Optimal proportional reinsurance and investment for stochastic factor models

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Abstract

In this work we investigate the optimal proportional reinsurance-investment strategy of an insurance company which wishes to maximize the expected exponential utility of its terminal wealth in a finite time horizon. Our goal is to extend the classical Cramér-Lundberg model introducing a stochastic factor which affects the intensity of the claims arrival process, described by a Cox process, as well as the insurance and reinsurance premia. The financial market is supposed not influenced by the stochastic factor, hence it is independent on the insurance market. Using the classical stochastic control approach based on the Hamilton-Jacobi-Bellman equation we characterize the optimal strategy and provide a verification result for the value function via classical solutions to two backward partial differential equations. Existence and uniqueness of these solutions are discussed. Results under various premium calculation principles are illustrated and a new premium calculation rule is proposed in order to get more realistic strategies and to better fit our stochastic factor model. Finally, numerical simulations are performed to obtain sensitivity analyses.

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1. Introduction

In this paper we investigate the optimal reinsurance-investment problem of an insurance company which wishes to maximize the expected exponential utility of its terminal wealth in a finite time horizon. In the actuarial literature there is an increasing interest in both optimal reinsurance and optimal investment strategies, because they allow insurance firms to increase financial results and to manage risks. In particular, reinsurance contracts help the reinsured to increase the business capacity, to stabilize operating results, to enter in new markets, and so on. Among the traditional reinsurance arrangements the excess-of-loss and the proportional treaties are of great importance. The former was studied in [Sheng et al., 2014], [Li et al., 2018] and references therein. The latter was intensively studied by many authors under the criterion of maximizing the expected utility of the terminal wealth. Beyond the references contained therein, let us recall some noteworthy papers: in [Liu and Ma, 2009] the authors considered a very general model, also including consumption, focusing on well posedness of the optimization problem and on existence of admissible strategies; in [Liang et al., 2011] a stock price with instantaneous rate of investment return described by

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an Ornstein-Uhlenbeck process has been considered; in [Liang and Bayraktar, 2014] the problem has been studied in a partially observable framework by introducing an unobservable Markov-modulated risk process; in [Zhu et al., 2015] the surplus is invested in a defaultable financial market; in [Liang and Yuen, 2016] and [Yuen et al., 2015] multiple dependent classes of insurance business are considered. All these works may be considered as attempts to extend both the insurance risk and the financial market models. In all these articles we can recognize two different approaches to deal with the surplus process of the insurance company: some authors considered it as a diffusion process approximating the pure-jump term of the Cramér-Lundberg model (see for example [Bai and Guo, 2008, Cao and Wan, 2009, Zhang et al., 2009, Gu et al., 2010, Li et al., 2018] and references therein). This approach is validated by means of the famous Cramér-Lundberg approximation (see [Grandell, 1991]). Other authors (see [Liu and Ma, 2009, Zhu et al., 2015, Liang et al., 2011, Sheng et al., 2014, Yuen et al., 2015] and references therein) took into account the jump term using a compound Poisson risk model with constant intensity, that is the classical Cramér-Lundberg model. On the one hand this is the standard model for nonlife insurance and it is simple enough to perform calculations, on the other it is too simple to be realistic (as noticed by [Hipp, 2004]).
As observed by Grandell, J. in [Grandell, 1991], more reasonable risk models should allow the insurance firm to consider the so called *size fluctuations* as well as the *risk fluctuations*, which refer to variations of the number of policyholders and to modifications of the underlying risks, respectively.

This paper aims at extending the classical risk model by modelling the claims arrival process as a doubly stochastic Poisson process with intensity affected by an exogenous stochastic process $\{Y_t\}_{t\in[0,T]}$. This environmental factor leads us to a reasonably realistic description of any risk movement (see [Grandell, 1991], [Schmidli, 2018]). For example, in automobile insurance Y may describe road conditions, weather conditions (foggy days, rainy days, ...), traffic volume, and so on. While in [Liang and Bayraktar, 2014] the authors considered a Markov-modulated compound Poisson process with the (unobservable) stochastic factor described by a finite state Markov chain, we consider a stochastic factor model where the exogenous process follows a general diffusion. However, as in that work, we suppose that the stochastic factor does not influence the financial market, which remains independent on the insurance market. An additional feature is that the insurance and the reinsurance premia are not evaluated using premium calculation principles, contrary to the majority of the literature; moreover, they turn out to be stochastic processes depending on Y. Furthermore, we highlight that under the most frequently used premium calculation principles (expected value and variance premium principles) some problems arise: firstly, the optimal reinsurance strategy turns out to be deterministic (this is a limiting factor because the main goal of our paper is to consider a stochastic factor model); secondly, the optimal reinsurance strategy does not explicitly depend on the claims intensity. In order to fix these problems, we will introduce a new premium calculation principle, which is called *intensity-adjusted variance premium principle*.

Finally, the financial market is more general than those usually considered in the literature, since it is composed by a risk-free bond and a risky asset described by a generalized Geometric Brownian Motion. For instance, in [Bai and Guo, 2008], [Cao and Wan, 2009], [Zhang et al., 2009] and [Liang and Bayraktar, 2014] the authors used a geometric Brownian model, in [Gu et al., 2010] and [Sheng et al., 2014] a CEV model. Nevertheless, some authors considered other general models: in [Irgens and Paulsen, 2004] and [Li et al., 2018] the risky asset follows a jump-diffusion process with constant parameters, in [Liang et al., 2011] the instantaneous rate of investment return follows an Ornstein-Uhlenbeck process, in [Zhu et al., 2015] the authors used the Heston model, in [Xu et al., 2017] the authors introduced a Markov-modulated model for the financial market. However, in these papers the authors considered the classical risk model with constant intensity for the claims arrival process.

Using the classical stochastic control approach based on the Hamilton-Jacobi-Bellman equation we characterize the optimal strategy and provide a verification result for the value function via classical solutions to two backward partial differential equations (see Theorem 6.1). Moreover we provide a class of sufficient conditions for existence and uniqueness of classical solutions to the PDEs involved (see Theorems 8.1 and 8.2). Results under various premium calculation principles are discussed, including the *intensity-adjusted variance premium principle*. Finally, numerical simulations are performed to obtain sensitivity analyses of the optimal strategies.

The paper is organized as follows: in Section 2 we formulate the main assumptions and describe the optimization problem; Section 3 contains the derivation of the Hamilton-Jacobi-Bellman equation. In Section 4 we characterize the optimal reinsurance strategy, discussing in Subsections 4.1 and 4.2 how the general results apply to special premium calculation principles (expected value, variance premium and intensity-adjusted variance principles). In Section 5 we provide the optimal investment strategy. Section 6 contains the Verification Theorem. In Section 7 we illustrate some numerical results and sensitivity analyses. In Section 8 existence and uniqueness theorems are discussed for the PDEs involved in the problem. Finally, in Appendix A the reader can find some proofs of secondary results.

2. Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a complete probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ (shortly denoted by $\{\mathcal{F}_t\}$) and $T > 0$ a fixed time horizon. We suppose that such a filtration satisfies the usual hypotheses of completeness and right continuity. We introduce the stochastic factor $Y = \{Y_t\}_{t\in[0,T]}$ as the solution to the following SDE:

$$dY_t = b(t, Y_t)\, dt + \gamma(t, Y_t)\, dW_t^{(Y)}, \qquad Y_0 \in \mathbb{R}, \tag{2.1}$$

where $\{W_t^{(Y)}\}_{t\in[0,T]}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$. This stochastic process represents any environmental factor resulting in risk fluctuations. For instance, as suggested by Grandell, J. (see [Grandell, 1991], Chapter 2), in automobile insurance, Y may describe road conditions, weather conditions (foggy days, rainy days, ...), traffic volume, and so on.

Assumption 2.1. *In the sequel we assume that $b(t, y)$ and $\gamma(t, y)$ are locally Lipschitz-continuous in $y \in \mathbb{R}$, uniformly in $t \in [0, T]$, i.e. for each $n = 1, \dots$ there exists a positive constant K_n such that*

$$|b(t, y) - b(t, y')| + |\gamma(t, y) - \gamma(t, y')| \le K_n|y - y'| \qquad \forall y, y' \in [-n, n], t \in [0, T].$$

Moreover, we assume the sub-linear growth condition in $y \in \mathbb{R}$, i.e. for some positive constant K_2 the following inequality holds true:

$$|b(t, y)| + |\gamma(t, y)| \le K_2(1 + |y|) \qquad \forall t \in [0, T], y \in \mathbb{R}.$$

Remark 2.1. *Under Assumption 2.1, from classical results (see [Gihman and Skorohod, 1972]) it follows that for any initial condition $(t, y) \in [0, T] \times \mathbb{R}$ there exists a unique strong solution $\{Y_{t,y}(s)\}_{s\in[t,T]}$ (starting from y at time t) such that for any $p \ge 1$*

$$\mathbb{E}[\sup_{s\in[t,T]} |Y_{t,y}(s)|^p] < \infty, \tag{2.2}$$

which in turn implies

$$\mathbb{P}[\sup_{s\in[t,T]} |Y_{t,y}(s)| < \infty] = 1.$$

In the sequel we will denote by $\{Y_t\}_{t\in[0,T]}$ the solution starting from $Y_0 \in \mathbb{R}$ at time $t = 0$. Let us observe that (2.2) implies

$$\mathbb{E}\left[\int_0^T |b(t, Y_t)|\, dt + \int_0^T \gamma(t, Y_t)^2\, dt\right] < \infty. \tag{2.3}$$

Let us denote by $\mathcal{L}^Y$ the infinitesimal generator of Y:

$$\mathcal{L}^Y f(t, y) = b(t, y)\frac{\partial f}{\partial y}(t, y) + \frac{1}{2}\gamma(t, y)^2 \frac{\partial^2 f}{\partial y^2}(t, y) \qquad f \in \mathcal{C}^{1,2}((0, T) \times \mathbb{R}).$$

Let us introduce a strictly positive measurable function $\lambda(t, y) : [0, T] \times \mathbb{R} \to (0, +\infty)$ and define the process $\{\lambda_t \doteq \lambda(t, Y_t)\}_{t\in[0,T]}$ for all $t \in [0, T]$. Under the hypothesis that

$$\mathbb{E}\left[\int_0^T \lambda_u \, du\right] < \infty, \tag{2.4}$$

we denote by $\{N_t\}_{t\in[0,T]}$ the claims arrival process, which is a conditional Poisson process having $\{\lambda_t\}_{t\in[0,T]}$ as intensity. More precisely, we have that for all $0 \le s \le t \le T$ and $k = 0, 1, \ldots$

$$\mathbb{P}[N_t - N_s = k \mid \mathcal{F}_T^Y \vee \mathcal{F}_s] = \frac{\left(\int_s^t \lambda_u \, du\right)^k}{k!} e^{-\int_s^t \lambda_u \, du},$$

where $\{\mathcal{F}_t^Y\}_{t\in[0,T]}$ denotes the filtration generated by Y. Then it is easy to show that

$$N_t - \int_0^t \lambda_s \, ds$$

is an $\{\mathcal{F}_t\}$-martingale[1].

Now we define the cumulative claims up to time t as follows:

$$C_t = \sum_{i=1}^{N_t} Z_i \qquad t \in [0, T],$$

where the sequence of i.i.d. strictly positive $\mathcal{F}_0$-random variables $\{Z_i\}_{i=1,\ldots}$ represents the amount of the claims. In the sequel we will assume that all the $\{Z_i\}_{i=1,\ldots}$ are distributed like a r.v. Z, independent on $\{N_t\}_{t\in[0,T]}$ and $\{Y_t\}_{t\in[0,T]}$, with distribution function $F_Z(dz)$ such that $F_Z(z) = 1 \quad \forall z \ge D$, with $D > 0$ (eventually $D = +\infty$). Moreover, Z satisfies some suitable integrability conditions (see Assumption 2.2 below).
Consider the random measure associated with the marked point process $\{C_t\}_{t\in[0,T]}$ defined as follows

$$\begin{aligned} m(dt, dz) &= \sum_{\substack{t\in[0,T]: \\ \Delta C_t \neq 0}} \delta_{(t, \Delta C_t)}(dt, dz) \\ &= \sum_{n\ge1} \delta_{(T_n, Z_n)}(dt, dz) \mathbb{1}_{\{T_n \le T\}}, \end{aligned} \tag{2.5}$$

where $\{T_n\}_{n=1,\ldots}$ denotes the sequence of jump times of $\{N_t\}_{t\in[0,T]}$ and $\delta_{(t,z)}$ the Dirac measure at point $(t, z) \in \mathbb{R}^+ \times \mathbb{R}$. Then the process $\{C_t\}_{t\in[0,T]}$ can be written as

$$C_t = \int_0^t \int_0^D z m(ds, dz). \tag{2.6}$$

The following Lemma will be useful in the sequel.

Lemma 2.1. *The random measure $m(dt, dz)$ given in (2.5) has $\{\mathcal{F}_t\}$-dual predictable projection ν given by the following expression:*

$$\nu(dt, dz) = dF_Z(z)\lambda_t \, dt, \tag{2.7}$$

i.e. for every nonnegative, $\{\mathcal{F}_t\}$-predictable and $[0, D]$-indexed process $\{H(t, z)\}_{t\in[0,T]}$

$$\mathbb{E}\left[\int_0^T \int_0^D H(t, z) \, m(dt, dz)\right] = \mathbb{E}\left[\int_0^T \int_0^D H(t, z) \, dF_Z(z)\lambda_t \, dt\right].$$

[1]See e.g. [Brémaud, 1981, II]

Proof. See Appendix A. □

Remark 2.2. *Let us observe that for any $\{\mathcal{F}_t\}$-predictable and $[0,D]$-indexed process $\{H(t,z)\}_{t\in[0,T]}$ such that*

$$\mathbb{E}\left[\int_0^T\int_0^D |H(t,z)|\, dF_Z(z)\lambda_t\, dt\right] < \infty,$$

the process

$$M_t = \int_0^t\int_0^D H(s,z)\left(m(ds,dz) - dF_Z(z)\lambda_s\, ds\right) \qquad t\in[0,T]$$

turns out to be an $\{\mathcal{F}_t\}$-martingale. If in addition

$$\mathbb{E}\left[\int_0^T\int_0^D |H(t,z)|^2\, dF_Z(z)\lambda_t\, dt\right] < \infty,$$

then $\{M_t\}_{t\in[0,T]}$ is a square integrable $\{\mathcal{F}_t\}$-martingale and

$$\mathbb{E}[M_t^2] = \mathbb{E}\left[\int_0^t\int_0^D |H(t,z)|^2\, dF_Z(z)\lambda_t\, dt\right] \qquad \forall t\in[0,T].$$

Moreover, the predictable covariation process of $\{M_t\}_{t\in[0,T]}$ is given by

$$\langle M\rangle_t = \int_0^T\int_0^D |H(t,z)|^2\, dF_Z(z)\lambda_t\, dt,$$

that is $\{M_t^2 - \langle M\rangle_t\}_{t\in[0,T]}$ is an $\{\mathcal{F}_t\}$-martingale[2].

Remark 2.3. *Let $\{\mathcal{G}_t\}_{t\in[0,T]}$ be the filtration defined by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_T^Y$. Then $m(dt,dz)$ defined in (2.5) has $\{\mathcal{G}_t\}$-dual predictable projection ν given in (2.7). In order to show it, observe that since $\{\lambda_t\}_{t\in[0,T]}$ is $\{\mathcal{F}_t\}$-adapted by definition and $\mathcal{F}_t\subseteq\mathcal{G}_t$ $\forall t\in[0,T]$, then $\{\lambda_t\}_{t\in[0,T]}$ is also $\{\mathcal{G}_t\}$-adapted. Now notice that $\{\lambda_t\}_{t\in[0,T]}$ is the $\{\mathcal{G}_t\}$-intensity of $\{N_t\}_{t\in[0,T]}$, because for any $0\le s\le t\le T$*

$$\begin{aligned}\mathbb{E}[N_t \mid \mathcal{G}_s] &= N_s + \mathbb{E}[N_t - N_s \mid \mathcal{G}_s]\\ &= N_s + \sum_{k\geq 1} k\frac{\left(\int_s^t \lambda_u\, du\right)^k}{k!} e^{-\int_s^t \lambda_u\, du}\\ &= N_s + \int_s^t \lambda_u\, du\end{aligned}$$

and this implies that

$$\mathbb{E}[N_t - \int_0^t \lambda_u\, du \mid \mathcal{G}_s] = N_s - \int_0^s \lambda_u\, du.$$

Then our statement follows by the proof of Lemma 2.1 (see Appendix A) by replacing $\{\mathcal{F}_t\}$-predictable and $[0,D]$-indexed processes with $\{\mathcal{G}_t\}$-predictable and $[0,D]$-indexed processes.

In this framework we suppose that the gross risk premium rate is affected by the stochastic factor, i.e. we describe the insurance premium as a stochastic process $\{c_t \doteq c(t,Y_t)\}_{t\in[0,T]}$, where $c:[0,T]\times\mathbb{R}\to(0,+\infty)$ is a nonnegative measurable function. The insurance company can continuously purchase a proportional reinsurance contract, transferring at each time $t\in[0,T]$ a percentage u_t of its own risks to the reinsurer, who receives a reinsurance premium q_t given by the definition below.

Definition 2.1. *(Proportional reinsurance premium) Let us define a function $q(t,y,u):[0,T]\times\mathbb{R}\times[0,1]\to[0,+\infty)$, continuous w.r.t. the triple (t,y,u), having continuous partial derivatives $\frac{\partial q(t,y,u)}{\partial u}$, $\frac{\partial^2 q(t,y,u)}{\partial u^2}$ in $u\in[0,1]$ and such that*

[2] For these results and other related topics see e.g. [Brémaud, 1981, II] and [Bass, 2004].

1. $q(t,y,0)=0$ for all $(t,y)\in[0,T]\times\mathbb{R}$, because a null protection is not expensive;

2. $\frac{\partial q(t,y,u)}{\partial u}\geq 0$ for all $(t,y,u)\in[0,T]\times\mathbb{R}\times[0,1]$, since the premium is increasing with respect to the protection;

3. $q(t,y,1)>c(t,y)$ for all $(t,y)\in[0,T]\times\mathbb{R}$, because the cedant is not allowed to gain a profit without risk.

In the rest of the paper $\frac{\partial q(t,y,0)}{\partial u}$ and $\frac{\partial q(t,y,1)}{\partial u}$ should be intended as right and left derivatives, respectively. Moreover, we assume the following integrability condition:

$$\mathbb{E}\left[\int_0^T q(t,Y_t,u)\,dt\right]<\infty \quad \forall u\in[0,1]. \tag{2.8}$$

Then the reinsurance premium associated with a reinsurance strategy $\{u_t\}_{t\in[0,T]}$ (which is the protection level chosen by the insurer) is defined as $\{q_t \doteq q(t,Y_t,u_t)\}_{t\in[0,T]}$.

In addition, we will use the hypothesis that the insurance gross premium and the reinsurance premium will never diverge too much (being approximately influenced by the stochastic factor in the same way), that is there exists a positive constant K such that

$$|q(t,Y_t,u)-c(t,Y_t)|\leq K \quad \mathbb{P}\text{-a.s.} \quad \forall t\in[0,T], u\in[0,1]. \tag{2.9}$$

Under these hypotheses the surplus (or reserve) process associated with a given reinsurance strategy $\{u_t\}_{t\in[0,T]}$ is described by the following SDE:

$$\begin{aligned} dR_t^u &= \Big[c(t,Y_t)-q(t,Y_t,u_t)\Big]dt-(1-u_t)dC_t \\ &= \Big[c(t,Y_t)-q(t,Y_t,u_t)\Big]dt-\int_0^D(1-u_t)z\,m(dt,dz) \qquad R_0^u=R_0\in\mathbb{R}^+. \end{aligned} \tag{2.10}$$

Let us observe that by Remark 2.2, since

$$\mathbb{E}\left[\int_0^T\int_0^D u_r z\lambda_r\,dF_Z(z)\,dr\right]\leq \mathbb{E}[Z]\mathbb{E}\left[\int_0^T\lambda_r\,dr\right]<\infty,$$

the process $\int_0^t\int_0^D(1-u_s)z(m(ds,dz)-\lambda_s\,dF_Z(z)\,ds)$ turns out to be an $\{\mathcal{F}_t\}$-martingale.

Furthermore, we allow the insurer to invest its surplus in a financial market consisting of a risk-free bond $\{B_t\}_{t\in[0,T]}$ and a risky asset $\{P_t\}_{t\in[0,T]}$, whose dynamics are

$$dB_t=RB_t\,dt \qquad B_0=1, \tag{2.11}$$

with constant risk-less interest rate $R>0$, and

$$dP_t=P_t\Big[\mu(t,P_t)\,dt+\sigma(t,P_t)\,dW_t^{(P)}\Big] \qquad P_0>0, \tag{2.12}$$

respectively, where $\{W_t^{(P)}\}_{t\in[0,T]}$ is a standard Brownian motion independent of $\{W^{(Y)}\}_{t\in[0,T]}$ and the random measure $m(dt,dz)$. As a consequence, we assume that the financial and the insurance markets are independent. Even if a more general formulation may be achieved introducing the stochastic factor in the risky asset dynamic, the independence is very reasonable in many cases. For instance, whenever the insurance policies are subscribed in a country and the risky asset is traded in another (possibly distant) country, this is not so far from the reality.

Let us assume that there exists a unique strong solution to (2.12) such that

$$\mathbb{E}\left[\int_0^T|P_t\mu(t,P_t)|\,dt+\int_0^T P_t^2\sigma(t,P_t)^2\,dt\right]<\infty, \tag{2.13}$$

$$\sup_{t\in[0,T]}\mathbb{E}[P_t^2]<\infty, \tag{2.14}$$

(for example, it is true if the coefficients of the SDE (2.12) satisfy the classical Lipschitz and sub-linear growth conditions, see [Gihman and Skorohod, 1972]). Furthermore, we assume the Novikov condition:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |\frac{\mu(t,P_t)-R}{\sigma(t,P_t)}|^2\,dt}\right] < \infty, \tag{2.15}$$

which implies the existence of a risk-neutral measure for $\{P_t\}_{t\in[0,T]}$ and ensures that the financial market does not admit arbitrage.

We will denote by w_t the total amount invested in the risky asset at time $t \in [0,T]$, so that $X_t - w_t$ will be the capital invested in the risk-free asset (now X_t indicates the total wealth, but it will be defined more accurately below, see equation (2.17)). We also allow the insurer to short-sell and to borrow/lend any infinitesimal amount, so that $w_t \in \mathbb{R}$.
Finally, we only consider self-financing strategies: the insurer company only invests the surplus obtained with the core business, neither subtracting anything from the gains, nor adding something from another business.

The insurer's wealth $\{X_t^\alpha\}_{t\in[0,T]}$ associated with a given strategy $\alpha_t = (u_t, w_t)$ is described by the following SDE:

$$\begin{aligned} dX_t^\alpha &= dR_t^u + w_t\frac{dP_t}{P_t} + \big(X_t^\alpha - w_t\big)\frac{dB_t}{B_t} \\ &= \Big[c(t,Y_t) - q(t,Y_t,u_t)\Big]dt + w_t\Big[\mu(t,P_t)\,dt + \sigma(t,P_t)\,dW_t^{(P)}\Big] \\ &\quad + \big(X_t^\alpha - w_t\big)R\,dt - \int_0^D (1-u_t)z\,m(dt,dz), \end{aligned} \tag{2.16}$$

with $X_0^\alpha = R_0 \in \mathbb{R}^+$. Remember that $\{u_t\}_{t\in[0,T]}$ and $\{w_t\}_{t\in[0,T]}$ are the proportion of reinsured claims and the total amount invested in the risky asset $\{P_t\}_{t\in[0,T]}$, respectively.

Remark 2.4. *It can be verified that the solution to the SDE* (2.16) *is given by the following:*

$$\begin{aligned} X_t^\alpha = X_0^\alpha e^{Rt} &+ \int_0^t e^{R(t-r)}\big[c(r,Y_r) - q(r,Y_r,u_r)\big]\,dr + \int_0^t e^{R(t-r)}w_r[\mu(r,P_r) - R]\,dr \\ &+ \int_0^t e^{R(t-r)}w_r\sigma(r,P_r)\,dW_r^{(P)} - \int_0^t\int_0^D e^{R(t-r)}(1-u_r)z\,m(dr,dz). \end{aligned} \tag{2.17}$$

Now we are ready to formulate the optimization problem of an insurance company which subscribes a proportional reinsurance contract and invests its surplus in the financial market described above. Its main goal is to choose a strategy $\{\alpha_t = (u_t, w_t)\}_{t\in[0,T]}$ in order to maximize the expected utility of the terminal wealth:

$$\sup_{\alpha\in\mathcal{U}} \mathbb{E}\big[U(X_T^\alpha)\big],$$

where $\mathcal{U}$ denotes a suitable class of admissible controls (see Definition 2.2 below) and $U : \mathbb{R} \to [0,+\infty)$ is the utility function representing the insurer preferences. We focus on CARA (*Constant Absolute Risk Aversion*) utility functions, whose general expression is given by

$$U(x) = 1 - e^{-\eta x} \qquad x \in \mathbb{R},$$

where $\eta > 0$ is the risk-aversion parameter. This utility function is highly relevant in economic science and in particular in insurance theory, in fact it is commonly used for reinsurance problems (e.g. see [Bai and Guo, 2008], [Cao and Wan, 2009], [Sheng et al., 2014], and many others).
Using the dynamic programming principle we will consider a dynamic problem which consists in

finding the optimal strategy α_s, for $s \in [t,T]$, for the following optimization problem given the information available at the time $t \in [0,T]$:

$$\sup_{\alpha \in \mathcal{U}_t} \mathbb{E}\left[U(X^{\alpha}_{t,x}(T)) \mid \mathcal{F}_t \right] \qquad t \in [0,T],$$

where $\mathcal{U}_t$ denotes the class of admissible controls in the time interval $[t,T]$ (see Definition 2.2 below). Here $\{X^{\alpha}_{t,x}(s)\}_{s\in[t,T]}$ denotes the solution to equation (2.16) with initial condition $X^{\alpha}_t = x$.

For the sake of simplicity, we will reduce ourselves to study the function $-e^{-\eta x}$. Another possible choice is to focus on the corresponding minimizing problem for the function $e^{-\eta x}$, but the first choice is usually preferred in the literature.

Definition 2.2. *We will denote by $\mathcal{U}$ the set of all admissible strategies, which are all the $\{\mathcal{F}_t\}$-predictable processes $\alpha_t = (u_t, w_t)$, $t \in [0,T]$, with values in $[0,1] \times \mathbb{R}$, such that*

$$\mathbb{E}\left[\int_0^T |w_r||\mu(r,P_r) - R| \, dr \right] < \infty, \quad \mathbb{E}\left[\int_0^T w_r^2 \sigma(r,P_r)^2 \, dr \right] < \infty.$$

When we want to restrict the controls to the time interval $[t,T]$, we will use the notation $\mathcal{U}_t$.

From now on the following assumptions are fulfilled.

Assumption 2.2.

$$\mathbb{E}[e^{\eta Z e^{RT}}] < \infty, \qquad \mathbb{E}[Z e^{\eta Z e^{RT}}] < \infty, \qquad \mathbb{E}[Z^2 e^{\eta Z e^{RT}}] < \infty, \tag{2.18}$$

$$\mathbb{E}\left[e^{(\mathbb{E}[e^{\eta e^{RT} Z}] - 1) \int_t^T \lambda_s \, ds} \mid \mathcal{F}_t \right] < \infty, \qquad \langle \mathbb{P} = 1 \rangle \quad \forall t \in [0,T]. \tag{2.19}$$

Proposition 2.1. *Under the Assumption 2.2 the control $(0,0)$ is admissible and such that*

$$\mathbb{E}[e^{-\eta X^{(0,0)}_{t,x}(T)} \mid \mathcal{F}_t] < \infty \qquad \langle \mathbb{P} = 1 \rangle \qquad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Proof. See Appendix A. $\square$

Remark 2.5. *Let us observe that Proposition 2.1 implies that*

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{U}_t} \mathbb{E}\left[U(X^{\alpha}_{t,x}(T)) \mid \mathcal{F}_t \right] > -\infty \qquad \langle \mathbb{P} = 1 \rangle \qquad \forall t \in [0,T]$$

and as a consequence that

$$\sup_{\alpha \in \mathcal{U}} \mathbb{E}\big[U(X^{\alpha}_T) \big] > -\infty.$$

In order to solve this dynamic problem we introduce the *value function* associated with it:

$$v(t,x,y,p) = \sup_{\alpha \in \mathcal{U}_t} \mathbb{E}\left[-e^{-\eta X^{\alpha}_{t,x}(T)} \mid Y_t = y, P_t = p \right], \tag{2.20}$$

where the function $v : V \to \mathbb{R}$ is defined in the domain

$$V \doteq [0,T] \times \mathbb{R}^2 \times (0,+\infty).$$

3. Hamilton-Jacobi-Bellman equation

Let us consider the Hamilton-Jacobi-Bellman equation that the value function is expected to solve if sufficiently regular:

$$\begin{cases} \sup_{(u,w)\in[0,1]\times\mathbb{R}} \mathcal{L}^{\alpha} v(t,x,y,p) = 0 & \forall (t,x,y,p) \in [0,T) \times \mathbb{R}^2 \times (0,+\infty) \\ v(T,x,y,p) = -e^{-\eta x} & \forall (x,y,p) \in \mathbb{R}^2 \times (0,+\infty), \end{cases} \tag{3.1}$$

where $\mathcal{L}^\alpha$ denotes the Markov generator of the triplet (X_t^α, Y_t, P_t) associated with a constant control $\alpha = (u, w)$. In what follows, we denote by $\mathcal{C}_b^{1,2}$ all bounded functions $f(t, x_1, \dots, x_n)$, with $n \geq 1$, with bounded first order derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ and bounded second order derivatives w.r.t. the spatial variables $\frac{\partial^2 f}{\partial x_1^2}, \dots, \frac{\partial f}{\partial x_n^2}$.

Lemma 3.1. *Let $f : V \to \mathbb{R}$ be a function in $\mathcal{C}_b^{1,2}$. Then the Markov generator of the stochastic process (X_t^α, Y_t, P_t) for all constant strategies $\alpha = (u, w) \in [0,1] \times \mathbb{R}$ is given by the following expression:*

$$\begin{aligned}
\mathcal{L}^\alpha f(t,x,y,p) &= \frac{\partial f}{\partial t}(t,x,y,p) + \frac{\partial f}{\partial x}(t,x,y,p)\big[Rx + c(t,y) - q(t,y,u) + w(\mu(t,p) - R)\big] \\
&\quad + \frac{1}{2} w^2 \sigma(t,p)^2 \frac{\partial^2 f}{\partial x^2}(t,x,y,p) + b(t,y)\frac{\partial f}{\partial y}(t,x,y,p) + \frac{1}{2}\gamma(t,y)^2 \frac{\partial^2 f}{\partial y^2}(t,x,y,p) \\
&\quad + p\mu(t,p)\frac{\partial f}{\partial p}(t,x,y,p) + \frac{1}{2}p^2\sigma(t,p)^2\frac{\partial^2 f}{\partial p^2}(t,x,y,p) + w\sigma(t,p)^2 p \frac{\partial^2 f}{\partial x \partial p}(t,x,y,p) \\
&\quad + \int_0^D \Big[f(t, x - (1-u)z, y, p) - f(t,x,y,p)\Big]\lambda(t,y)\, dF_Z(z). \qquad (3.2)
\end{aligned}$$

Proof. See Appendix A. □

Remark 3.1. *Since the triplet (X_t^α, Y_t, P_t) is a Markov process, any Markovian control is of the form $\alpha_t = \alpha(t, X_t^\alpha, Y_t, P_t)$, where α denotes a suitable function such that $\alpha(t,x,y,p) = (u(t,x,y,p), w(t,x,y,p))$. The generator $\mathcal{L}^\alpha f(t,x,y,p)$ associated to a general Markovian strategy can be easily obtained by replacing α with α_t in* (3.2)*.*

Now let us introduce the following ansatz:

$$v(t,x,y,p) = -e^{-\eta x e^{R(T-t)}} \varphi(t,y,p),$$

where φ does not depend on x and it is a positive function[3]. Then the original HJB problem given in (3.1) reduces to the simpler one given by

$$\begin{aligned}
&-\frac{\partial \varphi}{\partial t}(t,y,p) - b(t,y)\frac{\partial \varphi}{\partial y}(t,y,p) - \frac{1}{2}\gamma(t,y)^2\frac{\partial^2 \varphi}{\partial y^2}(t,y,p) + \eta e^{R(T-t)} c(t,y)\varphi(t,y,p) \\
&\qquad - p\mu(t,p)\frac{\partial \varphi}{\partial p}(t,y,p) - \frac{1}{2}\sigma(t,p)^2 p^2 \frac{\partial^2 \varphi}{\partial p^2}(t,y,p) \\
&\qquad + \sup_{u \in [0,1]} \Psi^u(t,y)\varphi(t,y,p) + \sup_{w \in \mathbb{R}} \Psi^w(t,y,p) = 0, \qquad (3.3)
\end{aligned}$$

with final condition $\varphi(T,y,p) = 1$ for all $(y,p) \in \mathbb{R} \times (0, +\infty)$, defining

$$\Psi^u(t,y) \doteq -\eta e^{R(T-t)} q(t,y,u) + \lambda(t,y) \int_0^D \left[1 - e^{\eta(1-u)z e^{R(T-t)}}\right] dF_Z(z) \qquad (3.4)$$

and

$$\begin{aligned}
\Psi^w(t,y,p) &\doteq \eta e^{R(T-t)} \left((\mu(t,p) - R)\varphi(t,y,p) + p\sigma(t,p)^2 \frac{\partial \varphi}{\partial p}(t,y,p) \right) w \\
&\quad - \frac{1}{2}\sigma(t,p)^2 \eta^2 e^{2R(T-t)} \varphi(t,y,p) w^2. \qquad (3.5)
\end{aligned}$$

It should make it clear that we can split the optimal control research in two distinct problems: the optimization of Ψ^u will give us the optimal level of reinsurance (see Section 4), while working with Ψ^w we will find the optimal investment policy (see Section 5).

[3] Intuitively, we note that $X_{t,x}^\alpha(T) = X_{t,0}^\alpha(T) + xe^{R(T-t)}$ and we use the exponential form of the function v.

4. Optimal reinsurance strategy

In this section we discuss the problem

$$\sup_{u\in[0,1]} \Psi^u(t,y), \quad (t,y) \in [0,T] \times \mathbb{R}, \tag{4.1}$$

with $\Psi^u(t,y)$ given in (3.4).

First, let us observe that $\Psi^u(t,y)$ is continuous w.r.t. $u \in [0,1]$, for any $(t,y) \in [0,T] \times \mathbb{R}$ and admits continuous first and the second order derivatives w.r.t. $u \in [0,1]$

$$\begin{aligned}
\frac{\partial \Psi^u(t,y)}{\partial u} &= -\eta e^{R(T-t)} \left[\frac{\partial q(t,y,u)}{\partial u} - \lambda(t,y) \int_0^D z e^{\eta(1-u)ze^{R(T-t)}} \, dF_Z(z) \right], \\
\frac{\partial^2 \Psi^u(t,y)}{\partial u^2} &= -\eta e^{R(T-t)} \left[\frac{\partial^2 q(t,y,u)}{\partial u^2} + \eta e^{R(T-t)} \lambda(t,y) \int_0^D z^2 e^{\eta(1-u)ze^{R(T-t)}} \, dF_Z(z) \right].
\end{aligned}$$

Notice that these derivatives are well defined thanks to (2.18).

Now we are ready for the main result of this section.

Proposition 4.1. *Let us suppose that $\Psi^u(t,y)$ given in (3.4) is strictly concave in $u \in [0,1]$ $\forall (t,y) \in [0,T] \times \mathbb{R}$. Then there exists a unique measurable function $u^*(t,y)$ for all $(t,y) \in [0,T] \times \mathbb{R}$ solution to (4.1). Moreover, it is given by*

$$u^*(t,y) = \begin{cases} 0 & (t,y) \in A_0 \\ \hat{u}(t,y) & (t,y) \in (A_0 \cup A_1)^C \\ 1 & (t,y) \in A_1, \end{cases} \tag{4.2}$$

where

$$\begin{aligned}
A_0 &\doteq \left\{ (t,y) \in [0,T] \times \mathbb{R} \mid \lambda(t,y) \mathbb{E}[Z e^{\eta Z e^{R(T-t)}}] \leq \frac{\partial q(t,y,0)}{\partial u} \right\}, \\
A_1 &\doteq \left\{ (t,y) \in [0,T] \times \mathbb{R} \mid \frac{\partial q(t,y,1)}{\partial u} \leq \mathbb{E}[Z] \lambda(t,y) \right\},
\end{aligned}$$

and $\hat{u}(t,y)$ is the unique solution to the following equation:

$$\frac{\partial q(t,y,u)}{\partial u} = \lambda(t,y) \int_0^D z e^{\eta(1-u)ze^{R(T-t)}} \, dF_Z(z). \tag{4.3}$$

Proof. Since $\Psi^u(t,y)$ is continuous and strictly concave in $u \in [0,1]$, there exists a unique maximizer $u^*(t,y)$ of (4.1), whose measurability follows by classical selection theorems.
Now observe that $A_0 \cap A_1 = \emptyset$. In fact, we have that

$$\begin{aligned}
A_0 &= \left\{ (t,y) \in [0,T] \times \mathbb{R} \mid \frac{\partial \Psi^0(t,y)}{\partial u} \leq 0 \right\}, \\
A_1 &= \left\{ (t,y) \in [0,T] \times \mathbb{R} \mid \frac{\partial \Psi^1(t,y)}{\partial u} \geq 0 \right\}.
\end{aligned}$$

Let us assume for sake of contradiction that $(t,y) \in A_0 \cap A_1$. Since $\frac{\partial \Psi^u(t,y)}{\partial u}$ is decreasing in $u \in [0,1]$, we have that

$$\frac{\partial \Psi^0(t,y)}{\partial u} = \frac{\partial \Psi^u(t,y)}{\partial u} = \frac{\partial \Psi^1(t,y)}{\partial u} = 0, \forall u \in (0,1),$$

hence $\Psi^u(t,y)$ is constant $\forall u \in [0,1]$, but this is absurd, because $\Psi^u(t,y)$ is strictly concave.

As a consequence, $A_0 \dot\cup (A_0 \cup A_1)^C \dot\cup A_1 = [0,T] \times \mathbb{R}$.

Now we have three cases. If $(t,y) \in A_0$ then $\Psi^u(t,y)$ is decreasing in $u \in [0,1]$, hence no reinsurance is chosen, i.e. $u^*(t,y) = 0$.

If $(t,y) \in \hat{A}$ then there exists a unique $u^*(t,y) \in (0,1)$ such that $\frac{\partial \Psi^u(t,y)}{\partial u} = 0$, and it is the unique solution to equation (4.3).

Finally, if $(t,y) \in A_1$ then $\Psi^u(t,y)$ is increasing in $u \in [0,1]$, hence $u^*(t,y) = 1$.

$\square$

Remark 4.1. *In our model $\lambda(t,y) > 0$. Nevertheless, for the sake of completeness we observe that if $\lambda(t,y)$ had been vanished for some (t,y), then we would have obtained*

$$\{\ (t,y) \in [0,T] \times \mathbb{R} \mid \lambda(t,y) = 0\ \} \subseteq A_0,$$

i.e. $u^*(t,y) = 0$. In fact, $\lambda(t,y) = 0$ corresponds to a degenerate situation: the risk premia are paid, but there is no "real" risk to be insured.

From the economic point of view, we could say that if the reinsurance is not too much expensive (more precisely, if the price of an infinitesimal protection is below a certain dynamic threshold) and if full reinsurance is not optimal, then the optimal strategy is provided by (4.3), i.e. by equating the marginal cost and the marginal gain.

Now we provide some sufficient conditions in order to guarantee existence and uniqueness of a solution to (4.1).

Lemma 4.1. *Suppose that the following condition holds true:*

$$\frac{\partial^2 q(t,y,u)}{\partial u^2} \geq 0 \qquad \forall (t,y,u) \in [0,T] \times \mathbb{R} \times [0,1].$$

Then $\Psi^u(t,y)$ given in (3.4) is strictly concave in $u \in [0,1]$ $\forall (t,y) \in [0,T] \times \mathbb{R}$. As a consequence, there exists a unique solution to (4.1).

Proof. Since $\eta > 0$ and $\lambda(t,y) > 0$ $\forall (t,y) \in [0,T] \times \mathbb{R}$, we have that

$$\frac{\partial^2 \Psi^u(t,y)}{\partial u^2} = -\eta e^{R(T-t)} \left[\frac{\partial^2 q(t,y,u)}{\partial u^2} + \eta e^{R(T-t)} \lambda(t,y) \int_0^D z^2 e^{\eta(1-u) z e^{R(T-t)}}\, dF_Z(z) \right] < 0,$$

which implies that $\Psi^u(t,y)$ is strictly concave in $u \in [0,1]$. Now we use Proposition 4.1. $\square$

Remark 4.2. *Under the hypotheses that $\frac{\partial^2 q(t,y,u)}{\partial u^2} \geq 0$ and $c(t,y) > \mathbb{E}[Z]\lambda(t,y)$ for all $(t,y,u) \in [0,T] \times \mathbb{R} \times (0,1)$, the full reinsurance is never optimal. In fact, for any arbitrary couple (t,y) we have that*

$$q(t,y,1) = q(t,y,0) + \int_0^1 \frac{\partial q(t,y,u)}{\partial u}\, du.$$

Being $q(t,y,0) = 0$ and $q(t,y,1) > c(t,y) > \mathbb{E}[Z]\lambda(t,y)$ (because the reinsurance is not cheap and using the net-profit condition for the insurance premium), we obtain that

$$\int_0^1 \frac{\partial q(t,y,u)}{\partial u}\, du > \mathbb{E}[Z]\lambda(t,y).$$

Since $\frac{\partial q(t,y,u)}{\partial u}$ is continuous in $u \in [0,1]$ by hypothesis, from the mean value theorem we know that there exists $u_0 \in (0,1)$ such that

$$\frac{\partial q(t,y,u_0)}{\partial u} > \mathbb{E}[Z]\lambda(t,y).$$

Under the hypothesis that $\frac{\partial^2 q(t,y,u)}{\partial u^2} \geq 0$ for all $u \in (0,1)$, $\frac{\partial q(t,y,u)}{\partial u}$ is an increasing function of u, and this implies that

$$\frac{\partial q(t,y,1)}{\partial u} \geq \frac{\partial q(t,y,u_0)}{\partial u} > \mathbb{E}[Z]\lambda(t,y).$$

From this result we deduce that

$$\frac{\partial \Psi^1(t,y)}{\partial u} = -\eta e^{R(T-t)} \left[\frac{\partial q(t,y,1)}{\partial u} - \mathbb{E}[Z]\lambda(t,y) \right] < 0, \quad (t,y) \in [0,T] \times \mathbb{R},$$

which implies that $A_1 = \emptyset$, i.e. the full reinsurance is never optimal.

Let us observe that the preceding remark requires two special conditions. The first one concerns the concavity of the reinsurance premium and in Subsection 4.1 we will show that it is fulfilled by the most famous premium calculation principles. The second hypothesis is the so called *net-profit condition* (e.g. see [Grandell, 1991]) and it is usually assumed in insurance risk models to ensure that the expected gross risk premium covers the expected losses.

In the following remark we point out a consequence of the preceding result in order to better appreciate the generality of Proposition 4.1.

Remark 4.3. *In the actuarial literature, full reinsurance is mostly considered never optimal. The main reason is that most authors use premium calculation principles. In consequence, the reinsurance premium turns out to be convex with respect to the protection level. By Remark 4.2, this property leads to neglect the full reinsurance case. Our result in Proposition 4.1 (see the third case in (4.2)) allows the insurer to choose full reinsurance as long as $(t,y) \in A_1$. From a technical point of view, this result follows from the generality of Definition 2.1. From the economic point of view, it is reasonable that the insurance firm could regard full reinsurance as convenient for a limited period of time and in some particular scenarios, because the objective is to maximize the expected utility of the wealth at the end of the period.*

Now we investigate how Proposition 4.1 applies to a special case.

Example 4.1. *(Exponentially distributed claims)*
Let Z to be an exponential r.v. with parameter $\zeta > 0$, then for any fixed $(t,y) \in [0,T] \times \mathbb{R}$ equation (4.3) becomes

$$\lambda(t,y) \int_0^\infty z e^{\eta(1-u) z e^{R(T-t)}} \zeta e^{-\zeta z}\, dz = \frac{\partial q(t,y,u)}{\partial u}.$$

Taking $k = \eta(1-u)e^{R(T-t)} - \zeta$ it can be written as

$$\lambda(t,y) \int_0^\infty z e^{kz} \zeta\, dz = \frac{\partial q(t,y,u)}{\partial u}$$

and requiring that

$$\frac{\zeta}{\eta} > e^{RT}, \tag{4.4}$$

which implies $k < 0$, finally equation (4.3) reads as

$$\lambda(t,y) \frac{\zeta}{(\eta(1-u)e^{R(T-t)} - \zeta)^2} = \frac{\partial q(t,y,u)}{\partial u}. \tag{4.5}$$

Summarizing, if Z is an exponential r.v. with parameter $\zeta > \eta e^{RT}$, if $\Psi^u(t,y)$ given in (3.4) is strictly concave, then we have that expression (4.2) holds true with

$$A_0 \doteq \left\{ (t,y) \in [0,T] \times \mathbb{R} \mid \lambda(t,y) \frac{\zeta}{(\eta e^{R(T-t)} - \zeta)^2} \leq \frac{\partial q(t,y,0)}{\partial u} \right\},$$
$$A_1 \doteq \left\{ (t,y) \in [0,T] \times \mathbb{R} \mid \frac{\partial q(t,y,1)}{\partial u} \leq \frac{\lambda(t,y)}{\zeta} \right\},$$

and with $\hat{u}(t,y)$ being the unique solution to equation (4.5).

4.1. Expected value and variance premium principles

Proposition 4.1 clarifies that the optimal reinsurance strategy crucially depends on the reinsurance premium. In this subsection we specialize that result using two of the most famous premium calculation principles: the *expected value principle* and the *variance premium principle*. We will show that in both cases we loose the dependence of the optimal reinsurance strategy on the stochastic factor. Moreover, the optimal reinsurance strategy does not explicitly depend on the claims intensity. These will be our motivations for introducing the *intensity-adjusted variance premium principle* in Subsection 4.2.

Lemma 4.2. *Under the expected value principle, i.e. if the reinsurance premium admits the following expression*

$$q(t,y,u) = (1+\theta_r)\mathbb{E}[Z]\lambda(t,y)u \qquad \forall (t,y,u) \in [0,T]\times\mathbb{R}\times[0,1] \tag{4.6}$$

for some constant $\theta_r > 0$ (which is called the reinsurance safety loading), there exists a unique maximizer $u^*(t)$ for all $(t,y) \in [0,T]\times\mathbb{R}$ for the problem (4.1)*. In particular,*

$$u^*(t) = \begin{cases} 0 & t \in A_0 \\ \hat{u}(t) & t \in [0,T]\setminus A_0, \end{cases} \tag{4.7}$$

where

$$A_0 \doteq \Big\{ t\in[0,T] \mid \mathbb{E}[Ze^{\eta Z e^{R(T-t)}}] \le (1+\theta_r)\mathbb{E}[Z] \Big\}$$

and $\hat{u}(t)$ is the unique solution to the following equation:

$$(1+\theta_r)\mathbb{E}[Z] = \int_0^D z e^{\eta(1-u)ze^{R(T-t)}}\, dF_Z(z). \tag{4.8}$$

Proof. From (4.6) we get

$$\frac{\partial q(t,y,u)}{\partial u} = (1+\theta_r)\mathbb{E}[Z]\lambda(t,y), \frac{\partial^2 q(t,y,u)}{\partial u^2} = 0 \qquad \forall u\in(0,1),$$

which implies that $\Psi^u(t,y)$ is strictly concave in $u\in(0,1)$ thanks to Lemma 4.1. Moreover, by the means of Remark 4.2 we know that the full reinsurance is always sub-optimal, in fact the set A_1 in Proposition 4.1 is empty. Now we only have to apply Proposition 4.1. □

Note that we always have $\mathbb{E}[Ze^{\eta Z e^{R(T-t)}}] > \mathbb{E}[Z]$ for each $t\in[0,T]$, thus A_0 could be an empty set when the reinsurer's safety loading is close to 0.

Example 4.2. *(Exponentially distributed claims under the expected value principle)*
Let us come back to example 4.1. Under the expected value principle (4.6) *the result for exponential claims is even more simplified, in fact we find the following explicit solution:*

$$u^*(t) = \begin{cases} 1 - \frac{\zeta}{\eta}\Big(1-\frac{1}{\sqrt{1+\theta_r}}\Big)e^{-R(T-t)} & t\in[0,t_0\wedge T) \\ 0 & t\in[t_0\wedge T, T], \end{cases} \tag{4.9}$$

where

$$t_0 = T - \frac{1}{R}\log\Big[\frac{\zeta}{\eta}\Big(1-\frac{1}{\sqrt{1+\theta_r}}\Big)\Big]. \tag{4.10}$$

The expression for t_0 can be derived from the characterization of the set $[0,T]\times\mathbb{R}\setminus A_0$, which in this case reads as follows:

$$\frac{\zeta - \sqrt{\frac{\zeta}{(1+\theta_r)\mathbb{E}[Z]}}}{\eta} < e^{R(T-t)} < \frac{\zeta + \sqrt{\frac{\zeta}{(1+\theta_r)\mathbb{E}[Z]}}}{\eta},$$

where the second inequality is always fulfilled in view of (4.4)*, hence we get t_0 only from the first inequality.*

Lemma 4.3. *Under the variance premium principle, i.e. if the reinsurance premium admits the following expression*

$$q(t,y,u) = \mathbb{E}[Z]\lambda(t,y)u + \theta_r \mathbb{E}[Z^2]\lambda(t,y)u^2 \tag{4.11}$$

for some constant reinsurance safety loading $\theta_r > 0$, the optimization problem (4.1) *admits a unique maximizer $u^*(t) \in (0,1)$ for all $(t,y) \in [0,T] \times \mathbb{R}$, which is the solution to the following equation:*

$$2\theta_r \mathbb{E}[Z^2]u = \int_0^D z e^{\eta(1-u)ze^{R(T-t)}}\, dF_Z(z) - \mathbb{E}[Z]. \tag{4.12}$$

Proof. Using the expression (4.11) we get that

$$\frac{\partial q(t,y,u)}{\partial u} = \mathbb{E}[Z]\lambda(t,y) + 2\theta_r \mathbb{E}[Z^2]\lambda(t,y)u \qquad \forall u \in (0,1)$$

and

$$\frac{\partial^2 q(t,y,u)}{\partial u^2} = 2\theta_r \mathbb{E}[Z^2]\lambda(t,y) > 0 \qquad \forall u \in (0,1).$$

By Lemma 4.1 $\Psi^u(t,y)$ is strictly concave w.r.t. u and the full reinsurance is never optimal because of Remark 4.2. Moreover, in order to apply Proposition 4.1 we notice that

$$\mathbb{E}[Ze^{\eta Z e^{R(T-t)}}] > \mathbb{E}[Z] \Rightarrow A_0 = \emptyset,$$

thus the optimal strategy is unique and it belongs to $(0,1)$. In order to find such a solution, we turn the attention to the first order condition, which is exactly the equation (4.12). □

The same result was obtained in [Liang and Bayraktar, 2014], Lemma 3.1.

Example 4.3. *(Exponentially distributed claims under the variance premium principle) Under the variance premium principle* (4.11)*, suppose that the claims are exponentially distributed with parameter $\zeta > \eta e^{RT}$. Then it is easy to show that the optimal strategy is given by*

$$u^*(t) = 1 - \frac{\zeta}{\eta}\left(1 - \sqrt{\frac{\zeta}{\zeta + 4\theta_r}}\right)e^{-R(T-t)} \qquad t \in [0,T]. \tag{4.13}$$

4.2. Intensity-adjusted variance premium principle

We have shown that both the *expected value principle* (see Lemma 4.2) and the *variance premium principle* (see Lemma 4.3) lead to deterministic optimal reinsurance strategies, which do not depend on the stochastic factor. Since the main objective of our paper is to analyze the maximization problem under a stochastic factor model, we would like to keep that dependence. In addition, in both cases the optimal reinsurance strategy does not explicitly depend on the claims intensity. As a consequence, there is a paradox that we clarify with the following example. Let us consider two identical insurers (i.e. with the same risk-aversion, time horizon, and so on) who work in the same insurance business line, for example in automobile insurance, but in two distinct territories with different riskiness. More precisely, let us assume that the two companies insure claims which have the same distribution F_Z but occur with different probabilities. Hence it is a reasonable assumption that the claims arrival processes have two different intensities. Now let us suppose that both the insurers use Lemma 4.2 (or Lemma 4.3) in order to solve the maximization problem (4.1). Then they will obtain the same reinsurance strategy, but this is not what we expect. Hence the optimal reinsurance strategy should explicitly depend on the claims intensity.
In order to fix these two problems, in this subsection we introduce a new premium calculation principle, which will be referred as the *intensity-adjusted variance premium principle.*

Let us first formalize that there exists a special class of premium calculation principles that lead us to deterministic strategies which do not depend on the claims intensity.

Remark 4.4. *For any reinsurance premium $\{q_t\}_{t\in[0,T]}$ admitting the following representation*

$$q(t,y,u) = \lambda(t,y)Q(t,u) \tag{4.14}$$

for a suitable[4] *function $Q : [0,T]\times[0,1] \to [0,+\infty)$, the optimal reinsurance strategy $u_t^* = u^*(t,Y_t)$ given in Proposition 4.1 turns out to be deterministic. Moreover, it does not explicitly depend on the claims intensity. For example, the expected value principle and the variance premium principle admit the factorization* (4.14) *with $Q(t,u) = (1+\theta_r)\mathbb{E}[Z]u$ and $Q(t,u) = \mathbb{E}[Z]u + \theta_r\mathbb{E}[Z^2]u^2$, respectively.*

Now the basic idea is to find a reinsurance premium $\{q_t\}_{t\in[0,T]}$ (see Definition 2.1) such that

$$\mathbb{E}\left[\int_0^t q(s,Y_s,u_s)\,ds\right] = \mathbb{E}\left[\int_0^t u_s\,dC_s\right] + \theta_r \operatorname{var}\left[\int_0^t u_s\,dC_s\right] \qquad \forall t\in[0,T] \tag{4.15}$$

for a given reinsurance safety loading θ_r in order to dynamically satisfy the original formulation of the *variance premium principle*[5]. For this purpose, we give the following result.

Lemma 4.4. *For any $\{\mathcal{F}_t^Y\}_{t\in[0,T]}$-predictable reinsurance strategy $\{u_t\}_{t\in[0,T]}$ we have that for any $t\in[0,T]$*

$$\operatorname{var}\left[\int_0^t u_s\,dC_s\right] = \mathbb{E}[Z^2]\mathbb{E}\left[\int_0^t u_s^2\lambda_s\,ds\right] + \mathbb{E}[Z]^2\operatorname{var}\left[\int_0^t u_s\lambda_s\,ds\right]. \tag{4.16}$$

Proof. Let us denote by $\{M_t^u\}_{t\in[0,T]}$ the following $\{\mathcal{F}_t\}$-martingale:

$$M_t^u = \int_0^D\int_0^t u_s z\big(m(ds,dz) - dF_Z(z)\lambda_s\,ds\big).$$

Recalling that $\{C_t\}_{t\in[0,T]}$ is defined in (2.6), the variance of the reinsurer's cumulative losses at the time $t\in[0,T]$ is given by

$$\begin{aligned}\operatorname{var}\left[\int_0^t u_s\,dC_s\right] &= \mathbb{E}\left[\left(\int_0^t u_s\,dC_s\right)^2\right] - \mathbb{E}\left[\int_0^t u_s\,dC_s\right]^2\\ &= \mathbb{E}\left[|M_t^u|^2 + \left(\int_0^t u_s\lambda_s\mathbb{E}[Z]\,ds\right)^2 + 2M_t^u\int_0^t u_s\lambda_s\mathbb{E}[Z]\,ds\right] - \mathbb{E}\left[\int_0^t u_s\,dC_s\right]^2.\end{aligned}$$

Denoting with $\langle M^u\rangle_t$ the predictable covariance process of M_t^u, using Remark 2.2, we find that

$$\begin{aligned}\operatorname{var}\left[\int_0^t u_s\,dC_s\right] &= \mathbb{E}[\langle M^u\rangle_t] + \mathbb{E}[Z]^2\mathbb{E}\left[\left(\int_0^t u_s\lambda_s\,ds\right)^2\right] - \mathbb{E}[Z]^2\mathbb{E}\left[\int_0^t u_s\lambda_s\,ds\right]^2\\ &= \mathbb{E}[Z^2]\mathbb{E}\left[\int_0^t u_s^2\lambda_s\,ds\right] + \mathbb{E}[Z]^2\operatorname{var}\left[\int_0^t u_s\lambda_s\,ds\right] \qquad \forall t\in[0,T].\end{aligned} \tag{4.17}$$

Here we have used that $\mathbb{E}\left[M_t^u\int_0^t u_s\lambda_s\mathbb{E}[Z]\,ds\right] = 0$. In fact we notice that

$$\begin{aligned}\mathbb{E}\left[M_t^u\int_0^t u_s\lambda_s\mathbb{E}[Z]\,ds\right] &= \mathbb{E}\left[\mathbb{E}\left[M_t^u\int_0^t u_s\lambda_s\mathbb{E}[Z]\,ds \mid \mathcal{F}_T^Y\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\big[M_t^u\mid\mathcal{F}_T^Y\big]\int_0^t u_s\lambda_s\mathbb{E}[Z]\,ds\right]\end{aligned}$$

and being $\mathcal{G}_0 = \mathcal{F}_0\vee\mathcal{F}_T^Y \supseteq \mathcal{F}_T^Y$ (see Remark 2.3) we have that

$$\mathbb{E}\big[M_t^u\mid\mathcal{F}_T^Y\big] = \mathbb{E}\big[\mathbb{E}\big[M_t^u\mid\mathcal{G}_0\big]\mathcal{F}_T^Y\big] = \mathbb{E}\big[M_0^u\mid\mathcal{F}_T^Y\big] = 0$$

and the proof is complete. □

[4] I.e. Q is such that q fulfills the Definition 2.1.

[5] See e.g. [Young, 2006].

Remark 4.5. *We highlight that Lemma 4.4 applies to $\{\mathcal{F}_t^Y\}_{t\in[0,T]}$-predictable reinsurance strategies, but this is not restrictive. In fact, from Proposition 4.1 we know that the optimal strategy belongs to the class of $\{\mathcal{F}_t^Y\}_{t\in[0,T]}$-predictable processes.*

Remark 4.6. *In the classical Cramér-Lundberg model, i.e. $\lambda(t,y) = \lambda$, for any deterministic strategy $u_t = u(t)$*

$$\mathrm{var}\left[\int_0^t u_s \lambda \, ds\right] = 0,$$

thus in this case we choose expression (4.11) *and the equation* (4.15) *is satisfied.*

Under any risk model with stochastic intensity the formula (4.11) neglects the term

$$\mathbb{E}[Z]^2 \, \mathrm{var}\left[\int_0^t u_s \lambda_s \, ds\right]$$

in the equation (4.16). In order to capture the effect of this term, we can find the following estimate:

$$\begin{aligned}\mathrm{var}\left[\int_0^t u_s \lambda_s \, ds\right] &\le \mathbb{E}\left[\left(\int_0^t u_s \lambda_s \, ds\right)^2\right] \\ &\le \mathbb{E}\left[T \int_0^t u_s^2 \lambda_s^2 \, ds\right].\end{aligned}$$

As a consequence, we can choose as premium calculation rule

$$q(t,y,u) = \mathbb{E}[Z]\lambda(t,y)u + \theta_r \mathbb{E}[Z^2]\big[\lambda(t,y) + T\lambda(t,y)^2\big]u^2, \tag{4.18}$$

which will be called *intensity-adjusted variance principle* in this work; using this formula, we ensure that

$$\mathbb{E}\left[\int_0^t q(s, Y_s, u_s)\, ds\right] \ge \mathbb{E}\left[\int_0^t u_s \, dC_s\right] + \theta_r \, \mathrm{var}\left[\int_0^t u_s \, dC_s\right] \qquad \forall t \in [0,T]$$

for all $\{\mathcal{F}_t^Y\}_{t\in[0,T]}$-predictable reinsurance strategies and for any arbitrary level of reinsurance safety loading $\theta_r > 0$.

Lemma 4.5. *Under the intensity-adjusted variance premium principle* (4.18)*, the optimization problem* (4.1) *admits a unique maximizer $u^*(t,y) \in (0,1)$ for all $(t,y) \in [0,T]\times\mathbb{R}$, which is the solution to the following equation:*

$$2\theta_r \mathbb{E}[Z^2]\big[1 + T\lambda(t,y)\big]u = \int_0^D z e^{\eta(1-u)z e^{R(T-t)}} \, dF_Z(z) - \mathbb{E}[Z]. \tag{4.19}$$

Proof. From the expression (4.18) we get

$$\frac{\partial q(t,y,u)}{\partial u} = \mathbb{E}[Z]\lambda(t,y) + 2\theta_r \mathbb{E}[Z^2]\big[\lambda(t,y) + T\lambda(t,y)^2\big]u \qquad \forall u \in (0,1)$$

and

$$\frac{\partial^2 q(t,y,u)}{\partial u^2} = 2\theta_r \mathbb{E}[Z^2]\big[\lambda(t,y) + T\lambda(t,y)^2\big] > 0 \qquad \forall u \in (0,1).$$

By Lemma 4.1 $\Psi^u(t,y)$ is strictly concave w.r.t. u and full reinsurance is never optimal because of Remark 4.2. Moreover, we notice that $A_0 = \emptyset$ as in Lemma 4.3, thus the optimal strategy is unique and it belongs to $(0,1)$. In order to find such a solution, we turn the attention to the first order condition, which is exactly equation (4.19). $\square$

Through the numerical simulations in Section 7 we will show that the *intensity-adjusted variance premium principle* leads to optimal strategies which are consistent with the desired properties obtained under the other premium calculation principles. Moreover, the reinsurance strategies under the *intensity-adjusted variance premium principle* are not deterministic and explicitly depend on the (stochastic) intensity. Hence the problems described in the beginning of this subsection are fixed.

Using the result given in Example 4.3, it is easy to specialize Lemma 4.5 to the case of exponentially distributed claims.

Example 4.4. *(Exponentially distributed claims under the intensity-adjusted variance premium principle)*
Under the intensity-adjusted variance premium principle (4.18)*, suppose that the claims are exponentially distributed with parameter $\zeta > \eta e^{RT}$. Then the optimal strategy $u^*(t,y) \in (0,1)$ is given by*

$$u^*(t,y) = 1 - \frac{\zeta}{\eta}\left(1 - \sqrt{\frac{\zeta}{\zeta + 4\theta_r\big[1 + T\lambda(t,y)\big]}}\right)e^{-R(T-t)} \qquad (t,y) \in [0,T] \times \mathbb{R}. \tag{4.20}$$

Remark 4.7. *In [Liang and Yuen, 2016] and [Yuen et al., 2015] the authors used the variance premium and the expected value principles, respectively, to obtain optimal reinsurance strategies in a risk model with multiple dependent classes of insurance business. In those papers the optimal strategies explicitly depend on the claims intensities, but it is due to the presence of more than one business line, hence our arguments are not valid there. Nevertheless, in [Yuen et al., 2015] the authors realized that in the diffusion approximation of the classical risk model the variance premium principle leads to optimal strategies which do not depend on the claims intensities. In fact, this was the main motivation of their work. Their observation confirms our perplexities of strategies independent on the claims intensity.*

5. Optimal investment policy

Lemma 5.1. *The problem*

$$\sup_{w(t,y,p)\in\mathbb{R}} \Psi^w(t,y,p),$$

where $\Psi^w(t,y,p)$ is defined in (3.5)*, admits a unique solution $w^*(t,y,p)$ for all $(t,y,p) \in [0,T] \times \mathbb{R} \times (0,+\infty)$ given by*

$$w^*(t,y,p) = \frac{\mu(t,p) - R}{\eta\sigma(t,p)^2 e^{R(T-t)}} + \frac{p}{\eta e^{R(T-t)}}\frac{\frac{\partial\varphi}{\partial p}(t,y,p)}{\varphi(t,y,p)}. \tag{5.1}$$

Proof. Since $\varphi(t,y,p) > 0$, $\Psi^w(t,y,p)$ is strictly concave w.r.t. w and the result follows from the first order condition. □

We emphasize that the optimal w^* is the sum of the classical solution[6] plus an adjustment term due to the dependence of the risky asset price coefficients on the stochastic process $\{P_t\}$.

Remark 5.1. *If μ, σ are continuous function and $\varphi \in \mathcal{C}^{1,2}$, then w^* is a continuous function w.r.t. (t,y,p).*

Corollary 5.1. *Suppose that there exist two functions $f(t,y) : [0,T] \times \mathbb{R} \to (0,+\infty)$ and $g(t,p) : [0,T] \times (0,+\infty) \to \mathbb{R}$ such that $\varphi(t,y,p) = f(t,y)e^{g(t,p)}$ for all $(t,y,p) \in [0,T] \times \mathbb{R} \times (0,+\infty)$, with $f(t,y) > 0$. Then the optimal investment strategy* (5.1) *reads as follows:*

$$w^*(t,p) = \frac{\mu(t,p) - R}{\eta\sigma(t,p)^2 e^{R(T-t)}} + \frac{p}{\eta e^{R(T-t)}}\frac{\partial g}{\partial p}(t,p). \tag{5.2}$$

[6] See e.g. [Merton, 1969].

Remark 5.2. *Different dynamics for the risky-asset prices could be considered. For instance, in the case of pure jump processes, explicit formulas for optimal investment strategies can be found in [Ceci, 2009] and [Ceci and Gerardi, 2009]. See also [Ceci and Gerardi, 2010] in the case of power utility.*

6. Verification Theorem

Now we conjecture a solution to equation (3.3) of the form $\varphi(t,y,p) = f(t,y)e^{g(t,p)}$, with $f(t,y) > 0$. Using Lemma 5.1, replacing all the derivatives and performing some calculations, the equation (3.3) reads as follows

$$\begin{cases} -\dfrac{\partial f}{\partial t}(t,y) - b(t,y)\dfrac{\partial f}{\partial y}(t,y) - \dfrac{1}{2}\gamma(t,y)^2\dfrac{\partial^2 f}{\partial y^2}(t,y) + \left[\eta e^{R(T-t)}c(t,y) + \displaystyle\max_{u(t,y)\in[0,1]} \Psi^u(t,y)\right] f(t,y) \\ + f(t,y)\left[-\dfrac{\partial g}{\partial t}(t,p) - pR\dfrac{\partial g}{\partial p}(t,p) - \dfrac{1}{2}p^2\sigma(t,p)^2\dfrac{\partial^2 g}{\partial p^2}(t,p) + \dfrac{1}{2}\dfrac{\big(\mu(t,p)-R\big)^2}{\sigma(t,p)^2}\right] = 0 \\ f(T,y)e^{g(T,p)} = 1 \qquad \forall (y,p) \in \mathbb{R}\times(0,+\infty) \end{cases} \tag{6.1}$$

It is easy to show that if f, g are two solutions to the following Cauchy problems

$$\begin{cases} -\dfrac{\partial f}{\partial t}(t,y) - b(t,y)\dfrac{\partial f}{\partial y}(t,y) - \dfrac{1}{2}\gamma(t,y)^2\dfrac{\partial^2 f}{\partial y^2}(t,y) \\ + \left[\eta e^{R(T-t)}c(t,y) + \displaystyle\max_{u(t,y)\in[0,1]} \Psi^u(t,y)\right] f(t,y) = 0 \\ f(T,y) = 1, \end{cases} \tag{6.2}$$

$$\begin{cases} -\dfrac{\partial g}{\partial t}(t,p) - pR\dfrac{\partial g}{\partial p}(t,p) - \dfrac{1}{2}p^2\sigma(t,p)^2\dfrac{\partial^2 g}{\partial p^2}(t,p) + \dfrac{1}{2}\dfrac{\big(\mu(t,p)-R\big)^2}{\sigma(t,p)^2} = 0 \\ g(T,p) = 0, \end{cases} \tag{6.3}$$

then they solve the Cauchy problem (6.1) and $v(t,x,y,p) = -e^{-\eta x e^{R(T-t)}} f(t,y)e^{g(t,p)}$ solves the original HJB equation given in (3.1).
Before we prove a verification theorem, we must show that our proposed optimal controls are admissible strategies.

Lemma 6.1. *Suppose that (6.2) and (6.3) admit classical solutions with $\frac{\partial g}{\partial p}$ satisfying the following growth condition:*

$$\left|\frac{\partial g}{\partial p}(t,p)\right| \le C(1+|p|^\beta) \qquad \forall (t,p) \in [0,T]\times(0,+\infty) \tag{6.4}$$

for some constants $\beta > 0$ and $C > 0$. Moreover, assume that

$$\mathbb{E}\left[\int_0^T |\mu(t,P_t)| P_t^{\beta+1}\, dt + \int_0^T \sigma(t,P_t)^2 P_t^{2\beta+2}\, dt\right] < \infty. \tag{6.5}$$

Let be $u^*(t,y)$ as given in Proposition 4.1 and $w^*(t,p)$ in Lemma 5.1. Let us define the processes $u^*_t \doteq u^*(t,Y_t)$ and $w^*_t \doteq w^*(t,P_t)$; then the pair (u^*_t, w^*_t) is an admissible strategy, i.e. $(u^*_t, w^*_t) \in \mathcal{U}$.

Proof. First let us observe that both u^*_t, w^*_t are $\{\mathcal{F}_t\}$-predictable processes since $u^*(t,u)$ and $w^*(t,p)$ are measurable functions of their arguments and Y is $\{\mathcal{F}_t\}$-adapted. Moreover, they take values in $[0,1]$ and in $\mathbb{R}$, respectively. Furthermore, using the expression (5.2) we have that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T |w_t^*||\mu(t,P_t)-R|\,dt\right] &\le \mathbb{E}\left[\int_0^T \frac{(\mu(t,P_t)-R)^2}{\eta\sigma(t,P_t)^2 e^{R(T-t)}}\,dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \frac{|\mu(t,P_t)-R|P_t}{\eta e^{R(T-t)}}\left|\frac{\partial g}{\partial p}(t,P_t)\right|dt\right] \\
&\le \mathbb{E}\left[\int_0^T \frac{(\mu(t,P_t)-R)^2}{\eta\sigma(t,P_t)^2 e^{R(T-t)}}\,dt\right] \\
&\quad + C\,\mathbb{E}\left[\int_0^T |\mu(t,P_t)|(1+P_t^\beta)P_t\,dt\right] < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T (w_t^*\sigma(t,P_t))^2\,dt\right] &= \mathbb{E}\left[\int_0^T \frac{(\mu(t,p)-R)^2}{\eta^2\sigma(t,p)^2 e^{2R(T-t)}}\,dt\right] \\
&\quad + \mathbb{E}\left[\int_0^T \frac{\sigma(t,P_t)^2 P_t^2}{\eta^2 e^{2R(T-t)}}\left(\frac{\partial g}{\partial p}(t,P_t)\right)^2 dt\right] \\
&\quad + 2\mathbb{E}\left[\int_0^T \frac{(\mu(t,p)-R)P_t}{\eta^2 e^{2R(T-t)}}\frac{\partial g}{\partial p}(t,P_t)\,dt\right] \\
&\le \mathbb{E}\left[\int_0^T \frac{(\mu(t,p)-R)^2}{\eta^2\sigma(t,p)^2 e^{2R(T-t)}}\,dt\right] \\
&\quad + C\,\mathbb{E}\left[\int_0^T \frac{\sigma(t,P_t)^2 P_t^2}{\eta^2 e^{2R(T-t)}}\left(1+P_t^\beta\right)^2 dt\right] \\
&\quad + C\,\mathbb{E}\left[\int_0^T \frac{|\mu(t,p)-R|P_t}{\eta^2 e^{2R(T-t)}}(1+P_t^\beta)\,dt\right] < \infty,
\end{aligned}
$$

where C denotes any positive constant and the expectations are finite because of the Novikov condition (2.15) together with (6.4) and (6.5). ☐

Now we are ready for the verification argument.

Theorem 6.1 (Verification Theorem)**.** *Suppose that* (6.2) *and* (6.3) *admit bounded classical solutions* $f \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R}))\cap\mathcal{C}([0,T]\times\mathbb{R}))$ *and* $g \in \mathcal{C}^{1,2}((0,T)\times(0,+\infty))\cap\mathcal{C}([0,T]\times(0,+\infty))$, *respectively.*
Let us assume that the conditions (6.4) *and* (6.5) *hold and suppose that*

$$
\left|\frac{\partial f}{\partial y}(t,y)\right| \le \tilde{C}(1+|y|^\beta) \qquad \forall (t,y)\in[0,T]\times\mathbb{R} \tag{6.6}
$$

for some constants $\beta > 0$ *and* $\tilde{C} > 0$*. As an alternative, the conditions* (6.4), (6.5) *and* (6.6) *may be replaced by the boundedness of* $\frac{\partial g}{\partial p}$ *and* $\frac{\partial f}{\partial y}$*.*
Then the function $v : V \to \mathbb{R}$ *defined by the following*

$$
v(t,x,y,p) = -e^{-\eta x e^{R(T-t)}} f(t,y) e^{g(t,p)} \tag{6.7}
$$

is the value function of the reinsurance-investment problem and

$$
\alpha^*(t,Y_t,P_t) = (u^*(t,Y_t), w^*(t,P_t))
$$

with $u^*(t,y)$ *given in* (4.2) *and* $w^*(t,p)$ *in* (5.2) *is an optimal control.*

Proof. Let $f(t,y) : [0,T]\times\mathbb{R} \to (0,+\infty)$ and $g(t,p) : [0,T]\times(0,+\infty) \to \mathbb{R}$ be functions satisfying the assumptions required by Theorem 6.1 and suppose that they are solutions to the Cauchy problems (6.2) and (6.3). Now consider the function $\varphi(t,y,p) = f(t,y)e^{g(t,p)}$. As already observed, it satisfies equation (3.3), i.e. it is a solution to the problem

$$\begin{cases} \sup\limits_{(u,w)\in[0,1]\times\mathbb{R}} \mathcal{H}^\alpha \varphi(t,y,p) = 0 \\ \varphi(t,y,p) = 1 \qquad \forall (y,p) \in \mathbb{R}\times(0,+\infty). \end{cases} \tag{6.8}$$

Now, taking $v(t,x,y,p) = -e^{-\eta x e^{R(T-t)}}\varphi(t,y,p)$, we have that v is a solution to the Cauchy problem (3.1). This implies that, for any $(t,x,y,p) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times(0,+\infty)$

$$\mathcal{L}^\alpha v(s, X^\alpha_{t,x}(s), Y_{t,y}(s), P_{t,p}(s)) \le 0 \quad \forall s \in [t,T]$$

for all $\alpha \in \mathcal{U}_t$, where $\{Y_{t,y}(s)\}_{s\in[t,T]}$ denotes the solution to equation (2.1) with initial condition $Y_t = y$ and, similarly, $\{P_{t,p}(s)\}_{s\in[t,T]}$ denotes the solution to equation (2.12) with initial condition $P_t = p$.
Now, from Itô's formula we have that

$$v(T, X^\alpha_{t,x}(T), Y_{t,y}(T), P_{t,p}(T)) - v(t,x,y,p) = \int_t^T \mathcal{L}^\alpha v(s, X^\alpha_{t,x}(s), Y_{t,y}(s), P_{t,p}(s))\,ds + M_T, \tag{6.9}$$

where $\{M_r\}_{r\in[t,T]}$ is the following stochastic process:

$$\begin{aligned} M_r = &\int_t^r w_s \sigma(s,P_s)\frac{\partial v}{\partial x}(s, X^\alpha_s, Y_s, P_s)\, dW^{(P)}_s \\ &+ \int_t^r P_s\sigma(s,P_s)\frac{\partial v}{\partial p}(s, X^\alpha_s, Y_s, P_s)\, dW^{(P)}_s + \int_t^r \gamma(s,Y_s)\frac{\partial v}{\partial y}(s, X^\alpha_s, Y_s, P_s)\, dW^{(Y)}_s \\ &+ \int_0^D \int_t^r \Big[v(s, X^\alpha_s - (1-u_s)z, Y_s, P_s) - v(s, X^\alpha_s, Y_s, P_s)\Big]\Big(m(ds,dz) - dF_Z(z)\,\lambda(s,Y_s)ds\Big). \end{aligned} \tag{6.10}$$

Now we prove that $\{M_r\}_{r\in[t,T]}$ is an $\{\mathcal{F}_r\}$-local martingale. In particular, we need to show that

$$\begin{aligned} &\mathbb{E}\Bigg[\int_t^{T\wedge\tau_n}\Big(w_s\sigma(s,P_s)\frac{\partial v}{\partial x}(s, X^\alpha_s, Y_s, P_s)\Big)^2 ds\Bigg] < \infty, \\ &\mathbb{E}\Bigg[\int_t^{T\wedge\tau_n}\Big(P_s\sigma(s,P_s)\frac{\partial v}{\partial p}(s, X^\alpha_s, Y_s, P_s)\Big)^2 ds\Bigg] < \infty, \\ &\mathbb{E}\Bigg[\int_t^{T\wedge\tau_n}\Big(\gamma(s,Y_s)\frac{\partial v}{\partial y}(s, X^\alpha_s, Y_s, P_s)\Big)^2 ds\Bigg] < \infty, \\ &\mathbb{E}\Bigg[\int_0^D\int_t^{T\wedge\tau_n} |v(s, X^\alpha_s - (1-u_s)z, Y_s, P_s) - v(s, X^\alpha_s, Y_s, P_s)|\, dF_Z(z)\,\lambda(s,Y_s)ds\Bigg] < \infty, \end{aligned}$$

for a suitable non-decreasing sequence of stopping times $\{\tau_n\}_{n=1,\dots}$ such that $\lim_{n\to+\infty}\tau_n = +\infty$. Taking into account the expression (6.7), we note that

$$\begin{aligned} \frac{\partial v}{\partial x}(t,x,y,p) &= \eta e^{R(T-t)} e^{-\eta x e^{R(T-t)}} f(t,y) e^{g(t,p)}, \\ \frac{\partial v}{\partial y}(t,x,y,p) &= -e^{-\eta x e^{R(T-t)}} e^{g(t,p)} \frac{\partial f}{\partial y}(t,y), \\ \frac{\partial v}{\partial p}(t,x,y,p) &= -e^{-\eta x e^{R(T-t)}} f(t,y) e^{g(t,p)} \frac{\partial g}{\partial p}(t,p). \end{aligned}$$

Let us define a sequence of random times $\{\tau_n\}_{n=1,\dots}$ as follows:

$$\tau_n \doteq \inf\{s \in [t,T] \mid X^\alpha_s < -n \vee |Y_s| > n\} \qquad n = 1,\dots$$

In the sequel of the proof we denote by C_n any constant depending on $n = 1, \dots$.
Then we have that

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^{T\wedge\tau_n} \Big(w_s\sigma(s,P_s)\frac{\partial v}{\partial x}(s,X_s^\alpha,Y_s,P_s)\Big)^2\,ds\bigg]\\
&= \mathbb{E}\bigg[\int_0^{T\wedge\tau_n} \Big(w_s\sigma(s,P_s)\eta e^{R(T-s)}e^{-\eta X_s^\alpha e^{R(T-s)}} f(s,Y_s)e^{g(s,P_s)}\Big)^2\,ds\bigg]\\
&\le C_n\,\mathbb{E}\bigg[\int_0^{T} \Big(w_s\sigma(s,P_s)\Big)^2\,ds\bigg] < \infty \qquad \forall n = 1,\dots,
\end{aligned}
$$

because w_t is admissible and f and g are bounded by hypothesis. Moreover, we have that

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^{T\wedge\tau_n} \Big(\gamma(s,Y_s)\frac{\partial v}{\partial y}(s,X_s^\alpha,Y_s,P_s)\Big)^2\,ds\bigg]\\
&= \mathbb{E}\bigg[\int_0^{T\wedge\tau_n} \Big(\gamma(s,Y_s)e^{-\eta X_s^\alpha e^{R(T-s)}} e^{g(s,P_s)}\frac{\partial f}{\partial y}(s,Y_s)\Big)^2\,ds\bigg]\\
&\le \tilde{C}\mathbb{E}\bigg[\int_0^{T\wedge\tau_n} \Big(\gamma(s,Y_s)e^{-\eta X_s^\alpha e^{R(T-s)}} e^{g(s,P_s)}\Big)^2(1+|Y_s|^\beta)^2\,ds\bigg]\\
&\le C_n\,\mathbb{E}\bigg[\int_0^{T} \gamma(s,Y_s)^2\,ds\bigg] < \infty \qquad \forall n = 1,\dots,
\end{aligned}
$$

because g is bounded and using the assumptions (2.3) and (6.6). Further, we obtain that

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^{T\wedge\tau_n} \Big(P_s\sigma(s,P_s)\frac{\partial v}{\partial p}(s,X_s^\alpha,Y_s,P_s)\Big)^2\,ds\bigg]\\
&= \mathbb{E}\bigg[\int_0^{T\wedge\tau_n} P_s^2\sigma(s,P_s)^2\Big(e^{-\eta X_s^\alpha e^{R(T-s)}} f(s,Y_s)e^{g(s,P_s)}\frac{\partial g}{\partial p}(s,P_s)\Big)^2\,ds\bigg]\\
&\le C\mathbb{E}\bigg[\int_0^{T\wedge\tau_n} P_s^2\sigma(s,P_s)^2\Big(e^{-\eta X_s^\alpha e^{R(T-s)}} f(s,Y_s)e^{g(s,P_s)}\Big)^2(1+|P_s|^\beta)^2\,ds\bigg]\\
&\le C_n\mathbb{E}\bigg[\int_0^{T} \sigma(s,P_s)^2(P_s^2+P_s^{2\beta+2})\,ds\bigg] < \infty \qquad \forall n = 1,\dots,
\end{aligned}
$$

because f and g are bounded by hypothesis and using conditions (2.13), (6.4) and (6.5). Finally,

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^{D}\int_t^{T\wedge\tau_n} |v(s,X_s^\alpha-(1-u_s)z,Y_s,P_s)-v(s,X_s^\alpha,Y_s,P_s)|\,dF_Z(z)\,\lambda(s,Y_s)ds\bigg]\\
&\le \mathbb{E}\bigg[\int_0^{D}\int_t^{T\wedge\tau_n} e^{-\eta X_s^\alpha e^{R(T-s)}} f(s,Y_s,P_s)e^{g(s,P_s)}|e^{(1-u_s)\eta z e^{R(T-s)}}-1|\,dF_Z(z)\,\lambda(s,Y_s)ds\bigg]\\
&\le C_n\mathbb{E}\bigg[\int_0^{D}\int_t^{T\wedge\tau_n} e^{\eta z e^{RT}}dF_Z(z)\,\lambda(s,Y_s)ds\bigg]\\
&\le C_n\mathbb{E}\Big[e^{\eta Z e^{RT}}\Big]\mathbb{E}\bigg[\int_t^{T}\lambda(s,Y_s)ds\bigg] < \infty,
\end{aligned}
$$

thanks to (2.4) and (2.18). Thus $\{M_r\}_{r\in[t,T]}$ is an $\{\mathcal{F}_r\}$-local martingale and $\{\tau_n\}_{n=1,\dots}$ is a localizing sequence for $\{M_r\}_{r\in[t,T]}$.

Taking the expected value of both sides of (6.9) with T replaced by $T\wedge\tau_n$, we obtain that

$$
\mathbb{E}[v(T\wedge\tau_n, X_{t,x}^\alpha(T\wedge\tau_n), Y_{t,y}(T\wedge\tau_n), P_{t,p}(T\wedge\tau_n)) \mid \mathcal{F}_t] \le v(t,x,y,p)
$$

for any $\alpha \in \mathcal{U}_t, t \in [0, T \wedge \tau_n], n \geq 1$. Now notice that

$$
\begin{aligned}
&\mathbb{E}[v(T\wedge\tau_n, X^\alpha_{t,x}(T\wedge\tau_n), Y_{t,y}(T\wedge\tau_n), P_{t,p}(T\wedge\tau_n))^2] \\
&= \mathbb{E}[e^{-2\eta X^\alpha_{t,x}(T\wedge\tau_n)e^{R(T\wedge\tau_n-t)}} f(T\wedge\tau_n, Y_{T\wedge\tau_n})^2 e^{2g(T\wedge\tau_n, P_{T\wedge\tau_n})}] \\
&\leq C\, e^{-2\eta n e^{R(T\wedge\tau_n)}} \leq C,
\end{aligned}
$$

thus $\{v(T\wedge\tau_n, X^\alpha_{t,x}(T\wedge\tau_n), Y_{t,y}(T\wedge\tau_n), P_{t,p}(T\wedge\tau_n))\}_{n=1,\dots}$ is a family of uniformly integrable random variables. Hence it converges almost surely. Observing that $\{\tau_n\}_{n=1,\dots}$ is a bounded and non-decreasing sequence, since $\mathbb{P}[|X^\alpha_t| < +\infty] = 1$ (see (2.17)) and using (2.2) and (2.14), taking the limit for $n \to +\infty$, we conclude that

$$
\begin{aligned}
&\mathbb{E}[v(T, X^\alpha_{t,x}(T), Y_{t,y}(T), P_{t,p}(T)) \mid \mathcal{F}_t] \\
&= \lim_{n\to+\infty} \mathbb{E}[v(T\wedge\tau_n, X^\alpha_{t,x}(T\wedge\tau_n), Y_{t,y}(T\wedge\tau_n), P_{t,p}(T\wedge\tau_n)) \mid \mathcal{F}_t] \\
&\leq v(t,x,y,p) \qquad \forall \alpha \in \mathcal{U}_t, t \in [0,T].
\end{aligned}
\tag{6.11}
$$

To be precise, we have that

$$
\lim_{n\to+\infty} X^\alpha_{t,x}(T\wedge\tau_n) = X^\alpha_{t,x}(T-) = X^\alpha_{t,x}(T) \qquad \mathbb{P}\text{-a.s.},
$$

since the jump of $\{N_t\}_{t\in[0,T]}$ occurs at time T with probability zero. Using the final condition of the HJB equation (3.1), from (6.11) we get

$$
\mathbb{E}[U(X^\alpha_{t,x}(T))] \leq v(t,x,y,p) \qquad \forall \alpha \in \mathcal{U}_t, t \in [0,T].
$$

Now note that $\alpha^*(t,y,p)$ was calculated in order to obtain $\mathcal{L}^{\alpha^*} v(t,x,y,p) = 0$; replicating the calculations above, replacing $\mathcal{L}^\alpha$ with $\mathcal{L}^{\alpha^*}$, we find the equality:

$$
\sup_{\alpha\in\mathcal{U}_t} \mathbb{E}[U(X^\alpha_{t,x}(T)) \mid Y_t = y, P_t = p] = v(t,x,y,p),
$$

thus $\alpha^*(t, Y_t, P_t)$ is an optimal control. $\square$

After the characterization of the value function, we provide a probabilistic representation by means of the Feynman-Kac formula. In preparation for this result, let us introduce a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Novikov condition (2.15) implies that the process $\{L_t\}_{t\in[0,T]}$ defined by

$$
L_t = e^{-\left(\frac{1}{2}\int_0^t |\frac{\mu(s,P_s)-R}{\sigma(s,P_s)}|^2 ds + \int_0^t \frac{\mu(s,P_s)-R}{\sigma(s,P_s)} dW^{(P)}_s\right)}
$$

is an $\{\mathcal{F}_t\}$-martingale and we can introduce the following probability measure $\mathbb{Q}$:

$$
\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t \qquad t \in [0,T].
\tag{6.12}
$$

By Girsanov theorem we know that $\tilde{W}^{(P)}_t = W^{(P)}_t + \int_0^t \frac{\mu(s,P_s)-R}{\sigma(s,P_s)} ds$ is a $\mathbb{Q}$-Brownian motion and we can rewrite the risky asset dynamic as

$$
dP_t = P_t\left[R\,dt + \sigma(t,P_t)\,d\tilde{W}^{(P)}_t\right].
\tag{6.13}
$$

Since the discounted price $\{\tilde{P}_t = P_t e^{-Rt}\}_{t\in[0,T]}$ turns out to be an $\{\mathcal{F}_t\}$-martingale, then $\mathbb{Q}$ is a martingale or risk-neutral measure for $\{P_t\}$[7]. We will denote by $\mathbb{E}^{\mathbb{Q}}$ the conditional expectation with respect to $\mathbb{Q}$.

[7]Let us observe that under $\mathbb{Q}$ the dynamics of $\{Y_t\}$ and $\{R_t\}$ do not change.

Proposition 6.1. *Suppose that* (6.2) *and* (6.3) *admit classical solutions* $f \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R}) \cap \mathcal{C}([0,T]\times\mathbb{R}))$ *and* $g \in \mathcal{C}^{1,2}((0,T)\times(0,+\infty)) \cap \mathcal{C}([0,T]\times(0,+\infty))$*, respectively, both bounded with* $\frac{\partial f}{\partial y}$ *and* $\frac{\partial g}{\partial p}$ *satisfying the growth conditions* (6.6) *and* (6.4)*. Then* f *and* g *admit the following Feynman-Kac representations:*

$$f(t,y) = \mathbb{E}\left[e^{-\int_t^T \left(\eta e^{R(T-s)} c(s,Y_s) + \Psi^{u^*}(s,Y_s)\right) ds} \mid Y_t = y\right], \tag{6.14}$$

$$g(t,p) = -\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{1}{2}\frac{\left(\mu(s,P_s)-R\right)^2}{\sigma(s,P_s)^2}\, ds \mid P_t = p\right], \tag{6.15}$$

where $\Psi^{u^*}(t,y)$ *is the function defined by* (3.4)*, replacing* u *with* $u^*(t,y)$*, and* $\mathbb{Q}$ *is the probability measure introduced in* (6.12)*.*

Proof. The result is a simple consequence of the Feynman-Kac theorem. □

In Section 8 we will provide sufficient conditions which ensure that the functions f and g given in (6.14) and (6.15) are $\mathcal{C}^{1,2}((0,T)\times\mathbb{R})$ and $\mathcal{C}^{1,2}((0,T)\times(0,+\infty))$ solutions to the Cauchy problems (6.2) and (6.3), respectively.

7. Simulations and numerical results

Here we illustrate some numerical results based on the theoretical framework developed in the previous sections. In particular, we perform sensitivity analysis of the optimal reinsurance-investment strategy in order to study the effect of the model parameters on the insurer's decision.

7.1. Reinsurance strategy

In this subsection we compare the optimal reinsurance strategy under the *variance premium principle* (see Lemma 4.3) and the *intensity-adjusted variance premium principle* (see Lemma 4.5). They will be shortly referred as VP and IAVP, respectively. The main difference is that under VP we loose the dependence on the stochastic factor, while under IAVP we keep this dependence.

In what follows we assume that $\{Z_i\}_{i=1,\dots}$ is a sequence of i.i.d. positive random variables Pareto distributed with shape parameter 1.8182 and scale parameter 0.0545. The stochastic factor is described by the SDE (2.1) with constant parameters $b = 0.3, \gamma = 0.3$ and initial condition $Y_0 = 1$. For the sake of simplicity, we assume that $\lambda(t,y) = \lambda_0 e^{\frac{1}{2}y}$, that is $\{\lambda_t = \lambda(t,Y_t)\}_{t\in[0,T]}$ solves

$$d\lambda_t = \lambda_t \frac{1}{2} dY_t \qquad \lambda_0 = 0.1,$$

which guarantees that the intensity is positive. Finally, we consider the model parameters in Table 1, using the notation introduced in Section 2.

Table 1: *Simulation parameters*

Parameter	**Value**
T	5 Y
η	0.5
θ_r	0.1
R	5%

In the sequel, the dashed line refers to the VP case, while the normal line represents the IAVP case.

From Figure 1 we observe that the optimal reinsurance strategy is positively correlated to the risk-aversion parameter; moreover, the strategy under VP seems to be more sensitive to any variation of the risk-aversion.

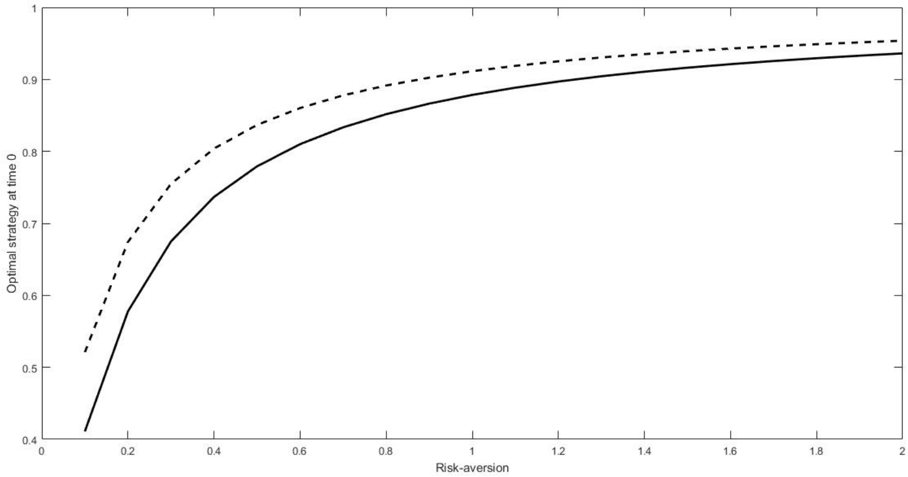

Figure 1: *The effect of the risk-aversion parameter η on the optimal initial strategy under VP (dashed) and IAVP (normal).*

Moreover, we observe that under VP the strategy is always more conservative (i.e. the protection level is higher) than under IAVP. There are at least two reasons: firstly, in Subsection 4.2 we noticed that under VP the reinsurer underestimates the variance of her losses, hence the premium will be underestimated; secondly, under IAVP the insurer overestimates the reinsurance premium. As a consequence of the well-known law of demand, under IAVP the insurer will buy a lower protection level. This fact keeps happening in the next figures, but it will not be pointed out again.

In Figure 2 we notice that any increase in the reinsurance safety loading leads to a decrease of the reinsured risks. It is a simple consequence of the well-known law of demand: the higher the price, the lower the quantity demanded. It is worth noting that under our assumptions the strategy under IAVP is more sensitive than under VP.

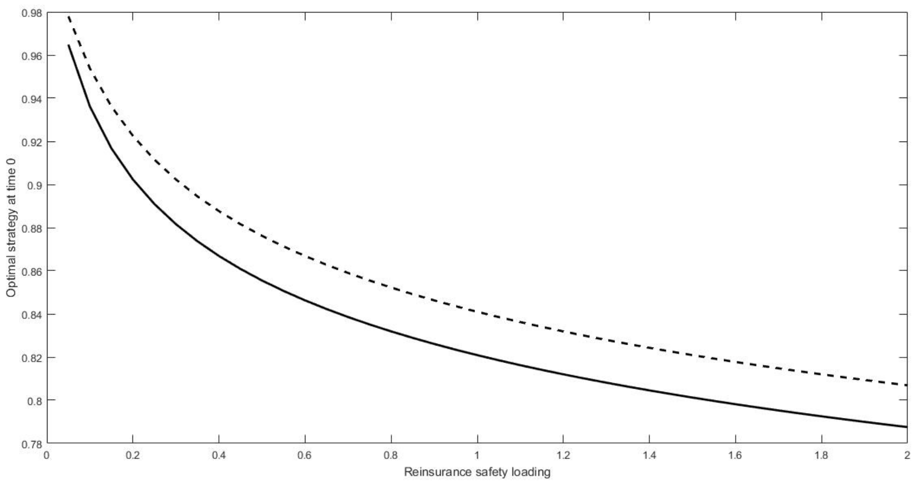

Figure 2: *The effect of the reinsurance safety loading θ_r on the optimal initial strategy under VP (dashed) and IAVP (normal).*

Finally, in Figure 3 we can see that the insurer increases the protection when the time horizon is higher. Again, the strategy under VP turns out to be more sensitive to any change of the time horizon.

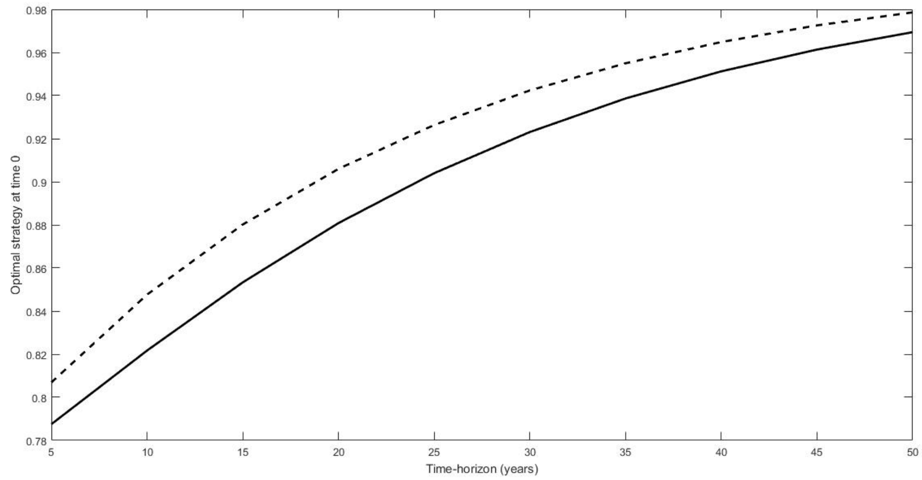

Figure 3: *The effect of the time horizon T on the optimal initial strategy under VP (dashed) and IAVP (normal).*

We conclude this subsection investigating the dynamical properties of the reinsurance strategies under VP and IAVP[8]. Figure 4 shows that the mean behavior of the optimal reinsurance strategy is decreasing over the time interval (as expected from Figure 3); nevertheless, under IAVP the strategy crucially depends on the stochastic factor, hence the insurer will react to any movement of the claims intensity, while under VP she will follow a deterministic strategy.

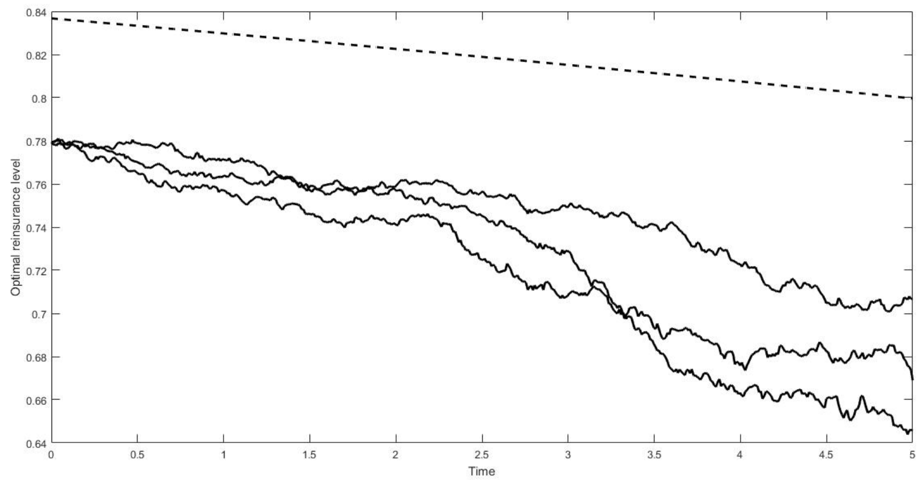

Figure 4: *Dynamical reinsurance strategies in $[0, T]$ under VP (dashed) and IAVP (normal).*

Summarizing the main results of our numerical simulations, we can conclude that, from a qualitative point of view, any variation of the model parameters has the same effect on the optimal strategy under VP and IAVP. Using our model parameters, regarding modifications of risk-aversion and time-horizon, we observed that under VP the strategy seems to be more

[8]Under a practical point of view, we simulated the stochastic processes using the classical Euler's approximation method, with $dt = \frac{T}{500}$.

sensitive than under IAVP; for the safety loading, we observed the opposite behavior. In addition, under VP the strategy is dynamically more stable over the time interval $[0, T]$, because it does not take into account any variation of the claims intensity. Finally, under VP the insurer will follow a more conservative strategy than under IAVP, i.e. she will buy more protection.

7.2. Investment strategy

Now we illustrate a sensitivity analysis for the investment strategy based on the Corollary 5.1. In our simulations we assumed that the risky asset follows a CEV model, that is

$$dP_t = P_t \left[\mu \, dt + \sigma P_t^{\beta} \, dW_t^{(P)} \right] \qquad P_0 = 1,$$

with $\mu = 0.1, \sigma = 0.1, \beta = 0.5$, while the risk-free interest rate is $R = 5\%$ as in the previous subsection. Let us observe that this model corresponds to (2.12) assuming that $\mu(t, p) = \mu$ and $\sigma(t, p) = \sigma p^{\beta}$, with constant $\mu, \sigma > 0$. The numerical computation of the function $g(t, p)$ and its partial derivative $\frac{\partial g}{\partial p}(t, p)$ is required by the equation (5.2); for this purpose we used the Feynman-Kac representation given in (6.15) evaluated through the standard Monte Carlo method.

In figure 5 we show that the higher is the insurer's risk aversion, the lower is the total amount invested in the risky asset.

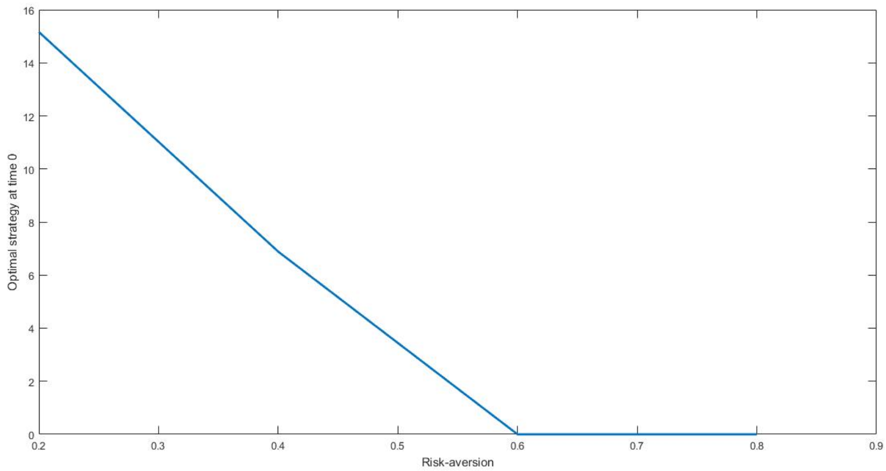

Figure 5: *The effect of the risk-aversion parameter η on the optimal initial strategy.*

Figure 6 illustrates that if the volatility increases, then an increasing portion of the insurer's wealth is invested in the risk-free asset.

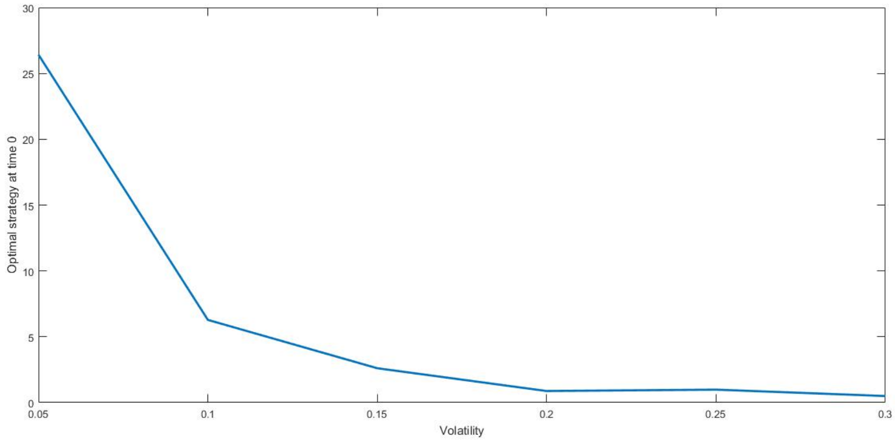

Figure 6: *The effect of the volatility parameter σ on the optimal initial strategy.*

Finally, if the risk-free interest rate grows up, then the insurer will find it more convenient to invest its surplus in the risk-free asset, as shown in figure 7.

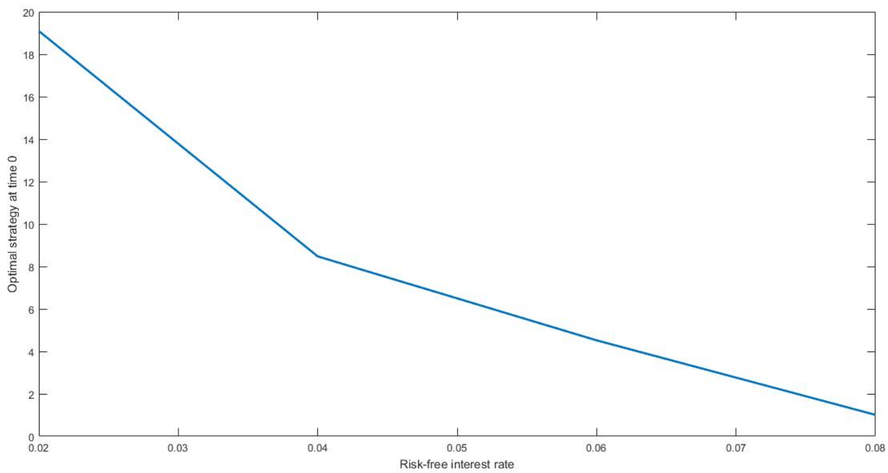

Figure 7: *The effect of the risk-free interest rate R on the optimal initial strategy.*

Similar results can be found in [Sheng et al., 2014]. In particular, figure 6 confirms the result obtained in Figure 3a of that paper; in addition, figures 5 and 7 complete the sensitivity analyses performed there.

8. Existence and uniqueness of classical solutions

In this section we are interested in providing sufficient conditions for existence and uniqueness of the solutions to the PDEs involved in the reinsurance-investment problem, see the Cauchy problems (6.2) and (6.3) and as a consequence of a classical solution to HJB equation associated with our problem.
Let us start from the PDE (6.3).

Theorem 8.1. *Suppose that $\sigma(t,p)p$ is locally Lipschitz-continuous in p, uniformly in $t \in [0,T]$ and for any $n \in \mathbb{N}$ the PDE*

$$\begin{cases} -\dfrac{\partial w}{\partial t}(t,p) - pR\dfrac{\partial w}{\partial p}(t,p) - \dfrac{1}{2}p^2\sigma(t,p)^2\dfrac{\partial^2 w}{\partial p^2}(t,p) + \dfrac{1}{2}\dfrac{\big(\mu(t,p)-R\big)^2}{\sigma(t,p)^2} = 0 \quad \forall (t,p) \in (0,T) \times D_n \\ w(t,p) = g(t,p) \quad \forall (t,p) \in (0,T) \times \partial D_n \cup \{T\} \times D_n, \end{cases}$$

with $D_n \doteq (\frac{1}{n}, n)$ and $g(t,p)$ given in (6.15), *has a classical solution $w_n(t,p)$.*

Then the function $g(t,p)$ satisfies the Cauchy problem (6.3) *and there exists a unique classical solution to* (6.3). *Moreover, we have that $g \in \mathcal{C}^{1,2}((0,T) \times (0,+\infty))$.*

Proof. The proof is based on [Heath and Schweizer, 2000, Theorem 1]. Let us recall that under the martingale measure $\mathbb{Q}$ (defined in (6.12)) the risky asset dynamic is given by (6.13). Thanks to our assumption on $\sigma(t,p)p$, the SDE (6.13) admits a unique solution up to a possibly finite explosion time. Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$ by definition, by (2.14) we get that $\mathbb{Q}[sup_{s\in[t,T]} P_{t,p}(s) < +\infty] = 1$. It implies that the expectation in (6.15) is well posed.

Now fix arbitrary $(t,p) \in (0,T) \times (0,+\infty)$. Since $\bigcup_{n\in\mathbb{N}} D_n = (0,+\infty)$, then $\exists n \in \mathbb{N}$ such that $p \in D_n$. Let us denote by $\tau_n \doteq \inf\{s \geq t \mid P_{t,p}(s) \notin D_n\} \wedge T$ the first exit time from D_n before T. We have that $(\tau_n, P_{t,p}(\tau_n)) \in (0,T) \times \partial D_n \cup \{T\} \times D_n$, because of the continuity of $\{P_{t,p}(s)\}_{s\in[t,T]}$. It turns out that $g(\tau_n, P_{t,p}(\tau_n)) < \infty$.

By a simple application of the Itô's formula to the process $\{w_n(s, P_{t,p}(s))\}_{s\in[t,T]}$, using the PDE above, it is easy to show that[9]

$$w_n(t,p) = \mathbb{E}^{\mathbb{Q}}\left[g(\tau_n, P_{t,p}(\tau_n)) - \int_t^{\tau_n} \frac{1}{2}\frac{\big(\mu(s,P_{t,p}(s))-R\big)^2}{\sigma(s,P_{t,p}(s))^2}\,ds\right].$$

Taking into account the expression of g, given in (6.15), we have that

$$\begin{aligned} w_n(t,p) &= \mathbb{E}^{\mathbb{Q}}\left[-\int_{\tau_n}^{T} \frac{1}{2}\frac{\big(\mu(s,P_{t,p}(s))-R\big)^2}{\sigma(s,P_{t,p}(s))^2}\,ds - \int_t^{\tau_n} \frac{1}{2}\frac{\big(\mu(s,P_{t,p}(s))-R\big)^2}{\sigma(s,P_{t,p}(s))^2}\,ds\right] \\ &= -\mathbb{E}^{\mathbb{Q}}\left[\int_t^{T} \frac{1}{2}\frac{\big(\mu(s,P_{t,p}(s))-R\big)^2}{\sigma(s,P_{t,p}(s))^2}\,ds\right] = g(t,p), \end{aligned}$$

Hence w_n and g coincide on $(0,T) \times D_n$ $\forall n \in \mathbb{N}$ and this implies that g satisfies (6.3) on $(0,T) \times (0,+\infty)$. The boundary condition $g(T,p) = 0$ immediately follow by the definition of g. □

Remark 8.1. *In [Sheng et al., 2014] the authors found an explicit solution to the Cauchy problem* (6.3) *in the particular case of the CEV model, i.e. when $\mu(t,p) = \mu$ and $\sigma(t,p) = kp^\beta$.*

Let us observe that it is not easy to check whether the main hypothesis of Theorem 8.1, i.e. the existence of w_n for any n, is fulfilled or not. Thus we need to provide more palatable assumptions. This is the motivation of Corollary 8.1, which is preceded by a preparation result.

Lemma 8.1. *Let us define the set $D_n \doteq (\frac{1}{n}, n)$ for $n = 1, \dots$ and assume that the functions $\mu(t,p), \sigma(t,p)$ are continuous in $(t,p) \in [0,T] \times (0,+\infty)$ and Lipschitz-continuous in $p \in \overline{D_n}$,*

[9] For details, see [Friedman, 1975, Theorem 6.5.2].

uniformly in $t \in [0,T]$. Moreover, assume that $\sigma(t,p)$ is locally bounded from below, i.e. there exists a constant $\delta_\sigma(n) > 0$ such that $\sigma(t,p) \geq \delta_\sigma(n)$ $\forall (t,p) \in [0,T] \times D_n$.
Then for each $n = 1, \ldots$ the function $k : [0,T] \times (0,+\infty) \to \mathbb{R}$ defined by

$$k(t,p) = \frac{\big(\mu(t,p) - R\big)^2}{\sigma(t,p)^2} \tag{8.1}$$

is uniformly Lipschitz-continuous on $[0,T] \times \overline{D_n}$.

Proof. Since $\frac{\mu(t,p)-R}{\sigma(t,p)}$ is bounded, we have that

$$\begin{aligned} |k(t,p) - k(t',p')| &= \left| \left(\frac{\mu(t,p) - R}{\sigma(t,p)} \right)^2 - \left(\frac{\mu(t',p') - R}{\sigma(t',p')} \right)^2 \right| \\ &\leq K_n \left| \frac{\mu(t,p) - R}{\sigma(t,p)} - \frac{\mu(t',p') - R}{\sigma(t',p')} \right| \\ &= K_n \left| \frac{\sigma(t',p')[\mu(t,p) - R] - \sigma(t,p)[\mu(t',p') - R]}{\sigma(t,p)\sigma(t',p')} \right| \end{aligned}$$

for a positive constant $K_n > 0$ which depends on n. Now, $\sigma(t,p)$ being bounded from below, setting $\tilde{K_n} = \frac{K_n}{\delta_\sigma^2}$ we have that

$$\begin{aligned} |k(t,p) - k(t',p')| &\leq \tilde{K_n} |\sigma(t',p')[\mu(t,p) - R] - \sigma(t,p)[\mu(t',p') - R]| \\ &\leq \tilde{K_n} R |\sigma(t,p) - \sigma(t',p')| + \tilde{K_n} |\sigma(t',p')\mu(t,p) - \sigma(t,p)\mu(t',p')| \\ &\leq \tilde{K_n} R |\sigma(t,p) - \sigma(t',p')| + \tilde{K_n} |\sigma(t',p')\mu(t,p) - \sigma(t',p')\mu(t',p')| \\ &\quad + \tilde{K_n} \mu(t',p') |\sigma(t',p') - \sigma(t,p)| \end{aligned}$$

and, observing that any Lipschitz-continuous function on a bounded domain is also bounded, the result is a consequence of our hypotheses. □

Corollary 8.1. *Let us assume that $\mu(t,p), \sigma(t,p)$ are bounded, continuous in $(t,p) \in [0,T] \times (0,+\infty)$ and Lipschitz-continuous in $p \in \overline{D_n}$, uniformly in $t \in [0,T]$. In addition, let $\sigma(t,p)$ be Lipschitz-continuous in $t \in [0,T]$ and locally bounded from below, i.e. $\exists\, \delta_\sigma(n) > 0$ such that $\sigma(t,p) \geq \delta_\sigma(n)$ $\forall (t,p) \in [0,T] \times D_n$. Then $g(t,p)$ given in* (6.15) *solves the Cauchy problem* (6.3) *and there exists a unique classical solution to* (6.3). *Moreover, we have that $g \in \mathcal{C}^{1,2}((0,T) \times (0,+\infty))$.*

Proof. As discussed in [Heath and Schweizer, 2000] after Theorem 1, the main hypothesis of our Theorem 8.1 is implied by the combination of the following conditions:

- there exists a sequence of bounded sets $\{D_n\}_{n \in \mathbb{N}}$, with $D_n \subseteq (0,+\infty)$ $\forall n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} D_n = (0,+\infty)$; in our case we define $D_n \doteq (\frac{1}{n}, n)$;

- pR and $\sigma(t,p)p$ are uniformly Lipschitz-continuous in $(t,p) \in [0,T] \times \overline{D_n}$; it is implied by our hypotheses;

- $\sigma(t,p)$ is bounded from below, i.e. there exists a constant $\delta_\sigma > 0$ such that $\sigma(t,p) \geq \delta_\sigma$ for all $(t,p) \in [0,T] \times (0,+\infty)$, which is fulfilled by hypothesis in this corollary;

- the integrand function in (6.15) is Hölder-continuous on $[0,T] \times \overline{D_n}$; this is implied by Lemma 8.1, whose assumptions are clearly satisfied here;

- the function $g(t,p)$ defined by equation (6.15) is finite and continuous on $(0,T) \times \partial D_n \cup \{T\} \times \bar{D_n}$.

We only need to check the last requirement. By [Heath and Schweizer, 2000, Lemma 2], it is sufficient to prove that the function $k(t,p)$ given in (8.1), i.e. the integrand of (6.15), is continuous and bounded. By the continuity of $\mu(t,p), \sigma(t,p)$, we get the former requirement. Moreover, k is bounded because $\mu(t,p), \sigma(t,p)$ are bounded and $\sigma(t,p)$ is bounded from below. $\square$

Now we turn the attention to the second PDE involved in the reinsurance-investment problem, see the Cauchy problem (6.2). Before proving the existence theorem, let us state some preliminary results.

Lemma 8.2. *Given a compact set $K \in \mathbb{R}$ let us assume that $H(t,y,u) : [0,T] \times \mathbb{R} \times K \to \mathbb{R}$ is continuous in $u \in K$ and Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ uniformly in $u \in K$ with exponent $0 < \xi \leq 1$. Then $\max_{u \in K} H(t,y,u)$ is Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ with exponent $0 < \xi \leq 1$.*

Proof. Given $t, t' \in [0,T]$ and $y, y' \in \mathbb{R}$, let us define

$$h_1(u) = H(t,y,u) \qquad h_2(u) = H(t',y',u).$$

Then we have that

$$|\max_{u \in K} h_1(u) - \max_{u \in K} h_2(u)| \leq \max_{u \in K} |h_1(u) - h_2(u)|. \tag{8.2}$$

In fact, observing that

$$|\max_{u \in K} h_1(u) - \max_{u \in K} h_2(u)| = \begin{cases} \max_{u \in K} h_1(u) - \max_{u \in K} h_2(u) & \text{if} \quad \max_{u \in K} h_1(u) \geq \max_{u \in K} h_2(u) \\ \max_{u \in K} h_2(u) - \max_{u \in K} h_1(u) & \text{if} \quad \max_{u \in K} h_1(u) < \max_{u \in K} h_2(u), \end{cases}$$

we notice that in the first case

$$\begin{aligned} \max_{u \in K} h_1(u) - \max_{u \in K} h_2(u) &= \max_{u \in K} [h_1(u) - h_2(u) + h_2(u)] - \max_{u \in K} h_2(u) \\ &\leq \max_{u \in K} [h_1(u) - h_2(u)] \\ &\leq \max_{u \in K} |h_1(u) - h_2(u)|, \end{aligned}$$

and in the second case we have that

$$\begin{aligned} \max_{u \in K} h_2(u) - \max_{u \in K} h_1(u) &\leq \max_{u \in K} [h_2(u) - h_1(u)] \\ &\leq \max_{u \in K} |h_1(u) - h_2(u)|. \end{aligned}$$

Now, using inequality (8.2), we have that

$$\begin{aligned} |\max_{u \in K} H(t,y,u) - \max_{u \in K} H(t',y',u)| &\leq \max_{u \in K} |H(t,y,u) - H(t',y',u)| \\ &\leq L(|t-t'|^\xi + |y-y'|^\xi) \end{aligned}$$

and this completes the proof. $\square$

Corollary 8.2. *Let us assume that the following hypotheses hold:*

- *$q(t,y,u)$ is bounded and Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ uniformly in $u \in [0,1]$ with exponent $0 < \xi \leq 1$;*
- *$\lambda(t,y)$ is bounded and Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ with exponent $0 < \xi \leq 1$.*

Then $\max_{u(t,y) \in [0,1]} \Psi^u(t,y)$ is Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ with exponent $0 < \xi \leq 1$.

Proof. In view of Lemma 8.2, it is sufficient to show that $\Psi^u(t,y)$ is Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ uniformly in $u \in [0,1]$ with exponent $0 < \xi \leq 1$. Let us recall equation (3.4):

$$\Psi^u(t,y) = -\eta e^{R(T-t)} q(t,y,u) + \lambda(t,y) \int_0^D \left[1 - e^{\eta(1-u)ze^{R(T-t)}}\right] dF_Z(z).$$

Since $e^{R(T-t)}$ is differentiable and bounded on $t \in [0,T]$, our first hypothesis ensures that the first term $\eta e^{R(T-t)} q(t,y,u)$ is Hölder-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ uniformly in $u \in [0,1]$ with exponent $0 < \xi \leq 1$. For the second term we notice that it is a product of two bounded and Hölder-continuous functions, in fact

$$\begin{aligned}&\left|\int_0^D e^{\eta(1-u)ze^{R(T-t)}}\, dF_Z(z) - \int_0^D e^{\eta(1-u)ze^{R(T-t')}}\, dF_Z(z)\right| \\ &\leq \mathbb{E}\left[\left|e^{\eta(1-u)Ze^{R(T-t)}} - e^{\eta(1-u)Ze^{R(T-t')}}\right|\right].\end{aligned}$$

Using Lagrange's theorem, there exists $\bar{t} \in [0,T]$ such that

$$\begin{aligned}&\mathbb{E}\left[\left|e^{\eta(1-u)Ze^{R(T-t)}} - e^{\eta(1-u)Ze^{R(T-t')}}\right|\right] \\ &\leq \mathbb{E}\left[\left|R\eta(1-u)Ze^{R(T-\bar{t})} e^{\eta(1-u)Ze^{R(T-\bar{t})}}\right|\right]|t-t'| \\ &\leq R\eta e^{RT}\mathbb{E}\left[Ze^{\eta Z e^{RT}}\right]|t-t'|\end{aligned}$$

and the proof is complete. □

The following theorem is based on the main result of [Heath and Schweizer, 2000].

Theorem 8.2. *Suppose that for any $n \in \mathbb{N}$ the following PDE*

$$\begin{cases} -\dfrac{\partial w}{\partial t}(t,y) - b(t,y)\dfrac{\partial w}{\partial y}(t,y) - \dfrac{1}{2}\gamma(t,y)^2\dfrac{\partial^2 w}{\partial y^2}(t,y) \\ +\left[\eta e^{R(T-t)}c(t,y) + \max\limits_{u(t,y)\in[0,1]} \Psi^u(t,y)\right] w(t,y) = 0 \qquad \forall (t,y) \in (0,T) \times D_n \\ w(t,y) = f(t,y) \qquad \forall (t,y) \in (0,T) \times \partial D_n \cup \{T\} \times D_n, \end{cases}$$

with $D_n \doteq (-n,n)$, admits a classical solution $w_n(t,y)$. Then the function $f(t,y)$ defined in (6.14) *satisfies the Cauchy problem* (6.2) *and there exists a unique classical solution to* (6.2). *Moreover, we have that $f \in \mathcal{C}^{1,2}((0,T) \times \mathbb{R})$.*

Proof. By Assumption 2.1, the stochastic process $\{Y_t\}_{t\in[0,T]}$ does not explode and the expectaion (6.14) is well defined.

Now fix arbitrary $(t,y) \in (0,T) \times \mathbb{R}$. Since $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{R}$ by construction, $\exists n \in \mathbb{N}$ such that $y \in D_n$. Let us denote by $\tau_n \doteq \inf\{s \geq t \mid Y_{t,y}(s) \notin D_n\} \wedge T$ the first exit time from D_n before T. We have that $(\tau_n, Y_{t,y}(\tau_n)) \in (0,T) \times \partial D_n \cup \{T\} \times D_n$, because of the continuity of $\{Y_{t,y}(s)\}_{s\in[t,T]}$. It turns out that $f(\tau_n, Y_{t,y}(\tau_n)) < \infty$.

By a simple application of the Itô's formula to the process $\{w_n(s, Y_{t,y}(s))\}_{s\in[t,T]}$, using the PDE above, we can verify that

$$w_n(t,y) = \mathbb{E}\left[e^{-\int_t^{\tau_n}\left(\eta e^{R(T-s)}c(s,Y_s) + \Psi^{u^*}(s,Y_s)\right)ds} \mid Y_t = y\right].$$

Using the Markov property of $\{Y_t\}_{t\in[0,T]}$ and the expression of $f(t,y)$ given in (6.14), it immediately follows that

$$w_n(t,y) = f(t,y) \qquad \forall (t,y) \in (0,T) \times D_n.$$

The boundary condition is evident and this completes the proof. □

Similarly to Theorem 8.1, even in Theorem 8.2 the main assumption is not so clear. In practice, it may be difficult to check whether the solution w_n exists or not. Nevertheless, in the following result we provide sufficient conditions in order to guarantee that it is fulfilled.

Corollary 8.3. *Suppose that $c(t,y)$ and $\lambda(t,y)$ are bounded and Lipschitz-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ and $q(t,y,u)$ is bounded and Lipschitz-continuous in $(t,y) \in [0,T] \times \mathbb{R}$ uniformly in $u \in [0,1]$. In addition, let us assume that $\gamma(t,y)$ is bounded from below, i.e. there exists a constant $\delta_\gamma > 0$ such that $\gamma(t,y) \geq \delta_\gamma$ for all $(t,y) \in [0,T] \times \mathbb{R}$. Then $f(t,y)$ defined in (6.14) satisfies the Cauchy problem (6.2) and there exists a unique classical solution to (6.2). Moreover, we have that $f \in \mathcal{C}^{1,2}((0,T) \times \mathbb{R})$.*

Proof. As discussed in [Heath and Schweizer, 2000] after Theorem 1, the main hypothesis of our Theorem 8.2 is implied by the combination of these conditions:

- there exists a sequence of bounded sets $\{D_n\}_{n\in\mathbb{N}}$, with $D_n \subseteq \mathbb{R}\ \forall n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{R}$; in our case we define $D_n \doteq (-n,n)$;
- $b(t,y), \gamma(t,y)$ satisfy the (local) Lipschitz continuity given in our Assumption 2.1;
- $\gamma(t,y)$ is bounded from below; this is true by hypothesis in this corollary;
- the integrand function in (6.15) is Hölder-continuous on $[0,T] \times \overline{D_n}$; this is implied by Lemma 8.1, whose assumption are fulfilled by hypotheses here;
- the function $f(t,y)$ defined by equation (6.14) is finite and continuous on $(0,T) \times \partial D_n \cup \{T\} \times \bar{D}_n$.

In order to complete the proof, we need to check the last requirement. In particular, by the cited [Heath and Schweizer, 2000, Lemma 2], we need to prove that the integrand function of (6.14), i.e.

$$\eta e^{R(T-t)} c(t,y) + \max_{u(t,y)\in[0,1]} \Psi^u(t,y),$$

is continuous and bounded from above. By our hypotheses it is clearly bounded. Moreover, by Corollary 8.2, it is also continuous. □

A. Appendix

Proof of Lemma 2.1. First, let us start considering all the $[0,D]$-indexed processes $\{H(t,z)\}_{t\in[0,T]}$ of this type:

$$H(t,z) = \tilde{H}_t \mathbb{1}_A(z) \qquad t \in [0,T], A \in [0,D],$$

where $\{\tilde{H}_t\}_{t\in[0,T]}$ is a nonnegative and $\{\mathcal{F}_t\}$-predictable process. By the independence between $\{N_t\}_{t\in[0,T]}$ and $\{Z_n\}_{n\geq 1}$ we have that

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \int_0^D H(t,z)\, m(dt,dz)\right] &= \mathbb{E}\left[\sum_{n\geq 1} \tilde{H}_{T_n} \mathbb{1}_A(Z_n) \mathbb{1}_{\{T_n \leq T\}}\right] \\
&= \sum_{n\geq 1} \mathbb{P}[Z_n \in A] \mathbb{E}\left[\tilde{H}_{T_n} \mathbb{1}_{\{T_n \leq T\}}\right] \\
&= \mathbb{P}[Z \in A] \mathbb{E}\left[\sum_{n\geq 1} \tilde{H}_{T_n} \mathbb{1}_{\{T_n \leq T\}}\right] \\
&= \mathbb{P}[Z \in A] \mathbb{E}\left[\int_0^T \tilde{H}_t \lambda_t \, dt\right] \\
&= \mathbb{E}\left[\int_0^D \int_0^T H(t,z)\, dF_Z(z) \lambda_t \, dt\right].
\end{aligned}$$

Using [Brémaud, 1981, App. A1, T4 Theorem, p.263] this result can be extended to all non-negative, $\{\mathcal{F}_t\}$-predictable and $[0, D]$-indexed process $\{H(t, z)\}_{t\in[0,T]}$ and this completes the proof. $\square$

Proof of Proposition 2.1. For any constant strategy $\alpha_t = (u, w)$ with $u \in [0, 1]$ and $w \in \mathbb{R}$ we have that

$$\begin{aligned}
&\mathbb{E}[e^{-\eta X^{\alpha}_{t,x}(T)} \mid \mathcal{F}_t] = \\
&= e^{-\eta x e^{R(T-t)}} \mathbb{E}[e^{-\eta \int_t^T e^{R(T-s)} [c(s,Y_s) - q(s,Y_s,u)]\, ds} e^{\eta \int_t^T \int_0^D e^{R(T-r)}(1-u) z\, m(dr,dz)} \mid \mathcal{F}_t] \times \\
&\times \mathbb{E}[e^{-\eta \int_t^T e^{R(T-s)} w[\mu(s,P_s) - R]\, ds} e^{-\eta \int_t^T e^{R(T-s)} w\sigma(s,P_s)\, dW^{(P)}_s} \mid \mathcal{F}_t],
\end{aligned} \tag{A.1}$$

because of the independence between the financial and the insurance markets. In particular, for the null strategy $\alpha_t = (0, 0)$, using the inequality (2.9), we have that

$$\begin{aligned}
&\mathbb{E}[e^{-\eta X^{(0,0)}_{t,x}(T)} \mid \mathcal{F}_t] \le \\
&\le e^{-\eta x e^{R(T-t)}} e^{\eta \frac{K}{R}(e^{R(T-t)}-1)} \mathbb{E}[e^{\eta \int_t^T \int_0^D e^{R(T-r)} z\, m(dr,dz)} \mid \mathcal{F}_t].
\end{aligned}$$

Now let us notice that

$$\begin{aligned}
\mathbb{E}[e^{\eta e^{RT} \int_t^T \int_0^D z\, m(dr,dz)} \mid \mathcal{F}_t] &= \mathbb{E}[e^{\eta e^{RT} \sum_{i=N_t}^{N_T} Z_i} \mid \mathcal{F}_t] \\
&= \sum_{n \ge N_t} \mathbb{E}[e^{\eta e^{RT} \sum_{i=N_t}^{n} Z_i} \mid \mathcal{F}_t]\, \mathbb{P}[N_T = n \mid \mathcal{F}_t] \\
&= \sum_{n \ge N_t} \mathbb{E}\Big[\prod_{i=N_t}^{n} e^{\eta e^{RT} Z_i} \mid \mathcal{F}_t\Big]\, \mathbb{P}[N_T = n \mid \mathcal{F}_t] \\
&= \sum_{n \ge N_t} \mathbb{E}[e^{\eta e^{RT} Z} \mid \mathcal{F}_t]^{(n-N_t)}\, \mathbb{P}[N_T = n \mid \mathcal{F}_t] \\
&= \sum_{n \ge 0} \mathbb{E}[e^{\eta e^{RT} Z}]^n\, \mathbb{P}[N_T - N_t = n \mid \mathcal{F}_t] \\
&= \sum_{n \ge 0} \mathbb{E}[e^{\eta e^{RT} Z}]^n\, \mathbb{E}\Big[\frac{\big(\int_t^T \lambda_s\, ds\big)^n}{n!} e^{-\int_t^T \lambda_s\, ds} \mid \mathcal{F}_t\Big] \\
&= \mathbb{E}\big[e^{(\mathbb{E}[e^{\eta e^{RT} Z}]-1) \int_t^T \lambda_s\, ds} \mid \mathcal{F}_t\big] < \infty \qquad \langle \mathbb{P} = 1 \rangle,
\end{aligned}$$

because of the Assumption 2.2. $\square$

Proof of Lemma 3.1. Looking at (2.1), (2.12) and (2.16), we apply Itô's formula to the stochastic process $f(t, X^{\alpha}_t, Y_t, P_t)$:

$$f(t, X^{\alpha}_t, Y_t, P_t) = f(0, X^{\alpha}_0, Y_0, P_0) + \int_0^t \mathcal{L}^{\alpha} f(s, X^{\alpha}_s, Y_s, P_s)\, ds + m_t,$$

where

$$\begin{aligned}
m_t = &\int_0^t w_s \sigma(s, P_s) \frac{\partial f}{\partial x}(s, X^{\alpha}_s, Y_s, P_s)\, dW^{(P)}_s \\
&+ \int_0^t P_s \sigma(s, P_s) \frac{\partial f}{\partial p}(s, X^{\alpha}_s, Y_s, P_s)\, dW^{(P)}_s + \int_0^t \gamma(s, Y_s) \frac{\partial f}{\partial y}(s, X^{\alpha}_s, Y_s, P_s)\, dW^{(Y)}_s \\
&+ \int_0^D \int_0^t \Big[f(s, X^{\alpha}_s - (1-u)z, Y_s, P_s) - f(s, X^{\alpha}_s, Y_s, P_s)\Big]\Big(m(ds, dz) - \lambda(s, Y_s)\, dF_Z(z)\Big).
\end{aligned} \tag{A.2}$$

We only need to prove that this is an $\{\mathcal{F}_t\}$-martingale. Let us observe that

$$\mathbb{E}\left[\int_0^T \left(w_s\sigma(s,P_s)\frac{\partial f}{\partial x}(s,X_s^\alpha,Y_s,P_s)\right)^2 ds\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T \left(P_s\sigma(s,P_s)\frac{\partial f}{\partial p}(s,X_s^\alpha,Y_s,P_s)\right)^2 ds\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T \left(\gamma(s,Y_s)\frac{\partial f}{\partial y}(s,X_s^\alpha,Y_s,P_s)\right)^2 ds\right] < \infty,$$

because all the partial derivatives are bounded and using the definition of the set $\mathcal{U}$, (2.13) and (2.3), respectively.
Thus the first three integrals in (A.2) are well defined and, according to the Itô integral theory, they are martingales. Finally, the jump term in (A.2) is a martingale too, being the function f bounded. $\square$

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