Structure of Uniformly Continuous Quantum Markov Semigroups

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Abstract

The structure of uniformly continuous quantum Markov semigroups with atomic decoherence-free subalgebra is established providing a natural decomposition of a Markovian open quantum system into its noiseless (decoherence-free) and irreducible (ergodic) components. This leads to a new characterisation of the structure of invariant states and a new method for finding decoherence-free subsystems and subspaces. Examples are presented to illustrate these results.

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1 Introduction

Quantum Markov Semigroups (QMS) $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on the von Neumann algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on a complex separable Hilbert space h describe the evolution of open quantum systems in quantum optics and quantum information processing.

The structure of uniformly continuous QMS and their generators has been analyzed by several authors starting from the early works by Davies[17], Spohn[37], Lindblad[28], Christensen and Evans[16] (see, for instance, [4, 32, 33] and the references therein). In most of these investigations, concern has been focused on the structure of the generator and the relationships between its algebraic properties and structural properties of the underlying open quantum system.

In recent years, there has been a growing interest in the use of QMSs to model open quantum systems having subsystems which are not affected by decoherence (see Lidar and Whaley [27], Knill and Laflamme[29], Olkiewicz[8, 30, 31], Ticozzi and Viola[35], see also [12, 13, 14] and the references therein). In these applications the QMS (in the Heisenberg picture) acts as a semigroup of automorphisms of a von Neumann subalgebra $\mathcal{N}(\mathcal{T})$ of $\mathcal{B}(\mathsf{h})$, called the *decoherence-free subalgebra*. This subalgebra allows identification of noise protected subsystems where states evolve unitarily, moreover, its structure and relationship with the set of fixed points also has important consequences on the asymptotic behaviour of the QMS (see [18, 22, 23, 37]).

In this paper, exploiting the explicit structure of purely atomic von Neumann algebras, we give a full description of the structure of uniformly continuous QMSs with atomic decoherence-free subalgebra.

Our first result, Theorem 11, shows that, when $\mathcal{N}(\mathcal{T})$ is a type I factor, the Hilbert space h is (isomorphic to) the tensor product of two Hilbert spaces k and m, $\mathcal{N}(\mathcal{T})$ is isomorphic to $\mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$ where $\mathbb{1}_{\mathsf{m}}$ is the identity operator on m and the operators in a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) representation of the generator factorise accordingly. Linear maps $\mathcal{T}_t$ (up to unitary isomorphism) factorise as $\mathcal{T}_t^{\mathsf{k}} \otimes \mathcal{T}_t^{\mathsf{m}}$ where $\mathcal{T}^{\mathsf{k}}$ and $\mathcal{T}^{\mathsf{m}}$ are QMS on $\mathcal{B}(\mathsf{k})$ and $\mathcal{B}(\mathsf{m})$ respectively and the QMS $\mathcal{T}^{\mathsf{k}}$ acts as a semigroup of automorphisms ($\mathcal{T}_t^{\mathsf{k}}(x) = \mathrm{e}^{\mathrm{i}tK} x \mathrm{e}^{-\mathrm{i}tK}$ for some self-adjoint K on k). In this way the decoherence-free (noise protected) part of the system turns out to be essentially independent of the noisy part of the system. This result shows, roughly speaking, that the only way of maintaining a subsystem free from decoherence is by keeping it isolated.

The main result, Theorem 12, concerns the case where $\mathcal{N}(\mathcal{T})$ is an atomic algebra and so it is a direct sum of type I factors and the above considerations apply to each term of the direct sum.

Our result has important consequences. The first concerns the structure of all invariant states of QMSs with a faithful invariant state, which is completely characterised by Theorem 21 extending to infinite dimensional Hilbert space h a result by Baumgartner and Narnhofer [6]. The second, Theorem 22, is a simple sufficient condition for establishing environment induced decoherence ([30, 31, 12]). Moreover, the decomposition $\mathsf{h} = \oplus_{i \in I} (\mathsf{k}_i \otimes \mathsf{m}_i)$ of the Hilbert space h as in Theorem 12 allows us to identify immediately decoherence-free quantum subsystems, in the sense of Ticozzi and Viola [35], and decoherence-free subspaces, as defined by Lidar et al. [27] (see also [3]), of a given quantum Markovian system.

The decoherence-free subalgebra plays a key role in all the above decompositions. Indeed, starting from the leading idea that an atomic subalgebra has a special structure, we undertake the analysis of the structure of the generator of an arbitrary uniformly continuous QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ and find an infinite dimensional generalisation of all the main known results.

The structure of the paper is as follows. In section 2 we recall the definitions and review some basic properties of the decoherence-free subalgebra

$\mathcal{N}(\mathcal{T})$ and the set of fixed points $\mathcal{F}(\mathcal{T})$ for the maps $\mathcal{T}_t$. In order to make our exposition self-contained we collect there several preliminary results scattered in the literature. Moreover, we also prove (Proposition 5) that the center of $\mathcal{N}(\mathcal{T})$ is contained in $\mathcal{F}(\mathcal{T})$. In section 3 we establish our main results Theorems 11 and 12. In the next section we prove our result on the structure of invariant states also deducing spectral properties of the Hamiltonian K in the decoherence-free part of the QMS (Lemma 19) and showing that the decoherence-free subalgebra of irreducible QMSs is trivial (Proposition 17). In section 5 we discuss the applications to environment induced decoherence and decoherence-free subsystems and subspaces. In the final section we present two examples, generic and circulant QMSs, to illustrate how our results work in a concrete set-up. The appendix discusses a known characterisation of atomic von Neumann algebras that we have been unable to find in the wealthy literature on the subject.

2 The decoherence-free subalgebra of a QMS

Let h be a complex separable Hilbert space. A QMS on the algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on h is a semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of completely positive, identity preserving normal maps which is also weakly* continuous. We will make the assumption from now on that $\mathcal{T}$ is indeed uniformly continuous i.e. $\lim_{t\to 0^+} \sup_{\|x\|\leq 1} \|\mathcal{T}_t(x) - x\| = 0$. Its generator $\mathcal{L}$ can be represented in the well-known (see [32, 33]) Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form as

$$\mathcal{L}(x) = \mathrm{i}[H, x] - \frac{1}{2}\sum_{\ell\geq 1} \left(L_\ell^* L_\ell x - 2L_\ell^* x L_\ell + x L_\ell^* L_\ell\right), \qquad (1)$$

where $H = H^*$ and $(L_\ell)_{\ell\geq 1}$ are operators on h such that the series $\sum_{\ell\geq 1} L_\ell^* L_\ell$ is strongly convergent and $[\cdot,\cdot]$ denotes the commutator $[x, y] = xy - yx$. The choice of operators H and $(L_\ell)_{\ell\geq 1}$ is not unique (see Parthasarathy [32] Theorem 30.16), however, this will not create any inconvenience in this paper.

Given a GKSL representation of $\mathcal{L}$ we call $\mathcal{L}_0$

$$\mathcal{L}_0(x) := -\frac{1}{2}\sum_{\ell\geq 1} \left(L_\ell^* L_\ell x - 2L_\ell^* x L\ell + x L_\ell^* L_\ell\right), \qquad x \in \mathcal{B}(\mathsf{h}),$$

dissipative part of $\mathcal{L}$ and $\mathrm{i}\delta_H(x) := \mathrm{i}[H, x]$ Hamiltonian part of $\mathcal{L}$ by abuse of language. Clearly, we have $\mathcal{L} = \mathrm{i}\delta_H + \mathcal{L}_0$.

The *decoherence-free (DF) subalgebra* of $\mathcal{T}$ is defined by

$$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x^* x) = \mathcal{T}_t(x)^* \mathcal{T}_t(x),\ \mathcal{T}_t(x x^*) = \mathcal{T}_t(x)\mathcal{T}_t(x)^* \ \ \forall\, t \geq 0\}. \qquad (2)$$

It is a well known fact that $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ on which every $\mathcal{T}_t$ acts as a $*$-automorphism by the following (see e.g. Evans[19] Theorem 3.1, [18] Proposition 2.1).

Proposition 1 *Let $\mathcal{T}$ be a quantum Markov semigroup on $\mathcal{B}(\mathsf{h})$ and let $\mathcal{N}(\mathcal{T})$ be the set defined by (2). Then*

1. *$\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant for all $t \geq 0$,*
2. *for all $x \in \mathcal{N}(\mathcal{T})$, $y \in \mathcal{B}(\mathsf{h})$ and $t \geq 0$ we have $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y)$ and $\mathcal{T}_t(y^*x) = \mathcal{T}_t(y^*)\mathcal{T}_t(x)$,*
3. *$\mathcal{N}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$.*

Proof. (1) Let $x \in \mathcal{N}(\mathcal{T})$ and $t > 0$. For all $s > 0$ we have

$$\mathcal{T}_s\left(\mathcal{T}_t(x^*x)\right) = \mathcal{T}_{s+t}(x^*x) = \mathcal{T}_{s+t}(x^*)\mathcal{T}_{s+t}(x) = \mathcal{T}_s\left(\mathcal{T}_t(x)^*\right)\mathcal{T}_s\left(\mathcal{T}_t(x)\right).$$

Exchanging x and x^* we find the identity

$$\mathcal{T}_s\left(\mathcal{T}_t(xx^*)\right) = \mathcal{T}_s\left(\mathcal{T}_t(x)\right)\mathcal{T}_s\left(\mathcal{T}_t(x)^*\right).$$

Thus, $\mathcal{T}_t(x)$ belongs to $\mathcal{N}(\mathcal{T})$.

(2) For all $t \geq 0$ and $x, y \in \mathcal{B}(\mathsf{h})$ define $\mathcal{D}_t(x,y) = \mathcal{T}_t(x^*y) - \mathcal{T}_t(x^*)\mathcal{T}_t(y)$. For every state ω on $\mathcal{B}(\mathsf{h})$ and every complex number z, by the complete positivity of $\mathcal{T}_t$, we have $\omega\left(\mathcal{D}_t(zx+y, zx+y)\right) \geq 0$. Now, if $x \in \mathcal{N}(\mathcal{T})$, then $\omega\left(\mathcal{D}_t(x,x)\right) = 0$ so that

$$0 \leq \omega\left(\mathcal{D}_t(zx+y, zx+y)\right) = 2\Re e\left(\bar{z}\omega\left(\mathcal{D}_t(x,y)\right)\right) + \omega\left(\mathcal{D}_t(y,y)\right)$$

for all complex number z. It follows that $\omega\left(\mathcal{D}_t(x,y)\right) = 0$ i.e. $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y)$, by the arbitrarity of ω, and (2) is proved.

(3) $\mathcal{N}(\mathcal{T})$ is a vector space by (2). Moreover, for all $x, y \in \mathcal{N}(\mathcal{T})$, we have

$$\mathcal{T}_t((xy)^*(xy)) = \mathcal{T}_t(y^*)\mathcal{T}_t(x^*)\mathcal{T}_t(x)\mathcal{T}_t(y) = \mathcal{T}_t((xy)^*)\mathcal{T}_t(xy).$$

The invariance of $\mathcal{N}(\mathcal{T})$ for the adjoint is obvious. Finally, for any net $(x_\gamma)_\gamma$ of elements of $\mathcal{N}(\mathcal{T})$ converging σ-strongly to a x in $\mathcal{B}(\mathsf{h})$ we have

$$\mathcal{T}_t(x^*x) = \lim_\gamma \mathcal{T}_t(x^*x_\gamma) = \lim_\gamma \mathcal{T}_t(x^*)\mathcal{T}_t(x_\gamma) = \mathcal{T}_t(x^*)\mathcal{T}_t(x).$$

Therefore x belongs to $\mathcal{N}(\mathcal{T})$ and (3) is proved.

The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ can be characterised as follows.

Proposition 2 *For all self-adjoint H in a GKSL representation of the $\mathcal{L}$ as in (1) we have*

$$\mathcal{N}(\mathcal{T}) \subseteq \left\{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH} x\, \mathrm{e}^{-\mathrm{i}tH} \quad \forall\, t \geq 0\right\}.$$

Proof. If x belongs to $\mathcal{N}(\mathcal{T})$, then, differentiating the identity $\mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x)$ at $t=0$, we have $\mathcal{L}(x^*x) = x^*\mathcal{L}(x) + \mathcal{L}(x^*)x$. Computing

$$\mathcal{L}(x^*x) - x^*\mathcal{L}(x) - \mathcal{L}(x^*)x = \sum_{\ell\geq 1} [L_\ell, x]^* [L_\ell, x] \tag{3}$$

for an arbitrary $x \in \mathcal{B}(\mathsf{h})$, we find $[L_\ell, x] = 0$ for $x \in \mathcal{N}(\mathcal{T})$. Moreover, since $\mathcal{N}(\mathcal{T})$ is a *-algebra, $x^* \in \mathcal{N}(\mathcal{T})$ so that $[L_\ell, x^*] = 0$ and, taking the adjoint, $[L_\ell^*, x] = 0$. It follows that $\mathcal{L}(x) = \mathrm{i}\,[H, x]$ for all $x \in \mathcal{N}(\mathcal{T})$.

Now fix $t > 0$ and $x \in \mathcal{N}(\mathcal{T})$. For all $0 \leq s \leq t$, $\mathcal{T}_s(x) \in \mathcal{N}(\mathcal{T})$ and, differentiating with respect to s,

$$\begin{aligned}\frac{d}{ds}\,\mathrm{e}^{\mathrm{i}(t-s)H}\mathcal{T}_s(x)\mathrm{e}^{-\mathrm{i}(t-s)H} &= \mathrm{i}\mathrm{e}^{\mathrm{i}(t-s)H}\,[H, \mathcal{T}_s(x)]\,\mathrm{e}^{-\mathrm{i}(t-s)H} \\ &- \mathrm{i}\mathrm{e}^{\mathrm{i}(t-s)H} H\mathcal{T}_s(x)\mathrm{e}^{-\mathrm{i}(t-s)H} \\ &- \mathrm{i}\mathrm{e}^{\mathrm{i}(t-s)H}\mathcal{T}_s(x)H\mathrm{e}^{-\mathrm{i}(t-s)H} \\ &= 0.\end{aligned}$$

We thus deduce that the function $s \mapsto \mathrm{e}^{\mathrm{i}(t-s)H}\mathcal{T}_s(x)\mathrm{e}^{-\mathrm{i}(t-s)H}$ is constant on $[0,t]$, and taking its values at $s=0$ and $s=t$, we find $\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH}x\mathrm{e}^{-\mathrm{i}tH}$. $\square$

In addition, we can characterise ([22] Theorem 2.1) $\mathcal{N}(\mathcal{T})$ in terms of operators H, L_ℓ in any GKSL representation of $\mathcal{L}$. First define iterated commutators $\delta_H^n(X)$ recursively by $\delta_H^0(X) = X, \delta_H^1(X) = [H, X]$, $\delta_H^{n+1}(X) = [H, \delta_H^n(X)]$.

Proposition 3 *The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the commutant of the set of operators*

$$\{\delta_H^n(L_\ell), \delta_H^n(L_\ell^*) \mid n \geq 0, \ell \geq 1\}. \tag{4}$$

Proof. If $x \in \mathcal{N}(\mathcal{T})$, then $\mathcal{T}_t(x) \in \mathcal{N}(\mathcal{T})$ by Proposition 1 (1), and so $\mathcal{L}(x) = \lim_{t\to 0^+} t^{-1}(\mathcal{T}_t(x) - x) \in \mathcal{N}(\mathcal{T})$. Moreover, arguing as in the proof of Proposition 2, we find $[L_\ell, x] = 0 = [L_\ell^*, x]$ and $\mathcal{L}(x) = \mathrm{i}[H, x] = \mathrm{i}\delta_H(x) \in \mathcal{N}(\mathcal{T})$. We now proceed by induction. Clearly all elements of $\mathcal{N}(\mathcal{T})$ commute with $\delta_H^0(L_\ell) = L_\ell$ and $\delta_H^0(L_\ell^*) = L_\ell^*$. Suppose that they commute with $\delta_H^n(L_\ell)$ and $\delta_H^n(L_\ell^*)$ for some n, then, by the Jacobi identity

$$\left[x, \delta_H^{n+1}(L_\ell)\right] = -\left[H, [\delta_H^n(L_\ell), x]\right] - [\delta_H^n(L_\ell), [x, H]] = 0$$

because $[x, H] = \mathrm{i}\mathcal{L}(x) \in \mathcal{N}(\mathcal{T})$. Thus, all elements of $\mathcal{N}(\mathcal{T})$ commute with $\delta_H^{n+1}(L_\ell)$ and, also with $\delta_H^{n+1}(L_\ell^*) = -\delta_H^{n+1}(L_\ell)^*$ because $\mathcal{N}(\mathcal{T})$ is a *-algebra. This shows that $\mathcal{N}(\mathcal{T})$ is contained in the commutant of the set (4).

Conversely, if x belongs to the commutant of the set (4), then it commutes with L_ℓ, L_ℓ^* and so $\mathcal{L}(x) = \mathrm{i}\delta_H(x)$. Moreover, $\delta_H(x)$ commutes with L_ℓ and L_ℓ^*

because, by the Jacobi identity, $[L_\ell, \delta_H(x)] = -[H,[x,L_\ell]] - [x, \delta_H(L_\ell)] = 0$ and, similarly, $[L_\ell^*, \delta_H(x)] = 0$. Suppose, by induction, that $\mathcal{L}^n(x) = \mathrm{i}^n \delta_H^n(x)$ and $\delta_H^k(x)$ commutes with $\delta_H^{n-k}(L_\ell)$ and $\delta_H^{n-k}(L_\ell^*)$ for all $k \leq n$, for some n. Then, $\mathcal{L}^{n+1}(x) = \mathrm{i}^n \mathcal{L}(\delta_H^n(x))$ and so

$$\begin{aligned}\mathcal{L}^{n+1}(x) &= \mathrm{i}^{n+1}\delta_H^{n+1}(x) + \frac{1}{2}\sum_{\ell\geq 1}\left(L_\ell^*\left[\delta_H^n(x), L_\ell\right] + \left[L_\ell^*, \delta_H^n(x)\right]L_\ell\right)\\ &= \mathrm{i}^{n+1}\delta_H^{n+1}(x)\end{aligned}$$

Moreover, by repeated use of the Jacobi identity, we have

$$\begin{aligned}\left[\delta_H^k(x), \delta_H^{n+1-k}(L_\ell)\right] &= -\left[\left[\delta_H^{k-1}(x), \delta_H^{n+1-k}(L_\ell)\right], H\right]\\ &\quad - \left[\left[\delta_H^{n+1-k}(L_\ell), H\right], \delta_H^{k-1}(x)\right]\\ &= \left[\delta_H^{k-1}(x), \delta_H^{n+2-k}(L_\ell)\right]\\ &= \ldots = \left[x, \delta_H^{n+1}(L_\ell)\right] = 0,\end{aligned}$$

and, similarly, $\left[\delta_H^k(x), \delta_H^{n+1-k}(L_\ell^*)\right] = 0$. It follows that $\mathcal{L}^n(x) = \mathrm{i}^n\delta_H^n(x)$ for all $n \geq 0$ and so $\mathcal{T}_t(x) = e^{\mathrm{i}tH} x\, e^{-\mathrm{i}tH}$, thus $x \in \mathcal{N}(\mathcal{T})$ by Proposition 2. □

It is worth noticing here that Proposition 3 holds for any GKSL representation of the generator $\mathcal{L}$. Indeed, even if the operators L_ℓ, H are not uniquely determined by $\mathcal{L}$ (see [32] Theorem 30.6) all other possible choices are of the form

$$L_\ell' = \sum_m u_{\ell m} L_m + z_m \mathbb{1}, \qquad H' = H + c + \frac{1}{2\mathrm{i}}\left(X - X^*\right)$$

where $(u_{\ell m})_{\ell,m\geq 1}$ is a unitary matrix, $(z_m)_{m\geq 1}$ is a sequence of complex scalars such that $\sum_m |z_m|^2 < \infty$, $c \in \mathbb{R}$ and $X = \sum_{m,j} \overline{z}_m u_{mj} L_j$. As a consequence, the commutant of the set of operators in Proposition 3 does not change replacing L_ℓ, H by L_ℓ', H.

Propositions 2 and 3 have been extended to weakly* continuous QMS with generators in a generalised GKSL form in [18].

Our investigation is concerned with the implications of the structure of $\mathcal{N}(\mathcal{T})$, as a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$, on the structure of $\mathcal{T}$. Let $\mathbb{1}_{\mathsf{k}}$ denote the identity operator on a Hilbert space k. We begin by recalling some basic definitions on operator algebras (see Takesaki [34]).

Definition 4 *Let $\mathcal{M}$ be a von Neumann algebra acting on h.*

(a) The center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ is the von Neumann algebra of elements x of $\mathcal{M}$ commuting with all $y \in \mathcal{M}$,

(b) $\mathcal{M}$ is a factor *if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbb{1}_{\mathsf{h}}$.*

(c) $\mathcal{M}$ *is a* type I factor *if it is a factor and possesses a non-zero minimal projection.*

Throughout the paper we will assume that

$$\mathcal{N}(\mathcal{T}) = \oplus_{i\in I} p_i \mathcal{N}(\mathcal{T}) p_i \tag{5}$$

where $(p_i)_{i\in I}$ is an (at most countable) family of mutually orthogonal non-zero projections, which are minimal projections in the center of $\mathcal{N}(\mathcal{T})$, such that $\sum_{i\in I} p_i = \mathbb{1}$ and each von Neumann algebra $p_i\mathcal{N}(\mathcal{T})p_i$ is a type I factor.

It is known that this property characterises atomic von Neumann algebras. We include a proof in the Appendix for completeness.

Proposition 5 *Let $\mathcal{M}$ be an atomic von Neumann algebra and let $(\alpha_t)_{t\geq 0}$ be a weak* continuous semigroup of $*$-automorphisms on $\mathcal{M}$. Then $\alpha_t(x) = x$ for all $x \in \mathcal{Z}(\mathcal{M})$ and $t \geq 0$.*

Proof. Let $(p_i)_{i\in I}$ be a family of mutually orthogonal projections which are minimal in $\mathcal{Z}(\mathcal{M})$ such that $\sum_{i\in I} p_i = \mathbb{1}$. Given $x \in \mathcal{Z}(\mathcal{M})$, every $p_i x p_i$ belongs to $p_i\mathcal{Z}(\mathcal{M})p_i = \mathbb{C}p_i$ by minimality; hence, it is enough to prove that every p_i is a fixed point for α.

Since α_t is a $*$-automorphism, clearly $\{\, \alpha_t(p_i) \mid i \in I \,\}$ is a family of mutually orthogonal projections; in particular this family is contained in $\mathcal{Z}(\mathcal{M})$, because for all $x \in \mathcal{M}$ we have $x = \alpha_t(y)$ for some $y \in \mathcal{M}$, and so

$$x\alpha_t(p_i) = \alpha_t(yp_i) = \alpha_t(p_i y) = \alpha_t(p_i)x$$

for all $i \in I$. Moreover, every $p_j\alpha_t(p_i)p_j$ is clearly a projection in $\mathcal{Z}(\mathcal{M})$ for each $j \in I$, because

$$(p_j\alpha_t(p_i)p_j)^2 = p_j\alpha_t(p_i)p_j\alpha_t(p_i)p_j = p_j\alpha_t(p_i)^2 p_j = p_j\alpha_t(p_i)p_j$$

and $(p_j\alpha_t(p_i)p_j)^* = p_j\alpha_t(p_i)p_j$. In addition, we have also $p_j\alpha_t(p_i)p_j \leq p_j$ since $\alpha_t(p_i) \leq \alpha_t(\mathbb{1}) = \mathbb{1}$. Therefore, by the minimality of projections p_j, for every $t \geq 0$, either $p_j\alpha_t(p_i)p_j = 0$ or $p_j\alpha_t(p_i)p_j = p_j$. By the weak* continuity of the map $t \mapsto \alpha_t(p_i)$, we find $p_i\alpha_t(p_i)p_i = p_i$ and $p_j\alpha_t(p_i)p_j = 0$ for $j \neq i$. It follows that $p_i\alpha_t(p_i^\perp)p_i = p_i\alpha_t(\mathbb{1} - p_i)p_i = 0$, so that $p_i\alpha_t(p_i^\perp) = \alpha_t(p_i^\perp)p_i = 0$, by the positivity of $\alpha_t(p_i)$, and $\alpha_t(p_i) = p_i$. □

We now study the structure of $\mathcal{N}(\mathcal{T})$.

Proposition 6 *If $\mathcal{N}(\mathcal{T})$ is an atomic algebra, then its center $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ is contained in the set of fixed points of all the maps $\mathcal{T}_t$.*

Proof. We know from Proposition 2 that every $\mathcal{T}_t$ acts as a $*$-automorphism on $\mathcal{N}(\mathcal{T})$ with inverse $x \mapsto \mathrm{e}^{-\mathrm{i}tH} x\, \mathrm{e}^{\mathrm{i}tH}$. Defining α_t as the restriction of $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ for all $t \geq 0$, we obtain a weak* continuous group of $*$-automorphism on $\mathcal{N}(\mathcal{T})$ and the conclusion follows from Proposition 5. □

We now briefly recall some results on the set

$$\mathcal{F}(\mathcal{T}) = \{\, x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = x, \forall t \geq 0\} . \tag{6}$$

of fixed points for the linear maps $\mathcal{T}_t$ that will be useful in the sequel. Clearly, $\mathcal{F}(\mathcal{T})$ is a vector space containing $\mathbb{1}$, and $a \in \mathcal{F}(\mathcal{T})$ if and only if $a^* \in \mathcal{F}(\mathcal{T})$; moreover it is norm-closed and weakly*-closed, and so it is an operator system.

Lemma 7 *An orthogonal projection $p \in \mathcal{B}(\mathsf{h})$ belongs to $\mathcal{F}(\mathcal{T})$ if and only if it commutes with the operators L_ℓ and H of any GKSL representation of $\mathcal{L}$.*

Proof. Clearly, if p commutes with the operators L_ℓ and H, then $\mathcal{L}(p) = 0$ and $\mathcal{T}_t(p) = p$ for all $t \geq 0$.

Conversely, if $\mathcal{T}_t(p) = p$ for all $t \geq 0$, then $\mathcal{L}(p) = 0$. Left and right multiplying by the orthogonal projection $p^\perp = \mathbb{1} - p$, we have

$$0 = p^\perp \mathcal{L}(p) p^\perp = p^\perp \sum_{\ell \geq 1} L_\ell^* p L_\ell p^\perp$$

and so $pL_\ell p^\perp = 0$. Similarly, starting from $\mathcal{L}(p^\perp) = \mathcal{L}(\mathbb{1} - p) = \mathcal{L}(\mathbb{1}) - \mathcal{L}(p) = 0$, we find $p^\perp L_\ell p = 0$. Taking the adjoints we also obtain $pL_\ell^* p^\perp = p^\perp L_\ell^* p = 0$, and so p commutes with L_ℓ and L_ℓ^*. As a results $\mathcal{L}(p) = \mathrm{i}[H, p] = 0$ and p also commutes with H. □

The following example shows that $\mathcal{F}(\mathcal{T})$, unlike $\mathcal{N}(\mathcal{T})$, may not be an algebra. We refer to [12], section 4, for additional examples.

Example 1 Let $\mathsf{h} = \mathbb{C}^3$ with canonical orthonormal basis $(e_i)_{0 \leq i \leq 2}$ and let $\mathcal{L}$ be the generator in the GKSL form with a single non-zero operator $L = |e_0\rangle\langle e_2|$ and $H = L^* L = |e_2\rangle\langle e_2|$. A straightforward computation yields, for a 3×3 matrix $a = (a_{ij})_{0 \leq i,j \leq 2}$ we have

$$\begin{aligned} \mathcal{L}(a) &= (a_{00} - a_{22})|e_2\rangle\langle e_2| \\ &- \left(\frac{1}{2} + \mathrm{i}\right)(a_{02}|e_0\rangle\langle e_2| + a_{12}|e_1\rangle\langle e_2|) \\ &- \left(\frac{1}{2} - \mathrm{i}\right)(a_{20}|e_2\rangle\langle e_0| + a_{21}|e_2\rangle\langle e_1|) . \end{aligned}$$

Thus, a is a fixed point for the QMS generated by $\mathcal{L}$ if and only if $a_{02} = a_{12} = a_{20} = a_{21} = 0$ and $a_{00} = a_{22}$. Now, it is easy to see that, for such an a, the matrix a^*a also satisfies $\mathcal{L}(a^*a) = 0$ if and only if, by (3), a commutes with L, namely $a_{10} = 0$.

Since by Proposition 3 every element in N(T) commutes with L and L^*, another computation shows that, if $a \in \mathcal{N}(\mathcal{T})$, then it commutes with L and L^*, therefore $a_{ij} = 0$ for $i \neq j$ and $a_{00} = a_{22}$. In this case, since $\delta_H(L) = L$, it also commutes also with all the iterated commutators $\delta_H^n(L) = L, \delta_H^n(L^*) = L^*$. In other words, a belongs to $\mathcal{N}(\mathcal{T})$ if and only if $a_{ij} = 0$ for $i \neq j$ and $a_{00} = a_{22}$. Hence, in this example $\mathcal{N}(\mathcal{T}) \subseteq \mathcal{F}(\mathcal{T})$.

In many situations, however, $\mathcal{F}(\mathcal{T})$ is an algebra and is contained in $\mathcal{N}(\mathcal{T})$. Further simple but useful properties (see [19, 22, 23]) are collected in the following proposition.

Proposition 8 *The following hold:*

1. *the fixed points set $\mathcal{F}(\mathcal{T})$ is a *-algebra if and only if it is contained in the decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$,*
2. *if the QMS $\mathcal{T}$ has a faithful normal invariant state, then $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$,*
3. *if $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$, then it coincides with the commutant of the set of operators $\{ L_\ell, L_\ell^*, H \mid \ell \geq 1 \}$.*

Proof. (1) If $\mathcal{F}(\mathcal{T})$ is contained in $\mathcal{N}(\mathcal{T})$, then, for all $x \in \mathcal{F}(\mathcal{T}) \subseteq \mathcal{N}(\mathcal{T})$, we have $\mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x) = x^*x$ and $x^*x \in \mathcal{F}(\mathcal{T})$. Conversely, if $\mathcal{F}(\mathcal{T})$ is a *-algebra, then, for all $a \in \mathcal{F}(\mathcal{T})$, $a^*a \in \mathcal{F}(\mathcal{T})$, thus $\mathcal{T}_t(a^*a) = a^*a = \mathcal{T}_t(a^*)\mathcal{T}_t(a)$ and a belongs to $\mathcal{N}(\mathcal{T})$.

(2) Let ρ be a faithful invariant state for $\mathcal{T}$. If $\mathcal{T}_t(x) = x$ for all $t \geq 0$, then, by complete positivity $x^*x = \mathcal{T}_t(x^*)\mathcal{T}_t(x) \leq \mathcal{T}_t(x^*x)$, and $\mathrm{tr}\,(\rho(\mathcal{T}_t(x^*x) - x^*x)) = 0$ by the invariance of ρ. Thus $\mathcal{T}_t(x^*x) = x^*x$ for all $t \geq 0$ because ρ is faithful and so $x^*x \in \mathcal{F}(\mathcal{T})$.

(3) If $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$, then it is generated by its projections which belong to the commutant of $\{L_\ell, L_\ell^*, H \mid \ell \geq 1\}$ by Lemma 7. Thus $\mathcal{F}(\mathcal{T})$ is contained in this commutant. Conversely, any x commuting with L_ℓ, L_ℓ^*, H satisfies $\mathcal{L}(x) = 0$ and so $\mathcal{T}_t(x) = x$. □

We finish this section by recalling two results on the asymptotic behaviour of a QMS related with $\mathcal{F}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})$. The first one follows from an application of the mean ergodic theorem (see [23] (Theorem 1.1)).

Theorem 9 *For a QMS $\mathcal{T}$ with a faithful invariant state the limit*

$$\mathcal{E}(x) = w^* - \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathcal{T}_s(x) ds$$

exists for all $x \in \mathcal{B}(\mathsf{h})$ and defines a $\mathcal{T}$-invariant normal conditional expectation $\mathcal{E}$ onto the von Neumann algebra $\mathcal{F}(\mathcal{T})$ of fixed points for $\mathcal{T}$.

The second one, proved in [23] Theorem 3.3, ensures that maps $\mathcal{T}_t$ converge to the above conditional expectation as t tends to infinity.

Theorem 10 *Suppose that there exists a faithful family of normal invariant states for the QMS $\mathcal{T}$. Then $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ implies that*

$$w^* - \lim_{t\to\infty} \mathcal{T}_t(x) = \mathcal{E}(x)$$

The idea behind this result is quite simple. If $x \in \mathcal{N}(\mathcal{T})$, then $\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH} x\, \mathrm{e}^{-\mathrm{i}tH}$, thus we may find some x for which $\mathcal{T}_t(x)$ oscillates (for instance an eigenvector of $\mathcal{L}$ with purely imaginary eigenvalue). This can not happen if $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$.

3 The structure of QMS with atomic decoherence-free subalgebra

In this section we prove our main results on the structure of QMS with atomic decoherence-free subalgebra. The starting point of our analysis is Proposition 6. Since the central projections p_i in (5) are fixed points for $\mathcal{T}_t$, by Lemma 7, we have $\mathcal{L}^n(p_i x p_i) = p_i \mathcal{L}^n(x) p_i$, for all $n \geq 0$ and so

$$\mathcal{T}_t(p_i x p_i) = \mathcal{T}_t(p_i)\mathcal{T}_t(x)\mathcal{T}_t(p_i) = p_i \mathcal{T}_t(x) p_i \tag{7}$$

for each $x \in \mathcal{N}(\mathcal{T})$ and each factor $p_i\mathcal{N}(\mathcal{T})p_i$ is $\mathcal{T}_t$-invariant.

Moreover, it would not be difficult to see that the decoherence-free subalgebra of the restriction of $\mathcal{T}$ to bounded operators on $p_i\mathsf{h}$ is $p_i\mathcal{N}(\mathcal{T})p_i$. Thus, we begin by considering the case where $\mathcal{N}(\mathcal{T})$ is a type I factor and investigate the implications on the structure of $\mathcal{T}$.

We recall that, by well known results on the structure of type I factors (see e.g. Jones [25], Theorem 4.2.1), in this case, there exist two Hilbert spaces k and m and a unitary operator $U : \mathsf{h} \to \mathsf{k} \otimes \mathsf{m}$ such that

$$U\mathcal{N}(\mathcal{T})U^* = \mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}. \tag{8}$$

Exploiting this structure we can prove our first result.

Theorem 11 *Suppose that $\mathcal{N}(\mathcal{T})$ is a type I factor, and let k, m be Hilbert spaces and $U : \mathsf{h} \to \mathsf{k} \otimes \mathsf{m}$ a unitary operator satisfying (8). Then:*

1. *for every GKSL representation (1) of the generator $\mathcal{L}$ by means of operators L_ℓ, H, we have*

$$UL_\ell U^* = \mathbb{1}_{\mathsf{k}} \otimes M_\ell \quad \forall\, \ell \geq 1, \qquad UHU^* = K \otimes \mathbb{1}_{\mathsf{m}} + \mathbb{1}_{\mathsf{k}} \otimes M_0,$$

where M_ℓ are operators on m such that the series $\sum_\ell M_\ell^* M_\ell$ is strongly convergent and K (resp. M_0) is a self-adjoint operator on k (resp. m),

2. *defining the GKSL generators $\mathcal{L}^{\mathsf{k}}$ on $\mathcal{B}(\mathsf{k})$ and $\mathcal{L}^{\mathsf{m}}$ on $\mathcal{B}(\mathsf{m})$ by*

$$\begin{aligned}\mathcal{L}^{\mathsf{k}}(a) &= \mathrm{i}\,[K,a]\,, &(9)\\ \mathcal{L}^{\mathsf{m}}(y) &= \mathrm{i}\,[M_0,y]\\ &- \frac{1}{2}\sum_{\ell\geq 1}\left(M_\ell^* M_\ell\, y - 2M_\ell^* y M_\ell + y\, M_\ell^* M_\ell\right) &(10)\end{aligned}$$

the QMSs $\mathcal{T}^{\mathsf{k}}$ on $\mathcal{B}(\mathsf{k})$ generated by $\mathcal{L}^{\mathsf{k}}$ and $\mathcal{T}^{\mathsf{m}}$ on $\mathcal{B}(\mathsf{m})$ generated by $\mathcal{L}^{\mathsf{m}}$ satisfy $U\mathcal{T}_t(x)U^* = (\mathcal{T}_t^{\mathsf{k}}\otimes\mathcal{T}_t^{\mathsf{m}})(UxU^*)$ for all $x\in\mathcal{B}(\mathsf{h})$,

3. *we have $\mathcal{T}_t^{\mathsf{k}}(a) = \mathrm{e}^{\mathrm{i}tK}a\,\mathrm{e}^{-\mathrm{i}tK}$ for all $a\in\mathcal{B}(\mathsf{k})$, $t\geq 0$; moreover, $\mathcal{N}(\mathcal{T}^{\mathsf{k}}) = \mathcal{B}(\mathsf{k})$ and $\mathcal{N}(\mathcal{T}^{\mathsf{m}}) = \mathbb{C}\mathbb{1}_{\mathsf{m}}$.*

Proof. Let L_ℓ, H be the operators of a GKSL representation (1) of the generator $\mathcal{L}$. Since $\mathcal{N}(\mathcal{T})$ is contained in the commutant of L_ℓ and L_ℓ^* by Propostion 3, it follows that $UL_\ell U^*$ and $UL_\ell^* U^*$ belong to the commutant of $\mathcal{B}(\mathsf{k})\otimes\mathbb{1}_{\mathsf{m}}$ and so they are operators of the form $\mathbb{1}_{\mathsf{k}}\otimes M_\ell$ and $\mathbb{1}_{\mathsf{k}}\otimes M_\ell^*$ for some bounded operator M_ℓ on m. The series $\sum_{\ell\geq 1} M_\ell^* M_\ell$ is strongly convergent on m because, if we fix a vector $u\in\mathsf{k}$, then, for each vector $v\in\mathsf{m}$ we have

$$\begin{aligned}u\otimes\left(\sum_{1\leq\ell\leq n} M_\ell^* M_\ell\, v\right) &= \sum_{1\leq\ell\leq n}\left(\mathbb{1}_{\mathsf{k}}\otimes M_\ell\right)^*\left(\mathbb{1}_{\mathsf{k}}\otimes M_\ell\right)(u\otimes v)\\ &= U\left(\sum_{1\leq\ell\leq n} L_\ell^* L_\ell\right)U^*(u\otimes v)\end{aligned}$$

for all $n\geq 1$, and the series $\sum_\ell L_\ell^* L_\ell$ is strongly convergent on h.

We now turn to UHU^*. For any $x\in\mathcal{N}(\mathcal{T})$ we have $UxU^* = x_0\otimes\mathbb{1}_{\mathsf{m}}$ with $x_0\in\mathcal{B}(\mathsf{k})$ and $\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH}x\,\mathrm{e}^{-\mathrm{i}tH}$, so that

$$U\mathcal{T}_t(U^*(x_0\otimes\mathbb{1}_{\mathsf{m}})U)U^* = (U\mathrm{e}^{\mathrm{i}tH}U^*)\,(x_0\otimes\mathbb{1}_{\mathsf{m}})\,(U\mathrm{e}^{\mathrm{i}tH}U^*)^*.$$

By the $\mathcal{T}_t$-invariance of $\mathcal{N}(\mathcal{T})$, the right-hand side has the form

$$(W_t\otimes\mathbb{1}_{\mathsf{m}})(x_0\otimes\mathbb{1}_{\mathsf{m}})(W_t\otimes\mathbb{1}_{\mathsf{m}})^*$$

for a one-parameter group $(W_t)_{t\in\mathbb{R}}$ of unitary operators on k. Thus, by defining $V_t = U\mathrm{e}^{\mathrm{i}tH}U^*$, we have

$$V_t(x_0\otimes\mathbb{1}_{\mathsf{m}})V_t^* = (W_t\otimes\mathbb{1}_{\mathsf{m}})(x_0\otimes\mathbb{1}_{\mathsf{m}})(W_t\otimes\mathbb{1}_{\mathsf{m}})^*$$

namely, for all $x_0\in\mathcal{B}(\mathsf{k})$,

$$\left((W_t\otimes\mathbb{1}_{\mathsf{m}})^*V_t\right)(x_0\otimes\mathbb{1}_{\mathsf{m}}) = (x_0\otimes\mathbb{1}_{\mathsf{m}})\left((W_t\otimes\mathbb{1}_{\mathsf{m}})^*V_t\right).$$

It follows that $(W_t \otimes \mathbb{1}_{\mathsf{m}})^* V_t$ must be of the form $\mathbb{1}_{\mathsf{k}} \otimes R_t$ with unitaries R_t on m and, by the group property of $(V_t)_{t\in\mathbb{R}}$, also $(R_t)_{t\in\mathbb{R}}$ must be a group. Denoting $\mathrm{i}K$ and $\mathrm{i}M_0$ the generators of the unitary groups $(W_t)_{t\in\mathbb{R}}$ and $(R_t)_{t\in\mathbb{R}}$ respectively, we find

$$UHU^* = K \otimes \mathbb{1}_{\mathsf{m}} + \mathbb{1}_{\mathsf{k}} \otimes M_0. \tag{11}$$

This proves 1.

To prove 2, note first that $\mathcal{L}^{\mathsf{k}}$ and $\mathcal{L}^{\mathsf{m}}$ generate QMSs $\mathcal{T}^{\mathsf{k}}$ and $\mathcal{T}^{\mathsf{m}}$ and

$$U\mathcal{L}(U^*(a \otimes y)U)U^* = \mathcal{L}^{\mathsf{k}}(a) \otimes y + a \otimes \mathcal{L}^{\mathsf{m}}(y),$$

for all $a \in \mathcal{B}(\mathsf{k})$, $y \in \mathcal{B}(\mathsf{m})$, so that, by the weak* density of the linear span of operators $a \otimes y$ in $\mathcal{B}(\mathsf{h})$, the QMSs $(U\mathcal{T}_t(U^* \cdot U)U^*)_{t\geq 0}$ and $(\mathcal{T}^{\mathsf{k}}_t \otimes \mathcal{T}^{\mathsf{m}}_t)_{t\geq 0}$ have the same generator.

Clearly $\mathcal{T}^{\mathsf{k}}_t(a) = \mathrm{e}^{\mathrm{i}tK} a\, \mathrm{e}^{\mathrm{i}tK}$ for all $t \geq 0$, and so $\mathcal{N}(\mathcal{T}^{\mathsf{k}}) = \mathcal{B}(\mathsf{k})$. Moreover, if p is a projection in $\mathcal{N}(\mathcal{T}^{\mathsf{m}})$, then, by Proposition 3 of the decoherence-free subalgebra, p commutes with all iterated commutators $\delta^n_{M_0}(M_\ell)$, $\delta^n_{M_0}(M^*_\ell)$ ($n \geq 0, \ell \geq 1$). Thus, recalling (11), $\mathbb{1}_{\mathsf{k}} \otimes p$ commutes with all iterated commutators

$$\delta^n_{UHU^*}(\mathbb{1}_{\mathsf{k}} \otimes M_\ell) = U\delta^n_H(L_\ell)U^* \qquad \delta^n_{UHU^*}(\mathbb{1}_{\mathsf{k}} \otimes M^*_\ell) = U\delta^n_H(L^*_\ell)U^*,$$

i.e. it belongs to $U\mathcal{N}(\mathcal{T})U^* = \mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$ and so $p = \mathbb{1}_{\mathsf{m}}$.

This proves 3.

Theorem 11 shows that maps $\mathcal{T}_t$ factorise as the composition of the commuting maps $\mathcal{T}^{\mathsf{k}}_t \otimes I_{\mathcal{B}(\mathsf{m})}$ and $I_{\mathcal{B}(\mathsf{k})} \otimes \mathcal{T}^{\mathsf{m}}_t$ where $I_{\mathcal{B}(\mathsf{m})}$ (resp. $I_{\mathcal{B}(\mathsf{k})}$) is the identity map on $\mathcal{B}(\mathsf{m})$ (resp. $\mathcal{B}(\mathsf{k})$). The former is the decoherence-free factor and the latter is the decoherence-affected factor by item 3. The generator $\mathcal{L}$ of $\mathcal{T}$ is the sum of two commuting generators $I_{\mathcal{B}(\mathsf{k})} \otimes \mathcal{L}^{\mathsf{m}}$ and $\mathcal{L}^{\mathsf{k}} \otimes I_{\mathcal{B}(\mathsf{m})} = \mathrm{i}\,[K \otimes \mathbb{1}_{\mathsf{m}}, \cdot\,]$. This result can be be interpreted as the independence of the decoherence-free (noiseless) and the noisy part of the system.

Remark 1 If $\mathcal{N}(\mathcal{T})$ is an atomic algebra, by Proposition 31, we can find a family $(p_i)_{i\in I}$ of mutually orthogonal projections which are minimal in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ such that $\sum_{i\in I} p_i = \mathbb{1}$ and satisfying (5). Moreover, each $p_i\mathcal{N}(\mathcal{T})p_i$ a type I factor acting on the Hilbert space $p_i\mathsf{h}$; thus, there exist two countable sequences of Hilbert spaces $(\mathsf{k}_i)_{i\in I}$, $(\mathsf{m}_i)_{i\in I}$, and unitary operators $U_i : p_i\mathsf{h} \to \mathsf{k}_i \otimes \mathsf{m}_i$ such that

$$U_i p_i \mathcal{N}(\mathcal{T}) p_i U^*_i = \mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i}, \qquad U_i p_i \mathcal{B}(\mathsf{h}) p_i U^*_i = \mathcal{B}(\mathsf{k}_i \otimes \mathsf{m}_i). \tag{12}$$

Therefore, defining $U = \oplus_{i\in I} U_i$, we obtain a unitary operator $U : \mathsf{h} \to \oplus_{i\in I}(\mathsf{k}_i \otimes \mathsf{m}_i)$ such that

$$U\mathcal{N}(\mathcal{T})U^* = \oplus_{i\in I} \left(\mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i}\right). \tag{13}$$

We now establish our main result using the structure of $\mathcal{N}(\mathcal{T})$ given by (13).

Theorem 12 *Suppose that $\mathcal{N}(\mathcal{T})$ is an atomic algebra and let $(\mathsf{k}_i)_{i\in I}$, $(\mathsf{m}_i)_{i\in I}$ be two countable sequences of Hilbert spaces and $U = \oplus_{i\in I} U_i$ be a unitary operator associated with a family $(p_i)_{i\in I}$ as in Remark 1. Then:*

1. *for every GSKL representation of $\mathcal{L}$ by means of operators $H, (L_\ell)_{\ell\geq 1}$, we have*
$$UL_\ell U^* = \oplus_{i\in I}\left(\mathbb{1}_{\mathsf{k}_i}\otimes M_\ell^{(i)}\right)$$
for a collection $(M_\ell^{(i)})_{\ell\geq 1}$ of operators in $\mathcal{B}(\mathsf{m}_i)$, such that the series $\sum_{\ell\geq 1} M_\ell^{(i)*}M_\ell^{(i)}$ strongly convergent for all $i\in I$, and
$$UHU^* = \oplus_{i\in I}\left(K_i\otimes \mathbb{1}_{\mathsf{m}_i} + \mathbb{1}_{\mathsf{k}_i}\otimes M_0^{(i)}\right)$$
for self-adjoint operators $K_i\in\mathcal{B}(\mathsf{k}_i)$ and $M_0^{(i)}\in\mathcal{B}(\mathsf{m}_i)$, $i\in I$,
2. *defining on the algebra $\mathcal{B}\left(\oplus_{i\in I}\left(\mathsf{k}_i\otimes\mathsf{m}_i\right)\right)$*
$$\mathcal{L}^{\mathsf{df}} = \mathrm{i}\left[\oplus_{i\in I}(K_i\otimes\mathbb{1}_{\mathsf{m}_i}),\cdot\right] \tag{14}$$
and $\mathcal{L}^{\mathsf{da}}$ as the Lindblad operator given by
$$\{\oplus_{i\in I}\left(\mathbb{1}_{\mathsf{k}_i}\otimes M_\ell^{(i)}\right), \oplus_{i\in I}(\mathbb{1}_{\mathsf{k}_i}\otimes M_0^{(i)})\,|\, l\geq 1\},$$
we find the commuting generators $\mathcal{L}^{\mathsf{df}}$ and $\mathcal{L}^{\mathsf{da}}$ of two commuting QMSs $\mathcal{T}^{\mathsf{df}}$ (the decoherence-free semigroup) and $\mathcal{T}^{\mathsf{da}}$ (the decoherence-affected semigroup) such that $\widetilde{\mathcal{T}}_t = \mathcal{T}_t^{\mathsf{da}}\circ\mathcal{T}_t^{\mathsf{df}} = \mathcal{T}_t^{\mathsf{df}}\circ\mathcal{T}_t^{\mathsf{da}}$, where $\widetilde{\mathcal{T}}$ is the QMS defined by
$$\widetilde{\mathcal{T}}_t(UxU^*) = U\mathcal{T}_t(x)U^* \qquad \forall\, x\in\mathcal{B}(\mathsf{h}). \tag{15}$$
In particular, we have
$$\mathcal{L}^{\mathsf{da}}(x) = \oplus_{i\in I}\left(I_{\mathcal{B}(\mathsf{k}_i)}\otimes\mathcal{L}^{\mathsf{m}_i}\right)(x) = \oplus_{i\in I}\left(a_i\otimes\mathcal{L}^{\mathsf{m}_i}(y_i)\right)$$
for all $x = \oplus_{i\in I}(a_i\otimes y_i)$ with $a\in\mathcal{B}(\mathsf{k}_i)$ and $y\in\mathcal{B}(\mathsf{m}_i)$, where $\mathcal{L}^{\mathsf{m}_i}$ is given by (10),
3. *the action of $\mathcal{T}^{\mathsf{df}}$ is explicitly given by $\mathcal{T}_t^{\mathsf{df}}(x) = \mathrm{e}^{\mathrm{i}tK}x\mathrm{e}^{-\mathrm{i}tK}$ for all $x\in\mathcal{B}\left(\oplus_{i\in I}\left(\mathsf{k}_i\otimes\mathsf{m}_i\right)\right)$, where K is the self-adjoint operator $\oplus_{i\in I}(K_i\otimes\mathbb{1}_{\mathsf{m}_i})$; moreover $\mathcal{N}(\mathcal{T}^{\mathsf{df}}) = \mathcal{B}\left(\oplus_{i\in I}\left(\mathsf{k}_i\otimes\mathsf{m}_i\right)\right)$ and $\mathcal{N}(\mathcal{T}^{\mathsf{da}}) = U\mathcal{N}(\mathcal{T})U^*$.*

Proof. Note that, since each p_i is a fixed point for $\mathcal{T}_t$, by Proposition 6, the algebra $p_i\mathcal{B}(\mathsf{h})p_i = \mathcal{B}(p_i\mathsf{h})$ is preserved by the action of every map $\mathcal{T}_t$, and so we can consider the restriction of $\mathcal{T}$ to this algebra, getting a

QMS on $\mathcal{B}(p_i\mathsf{h})$ denoted by $\mathcal{T}^{(i)}$. Since, for all $x \in \mathcal{N}(\mathcal{T})$, by Proposition 1 and Lemma 7 we have $\mathcal{T}_t(p_i x^* p_i x p_i) = p_i \mathcal{T}_t(x^* p_i x) p_i = p_i \mathcal{T}_t(x^*) \mathcal{T}_t(p_i x) p_i = p_i \mathcal{T}_t(x^*) p_i \mathcal{T}_t(x) p_i = \mathcal{T}_t(p_i x^* p_i) \mathcal{T}_t(p_i x p_i)$, namely $p_i x p_i \in \mathcal{N}(\mathcal{T})^{(i)}$, it is easy to see that the decoherence-free subalgebra $\mathcal{N}(\mathcal{T}^{(i)})$ of $\mathcal{T}^{(i)}$ is exactly $p_i \mathcal{N}(\mathcal{T}) p_i$. Moreover, given a GSKL representation of $\mathcal{L}$ by means of operators $H, (L_\ell)_{\ell \geq 1}$, since $p_i \in \mathcal{N}(\mathcal{T})$ commutes with every L_ℓ by Proposition 3, and consequently also with H (being p_i a fixed point), the operators $p_i H p_i, (p_i L_\ell p_i)_{\ell \geq 1}$ provide a GSKL representation of the generator $\mathcal{L}^{(i)}$ of $\mathcal{T}^{(i)}$. Therefore, applying Theorem 11 to $\mathcal{T}^{(i)}$, we get

$$U_i p_i L_\ell p_i U_i^* = \mathbb{1}_{\mathsf{k}_i} \otimes M_\ell^{(i)} \qquad U_i p_i H p_i U_i^* = K_i \otimes \mathbb{1}_{\mathsf{m}_i} + \mathbb{1}_{\mathsf{k}_i} \otimes M_0^{(i)}$$

for some operators $K_i = K_i^*$ in $\mathcal{B}(\mathsf{k}_i)$ and $M_0^{(i)} = (M_0^{(i)})^*, (M_\ell^{(i)})_{\ell \geq 1}$ in $\mathcal{B}(\mathsf{m}_i)$. Since $U = \oplus_{i \in I} U_i$, item 1 is proved.

The claim 2 follows by the same argument of Theorem 11 claim 2.

The explicit action of $\mathcal{T}^{\mathsf{df}}$ is also clear, and so $\mathcal{N}(\mathcal{T}^{\mathsf{df}}) = \mathcal{B}\left(\oplus_{i \in I} \left(\mathsf{k}_i \otimes \mathsf{m}_i\right)\right)$.

Finally, by Proposition 3, an operator x is in $\mathcal{N}(\mathcal{T}^{\mathsf{da}})$ if and only if it commutes with all iterated commutators

$$\begin{aligned}
\delta^n_{\oplus_{i \in I}\left(\mathbb{1}_{\mathsf{k}_i} \otimes M_0^{(i)}\right)} \left(\oplus_{i \in I}\left(\mathbb{1}_{\mathsf{k}_i} \otimes M_\ell^{(i)}\right)\right) &= \delta^n_{UHU^*}\left(\oplus_{i \in I}\left(\mathbb{1}_{\mathsf{k}_i} \otimes M_\ell^{(i)}\right)\right) \\
&= U \delta^n_H(L_\ell) U^*, \\
\delta^n_{\oplus_{i \in I}\left(\mathbb{1}_{\mathsf{k}_i} \otimes M_0^{(i)}\right)} \left(\oplus_{i \in I}\left(\mathbb{1}_{\mathsf{k}_i} \otimes M_\ell^{(i)*}\right)\right) &= \delta^n_{UHU^*}\left(\oplus_{i \in I}\left(\mathbb{1}_{\mathsf{k}_i} \otimes M_\ell^{(i)*}\right)\right) \\
&= U \delta^n_H(L_\ell^*) U^*,
\end{aligned}$$

because all $K_j \otimes \mathbb{1}_{\mathsf{m}_j}$ and $\mathbb{1}_{\mathsf{k}_i} \otimes M_\ell^{(i)}$ commute, and so $x \in U \mathcal{N}(\mathcal{T}) U^*$.

Remark 2 Note that, in particular, the central projection p_i is minimal in $\mathcal{N}(\mathcal{T})$ if and only if k_i is a one-dimensional space, i.e. $U_i p_i \mathcal{N}(\mathcal{T}) p_i U_i^* = \mathbb{C}\mathbb{1}_{\mathsf{m}_i}$. Moreover, defining the QMS $\widetilde{\mathcal{T}}$ as in (15), we have

$$\widetilde{\mathcal{T}}_t(a \otimes b) = \mathrm{e}^{\mathrm{i}tK_i} a \mathrm{e}^{-\mathrm{i}tK_i} \otimes \mathcal{T}_t^{\mathsf{m}_i}(b)$$

for all $a \in \mathcal{B}(\mathsf{k}_i),\ b \in \mathcal{B}(\mathsf{m}_i),\ i \in I,\ t \geq 0$, where $\mathcal{T}^{\mathsf{m}_i}$ is the QMS on $\mathcal{B}(\mathsf{m}_i)$ generated by $\mathcal{L}^{\mathsf{m}_i}$. Finally, $\mathcal{N}(\mathcal{T}^{\mathsf{m}_i}) = \mathbb{C}\mathbb{1}_{\mathsf{m}_i}$ for all $i \in I$.

Theorem 12 also provides a constructive method for finding the decoherence-free part of a quantum Markovian evolution starting from the decomposition (5). The following proposition turns out to be useful when we want to identify $\mathcal{N}(\mathcal{T})$.

4 Structure of normal invariant states

In this section we give a complete description of invariant states of a QMS with atomic decoherence-free subalgebra. We omit the word normal in order to simplify the terminology since we are interested only in normal states; moreover states will be often identified with their densities.

We begin by recalling some well-known properties of invariant states. The *support projection* $s(\rho)$ of a state ρ is defined as the orthogonal projection onto its range. More precisely, if $\rho = \sum_{j\in J} \lambda_j |e_j\rangle\langle e_j|$ with $(e_j)_j$ orthonormal vectors in h and $\lambda_j > 0$ for all $j \in J$, then $s(\rho) = \sum_{j\in J} |e_j\rangle\langle e_j|$. In particular, ρ is faithful if and only if $s(\rho) = \mathbb{1}$.

The support projection p of an invariant state ρ is *subharmonic* ([20]) Theorem II.1, [38] Theorem 1) i.e. $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$. Useful properties of subharmonic projections are collected in the following proposition (see e.g. [23]).

Proposition 13 *Let $p \in \mathcal{B}(\mathsf{h})$ be a suhbarmonic projection. Then:*

1. *for all state σ with $s(\sigma) \leq p$, the support of the normal state $\mathcal{T}_{*t}(\sigma)$ also satisfies $s(\mathcal{T}_{*t}(\sigma)) \leq p$ for all $t \geq 0$,*
2. *$p\mathcal{T}_t(pxp)p = p\mathcal{T}_t(x)p$ for all $x \in \mathcal{B}(\mathsf{h})$, $t \geq 0$*
3. *the one-parameter family of linear maps $(\mathcal{T}^p_t)_{t\geq 0}$ defined by $\mathcal{T}^p_t(x) = p\mathcal{T}_t(x)p$ for $x \in p\mathcal{B}(\mathsf{h})p$ is a QMS on $p\mathcal{B}(\mathsf{h})p$, called the* reduced *QMS,*
4. *p is harmonic, i.e. $\mathcal{T}_t(p) = p$ for all $t \geq 0$, if and only if it belongs to the commutant $\{L_k, L_k^*, H \mid k \geq 1\}'$; in this case if ρ is a $\mathcal{T}$-invariant state such that $\mathrm{tr}\,(\rho p) \neq 0$, then*

$$\rho_p := p\rho p/\mathrm{tr}\,(\rho p) \tag{16}$$

   is an invariant state for the reduced QMS $\mathcal{T}^p$; moreover, if ρ is faithful, then ρ_p is faithful on $p\mathcal{B}(\mathsf{h})p$ (i.e. $s(\rho_p) = p$).

Proof. 1. If $p^\perp$ is the orthogonal projection $\mathbb{1} - p$, for all $t \geq 0$ we find $0 \leq \mathrm{tr}\left(\mathcal{T}_{*t}(\sigma)p^\perp\right) = \mathrm{tr}\left(\sigma\mathcal{T}_t(p^\perp)\right) \leq \mathrm{tr}\left(\sigma p^\perp\right) = 0$. It follows that $p^\perp\mathcal{T}_{*t}(\sigma)p^\perp = 0$ and so, by positivity of $\mathcal{T}_{*t}(\sigma)$, we have $\mathcal{T}_{*t}(\sigma) = p\mathcal{T}_{*t}(\sigma)p$.

2. Let x be a positive operator in $\mathcal{B}(\mathsf{h})$. Every state ω with support smaller than p satisfies $\omega = p\omega = \omega p$, therefore we have $\mathrm{tr}\,(\omega p\mathcal{T}_t(pxp)p) = \mathrm{tr}\,(\mathcal{T}_{*t}(\omega)pxp)$. Now, since also the support of $\mathcal{T}_{*t}(\omega)$ is smaller than p, we find

$$\mathrm{tr}\,(\omega p\mathcal{T}_t(pxp)p) = \mathrm{tr}\,(\mathcal{T}_{*t}(\omega)x) = \mathrm{tr}\,(\omega\mathcal{T}_t(x)) = \mathrm{tr}\,(\omega p\mathcal{T}_t(x)p)$$

and the conclusion follows.

3. For all $x \in p\mathcal{B}(\mathsf{h})p$ and $t, s \geq 0$ we have from 2

$$\mathcal{T}^p_{t+s}(x) = p\mathcal{T}_t\left(\mathcal{T}_s(x)\right)p = p\mathcal{T}_t\left(p\mathcal{T}_s(x)p\right)p = p\mathcal{T}_t\left(\mathcal{T}^p_s(x)\right)p = \mathcal{T}^p_t\left(\mathcal{T}^p_s(x)\right).$$

Moreover, since p is subharmonic and smaller than $\mathbb{1}$, we have also $\mathcal{T}_t^p(p) = p$. Complete positivity and continuity properties are immediate.

4. The first part of the claim follows from Lemma 7.

It is clear that $s(\rho_p) \leq p$. Since p commutes with each L_k, L_k^* and with H, we have $\mathcal{T}_t(x) = \mathcal{T}_t(pxp) = p\mathcal{T}_t(x)p$ for all $x \in p\mathcal{B}(\mathsf{h})p$ and $t \geq 0$. Hence, we find

$$\operatorname{tr}\left(\rho_p \mathcal{T}_t^p(x)\right) = \frac{\operatorname{tr}\left(\rho\, p\mathcal{T}_t(pxp)p\right)}{\operatorname{tr}\left(\rho p\right)} = \frac{\operatorname{tr}\left(\rho \mathcal{T}_t(x)\right)}{\operatorname{tr}\left(\rho p\right)} = \frac{\operatorname{tr}\left(\rho x\right)}{\operatorname{tr}\left(\rho p\right)} = \operatorname{tr}\left(\rho_p x\right)$$

and so ρ_p is a $\mathcal{T}^p$-invariant state. Assume now that ρ faithful. Given $x \in \mathcal{B}(\mathsf{h})$ such that $pxp \geq 0$, the equality $0 = \operatorname{tr}\left(\rho_p x\right) = \operatorname{tr}\left(\rho pxp\right)/\operatorname{tr}\left(\rho p\right)$ implies $pxp = 0$ by the faithfulness of ρ. Hence, ρ_p is faithful on $p\mathcal{B}(\mathsf{h})p$. □

We refer the interested reader to the recent paper [24] for additional information on the support of states evolving under the action of a QMS.

If $(q_i)_{i\in I}$ is a collection of subharmonic projections, the projection p onto the linear span of subspaces $q_i\mathsf{h}$ is also subharmonic ([38] Proposition 3). We can then define the *(fast) recurrent projection* p_R as the smallest projection in h containing the support of all invariant states

$$p_R := \sup\{s(\sigma) \mid \sigma \text{ invariant state}\}.$$

Moreover, we can always find an invariant state having p_R as support (see Theorem 4 of [38]). As a consequence, the reduced QMS $\mathcal{T}^{p_R}$ on $p_R\mathcal{B}(\mathsf{h})p_R = \mathcal{B}(p_R(\mathsf{h}))$ has a faithful invariant state.

Since this section is devoted to the description of invariant states, in the sequel, we consider this reduced semigroup dropping the exponent p_R and *assuming the existence of a faithful invariant state.*

As a consequence we have the following

Proposition 14 *Let $\mathcal{T}$ be a QMS with a faithful invariant state ρ and let $\mathcal{N}(\mathcal{T})$ be as in (5) with $(p_i)_{i\in I}$ minimal projections in the center of $\mathcal{N}(\mathcal{T})$. Then $p_i\sigma p_j = 0$ for all $i \neq j$ and for every invariant state σ.*

Proof. Since central projections p_i, p_j are in $\mathcal{F}(\mathcal{T})$, for all $t \geq 0$ and $x \in \mathcal{B}(\mathsf{h})$ we have $\mathcal{T}_t(p_j x p_i) = p_j \mathcal{T}_t(x) p_i$ and also $\mathcal{T}_{*t}(p_i\sigma p_j) = p_i \mathcal{T}_{*t}(\sigma) p_j$ for all trace class operator σ. It follows that

$$\operatorname{tr}\left(p_i\sigma p_j x\right) = \operatorname{tr}\left(p_i \mathcal{T}_{*s}(\sigma) p_j x\right) = \operatorname{tr}\left(\sigma \mathcal{T}_s(p_j x p_i)\right) = \operatorname{tr}\left(\sigma p_j \mathcal{T}_s(x) p_i\right)$$

for all invariant state σ and $x \in \mathcal{B}(\mathsf{h})$. Integrating on $[0, t]$ and dividing by t we find

$$\operatorname{tr}\left(p_i\sigma p_j x\right) = \operatorname{tr}\left(\sigma p_j \left(t^{-1}\int_0^t \mathcal{T}_s(x) ds\right) p_i\right),$$

and, taking the limit as $t \to \infty$, by Theorem 9, we have,

$$\mathrm{tr}\left(p_i \sigma p_j x\right) = \mathrm{tr}\left(p_i \sigma p_j \mathcal{E}(x)\right)$$

where $\mathcal{E}(x) \in \mathcal{F}(\mathcal{T})$. Now, since $\mathcal{F}(\mathcal{T})$ is also contained in $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I} p_i \mathcal{N}(\mathcal{T}) p_i$, we get $p_j \mathcal{E}(x) p_i = 0$ for $i \neq j$ as well as $\mathrm{tr}\left(p_i \sigma p_j x\right) = 0$, and so $p_i \sigma p_j = 0$ by the arbitrarity of x. □

Item 4 of Proposition 13 and Proposition 14 show that, for studying the structure of invariant states, (with a unitary transformation as in Theorem 11) we can restrict ourselves to the case where we are given a QMS $\mathcal{T}$ with $\mathcal{N}(\mathcal{T}) = \mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$ with a faithful invariant state ρ. In other words, we can now identify $\widetilde{\mathcal{T}}$ and $\mathcal{T}$ and suppose that $\mathcal{N}(\mathcal{T})$ is a type I factor.

Before we begin our study of this case, it will be useful to remind ourselves of some properties of partial traces. We refer to S. Attal's lecture notes [5] for proofs. Given two Hilbert spaces k and m, for every $f \in \mathsf{m}$ we define the bounded operator

$$\left|f\right\rangle_{\mathsf{m}} : \mathsf{k} \to \mathsf{k} \otimes \mathsf{m}, \qquad \left|f\right\rangle_{\mathsf{m}} e = e \otimes f$$

with adjoint operator

$$_{\mathsf{m}}\left\langle f\right| : \mathsf{k} \otimes \mathsf{m} \to \mathsf{k}, \qquad _{\mathsf{m}}\left\langle f\right| u \otimes v = \langle f, v\rangle\, u.$$

For a trace-class operator σ on $\mathsf{k} \otimes \mathsf{m}$ the partial trace of σ with respect to m is the trace-class operator on k defined by

$$\mathrm{tr}_{\mathsf{m}}\left(\sigma\right) = \sum_{n\geq 1} {}_{\mathsf{m}}\left\langle f_n\right| \sigma \left|f_n\right\rangle_{\mathsf{m}},$$

where $(f_n)_{n\geq 1}$ is an orthonormal basis of m. It can be shown that the above series is convergent with respect to the trace norm and its sum does not depend on the choice of the orthonormal basis of m. Moreover, the partial trace $\mathrm{tr}_{\mathsf{m}}\left(\sigma\right)$ is the only trace-class operator on k satisfying $\mathrm{tr}\left(\sigma a \otimes \mathbb{1}_{\mathsf{m}}\right) = \mathrm{tr}\left(\mathrm{tr}_{\mathsf{m}}\left(\sigma\right) a\right)$ for all $a \in \mathcal{B}(\mathsf{k})$.

Lemma 15 *Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathsf{k} \otimes \mathsf{m})$ with an invariant state ρ such that $\mathcal{T}_t(a \otimes b) = \mathcal{T}_t^{\mathsf{k}}(a) \otimes \mathcal{T}_t^{\mathsf{m}}(b)$ for all $t \geq 0$, $a \in \mathcal{B}(\mathsf{k}), b \in \mathcal{B}(\mathsf{m})$ where $\mathcal{T}^{\mathsf{k}}$ and $\mathcal{T}^{\mathsf{m}}$ are QMS on $\mathcal{B}(\mathsf{k})$ and $\mathcal{B}(\mathsf{m})$ respectively. The partial trace $\mathrm{tr}_{\mathsf{m}}\left(\rho\right)$ (resp. $\mathrm{tr}_{\mathsf{k}}\left(\rho\right)$) is an invariant state for the QMS $\mathcal{T}^{\mathsf{k}}$ (resp. $\mathcal{T}^{\mathsf{m}}$). Furthermore, if ρ is faithful, then also $\mathrm{tr}_{\mathsf{m}}\left(\rho\right)$ and $\mathrm{tr}_{\mathsf{k}}\left(\rho\right)$ are faithful.*

Proof. For all $a \in \mathcal{B}(\mathsf{k})$, by the properties of the partial trace, we have

$$\mathrm{tr}\left(\mathrm{tr}_{\mathsf{m}}\left(\rho\right) \mathcal{T}_t^{\mathsf{k}}(a)\right) = \mathrm{tr}\left(\rho\left(\mathcal{T}_t^{\mathsf{k}}(a) \otimes \mathbb{1}_{\mathsf{m}}\right)\right) = \mathrm{tr}\left(\rho \mathcal{T}_t(a \otimes \mathbb{1}_{\mathsf{m}})\right)$$

so that, by the invariance of ρ,

$$\mathrm{tr}\left(\mathrm{tr}_{\mathsf{m}}(\rho)\,\mathcal{T}_t^{\mathsf{k}}(a)\right) = \mathrm{tr}\left(\rho(a\otimes \mathbb{1}_{\mathsf{m}})\right) = \mathrm{tr}\left(\mathrm{tr}_{\mathsf{m}}(\rho)\,a\right).$$

This proves that $\mathrm{tr}_{\mathsf{m}}(\rho)$ is an invariant state for the QMS $\mathcal{T}^{\mathsf{k}}$. Clearly, we can prove that $\mathrm{tr}_{\mathsf{k}}(\rho)$ is an invariant state for the $\mathcal{T}^{\mathsf{m}}$ in the same way.

Finally, if ρ is faithful on $\mathcal{B}(\mathsf{k}\otimes\mathsf{m})$, then also $\mathrm{tr}_{\mathsf{m}}(\rho)$ is faithful on $\mathcal{B}(\mathsf{m})$ because, for all positive $b\in\mathcal{B}(\mathsf{m})$, $\mathbb{1}_{\mathsf{k}}\otimes b$ is positive and we have $\mathrm{tr}\left(\mathrm{tr}_{\mathsf{m}}(\rho)\,b\right) = \mathrm{tr}\left(\rho(\mathbb{1}_{\mathsf{k}}\otimes b)\right)$. We can check in the same way that $\mathrm{tr}_{\mathsf{k}}(\rho)$ is faithful. □

We now study invariant states for the QMS $\mathcal{T}^{\mathsf{m}}$. We begin by recalling the notion of irreducibility and highlighting its relationship with the structure of $\mathcal{N}(\mathcal{T})$.

Definition 16 *A QMS $\mathcal{T}$ on a von Neumann algebra $\mathcal{M}$ is said to be* irreducible *if there exist no non-trivial projection $p\in\mathcal{M}$ satisfying $\mathcal{T}_t(p)\geq p$ for all $t\geq 0$.*

Proposition 17 *Assume that $\mathcal{N}(\mathcal{T})$ is atomic and there exists a faithful invariant state ρ. If $\mathcal{T}$ is irreducible, then both $\mathcal{N}(\mathcal{T})$ and $\mathcal{F}(\mathcal{T})$ are trivial.*

Proof. Since $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$, if it were non-trivial, it would contain a non-trivial projection p so that $\mathcal{T}_t(p)=p$ contradicting irreducibility.
As a consequence, by Proposition 6, the center of $\mathcal{N}(\mathcal{T})$ is trivial, i.e. $\mathcal{N}(\mathcal{T})$ is a type I factor and we can apply Theorem 11. Let k and K, be as in Theorem 11. If K is not a multiple of the identity operator on k, considering a non-trivial projection p on k commuting with K, the operator $p\otimes\mathbb{1}_{\mathsf{m}}$ is a non-trivial projection p which is a fixed point for $\mathcal{T}$ contradicting irreducibility. Thus, since K is a multiple of the identity operator, for all $a\in\mathcal{N}(\mathcal{T})$, we have $\mathcal{T}_t(a)=a$ so that $\mathcal{N}(\mathcal{T})=\mathcal{F}(\mathcal{T})$ is trivial. □

We now exploit properties of irreducible QMS for characterising invariant states of semigroups $\mathcal{T}^{\mathsf{m}_i}$.

Theorem 18 *Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathsf{k}\otimes\mathsf{m})$ with a faithful invariant state ρ and $\mathcal{N}(\mathcal{T})=\mathcal{B}(\mathsf{k})\otimes\mathbb{1}_{\mathsf{m}}$. Then the QMS $\mathcal{T}^{\mathsf{m}}$ on $\mathcal{B}(\mathsf{m})$ is irreducible, has a unique invariant state τ_{m} and, for all trace-class operator η on m, we have*

$$w-\lim_{t\to\infty}\mathcal{T}_{*t}^{\mathsf{m}}(\eta) = \mathrm{tr}(\eta)\,\tau_{\mathsf{m}} \qquad (17)$$

Proof. Let p be a non-zero subharmonic projection for $\mathcal{T}^{\mathsf{m}}$, i.e. $\mathcal{T}_t^{\mathsf{m}}(p)\geq p$ for all $t\geq 0$, then $\mathbb{1}_{\mathsf{k}}\otimes p$ is a subharmonic projection for $\mathcal{T}$. By the invariance of ρ we have $\mathrm{tr}\,(\rho(\mathcal{T}_t(\mathbb{1}_{\mathsf{k}}\otimes p) - \mathbb{1}_{\mathsf{k}}\otimes p)) = 0$, and so $\mathcal{T}_t(\mathbb{1}_{\mathsf{k}}\otimes p) = \mathbb{1}_{\mathsf{k}}\otimes p$ since ρ is faithful. This means that

$$\mathbb{1}_{\mathsf{k}}\otimes p \in \mathcal{F}(\mathcal{T}) \subseteq \mathcal{N}(\mathcal{T}) = \mathcal{B}(\mathsf{k})\otimes \mathbb{1}_{\mathsf{m}},$$

i.e. $p = \mathbb{1}_{\mathsf{m}}$. Thus $\mathcal{T}^{\mathsf{m}}$ is irreducible.

Moreover, since ρ is a faithful invariant state for $\mathcal{T}$, its partial trace $\mathrm{tr}_{\mathsf{k}}\,(\rho)$ is a faithful invariant state for $\mathcal{T}^{\mathsf{m}}$ by Lemma 15, and so $\mathcal{N}(\mathcal{T}^{\mathsf{m}})$ is trivial thanks to Proposition 17. Therefore, $\mathcal{F}(\mathcal{T}^{\mathsf{m}}) = \mathcal{N}(\mathcal{T}^{\mathsf{m}}) = \mathbb{C}\mathbb{1}_{\mathsf{m}}$. It follows then from Theorem 10 that $w^* - \lim_{t\to\infty}\mathcal{T}^{\mathsf{m}}_t(b) \in \mathcal{F}(\mathcal{T}^{\mathsf{m}})$ exists and is a multiple of the identity operator. Taking the trace with respect to the invariant state $\tau_{\mathsf{m}} := \mathrm{tr}_{\mathsf{k}}\,(\rho)$ the limit is easily shown to be $\mathrm{tr}\,(\tau_{\mathsf{m}} b)$. It follows that, for all trace-class operator η on $\mathcal{B}(\mathsf{m})$ and all $b\in\mathcal{B}(\mathsf{m})$ we have then

$$\lim_{t\to\infty}\mathrm{tr}\,(\mathcal{T}^{\mathsf{m}}_{*t}(\eta) b) = \mathrm{tr}\,(\eta)\,\mathrm{tr}\,(\tau_{\mathsf{m}} b)$$

and (17) is proved.

In the proof of our result on the structure of invariant states we need the following

Lemma 19 *Let $\alpha = (\alpha_t)_{t\geq 0}$ be a semigroup of automorphisms of $\mathcal{B}(\mathsf{k})$ given by $\alpha_t(a) = \mathrm{e}^{\mathrm{i}tK} a \mathrm{e}^{-\mathrm{i}tK}$ for some bounded self-adjoint operator K on k. If ω is a faithful normal invariant state for α, then K has pure point spectrum.*

Proof. Let $\omega = \sum_{j\geq 1}\omega_j q_j$ be the spectral decomposition of ω with strictly positive eigenvalues in decreasing order $\omega_1 > \omega_2 > \dots$ and q_j finite-dimensional mutually orthogonal projections such that $\sum_{j\geq 1} q_j = \mathbb{1}$. Clearly, $\mathrm{e}^{-\mathrm{i}tK}\omega\,\mathrm{e}^{\mathrm{i}tK} = \omega$ since ω is an invariant state and $\mathrm{e}^{-\mathrm{i}tK}\omega^n\,\mathrm{e}^{\mathrm{i}tK} = \omega^n$ by the homomorphism property for all $n\geq 1$, and so

$$\sum_{j\geq 1}\omega_j^n\,\mathrm{e}^{-\mathrm{i}tK} q_j\,\mathrm{e}^{\mathrm{i}tK} = \sum_{j\geq 1}\omega_j^n\, q_j. \tag{18}$$

Dividing both sides by $\omega_1^n > 0$ and taking the limit as $n\to\infty$, we find $\mathrm{e}^{-\mathrm{i}tK} q_1 \mathrm{e}^{\mathrm{i}tK} = q_1$ for all $t\geq 0$ and so q_1 commutes with K. Removing the term $j = 1$ in (18) we can prove by the same argument that q_2 commutes with K and so on recursively. It follows that $K = \sum_j q_j K q_j$. Clearly, each $q_j K q_j$ is a self-adjoint operator on the finite-dimensional space $q_j\mathsf{h}$, and so we can find an orthonormal basis of eigenvectors of K of each of these subspaces. Vectors of these orthonormal bases form an orthonormal basis of h by the faithfulness of ω. □

We can now prove the main theorem characterising the structure of $\mathcal{T}$-invariant states.

Theorem 20 *Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathsf{k}\otimes\mathsf{m})$ with a faithful invariant state ρ and $\mathcal{N}(\mathcal{T})=\mathcal{B}(\mathsf{k})\otimes\mathbb{1}_{\mathsf{m}}$ and let τ_{m} be the unique invariant state of the partially traced semigroup $\mathcal{T}^{\mathsf{m}}$. If η is a $\mathcal{T}$-invariant state, then*

$$\eta=\sigma\otimes\tau_{\mathsf{m}} \tag{19}$$

where σ is a state on $\mathcal{B}(\mathsf{k})$ whose density commutes with K.

Proof. By Lemma 19 we can find an orthonormal basis $(e_j)_{j\geq 1}$ of eigenvectors of K so that $Ke_j=\kappa_j e_j$ for some $\kappa_j\in\mathbb{R}$. Moreover, if η is an invariant state, we can define trace-class operators on m by products of bounded and trace-class operators, as $\eta_{jk}={}_{\mathsf{k}}\langle e_j|\eta|e_k\rangle_{\mathsf{k}}$ so that

$$\eta=\sum_{j,k\geq 1}|e_j\rangle\langle e_k|\otimes\eta_{jk}.$$

By Theorem 11

$$\eta=\mathcal{T}_{*t}(\eta)=\sum_{j,k\geq 1}\mathrm{e}^{\mathrm{i}(\kappa_k-\kappa_j)t}|e_j\rangle\langle e_k|\otimes\mathcal{T}^{\mathsf{m}}_{*t}(\eta_{jk})$$

and so, by the linear independence of rank one operators $|e_j\rangle\langle e_k|$,

$$\mathrm{e}^{\mathrm{i}(\kappa_j-\kappa_k)t}\eta_{jk}=\mathcal{T}^{\mathsf{m}}_{*t}(\eta_{jk})$$

for all j,k. Each operator $\mathcal{T}^{\mathsf{m}}_{*t}(\eta_{jk})$ tends to $\mathrm{tr}\,(\eta_{jk})\,\tau_{\mathsf{m}}$ as $t\to\infty$ (in the weak topology) by Theorem 18. Thus, if $\kappa_j\neq\kappa_k$, we find $\mathrm{tr}\,(\eta_{jk})=0$, while, if $\kappa_j=\kappa_k$ we have $\mathrm{tr}\,(\eta_{jk})\,\tau_{\mathsf{m}}=\eta_{jk}$. It follows that

$$\eta=\sum_{j,k}\left(\mathrm{tr}\,(\eta_{jk})\,|e_j\rangle\langle e_k|\right)\otimes\tau_{\mathsf{m}}.$$

Defining $\sigma:=\sum_{j,k}\mathrm{tr}\,(\eta_{jk})\,|e_j\rangle\langle e_k|$, (19) follows. Finally, a straightforward computation yields

$$K\sigma-\sigma K=\sum_{j,k}\left(\mathrm{tr}\,(\eta_{jk})\,(\kappa_j-\kappa_k)\right)|e_j\rangle\langle e_k|=0,$$

since $\mathrm{tr}\,(\eta_{jk})=0$ for $\kappa_j\neq\kappa_k$, and so K commutes with σ.

If $\mathcal{N}(\mathcal{T})$ is not a type I factor, but it is atomic, then from Theorem 20 and Proposition 14 we have immediately the following

Theorem 21 *Assume that $\mathcal{N}(\mathcal{T})$ is atomic and there exists a faithful $\mathcal{T}$-invariant state. Let $(p_i)_{i\in I}$, $(\mathsf{k}_i)_{i\in I}$, $(\mathsf{m}_i)_{i\in I}$, $(K_i)_{i\in I}$, $U:\mathsf{h}\to\oplus_{i\in I}(\mathsf{k}_i\otimes\mathsf{m}_i)$ be as in Theorem 12. A $\mathcal{T}$-invariant state η can be written in the form*

$$U\eta U^*=\sum_{i\in I}\mathrm{tr}\,(\eta p_i)\,\sigma_i\otimes\tau_{\mathsf{m}_i}$$

where, for every $i\in I$,

1. τ_{m_i} *is the unique* $\mathcal{T}^{\mathsf{m}_i}$*-invariant state which is also faithful,*
2. σ_i *is a density on* k_i *commuting with* K_i*.*

Remark 3 Under the conditions of Theorem 21, all K_i have pure point spectrum by Lemma 19. By considering the spectral decomposition $K_i = \sum_j \kappa_j q_{ij}$ with $(q_{ij})_{j\in J_i}$ mutually orthogonal projections such that $\sum_{j\in J_i} q_{ij} = \mathbb{1}_{\mathsf{k}_i}$ (and $\kappa_j \neq \kappa_{j'}$ for $j \neq j'$), we can write the unitary isomorphism

$$\mathsf{k}_i \otimes \mathsf{m}_i = (\oplus_{j\in J_i} q_{ij}\mathsf{k}_i) \otimes \mathsf{m}_i.$$

Now every density σ commuting with K_i can be written in the form $\sigma = \sum_{j\in J_i} \sigma_j$ where $(\sigma_j)_{j\in J_i}$ is a collection of positive trace-class operators on subspaces $q_{ij}\mathsf{k}_i$ normalized by $\sum_{j\in J_i} \mathrm{tr}\,(\sigma_j) = 1$. Clearly, if the projection q_{ij} is one-dimensional, i.e. the eigenvalue κ_j is simple, then σ_j is a scalar r_j, say, in $[0,1]$. As a consequence, every invariant state supported in $\mathsf{k}_i \otimes \mathsf{m}_i$ turns out to be written (up to a unitary isomorphism) in the form

$$\sigma \otimes \tau_{\mathsf{m}_i} = \sum_{j\in J_i} \sigma_j \otimes \tau_{\mathsf{m}_i} = \sum_{j\in J_i,\, \dim(q_{ij})=1} r_j \tau_{\mathsf{m}_i} + \sum_{j\in J_i,\, \dim(q_{ij})>1} \sigma_j \otimes \tau_{\mathsf{m}_i}$$

for positive constants r_j and arbitrary trace-class operators σ_j on eigenspaces of K_i with dimension strictly bigger than 1.

From Theorem 21 it follows that each invariant state can be written as

$$\sum_k c_k \tau_k + \sum_m d_m \eta_m \otimes \tau_m$$

where c_k and d_m are non-negative numbers, $\sum_k c_k + \sum_m d_m = 1$, η_m can be any density matrix on an eigenspace of some K_i with dimension strictly bigger than 1, and τ_m is the unique invariant state of some $\mathcal{T}^{\mathsf{m}_i}$.

The same result holds for any QMS with atomic decoherence-free subalgebra $\mathcal{N}(\mathcal{T}^R)$ of the semigroup $\mathcal{T}^R$ reduced by the fast recurrent projection p_R.

This generalises the result proved by Baumgartner and Narnhofer in [6] (Theorem 7) in the finite dimensional case.

5 Applications to decoherence

In this section we apply our results to the study of environment induced decoherence ([8, 12, 30, 31]) and to the identification of subsystems of an open quantum system which are not affected by decoherence ([3, 27, 29, 35]).

5.1 Environment induced decoherence

We say that there is *environmental induced decoherence* (EID) on the system described by $\mathcal{T}$ if there exists a $\mathcal{T}_t$-invariant and $*$-invariant weak* closed subspace $\mathcal{M}_2$ of $\mathcal{B}(\mathsf{h})$ such that:

(EID1) $\mathcal{B}(\mathsf{h}) = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$,

(EID2) $w^* - \lim_{t\to\infty} \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$.

We refer to [12] Section 2 for a discussion of this concept. As an application of our result on the structure of QMS we will now give a sufficient condition for EID.

Theorem 22 *Assume that $\mathcal{N}(\mathcal{T})$ is atomic and $\mathcal{T}$ possesses a faithful normal invariant state. Then EID holds.*

Proof. Since $\mathcal{T}$ possesses a faithful normal invariant state ρ, say, it is enough to establish the existence of a normal conditional expectation $\mathcal{E}$ onto $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I} (\mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i})$ which is compatible with ρ, i.e. such that $\rho\circ\mathcal{E} = \rho$ (Theorem 18 of [15]).

Given $x \in \mathcal{B}(\mathsf{h})$, we write $x = \sum_{l,m} p_l x p_m$ with minimal projections in the center of $\mathcal{N}(\mathcal{T})$ as in (5), and identify each $p_l x p_m$ with a bounded operator $x_{lm} : \mathsf{k}_m \otimes \mathsf{m}_m \to \mathsf{k}_l \otimes \mathsf{m}_l$. Identifying an operator on $\mathsf{k}_i \otimes \mathsf{m}_i$ with its extension as the zero operator on the orthogonal subspace, we then define

$$\mathcal{E} : \mathcal{B}(\mathsf{h}) \to \mathcal{N}(\mathcal{T}), \qquad \mathcal{E}(x) := \sum_{i\in I} \mathcal{E}_i(x_{ii})$$

with $\mathcal{E}_i : \mathcal{B}(\mathsf{k}_i \otimes \mathsf{m}_i) \to \mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i}$ given by

$$\mathcal{E}_i(a) = \sum_j {}_{\mathsf{m}_i}\langle f_j \,|\, (\mathbb{1}_{\mathsf{k}_i} \otimes \tau_{\mathsf{m}_i}) a \,|\, f_j \rangle_{\mathsf{m}_i} \otimes \mathbb{1}_{\mathsf{m}_i},$$

for each $a \in \mathcal{B}(\mathsf{k}_i \otimes \mathsf{m}_i)$, where τ_{m_i} is the unique faithful invariant state for $\mathcal{T}^{\mathsf{m}_i}$ and $(f_j)_j$ is an orthonormal basis of m_i diagonalizing τ_{m_i}. It is easy to see that every $\mathcal{E}_i$ is a positive normal map such that $\mathcal{E}_i^2 = \mathcal{E}_i$ and $\mathcal{E}_i(\mathbb{1}_{\mathsf{k}_i\otimes\mathsf{m}_i}) = \mathbb{1}_{\mathsf{k}_i\otimes\mathsf{m}_i}$, so that each $\mathcal{E}_i$ is a normal conditional expectation onto $\mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i}$. Consequently, $\mathcal{E}$ is a normal conditional expectation onto $\mathcal{N}(\mathcal{T}) = \oplus_{i\in I} (\mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i})$.

Now, we have to show that $\mathcal{E}$ is compatible with ρ. First, note that, since ρ is invariant, by Theorem 21 we have $\rho = \sum_{i\in I} \sigma_i \otimes \tau_i$ for some trace-class operators σ_i on k_i commuting with K_i. Therefore, $\mathrm{tr}\,(\rho x) = \sum_{i\in I} \mathrm{tr}\,((\sigma_i \otimes \tau_i) x_{ii})$, with $x_{ii} = p_i x p_i$, for all $x \in \mathcal{B}(\mathsf{h})$, and $\mathrm{tr}\,(\rho\mathcal{E}(x)) = \sum_{i\in I} \mathrm{tr}\,(\rho\mathcal{E}_i(x_{ii}))$, so that it is enough to prove that every $\mathcal{E}_i$ is compatible with $\sigma_i \otimes \tau_i$.

Now, for $a \in \mathcal{B}(\mathsf{k}_i \otimes \mathsf{m}_i)$ we easily compute

$$\begin{aligned}
\operatorname{tr}((\sigma_i \otimes \tau_i)\mathcal{E}_i(a)) &= \sum_j \operatorname{tr}((\sigma_i \otimes \tau_i)({}_{\mathsf{m}_i}\langle f_j \,|\, (\mathbb{1}_{\mathsf{k}_i} \otimes \tau_i) a \,|\, f_j\rangle_{\mathsf{m}_i} \otimes \mathbb{1}_{\mathsf{m}_i})) \\
&= \sum_j \operatorname{tr}(\sigma_i \, {}_{\mathsf{m}_i}\langle f_j \,|\, (\mathbb{1}_{\mathsf{k}_i} \otimes \tau_i) a \,|\, f_j\rangle_{\mathsf{m}_i} \otimes \tau_i) \\
&= \sum_j \operatorname{tr}(\sigma_i \, {}_{\mathsf{m}_i}\langle f_j \,|\, (\mathbb{1}_{\mathsf{k}_i} \otimes \tau_i) a \,|\, f_j\rangle_{\mathsf{m}_i}) \\
&= \operatorname{tr}((\sigma_i \otimes \mathbb{1}_{\mathsf{m}_i})(\mathbb{1}_{\mathsf{k}_i} \otimes \tau_i) a) = \operatorname{tr}((\sigma_i \otimes \tau_i) a),
\end{aligned}$$

from which the required result follows.

5.2 Decoherence-free subsystems and subspaces

A quantum subsystem can be thought of intuitively as "portion" of the full system, whose states, in the simplest setting, faithfully embody quantum information. More precisely, following Ticozzi and Viola ([35] Definition 4), we call *quantum subsystem* of a system on h a system whose Hilbert space is a tensor factor h_{s} of a subspace h_{sf} of h, i.e.

$$\mathsf{h} = \mathsf{h}_{\mathsf{sf}} \oplus \mathsf{h}_{\mathsf{r}} = (\mathsf{h}_{\mathsf{s}} \otimes \mathsf{h}_{\mathsf{f}}) \oplus \mathsf{h}_{\mathsf{r}} \tag{20}$$

for some factor h_{f} and remainder space h_{r}.

Definition 23 *Let h be decomposed as in (20). We say that h_{s} supports a* decoherence-free (or a noiseless) subsystem *for some QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ if and only the evolution of a factorised initial state $\rho = \rho_{\mathsf{s}} \otimes \rho_{\mathsf{f}}$, with ρ_{s} state on $\mathcal{B}(\mathsf{h}_{\mathsf{s}})$ and ρ_{f} state on $\mathcal{B}(\mathsf{h}_{\mathsf{f}})$, is given by*

$$\mathcal{T}_{*t}(\rho) = U_t \rho_{\mathsf{s}} U_t^* \otimes \mathcal{T}^{\mathsf{f}}_{*t}(\rho_{\mathsf{f}})$$

for $t \geq 0$, where U_t is a unitary operator on h_{f} and $\mathcal{T}^{\mathsf{f}}$ is a QMS on $\mathcal{B}(\mathsf{h}_{\mathsf{f}})$. We say that h_{s} supports a decoherence-free subspace *if h_{f} is one-dimensional, i.e. $\mathsf{h} \simeq \mathsf{h}_{\mathsf{s}} \oplus \mathsf{h}_{\mathsf{r}}$.*

Note that the above definition of decoherence-free subspace is clearly equivalent to the usual one (see [27] Definition 1).

Applying Theorem 12 we can easily identify quantum subsystems for a given QMS. Indeed, using the same notation as in Theorem 12, we have the following

Proposition 24 *If $\mathcal{N}(\mathcal{T})$ is an atomic algebra then:*

1. *every subspace* k_i *with factor* m_i *and remainder space* $\oplus_{j\in I\setminus\{i\}}(\mathsf{k}_j\otimes\mathsf{m}_j)$, *supports a decoherence-free subsystem for* $\mathcal{T}$,
2. *every* k_i *with* $\dim\mathsf{m}_i = 1$ *supports a decoherence-free subspace for* $\mathcal{T}$*; in particular, we have* $M_\ell^{(i)} = \lambda_\ell^{(i)}\mathbb{1}_{\mathsf{m}_i}$ *for all* ℓ *and* $M_0^{(i)} = \alpha_i\mathbb{1}_{\mathsf{m}_i}$ *for some* $\lambda_\ell^{(i)}$, $\alpha_i\in\mathbb{R}$,
3. *every subspace* $\oplus_{j\in J}\mathsf{k}_j$, *where* $J\subseteq I_1 := \{i\in I \mid \dim\mathsf{m}_i = 1, \lambda_\ell^{(i)} = \lambda_\ell\ \forall\,\ell\}$ *(for some collection* $(\lambda_\ell)_{\ell\ge 1}\subseteq\mathbb{R}$*) with trivial factor* $\mathsf{h}_\mathsf{f} = \mathbb{C}$ *and remainder space* $\oplus_{j\in I\setminus J}(\mathsf{k}_j\otimes\mathsf{m}_j)$, *supports a decoherence-free subspace for* $\mathcal{T}$.

Proof. All the above can be proved straightforwardly applying Theorem 12. We check, for instance, item 3.

We have to show that

$$\mathcal{T}_{*t}(|u\rangle\langle u|\otimes\mathbb{1}) = \mathrm{e}^{-\mathrm{i}tK}|u\rangle\langle u|\mathrm{e}^{\mathrm{i}tK}\otimes\mathbb{1} \tag{21}$$

for some self-adjoint operator K on $\oplus_{j\in J}\mathsf{k}_j$ and for all $u\in\oplus_{j\in J}\mathsf{k}_j$.

So, let $u = \sum_{j\in J}u_j$ with $u_j\in\mathsf{k}_j$ for $j\in J$, and let K_j be the self-adjoint operator on k_j in Theorem 12(2), identified with its standard ampliation to h. Theorem 12 gives

$$\mathcal{T}_{*t}(|u_j\rangle\langle u_j|\otimes\mathbb{1}) = \mathrm{e}^{-\mathrm{i}tK_j}|u_j\rangle\langle u_j|\mathrm{e}^{\mathrm{i}tK_j}\otimes\mathbb{1}.$$

Now, given $j,h\in J$, we have

$$\begin{aligned}\mathcal{L}_*(|u_j\rangle\langle u_h|\otimes\mathbb{1}) &= -\mathrm{i}\left(|(K_j+\alpha_j)u_j\rangle\langle u_h| - |u_j\rangle\langle(K_h+\alpha_h)u_h|\right)\otimes\mathbb{1}\\ &= -\mathrm{i}[(K_j+\alpha_j)\oplus(K_h+\alpha_h), |u_j\rangle\langle u_h|]\otimes\mathbb{1},\end{aligned}$$

since the action of the dissipative part on $|u_j\rangle\langle u_h|$ is 0 being $M_l^{(j)} = \lambda_\ell = M_\ell^{(h)}$ for $j,h\in J$ and for every ℓ. Therefore, we obtain

$$\begin{aligned}\mathcal{L}_*(|u\rangle\langle u|) &= \sum_{j,h\in J}\mathcal{L}_*(|u_j\rangle\langle u_h|)\\ &= -\mathrm{i}\sum_{j,h\in J}[(K_j+\alpha_j)\oplus(K_h+\alpha_h), |u_j\rangle\langle u_h|]\otimes\mathbb{1}\\ &= -\mathrm{i}\sum_{j,h\in J}[\oplus_{l\in J}(K_l+\alpha_l), |u_j\rangle\langle u_h|]\otimes\mathbb{1}\\ &= -\mathrm{i}\,[\oplus_{j\in J}(K_j+\alpha_j), |u\rangle\langle u|]\otimes\mathbb{1}\end{aligned}$$

and, consequently, equation (21) is satisfied with $K := \oplus_{j\in J}(K_j+\alpha_j)$.

6 Examples

6.1 Generic semigroups

Take $\mathsf{h} = \ell^2(I)$ the Hilbert space of square-summable, complex-valued sequences, with $I \subseteq \mathbb{N}$ a finite or infinite set, and denote by $(e_n)_{n\in I}$ the canonical orthonormal basis of h.

We consider a class of uniformly continuous QMSs on $\mathcal{B}(\mathsf{h})$ whose generators can be represented in the canonical GKSL form

$$\mathcal{L}(x) = G^* x + \sum_{j\neq m} L_{mj}^* x L_{mj} + xG$$

where

$$G = -\sum_{m\in\mathbb{N}} \left(\frac{\gamma_{mm}}{2} + i\kappa_m\right) |e_m\rangle\langle e_m|, \qquad L_{mj} = \sqrt{\gamma_{mj}}\, |e_j\rangle\langle e_m|,$$

for $j \neq m$ with $\kappa_m \in \mathbb{R}$, $\gamma_{mj} \geq 0$ for every $m \neq j$, $\gamma_{mm} := -\sum_{j\neq m} \gamma_{mj} < +\infty$ for any m, and

$$\sup_i |\kappa_i| < +\infty, \qquad \sup_i |\gamma_{ii}| < +\infty. \tag{22}$$

Note that $\mathcal{L}$ is bounded thanks to condition (22), and can be written in the form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{j\neq m} \left(L_{jm}^* L_{jm} x - 2L_{jm}^* x L_{jm} + x L_{jm}^* L_{jm}\right)$$

with $H = \sum_{j\in I} \kappa_j |e_j\rangle\langle e_j|$.

These semigroups, called *generic*, were introduced by Accardi and Kozyrev in [2]; they arise in the stochastic limit of a open quantum system with generic Hamiltonian, interacting with a zero mean, gauge invariant, Gaussian field (see also [1, 10]).

The restriction of $\mathcal{L}$ to the diagonal algebra $\mathcal{D}$, generated by rank one projections $|e_n\rangle\langle e_n|$, is the generator Γ of a classical time continuous Markov chain $(X_t)_{t\geq 0}$ with states I (see [10]). In particular, denoted by $\mathcal{T}$ the QMS generated by $\mathcal{L}$, for every $x = \sum_{n\in I} f(n)|e_n\rangle\langle e_n| \in \mathcal{D}$, we have

$$\mathcal{L}(x) = \sum_{n\in I} (\Gamma f)(n) |e_n\rangle\langle e_n|, \quad \Gamma f = \sum_{j\in I} \gamma_{nj} f(j) e_j, \tag{23}$$

$$\mathcal{T}_t(|e_n\rangle\langle e_n|) = \sum_{k\in I} \mathbb{P}\left\{X_t = n \,|\, X_0 = k\right\} |e_k\rangle\langle e_k| \quad \forall\, n \in I. \tag{24}$$

For these semigroups, we recall the characterisation of the decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ (see [14] Theorem 9 and [11] Theorem 26). Note first of all that $[H, L_{mj}] = (\kappa_j - \kappa_m) L_{mj}$ and so $\delta_H^n(L_{mj}) = (\kappa_j - \kappa_m)^n L_{mj}$ for all $n \geq 0$. It follows immediately from Proposition 3 that $\mathcal{N}(\mathcal{T})$ is the commutant of the set $\{L_{mj}, L_{mj}^* \mid j, m \in I\}$, i.e. the commutant of the set of rank-one operators $\{\, |e_j\rangle\langle e_m|, |e_m\rangle\langle e_j| \;:\; j, m \in I,\ \gamma_{jm} + \gamma_{mj} > 0\,\}$. Clearly a self-adjoint x belongs to this commutant if and only if $|xe_j\rangle\,\langle e_m| = |e_j\rangle\,\langle xe_m|$, namely $xe_j = \chi_j e_j$ and $xe_m = \chi_m e_m$ with $\chi_j = \chi_m = \chi \in \mathbb{R}$. Moreover, if $\gamma_{jm} \neq 0$, but there exists $k_1, \ldots k_n$ in I with $\gamma_{jk_1}\gamma_{k_1k_2} \cdots \gamma_{k_n m} \neq 0$, then $xe_j = \chi e_j, xe_{k_1} = \chi e_{k_1}, \ldots, xe_m = \chi e_m$.

Thus, denoting by $\mathcal{C}_n$ $n \geq 1$, communication classes of states i such that $\gamma_{ij} + \gamma_{ji} > 0$ for some $j \neq i$ with respect to the standard equivalence relation on states associated with the rates matrix Q obtained from Γ, for example, in the following way $Q_{ij} = \gamma_{ij} + 2^{-j} 1_{\{\gamma_{ji} > 0\}}$ for $i \neq j$, $Q_{ii} = -\sum_{j \neq i} Q_{ij}$, and denoting by

$$\mathrm{Iso} := \{ i \in I : \gamma_{ij} = \gamma_{ji} = 0 \ \ \forall\, j \neq i \}$$

the set of the isolated states of the Markov chain $(X_t)_t$, we have the following result:

Proposition 25 *The decoherence-free subalgebra of the generic QMS generated by (22) is the von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ generated by projections $p_n = \sum_{k \in \mathcal{C}_n} |e_k\rangle\langle e_k|$ with $n \geq 1$, corresponding to the above communication classes, and by rank-one operators $|e_i\rangle\langle e_j|$ with $i, j \in \mathrm{Iso}$.*

Proof. We want now to find a maximal family of mutually orthogonal minimal projections in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$, in order to apply Theorem 12.

An operator $x \in \mathcal{N}(\mathcal{T})$ can be represented as $x_0 + \sum_{n \geq 1} \chi_n p_n$ with $x_0 = p_0 x p_0 \in \mathcal{B}(p_0 \mathsf{h})$ and $\chi_n \in \mathbb{C}$. Therefore, an element $y = y_0 + \sum_{n \geq 1} z_n p_n$ in $\mathcal{N}(\mathcal{T})$, with $y_0 = p_0 y p_0 \in \mathcal{B}(p_0 \mathsf{h})$ and $z_n \in \mathbb{C}$, commutes with all $x \in \mathcal{N}(\mathcal{T})$ if and only if y_0 commutes with all operators $x_0 \in \mathcal{B}(p_0 \mathsf{h})$, namely $y_0 = z_0 p_0$ for some $z_0 \in \mathbb{C}$. Consequently, since every p_n with $n \geq 1$ is minimal in $\mathcal{N}(\mathcal{T})$, Remark 2 formula (12), yields

$$p_n \mathsf{h} \simeq \mathsf{m}_n \quad \forall n \geq 1, \qquad p_0 \mathsf{h} \simeq \mathsf{k}_0$$

for some complex Hilbert spaces $(\mathsf{m}_n)_{n \geq 1}$ and k_0, while Hilbert spaces m_0 and $(\mathsf{k}_n)_{n \geq 1}$ are one-dimensional. By Theorem 12, the decoherence-free and decoherence-affected semigroups are generated by

$$\mathcal{L}^{\mathsf{df}} = \mathrm{i}[\, p_0 H p_0 \,, \cdot\,] \qquad \mathcal{L}^{\mathsf{da}} = \mathcal{L} - \mathcal{L}^{\mathsf{df}}.$$

6.2 Circulant QMS

In the previous example for each minimal central projection p_i either k_i or m_i is trivial, i.e. one-dimensional. We now consider a paradigm case exhibiting non-trivial k_i and m_i for the same index i.

Let $\mathsf{h} = \mathbb{C}^d$ $(d \geq 2)$, let $n \in \{1, \ldots, d-1\}$ and let J the unitary circular shift defined by $Je_i = e_{i-1}$ (sum modulo d) with respect to an orthonormal basis of h. We consider the QMS on $\mathcal{B}(\mathsf{h}) = M_d(\mathbb{C})$ generated by (1) with

$$L_1 = z_1 J^n, \qquad L_2 = z_2 J^{*n} = z_2 J^{-n}$$

where z_1, z_2 are complex constants with $z_1 \cdot z_2 \neq 0$, $L_\ell = 0$ for $\ell \geq 2$. We begin by considering $H = 0$. This is a circulant QMS of those studied by Bolaños and Quezada in [9].

Let $k = GCD(n, d)$ (Greatest Common Divisor) and let $d = km$. The decoherence-free subalgebra $\mathcal{N}(\mathcal{T}) = \{\, J^n, J^{n*}\}'$ is characterised as follows.

Proposition 26 *The algebra $\mathcal{N}(\mathcal{T})$ is the commutant of $\{J^k, J^{-k}\}'$.*

Proof. The set of powers $hn \pmod d$ with $h \in \mathbb{Z}$ is $\{0, k, 2k, \ldots, (m-1)k\}$. Indeed, since nk^{-1} and dk^{-1} are coprime, we can find integers a, b such that $ank^{-1} + bdk^{-1} = 1$ i.e. $k = an + bd$, and so $k = an \mod d$. Let $x \in \mathcal{N}(\mathcal{T}) = \{J^n, J^{-n}\}'$, then $[x, J^k] = [x, J^{an}] = 0$ and $[x, J^{-k}] = [x, J^{-an}] = 0$, thus $\mathcal{N}(\mathcal{T}) \subseteq \{J^k, J^{-k}\}'$. On the other hand, since $n = hk$ for some natural number h, if $[x, J^k] = 0 = [x, J^{-k}]$, by induction on j we have also $[x, J^{kj}] = 0 = [x, J^{-kj}]$ and so $[x, J^n] = [x, J^{hk}] = 0$, $[x, J^{-n}] = [x, J^{-hk}] = 0$, namely $\{J^k, J^{-k}\}' \subseteq \mathcal{N}(\mathcal{T})$.

Proposition 27 *The center $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ of $\mathcal{N}(\mathcal{T})$ is the double commutant $\{J^k, J^{-k}\}''$ (the abelian *-algebra generated by J^k), namely the linear space $\mathsf{P}(J^k)$ of complex polynomials in J^{kh} for $0 \leq h \leq m-1$.*

Proof. Clearly

$$\mathcal{Z}(\mathcal{N}(\mathcal{T})) = \mathcal{N}(\mathcal{T}) \cap \mathcal{N}(\mathcal{T})' = \{J^k, J^{-k}\}' \cap \{J^k, J^{-k}\}''.$$

Note that $J^k, J^{-k} \in \{J^k, J^{-k}\}'$ therefore the double commutant of $\{J^k, J^{-k}\}$ is contained in the commutant of $\{J^k, J^{-k}\}$ and $\mathcal{Z}(\mathcal{N}(\mathcal{T})) = \{J^k, J^{-k}\}''$. Since J is a normal operator this is the algebra generated by J^k. Indeed, by definition, all powers J^{kh} are contained in $\{J^k, J^{-k}\}''$ because they commute with every operator commuting with J^k and J^{-k} and so $\mathsf{P}(J^k) \subseteq \{J^k, J^{-k}\}''$. Conversely, $\mathsf{P}(J^k)$ is a *-algebra and contains J^k and $J^{-k} = J^{k(d-1)}$, therefore it coincides with $\{J^k, J^{-k}\}''$. Finally we can restrict exponents to kh with $0 \leq h \leq m-1$ by the Cayley-Hamilton theorem. $\square$

We now identify minimal central projections. Since the *-algebra generated by the normal operator J^k coincides with the vector space generated by its spectral projections, these turn out to be minimal central projections in $\mathcal{N}(\mathcal{T})$.

Spectral projections of J^k can be found explicitly from the well-known spectral decomposition of the circulant matrix J. Let $\omega := \mathrm{e}^{2\pi\mathrm{i}/d}$ be a primitive d-th root of unit; eigenvalues of J and corresponding eigenvectors are

$$\lambda_j = \omega^j \qquad v_j = \frac{1}{\sqrt{d}}(1, \omega^j, \ldots, \omega^{j(d-1)}).$$

for $j = 0, \ldots, d-1$. It follows that

$$J = \sum_{j=0}^{d-1} \omega^j |v_j\rangle\langle v_j| \qquad \text{and so} \qquad J^k = \sum_{j=0}^{d-1} \omega^{jk} |v_j\rangle\langle v_j|.$$

Now, writing $j \in \{0, \ldots, d-1\}$ as $j = mr + h$ with $0 \le h \le m-1 = d\,k^{-1} - 1$, we find that r must belong to $\{0, \ldots, k-1\}$ and the eigenvalues of J^k are

$$\omega^{jk} = \mathrm{e}^{2\pi\mathrm{i}(r+h/m)} = \mathrm{e}^{2\pi\mathrm{i}h/m}, \qquad h = 0, \ldots, m-1,$$

since $d = km$. Moreover, for each $h = 0, \ldots, m-1$, eigenvectors of $\omega^{jk} = \mathrm{e}^{2\pi\mathrm{i}(r+h/m)}$ are vectors v_{mr+h} with $r \in \{0, \ldots, k-1\}$. As a result of these computations defining

$$p_h = \sum_{r=0}^{k-1} |v_{mr+h}\rangle\langle v_{mr+h}|, \qquad h = 0, \ldots, m-1, \tag{25}$$

we have

$$\begin{aligned} J^k = \sum_{j=0}^{d-1} \omega^{jk} |v_j\rangle\langle v_j| &= \sum_{h=0}^{m-1} \sum_{r=0}^{k-1} \omega^{(mr+h)k} |v_{mr+h}\rangle\langle v_{mr+h}| \\ &= \sum_{h=0}^{m-1} \sum_{r=0}^{k-1} \mathrm{e}^{2\pi\mathrm{i}h/m} |v_{mr+h}\rangle\langle v_{mr+h}| \\ &= \sum_{h=0}^{m-1} \mathrm{e}^{2\pi\mathrm{i}h/m} p_h. \end{aligned}$$

and $(p_h)_{0 \le h \le m-1}$ is the collection of all *minimal central projections* in $\mathcal{N}(\mathcal{T})$.

We can now read off the factorisations of Theorems 11 and 12. Let $(f_r)_{0 \le r \le k-1}$ be an orthonormal basis of $\mathbb{C}^k$. For each $h = 0, \ldots, m-1$ define the unitary operator

$$U_h : p_h \mathbb{C}^d \to \mathbb{C}^k, \qquad U_h v_{mr+h} = f_r, \quad r = 0, \ldots, k-1,$$

so that we obtain the unitary $U := \oplus_{h=0}^{m-1} U_h$

$$U : \mathbb{C}^d \to \oplus_{h=0}^{m-1} \mathbb{C}^k = \oplus_{h=0}^{m-1} \left(\mathsf{k}_h \otimes \mathsf{m}_h \right)$$

with $\mathsf{k}_h = \mathbb{C}^k, \mathsf{m}_h = \mathbb{C}$ for $h = 0, \dots, m-1$.
It turns out that $U_h J^k U_h^* f_r = \omega^{(mr+h)k} f_r = \mathrm{e}^{2\pi \mathrm{i} h/m} f_r$ for all $r = 0, \dots, k-1$, i.e. $U_h J^k U_h^* = S_h$ with $S_h := \mathrm{e}^{2\pi \mathrm{i} h/m} \mathbb{1}_{\mathbb{C}^k}$. Hence, we have

$$U_h J^n U_h^* = S_h^{n/k} = \mathrm{e}^{2\pi \mathrm{i} hn/d} \mathbb{1}_{\mathbb{C}^k} \quad \text{and} \quad U J^n U^* = \oplus_{h=0}^{m-1} \left(\mathrm{e}^{2\pi \mathrm{i} hn/d} \mathbb{1}_{\mathbb{C}^k} \right).$$

Finally, since $U \mathcal{N}(\mathcal{T}) U^* = \oplus_{h=0}^{m-1} \{ U_h J^n U_h^*, U_h J^{-n} U_h^* \}'$, we immediately get

$$U \mathcal{N}(\mathcal{T}) U^* = \oplus_{h=0}^{m-1} M_k(\mathbb{C}).$$

We now propose another formulation with a slightly different unitary operator in order to choose a non-trivial Hamiltonian leading to a decoherence-free subalgebra with non-trivial Hilbert space k and non-trivial multiplicity space m. Let $(f_r)_{0 \le r \le k-1}$ and $(g_h)_{0 \le h \le m-1}$ be orthonormal bases of $\mathbb{C}^k$ and $\mathbb{C}^m$. For each j let $j = mr + h$ the division of j by m with remainder of h and define the unitary

$$F : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^m, \qquad F v_j = F v_{mr+h} = f_r \otimes g_h.$$

It turns out that $F J^k F^* = \mathbb{1}_{\mathbb{C}^k} \otimes S$ where S is the unitary operator on $\mathbb{C}^m$ defined by $S g_h = \omega^{hk} g_h = \mathrm{e}^{2\pi \mathrm{i} h/m} g_h$, and also $F J^n F^* = \mathbb{1}_{\mathbb{C}^k} \otimes S^{n/k}$.

The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the commutant of $\{J^k, J^{-k}\}$ by Proposition 26, therefore it is isomorphic to the algebra operators $x \otimes y$ where $x \in M_k(\mathbb{C})$ and y belongs to the commutant of S. This is generated by operators $x \otimes |g_h\rangle \langle g_h|$ $0 \le h \le m-1$, therefore we recover the previous decomposition

$$F \mathcal{N}(\mathcal{T}) F^* \simeq \oplus_{i=0}^{m-1} M_k(\mathbb{C})$$

and $\mathsf{k}_i = \mathbb{C}^k, \mathsf{m}_i = \mathbb{C}$ for $i = 0, \dots, m-1$. Moreover, since $H = 0$, we have $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) = \{ L_k, L_k^* \mid k = 1, 2 \}'$ and the decoherence-free semigroup is the trivial semigroup of identity maps.

However, if we consider the GKSL generator with the above L_1, L_2 and Hamiltonian

$$H = F^* \left(K \otimes \mathbb{1}_{\mathbb{C}^m} + \mathbb{1}_{\mathbb{C}^k} \otimes M_0 \right) F$$

with K and M_0 self-adjoint on $\mathbb{C}^k$ and $\mathbb{C}^m$ respectively and such that the commutant of $\{ \delta_{M_0}^l (S^{n/k}), \delta_{M_0}^l (S^{n/k}) \mid l \ge 0 \}$ in $M_m(\mathbb{C})$ is trivial, then

$$F \mathcal{N}(\mathcal{T}) F^* \simeq M_k(\mathbb{C}) \otimes \mathbb{1}_{\mathbb{C}^m}$$

is a factor and $\mathsf{k}_1 = \mathbb{C}^k$, $\mathsf{m}_1 = \mathbb{C}^m$.

If n/k and m are coprime (e.g. for $n=10$, $d=15$ so that $k=5, m=3$ and $n/k=2$), then $S^{n/k}$ has non nondegenerate spectrum, namely eigenvalues $\omega^{hn} = \mathrm{e}^{2\pi\mathrm{i}hn/mk} = \mathrm{e}^{2\pi\mathrm{i}hn/d}$, for $h=0,\ldots,m-1$, are simple and we can write

$$S^{n/k} = \sum_{h=0}^{m-1} \omega^{hn} |g_h\rangle\langle g_h|.$$

Since the subalgebra of $M_m(\mathbb{C})$ generated by $S^{n/k}$ is maximal abelian, any X in the commutant of $S^{n/k}$ which contains $\{\delta^l_{M_0}(S^{n/k}), \delta^l_{M_0}(S^{n/k}) \mid l \geq 0\}'$ must be diagonal in the basis $(g_h)_{0\leq h\leq m}$. We take, for instance,

$$M_0 = \sum_{h=0}^{m-1} \left(|g_{h+1}\rangle\langle g_h| + |g_{h-1}\rangle\langle g_h|\right)$$

so that $F^* \mathbb{1}_{\mathbb{C}^k} \otimes M_0 F v_{mr+h} = v_{mr+h+1} + v_{mr+h-1}$. Computing the commutator $[M_0, S^{n/k}]$ we find

$$\begin{aligned}
&= \sum_{h,h'=0}^{m-1} \left(\omega^{h'n}|g_{h+1}\rangle\langle g_h||g_{h'}\rangle\langle g_{h'}| + \omega^{h'n}|g_{h-1}\rangle\langle g_h||g_{h'}\rangle\langle g_{h'}|\right) \\
&- \sum_{h,h'=0}^{m-1} \left(\omega^{h'n}|g_{h'}\rangle\langle g_{h'}||g_{h+1}\rangle\langle g_h| + \omega^{h'n}|g_{h'}\rangle\langle g_{h'}||g_{h-1}\rangle\langle g_h|\right) \\
&= \sum_{h=0}^{m-1} \omega^{hn}|g_{h+1}\rangle\langle g_h| + \sum_{h=0}^{m-1} \omega^{hn}|g_{h-1}\rangle\langle g_h| \\
&- \sum_{h=0}^{m-1} \omega^{(h+1)n}|g_{h+1}\rangle\langle g_h| - \sum_{h=0}^{m-1} \omega^{(h-1)n}|g_{h-1}\rangle\langle g_h| \\
&= \sum_{h=0}^{m-1} \left(\omega^{hn} - \omega^{(h+1)n}\right)|g_{h+1}\rangle\langle g_h| - \sum_{h=0}^{m-1} \left(\omega^{(h-1)n} - \omega^{hn}\right)|g_{h-1}\rangle\langle g_h|
\end{aligned}$$

An operator $X = \sum_h z_h |g_h\rangle\langle g_h|$ commutes with $[M_0, S^{n/k}]$ if and only if

$$\begin{aligned}
[X,[M_0,S^{n/k}]] &= \sum_{h=0}^{m-1} \left(\omega^{hn} - \omega^{(h+1)n}\right)(z_{h+1} - z_h)\,|g_{h+1}\rangle\langle g_h| \\
&- \sum_{h=0}^{m-1} \left(\omega^{(h-1)n} - \omega^{hn}\right)(z_{h-1} - z_h)\,|g_{h-1}\rangle\langle g_h| = 0,
\end{aligned}$$

By the linear independence of rank-one operators $|g_{h+1}\rangle\langle g_h|, |g_{h-1}\rangle\langle g_h|$ ($h = 0,\ldots,d-1$), X commutes with $[M_0, S^{n/k}]$ if and only if $z_h = z_{h+1}$ for all h, i.e. X is a multiple of the identity operator.

By Theorem 12, the decoherence-free and decoherence-affected semigroups are now generated by

$$\mathcal{L}^{\mathsf{df}} = \mathrm{i}[\, F^*(K \otimes \mathbb{1}_{\mathbb{C}^m})F, \cdot\,] \qquad \mathcal{L}^{\mathsf{da}} = \mathcal{L} - \mathcal{L}^{\mathsf{df}}.$$

Appendix

First of all, we recall some preliminary definitions and results. Given a von Neumann algebra $\mathcal{M}$, we denote by $\mathcal{P}_{\min}(\mathcal{M})$ the set of its minimal projections. If p is a projection in $\mathcal{M}$, its *central support* z_p is the smallest projection in the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ such that $p \leq z_p$. We refer to Takesaki ([34] Definition 5.9 p.155) for the following definition.

Definition 28 *Let $\mathcal{M}$ be a von Neumann algebra acting on h. $\mathcal{M}$ is called* atomic *if for every non-zero projection $p \in \mathcal{M}$ there exists $q \in \mathcal{P}_{\min}(\mathcal{M})$, $q \neq 0$, such that $q \leq p$.*

Note that, since every projection $q \in p\mathcal{M}p$ is smaller than p, we have $p \in \mathcal{P}_{\min}(\mathcal{M})$ if and only if $p\mathcal{M}p = \mathbb{C}p$.

Lemma 29 *Let $\mathcal{M}$ be a type I factor. Then $\mathcal{M}$ is atomic.*

Proof. By Theorem 4.2.1 in [25] we know that $\mathcal{M}$ is unitarily equivalent to $\mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$ for some k and m Hilbert spaces. Since $\mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$ is clearly an atomic algebra and every unitary isomorphism maps minimal projections into minimal projections, we can conclude that $\mathcal{M}$ is atomic too.

Lemma 30 *If p is a non-zero minimal projection in $\mathcal{M}$, then its central support z_p is a non-zero minimal projection in $\mathcal{Z}(\mathcal{M})$.*

Proof. By definition, z_p is different from 0. Let $q \in \mathcal{Z}(\mathcal{M})$ be a non-zero projection such that $q \leq z_p$. Then $qp = pq = pqp \leq pz_pp = p$, since p and q commute and $p \leq z_p$ by definition of central support. Now, the minimality of p in $\mathcal{M}$ implies either $qp = 0$, i.e. $p \leq q^\perp$, or $qp = p$, i.e. $p \leq q \leq z_p$. In this latter case we can conclude that $q = z_p$ by definition of z_p. Otherwise, if $p \leq q^\perp$, then $p = pz_p \leq z_pq^\perp \leq z_p$, so that $z_p - q = z_pq^\perp = z_p$, since $z_pq^\perp$ is a projection in $\mathcal{Z}(\mathcal{M})$. As a consequence we have $q = 0$, which is a contradiction. □

Proposition 31 *Let $\mathcal{M}$ be a von Neumann algebra. The following are equivalent:*

1. *$\mathcal{M}$ is atomic;*

2. *there exists a collection* $(p_i)_{i\in I}$ *of mutually orthogonal projections in* $\mathcal{P}_{\min}(\mathcal{Z}(\mathcal{M}))$ *such that* $\sum_{i\in I} p_i = \mathbb{1}$ *and each* $p_i\mathcal{M}p_i$ *is a type I factor.*

Proof. 1. $\Rightarrow$ 2. Thanks to Lemma 30 we can find a maximal family $(p_i)_{i\in I}$ of mutually orthogonal minimal projections in $\mathcal{Z}(\mathcal{M})$. Let $p = \sum_{i\in I} p_i$. If $p \neq \mathbb{1}$, by the atomicity of $\mathcal{M}$ we can find a non-zero minimal projection $q \in \mathcal{M}$ such that $q \leq p^\perp$. Denoting by z_q the central support of q, by definition of z_q we have $q \leq z_q \leq p^\perp$, since $p^\perp$ is a projection in $\mathcal{Z}(\mathcal{M})$ which majorizes q. Finally, z_q is minimal in $\mathcal{Z}(\mathcal{M})$ by Lemma 30, contradicting the maximality of $(p_i)_{i\in I}$ and proving that $\sum_{i\in I} p_i = \mathbb{1}$.
In order to check that each $p_i\mathcal{M}p_i$ is a factor it is enough to prove that its center, which is a von Neumann algebra, contains only trivial projections. So, let $q \in \mathcal{Z}(p_i\mathcal{M}p_i)$ be a non-zero projection; since we have $0 = [q, p_i x] = [p_i q, x]$ for all $x \in \mathcal{M}$, $p_i q$ belongs to $\mathcal{Z}(\mathcal{M})$, and so $q = p_i q = p_i q p_i \in p_i\mathcal{Z}(\mathcal{M})p_i = \mathbb{C}p_i$ by minimality of p_i in $\mathcal{Z}(\mathcal{M})$. We thus conclude that $q = p_i$, i.e. $\mathcal{Z}(p_i\mathcal{M}p_i) = \mathbb{C}p_i$ and $p_i\mathcal{M}p_i$ is a factor.
Finally, since $\mathcal{M}$ is atomic, for every $i \in I$ there exists a non-zero minimal projection $q_i \in \mathcal{M}$ such that $q_i \leq p_i$; therefore, each q_i is a non-zero minimal projection in $p_i\mathcal{M}p_i$ and the factor $p_i\mathcal{M}p_i$ is type I.

2. $\Rightarrow$ 1. First of all we note that $\mathcal{M} = \oplus_{i\in I} p_i\mathcal{M}p_i$ since every $x \in \mathcal{M}$ can be written as $x = \sum_{i\in I} p_i x p_i$, because p_i in $\mathcal{Z}(\mathcal{M})$ and $\sum_i p_i = \mathbb{1}$.
We now show that $\mathcal{M}$ is atomic (Definition 28). Let $p \in \mathcal{M}$ be a non-zero projection, so that $p = \sum_{i\in I} p_i p p_i$ with $p_i p p_i$ a projection in $p_i\mathcal{M}p_i$ for all $i \in I$. Since every $p_i\mathcal{M}p_i$ is a type I factor, it is atomic by Lemma 29; hence, given $j \in I$ such that $p_j p p_j$ is not trivial, we can find a non-zero minimal projection $q \in p_j\mathcal{M}p_j$ with $q \leq p_j p p_j$. But q is minimal in $\mathcal{M}$ too, and so we can conclude that $q \leq p_j p p_j \leq \sum_{i\in I} p_i p p_i = p$. □

Remark 4 *Note that, if* h *is separable, then the index set* I *introduced in the previous Lemma is necessarily countable, having* $\mathsf{h} = \oplus_{i\in I} p_i\mathsf{h}$.

A necessary and sufficient condition for $\mathcal{M}$ to be atomic is given by the following result due to Tomiyama [36], Theorem 5.

Theorem 32 *Let* $\mathcal{M}$ *be a von Neumann algebra acting on* h*. Then* $\mathcal{M}$ *is atomic if and only if there exists a normal conditional expectation* $\mathcal{E} : \mathcal{B}(\mathsf{h}) \to \mathcal{M}$ *such that* $\operatorname{Ran}\mathcal{E} = \mathcal{M}$.

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