)/\alpha$ and $\lim_{k_t\to+\infty} G(k_t) = 0$.

(i) Assume $0 < s_r < ((\rho-1)/\rho)s_w$. Then $G(k_t)$ is bimodal with $k_m < k_M$. Then

  (a) if $G(k_m) > H(k_m)$ no intersection exists in $(0, k_m)$. Moreover, if $G(k_M) \geq H(k_M)$, one intersection exists in $[k_M, +\infty)$ and at most two intersections may exist in $[k_m, k_M)$, whereas if $G(k_M) < H(k_M)$, no intersection exists in $[k_M, +\infty)$ and from one up to three intersections exist in (k_m, k_M);

  (b) if $G(k_m) \leq H(k_m)$, one intersection exists in $(0, k_m]$. Moreover, if $G(k_M) \geq H(k_M)$, one intersection exists in (k_m, k_M) and one in $[k_M, +\infty)$, whereas if $G(k_M) < H(k_M)$, at most two intersections exist in (k_m, k_M) and no intersection exists in $[k_M, +\infty)$.

(ii) Assume $((\rho-1)/\rho)s_w \leq s_r < 1$. Then $G(k_t)$ is unimodal with maximum point k_M. Then

  (a) if $G(k_M) < H(k_M)$, from one up to three intersections exist in $(0, k_M)$ and no intersection exists in $[k_M, +\infty)$;

  (b) if $G(k_M) \geq H(k_M)$, one intersection exists in $[k_M, +\infty)$. Moreover at most two intersections exist in $(0, k_M)$.

□

Notice that points $G(k_M)$ and $H(k_M)$ determine the position of the higher equilibrium: whatever the saving propensity of shareholders, when $G(k_M) > H(k_M)$, one fixed point exists in the interval $[k_m, +\infty)$, while no fixed points exist in that interval otherwise.

Differently from previous models using this initial framework, when production is possible even without capital, the origin is not a fixed point. Moreover, up to three fixed points may exist (see Figure 1(b)). Notice that the number of fixed points is in agreement with that obtained by Brianzoni et al. [8] and Grassetti et al. [21] considering different types of VES production functions, but, whereas in previous works the condition $s_r \geq s_w$ was required, in this case multiple equilibria may exist even if workers save more than shareholders.

3.2. Local Dynamics. Function $F(k_t)$ as given by (11) can be written in terms of function $G(k_t)$ as follows:

$$F(k_t) = \frac{1}{1+n}\left[(1-\delta)k_t + \alpha k_t G(k_t)\right] + \frac{s_w c}{1+n} \tag{15}$$

Hence

$$F'(k_t) = \frac{1}{1+n}\left\{1 - \delta + \alpha S(k_t)\right\}, \tag{16}$$

where

$$\begin{aligned} S(k_t) &= G(k_t) + k_t G'(k_t) \\ &= \frac{\rho k_t^{\rho-1}\left[(s_w - \rho\Delta_s)\beta k_t^{\rho} + \rho\Delta_s + s_w\right]}{\left(1+\beta k_t^{\rho}\right)^3}. \end{aligned} \tag{17}$$

Notice that $G(0) = \lim_{k_t\to 0^+} k_t G'(k_t) = 0$; hence $F'(0) \in (0,1)$, that is, $\exists I_+ = (0, r_1)$ s.t. $F(k_t)$ is strictly increasing $\forall k_t \in (0, r_1)$. Since $\lim_{k_t\to+\infty} F'(k_t) = (1-\delta)/(1+n) \in (0,1)$, then $\exists I_\infty = (r_2, +\infty)$ s.t. $F(k_t)$ is strictly increasing $\forall k_t \in (r_2, +\infty)$. Hence, taking into account also the properties of function $G(k_t)$, two cases may occur: $F(k_t)$ is strictly increasing or $F(k_t)$ is bimodal; i.e., it admits a minimum point $k_{\min}$ and a maximum point $k_{\max}$. The following proposition holds.

Proposition 4. *Consider $F(k_t)$ as given by (11). If $s_r \in [((\rho - 1)/\rho)s_w, ((\rho+1)/\rho)s_w)$, then $F(k_t)$ is strictly increasing.*

Proof. If $s_w - \rho\Delta_s > 0$ and $s_w + \rho\Delta_s \geq 0$, then $F'(k_t) > 0\ \forall k_t > 0$. □

Proposition 4 establishes a sufficient condition for the strict monotonicity of $F(k_t)$: if this condition is not fulfilled, map F can be strictly increasing or bimodal. Notice that, when the condition of Proposition 4 holds and only one equilibrium exists, the long-run behaviour of an economy can be easily understood, as the following proposition states.

Proposition 5. *Assume $s_r \in [((\rho-1)/\rho)s_w, ((\rho+1)/\rho)s_w)$, and let k^* be the unique fixed point of map F. Then k^* is globally stable.*

Proof. From Proposition 4 it is known that, when $s_r \in [((\rho - 1)/\rho)s_w, ((\rho+1)/\rho)s_w)$, $F'(k_t) > 0\ \forall k_t > 0$. Moreover, $F(0) > 0$ and $\lim_{k_t \longrightarrow +\infty} F(k_t) - k_t < 0$. Being k^* the only fixed point of F, i.e., $F(k^*) = k^*$, then $F(k_t) > (<) k^*\ \forall 0 \geq k_t < k^* (k_t > k^*)$ and hence k^* is globally stable. □

Recall from Proposition 3 that map F may have from one up to three fixed points. When the condition of Proposition 4 holds and F has one fixed point, then it is *globally stable*. Modifying the parameter values, a *fold bifurcation* occurs (when parameters reach the bifurcation values, one nonhyperbolic fixed point is created) and a pair of fixed points emerges so map F has three fixed points $k_1 < k_2 < k_3$ (see Figure 1(b)). Being $F(k_t)$ increasing, the fixed points are alternatively stable and unstable, with the unstable equilibrium k_2 that separates the basins of attraction of k_1 and k_3. Notice that, when 3 fixed points exist the model simultaneously explains the behaviour of economies that, starting from different initial conditions, reach different equilibria level of capita per capita in the long-run.

We now focus on the role of labour productivity on growth. Notice that the productivity of labour is the measure of the amount of output produced with one hour's labour. Moreover, the existence of a positive relation between labour productivity and wage has been demonstrated (see Meager and Speckesser [25]). For the SS production function it has $F(0) = c$; therefore parameter c influences the production without capital: when the only input of production is labour, the higher the value of c, the higher the value of output for the economy. Moreover, considering the wage equation as given by (3), it follows that $\partial w(k_t)/\partial c > 0$, so a positive correlation between c and $w(k_t)$ is evident. Given these considerations, c can be seen as an indirect measure of labour productivity. The following proposition highlights the role played by parameter c on the existence and position of the equilibrium level of an economy.

Proposition 6. *Consider two economies described by SS technology and differing only in the constant c, both with only one equilibrium level for capital per capita. Then the one with higher production without capital (i.e., higher value of c) will experience a higher level of capital per capita in the steady states.*

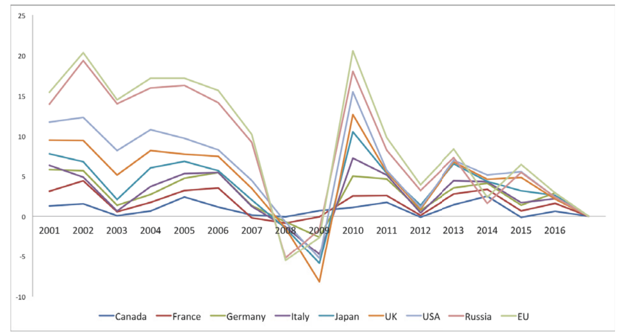


Figure 2: Labour productivity, annual growth rate (%), 2000–2016. Source: OECD.

Proof. As the proof of Proposition 3 showed, the locations of the steady states depend on the intersection between $G(k_t)$ and $H(k_t)$. Recall that $G(k_t)$ has a maximum point k_M and can have a minimum point $k_m < k_M$. Moreover, $\lim_{k_t \longrightarrow 0^+} G(k_t) = \lim_{k_t \longrightarrow +\infty} G(k_t) = 0$ while $H(k_t)$ is strictly increasing with $\lim_{k_t \longrightarrow 0^+} H(k_t) = -\infty$ and $\lim_{k_t \longrightarrow +\infty} H(k_t) = (n+\delta)/\alpha$. Notice that parameter c has no effect on the shape of function $G(k_t)$, while $\partial H(k_t)/\partial c < 0$, so that $H(k_t)$ shifts downward as c increases. Therefore, when only one intersection between G and H exists, a higher value of c determines a higher value of the intersection point. □

The previous proposition demonstrates a positive correlation between labour productivity and the value of the capital stock for the higher steady state. This result is in agreement with the Organisation for Economic Cooperation and Development (OECD) data: as it can be seen in Figure 2, the years of economic crisis are related to lower levels of labour productivity.

Notice that the condition of Proposition 4 is fulfilled when $s_r \in I(s_w, \epsilon)$. Therefore, when the saving behaviours of the two income groups are similar, the economy does not experience boom and bust periods (since F is strictly increasing, no fluctuation can be generated). To investigate the appearance of fluctuations, we now prove that function F may be unimodal, depending on parameter values.

Proposition 7. *Consider F and S as given, respectively, by (11) and (17). Define $r = \rho\sqrt{[(3\rho^2+1)\Delta_s + 4s_w]\Delta_s + s_w^2}$ and assume $k_{S1} = [(2\rho^2\Delta_s + s_w + r)/\beta(\rho+1)(\rho\Delta_s - s_w)]^{1/\rho}$ and $k_{S2} = [(2\rho^2\Delta_s + s_w - r)/\beta(\rho+1)(\rho\Delta_s - s_w)]^{1/\rho}$.*

Map F is bimodal with $k_{\max} < k_{\min}$ if one of the following conditions holds:

(i) $s_r \geq ((\rho+1)/\rho)s_w \wedge S(k_{S2}) < (\delta - 1)/\alpha$;

(ii) $s_r < ((\rho-1)/\rho)s_w \wedge S(k_{S1}) < (\delta - 1)/\alpha$.

Map F is strictly increasing otherwise.

Proof. Recall (16). The critical points of F are solutions of $S(k_t) = (\delta - 1)/\alpha$. Notice that $S(0) = \lim_{k_t \longrightarrow +\infty} S(k_t) = 0$. Moreover, $S'(k_t) = \rho k_t^{\rho-2} Q(k_t)/(\beta k_t^{\rho} + 1)^4$, where

$$Q(k_t) = (\rho + 1)(\rho\Delta_s - s_w)\beta^2 k_t^{2\rho} - 2(2\rho^2\Delta_s + s_w)\beta k_t^{\rho} + (\rho - 1)(\rho\Delta_s + s_w), \quad (18)$$

and therefore $S'(k_t) = 0$ iff $Q(k_t) = 0$ that is for $k_t = k_{S1}$ and $k_t = k_{S2}$. Assume $k_S = [(\rho\Delta_s + s_w)/\beta(\rho\Delta_s - s_w)]^{1/\rho}$ and notice that $k_{S1} < k_S < k_{S2}$. Then

(i) if $s_r \geq ((\rho + 1)/\rho)s_w$, function $S(k_t)$ is negative $\forall k_t > k_S$ and it has one minimum point given by k_{S_2}. Therefore $S(k_t)$ intersects the negative and constant function $(\delta - 1)/\alpha$ iff $S(k_{S2}) < (\delta - 1)/\alpha$;

(ii) if $s_r < ((\rho - 1)/\rho)s_w$, function $S(k_t)$ is negative $\forall k_t < k_S$ and it has one minimum point given by k_{S_1}. Therefore $S(k_t)$ intersects the negative and constant function $(\delta - 1)/\alpha$ iff $S(k_{S1}) < (\delta - 1)/\alpha$.

□

If neither conditions of Proposition 7, (i) or (ii) are fulfilled, map F is strictly increasing and the same considerations derived from Proposition 4 hold: only convergence to a positive equilibrium is possible and no complex dynamics may emerge. Notice that when the system has three fixed points, a policy increasing the capital per capita would push the economy to the higher equilibrium level. On the other hand, when F is bimodal, parameter c influences the existence of a globally stable fixed point, as the following proposition states.

Proposition 8. *Let F as given by (11) be bimodal with minimum point k_{min}. Then $\exists \overline{c}$ s.t., if $c > \overline{c}$, map F has one fixed point $k^* > k_{min}$ and it is globally stable.*

Proof. When one of the conditions of Proposition 7 holds, map F is bimodal with $k_{\max} < k_{\min}$. Notice that $\partial F(k_t)/\partial c > 0$, therefore $\exists \overline{c}$ s.c. $F(k_t) > k_t \ \forall k_t < k_{\min}$ if $c > \overline{c}$. Being $F(0) > 0$ and $\lim_{k_t \longrightarrow +\infty} F(k_t) - k_t < 0$, it follows that one fixed point $k^* > k_{\min}$ exists and it is globally stable. □

As demonstrated by Proposition 8, when the output productivity is sufficiently high, the convergence to the equilibrium level $k^* > k_{\min}$ is guaranteed. This result is in line with that obtained by Tugcu [26]: considering a binary dependent variable model, he found that a higher expenditure in secondary education, health, and R&D (i.e., policies that increase labour productivity) pushes the economy out of the middle trap (the phenomenon for which an economy gets stuck when it achieves the middle-income status) and towards higher levels of income.

In order to observe fluctuations and more complex dynamics, map F has to be nonmonotonic. Notice that several works (see, among all, Brianzoni et al. [8] and Tramontana et al. [27]) demonstrate that cycles and chaotic dynamics appear when the ES between production factors is sufficiently low. Therefore, we now investigate the role of the output elasticity of capital ρ on the shape of F.

Proposition 9. *Let F be as given by (11) and assume $s_r \geq ((\rho+1)/\rho)s_w$ or $s_r < ((\rho - 1)/\rho)s_w$. Then $\exists \widetilde{\rho}$ s.t. map F is bimodal $\forall \rho \geq \widetilde{\rho}$.*

Proof. Recall from Proposition 7 that F is bimodal if $s_r \geq ((\rho+1)/\rho)s_w \wedge S(k_{S2}) < (\delta - 1)/\alpha$ or $s_r < ((\rho - 1)/\rho)s_w \wedge S(k_{S1}) < (\delta - 1)/\alpha$, where $(\delta - 1)/\alpha < 0$. Moreover $\partial S(k_{S1})/\partial\rho < 0$, $\partial S(k_{S2})/\partial\rho < 0$, and $\lim_{\rho \longrightarrow +\infty} S(k_{S1}) = \lim_{\rho \longrightarrow +\infty} S(k_{S2}) = -\infty$; therefore $\exists \widetilde{\rho}$ s.t. the minimum point of $S(k_t)$ is below the constant function q for all $\rho \geq \widetilde{\rho}$. □

Proposition 9 establishes a necessary condition for the arising of fluctuations, cycles, and complex dynamics, that is, a sufficiently high ρ and sufficiently different saving behaviours between workers and shareholders. In this section we showed that, when the SS production function is considered in the Solow-type growth model with different saving propensities between workers and shareholder, from one up to three equilibria exist for the capital per capita level. Differently from previous works, in which production is not possible without capital, the origin is not a fixed point: thanks to labour productivity, even a country in which only labour is available for production can experience economic growth. Notice that in neoclassical growth models the stock of capital K (from which the capital per capita k derives as a ration between capital and labour) is defined as the stock of the already-produced commodities (see Solow [1]). Given this definition, an initial condition in which the first unit of capital was produced must exist. In contrast to this requirement, previous literature considers the condition $k_t = 0$ as a fixed point, implying that if no "already-produced" goods exist, no goods can be produced. Our result overcomes the lack of previous works. Furthermore, we found that the productivity of labour positively influences the existence and stability of a high equilibrium level for the economy. In the next section, thanks to numerical simulations, we analyse the influence of the elasticity of substitution between production factors and labour productivity on the existence of fluctuations for an economy.

4. Fluctuations and the Role of Labour Productivity

The aim of this section is to investigate the factors that may influence the generation of boom and bust periods for an economy. As discussed in the previous section, fluctuations and complex dynamics may arise only if the difference between saving propensities of workers and shareholders is sufficiently high or ρ is high enough. Therefore, in the following we consider the case in which one of the conditions (i) or (ii) given in Proposition 7 holds.

Notice that cycles and chaotic dynamics arise when a fixed point k_n^* loses stability via flip bifurcation; i.e., $F'(k_n^*)$ passes the value -1, since a fixed point placed on the decreasing branch of map F is required. Thus, a necessary condition for fluctuations is $F(k_{\max}) > k_{\max} \wedge F(k_{\min}) <$

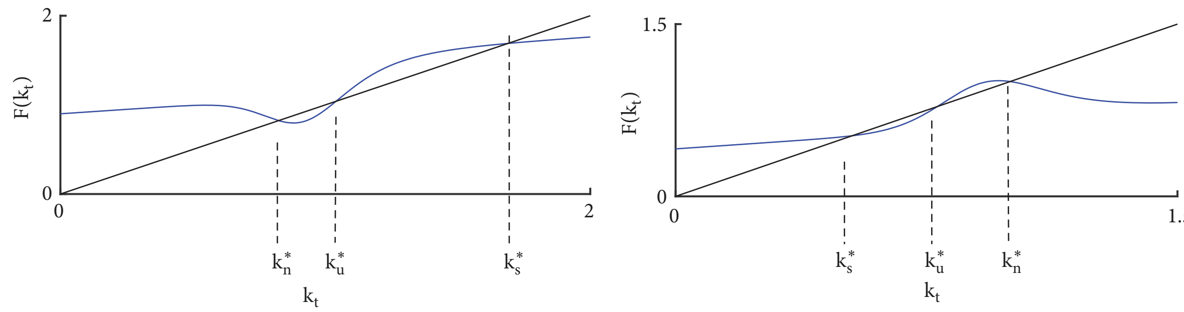


Figure 3: Parameter values $n = 1.72$, $\delta = .46$, $s_r = .44$, $\alpha = 2.65$, $\beta = 1.73$, and $\rho = 8.82$. (a) $k_n^* < k_u^* < k_s^*$ for $s_w = .83$ and $c = 2.95$, (b) $k_s^* < k_u^* < k_n^*$ for $s_w = .15$ and $c = 7.5$.

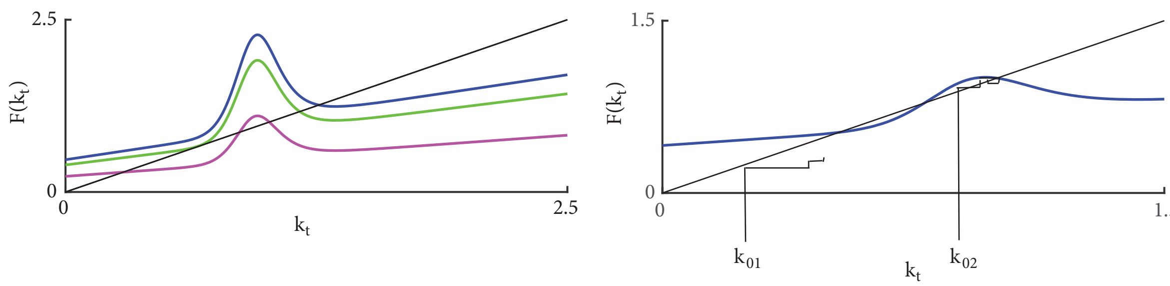


Figure 4: (a) Cases (a) in blue, (b) in green, and (c) in purple of Proposition 10. (b) Two coexisting attractors.

$k_{\min}$. Assume $k_n^* \in (k_{\max}, k_{\min})$ to be a fixed point, then, considering Proposition 3 and being $F(0) > 0$ and $F'(k_t) > 0\ \forall k < k_{\max}$, it follows that up to two more fixed points may exist and, given the monotonicity of F for $k < k_{\max}$ and $k > k_{\min}$, the number of fixed points determines their stability: if one fixed point k_{nh}^* exists, it is nonhyperbolic, while if two fixed points k_s^* and k_u^* exist, they are, respectively, stable and unstable (in Figure 3 the cases $k_n^* < k_u^* < k_s^*$ and $k_s^* < k_u^* < k_n^*$ are depicted, respectively, in panels (a) and (b)).

The previous considerations are summarised in the following proposition.

Proposition 10. *Assume F bimodal with $F(k_{max}) > k_{max} \vee F(k_{min}) < k_{min}$. Then map F has one fixed point $k_{max} < k_n^* < k_{min}$ with $F'(k_n^*) < 0$. Moreover three cases may occur:*

(a) *no other fixed point exists;*

(b) *a nonhyperbolic fixed point k_{nh}^* exists;*

(c) *two fixed points k_s^* and k_u^* exist, with k_s^* stable and k_u^* unstable.*

Therefore, a multistability phenomenon may occur (see Figure 4 panel (b)) and the model is able to explain why economies starting from different capital per capita levels may experience different long-run behaviours. This situation is depicted in Figure 5: starting from different initial conditions, two different attractors (in black and blue) may be reached.

Given the form of F, an analytical study of the influence that parameters have on the loss of stability for the equilibrium k_n is not possible. Therefore, we analyse this influence thanks to numerical simulations.

Figure 5(a) shows that the elasticity of substitution between production factors influences the generation of chaotic dynamics, as previous works demonstrated while considering economies in which capital is necessary for production. A different role is, instead, played by parameter c, which is related to labour productivity. Numerical simulations (an example is given in panel (b) of Figure 5) showed that fluctuations arise for low values of c. Moreover when the productivity of labour is sufficiently high, the attractor is a fixed point, no cycles can be generated, and the value of capital per capita in equilibrium increases if c increases, since if c is sufficiently high and F is bimodal, $F(k_{\min}) > k_{\min}$. Given these results, an economic policy that increases the productivity of labour would push an economy out of boom and bust periods and increase the level of capital per capita in equilibrium.

5. Conclusions

In this work we introduced a technology that considers production possible even if the only input is labour, making criticism of the production functions previously used in economic growth models. We then investigated the long-run dynamics of the Solow-type growth model with differential

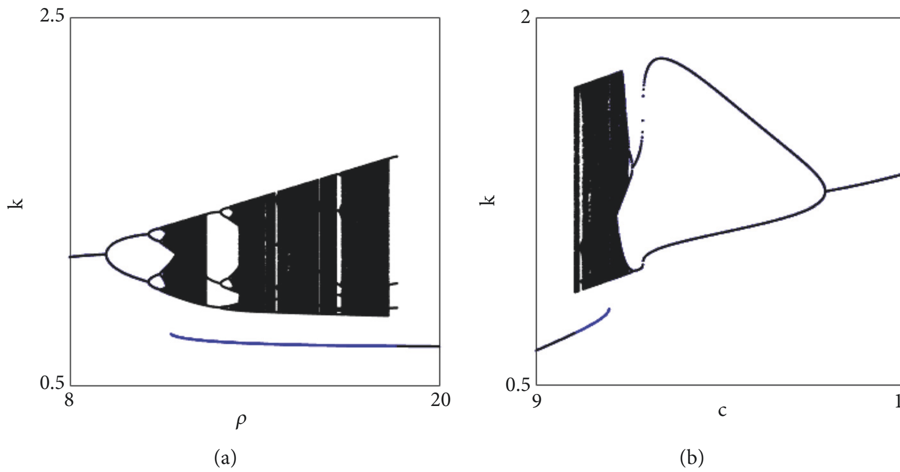


Figure 5: Bifurcation diagrams. Parameter values $\alpha = .2$, $\beta = .4$, $\delta = n = .7$, $s_r = .8$, and $s_w = .1$, initial conditions $k_{\min}$ (black) and $k_0 = .02$ (blue): (a) with respect to ρ for $c = 10$; (b) with respect to c for $\rho = 18$.

savings between workers and shareholders, considering the function presented. Differently from previous works, we found that the origin is not a fixed point: when in the early stage of development an economy may produce only with labour, a positive evolution of capital per capita is observed. We showed that fluctuations may arise if the difference between saving propensities is sufficiently high. Moreover, we showed that an economic policy intended to increase labour productivity would push an economy out of boom and bust periods and increase the level of capita per capita in the equilibrium.

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

This paper did not receive specific funding and was performed as part of the employment of the authors to University of Macerata.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. M. Solow, "A contribution to the theory of economic growth," *The Quarterly Journal of Economics*, vol. 70, no. 1, pp. 65–94, 1956.

[2] T. W. Swan, "Economic growth and capital accumulation," *Economic Record*, vol. 32, no. 2, pp. 334–361, 1956.

[3] V. Böhm and L. Kaas, "Differential savings, factor shares, and endogenous growth cycles," *Journal of Economic Dynamics & Control*, vol. 24, no. 5-7, pp. 965–980, 2000.

[4] S. Brianzoni, C. Mammana, and E. Michetti, "Complex dynamics in the neoclassical growth model with differential savings and non-constant labor force growth," *Studies in Nonlinear Dynamics and Econometrics*, vol. 11, no. 3, 2007.

[5] S. Brianzoni, C. Mammana, and E. Michetti, "Nonlinear dynamics in a business-cycle model with logistic population growth," *Chaos, Solitons & Fractals*, vol. 40, no. 2, pp. 717–730, 2009.

[6] W. H. Masanjala and C. Papageorgiou, "The solow model with ces technology: Nonlinearities and parameter heterogeneity," *Journal of Applied Econometrics*, vol. 19, no. 2, pp. 171–201, 2004.

[7] C. Papageorgiou and M. Saam, "Two-level CES production technology in the solow and diamond growth models," *The Scandinavian Journal of Economics*, vol. 110, no. 1, pp. 119–143, 2008.

[8] S. Brianzoni, C. Mammana, and E. Michetti, "Variable elasticity of substituition in a discrete time Solow-Swan growth model with differential saving," *Chaos, Solitons & Fractals*, vol. 45, no. 1, pp. 98–108, 2012.

[9] D. Cheban, C. Mammana, and E. Michetti, "Global attractors of quasi-linear non-autonomous difference equations: a growth model with endogenous population growth," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 3, pp. 1716–1731, 2013.

[10] G. Karagiannis, T. Palivos, and C. Papageorgiou, "Variable elasticity variable elasticity of substitution and economic growth," in *New Trends in Macroeconomics*, Diebolt, Claude, and C. Kyrtsou, Eds., pp. 21–37, 2005.

[11] S. Brianzoni, C. Mammana, and E. Michetti, "Local and global dynamics in a discrete time growth model with nonconcave production function," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 536570, 22 pages, 2012.

[12] S. Brianzoni, C. Mammana, and E. Michetti, "Local and global dynamics in a neoclassical growth model with nonconcave production function and nonconstant population growth rate," *SIAM Journal on Applied Mathematics*, vol. 75, no. 1, pp. 61–74, 2015.

[13] C. Azariadis and J. Stachurski, "Poverty traps," in *Handbook of Economic Growth*, P. Aghion, Ed., vol. 1, chapter 5, pp. 294–384, Elsevier, 2005.

[14] A. K. Skiba, "Optimal growth with a convex-concave production function," *Econometrica*, vol. 46, no. 3, pp. 527–539, 1978.

[15] V. Capasso, R. Engbers, and D. La Torre, "On a spatial Solow model with technological diffusion and nonconcave production function," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3858–3876, 2010.

[16] R. J. Barro and X. Sala-i Martin, *Economic Growth*, MIT Press, 2004.

[17] R. Sato and R. F. Hoffman, "Production Functions with Variable Elasticity of Factor Substitution: Some Analysis and Testing," *The Review of Economics and Statistics*, vol. 50, no. 4, p. 453, 1968.

[18] M. Prywes, "A nested CES approach to capital-energy substitution," *Energy Economics*, vol. 8, no. 1, pp. 22–28, 1986.

[19] P. Thompson and T. G. Taylor, "The Capital-Energy Substitutability Debate: A New Look," *The Review of Economics and Statistics*, vol. 77, no. 3, pp. 565–569, 1995.

[20] A. Jurgen, *Technical change and the elasticity of factor substitution*, vol. 147, Beitrage der Hochschule Pforzheim, 2014.

[21] F. Grassetti, C. Mammana, and E. Michetti, "Poverty trap, boom and bust periods and growth. A nonlinear model for non-developed and developing countries," *Decisions in Economics and Finance*, 2018.

[22] N. Paterson, *Elasticities of substitution in computable general equilibrium models. Working paper*, Department of Finance Canada, 2012.

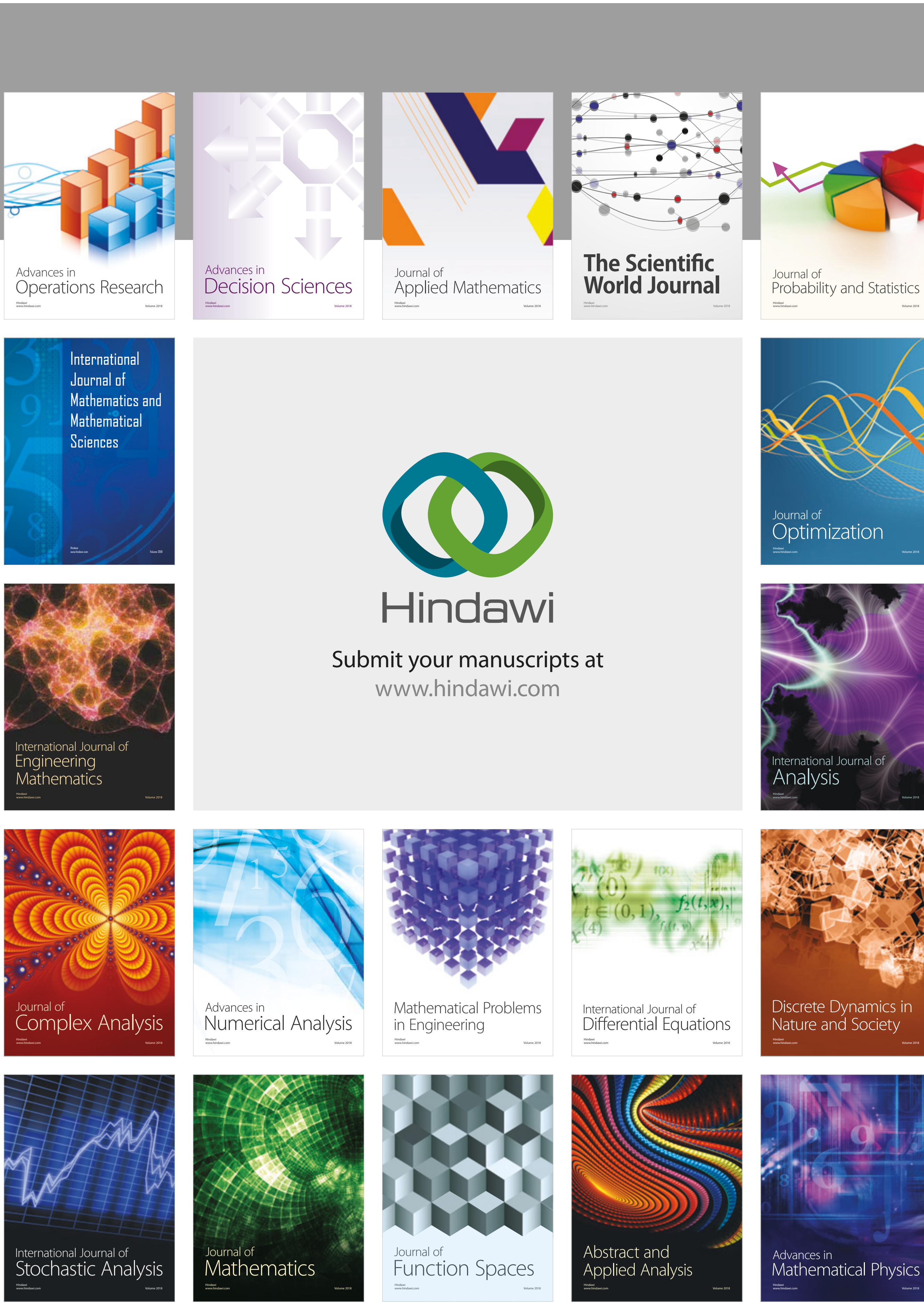
[23] R. Klump and O. De La Grandville, "Economic growth and the elasticity of substitution: Two theorems and some suggestions," *American Economic Review*, vol. 90, no. 1, pp. 282–291, 2000.

[24] K. Miyagiwa and C. Papageorgiou, "Elasticity of substitution and growth: normalized CES in the Diamond model," *Economic Theory*, vol. 21, no. 1, pp. 155–165, 2003.

[25] N. Meager and S. Speckesser, "Wages, productivity and employment: A review of theory and international data," *European Employment Observatory*, 2011.

[26] C. T. Tugcu, "How to escape the middle income trap: international evidence from a binary dependent variable model," *Theoretical and Applied Economics*, vol. 1, no. 206, pp. 49–56, 2015.

[27] F. Tramontana, L. Gardini, and A. Agliari, "Endogenous cycles in discontinuous growth models," *Mathematics and Computers in Simulation*, vol. 81, no. 8, pp. 1625–1639, 2011.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
The Scientific
World Journal
Journal of
Probability and Statistics
International
Journal of
Mathematics and
Mathematical
Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of
Optimization
International Journal of
Engineering
Mathematics
International Journal of
Analysis
Journal of
Complex Analysis
Advances in
Numerical Analysis
Mathematical Problems
in Engineering
International Journal of
Differential Equations
Discrete Dynamics in
Nature and Society
International Journal of
Stochastic Analysis
Journal of
Mathematics
Journal of
Function Spaces
Abstract and
Applied Analysis
Advances in
Mathematical Physics