COMBINATORIAL IDENTITIES INVOLVING THE CENTRAL COEFFICIENTS OF A SHEFFER MATRIX

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Given $m \in \mathbb{N}$, $m \geq 1$, and a Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0}$, we obtain the exponential generating series for the coefficients $\binom{a+(m+1)n}{a+mn}^{-1} s_{a+(m+1)n,a+mn}$. Then, by using this series, we obtain two general combinatorial identities, and their specialization to r-Stirling, r-Lah and r-idempotent numbers. In particular, using this approach, we recover two well known binomial identities, namely *Gould's identity* and *Hagen-Rothe's identity*. Moreover, we generalize these results obtaining an *exchange identity* for a cross sequence (or for two Sheffer sequences) and an *Abel-like identity* for a cross sequence (or for an s-Appell sequence). We also obtain some new Sheffer matrices.

1. INTRODUCTION

A *Sheffer matrix* [**2, 26**] is an infinite lower triangular matrix $S = [s_{n,k}]_{n,k\geq 0}$ whose columns have exponential generating series

$$s_k(t) = \sum_{n\geq k} s_{n,k} \frac{t^n}{n!} = g(t)\frac{f(t)^k}{k!} \qquad (k \in \mathbb{N})$$

for two exponential series $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$, with $g_0 = 1$ and $f_0 = 0$, $f_1 \neq 0$. In this case, we also write $S = (g(t), f(t))$ and we say that the pair $(g(t), f(t))$ is the *spectral representation* of S, or simply the *spectrum* of S.

Given $m \in \mathbb{N}$, $m \geq 1$, and $a \in \mathbb{N}$, the *m-central coefficients* of S are the entries $c_n^{(m)} = s_{(m+1)n,mn}$, while the *shifted m-central coefficients* are the entries $c_n^{(a,m)} = s_{a+(m+1)n,a+mn}$.

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A *Sheffer sequence* with *spectrum* $(g(t), f(t))$, [**2, 4, 8, 29, 32, 33, 34, 36, 37**], is the polynomials sequence $\{s_n(x)\}_{n\in\mathbb{N}}$ of the row polynomials of the Sheffer matrix $S = (g(t), f(t))$, with generating exponential series

$$\sum_{n\geq 0} s_n(x)\,\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xf(t)}\,. \tag{1}$$

An s-*Appell sequence* [**26**], with $s \neq 0$, is a Sheffer sequence with spectrum $(g(t), st)$. For $s = 1$, we have the ordinary *Appell polynomials* [**3**] [**32**, p. 86] [**34**].

Several classical combinatorial sequences, such as the *binomial coefficients*, the *Stirling numbers* of the first and the second kind and the *Lah numbers*, form a Sheffer matrix. Similarly, several classical polynomial sequences, such as the *falling* and *rising factorial powers*, the *generalized Hermite polynomials*, the *generalized Laguerre polynomials*, the *generalized Bernoulli* and *Euler polynomials*, the *exponential polynomials*, the *actuarial polynomials*, the *Cayley continuants*, the *Abel polynomials*, form a Sheffer sequence.

The theory of Sheffer matrices (or sequences) provides a powerful tool for studying and deriving combinatorial identities [**25, 26, 39**]. In this paper, we start by deriving the exponential generating series for the coefficients

$$\binom{a+(m+1)n}{a+mn}^{-1} c_n^{(a,m)} = \binom{a+(m+1)n}{a+mn}^{-1} s_{a+(m+1)n,a+mn} \tag{2}$$

(Theorem 1). Then, by using this result, we obtain the spectral representation of the Sheffer matrix generated by these coefficients (Theorem 2) and two general combinatorial identities (Theorems 3 and 5). In particular, we specialize these identities to some combinatorial families of numbers, such as the r-*Stirling numbers* of the first and second kind, the r-*Lah numbers*, the r-*idempotent numbers*. In particular, using this approach, we recover two well known binomial identities, namely *Gould's identity* [**16, 17**] and *Hagen-Rothe's identity* [**35, 21**]. Moreover, we generalize the results obtained for the r-idempotent numbers by determining an *exchange identity* for an arbitrary cross sequence (Theorem 18) or for two Sheffer sequences (Theorem 19), and an *Abel-like identity* for an arbitrary cross sequence (Theorem 20) or for an arbitrary s-Appell sequence (Theorem 21). These Abel-like identities generalize the classical *Abel's binomial identity* [**1**] (see formula (28)). Finally, we also obtain some new Sheffer matrices.

2. MAIN RESULTS

We start by determining the generating series for the coefficients (2).

Theorem 1. *Let* $S = [s_{n,k}]_{n,k\in\mathbb{N}} = (g(t), f(t))$ *be a Sheffer matrix, and let* $c_n^{(a,m)} = s_{a+(m+1)n,a+mn}$ *be the shifted* m*-central coefficients. Let* $F = [f_{n,k}]_{n,k\in\mathbb{N}} = (1, f(t))$*. Then, we have the exponential generating series*

$$c^{(a,m)}(t) = \sum_{n\geq 0} \frac{c_n^{(a,m)}}{\binom{a+(m+1)n}{a+mn}}\,\frac{t^n}{n!} = \frac{t\varphi'(t)}{\varphi(t)}\left(\frac{\varphi(t)}{t}\right)^{a/m} g(\varphi(t)) \tag{3}$$

where

$$\varphi(t) = \sum_{n\geq 1} \frac{f_{(m+1)n-1,mn}}{\binom{(m+1)n-1}{n-1}} \frac{t^n}{n!} \tag{4}$$

is the unique invertible exponential series satisfying the equation

$$\varphi(t)^{m+1} = t f(\varphi(t))^m \,. \tag{5}$$

Equivalently, $\varphi(t)$ is the exponential series whose compositional inverse is

$$\widehat{\varphi}(t) = \frac{t^{m+1}}{f(t)^m} = t\left(\frac{t}{f(t)}\right)^m \,. \tag{6}$$

Proof. Consider the bivariate series

$$\begin{aligned}
H(t,u) &= \sum_{n,k\geq 0} H_{n,k}\, t^n u^k \\
&= \sum_{n,k\geq 0} \frac{(a+mk)!}{k!(a+n+mk)!}\, s_{a+n+mk,a+mk}\, t^n u^k \\
&= \sum_{k\geq 0} \sum_{n\geq a+mk} \frac{(a+mk)!}{n!k!}\, s_{n,a+mk}\, t^{n-a-mk} u^k \\
&= \sum_{k\geq 0} \frac{(a+mk)!}{k!} \left[\sum_{n\geq a+mk} s_{n,a+mk} \frac{t^n}{n!}\right] \frac{u^k}{t^{a+mk}} \\
&= \sum_{k\geq 0} \frac{(a+mk)!}{k!}\, g(t) \frac{f(t)^{a+mk}}{(a+mk)!} \frac{u^k}{t^{a+mk}} \\
&= g(t) \sum_{k\geq 0} \left(\frac{f(t)}{t}\right)^{a+mk} \frac{u^k}{k!} \\
&= g(t) \left(\frac{f(t)}{t}\right)^a \mathrm{e}^{u\left(\frac{f(t)}{t}\right)^m}
\end{aligned}$$

whose diagonal series is

$$h(t) = \sum_{n\geq 0} H_{n,n}\, t^n = \sum_{n\geq 0} \frac{(a+mn)!}{n!(a+(m+1)n)!}\, s_{a+(m+1)n,a+mn}\, t^n \,.$$

By Cauchy's integral formula [**11**] [**22**], we have

$$h(t) = \frac{1}{2\pi \mathrm{i}} \oint H(z,t/z) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi \mathrm{i}} \oint g(z) \left(\frac{f(z)}{z}\right)^a \mathrm{e}^{t\frac{f(z)^m}{z^{m+1}}} \frac{\mathrm{d}z}{z} \,.$$

Let $z = \varphi(w)$, where φ is the unique invertible exponential series[1] defined by equation (5). Then $w = \frac{z^{m+1}}{f(z)^m} = \widehat{\varphi}(z)$, $\left(\frac{f(z)}{z}\right)^m = \frac{z}{\widehat{\varphi}(z)} = \frac{\varphi(w)}{w}$, $\frac{f(z)}{z} = \left(\frac{\varphi(w)}{w}\right)^{1/m}$,

[1]Notice that equation (5) implies $\varphi(0) = 0$ and $\varphi'(0) = f_1^m \neq 0$.

$\mathrm{d}z = \varphi'(w)\,\mathrm{d}w$ and

$$h(t) = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{e}^{\frac{t}{w}} \frac{w\varphi'(w)}{\varphi(w)} \left(\frac{\varphi(w)}{w}\right)^{a/m} g(\varphi(w)) \frac{\mathrm{d}w}{w} .$$

Since the *Hadamard product* [**11**] of two ordinary series $a(t) = \sum_{n\geq} a_n t^n$ and $b(t) = \sum_{n\geq} b_n t^n$ is given by

$$a(t) \odot b(t) = \sum_{n\geq 0} a_n b_n\, t^n = \frac{1}{2\pi\mathrm{i}} \oint a(t/z)\, b(z) \frac{\mathrm{d}z}{z} ,$$

then we have

$$h(t) = \mathrm{e}^t \odot \left[\frac{t\varphi'(t)}{\varphi(t)} \left(\frac{\varphi(t)}{t}\right)^{a/m} g(\varphi(t))\right] .$$

Hence, if we set

$$c^{(a,m)}(t) = \frac{t\varphi'(t)}{\varphi(t)} \left(\frac{f(\varphi(t))}{\varphi(t)}\right)^{a/m} g(\varphi(t)) = \sum_{n\geq 0} C_n \frac{t^n}{n!} ,$$

then we have

$$h(t) = \mathrm{e}^t \odot c^{(a,m)}(t) = \sum_{n\geq 0} \frac{t^n}{n!} \odot \sum_{n\geq 0} C_n \frac{t^n}{n!} = \sum_{n\geq 0} C_n \frac{t^n}{(n!)^2}$$

and consequently

$$\frac{(a+mn)!}{n!(a+(m+1)n)!}\, c_n^{(a,m)} = \frac{C_n}{(n!)^2}$$

from which we obtain

$$C_n = \frac{n!(a+mn)!}{(a+(m+1)n)!}\, c_n^{(a,m)} = \frac{c_n^{(m)}}{\binom{a+(m+1)n}{a+mn}} .$$

This proves identity (3). Finally, by the Lagrange Inversion Formula [**40**, p. 38], we have

$$\begin{aligned} [t^n]\varphi(t) &= \frac{1}{n}\, [t^{n-1}] \left(\frac{t}{\widehat{\varphi}(t)}\right)^n = \frac{1}{n}\, [t^{n-1}] \left(\frac{f(t)}{t}\right)^{mn} \\ &= \frac{(mn)!}{n}\, [t^{(m+1)n-1}] \frac{f(t)^{mn}}{(mn)!} = \frac{(mn)!}{n} \frac{f_{(m+1)n-1,mn}}{((m+1)n-1)!} \\ &= \frac{(mn)!(n-1)!}{((m+1)n-1)!}\, f_{(m+1)n-1,mn} \frac{1}{n!} = \frac{f_{(m+1)n-1,mn}}{\binom{(m+1)n-1}{mn}} \frac{1}{n!} . \end{aligned}$$

This proves identity (4).

As an immediate consequence of Theorem 1, we have the following result.

Theorem 2. *Let $m \in \mathbb{N}$, $m \geq 1$. Let $S = [s_{n,k}]_{n,k\in\mathbb{N}} = (g(t), f(t))$ be a Sheffer matrix, and let $\varphi(t)$ be series (4). Then, we have the Sheffer matrix*

$$\left[\binom{n}{k} \frac{s_{(m+1)n-k,mn}}{\binom{(m+1)n-k}{n-k}}\right]_{n,k\geq 0} = \left(\frac{t\varphi'(t)}{\varphi(t)}\, g(\varphi(t)), \varphi(t)\right) \tag{7}$$

Proof. Let $k \in \mathbb{N}$ and $a = mk$. Then, by series (3), we have the identity

$$\begin{aligned}
\frac{t\varphi'(t)}{\varphi(t)}\, g(\varphi(t))\, \frac{\varphi(t)^k}{k!} &= \frac{t^k}{k!} \cdot \frac{t\varphi'(t)}{\varphi(t)}\, g(\varphi(t)) \left(\frac{\varphi(t)}{t}\right)^k \\
&= \frac{t^k}{k!} \sum_{n\geq 0} \frac{c_n^{(mk,m)}}{\binom{mk+(m+1)n}{mk+mn}}\, \frac{t^n}{n!} = \sum_{n\geq 0} \binom{n+k}{k} \frac{c_n^{(mk,m)}}{\binom{m(n+k)+n}{m(n+k)}}\, \frac{t^{n+k}}{(n+k)!} \\
&= \sum_{n\geq k} \binom{n}{k} \frac{c_{n-k}^{(mk,m)}}{\binom{(m+1)n-k}{mn}}\, \frac{t^n}{n!} = \sum_{n\geq k} \binom{n}{k} \frac{s_{(m+1)n-k,mn}}{\binom{(m+1)n-k}{n-k}}\, \frac{t^n}{n!}\,.
\end{aligned}$$

This means that we have the Sheffer matrix (7). □

Another consequence of Theorem 1 is the next property, giving our first main identity.

Theorem 3. *Let $a, b, m \in \mathbb{N}$, $m \geq 1$. Given two Sheffer matrices*

$$\begin{aligned}
S_1 &= [s'_{n,k}]_{n,k\in\mathbb{N}} = (g_1(t), f(t)) \\
S_2 &= [s''_{n,k}]_{n,k\in\mathbb{N}} = (g_2(t), f(t))\,,
\end{aligned}$$

let $c_1^{(a,m)}(t)$ and $c_2^{(b,m)}(t)$ be the respective exponential generating series defined by (3). Then, we have the relation

$$c_1^{(a,m)}(t)\; c_2^{(b,m)}(t) = c_1^{(0,m)}(t)\; c_2^{(a+b,m)}(t) \tag{8}$$

or, equivalently, the identity

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \frac{s'_{a+(m+1)k,a+mk}}{\binom{a+(m+1)k}{a+mk}}\, \frac{s''_{b+(m+1)(n-k),b+m(n-k)}}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \\
&\qquad = \sum_{k=0}^{n} \binom{n}{k} \frac{s'_{(m+1)k,mk}}{\binom{(m+1)k}{mk}}\, \frac{s''_{a+b+(m+1)(n-k),a+b+m(n-k)}}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}\,.
\end{aligned} \tag{9}$$

Proof. By Theorem 1, we have

$$\begin{aligned}
c_1^{(a,m)}(t)\; c_2^{(b,m)}(t) &= \left(\frac{t\varphi'(t)}{\varphi(t)}\right)^2 \left(\frac{\varphi(t)}{t}\right)^{(a+b)/m} g_1(\varphi(t))g_2(\varphi(t)) \\
&= \frac{t\varphi'(t)}{\varphi(t)}\, g_1(\varphi(t)) \cdot \frac{t\varphi'(t)}{\varphi(t)} \left(\frac{\varphi(t)}{t}\right)^{(a+b)/m} g_2(\varphi(t)) = c_1^{(0,m)}(t)\; c_2^{(a+b,m)}(t)\,.
\end{aligned}$$

This yields identity (8).

To prove Theorem 5, we need the following result.

Lemma 4. *Let $S = [s_{n,k}]_{n,k\in\mathbb{N}} = (g(t), f(t))$ be a Sheffer matrix and let $F = [f_{n,k}]_{n,k\in\mathbb{N}} = (1, f(t))$. Let $\varphi(t)$ be the exponential series defined by equation (5). For every $a \in \mathbb{N}$, $a \neq 0$, we have the exponential series*

$$\left(\frac{\varphi(t)}{t}\right)^a = \sum_{n\geq 0} \frac{f_{ma+(m+1)n,ma+mn}}{\binom{ma+(m+1)n}{ma+mn}} \frac{a}{a+n} \frac{t^n}{n!} . \tag{10}$$

Proof. By applying the Lagrange Inversion Formula [**40**, p. 38], we have

$$\begin{aligned}
[t^n]\left(\frac{\varphi(t)}{t}\right)^a &= [t^{a+n}]\varphi(t)^a = \frac{a}{a+n}\,[t^n]\left(\frac{t}{\widehat{\varphi}(t)}\right)^{a+n} = \frac{a}{a+n}\,[t^n]\left(\frac{f(t)}{t}\right)^{m(a+n)} \\
&= \frac{a}{a+n}\,(m(a+n))!\,[t^{ma+(m+1)n}]\frac{f(t)^{m(a+n)}}{(m(a+n))!} \\
&= \frac{a}{a+n}\,(m(a+n))!\,\frac{f_{ma+(m+1)n,m(a+n)}}{(ma+(m+1)n)!} \\
&= \frac{a}{a+n}\,\frac{n!(m(a+n))!}{(ma+(m+1)n)!}\,f_{ma+(m+1)n,m(a+n)}\frac{1}{n!} \\
&= \frac{f_{ma+(m+1)n,m(a+n)}}{\binom{ma+(m+1)n}{ma+mn}}\,\frac{a}{a+n}\,\frac{1}{n!} .
\end{aligned}$$

This proves the claim.

Now, we can prove our second main identity.

Theorem 5. *Let $S = [s_{n,k}]_{n,k\in\mathbb{N}} = (g(t), f(t))$ be a Sheffer matrix and let $F = [f_{n,k}]_{n,k\in\mathbb{N}} = (1, f(t))$. Then, for every $a, b \in \mathbb{N}$, $b \neq 0$, we have the identity*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{s_{a+(m+1)k,a+mk}}{\binom{a+(m+1)k}{a+mk}}\,\frac{f_{mb+(m+1)(n-k),mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}}\,\frac{b}{b+n-k} = \\
&\qquad = \frac{s_{a+mb+(m+1)n,a+mb+mn}}{\binom{a+mb+(m+1)n}{a+mb+mn}} .
\end{aligned} \tag{11}$$

In particular, if $g(t) = 1$, then

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{f_{a+(m+1)k,a+mk}}{\binom{a+(m+1)k}{a+mk}}\,\frac{f_{mb+(m+1)(n-k),mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}}\,\frac{b}{b+n-k} = \\
&\qquad = \frac{f_{a+mb+(m+1)n,a+mb+mn}}{\binom{a+mb+(m+1)n}{a+mb+mn}} .
\end{aligned} \tag{12}$$

Moreover, if $g(t) = f'(t)$*, then*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{f_{a+(m+1)k+1,a+mk+1}}{\binom{a+(m+1)k}{a+mk}} \frac{f_{mb+(m+1)(n-k),mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{f_{a+mb+(m+1)n+1,a+mb+mn+1}}{\binom{a+mb+(m+1)n}{a+mb+mn}} . \tag{13}$$

Proof. By series (3), we have

$$c^{(a,m)}(t) \left(\frac{\varphi(t)}{t} \right)^b = \frac{t\varphi'(t)}{\varphi(t)} \left(\frac{\varphi(t)}{t} \right)^{(a+mb)/m} g(\varphi(t)) = c^{(a+mb,m)}(t) .$$

By Theorem 1 and Lemma 4, this equation is equivalent to identity (11). In particular, if $g(t) = 1$, then $s_{n,k} = f_{n,k}$ and we have identity (12). If $g(t) = f'(t)$, then $s_{n,k} = f_{n+1,k+1}$ and we have identity (13). □

3. EXAMPLES

In this section, we exemplify the identities obtained in Theorems 3 and 5 by considering some specific Sheffer matrices of combinatorial interest. In some cases, we also give explicitly the spectral representation of the Sheffer matrix obtained in Theorem 2.

3.1 r-Stirling numbers of the second kind

The *r-Stirling numbers of the second kind* [**5, 13**] are the entries of the Sheffer matrix

$$S^{(r)} = \left[\left\{ n \atop k \right\}_r \right]_{n,k \geq 0} = (\mathrm{e}^{rt}, \mathrm{e}^t - 1) .$$

Equivalently, they have exponential generating series

$$\sum_{n \geq k} \left\{ n \atop k \right\}_r \frac{t^n}{n!} = \mathrm{e}^{rt} \frac{(\mathrm{e}^t - 1)^k}{k!} .$$

In particular, for $r = 0$ and $r = 1$, we have the ordinary *Stirling numbers of the second kind* [**11**, p. 310] [**30**, p. 48] [**38**, A008277]: $\left\{ n \atop k \right\}_0 = \left\{ n \atop k \right\}$ and $\left\{ n \atop k \right\}_1 = \left\{ n+1 \atop k+1 \right\}$. The row polynomials are related to the *actuarial polynomials* [**32**, p. 123] [**41**].

Theorem 6. *For every* $a, b, r, s, m, n \in \mathbb{N}$*,* $m \geq 1$*, we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left\{ a+(m+1)k \atop a+mk \right\}_r}{\binom{a+(m+1)k}{a+mk}} \frac{\left\{ b+(m+1)(n-k) \atop b+m(n-k) \right\}_s}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\left\{ (m+1)k \atop mk \right\}_r}{\binom{(m+1)k}{mk}} \frac{\left\{ a+b+(m+1)(n-k) \atop a+b+m(n-k) \right\}_s}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left\{ a+(m+1)k \atop a+mk \right\}_r}{\binom{a+(m+1)k}{a+mk}} \frac{\left\{ mb+(m+1)(n-k) \atop mb+m(n-k) \right\}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\left\{ a+mb+(m+1)n \atop a+mb+mn \right\}_r}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0) .$$

Proof. If $S_1 = S^{(r)}$ and $S_2 = S^{(s)}$, then identity (9) yields the first identity. Similarly, if $S = S^{(r)}$, then $F = S^{(0)}$ and (11) yields the second identity. $\square$

Remark 7. In particular, for $r = s = 0$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac\{\}{0pt}{}{a+(m+1)k}{a+mk}}{\binom{a+(m+1)k}{a+mk}} \frac{\genfrac\{\}{0pt}{}{b+(m+1)(n-k)}{b+m(n-k)}}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac\{\}{0pt}{}{(m+1)k}{mk}}{\binom{(m+1)k}{mk}} \frac{\genfrac\{\}{0pt}{}{a+b+(m+1)(n-k)}{a+b+m(n-k)}}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac\{\}{0pt}{}{a+(m+1)k}{a+mk}}{\binom{a+(m+1)k}{a+mk}} \frac{\genfrac\{\}{0pt}{}{mb+(m+1)(n-k)}{mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\genfrac\{\}{0pt}{}{a+mb+(m+1)n}{a+mb+mn}}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0)$$

and, for $r = s = 1$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac\{\}{0pt}{}{a+(m+1)k+1}{a+mk+1}}{\binom{a+(m+1)k}{a+mk}} \frac{\genfrac\{\}{0pt}{}{b+(m+1)(n-k)+1}{b+m(n-k)+1}}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac\{\}{0pt}{}{(m+1)k+1}{mk+1}}{\binom{(m+1)k}{mk}} \frac{\genfrac\{\}{0pt}{}{a+b+(m+1)(n-k)+1}{a+b+m(n-k)+1}}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac\{\}{0pt}{}{a+(m+1)k+1}{a+mk+1}}{\binom{a+(m+1)k}{a+mk}} \frac{\genfrac\{\}{0pt}{}{mb+(m+1)(n-k)}{mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\genfrac\{\}{0pt}{}{a+mb+(m+1)n+1}{a+mb+mn+1}}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0)\,.$$

3.2 r-Stirling numbers of the first kind

The *r-Stirling numbers of the first kind* [**5, 23**] are the entries of the Sheffer matrix

$$\mathfrak{S}^{(r)} = \left[\genfrac[]{0pt}{}{n}{k}_r \right]_{n,k \geq 0} = \left(\frac{1}{(1-t)^r}, \log \frac{1}{1-t} \right).$$

Equivalently, they have exponential generating series

$$\sum_{n \geq k} \genfrac[]{0pt}{}{n}{k}_r \frac{t^n}{n!} = \frac{1}{(1-t)^r} \frac{1}{k!} \left(\log \frac{1}{1-t} \right)^k .$$

In particular, for $r = 0$ and $r = 1$, we have the ordinary *Stirling numbers of the first kind* [**11**, p. 310] [**30**, p. 48] [**38**, A132393, A008275]: $\genfrac[]{0pt}{}{n}{k}_0 = \genfrac[]{0pt}{}{n}{k}$ and $\genfrac[]{0pt}{}{n}{k}_1 = \genfrac[]{0pt}{}{n+1}{k+1}$.

Theorem 8. *For every $a, b, r, s, m, n \in \mathbb{N}$, $m \geq 1$, we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac[]{0pt}{}{a+(m+1)k}{a+mk}_r}{\binom{a+(m+1)k}{a+mk}} \frac{\genfrac[]{0pt}{}{b+(m+1)(n-k)}{b+m(n-k)}_s}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac[]{0pt}{}{(m+1)k}{mk}_r}{\binom{(m+1)k}{mk}} \frac{\genfrac[]{0pt}{}{a+b+(m+1)(n-k)}{a+b+m(n-k)}_s}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\genfrac[]{0pt}{}{a+(m+1)k}{a+mk}_r}{\binom{a+(m+1)k}{a+mk}} \frac{\genfrac[]{0pt}{}{mb+(m+1)(n-k)}{mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\genfrac[]{0pt}{}{a+mb+(m+1)n}{a+mb+mn}_r}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0)\,.$$

Proof. If $S_1 = \mathfrak{S}^{(r)}$ and $S_2 = \mathfrak{S}^{(s)}$, then identity (9) yields the first identity. Similarly, if $S = \mathfrak{S}^{(r)}$, then $F = \mathfrak{S}^{(0)}$ and (11) yields the second identity. $\square$

Remark 9. In particular, for $r = s = 0$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left[{a+(m+1)k \atop a+mk}\right]}{\binom{a+(m+1)k}{a+mk}} \frac{\left[{b+(m+1)(n-k) \atop b+m(n-k)}\right]}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\left[{(m+1)k \atop mk}\right]}{\binom{(m+1)k}{mk}} \frac{\left[{a+b+(m+1)(n-k) \atop a+b+m(n-k)}\right]}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left[{a+(m+1)k \atop a+mk}\right]}{\binom{a+(m+1)k}{a+mk}} \frac{\left[{mb+(m+1)(n-k) \atop mb+m(n-k)}\right]}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\left[{a+mb+(m+1)n \atop a+mb+mn}\right]}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0)$$

and, for $r = s = 1$, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left[{a+(m+1)k+1 \atop a+mk+1}\right]}{\binom{a+(m+1)k}{a+mk}} \frac{\left[{b+(m+1)(n-k)+1 \atop b+m(n-k)+1}\right]}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\left[{(m+1)k+1 \atop mk+1}\right]}{\binom{(m+1)k}{mk}} \frac{\left[{a+b+(m+1)(n-k)+1 \atop a+b+m(n-k)+1}\right]}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left[{a+(m+1)k+1 \atop a+mk+1}\right]}{\binom{a+(m+1)k}{a+mk}} \frac{\left[{mb+(m+1)(n-k) \atop mb+m(n-k)}\right]}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\left[{a+mb+(m+1)n+1 \atop a+mb+mn+1}\right]}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0)\,.$$

3.3 r-Lah numbers

The *r-Lah numbers* [**28**], defined by $\left|{n \atop k}\right|_r = \binom{n+2r-1}{k+2r-1}\frac{n!}{k!}$, are the entries of the Sheffer matrix

$$L^{(r)} = \left[\left|{n \atop k}\right|_r\right]_{n,k\geq 0} = \left(\frac{1}{(1-t)^{2r}}, \frac{t}{1-t}\right). \tag{14}$$

Equivalently, they have exponential generating series

$$\sum_{n\geq k} \left|{n \atop k}\right|_r \frac{t^n}{n!} = \frac{1}{(1-t)^{2r}} \frac{1}{k!} \left(\frac{t}{1-t}\right)^k = \frac{1}{k!} \frac{t^k}{(1-t)^{2r+k}}\,.$$

In particular, for $r = 0$ and $r = 1$, we have the ordinary *Lah numbers* [**11**, p. 156] [**30**, p. 44] [**38**, A008297, A271703]: $\left|{n \atop k}\right|_0 = \left|{n \atop k}\right|$ and $\left|{n \atop k}\right|_1 = \left|{n+1 \atop k+1}\right|$.

Theorem 10. *For every $a, b, r, s, m, n \in \mathbb{N}$, $m \geq 1$, we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left|{a+(m+1)k \atop a+mk}\right|_r}{\binom{a+(m+1)k}{a+mk}} \frac{\left|{b+(m+1)(n-k) \atop b+m(n-k)}\right|_s}{\binom{b+(m+1)(n-k)}{b+m(n-k)}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\left|{(m+1)k \atop mk}\right|_r}{\binom{(m+1)k}{mk}} \frac{\left|{a+b+(m+1)(n-k) \atop a+b+m(n-k)}\right|_s}{\binom{a+b+(m+1)(n-k)}{a+b+m(n-k)}}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\left|{a+(m+1)k \atop a+mk}\right|_r}{\binom{a+(m+1)k}{a+mk}} \frac{\left|{mb+(m+1)(n-k) \atop mb+m(n-k)}\right|}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} \frac{b}{b+n-k} = \frac{\left|{a+mb+(m+1)n \atop a+mb+mn}\right|_r}{\binom{a+mb+(m+1)n}{a+mb+mn}} \qquad (b \neq 0)\,.$$

Proof. If $S_1 = L^{(r)}$ and $S_2 = L^{(s)}$, then identity (9) yields the first identity. Similarly, if $S = L^{(r)}$, then $F = L^{(0)}$ and (11) yields the second identity. $\square$

Theorem 11. *We have the Sheffer matrix*

$$\left[\binom{n}{k}\frac{\left|{2n-k \atop n}\right|_r}{\binom{2n-k}{n}}\right]_{n,k\geq 0} = \left(\frac{1+\sqrt{1-4t}}{2\sqrt{1-4t}}\left(\frac{1-\sqrt{1-4t}}{2t}\right)^{2r}, \frac{1-\sqrt{1-4t}}{2}\right). \tag{15}$$

Proof. Let $m = 1$. By identity (5), the series $\varphi(t)$ associated with the Sheffer matrix (14) is defined by the equation

$$\varphi(t)^2 = \frac{t\varphi(t)}{1-\varphi(t)} \qquad \text{or} \qquad \varphi(t)^2 - \varphi(t) + t = 0\,.$$

Hence, we have $\varphi(t) = \frac{1-\sqrt{1-4t}}{2}$ and $\varphi'(t) = \frac{1}{\sqrt{1-4t}}$. By Theorem 2, we obtain the Sheffer matrix (15).

3.4 Binomial coefficients

Consider the Sheffer matrix

$$B^{(\alpha)} = \left[B^{(\alpha)}_{n,k}\right]_{n,k\geq 0} = \left(\frac{1}{(1-4t)^{\alpha}\sqrt{1-4t}}, \frac{t}{1-4t}\right), \tag{16}$$

where

$$B^{(\alpha)}_{n,k} = 4^{n-k}\binom{n+\alpha-1/2}{n-k}\frac{n!}{k!} = \frac{\binom{2n+2\alpha}{2n-2k}}{\binom{n+\alpha}{n-k}}\binom{2n-2k}{n-k}\frac{n!}{k!}\,.$$

For $\alpha = 0$, we have sequence A048854 in [**38**] (see also [**32**, p. 25]). For $\alpha = 1$, we have sequence A286724 in [**38**]. For $\alpha = 2r - 1/2$, we have $B^{(\alpha)}_{n,k} = \left|{n \atop k}\right|_r 4^{n-k}$, where the coefficients $\left|{n \atop k}\right|_r$ are the r-Lah numbers.

In this case, we have the following result.

Theorem 12. *Let α, β, γ and δ arbitrary symbols. We have the identities*

$$\sum_{k=0}^{n}\binom{\alpha+k\beta}{k}\binom{\gamma+(n-k)\beta}{n-k} = \sum_{k=0}^{n}\binom{\alpha+\delta+k\beta}{k}\binom{\gamma-\delta+(n-k)\beta}{n-k} \tag{17}$$

$$\sum_{k=0}^{n}\binom{\alpha+k\beta}{k}\binom{\gamma+(n-k)\beta}{n-k}\frac{\gamma}{\gamma+(n-k)\beta} = \binom{\alpha+\gamma+n\beta}{n}. \tag{18}$$

Proof. Let $a, b, m, n \in \mathbb{N}$, $m \geq 1$. If $S_1 = B^{(\alpha)}$ and $S_2 = B^{(\beta)}$, then identity (9) becomes

$$\sum_{k=0}^{n}\binom{n}{k}\frac{B^{(\alpha)}_{a+(m+1)k,a+mk}}{\binom{a+(m+1)k}{a+mk}}\frac{B^{(\beta)}_{mb+(m+1)(n-k),mb+m(n-k)}}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}} =$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{B^{(\alpha)}_{(m+1)k,mk}}{\binom{(m+1)k}{mk}}\frac{B^{(\beta)}_{a+mb+(m+1)(n-k),a+mb+m(n-k)}}{\binom{a+mb+(m+1)(n-k)}{a+mb+m(n-k)}}.$$

This identity can be simplified noticing that

$$\frac{B^{(\alpha)}_{n,k}}{\binom{n}{k}} = 4^{n-k}(n-k)!\binom{n+\alpha-1/2}{n-k}. \tag{19}$$

Indeed, after some simplifications, we have

$$\sum_{k=0}^{n}\binom{a+(m+1)k+\alpha-1/2}{k}\binom{mb+(m+1)(n-k)+\beta-1/2}{n-k} =$$
$$= \sum_{k=0}^{n}\binom{(m+1)k+\alpha-1/2}{k}\binom{a+mb+(m+1)(n-k)+\beta-1/2}{n-k}.$$

Setting $A = a+\alpha-1/2$, $B = m+1$, $C = mb+\beta-1/2$ and $D = -a$, we obtain the identity

$$\sum_{k=0}^{n}\binom{A+kB}{k}\binom{C+(n-k)B}{n-k} = \sum_{k=0}^{n}\binom{A+D+kB}{k}\binom{C-D+(n-k)B}{n-k}.$$

For the arbitrariness of the parameters A, B, C and D, this identity is equivalent to identity (17).

Furthermore, if $S = L^{(\alpha)}$, then $F = \left(1, \frac{t}{1-4t}\right) = \left[\left|{n \atop k}\right| 4^{n-k}\right]_{n,k\geq 0}$ and identity (11) becomes

$$\sum_{k=0}^{n}\binom{n}{k}\frac{B^{(\alpha)}_{a+(m+1)k,a+mk}}{\binom{a+(m+1)k}{a+mk}}\frac{\left|{mb+(m+1)(n-k) \atop mb+m(n-k)}\right|}{\binom{mb+(m+1)(n-k)}{mb+m(n-k)}}\frac{4^{n-k}b}{b+n-k} = \frac{B^{(\alpha)}_{a+mb+(m+1)n,a+mb+mn}}{\binom{a+mb+(m+1)n}{a+mb+mn}}$$

provided that $b \neq 0$. Also this identity can be simplified using formula (19) and noticing that

$$\frac{\left|{n \atop k}\right|}{\binom{n}{k}} = (n-k)!\binom{n-1}{k-1} = (n-k)!\binom{n}{k}\frac{k}{n} \qquad (n \neq 0).$$

In this way, after some simplifications, the above identity becomes

$$\sum_{k=0}^{n}\binom{a+(m+1)k+\alpha-1/2}{k}\binom{mb+(m+1)(n-k)}{n-k}\frac{mb}{mb+(m+1)(n-k)} =$$
$$= \binom{a+mb+(m+1)n+\alpha)-1/2}{n}.$$

Setting $A = a+\alpha-1/2$, $B = m+1$ and $C = mb$, we have the identity

$$\sum_{k=0}^{n}\binom{A+kB}{k}\binom{C+(n-k)B}{n-k}\frac{C}{C+(n-k)B} = \binom{A+nB}{n}.$$

For the arbitrariness of the parameters A, B and C, this identity is equivalent to identity (18). $\square$

Notice that identities (17) and (18) are well known. More precisely, identity (17) is *Gould's identity* [**16, 17, 20, 39**] [**18**, p. 41, Formula (3.143)] [**24**, Formula (1.3)], while identity (18) is *Hagen-Rothe's identity* [**35, 21, 16, 17, 9, 10, 20**] [**31**, Section 4.5] [**19**, p. 202, Formula (5.62)].

A similar result can also be obtained from Theorem 10. More precisely, from the first identity we can recover identity (17), while from the second identity we can derive the identity

$$\sum_{k=0}^{n} \binom{\alpha + k\beta}{k} \binom{\gamma + (n-k)\beta}{n-k} \frac{\gamma+1}{\gamma+1+(n-k)(\beta-1)} = \binom{\alpha+\gamma+n\beta+1}{n}.$$

Theorem 13. *We have the Sheffer matrix*

$$\begin{aligned} &\left[\frac{\binom{4n-2k+2\alpha}{2n-2k}}{\binom{2n-k+\alpha}{n-k}} \binom{2n-2k}{n-k} \frac{n!}{k!} \right]_{n,k\in\mathbb{N}} = \\ &= \left(\frac{1+\sqrt{1-16t}}{2\sqrt{1-16t}} \left(\frac{1-\sqrt{1-16t}}{8t} \right)^{\alpha+\frac{1}{2}}, \frac{1-\sqrt{1-16t}}{8} \right). \end{aligned} \tag{20}$$

Proof. Let $m = 1$. By identity (5), the series $\varphi(t)$ associated with the Sheffer matrix (16) is defined by the equation

$$\varphi(t)^2 = \frac{t\varphi(t)}{1-4\varphi(t)} \qquad \text{or} \qquad 4\varphi(t)^2 - \varphi(t) + t = 0\,.$$

Hence, we have $\varphi(t) = \frac{1-\sqrt{1-16t}}{8}$ and $\varphi'(t) = \frac{1}{\sqrt{1-16t}}$. By Theorem 2, we obtain the Sheffer matrix (20).

3.5 r-idempotent numbers

We define the *r-idempotent numbers* as the entries of the Sheffer matrix

$$J^{(r)} = \left[\binom{n}{k} (r+k)^{n-k} \right]_{n,k\geq 0} = (\mathrm{e}^{rt}, t\mathrm{e}^{t})\,.$$

For $r = 0$, we have the *idempotent numbers* [**38**, A059297], and for $r = 1$ we have sequence A154372 in [**38**].

Remark 14. For $r \in \mathbb{N}$, the r-idempotent numbers admit the following simple combinatorial interpretation. An *idempotent map* $f : X \to X$ on a set X is a map such that $f^2 = f$, i.e. a map where each element of X is a fixed point or is mapped into a fixed point. If $X = \{1, 2, \ldots, r, r+1, \ldots, n+r\}$ and $R = \{1, 2, \ldots, r\}$, then the numbers $\binom{n}{k}(r+k)^{n-k}$ count the idempotent maps $f : X \to X$ with $r+k$ fixed points, where the first elements 1, 2, ..., r are all fixed points. Indeed, an idempotent map $f : X \to X$ with this property is equivalent to a pair (K, φ), where K is a k-subset of $X \setminus R$, so that $R \cup K$ is the set of all fixed points, and φ is a function from $X \setminus (R \cup K)$ to $R \cup K$. The subset K can be chosen in $\binom{n}{k}$ different ways, and the function φ can be chosen in $(r+k)^{n-k}$ different ways.

Theorem 15. *For every $a, b, r, m, n \in \mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} (a+r+mk)^k (b-mk)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (r+mk)^k (a+b-mk)^{n-k}$$
$$x \sum_{k=0}^{n} \binom{n}{k} (x+mk)^{k-1} (y-mk)^{n-k} = (x+y)^n \, .$$

Proof. Let $S_1 = J^{(r)}$ and $S_2 = J^{(s)}$, and apply Theorem 3. Replacing $s+b+mn$ by b, we have the first identity. If $S = J^{(r)}$, then $F = J^{(0)}$ and identity (11) becomes

$$mb \sum_{k=0}^{n} \binom{n}{k} (r+a+mk)^k (m(b+n)-mk)^{n-k-1} = (r+a+m(b+n))^n \, .$$

Setting $x = r+a$ and $y = m(b+n)$, we have

$$(y-mn) \sum_{k=0}^{n} \binom{n}{k} (x+mk)^k (y-mk)^{n-k-1} = (x+y)^n$$

or, equivalently,

$$(y-mn) \sum_{k=0}^{n} \binom{n}{k} (y-mn+mk)^{k-1} (x+mn-mk)^{n-k} = (x+y)^n \, .$$

Now, replacing $y-mn$ by x and $x+mn$ by y, we derive the second identity. □

Notice that the second identity is a particular case of *Abel's identity* (see Remark 22).

Theorem 16. *For any $m \in \mathbb{N}$, $m \neq 0$, we have the Sheffer matrix*

$$\left[\binom{n}{k} (r+mn)^{n-k} \right] = \left(\frac{1}{1-c(mt)} \left(\frac{c(mt)}{mt} \right)^r, \frac{c(mt)}{m} \right) \tag{21}$$

where

$$c(t) = \sum_{n \geq 1} n^{n-1} \frac{t^n}{n!} \, . \tag{22}$$

Proof. By identity (5), the series $\varphi(t)$ associated with the Sheffer matrix $J^{(r)}$ is defined by the equation $\varphi(t)^{m+1} = t\varphi(t)^m \mathrm{e}^{m\varphi(t)}$, that is $\varphi(t) = t\mathrm{e}^{m\varphi(t)}$. So $\varphi(t) = c(mt)/m = -W(-mt)/m$, where $c(t) = -W(-t)$ is the *Cayley series* [**30**, p. 128] [**40**, p. 25], giving the exponential generating series for the labeled rooted trees, and $W(t)$ is the *Lambert series* [**12**]. Since

$$\frac{t\varphi'(t)}{\varphi(t)} = \frac{1}{1-m\varphi(t)} = \frac{1}{1-c(mt)} \, ,$$

by applying Theorem 2, we obtain the Sheffer matrix (21).

4. SHEFFER POLYNOMIALS

The results obtained in Subsection 3.5, for the r-idempotent numbers, can be generalized as follows. Given a Sheffer sequence $\{s_n(x)\}_{n\in\mathbb{N}}$, with spectrum $(g(t), f(t))$, we can always consider the associated *cross sequence* $\{s_n^{(\lambda)}(x)\}_{n\in\mathbb{N}}$ of index λ [**32**, p.140] defined as the Sheffer sequence with spectrum $(g(t)^\lambda, f(t))$.

Lemma 17. *Given a Sheffer sequence $\{s_n(x)\}_{n\in\mathbb{N}}$, with spectrum $(g(t), f(t))$, we have the Sheffer matrix*

$$\Big(g(t)^\lambda\, e^{xf(t)}, tg(t)^\alpha\, \mathrm{e}^{zf(t)}\Big) = \left[\binom{n}{k} s_{n-k}^{(\lambda+k\alpha)}(x+kz)\right]_{n,k\geq 0}. \tag{23}$$

Proof. The exponential generating series for the kth column is

$$g(t)^\lambda\, \mathrm{e}^{xf(t)} \frac{t^k}{k!}\, g(t)^{k\alpha} \mathrm{e}^{kzf(t)} = \frac{t^k}{k!}\, g(t)^{\lambda+k\alpha} \mathrm{e}^{(x+kz)f(t)} = \frac{t^k}{k!} \sum_{n\geq 0} s_n^{(\lambda+k\alpha)}(x+kz) \frac{t^n}{n!}$$

$$= \sum_{n\geq 0} \binom{n+k}{k} s_n^{(\lambda+k\alpha)}(x+kz) \frac{t^{n+k}}{(n+k)!} = \sum_{n\geq k} \binom{n}{k} s_{n-k}^{(\lambda+k\alpha)}(x+kz) \frac{t^n}{n!}.$$

This implies identity (23). □

Then, we have the next *exchange identity* for a cross sequence.

Theorem 18. *Let $\{s_n(x)\}_{n\in\mathbb{N}}$ be a Sheffer sequence with spectrum $(g(t), f(t))$. Then, we have the identity*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} s_k^{(\lambda+\mu+k\alpha)}(w+x+kz)\, s_{n-k}^{(\nu-k\alpha)}(y-kz) = \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} s_k^{(\mu+k\alpha)}(x+kz)\, s_{n-k}^{(\lambda+\nu-k\alpha)}(w+y-kz) \end{aligned} \tag{24}$$

Proof. If $S_1 = (g(t)^\mu\, \mathrm{e}^{xf(t)}, tg(t)^\alpha \mathrm{e}^{zf(t)})$ and $S_2 = (g(t)^\nu\, \mathrm{e}^{yf(t)}, tg(t)^\alpha \mathrm{e}^{zf(t)})$, then identity (9), for $m = 1$ and $b = 0$, becomes

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} s_k^{(a\alpha+\mu+k\alpha)}(az+x+kz)\, s_{n-k}^{(\nu+n\alpha-k\alpha)}(y+nz-kz) = \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} s_k^{(\mu+k)\alpha)}(x+kz)\, s_{n-k}^{(a\alpha+\nu+n\alpha-k\alpha)}(az+y+nz-kz)\,. \end{aligned}$$

Setting $\lambda = a\alpha$ and $w = az$, and replacing $\nu + n\alpha$ by ν and $y + nz$ by y, we obtain identity (24). □

We also have the following *exchange identity* for two Sheffer sequences.

Theorem 19. *Let $\{s_n(x)\}_{n\in\mathbb{N}}$ and $\{t_n(x)\}_{n\in\mathbb{N}}$ be two Sheffer sequences, with spectrum $(g_1(t), f(t))$ and $(g_2(t), f(t))$, respectively. Then, we have the identity*

$$(25)\quad \sum_{k=0}^{n}\binom{n}{k}s_k(w+x+kz)\,t_{n-k}(y-kz) = \sum_{k=0}^{n}\binom{n}{k}s_k(x+kz)\,t_{n-k}(w+y-kz)\,.$$

Proof. If $S_1 = (g_1(t)\,\mathrm{e}^{xf(t)}, t\mathrm{e}^{zf(t)})$ and $S_2 = (g_2(t)\,\mathrm{e}^{yf(t)}, t\mathrm{e}^{zf(t)})$, then identity (9), for $m = 1$, becomes

$$\sum_{k=0}^{n}\binom{n}{k}s_k(az+x+kz)\,t_{n-k}(y+bz+nz-kz) =$$
$$= \sum_{k=0}^{n}\binom{n}{k}s_k(x+kz)\,t_{n-k}(az+y+bz+nz-kz)\,.$$

Setting $w = az$ and replacing $y + bz + nz$ by y, we obtain identity (25). □

Moreover, we have the following *Abel-like identity* for a cross sequence.

Theorem 20. *Let $\{s_n(x)\}_{n\in\mathbb{N}}$ be a Sheffer sequence with spectrum $(g(t), f(t))$. Then, for the associated cross sequence, we have the identity*

$$(26)\qquad x\sum_{k=0}^{n}\binom{n}{k}\frac{s_k^{(\lambda(x-kz))}(x-kz)}{x-kz}\;s_{n-k}^{(\nu+k\lambda z)}(y+kz) = s_n^{(\lambda x+\nu)}(x+y)\,.$$

Proof. Let S be the matrix (23). For $\lambda = 0$ and $x = 0$, we have

$$F = \left(1, tg(t)^{\alpha}\mathrm{e}^{zf(t)}\right) = \left[\binom{n}{k}s_{n-k}^{(k\alpha)}(kz)\right]_{n,k\geq 0}.$$

Hence, by identity (11) with $m = 1$ and $a = 0$ (but without loss of generality), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}s_k^{(\nu+k\alpha)}(x+kz)s_{n-k}^{(b+n)\alpha-k\alpha}((b+n)z-kz)\frac{b}{b+n-k} =$$
$$= s_n^{(\nu+(b+n)\alpha)}(x+(b+n)z)\,.$$

Setting $y = (b+n)z$ and $\mu = (b+n)\alpha$, we have $bz = y - nz$, $\alpha y - \mu z = 0$ and

$$\sum_{k=0}^{n}\binom{n}{k}s_k^{(\nu+k\alpha)}(x+kz)s_{n-k}^{(\mu-k\alpha)}(y-kz)\frac{y-nz}{y-kz} = s_n^{(\mu+\nu)}(x+y)\,.$$

Equivalently, we have

$$(y-nz)\sum_{k=0}^{n}\binom{n}{k}s_{n-k}^{(\nu+n\alpha-k\alpha)}(x+nz-kz)\frac{s_k^{(\mu-n\alpha+k\alpha)}(y-nz+kz)}{y-nz+kz} = s_n^{(\mu+\nu)}(x+y)\,.$$

Now, replacing $\nu + n\alpha$ by ν, $\mu - n\alpha$ by μ, $y - nx$ by x, $x + nz$ by y and z by $-z$, we obtain the identity

$$x \sum_{k=0}^{n} \binom{n}{k} \frac{s_k^{(\mu-k\alpha)}(x-kz)}{x-kz}\, s_{n-k}^{(\nu-k\alpha)}(y+kz) = s_n^{(\mu+\nu)}(x+y)$$

subject to the condition $\alpha x - \mu z = 0$. Finally, by setting $\alpha = \lambda z$ and $\mu = \lambda x$, we obtain identity (26). $\square$

Furthermore, we have the following *Abel-like identity* for s-Appell polynomials (not necessarily forming a cross sequence).

Theorem 21. *Let $\{a_n(x)\}_{n\in\mathbb{N}}$ be an s-Appell sequence with spectrum $(g(t), st)$. Then, we have the identity*

$$x \sum_{k=0}^{n} \binom{n}{k} s^k \,(x-kz)^{k-1} a_{n-k}(y+kz) = a_n(x+y)\,. \tag{27}$$

Proof. Let $\lambda = 0$ and $\nu = 1$ in identity (27). Then, we have

$$x \sum_{k=0}^{n} \binom{n}{k} \frac{a_k^{(0)}(x-kz)}{x-kz}\, a_{n-k}(y+kz) = a_n(x+y)\,.$$

Since $\sum_{n\geq 0} a_n^{(0)}(x)\,\frac{t^n}{n!} = \mathrm{e}^{sxt}$, we have $a_n^{(0)}(x) = (sx)^n$. Hence $a_k^{(0)}(x-kz) = s^k\,(x-kx)^k$, and we have identity (27). $\square$

Remark 22. For the ordinary powers x^n, which form an Appell sequence, identity (27) yields the original *Abel's binomial identity* [**1**] [**11**, p. 128] [**31**, p. 18] [**32**, p. 73]

$$x \sum_{k=0}^{n} \binom{n}{k} (x-kz)^{k-1}(y+kz)^{n-k} = (x+y)^n\,. \tag{28}$$

In this way, we also reobtain the second identity stated in Theorem 15.

Furthermore, we have the following Sheffer matrices.

Theorem 23. *Let $\{s_n(x)\}_{n\in\mathbb{N}}$ be a Sheffer sequence with spectrum $(g(t), f(t))$. Then, we have the Sheffer matrix*

$$\left[\binom{n}{k} s_{n-k}^{(\lambda+n\alpha)}(x+nz)\right]_{n,k\geq 0} =$$

$$= \left(\frac{g(\varphi(t))^{\lambda}}{1 - z\varphi(t)f'(\varphi(t)) - \alpha\frac{\varphi(t)}{g(\varphi(t))}} \left(\frac{\varphi(t)}{tg(\varphi(t))^{\alpha}}\right)^{x/z}, \varphi(t)\right)$$

where $\varphi(t)$ is the unique invertible solution of the equation

$$\varphi(t) = tg(\varphi(t))^{\alpha} e^{zf(\varphi(t))} .$$

In particular, if $\{a_n(x)\}_{n\in\mathbb{N}}$ is an s-Appell sequence, then we have the Sheffer matrix

$$\left[\binom{n}{k} a_{n-k}(x+nz)\right]_{n,k\geq 0} = \left(\frac{g(\frac{c(szt)}{sz})}{1-c(szt)} \left(\frac{c(szt)}{szt}\right)^{x/z}, \frac{c(szt)}{sz} \right)$$

where $c(t)$ is the Cayley series (22).

Proof. For $m = 1$, the series $\varphi(t)$ associated with the Sheffer matrix (23) is defined (by identity (5)) by the equation $\varphi(t) = tg(\varphi(t))^{\alpha} \mathrm{e}^{zf(\varphi(t))}$. Since

$$\frac{t\varphi'(t)}{\varphi(t)} = \frac{1}{1 - z\varphi(t)f'(\varphi(t)) - \alpha\frac{\varphi(t)}{g(\varphi(t))}} \qquad \text{and} \qquad \mathrm{e}^{xf(t)} = \left(\frac{\varphi(t)}{tg(\varphi(t))^{\alpha}}\right)^{x/z} ,$$

we obtain the first Sheffer matrix, by applying Theorem 2. For an s-Appell sequence, we have $\lambda = 1$, $\alpha = 0$, $f(t) = st$, $f'(t) = s$ and $\varphi(t) = t\mathrm{e}^{sz\varphi(t)}$. Hence, we have $\varphi(t) = \frac{c(szt)}{sz}$ and, consequently, the obtain the second Sheffer matrix. □

We conclude with some examples.

Examples

1. The *falling factorials* $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ form a Sheffer sequence [**32**, p. 57], with exponential generating series

$$\sum_{n\geq 0} x^{\underline{n}} \frac{t^n}{n!} = (1+t)^x .$$

Then, after some simplifications, the exchange identity (25) becomes

$$\sum_{k=0}^{n} \binom{w+x+kz}{k}\binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+kz}{k}\binom{w+y-kz}{n-k}.$$

Notice that, setting $\alpha = w + x$, $\beta = z$, $\gamma = y - nz$ and $\delta = -w$, this identity becomes *Gould's identity* (17).

A similar result can be obtained for the *rising factorials*, or *Pochhammer symbol*, $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ [**32**, p. 5].

2. The *generalized Hermite polynomials* $H_n^{(\nu)}(x)$ form a 2-Appell sequence and a cross sequence [**14**, Vol. 2, p.192] [**26**], with exponential generating series

$$\sum_{n\geq 0} H_n^{(\nu)}(x) \frac{t^n}{n!} = \mathrm{e}^{2xt-\nu t^2} = \mathrm{e}^{-\nu t^2} \mathrm{e}^{2xt} .$$

Then, the exchange identity (24) becomes

$$\sum_{k=0}^{n}\binom{n}{k}H_k^{(\lambda+\mu+k\alpha)}(w+x+kz)\,H_{n-k}^{(\nu-k\alpha)}(y-kz)=$$
$$=\sum_{k=0}^{n}\binom{n}{k}H_k^{(\mu+k\alpha)}(x+kz)\,H_{n-k}^{(\lambda+\nu-k\alpha)}(w+y-kz)$$

and the Abel-like identity (26) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}\frac{H_k^{(\lambda(x-kz))}(x-kz)}{x-kz}\;H_{n-k}^{(\nu+\lambda kz)}(y+kz)=H_n^{(\lambda x+\nu)}(x+y)\,.$$

In particular, the Abel-like identity (27) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}2^k(x-kz)^{k-1}H_{n-k}^{(\nu)}(y+kz)=H_n^{(\nu)}(x+y)\,.$$

3. The *generalized Laguerre polynomials* $L_n^{(\nu)}(x)$ form a Sheffer sequence (but not a cross sequence) [**14**, Vol. 2, p. 189], with exponential generating series

$$\sum_{n\geq 0}L_n^{(\nu)}(x)\,\frac{t^n}{n!}=\frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)^{\nu+1}}\,.$$

The polynomials $L_n^{(\nu-1)}(x)$ form a cross sequence. So, the exchange identity (24) becomes (replacing μ and ν by $\mu+1$ and $\nu+1$, respectively)

$$\sum_{k=0}^{n}\binom{n}{k}L_k^{(\lambda+\mu+k\alpha)}(w+x+kz)\,L_{n-k}^{(\nu-k\alpha)}(y-kz)=$$
$$=\sum_{k=0}^{n}\binom{n}{k}L_k^{(\mu+k\alpha)}(x+kz)\,L_{n-k}^{(\lambda+\nu-k\alpha)}(w+y-kz)$$

while the Abel-like identity (26) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}\frac{L_k^{(\lambda(x-kz)-1)}(x-kz)}{x-kz}\;L_{n-k}^{(\nu+\lambda kz-1)}(y+kz)=L_n^{(\lambda x+\nu-1)}(x+y)\,.$$

Since the sequence is not s-Appell, identity (27) does not hold.

The polynomials $L_n^{(\nu-n)}(x)$ form a (-1)-Appell sequence and a cross sequence [**14**, Vol. 2, p. 189, Formula (19)], with exponential generating series

$$\sum_{n\geq 0}L_n^{(\alpha-n)}(x)\,\frac{t^n}{n!}=(1+t)^{\alpha}\,\mathrm{e}^{-xt}\,.$$

Then, the Abel-like identity (26) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}\frac{L_k^{(\lambda(x-kz)-k)}(x-kz)}{x-kz}\;L_{n-k}^{(\nu+\lambda kz-n+k)}(y+kz)=L_n^{(\lambda x+\nu-n)}(x+y)\,.$$

In particular, the Abel-like identity (27) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}(-1)^k(x-kz)^{k-1}L_{n-k}^{(\nu-n+k)}(y+kz)=L_n^{(\nu-n)}(x+y)\,.$$

4. The *generalized Bernoulli polynomials* $B_n^{(\nu)}(x)$ and the *generalized Euler polynomials* $E_n^{(\nu)}(x)$ form two Appell sequences and two cross sequences, [**32**, p. 93, p. 100] and [**14**, Vol. 3, p. 252], with exponential generating series

$$\sum_{n\geq 0}B_n^{(\nu)}(x)\,\frac{t^n}{n!}=\left(\frac{t}{\mathrm{e}^t-1}\right)^{\nu}\mathrm{e}^{xt}$$
$$\sum_{n\geq 0}E_n^{(\nu)}(x)\,\frac{t^n}{n!}=\left(\frac{2}{\mathrm{e}^t+1}\right)^{\nu}\mathrm{e}^{xt}\,.$$

In this case, in addition to the exchange identity (24), we also have the exchange identity (25), that becomes

$$\sum_{k=0}^{n}\binom{n}{k}B_k^{(\mu)}(w+x+kz)\,E_{n-k}^{(\nu)}(y-kz)=$$
$$=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\mu)}(x+kz)\,E_{n-k}^{(\nu)}(w+y-kz)\,.$$

Moreover, the Abel-like identity (26) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}\frac{B_k^{(\lambda(x-kz))}(x-kz)}{x-kz}\;B_{n-k}^{(\nu+\lambda kz)}(y+kz)=B_n^{(\lambda x+\nu)}(x+y)$$
$$x\sum_{k=0}^{n}\binom{n}{k}\frac{E_k^{(\lambda(x-kz))}(x-kz)}{x-kz}\;E_{n-k}^{(\nu+\lambda kz)}(y+kz)=E_n^{(\lambda x+\nu)}(x+y)$$

and the Abel-like identity (27) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}(x-kz)^{k-1}B_{n-k}^{(\nu)}(y+kz)=B_n^{(\nu)}(x+y)$$
$$x\sum_{k=0}^{n}\binom{n}{k}(x-kz)^{k-1}E_{n-k}^{(\nu)}(y+kz)=E_n^{(\nu)}(x+y)\,.$$

5. The *actuarial polynomials* $a_n^{(\nu)}(x)$ form a cross sequence [**32**, p. 123] [**41**], with exponential generating series

$$\sum_{n\geq 0} a_n^{(\nu)}(x)\,\frac{t^n}{n!} = \mathrm{e}^{\nu t - x(\mathrm{e}^t - 1)}\,.$$

In this case, the exchange identity (24) becomes

$$\sum_{k=0}^{n} \binom{n}{k} a_k^{(\lambda+\mu+k\alpha)}(w+x+kz)\, a_{n-k}^{(\nu-k\alpha)}(y-kz) = \\ = \sum_{k=0}^{n} \binom{n}{k} a_k^{(\mu+k\alpha)}(x+kz)\, a_{n-k}^{(\lambda+\nu-k\alpha)}(w+y-kz)$$

and the Abel-like identity (26) becomes

$$x\sum_{k=0}^{n} \binom{n}{k} \frac{a_k^{(\lambda(x-kz))}(x-kz)}{x-kz}\; a_{n-k}^{(\nu+\lambda kz)}(y+kz) = a_n^{(\lambda x+\nu)}(x+y)\,.$$

Also in this case, identity (27) does not hold.

6. The *Cayley continuants* $U_n^{(\nu)}(x)$ form a cross sequence [**7, 27**], with exponential generating series

$$\sum_{n\geq 0} U_n^{(\nu)}(x)\,\frac{t^n}{n!} = (1-t^2)^{\nu/2}\left(\frac{1+t}{1-t}\right)^{x/2}.$$

In this case, the exchange identity (24) becomes

$$\sum_{k=0}^{n} \binom{n}{k} U_k^{(\lambda+\mu+k\alpha)}(w+x+kz)\, U_{n-k}^{(\nu-k\alpha)}(y-kz) = \\ = \sum_{k=0}^{n} \binom{n}{k} U_k^{(\mu+k\alpha)}(x+kz)\, U_{n-k}^{(\lambda+\nu-k\alpha)}(w+y-kz)$$

and the Abel-like identity (26) becomes

$$x\sum_{k=0}^{n} \binom{n}{k} \frac{U_k^{(\lambda(x-kz))}(x-kz)}{x-kz}\; U_{n-k}^{(\nu+\lambda kz)}(y+kz) = U_n^{(\lambda x+\nu)}(x+y)\,.$$

7. The *generalized rencontres polynomials*[2] $D_n^{(\nu)}(x)$ form an Appell sequence (but not a cross sequence), with exponential generating series

$$\sum_{n\geq 0} D_n^{(\nu)}(x)\frac{t^n}{n!} = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\nu+1}}\,.$$

[2]See [**6, 15**] for a slightly different generalization of the rencontres polynomials.

So, the Abel-like identity (27) becomes

$$x\sum_{k=0}^{n}\binom{n}{k}(x-kz)^{k-1}D_{n-k}^{(\nu)}(y+kz)=D_{n}^{(\nu)}(x+y)\,.$$

8. The *Abel polynomials* $A_n^{(\nu)}(x)=x(x-\nu n)^{n-1}$ form a Sheffer sequence (but not a cross sequence) [**32**, p. 72] , with exponential generating series

$$\sum_{n\geq 0}A_n^{(\nu)}(x)\,\frac{t^n}{n!}=\mathrm{e}^{x\frac{W(\nu t)}{\nu}}$$

where $W(t)$ is the Lambert series. In this case, we have only the exchange identity (25), that becomes

$$\sum_{k=0}^{n}\binom{n}{k}A_k^{(\nu)}(w+x+kz)\,A_{n-k}^{(\nu)}(y-kz)=$$
$$=\sum_{k=0}^{n}\binom{n}{k}A_k^{(\nu)}(x+kz)\,A_{n-k}^{(\nu)}(w+y-kz)\,.$$

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REFERENCES

1. N. H. Abel: *Beweis eines Ausdruckes, von welchem die Binomial-Formel ein einzelner Fall ist*, J. Reine Angew. Math. **1** (1826), 159–160.

2. M. Aigner: "*A Course in Enumeration*", Springer, Berlin, 2007.

3. P. Appell: *Sur une classe de polynomes*, Ann. Sci. École Norm. Sup. (2) **9** (1880), 119–144.

4. R. P. Boas, R. C. Buck: "*Polynomial Expansions of Analytic Functions*", Springer–Verlag, New York, 1964.

5. A. Z. Broder: *The r-Stirling numbers*, Discrete Math. **49** (1984), 241–259.

6. S. Capparelli, M. M. Ferrari, E. Munarini, N. Zagaglia Salvi: *A Generalization of the* "Problème des Rencontres", J. Integer Seq. **21** (2018), Article 18.2.8.

7. A. Cayley: *On the determination of the value of a certain determinant*, Quart. Journ. of Math. **ii** (1858), 163–166 . (Collected Math. Papers, Vol. 3, Cambridge U.P. 1919, 120–123.)

8. T. S. Chihara: "*An Introduction to Orthogonal Polynomials*", Gordon and Breach, New York-London-Paris, 1978.

9. W. Chu: *Elementary proofs for convolution identities of Abel and Hagen–Rothe*, Electron. J. Combin. **17** (2010), #N24.

10. W. Chu: *Reciprocal formulae on binomial convolutions of Hagen-Rothe type*, Boll. Unione Mat. Ital. (9) **6** (2013), 591–605.

11. L. Comtet: "*Advanced Combinatorics*", Reidel, Dordrecht-Holland, Boston, 1974.

12. R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth: *On the Lambert W function*, Adv. Comput. Math. **5** (1996), 329–359.

13. M. d'Ocagne: *Sur une classe de nombres remarquables*, Amer. J. Math. **9** (1887), 353–380.

14. A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, (eds.): *Higher Trascendental Functions*, The Bateman Manuscript project, Vols. I-III, McGraw-Hill, New York 1953.

15. M. M. Ferrari, E. Munarini: *Decomposition of some Hankel matrices generated by the generalized rencontres polynomials*, Linear Algebra Appl. **567** (2019), 180–201.

16. H. W. Gould: *Some generalizations of Vandermonde's convolution*, Amer. Math. Monthly **63** (1956), 84–91.

17. H. W. Gould: *Final analysis of Vandermonde's convolution*, Amer. Math. Monthly **64** (1957), 409–415.

18. H. W. Gould: "*Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*", second edition, Morgantown, W. Va. 1972.

19. R. L. Graham, D. E. Knuth, O. Patashnik: "Concrete Mathematics", Addion-Wesley Pubilshing Co., 1989.

20. V. J. W. Guo: *Bijective proofs of Gould's and Rothe's identities*, Discrete Math. **308** (2008), 1756–1759.

21. J. G. Hagen: "Synopsis der Höheren Mathematik", Berlin, 1891.

22. M. L. J. Hautus, D. A. Klarner: *The diagonal of a double power series*, Duke Math. J. **38** (1971), 229–235.

23. D. S. Mitrinović, R. S. Mitrinović: *Tableaux d'une classe de nombres reliés aux nombres de Stirling*, Univ. Beograd. Pubi. Elektrotehn. Fak. Ser. Mat. Fiz. **77** (1962).

24. H. B. Mittal: *Combinatorial identities of Engelberg and Jensen's formula*, J. Math. Anal. Appl. **66** (1978), 339–345.

25. E. Munarini: *Shifting property for Riordan, Sheffer and connection constants matrices*, J. Integer Seq. **20** (2017), Art. 17.8.2.

26. E. Munarini: *Combinatorial identities for Appell polynomials*, Appl. Anal. Discrete Math. **12** (2018), 362–388.

27. E. Munarini, D. Torri: *Cayley continuants*, Theoret. Comput. Sci. **347** (2005), 353–369.

28. G. Nyul, G. Rácz: *The r-Lah numbers*, Discrete Math. **338** (2015), 1660–1666.

29. E. D. Rainville: "*Special Functions*", Macmillan, New York, 1960.

30. J. Riordan: "*An Introduction to Combinatorial Analysis*", John Wiley & Sons, 1958.

31. J. Riordan: "*Combinatorial identities*", John Wiley & Sons, New York-London-Sydney, 1968.

32. S. Roman: "*The Umbral Calculus*", Academic Press, New York, 1984.

33. S. Roman: "*Advanced Linear Algebra*", Springer-Verlag, Berlin, 1992.

34. S. M. Roman, G.-C. Rota: *The umbral calculus*, Advances in Math. **27** (1978), 95–188.

35. H. A. Rothe: *Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita*, Leipzig, 1793.

36. I. M. Sheffer: *Concerning Appell sets and associated linear functional equations*, Duke Math. J. **3** (1937), 593–609.

37. I. M. Sheffer: *Some properties of polynomial sets of type zero*, Duke Math. J. **5** (1939), 590–622.

38. N. J. A. Sloane: "*On-Line Encyclopedia of Integer Sequences*", `http://oeis.org/`.

39. R. Sprugnoli: *Riordan arrays and the Abel-Gould identity*, Discrete Math. **142** (1995), 213-233.

40. R. P. Stanley: "*Enumerative Combinatorics*", Volume 2, Cambridge University Press, Cambridge, 1999.

41. L. Toscano: *Una classe di polinomi della matematica attuariale*, Rivista Mat. Univ. Parma **1** (1950), 459–470.

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