The Moore–Gibson–Thompson equation with memory in the critical case

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Abstract

We consider the following abstract version of the Moore–Gibson–Thompson equation with memory

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \gamma Au(t) - \int_0^t g(s)Au(t-s)\mathrm{d}s = 0$$

depending on the parameters $\alpha, \beta, \gamma > 0$, where A is strictly positive selfadjoint linear operator and g is a convex (nonnegative) memory kernel. In the subcritical case $\alpha\beta > \gamma$, the related energy has been shown to decay exponentially in [19]. Here we discuss the critical case $\alpha\beta = \gamma$, and we prove that exponential stability occurs if and only if A is a bounded operator. Nonetheless, the energy decays to zero when A is unbounded as well.

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1. Introduction

Let $(\mathrm{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let

$$A : \mathrm{H} \to \mathrm{H} \qquad \text{of domain} \qquad \mathfrak{D}(A) \subset \mathrm{H}$$

be a strictly positive selfadjoint linear operator, where the (dense) embedding $\mathfrak{D}(A) \subset \mathrm{H}$ need not be compact. In this work we consider the integrodifferential equation

$$\partial_{ttt} u(t) + \alpha \partial_{tt} u(t) + \beta A \partial_t u(t) + \gamma A u(t) - \int_0^t g(s) A u(t-s) \mathrm{d}s = 0. \tag{1.1}$$

Here α, β, γ are strictly positive constants subject to the further constraint

$$\varkappa := \beta - \frac{\gamma}{\alpha} \geq 0,$$

while the (nonnull) function $g \in W^{1,1}(\mathbb{R}^+)$ with g' absolutely continuous on $\mathbb{R}^+ = (0, \infty)$, usually called *memory kernel*, satisfies the following assumptions:

(g1) $g(s) \geq 0$ and $g'(s) \leq 0$ for every $s > 0$.
(g2) $g'' \geq 0$ almost everywhere.
(g3) $\varrho := \int_0^\infty g(s)\mathrm{d}s \in (0, \gamma)$.
(g4) There exists $\delta > 0$ such that $g'(s) + \delta g(s) \leq 0$ for every $s > 0$.

The equation is supplemented with the initial conditions assigned at time $t = 0$

$$\begin{cases} u(0) = u_0, \\ \partial_t u(0) = v_0, \\ \partial_{tt} u(0) = w_0, \end{cases}$$

being u_0, v_0, w_0 prescribed initial data.

1.1. Physical motivations and background

The integrodifferential equation (1.1) is a natural outgrowth of the Moore–Gibson–Thompson (MGT) equation arising in nonlinear acoustics, and accounting for the second sound effects and the associated thermal relaxation in viscous gases/fluids [10,24,32]. We also address the reader to the work of Stokes [30], where presumably the MGT equation appeared for the first time. Introducing additional nonlocal effects due to molecular relaxation, one arrives at the viscoelastic version (1.1) of the MGT equation [11,21,25].

In order to gain a better understanding of the MGT equation, we shall begin with the Westervelt (or more generally Kuznetsov) equation, one of the fundamental models for nonlinear acoustic waves (see [12,14,33] and references therein). Denoting by β the diffusivity coefficient and letting $\gamma = c^2$ (the square of the sound speed), the Westervelt equation written for the state variable $u = u(t)$, standing for acoustic pressure, is given by

$$\partial_{tt}u - \gamma\Delta u - \beta\Delta\partial_t u = \partial_{tt}\Big(\frac{\xi}{\gamma}u^2\Big), \tag{1.2}$$

where ξ is a nonlinearity parameter, with the associated Dirichlet, Neumann or Robin boundary conditions. Equation (1.2) is derived by using

- the Navier–Stokes equations for momentum conservation,
- the conservation of mass, and
- the conservation of energy (first law of thermodynamics) involving the heat flux, assumed to be proportional to the gradient of the temperature, according to the classical Fourier law.

However, such a model is known to exhibit an *infinite signal speed paradox*, which is often criticized on physical grounds. In order to remove this drawback, the Cattaneo law has been proposed: a first order differential perturbation of the Fourier law with a relaxation parameter $\tau > 0$. Thus, instead of using the Fourier–Navier–Stokes equations, one employs the Cattaneo–Navier–Stokes ones. The resulting model, sometimes referred to as *Extendable Irreversible Thermodynam-ics* [21], leads to the third order in time MGT equation

$$\tau\partial_{ttt}u + \partial_{tt}u - \gamma\Delta u - \beta\Delta\partial_t u = \partial_{tt}\Big(\frac{\xi}{\gamma}u^2\Big). \tag{1.3}$$

From the mathematical point of view, there is a markable difference between the dynamics (1.2) and (1.3). The linear part of the Westervelt equation is associated with an analytic semigroup, whereas the nature of the MGT dynamics is "hyperbolic-like" (see [22]). This has a major ef-fect on the techniques needed to attack the two problems. Indeed, while "maximal parabolic regularity" has been proven to be a successful tool in studying the quasilinear parabolic-like dy-namics (1.2) (see [12,14,23]), this is no longer an option in the case of (1.3), which requires a hyperbolic-like approach based on the consideration of several energy levels with the cor-responding estimates (to close the loop in establishing appropriate fixed points) along with a barrier's method [14].

It is also relevant to note that the MGT equation can be seen within the context of viscoelastic-ity, for it can be obtained by differentiating in time a second order viscoelastic equation with an exponential kernel. However, the relation between the two models is not one-to-one, and the dy-namics of (1.3) (compared with viscoelasticity with an exponential kernel) is richer, as explained in [7].

Quoting [21], *"Extendable Irreversible Thermodynamics has been applied to a variety of materials and processes, but it is particularly well suited for describing systems with long relaxation times (like viscoelastic fluids) and phenomena involving high frequencies and short wavelengths."* Nonetheless, it has been observed in [1] that this theory, in its classical formulation, does not agree with experimental data on the *propagation of sound waves in dilute gases at high wave frequencies*. This led to the introduction of a further parameter accounting for molecu-lar relaxation. Nonlocal effects are reflected in the constitutive equations, where both the heat flux and the pressure tensor are now governed by evolutions with appropriate relaxation parameters. The resulting model is the MGT equation with memory, whose abstract form reads

$$\tau\partial_{ttt}u + \big[\alpha - ku\big]\partial_{tt}u + \beta A\partial_t u + \gamma Au - \int_0^t g(s)Aw(t-s)\mathrm{d}s = k[\partial_t u]^2, \tag{1.4}$$

where $\alpha, k > 0$ are fixed parameters,

$$w(t) = u(t) + \lambda \partial_t u(t), \quad \text{for some } \lambda \geq 0,$$

and the memory kernel g fulfills suitable assumptions. Depending on the properties of the environment surrounding sound propagation, the memory kernel can exhibit several structures by selecting different values of λ, yielding different possible configurations.

1.2. Earlier literature

Well-posedness and stability for equation (1.4) has been the subject of recent studies by sev-eral authors. Both linear and nonlinear models have been analyzed. The solvability of nonlinear equations depends in an essential way on the following two features of linearization:

(a) uniform stability of the linear dynamics, and
(b) local well-posedness of solutions at several topological levels [14,16].

This is not surprising, since we are dealing with a quasilinear and potentially degenerate dynamics. Thus, both questions (a) and (b) have been extensively dealt with in the past. It is well known that the uniform stability of a linear model depends on the strength of dissipation. It has been shown that, for strictly positive values of

$$\varkappa := \beta - \frac{\tau\gamma}{\alpha},$$

the linearized version of (1.4) (i.e. for $k = 0$) is exponentially stable as long as the relaxation kernel g is of exponential type with natural constraints. This stability can also be transferred to higher topological levels, allowing to carry out a nonlinear analysis [14,16]. For the linearized equation, it is shown in [20] that the system exhibits uniform decay rates of the energy for a large class of kernels, including kernels g with weak decay properties, generally described by an ODE of monotone type

$$g' + H(g) \leq 0, \tag{1.5}$$

with $H(\cdot)$ convex. The decay properties of the kernel are shown to be transferred into the decay of the corresponding PDE solutions. Hence, in the subcritical case $\varkappa > 0$ uniform stability appears rather robust. More specifically, it is shown in [19] that under assumptions (g1)–(g4) the energy functional corresponding to the memory type $w = u$ decays exponentially. Similar results (with some additional restrictions on the size of the kernel g) have been obtained for the so-called memory of type II, involving the velocity term $\partial_t u$. Thus, in these subcritical cases, inserting memory term of a general type displays enough flexibility in maintaining uniformity of decay rates. The situation is different at the critical level $\varkappa = 0$, where there is no mechanical damp-ing in the model, and the presence of the memory becomes essential. Here an obvious question arises, namely, what kind of memory (generally depending on the pressure and its derivative) is beneficial in producing uniform decay rates. According to [19,20], it is possible to achieve the desired uniform stability by adding a memory term containing not only u but also $\partial_t u$, under vari-ous assumptions on the relaxation kernel, including very weak dissipation characterized by (1.5). However, the methods developed in [19,20] provide no results when considering the critical case for a memory term containing only the pressure u. This is precisely the situation addressed in the present work, that is, the linear version of (1.4) when $w = u$ for the critical value $\varkappa = 0$.

A word of warning. In the sequel, we will set

$$\tau = 1.$$

Clearly, this is no loss of generality, up to rescaling the other quantities.

1.3. Main results

Let us begin by recalling the exponential stability results from [19], proved for subcritical and critical values of the parameter $\varkappa$. For every choice of initial data

$$(u_0, v_0, w_0) \in \mathfrak{D}(A^{\frac{1}{2}}) \times \mathfrak{D}(A^{\frac{1}{2}}) \times \mathrm{H},$$

equation (1.1) has a unique (weak) solution

$$u \in \mathcal{C}^1([0,\infty), \mathfrak{D}(A^{\frac{1}{2}})) \cap \mathcal{C}^2([0,\infty), \mathrm{H}).$$

Defining the corresponding energy

$$\mathsf{F}(t) = \|A^{\frac{1}{2}} u(t)\|^2 + \|A^{\frac{1}{2}} \partial_t u(t)\|^2 + \|\partial_{tt} u(t)\|^2 - \int_0^t g'(s) \|A^{\frac{1}{2}} u(t) - A^{\frac{1}{2}} u(t-s)\|^2 \mathrm{d}s,$$

the following theorem from [19] holds.

Theorem 1.1. *In the subcritical case $\varkappa > 0$ the energy decays exponentially, that is, there exist constants $C \geq 1$ and $\omega > 0$ such that*

$$\mathsf{F}(t) \leq C \mathsf{F}(0) \mathrm{e}^{-2\omega t}. \tag{1.6}$$

Actually, in [19] a similar result has also been established (see Theorem 1.6 therein), but for the memory term depending only on $\partial_t u$. It is important to say that subcriticality is still assumed, along with some additional bounds imposed on ϱ.

For the critical case $\varkappa = 0$, the exponential decay of the energy is proved in [19, Theorem 1.10], but with an added velocity $\partial_t u$ to the memory term. The result, reported here below, does not depend on the convexity of g, and the energy functional G describing the decay rate depends on g itself (rather than g') and on

$$w(t) = u(t) + \frac{\beta}{\gamma} \partial_t u(t).$$

Theorem 1.2. *Let* (g1), (g3) *with $\varrho \in (0, \beta)$ and* (g4) *hold. Then in the critical case $\varkappa = 0$ the energy*

$$\mathsf{G}(t) = \|A^{\frac{1}{2}}u(t)\|^2 + \|A^{\frac{1}{2}}\partial_t u(t)\|^2 + \|\partial_{tt}u(t)\|^2 + \int_0^t g(s)\|A^{\frac{1}{2}}w(t) - A^{\frac{1}{2}}w(t-s)\|^2 \mathrm{d}s,$$

defined for all solutions to (1.1)*, decays exponentially.*

Our aim is to address the critical case $\varkappa = 0$ with the memory term depending only on u. As acknowledged in [19], the techniques developed there were inconclusive in that situation. In the present paper we shall provide a definite answer. In fact, for the MGT equation *without* memory, i.e. when g vanishes, $\varkappa = 0$ is the threshold value between exponential stability and energy conservation (see [13,22]). The addition of the memory term introduces further dissipation. Thus, in principle, one might expect that $\varkappa = 0$ could become in a sense subcritical. Here, we actually show that the situation is something in between. Namely, the memory contribution is capable to drive the system to equilibrium, but is not strong enough to do it exponentially fast, at least if A is unbounded. Indeed our main result can be subsumed as follows:

Let $\varkappa = 0$. Then (1.6) *occurs if and only if A is a bounded operator. Nonetheless, for every initial energy $\mathsf{F}(0)$, it is always true that*

$$\lim_{t\to\infty} \mathsf{F}(t) = 0. \tag{1.7}$$

Strong convergence to zero of single trajectories provides no information on the rate of convergence. However, for more regular initial data we can say more. Indeed, we show that if

$$(u_0, v_0, w_0) \in \mathfrak{D}(A) \times \mathfrak{D}(A) \times \mathfrak{D}(A^{1/2})$$

then the energy $\mathsf{F}(t)$ decays at the rate $1/t$, implying in turn (1.7) via a density argument.

Remark 1.3. (Open Questions). In view of the above results, it is clear that there are several more open issues unresolved when it comes to uniform stability of the third order Volterra equation in the critical case. The complexity of the matter is due to the fact that, in third order equations, the memory term may typically depend on both u and $\partial_t u$. Thus, the question which memory term enables to stabilize uniformly the dynamics is paramount. We know that certain combinations of u and $\partial_t u$ are effective. Now, we also know that u alone will not suffice to stabilize uniformly the dynamics. It would be interesting to see what can be said about $\partial_t u$, namely, if this term alone is able to provide a strong enough damping mechanism. Short calculations reveal that such a damping drives the problem into a subcritical regime with, however, an antidissipative memory. Thus, we deal with two competing mechanisms: frictional damping quantified by subcriticality, and negative damping due to memory effects. This brings up an interesting scenario to consider.

1.4. Outline of the paper

In the next Section 2 we introduce a generalized (infinite memory) version of (1.1) within the so-called Dafermos past history framework. Such an equation is shown to generate a contraction semigroup in a suitable Hilbert space accounting for memory terms. This is done in Section 3. In Section 4 we prove that exponential stability does not occur when A is an unbounded operator, while in Section 5 exponential stability is demonstrated for the case of A bounded. Section 6 is devoted to the semiuniform stability of the semigroup, while in the final Section 7 we prove the polynomial decay of the solutions for more regular initial data.

As we shall see, the semigroup framework in memory spaces is not really essential for the proofs of the main results of this paper on the Volterra equation. However, it does provide an elegant and unified method to treat energy solutions of both Volterra and infinite memory equations.

2. The equation in the memory space

We consider the following generalization of (1.1) in the critical case $\varkappa = 0$ (i.e. $\alpha\beta = \gamma$)

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \alpha\beta Au(t) - \int_0^\infty g(s)Au(t-s)\mathrm{d}s = 0, \tag{2.1}$$

where the variable u is understood to be an assigned datum for negative times $t \le 0$. Note that (1.1) corresponds to the particular case of *null past history* of u, namely, when $u(s)_{|s<0} = 0$.

2.1. Functional setting

In what follows, we will introduce several Hilbert spaces (with related inner products and norms) depending on a parameter $r \in \mathbb{R}$. To simplify the notation, the subscript r will be *always omitted* whenever zero.

First, we define the family of nested Hilbert spaces

$$\mathrm{H}_r = \mathfrak{D}(A^{\frac{r}{2}}), \qquad \langle u, v\rangle_r = \langle A^{\frac{r}{2}}u, A^{\frac{r}{2}}v\rangle, \qquad \|u\|_r = \|A^{\frac{r}{2}}u\|.$$

In particular, we have the Poincaré inequality

$$\lambda_0\|u\|^2 \le \|u\|_1^2, \quad \forall u \in \mathrm{H}_1, \tag{2.2}$$

where

$$\lambda_0 = \min\left\{\lambda : \lambda \in \sigma(A)\right\} > 0,$$

being $\sigma(A)$ the spectrum of A. The symbol $\langle\cdot,\cdot\rangle$ will also be used to denote the duality pairing between H_r and its dual space H_{-r}.

In order to view (2.1) as an ODE in the so-called past history framework of Dafermos [6], we introduce the spaces

$$\mathcal{M}_r = L^2_{-g'}(\mathbb{R}^+; \mathrm{H}_{1+r}),$$

that is, the spaces of square summable functions $\eta : \mathbb{R}^+ \to \mathrm{H}_{1+r}$ with respect to the measure $-g'(s)\mathrm{d}s$, endowed with the Hilbert norms

$$\|\eta\|^2_{\mathcal{M}_r} = -\int_0^\infty g'(s)\|\eta(s)\|^2_{1+r}\mathrm{d}s.$$

Then, we define the Hilbert spaces

$$\mathcal{H}_r = \mathrm{H}_{1+r} \times \mathrm{H}_{1+r} \times \mathrm{H}_r \times \mathcal{M}_r$$

with the usual Euclidean product norms

$$\|(u, v, w, \eta)\|^2_{\mathcal{H}_r} = \|u\|^2_{1+r} + \|v\|^2_{1+r} + \|w\|^2_r + \|\eta\|^2_{\mathcal{M}_r}.$$

Note that, for every $r_1 > r_2$, the embedding $\mathcal{M}_{r_1} \subset \mathcal{M}_{r_2}$ is continuous, but not compact (see [27] for a counterexample). Accordingly, so is the embedding $\mathcal{H}_{r_1} \subset \mathcal{H}_{r_2}$.

We will also consider the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, i.e. the linear operator T given by

$$T\eta = -\eta' \qquad \text{with domain} \qquad \mathfrak{D}(T) = \big\{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0\big\},$$

the *prime* standing for the distributional derivative with respect to the variable $s > 0$.

2.2. *The equation*

In the same spirit of [6], we consider the auxiliary variable $\eta = \eta^t(s)$, containing all the information on the past history of u, and formally defined as

$$\eta^t(s) = u(t) - u(t-s),\ t \geq 0,\ s > 0.$$

Then, equation (2.1) translates into the evolution system

$$\begin{cases} \partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + (\alpha\beta - \varrho)Au(t) + \displaystyle\int_0^\infty g(s)A\eta^t(s)\mathrm{d}s = 0, \\ \partial_t\eta^t(s) = T\eta^t(s) + \partial_t u(t). \end{cases} \tag{2.3}$$

Introducing the state vector

$$\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt}u(t), \eta^t),$$

we view (2.3) as the ODE in $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}(t) = \mathbb{A}\boldsymbol{U}(t), \tag{2.4}$$

where $\mathbb{A}$ is the linear operator given by

$$\mathbb{A}\begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\alpha w - A\big[\beta v + (\alpha\beta - \varrho)u + \int_0^\infty g(s)\eta(s)\mathrm{d}s\big] \\ T\eta + v \end{pmatrix}$$

with domain

$$\mathfrak{D}(\mathbb{A}) = \left\{ (u,v,w,\eta) \in \mathcal{H} \;\middle|\; \begin{array}{c} w \in \mathrm{H}_1 \\ \beta v + (\alpha\beta - \varrho)u + \int_0^\infty g(s)\eta(s)\mathrm{d}s \in \mathrm{H}_2 \\ \eta \in \mathfrak{D}(T) \end{array} \right\}.$$

The equation is supplemented with the initial condition

$$\boldsymbol{U}(0) = \boldsymbol{U}_0,$$

for an arbitrarily given $\boldsymbol{U}_0 = (u_0, v_0, w_0, \eta_0) \in \mathcal{H}$.

3. The solution semigroup

In this section, we show that equation (2.4) generates a solution semigroup $S(t)$ on the phase space $\mathcal{H}$. This will be done by introducing an equivalent Hilbert norm, specifically tailored to the structure of the problem.

3.1. The equivalent norm

For all vectors $\boldsymbol{U} = (u, v, w, \eta)$ and $\tilde{\boldsymbol{U}} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\eta})$ in $\mathcal{H}$, we define

$$\begin{aligned}(\boldsymbol{U}, \tilde{\boldsymbol{U}})_{\mathcal{H}} &= \Big(\beta - \frac{\varrho}{\alpha}\Big)\langle v + \alpha u, \tilde{v} + \alpha\tilde{u}\rangle_1 + \langle w + \alpha v, \tilde{w} + \alpha\tilde{v}\rangle + \frac{\varrho}{\alpha}\langle v, \tilde{v}\rangle_1 + \langle \eta, \tilde{\eta}\rangle_{\mathcal{M}} \\ &\quad + \alpha\int_0^\infty g(s)\langle \eta(s), \tilde{\eta}(s)\rangle_1 \mathrm{d}s + \int_0^\infty g(s)\big[\langle \eta(s), \tilde{v}\rangle_1 + \langle v, \tilde{\eta}(s)\rangle_1\big]\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}|\boldsymbol{U}|_{\mathcal{H}}^2 &= \Big(\beta - \frac{\varrho}{\alpha}\Big)\|v + \alpha u\|_1^2 + \|w + \alpha v\|^2 + \frac{\varrho}{\alpha}\|v\|_1^2 + \|\eta\|_{\mathcal{M}}^2 \\ &\quad + \alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s + 2\int_0^\infty g(s)\langle \eta(s), v\rangle_1 \mathrm{d}s.\end{aligned}$$

In particular, condition (g3) implies that

$$\beta - \frac{\varrho}{\alpha} > 0.$$

We claim that the above are an inner product and a norm on $\mathcal{H}$, respectively, equivalent to the original ones. This in an immediate consequence of the following lemma.

Lemma 3.1. *There exists a structural constant $c > 1$ such that*

$$\frac{1}{c}\|\boldsymbol{U}\|_{\mathcal{H}} \le |\boldsymbol{U}|_{\mathcal{H}} \le c\|\boldsymbol{U}\|_{\mathcal{H}}, \quad \forall \boldsymbol{U} \in \mathcal{H}.$$

Proof. We begin to prove the estimate from below. To this end, for every $\varepsilon > 0$, we infer from the Young inequality that

$$\left|2\int_0^\infty g(s)\langle\eta(s), v\rangle_1 \mathrm{d}s\right| \le \frac{\varrho}{\alpha(1+\varepsilon)}\|v\|_1^2 + (1+\varepsilon)\alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s.$$

Hence,

$$\begin{aligned}\alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s + 2\int_0^\infty g(s)\langle\eta(s), v\rangle_1 \mathrm{d}s \\ \ge -\frac{\varrho}{\alpha(1+\varepsilon)}\|v\|_1^2 - \varepsilon\alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s.\end{aligned}$$

Using the assumption (g4) on the memory kernel g, we estimate the latter integral as

$$-\varepsilon\alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s \ge \frac{\varepsilon\alpha}{\delta}\int_0^\infty g'(s)\|\eta(s)\|_1^2 \mathrm{d}s = -\frac{\varepsilon\alpha}{\delta}\|\eta\|_{\mathcal{M}}^2.$$

In summary, we get

$$\alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s + 2\int_0^\infty g(s)\langle\eta(s), v\rangle_1 \mathrm{d}s \ge -\frac{\varrho}{\alpha(1+\varepsilon)}\|v\|_1^2 - \frac{\varepsilon\alpha}{\delta}\|\eta\|_{\mathcal{M}}^2.$$

Thus, up to fixing $\varepsilon > 0$ suitably small, we arrive at

$$\begin{aligned}|\boldsymbol{U}|_{\mathcal{H}}^2 &\ge \Big(\beta - \frac{\varrho}{\alpha}\Big)\|v+\alpha u\|_1^2 + \|w+\alpha v\|^2 + \frac{\varepsilon\varrho}{(1+\varepsilon)\alpha}\|v\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 \\ &\ge \varepsilon^2\bigg[\frac{1}{\alpha^2}\|v+\alpha u\|_1^2 + \|w+\alpha v\|^2 + \|v\|_1^2 + \|\eta\|_{\mathcal{M}}^2\bigg].\end{aligned}$$

In order to conclude, it is enough noting that

$$\frac{1}{\alpha^2}\|v+\alpha u\|_1^2 + \|w+\alpha v\|^2 + \|v\|_1^2 \ge \nu\Big[\|u\|_1^2 + \|v\|_1^2 + \|w\|^2\Big],$$

for some $\nu > 0$. Indeed, exploiting once more the Young inequality,

$$\begin{aligned}\frac{1}{\alpha^2}\|v+\alpha u\|_1^2 + \|w+\alpha v\|^2 + \|v\|_1^2 \\ \ge \nu\|u\|_1^2 + \|v\|_1^2 - \frac{\nu}{1-\nu}\bigg[\frac{1}{\alpha^2}\|v\|_1^2 + \alpha^2\|v\|^2\bigg] + \nu\|w\|^2,\end{aligned}$$

for all $\nu \in (0, 1)$. Due to the Poincaré inequality (2.2), we finally obtain

$$\begin{aligned}\frac{1}{\alpha^2}\|v+\alpha u\|_1^2 + \|w+\alpha v\|^2 + \|v\|_1^2 \\ \geq \nu\|u\|_1^2 + \left[1 - \frac{\nu}{1-\nu}\Big(\frac{1}{\alpha^2} + \frac{\alpha^2}{\lambda_0}\Big)\|v\|_1^2\right] + \nu\|w\|^2,\end{aligned}$$

which proves the claim, up to taking $\nu > 0$ sufficiently small.

The estimate from above is much simpler. Indeed, from the Young inequality and (g4),

$$\alpha\int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s + 2\int_0^\infty g(s)\langle\eta(s), v\rangle_1 \mathrm{d}s \leq \frac{\varrho}{\alpha}\|v\|_1^2 + \frac{2\alpha}{\delta}\|\eta\|_{\mathcal{M}}^2,$$

and making use of (2.2) we get

$$|\boldsymbol{U}|_{\mathcal{H}}^2 \leq \Big(\beta - \frac{\varrho}{\alpha}\Big)\|v+\alpha u\|_1^2 + \|w+\alpha v\|^2 + \frac{2\varrho}{\alpha}\|v\|_1^2 + \Big(1 + \frac{2\alpha}{\delta}\Big)\|\eta\|_{\mathcal{M}}^2 \leq c^2\|\boldsymbol{U}\|_{\mathcal{H}}^2,$$

for some $c > 1$ large enough. □

3.2. Generation of the semigroup

We are now in a position to prove the existence of the solution semigroup $S(t)$ generated by (2.4).

Theorem 3.2. *The operator $\mathbb{A}$ is the infinitesimal generator of a linear $\mathcal{C}_0$-semigroup*

$$S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Moreover $S(t)$ is a contraction with respect to the (equivalent) norm $|\cdot|_{\mathcal{H}}$, i.e.

$$|S(t)\boldsymbol{U}_0|_{\mathcal{H}} \leq |\boldsymbol{U}_0|_{\mathcal{H}}$$

for every $\boldsymbol{U}_0 \in \mathcal{H}$ and every $t \geq 0$.

For every initial datum $\boldsymbol{U}_0 = (u_0, v_0, w_0, \eta_0) \in \mathcal{H}$, by applying the Duhamel formula to the second equation of system (2.3), the memory component η^t of the solution $S(t)\boldsymbol{U}_0$ is easily seen to have the explicit form

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \leq t, \\ \eta_0(s-t) + u(t) - u_0 & s > t. \end{cases} \tag{3.1}$$

Remark 3.3. In the critical case $\varkappa = 0$, the Volterra equation (1.1) is just a particular instance of (2.3), corresponding to an initial datum of the form $\boldsymbol{U}_0 = (u_0, v_0, w_0, u_0) \in \mathcal{H}$ (where the last component is a constant function of s). This follows immediately by plugging (3.1) into the first equation of (2.3).

The proof of Theorem 3.2 is based on the classical Lumer–Phillips Theorem (see [28]), which says that $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup $S(t)$ whenever:

(i) $\mathbb{A}$ is dissipative; and
(ii) $\operatorname{Ran}(\mathbb{I} - \mathbb{A}) = \mathcal{H}$.

We will address points (i)–(ii) in the next two lemmas.

Lemma 3.4. *For all $\boldsymbol{U} \in \mathfrak{D}(\mathbb{A})$, we have the equality*

$$(\mathbb{A}\boldsymbol{U}, \boldsymbol{U})_{\mathcal{H}} = -\alpha \|\eta\|_{\mathcal{M}}^2 - \int_0^\infty g''(s)\|\eta(s)\|_1^2 \mathrm{d}s \leq 0,$$

where the latter inequality follows from (g2). *This tells that $\mathbb{A}$ is a dissipative operator.*

Proof. By direct calculations we draw the identity

$$\begin{aligned}(\mathbb{A}\boldsymbol{U}, \boldsymbol{U})_{\mathcal{H}} &= \int_0^\infty \big[\alpha g(s) - g'(s)\big]\langle T\eta(s), \eta(s)\rangle_1 \mathrm{d}s \\ &\quad + \int_0^\infty g(s)\langle T\eta(s), v\rangle_1 \mathrm{d}s - \int_0^\infty g'(s)\langle v, \eta(s)\rangle_1 \mathrm{d}s.\end{aligned}$$

Integrating by parts in s (as shown in [26] the boundary terms vanish), we get

$$\int_0^\infty \big[\alpha g(s) - g'(s)\big]\langle T\eta(s), \eta(s)\rangle_1 \mathrm{d}s = -\alpha \|\eta\|_{\mathcal{M}}^2 - \int_0^\infty g''(s)\|\eta(s)\|_1^2 \mathrm{d}s$$

and

$$\int_0^\infty g(s)\langle T\eta(s), v\rangle_1 \mathrm{d}s = \int_0^\infty g'(s)\langle \eta(s), v\rangle_1 \mathrm{d}s,$$

as claimed. $\square$

Lemma 3.5. *The operator $\mathbb{I} - \mathbb{A}$ maps $\mathfrak{D}(\mathbb{A})$ onto $\mathcal{H}$.*

Proof. Given $\tilde{\boldsymbol{U}} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\eta}) \in \mathcal{H}$, we look for a solution $\boldsymbol{U} = (u, v, w, \eta) \in \mathfrak{D}(\mathbb{A})$ to the equation

$$\boldsymbol{U} - \mathbb{A}\boldsymbol{U} = \tilde{\boldsymbol{U}}$$

which, in components, becomes

$$u - v = \tilde{u}, \tag{3.2}$$

$$v - w = \tilde{v}, \tag{3.3}$$

$$(1+\alpha)w + A\Big[\beta v + (\alpha\beta - \varrho)u + \int_0^\infty g(s)\eta(s)\mathrm{d}s\Big] = \tilde{w}, \tag{3.4}$$

$$\eta - T\eta - v = \tilde{\eta}. \tag{3.5}$$

Integrating (3.5) with the position $\eta(0) = 0$ we find

$$\eta(s) = (1 - \mathrm{e}^{-s})v + \phi(s), \tag{3.6}$$

where we denoted

$$\phi(s) = \int_0^s \mathrm{e}^{-(s-y)}\tilde{\eta}(y)\mathrm{d}y.$$

Since $-g'$ is nonnegative and decreasing, due to (g1)–(g2), making use of the standard properties of the convolution [9, §21] we get

$$\|\phi\|^2_{\mathcal{M}} \le \int_0^\infty \bigg(\int_0^s \mathrm{e}^{-(s-y)}\sqrt{-g'(y)}\,\|\tilde{\eta}(y)\|_1\mathrm{d}y\bigg)^2 \le \|\tilde{\eta}\|^2_{\mathcal{M}}. \tag{3.7}$$

Substituting (3.2)–(3.3) and (3.6) into (3.4), we obtain

$$(1+\alpha)v + \kappa A v = q, \tag{3.8}$$

having set

$$\kappa = \beta + \alpha\beta - \varrho + \int_0^\infty g(s)\big[1 - \mathrm{e}^{-s}\big]\mathrm{d}s > 0$$

and

$$q = \tilde{w} + (1+\alpha)\tilde{v} - A\Big[(\alpha\beta - \varrho)\tilde{u} + \int_0^\infty g(s)\phi(s)\mathrm{d}s\Big].$$

We claim that the elliptic equation (3.8) admits a (unique) solution $v \in \mathrm{H}_1$. To infer that, it is enough showing that $q \in \mathrm{H}_{-1}$. Indeed,

$$\|q\|_{-1} \le \|\tilde{w}\|_{-1} + (1+\alpha)\|\tilde{v}\|_{-1} + (\alpha\beta - \varrho)\|\tilde{u}\|_1 + \int_0^\infty g(s)\|\phi(s)\|_1\mathrm{d}s,$$

and, due to (g4), (3.7) and the Hölder inequality,

$$\int_0^\infty g(s)\|\phi(s)\|_1 \mathrm{d}s \le \sqrt{\varrho}\left(\int_0^\infty g(s)\|\phi(s)\|_1^2 \mathrm{d}s\right)^{\frac{1}{2}} \le \sqrt{\frac{\varrho}{\delta}}\,\|\tilde{\eta}\|_{\mathcal{M}}.$$

Accordingly, we learn from (3.2)–(3.3) that $u,\ w \in \mathrm{H}_1$. Besides, owing to (3.6)–(3.7),

$$\|\eta\|_{\mathcal{M}}^2 \le 2\|v\|_{\mathcal{M}}^2 + 2\|\phi\|_{\mathcal{M}}^2 \le 2g(0)\|v\|_1^2 + 2\|\tilde{\eta}\|_{\mathcal{M}}^2,$$

i.e. $\eta \in \mathcal{M}$ and so $T\eta = \eta - \tilde{\eta} - v \in \mathcal{M}$ as well. It is also immediate to check that $\eta(0) = 0$ in H_1. Finally, by comparison in (3.4),

$$\beta v + (\alpha\beta - \varrho)u + \int_0^\infty g(s)\eta(s)\mathrm{d}s = A^{-1}\big[\tilde{w} - (1+\alpha)w\big] \in \mathrm{H}_2.$$

This finishes the proof. □

3.3. Basic energy estimates

For every initial datum $\boldsymbol{U}_0 = (u_0, v_0, w_0, \eta_0) \in \mathcal{H}$, the energy at time t of the solution

$$\boldsymbol{U}(t) = S(t)\boldsymbol{U}_0 = (u(t), \partial_t u(t), \partial_{tt} u(t), \eta^t)$$

to (2.4) is given by

$$\begin{aligned}
\mathsf{E}_0(t) &= \frac{1}{2}|S(t)\boldsymbol{U}_0|_{\mathcal{H}}^2 \\
&= \frac{1}{2}\bigg[\Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_t u(t) + \alpha u(t)\|_1^2 + \|\partial_{tt} u(t) + \alpha \partial_t u(t)\|^2 + \frac{\varrho}{\alpha}\|\partial_t u(t)\|_1^2 + \|\eta^t\|_{\mathcal{M}}^2 \\
&\quad + \alpha \int_0^\infty g(s)\|\eta^t(s)\|_1^2 \mathrm{d}s + 2\int_0^\infty g(s)\langle \eta^t(s), \partial_t u(t)\rangle_1 \mathrm{d}s\bigg].
\end{aligned}$$

Proposition 3.6. *For every $t > \tau \ge 0$, the following dissipativity relation holds*

$$\mathsf{E}_0(t) + \alpha \int_\tau^t \|\eta^y\|_{\mathcal{M}}^2 \mathrm{d}y \le \mathsf{E}_0(\tau).$$

In addition, E_0 is a strict Lyapunov function for the dynamical system $S(t)$ acting on $\mathcal{H}$.

Proof. On account of Lemma 3.4, for every $\boldsymbol{U}_0 \in \mathfrak{D}(\mathbb{A})$ a multiplication in $\mathcal{H}$ of (2.4) and $\boldsymbol{U}(t) = S(t)\boldsymbol{U}_0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_0(t) + \alpha\|\eta^t\|^2_{\mathcal{M}} + \int_0^\infty g''(s)\|\eta^t(s)\|_1^2\mathrm{d}s = 0. \tag{3.9}$$

Therefore, using (g2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_0(t) + \alpha\|\eta^t\|^2_{\mathcal{M}} \le 0. \tag{3.10}$$

An integration then yields the desired formula for all regular initial data. By density, the integral inequality holds for all $\boldsymbol{U}_0 \in \mathcal{H}$.

As for the second part, this follows from the fact that if $\mathsf{E}_0(t)$ is constant, then the corresponding solution $\boldsymbol{U}(t)$ is zero. Indeed, when $\eta^t \equiv 0$ we readily infer from (3.1) that $u(t)$ is constant, hence all its time derivatives vanish. Accordingly, the first equation of (2.3) reduce to

$$(\alpha\beta - \varrho)Au(t) = 0,$$

giving $u(t) \equiv 0$. In summary, we showed that $\boldsymbol{U}(t) \equiv 0$, proving that E_0 is a strict Lyapunov function. □

Remark 3.7. Proposition 3.6 provides a dissipative estimate for the energy F of the Volterra equation (1.1) (here we are dealing with the critical case $\varkappa = 0$). Indeed, for the particular choice $\boldsymbol{U}_0 = (u_0, v_0, w_0, u_0) \in \mathcal{H}$, we have the equality

$$\mathsf{F}(t) = \|S(t)\boldsymbol{U}_0\|^2_{\mathcal{H}}.$$

Recalling the equivalence of the norms (Lemma 3.1), and observing that (3.1) becomes

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \le t, \\ u(t) & s > t, \end{cases}$$

we easily see that there exists a constant $c > 1$ such that

$$\mathsf{F}(t) - \int_\tau^t \int_0^y g'(s)\|u(y) - u(y-s)\|_1^2\mathrm{d}s\mathrm{d}y \le c\mathsf{F}(\tau).$$

Remark 3.8. In the Volterra case, i.e. for initial data $\boldsymbol{U}_0 = (u_0, v_0, w_0, u_0)$, the energy $\mathsf{E}_0(t)$ is actually equivalent to the expression denoted by $E_1(t)$ in [19] for the critical case, which satisfies for regular initial data the differential identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + R_1(t) = 0,$$

where, in the notation of [19], the nonnegative function R_1 contains the memory dissipation. This is the Volterra analogue of the energy equality (3.9).

4. Lack of exponential decay: the unbounded operator case

We now give an answer to the question raised in [19] on the exponential decay of the energy F associated to the Volterra equation (1.1) in the critical case $\varkappa = 0$. To this end, let us consider the exponential kernel

$$g(s) = \varrho\delta \mathrm{e}^{-\delta s}$$

with $\delta > 0$, which trivially complies with (g1)–(g4). For this choice, (1.1) reads

$$\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \alpha\beta Au(t) - \varrho\delta \int_0^t \mathrm{e}^{-\delta s} Au(t-s)\mathrm{d}s = 0. \tag{4.1}$$

Theorem 4.1. *Let $\varkappa = 0$, and let A be an unbounded operator. Then, for the energy*

$$\mathsf{F}(t) = \|u(t)\|_1^2 + \|\partial_t u(t)\|_1^2 + \|\partial_{tt}u(t)\|^2 + \varrho\delta^2 \int_0^t \mathrm{e}^{-\delta s}\|u(t) - u(t-s)\|_1^2 \mathrm{d}s$$

associated to (4.1), there exist no $C \geq 1$ and $\omega > 0$ for which the inequality (1.6) holds for all initial data $(u_0, v_0, w_0) \in \mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H}$.

Remark 4.2. Choosing $\boldsymbol{U}_0 = (u_0, v_0, w_0, u_0) \in \mathcal{H}$, the result implies that the semigroup $S(t)$ on $\mathcal{H}$ fails to be exponentially stable as well.

The proof of Theorem 4.1 requires some steps. In order to simplify the argument, we assume that the spectrum $\sigma(A)$ of A contains a sequence of eigenvalues $\lambda_n \to \infty$, e.g. as in the concrete case of the Dirichlet–Laplace operator $A = -\Delta$ on $\mathrm{H} = L^2(\Omega)$, with $\Omega \subset \mathbb{R}^N$ smooth bounded domain.

Step I. We consider the 4th-order equation

$$\partial_{tttt}u + (\alpha+\delta)\partial_{ttt}u + \alpha\delta\partial_{tt}u + \beta A\partial_{tt}u + \beta(\alpha+\delta)A\partial_t u + \delta(\alpha\beta - \varrho)Au = 0, \tag{4.2}$$

formally obtained by taking the sum ∂_t(4.1) + δ(4.1).

Proposition 4.3. *Equation (4.2) admits a unique (weak) solution*

$$u \in \mathcal{C}^2([0,\infty), \mathrm{H}_1) \cap \mathcal{C}^3([0,\infty), \mathrm{H}).$$

Proof. The proof can be done, e.g. via a standard Galerkin scheme. Since (4.2) is linear, we limit ourselves to provide the basic energy estimate. To this end, for a regular solution u, multiply (4.2) by $\partial_{ttt}u + (\alpha+\delta)\partial_{tt}u + \alpha\delta\partial_t u$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + (\alpha+\delta)\delta\varrho\|\partial_t u\|_1^2 = 0, \tag{4.3}$$

where the functional Λ is given by

$$\Lambda = \frac{1}{2}\Big[\|\partial_{ttt}u + (\alpha+\delta)\partial_{tt}u + \alpha\delta\partial_t u\|^2 + \Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_{tt}u + (\alpha+\delta)\partial_t u + \alpha\delta u\|_1^2\Big]$$
$$+ \frac{\varrho}{2\alpha}\Big[\|\partial_{tt}u\|_1^2 + \big((\alpha+\delta)^2 + \alpha\delta\big)\|\partial_t u\|_1^2 + 2(\alpha+\delta)\langle\partial_{tt}u, \partial_t u\rangle_1\Big].$$

Since

$$2(\alpha+\delta)\big|\langle\partial_{tt}u(t), \partial_t u(t)\rangle_1\big| \le \varepsilon\|\partial_{tt}u(t)\|_1^2 + \frac{(\alpha+\delta)^2}{\varepsilon}\|\partial_t u(t)\|_1^2$$

for a fixed

$$\frac{(\alpha+\delta)^2}{\alpha\delta + (\alpha+\delta)^2} < \varepsilon < 1,$$

it is easy to see that there exists a constant $c > 1$ such that

$$\frac{1}{c}\mathsf{W} \le \Lambda \le c\mathsf{W},$$

where

$$\mathsf{W} = \frac{1}{2}\Big[\|u\|_1^2 + \|\partial_t u\|_1^2 + \|\partial_{tt}u\|_1^2 + \|\partial_{ttt}u\|^2\Big]$$

denotes the natural energy on the phase space $\mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H}$. An integration of (4.3) on the interval $[0, t]$ provides the estimate

$$\mathsf{W}(t) \le c^2\mathsf{W}(0),$$

proving the claim. $\square$

The link between (4.2) and (4.1) is established in the next proposition.

Proposition 4.4. *Let $u_0 \in \mathrm{H}_1$, $v_0 \in \mathrm{H}_1$ and $w_0 \in \mathrm{H}_1$. Moreover, suppose that*

$$z_0 := -\alpha w_0 - \beta A v_0 - \alpha\beta A u_0 \in \mathrm{H}.$$

Then the solution u to (4.2) with initial data

$$u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \partial_{tt}u(0) = w_0, \quad \partial_{ttt}u(0) = z_0$$

is the (unique) solution to (4.1) with initial data

$$u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \partial_{tt}u(0) = w_0.$$

Proof. Since u solves (4.2), the function

$$\psi = \partial_{ttt}u + \alpha\partial_{tt}u + \beta A\partial_t u + \alpha\beta Au$$

fulfills the identity

$$\partial_t\psi + \delta\psi - \varrho\delta Au = 0.$$

Multiplying by $\mathrm{e}^{\delta t}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathrm{e}^{\delta t}\psi(t)\big] - \varrho\delta\mathrm{e}^{\delta t}Au(t) = 0.$$

Since

$$\psi(0) = z_0 + \alpha w_0 + \beta Av_0 + \alpha\beta Au_0 = 0,$$

integrating the equality above we are led to

$$\mathrm{e}^{\delta t}\big[\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \beta A\partial_t u(t) + \alpha\beta Au(t)\big] - \varrho\delta\int_0^t \mathrm{e}^{\delta s}Au(s)\mathrm{d}s,$$

and a final multiplication by $\mathrm{e}^{-\delta t}$ yields (4.1). $\square$

Step II. For every n, we consider the 4th-order polynomial in the complex variable ζ

$$P_n(\zeta) = \zeta^4 + (\alpha+\delta)\zeta^3 + (\alpha\delta + \beta\lambda_n)\zeta^2 + \beta(\alpha+\delta)\lambda_n\zeta + \delta(\alpha\beta - \varrho)\lambda_n.$$

Lemma 4.5. *For every n large enough, the equation*

$$P_n(\zeta) = 0$$

has the two complex roots

$$\zeta_n^{\pm} = x_n \pm \mathrm{i}y_n,$$

where

$$x_n = -\frac{M_n}{\lambda_n} \qquad \textit{with} \qquad M_n \to \frac{\varrho\delta(\alpha+\delta)}{2\beta^2},$$

and

$$y_n = \sqrt{\beta\lambda_n + \frac{\varrho\delta}{\beta} + \varepsilon_n} \qquad \textit{with} \qquad \varepsilon_n \to 0.$$

Proof. Writing

$$\zeta = x + \mathrm{i}y, \qquad x, y \in \mathbb{R},$$

the equation $P_n(\zeta) = 0$ turns into

$$\begin{aligned}\alpha\beta\delta\lambda_n - \varrho\delta\lambda_n + \alpha\beta\lambda_n x + \beta\delta\lambda_n x + \alpha\delta x^2 + \beta\lambda_n x^2 + \alpha x^3 + \delta x^3 + x^4 \\ = y^2(\alpha\delta + \beta\lambda_n + 3\alpha x + 3\delta x + 6x^2) - y^4,\end{aligned} \tag{4.4}$$

and

$$y(\alpha\beta\lambda_n + \beta\delta\lambda_n + 2\alpha\delta x + 2\beta\lambda_n x + 3\alpha x^2 + 3\delta x^2 + 4x^3) = y^3(\alpha + \delta + 4x). \tag{4.5}$$

Assuming $y \neq 0$, and substituting (4.5) into (4.4), we are led to

$$R_n(x) = 0,$$

where

$$\begin{aligned}R_n(x) &= [P_n(x)]^2 - (\alpha\delta + \beta\lambda_n + 3\alpha x + 3\delta x + 6x^2)P_n(x) \\ &\quad + \alpha\beta\delta\lambda_n - \varrho\delta\lambda_n + \alpha\beta\lambda_n x + \beta\delta\lambda_n x + \alpha\delta x^2 + \beta\lambda_n x^2 + \alpha x^3 + \delta x^3 + x^4,\end{aligned}$$

having set

$$P_n(x) = \frac{\alpha\beta\lambda_n + \beta\delta\lambda_n + 2\alpha\delta x + 2\beta\lambda_n x + 3\alpha x^2 + 3\delta x^2 + 4x^3}{\alpha + \delta + 4x}.$$

Choosing now

$$M > \frac{\varrho\delta(\alpha+\delta)}{2\beta^2},$$

by direct calculations we see that

$$R_n(0) = -\varrho\delta\lambda_n < 0,$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n} R_n\Big(-\frac{M}{\lambda_n}\Big) = \frac{2\beta^2 M - \varrho\delta(\alpha+\delta)}{\alpha+\delta} > 0.$$

Hence, for every n large enough, the equation $R_n(x) = 0$ has a root

$$x_n = -\frac{M_n}{\lambda_n} \qquad \text{with} \qquad 0 < M_n < M.$$

From the limit above, it is also clear that it must be

$$\lim_{n\to\infty} M_n = \frac{\varrho\delta(\alpha+\delta)}{2\beta^2}.$$

Plugging x_n into (4.5) we find y_n, hence the desired $\zeta_n^\pm$. $\square$

Step III. We are now ready to complete the proof of Theorem 4.1. For n large enough, let u_n be an eigenvector corresponding to λ_n. Without loss of generality, we can take $\|u_n\|_1 = 1$. Then, consider the function

$$u(t) = u_n\Big[\mathrm{e}^{x_n t}\sin(y_n t) + Q_n \mathrm{e}^{x_n t}\cos(y_n t)\Big],$$

with x_n, y_n given by Lemma 4.5, and

$$Q_n = \frac{y_n^3 - 3x_n^2 y_n - 2\alpha x_n y_n - \beta\lambda_n y_n}{x_n^3 - 3x_n y_n^2 + \alpha(x_n^2 - y_n^2) + \beta\lambda_n x_n + \alpha\beta\lambda_n}.$$

We claim that Q_n is well defined for n large. Indeed, using the expressions of x_n and y_n, it is easily seen that

$$x_n^3 - 3x_n y_n^2 + \alpha(x_n^2 - y_n^2) + \beta\lambda_n x_n + \alpha\beta\lambda_n \to \frac{\varrho\delta^2}{\beta} > 0.$$

Observe that $u(t)$ satisfies (4.2) with initial data

$$\begin{aligned}
u(0) &= Q_n u_n,\\
\partial_t u(0) &= \big[Q_n x_n + y_n\big]u_n,\\
\partial_{tt} u(0) &= \big[Q_n x_n^2 + 2x_n y_n - Q_n y_n^2\big]u_n\\
\partial_{ttt} u(0) &= \big[Q_n x_n^3 + 3x_n^2 y_n - 3Q_n x_n y_n^2 - y_n^3\big].
\end{aligned}$$

Moreover, by the choice of Q_n,

$$\partial_{ttt} u(0) = -\alpha\big[Q_n x_n^2 + 2x_n y_n - Q_n y_n^2\big]u_n - \beta A\big[Q_n x_n + y_n\big]u_n - \alpha\beta A Q_n u_n$$
$$= -\alpha\partial_{tt}\, u(0) - \beta A\partial_t u(0) - \alpha\beta A u(0).$$

Therefore, by Proposition 4.4, the function $u(t)$ solves (4.1). Finally,

choosing

$$t = t_k = \frac{1}{y_n}\Big[\frac{\pi}{2} + k\pi\Big]$$

we get

$$\mathsf{F}(t_k) \geq \|u(t_k)\|_1^2 = \mathrm{e}^{2x_n t_k} \quad \forall k \in \mathbb{N}.$$

Letting $k \to \infty$, this tells that, if exponential decay of rate $\omega > 0$ occurs in (1.6), there is the restriction

$$\omega \le -x_n.$$

But this is true for every n large, and $x_n \to 0$, hence there is no such a $\omega > 0$. $\square$

5. Exponential decay: the bounded operator case

As expected, the picture is completely different when A is a bounded operator. Since there is a strong Lyapunov function for the system and the set of equilibria is zero (see Proposition 3.6), the Nagy–Foias theory implies at least weak stability of the entire dynamics (see [5]). If A is bounded, such a weak stability may be expected to upgrade to uniform stability (hence exponential for the semigroup). However, this conclusion is not entirely obvious, because the overall dynamics is still infinite-dimensional.

Theorem 5.1. *Let A be a bounded operator. Then $S(t)$ is exponentially stable, i.e. there exist constants $C \ge 1$ and $\omega > 0$ such that*

$$|S(t)\boldsymbol{U}_0|_{\mathcal{H}} \le C|\boldsymbol{U}_0|_{\mathcal{H}} \mathrm{e}^{-\omega t}, \quad \forall \boldsymbol{U}_0 \in \mathcal{H}.$$

As a byproduct, the energy F of the Volterra equation (1.1) satisfies the exponential decay property (1.6) also in the critical case $\varkappa = 0$.

We will give two different proofs of Theorem 5.1. The first one, obtained via semigroup techniques through a contradiction argument, is more immediate. However, as always the case with proofs by contradiction, the result does not lead to explicit estimates, which are always informative and may allow to analyze further generalizations of the dynamics. On the contrary, the second one is based on direct energy estimates, and (in principle) can be exported to cover semilinear variants of the model. In addition, as we shall see later, the semigroup (spectral) proof will be a convenient tool to establish in the next Section 6 the semiuniform stability of the semigroup $S(t)$, while the energy estimates will play a key role in proving the decay rates of Section 7. Thus, both proofs are needed and, when A is unbounded, lead to different results.

Notation. In the forthcoming proofs, we shall employ the further space

$$\mathcal{N} = L^2_g(\mathbb{R}^+; \mathrm{H}_1)$$

of square summable functions $\eta : \mathbb{R}^+ \to \mathrm{H}_1$ with respect to the measure $g(s)\mathrm{d}s$, endowed with the Hilbert norm

$$\|\eta\|^2_{\mathcal{N}} = \int_0^\infty g(s)\|\eta(s)\|_1^2 \mathrm{d}s.$$

On account of (g4), we have the continuous embedding

$$\mathcal{M} \subset \mathcal{N}.$$

5.1. Proof of Theorem 5.1 [via semigroup techniques]

We use a classical result of Prüss [29] (see also [8] for the precise statement used here).

Lemma 5.2. *The contraction semigroup $S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}$ is exponentially stable if and only if there exists $\sigma > 0$ such that*[1]

$$\inf_{\mu\in\mathbb{R}} |(\mathrm{i}\mu\mathbb{I} - \mathbb{A})\boldsymbol{U}|_{\mathcal{H}} \geq \sigma|\boldsymbol{U}|_{\mathcal{H}}, \quad \forall \boldsymbol{U} \in \mathfrak{D}(\mathbb{A}).$$

By contradiction, let then $S(t)$ be not exponentially stable. Then there exist sequences $\mu_n \in \mathbb{R}$ and $\boldsymbol{U}_n = (u_n, v_n, w_n, \eta_n) \in \mathfrak{D}(\mathbb{A})$ with

$$|\boldsymbol{U}_n|_{\mathcal{H}} = 1 \tag{5.1}$$

such that

$$|(\mathrm{i}\mu_n\mathbb{I} - \mathbb{A})\boldsymbol{U}_n|_{\mathcal{H}} \to 0. \tag{5.2}$$

Recalling the equivalence of the norms $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, the equality (5.1) implies that

$$\begin{aligned} u_n, v_n &\;\text{ are bounded in } \mathrm{H}_1, \\ w_n &\;\text{ is bounded in } \mathrm{H}, \\ \eta_n &\;\text{ is bounded in } \mathcal{M}, \end{aligned}$$

while the convergence (5.2) componentwise reads

$$\mathrm{i}\mu_n u_n - v_n \to 0 \quad \text{in } \mathrm{H}_1, \tag{5.3}$$

$$\mathrm{i}\mu_n v_n - w_n \to 0 \quad \text{in } \mathrm{H}_1, \tag{5.4}$$

$$\mathrm{i}\mu_n w_n + \alpha w_n + A\Big[\beta v_n + (\alpha\beta - \varrho)u_n + \int_0^\infty g(s)\eta_n(s)\mathrm{d}s\Big] \to 0 \quad \text{in } \mathrm{H}, \tag{5.5}$$

$$\mathrm{i}\mu_n \eta_n - T\eta_n - v_n \to 0 \quad \text{in } \mathcal{M}. \tag{5.6}$$

We will reach a contradiction by showing that every component of $\boldsymbol{U}_n$ goes to zero (up to a subsequence) in its own norm. Exploiting (5.1) and (5.2) we deduce the convergence

$$\big|\mathfrak{Re}\,(\mathbb{A}\boldsymbol{U}_n, \boldsymbol{U}_n)_{\mathcal{H}}\big| = \big|\mathfrak{Re}\,((\mathrm{i}\mu_n\mathbb{I} - \mathbb{A})\boldsymbol{U}_n, \boldsymbol{U}_n)_{\mathcal{H}}\big| \leq |(\mathrm{i}\mu_n\mathbb{I} - \mathbb{A})\boldsymbol{U}_n|_{\mathcal{H}} \to 0.$$

On the other hand, by Lemma 3.4 and (g2),

$$\big|\mathfrak{Re}\,(\mathbb{A}\boldsymbol{U}_n, \boldsymbol{U}_n)_{\mathcal{H}}\big| = \alpha\|\eta_n\|^2_{\mathcal{M}} + \int_0^\infty g''(s)\|\eta_n(s)\|^2_1\mathrm{d}s \geq \alpha\|\eta_n\|^2_{\mathcal{M}},$$

[1] It is understood that $\mathbb{A}$ and $S(t)$ denote the complexifications of $\mathbb{A}$ and $S(t)$, respectively.

and we conclude that

$$\|\eta_n\|_{\mathcal{M}} \to 0 \qquad \Rightarrow \qquad \|\eta_n\|_{\mathcal{N}} \to 0. \tag{5.7}$$

Multiplying (5.6) by v_n in $\mathcal{N}$ (note that v_n is bounded in $\mathcal{M}$, hence in $\mathcal{N}$), we get

$$\mathrm{i}\mu_n\langle\eta_n, v_n\rangle_{\mathcal{N}} - \varrho\|v_n\|_1^2 \to 0. \tag{5.8}$$

Indeed, by an integration by parts and using (5.7),

$$-\langle T\eta_n, v_n\rangle_{\mathcal{N}} = \langle\eta_n, v_n\rangle_{\mathcal{M}} \to 0.$$

At this point, we consider separately two cases.

Case I. Assume first that μ_n is bounded. In this situation, from (5.7)–(5.8) we infer that

$$\|v_n\|_1 \to 0. \tag{5.9}$$

Accordingly, (5.4) and the continuous embedding $\mathrm{H}_1 \subset \mathrm{H}$ yield

$$\|w_n\|_1 \to 0 \qquad \Rightarrow \qquad \|w_n\| \to 0. \tag{5.10}$$

Next, taking the inner product in H of (5.5) with u_n (note that u_n is bounded in $\mathcal{N}$), we find

$$\mathrm{i}\mu_n\langle w_n, u_n\rangle + \alpha\langle w_n, u_n\rangle + \beta\langle v_n, u_n\rangle_1 + (\alpha\beta - \varrho)\|u_n\|_1^2 + \langle\eta_n, u_n\rangle_{\mathcal{N}} \to 0$$

which, in the light of (5.7) and (5.9)–(5.10), provides the convergence

$$\|u_n\|_1 \to 0. \tag{5.11}$$

Collecting (5.7) and (5.9)–(5.11) we learn that $\boldsymbol{U}_n \to 0$ in $\mathcal{H}$, in contradiction with (5.1). So far, no use has been made of the boundedness of A. $\square$

Case II. Conversely, assume that μ_n is not bounded. Then there exists a subsequence μ_{n_k} such that $|\mu_{n_k}| \to \infty$. Here, we appeal to the boundedness of A, which implies that all the spaces H_r are the same, with equivalent norms (without loss of generality, we can always take the H-norm). Then (5.3) and (5.4) immediately give

$$\|u_{n_k}\| \to 0 \qquad \text{and} \qquad \|v_{n_k}\| \to 0.$$

Besides, by (5.7) together with the Hölder inequality, we see at once that

$$\int_0^\infty g(s)\eta_{n_k}(s)\mathrm{d}s \to 0 \quad \text{in H}.$$

Hence, the relation (5.5) reduces to

$$\mathrm{i}\mu_{n_k} w_{n_k} + \alpha w_{n_k} \to 0 \quad \text{in H},$$

which implies the remaining convergence

$$\|w_{n_k}\| \to 0.$$

Again, the contradiction is attained for the subsequence $\boldsymbol{U}_{n_k}$. □

5.2. Proof of Theorem 5.1 [via energy estimates]

By density, it is enough to prove the result for initial data $\boldsymbol{U}_0 \in \mathfrak{D}(\mathbb{A})$. In that follows, $c \geq 0$ will denote a *generic* constant depending only on the structural parameters of the problem. We will also make use, without explicit mention, of the Poincaré inequality (2.2), as well as of the Hölder and Young inequalities. Denoting the solution to (2.4) corresponding to an arbitrary initial datum $\boldsymbol{U}_0 \in \mathfrak{D}(\mathbb{A})$ by

$$\boldsymbol{U}(t) = S(t)\boldsymbol{U}_0 = (u(t), \partial_t u(t), \partial_{tt} u(t), \eta^t),$$

we introduce three auxiliary energy-type functionals:

$$\begin{aligned}
\Phi(t) &= -\langle \eta^t, \partial_t u(t)\rangle_{\mathcal{N}}, \\
\Psi(t) &= -\langle \partial_t u(t), \partial_{tt} u(t) + \alpha \partial_t u(t)\rangle, \\
\Upsilon(t) &= \langle \partial_{tt} u(t) + \alpha \partial_t u(t), \partial_t u(t) + \alpha u(t)\rangle.
\end{aligned}$$

We have the following lemmas (all the forthcoming computations are allowed since we are in the domain of $\mathbb{A}$).

Lemma 5.3. *For every $\varepsilon > 0$ sufficiently small, the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{\varrho}{2}\|\partial_t u\|_1^2 \leq \varepsilon \|\partial_{tt} u\|_1^2 + \frac{c}{\varepsilon}\|\eta\|_{\mathcal{M}}^2$$

holds.

Proof. By direct differentiation along the trajectories of the dynamics, the functional Φ fulfills the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \varrho\|\partial_t u\|_1^2 = -\langle T\eta, \partial_t u\rangle_{\mathcal{N}} - \langle \eta, \partial_{tt} u\rangle_{\mathcal{N}}.$$

Integrating by parts,

$$-\langle T\eta, \partial_t u\rangle_{\mathcal{N}} = \langle \eta, \partial_t u\rangle_{\mathcal{M}}.$$

Next, we estimate

$$\langle \eta, \partial_t u\rangle_{\mathcal{M}} \leq \sqrt{g(0)}\|\partial_t u\|_1 \|\eta\|_{\mathcal{M}} \leq \frac{\varrho}{2}\|\partial_t u\|_1^2 + c\|\eta\|_{\mathcal{M}}^2$$

Moreover, due to the continuous embedding $\mathcal{M} \subset \mathcal{N}$,

$$-\langle \eta, \partial_{tt} u \rangle_{\mathcal{N}} \leq \sqrt{\varrho}\|\partial_{tt} u\|_1 \|\eta\|_{\mathcal{N}} \leq c\|\partial_{tt} u\|_1 \|\eta\|_{\mathcal{M}} \leq \varepsilon \|\partial_{tt} u\|_1^2 + \frac{c}{\varepsilon}\|\eta\|_{\mathcal{M}}^2,$$

for all $\varepsilon > 0$. The proof is finished. □

Lemma 5.4. *The inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \frac{1}{2}\|\partial_{tt} u + \alpha \partial_t u\|^2 \leq \frac{1}{16}\Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_t u + \alpha u\|_1^2 + c\|\partial_t u\|_1^2 + c\|\eta\|_{\mathcal{M}}^2$$

holds.

Proof. By means of elementary computations which employ the state equation, the functional Ψ satisfies the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \|\partial_{tt} u + \alpha \partial_t u\|^2 = \alpha \langle \partial_t u, \partial_{tt} u + \alpha \partial_t u\rangle + \Big(\beta - \frac{\varrho}{\alpha}\Big)\langle \partial_t u, \partial_t u + \alpha u\rangle_1 \\ + \langle \eta, \partial_t u\rangle_{\mathcal{N}} + \frac{\varrho}{\alpha}\|\partial_t u\|_1^2.$$

It is immediate to see that

$$\alpha \langle \partial_t u, \partial_{tt} u + \alpha \partial_t u\rangle \leq \frac{1}{2}\|\partial_{tt} u + \alpha \partial_t u\|^2 + c\|\partial_t u\|_1^2$$

and

$$\Big(\beta - \frac{\varrho}{\alpha}\Big)\langle \partial_t u, \partial_t u + \alpha u\rangle_1 \leq \frac{1}{16}\Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_t u + \alpha u\|_1^2 + c\|\partial_t u\|_1^2.$$

In addition, using again the continuous embedding $\mathcal{M} \subset \mathcal{N}$,

$$\langle \eta, \partial_t u\rangle_{\mathcal{N}} \leq \sqrt{\varrho}\|\partial_t u\|_1 \|\eta\|_{\mathcal{N}} \leq c\|\partial_t u\|_1 \|\eta\|_{\mathcal{M}} \leq c\|\partial_t u\|_1^2 + c\|\eta\|_{\mathcal{M}}^2,$$

yielding the claim. □

Lemma 5.5. *The inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon + \frac{1}{2}\Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_t u + \alpha u\|_1^2 \leq \|\partial_{tt} u + \alpha \partial_t u\|^2 + c\|\partial_t u\|_1^2 + c\|\eta\|_{\mathcal{M}}^2$$

holds.

Proof. It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon + \Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_t u + \alpha u\|_1^2 = \|\partial_{tt} u + \alpha \partial_t u\|^2 - \frac{\varrho}{\alpha}\langle \partial_t u, \partial_t u + \alpha u\rangle_1 - \langle \eta, \partial_t u + \alpha u\rangle_{\mathcal{N}}.$$

Since

$$-\frac{\varrho}{\alpha}\langle\partial_t u, \partial_t u+\alpha u\rangle_1 \le \frac{1}{4}\Big(\beta-\frac{\varrho}{\alpha}\Big)\|\partial_t u+\alpha u\|_1^2 + c\|\partial_t u\|_1^2$$

and, as $\mathcal{M}\subset\mathcal{N}$,

$$\begin{aligned}-\langle\eta, \partial_t u+\alpha u\rangle_{\mathcal{N}} &\le \sqrt{\varrho}\|\partial_t u+\alpha u\|_1\|\eta\|_{\mathcal{N}}\\ &\le c\|\partial_t u+\alpha u\|_1\|\eta\|_{\mathcal{M}}\\ &\le \frac{1}{4}\Big(\beta-\frac{\varrho}{\alpha}\Big)\|\partial_t u+\alpha u\|_1^2 + c\|\eta\|_{\mathcal{M}}^2,\end{aligned}$$

we are finished. □

We can now prove Theorem 5.1. For every $\varepsilon>0$, we introduce the functional

$$\Theta_\varepsilon(t)=\mathsf{E}_0(t)+\varepsilon^2\Phi(t)+\varepsilon^2\sqrt{\varepsilon}\big[4\Psi(t)+\Upsilon(t)\big].$$

It is readily seen that, for all $\varepsilon>0$ small enough, $\Theta_\varepsilon(t)$ is equivalent to $\mathsf{E}_0(t)$, which in turn is equivalent to $\|\boldsymbol{U}(t)\|_{\mathcal{H}}^2$ (by Lemma 3.1). Collecting the three lemmas above, together with the energy inequality (3.10), we easily find $\kappa>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_\varepsilon + 2\kappa\varepsilon^2\sqrt{\varepsilon}\Big[\|\partial_t u+\alpha u\|_1^2+\|\partial_{tt}u+\alpha\partial_t u\|^2+\|\partial_t u\|_1^2+\|\eta\|_{\mathcal{M}}^2\Big]\le \varepsilon^3\|\partial_{tt}u\|_1^2,$$

for all $\varepsilon>0$ small. So far, we did not use the boundedness of the operator A. Indeed, if A is bounded, we have the further control

$$\varepsilon^3\|\partial_{tt}u\|_1^2\le c\varepsilon^3\|\partial_{tt}u\|^2\le c\varepsilon^3\|\partial_{tt}u+\alpha\partial_t u\|^2+c\varepsilon^3\|\partial_t u\|_1^2.$$

Hence, up to fixing $\varepsilon>0$ suitably small, the bad term $\varepsilon^3\|\partial_{tt}u\|_1^2$ can be absorbed in the left-hand side. Accordingly, exploiting again the equivalence of the functionals, we end up with the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_\varepsilon+\kappa\varepsilon^2\sqrt{\varepsilon}\,\Theta_\varepsilon\le 0,$$

and the Gronwall lemma gives the desired exponential decay. □

6. Semiuniform decay

Although $S(t)$ is not exponentially stable when A is unbounded, a (weaker) decay property still holds.

Theorem 6.1. *The semigroup $S(t)$ on $\mathcal{H}$ is semiuniformly stable.*

Recall that the (bounded) semigroup $S(t) = \mathrm{e}^{t\mathbb{A}}$ is said to be *semiuniformly stable* if $\mathbb{A}$ is invertible and [3, Definition 2.1]

$$\lim_{t\to\infty} |S(t)\mathbb{A}^{-1}|_{\mathfrak{L}(\mathcal{H})} = 0,$$

where $|\cdot|_{\mathfrak{L}(\mathcal{H})}$ denotes the norm on the space $\mathfrak{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. Semiuniform stability is a stronger notion than (strong) stability. Indeed, it ensures the convergence $S(t)\boldsymbol{U}_0 \to 0$ for all $\boldsymbol{U}_0 \in \mathfrak{D}(\mathbb{A})$. Since $S(t)$ is bounded, this yields at once the convergence (without uniform rate)

$$|S(t)\boldsymbol{U}_0|_{\mathcal{H}} \to 0, \quad \forall \boldsymbol{U}_0 \in \mathcal{H}.$$

Corollary 6.2. *Let $\varkappa = 0$. Then, for all fixed initial data $(u_0, v_0, w_0) \in \mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H}$, the energy F of the Volterra equation (1.1) satisfies the decay property*

$$\lim_{t\to\infty} \mathsf{F}(t) = 0.$$

Clearly, both the theorem and the corollary are significant only when A is an unbounded operator, otherwise, as shown in the previous section, exponential stability occurs.

Remark 6.3. Although Theorem 6.1 implies the decay with uniform rate for some initial data (those in the domain of $\mathbb{A}$), the same conclusion cannot be drawn for the solutions to the Volterra equation, corresponding to initial data $\boldsymbol{U}_0 = (u_0, v_0, w_0, u_0)$. Indeed, $(u_0, v_0, w_0, u_0) \notin \mathfrak{D}(\mathbb{A})$ except in the rather trivial case when $u_0 = 0$.

The proof of Theorem 6.1 is based on the next lemma. From now on, $\mathbb{A}$ and $S(t)$ will denote the complexifications of $\mathbb{A}$ and $S(t)$, respectively.

Lemma 6.4. *Assume that, for every $\mu \in \mathbb{R}$, the operator $\mathrm{i}\mu\mathbb{I} - \mathbb{A}$ is bounded below. Then $S(t)$ is semiuniformly stable.*

Lemma 6.4 is a combination of the following two well-known theorems. The first is a criterion due to Batty [4, pp. 40–41] (see also [2,3]); the second one is a classical operator-theoretical result (see e.g. [31, §5.1]).

Theorem 6.5. *Assume that the imaginary axis $\mathrm{i}\mathbb{R}$ belongs to the resolvent set $\rho(\mathbb{A})$. Then $S(t)$ is semiuniformly stable.*

Theorem 6.6. *If ζ belongs to the closure of $\rho(\mathbb{A})$ and the (closed) operator $\zeta\,\mathbb{I} - \mathbb{A}$ is bounded below, then $\zeta \in \rho(\mathbb{A})$.*

Indeed, being $S(t)$ a contraction semigroup, we know that the imaginary axis belongs to the closure of $\rho(\mathbb{A})$ Hence, for every $\mu \in \mathbb{R}$ we infer from Theorem 6.6 that

$$\mathrm{i}\mu \in \rho(\mathbb{A}) \qquad \Leftrightarrow \qquad \mathrm{i}\mu\mathbb{I} - \mathbb{A} \text{ is bounded below.}$$

Proof of Theorem 6.1. By Lemma 6.4, all we need is showing that the convergence

$$\mathrm{i}\mu \boldsymbol{U}_n - \mathbb{A}\boldsymbol{U}_n \to 0 \quad \text{in } \mathcal{H}$$

does not occur, for any $\mu \in \mathbb{R}$ and any sequence $\boldsymbol{U}_n \in \mathfrak{D}(\mathbb{A})$ of unit $\mathcal{H}$-norm. Arguing by contradiction, we are back to the situation of the proof of Theorem 5.1, where the argument needs to be applied to a single value $\mu \in \mathbb{R}$. Then, reasoning as in case I only (where the boundedness of A was not used), leads to the desired conclusion. □

7. Polynomial decay for regular initial data

In this final section, we derive explicit polynomial decay rates for the energy arising from more regular initial data (again, for the case A unbounded). To this end, we shall consider the solutions in the spaces $\mathcal{H}_r$ for $r = 1, 0, -1$, with the associated energies

$$\begin{aligned}
\mathsf{E}_r(t) &= \frac{1}{2}|S(t)\boldsymbol{U}_0|^2_{\mathcal{H}_r} \\
&= \frac{1}{2}\bigg[\Big(\beta - \frac{\varrho}{\alpha}\Big)\|\partial_t u(t) + \alpha u(t)\|^2_{1+r} + \|\partial_{tt} u(t) + \alpha \partial_t u(t)\|^2_r + \frac{\varrho}{\alpha}\|\partial_t u(t)\|^2_{1+r} + \|\eta^t\|^2_{\mathcal{M}_r} \\
&\quad + \alpha \int_0^\infty g(s)\|\eta^t(s)\|^2_{1+r}\mathrm{d}s + 2\int_0^\infty g(s)\langle \eta^t(s), \partial_t u(t)\rangle_{1+r}\mathrm{d}s\bigg].
\end{aligned}$$

The above is indeed an equivalent norm on $\mathcal{H}_r$, and this merely follows by Lemma 3.1, replacing the operator A with A^{1+r}.

Theorem 7.1. *There exists a universal constant $C > 0$, depending only on the structural parameters of the problem, such that the inequality*

$$\mathsf{E}_0(t) \leq \frac{C}{1+t}\,\mathsf{E}_1(0)$$

holds for all initial data $\boldsymbol{U}_0 \in \mathcal{H}_1$.

Remark 7.2. Observe that the theorem does not imply the semiuniform stability of $S(t)$, since $\mathcal{H}_1$ is not contained in $\mathfrak{D}(\mathbb{A})$.

Choosing initial data of the form $\boldsymbol{U}_0 = (u_0, v_0, w_0, u_0)$, we deduce the Volterra version of Theorem 7.1.

Corollary 7.3. *There exists a universal constant $C > 0$ such that, for every fixed initial data $(u_0, v_0, w_0) \in \mathrm{H}_2 \times \mathrm{H}_2 \times \mathrm{H}_1$, the energy F of the Volterra equation* (1.1) *in the critical case $\varkappa = 0$ satisfies*

$$\mathsf{F}(t) \leq \frac{C}{1+t}\big[\|u_0\|^2_2 + \|v_0\|^2_2 + \|w_0\|^2_1\big].$$

In order to prove Theorem 7.1, we follow the approach presented in [15, Theorem 3.2.2] based on the following principle:

weak observability + *interpolation* + *dissipativity* $\Rightarrow$ *rate of convergence to equilibria.*

From now on, we agree to work with (regular) solutions arising from sufficiently regular initial data. It is clear from the proof of Proposition 3.6 that the energy identity (3.9) holds for $r = 1, 0, -1$, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_r(t) + \mathsf{D}_r(t) = 0,$$

having set

$$\mathsf{D}_r(t) = \alpha\|\eta^t\|^2_{\mathcal{M}_r} + \int_0^\infty g''(s)\|\eta^t(s)\|^2_{1+r}\mathrm{d}s.$$

In turn, for every $t > \tau \geq 0$, an integration over (τ, t) yields the equality

$$\mathsf{E}_r(t) + \int_\tau^t \mathsf{D}_r(y)\mathrm{d}y = \mathsf{E}_r(\tau). \tag{7.1}$$

In particular, due to (g2),

$$\mathsf{E}_r(t) \leq \mathsf{E}_r(\tau).$$

The next step is proving a *weak observability inequality*.

Lemma 7.4. *There exists a universal constant* $\mathfrak{c} > 0$ *such that the inequality*

$$\int_\tau^t \mathsf{E}_{-1}(y)\mathrm{d}y \leq \mathfrak{c}\,\mathsf{E}_{-1}(t) + \mathfrak{c}\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y$$

holds for every $t > \tau \geq 0$.

Proof. In what follows, $c \geq 0$ will denote a *generic* constant depending only on the structural quantities of the problem. Moreover, the Poincaré inequality (2.2), as well as the Hölder and the Young inequalities, will be tacitly used several times. We divide the proof into four steps.

Step 0. (Notation). Paralleling what done in Section 5, we introduce the space

$$\mathcal{N}_{-1} = L^2_g(\mathbb{R}^+; \mathrm{H})$$

of square summable functions $\eta : \mathbb{R}^+ \to \mathrm{H}$ with respect to the measure $g(s)\mathrm{d}s$, endowed with the Hilbert norm

$$\|\eta\|^2_{\mathcal{N}_{-1}} = \int_0^\infty g(s)\|\eta(s)\|^2 \mathrm{d}s.$$

On account of (g4), we have the continuous embedding[2]

$$\mathcal{M}_{-1} \subset \mathcal{N}_{-1}.$$

Step 1. We consider the functional

$$\Phi_{-1}(t) = -\langle \eta^t, \partial_t u(t)\rangle_{\mathcal{N}_{-1}}.$$

Arguing as in the proof of Lemma 5.3, and recalling (g2), we easily obtain the differential in-equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{-1} + \frac{\varrho}{2}\|\partial_t u\|^2 \le \varepsilon\|\partial_{tt} u\|^2_{-1} + \frac{c}{\varepsilon}\|\eta\|^2_{\mathcal{M}} \le \varepsilon\|\partial_{tt} u\|^2_{-1} + \frac{c}{\varepsilon}\mathsf{D}_0,$$

for every $\varepsilon > 0$ sufficiently small. An integration on (τ, t) yields

$$\int_\tau^t \|\partial_t u(y)\|^2 \mathrm{d}y \le \frac{2}{\varrho}\big[\Phi_{-1}(\tau) - \Phi_{-1}(t)\big] + \frac{2\varepsilon}{\varrho}\int_\tau^t \|\partial_{tt} u(y)\|^2_{-1}\mathrm{d}y + \frac{c}{\varepsilon}\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y.$$

Since $|\Phi_{-1}|$ is controlled by the energy E_{-1}, we deduce that

$$\int_\tau^t \|\partial_t u(y)\|^2 \mathrm{d}y \le c\big[\mathsf{E}_{-1}(\tau) + \mathsf{E}_{-1}(t)\big] + c\varepsilon\int_\tau^t \|\partial_{tt} u(y)\|^2_{-1}\mathrm{d}y + \frac{c}{\varepsilon}\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y.$$

Finally, making use of the energy equality (7.1) for $r = -1$,

$$\mathsf{E}_{-1}(\tau) = \mathsf{E}_{-1}(t) + \int_\tau^t \mathsf{D}_{-1}(y)\mathrm{d}y \le \mathsf{E}_{-1}(t) + c\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y.$$

Thus, we end up with

$$\int_\tau^t \|\partial_t u(y)\|^2 \mathrm{d}y \le c\mathsf{E}_{-1}(t) + c\varepsilon\int_\tau^t \|\partial_{tt} u(y)\|^2_{-1}\mathrm{d}y + \frac{c}{\varepsilon}\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y. \tag{7.2}$$

Step 2. By Lemma 5.4, up to properly rescaling the spaces, the functional

$$\Psi_{-1}(t) = -\langle \partial_t u(t), \partial_{tt} u(t) + \alpha\partial_t u(t)\rangle_{-1}$$

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[2] The space $\mathcal{M}_{-1} = L^2_{-g'}(\mathbb{R}^+; \mathrm{H})$ has been introduced in Subsection 2.1.

fulfills the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_{-1}+\frac{1}{2}\|\partial_{tt}u+\alpha\partial_t u\|_{-1}^2\le\frac{1}{16}\Big(\beta-\frac{\varrho}{\alpha}\Big)\|\partial_t u+\alpha u\|^2+c\|\partial_t u\|^2+c\|\eta\|_{\mathcal{M}_{-1}}^2.$$

Integrating on (τ,t) and using (g2) (noting that D_{-1} is controlled by D_0), we obtain

$$\int_\tau^t\|\partial_{tt}u(y)+\alpha\partial_t u(y)\|_{-1}^2\mathrm{d}y\le 2\big[\Psi_{-1}(\tau)-\Psi_{-1}(t)\big]+\frac{1}{8}\Big(\beta-\frac{\varrho}{\alpha}\Big)\int_\tau^t\|\partial_t u(y)+\alpha u(y)\|^2\mathrm{d}y$$
$$+c\int_\tau^t\|\partial_t u(y)\|^2\mathrm{d}y+c\int_\tau^t\mathsf{D}_0(y)\mathrm{d}y.$$

Again, since $|\Psi_{-1}|$ is controlled by E_{-1}, from the integral inequality above and (7.1) for $r=-1$, we conclude that

$$\int_\tau^t\|\partial_{tt}u(y)+\alpha\partial_t u(y)\|_{-1}^2\mathrm{d}y\le c\mathsf{E}_{-1}(t)+\frac{1}{8}\Big(\beta-\frac{\varrho}{\alpha}\Big)\int_\tau^t\|\partial_t u(y)+\alpha u(y)\|^2\mathrm{d}y \tag{7.3}$$
$$+c\int_\tau^t\|\partial_t u(y)\|^2\mathrm{d}y+c\int_\tau^t\mathsf{D}_0(y)\mathrm{d}y.$$

Step 3. We introduce the further functional

$$\Upsilon_{-1}(t)=\langle\partial_{tt}u(t)+\alpha\partial_t u(t),\partial_t u(t)+\alpha u(t)\rangle_{-1}.$$

The rescaled version of Lemma 5.5 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_{-1}+\frac{1}{2}\Big(\beta-\frac{\varrho}{\alpha}\Big)\|\partial_t u+\alpha u\|^2\le\|\partial_{tt}u+\alpha\partial_t u\|_{-1}^2+c\|\partial_t u\|^2+c\|\eta\|_{\mathcal{M}_{-1}}^2.$$

Analogously to the previous steps, we arrive at

$$\Big(\beta-\frac{\varrho}{\alpha}\Big)\int_\tau^t\|\partial_t u(y)+\alpha u(y)\|^2\mathrm{d}y\le c\mathsf{E}_{-1}(t)+2\int_\tau^t\|\partial_{tt}u(y)+\alpha\partial_t u(y)\|_{-1}^2\mathrm{d}y \tag{7.4}$$
$$+c\int_\tau^t\|\partial_t u(y)\|^2\mathrm{d}y+c\int_\tau^t\mathsf{D}_0(y)\mathrm{d}y.$$

Step 4. At this point, for $\varepsilon>0$ small enough, we take the sum

$$(7.2)+4\sqrt{\varepsilon}\,(7.3)+\sqrt{\varepsilon}\,(7.4),$$

so to obtain

$$\int_\tau^t \|\partial_t u(y)\|^2 \mathrm{d}y + 2\sqrt{\varepsilon}\int_\tau^t \|\partial_{tt}u(y)+\alpha\partial_t u(y)\|_{-1}^2 \mathrm{d}y$$

$$+\frac{\sqrt{\varepsilon}}{2}\Big(\beta-\frac{\varrho}{\alpha}\Big)\int_\tau^t \|\partial_t u(y)+\alpha u(y)\|^2 \mathrm{d}y$$

$$\leq c\mathsf{E}_{-1}(t)+c\varepsilon\int_\tau^t \|\partial_{tt}u(y)\|_{-1}^2\mathrm{d}y + c\sqrt{\varepsilon}\int_\tau^t \|\partial_t u(y)\|^2\mathrm{d}y+\frac{c}{\varepsilon}\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y.$$

Finally, estimating

$$\|\partial_{tt}u\|_{-1}^2 \leq c\|\partial_{tt}u+\alpha\partial_t u\|_{-1}^2 + c\|\partial_t u\|^2,$$

and fixing a suitably small $\varepsilon > 0$, we draw the inequality

$$\int_\tau^t \Big[\|\partial_t u(y)\|^2 + \|\partial_{tt}u(y)+\alpha\partial_t u(y)\|_{-1}^2 + \|\partial_t u(y)+\alpha u(y)\|^2\Big]\mathrm{d}y$$

$$\leq c\mathsf{E}_{-1}(t) + c\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y.$$

The claim follows by adding to both sides the term

$$\int_\tau^t \|\eta^t(y)\|_{\mathcal{M}_{-1}}^2 \mathrm{d}y \leq c\int_\tau^t \mathsf{D}_0(y)\mathrm{d}y.$$

Indeed, the resulting left-hand side is equivalent to the energy E_{-1} integrated on $(\tau,\, t)$. The latter conclusion comes from the rescaled version of the norm equivalence stated in Lemma 3.1. □

We are now in the position to give the

Proof of Theorem 7.1. Assume $\mathsf{E}_1(0) > 0$, otherwise there is nothing to prove. Let $\theta \geq 2\mathfrak{c}$ be fixed (where $\mathfrak{c} > 0$ is the universal constant of Lemma 7.4). Since E_{-1} is nonincreasing, it is immediate to see that

$$\theta\mathsf{E}_{-1}(\theta) \leq \int_0^\theta \mathsf{E}_{-1}(y)\mathrm{d}y.$$

Hence, exploiting Lemma 7.4 with $\tau = 0$ and $t = \theta$, and the energy equality (7.1) for $r = 0$, we get the control

$$(\theta-\mathfrak{c})\mathsf{E}_{-1}(\theta) \leq \mathfrak{c}\int_0^\theta \mathsf{D}_0(y)\mathrm{d}y = \mathfrak{c}\big[\mathsf{E}_0(0)-\mathsf{E}_0(\theta)\big].$$

Since $\theta \geq 2\mathfrak{c}$, we arrive at

$$\mathsf{E}_{-1}(\theta) \leq \mathsf{E}_0(0)-\mathsf{E}_0(\theta).$$

This estimate plus interpolation yield

$$\mathsf{E}_0^2(\theta) \leq K\mathsf{E}_{-1}(\theta)\mathsf{E}_1(\theta) \leq K\big[\mathsf{E}_0(0)-\mathsf{E}_0(\theta)\big]\mathsf{E}_1(\theta)$$

for some $K>0$ depending only on the structural parameters of the problem. Since $\mathsf{E}_1(\theta) \leq \mathsf{E}_1(0)$, we end up with

$$K\mathsf{E}_1(0)\mathsf{E}_0(\theta)+\mathsf{E}_0^2(\theta) \leq K\mathsf{E}_1(0)\mathsf{E}_0(0).$$

Calling

$$\mathsf{Q}(t)=K\mathsf{E}_1(0)\mathsf{E}_0(t) \qquad \text{and} \qquad p(x)=\left[\frac{x}{K\mathsf{E}_1(0)}\right]^2,$$

such an inequality can be rewritten as

$$\mathsf{Q}(\theta)+p\big(\mathsf{Q}(\theta)\big) \leq \mathsf{Q}(0).$$

The argument can be obviously repeated on every interval $[n\theta,(n+1)\theta]$, with $n \in \mathbb{N}$, so to obtain

$$\mathsf{Q}((n+1)\theta)+p\big(\mathsf{Q}((n+1)\theta)\big) \leq \mathsf{Q}(n\theta), \quad \forall n \in \mathbb{N}.$$

At this point, a direct application of Lemma 3.3 from [17] (see also [18]) provides the estimate

$$\mathsf{Q}(n\theta) \leq \sigma(n), \quad \forall n \in \mathbb{N}, \tag{7.5}$$

where σ is a solution to the ODE

$$\begin{cases} \sigma'+q(\sigma)=0, \\ \sigma(0)=\mathsf{Q}(0), \end{cases}$$

with

$$q(x)=x-(\mathrm{I}+p)^{-1}(x).$$

Denoting for simplicity

$$\varpi = K^2\mathsf{E}_1^2(0),$$

by explicit calculations we have

$$q(x) = x + \frac{1}{2}\big(\varpi - \sqrt{\varpi^2 + 4\varpi x}\big).$$

Since $q(x) \sim x$ for large x and

$$q(x) \sim \frac{x^2}{\varpi} \quad \text{as } x \to 0,$$

we conclude that

$$\sigma(t) \leq \frac{B\varpi}{t}$$

for some $B \geq 1$. For any given $t \geq \theta$, let now n be such that $t \in [n\theta,\ (n+1)\theta]$. Since E_0 is nonincreasing, we learn from (7.5) that

$$\mathsf{E}_0(t) \leq \mathsf{E}_0(n\theta) \leq \frac{BK}{n}\,\mathsf{E}_1(0) \leq \frac{C}{1+t}\,\mathsf{E}_1(0),$$

having set

$$C = 2BK(\theta + 1).$$

Recalling that E_0 is nonincreasing and controlled by E_1, it is then apparent that the inequality holds for all $t \geq 0$ (possibly for a larger C). This proves the theorem for regular solutions. The general case follows by a standard density argument. $\square$

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