On the time-dependent Cattaneo law in space dimension one

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ABSTRACT

We consider the one-dimensional wave equation

$$\varepsilon u_{tt} - u_{xx} + [1 + \varepsilon f'(u)]u_t + f(u) = h$$

where $\varepsilon = \varepsilon(t)$ is a decreasing function vanishing at infinity, providing a model for heat conduction of Cattaneo type with thermal resistance decreasing in time. Within the theory of processes on time-dependent spaces, we prove the existence of an invariant time-dependent attractor, which converges in a suitable sense to the attractor of the classical Fourier equation

$$u_t - u_{xx} + f(u) = h$$

formally arising in the limit $t \to \infty$.

1. The physical model

Let $\Omega = (0, L)$ be a finite interval. For $\tau \in \mathbb{R}$, the thermal evolution in a homogenous isotropic rigid heat conductor occupying the space–time cylinder $\Omega_\tau = \Omega \times (\tau, \infty)$ is governed by the balance equation

$$e_t + q_x = g,$$

where the *internal energy* $e = e(x, t)$ is a function of the *relative temperature* $u : \Omega_\tau \to \mathbb{R}$, with respect to an equilibrium reference value, while $q : \Omega_\tau \to \mathbb{R}$ is the *heat flux*. The forcing term g is taken of the form

$$g(x, t) = -f(u(x, t)) + h(x)$$

for some $f : \mathbb{R} \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$, where $f(u)$ represents a nonlinearly temperature-dependent internal source, and h is an external time-independent heat supply. We also assume Dirichlet boundary conditions

$$u(0, t) = u(L, t) = 0$$

expressing the fact that the boundaries of the conductor are kept at null (i.e. equilibrium) temperature for all times. Considering only small variations of the temperature and its gradient, the internal energy fulfills with good approximation the equality

$$e(x, t) = e_0(x) + \alpha u(x, t),$$

where e_0 is the internal energy at equilibrium and $\alpha > 0$ is the *specific heat*. Accordingly, the balance equation becomes

$$\alpha u_t + q_x + f(u) = h. \tag{1.1}$$

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In order to obtain an evolution equation for the temperature, a further ingredient is needed: the so-called constitutive law for the heat flux, establishing a link between q and u. Here, we consider a differential perturbation of the classical Fourier law

$$q + \kappa u_x = 0, \quad \kappa > 0,$$

namely, the Cattaneo law [2] given by

$$q + \varepsilon q_t + \kappa u_x = 0, \quad \kappa \gg \varepsilon > 0.$$

In which case, the sum (1.1) $+\varepsilon\partial_t$(1.1) entails the hyperbolic reaction–diffusion equation

$$\varepsilon\alpha u_{tt} - \kappa u_{xx} + [\alpha + \varepsilon f'(u)]u_t + f(u) = h, \tag{1.2}$$

widely employed in the description of many interesting phenomena, such as chemical reacting systems, gene selection, population dynamics or forest fire propagation (see e.g. [9,17,18]). In the case when ε is a positive constant, the asymptotic behavior of solutions to Eq. (1.2) has been studied in [10,14–16,19,20] where the authors deal with the weakly damped wave equation with a general displacement dependent damping

$$u_{tt} - \Delta u + \sigma(u)u_t + f(u) = h,$$

in space dimensions one, two and three (see also [3,11] for the case of equations with memory). In particular, in dimension one, the problem is well-known to generate a strongly continuous semigroup $S(t)$ on the phase space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$$

and the existence of the compact global attractor of optimal regularity is achieved (under suitable assumptions) in the classical framework of semigroup theory.

The main feature of this work is that we allow ε to be a function of time. More precisely, we assume that ε is a (positive) decreasing function vanishing as $t \to \infty$. Accordingly, the Cattaneo law collapses into the Fourier law in the longtime. From the point of view of the dynamics, this means that Eq. (1.2) eventually loses its hyperbolic character, becoming more and more similar to the classical reaction–diffusion equation

$$\alpha u_t - \kappa u_{xx} + f(u) = h. \tag{1.3}$$

Since the delay effects predicted by the Cattaneo law represent the answer of the material to external solicitations, our model tries to describe a heat conductor whose thermal resistance decreases progressively. Our aim is to study the longtime behavior of the solutions to (1.2) with ε depending on time. This is done according to the abstract framework of [5,6], specifically developed to deal with evolution problems where the coefficients of the differential operators depend explicitly on time.

1.1. Outline of the paper

In the next section, we give the basic assumptions on the nonlinearity f and on the relaxation function $\varepsilon = \varepsilon(t)$, and we provide the suitable mathematical framework. Section 3 contains the well-posedness result, showing that the equation generates a Lipschitz continuous process $U(t,\tau)$ acting on a family of time-dependent spaces $\{\mathcal{H}_t\}$. The dissipativity of $U(t,\tau)$ is discussed in Section 4, while Section 5 is devoted to our main Theorem 5.2, where we prove the existence of a time-dependent global attractor for $U(t,\tau)$. Loosely speaking, this is the smallest family $\{A_t\}$, where $A_t \subset \mathcal{H}_t$ for every t, which attracts bounded subsets in a *pullback* way. Besides, the attractor turns out to be invariant and of optimal regularity, so it is a suitable object to describe the regime behavior of the system. In Section 6, by exploiting an abstract result from [5], we prove the closeness (in terms of Hausdorff semidistance) between A_t and the classical global attractor A_∞ associated with the limit Eq. (1.3) as $t \to \infty$. This somehow tells that (1.3) provides an accurate description of the asymptotic dynamics of the solutions to (1.2), and gives a rigorous mathematical meaning to the formal collapse of the time-dependent Cattaneo model (1.2) into the Fourier equation (1.3) in the longtime. Section 7 establishes the uniform boundedness of the t-sections A_t of the attractor under a mild restriction on the derivative of ε. The final Section 8 contains some concluding remarks.

1.2. Notation

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm on the (real) Hilbert space $H = L^2(\Omega)$. We will also consider the spaces $V = H_0^1(\Omega)$ and $W = H^2(\Omega) \cap H_0^1(\Omega)$. Due to the Poincaré inequality, we set

$$\|u\|_V = \|u_x\| \quad \text{and} \quad \|u\|_W = \|u_{xx}\|.$$

In particular, since we are in space dimension 1, we have the continuous embedding

$$V \subset L^\infty(\Omega). \tag{1.4}$$

Finally, we define the time-dependent product spaces

$$\mathcal{H}_t = V \times H \quad \text{and} \quad \mathcal{V}_t = W \times V$$

normed by

$$\|\{a,b\}\|^2_{\mathcal{H}_t} = \|a_x\|^2 + \varepsilon(t)\|b\|^2 \quad \text{and} \quad \|\{a,b\}\|^2_{\mathcal{V}_t} = \|a_{xx}\|^2 + \varepsilon(t)\|b_x\|^2.$$

Since ε is a (strictly) positive function, $\mathcal{H}_t$ and $\mathcal{V}_t$ are Hilbert spaces for every fixed t. For every $R \geqslant 0$ and every $t \in \mathbb{R}$, we denote

$$\mathbb{B}_t(R) = \left\{ z \in \mathcal{H}_t : \|z\|_{\mathcal{H}_t} \leqslant R \right\}.$$

Throughout the paper, $C \geqslant 0$ will stand for a *generic* positive constant and $\mathcal{Q}(\cdot)$ for a *generic* increasing positive function, depending only on Ω and on the parameters of the problem (in particular, independent of time).

2. The Cauchy problem

Setting for simplicity $\alpha = \kappa = 1$, let then $\varepsilon = \varepsilon(t)$ be a function of time whose properties will be specified in a while. Denoting

$$\sigma_\varepsilon(u) = 1 + \varepsilon f'(u),$$

we consider the one-dimensional problem in the unknown variable $u = u(x,t) : \Omega_\tau \to \mathbb{R}$

$$\begin{cases} \varepsilon u_{tt} - u_{xx} + \sigma_\varepsilon(u) u_t + f(u) = h, \quad t > \tau, \\ u(0,t) = u(L,t) = 0, \\ u(\tau) = u_0, \\ u_t(\tau) = v_0, \end{cases} \tag{2.1}$$

where $u_0, v_0 : \Omega \to \mathbb{R}$ are given data. Here, $h \in L^2(\Omega)$ is a time-independent external source, while the nonlinearity $f \in \mathcal{C}^2(\mathbb{R})$, with $f(0) = 0$, satisfies the dissipation condition

$$\liminf_{|u| \to \infty} f'(u) > -\lambda_1, \tag{2.2}$$

$\lambda_1 > 0$ being the first eigenvalue of the Dirichlet operator $-\partial_{xx}$. In particular, condition (2.2) implies the following inequalities:

$$2\langle F(u), 1 \rangle \geqslant -(1-\nu)\|u_x\|^2 - C, \tag{2.3}$$

$$2\langle f(u), u \rangle \geqslant 2\langle F(u), 1 \rangle - (1-\nu)\|u_x\|^2 - C \tag{2.4}$$

for some $0 < \nu < 1$, having defined the primitive

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

Besides, (2.2) ensures that $f'(u)$ is bounded below, namely,

$$f'(u) \geqslant -\ell, \tag{2.5}$$

where

$$\ell = \max\left\{0, -\inf_{u \in \mathbb{R}} f'(u)\right\} < \infty.$$

2.1. Assumptions on $\boldsymbol{\varepsilon}$

Let $\varepsilon \in \mathcal{C}^1(\mathbb{R}^+)$ be a nonincreasing bounded function with bounded derivative satisfying

$$\lim_{t \to -\infty} \varepsilon(t) = \varepsilon_\infty \quad \text{and} \quad \lim_{t \to \infty} \varepsilon(t) = 0.$$

In particular,

$$\sup_{t \in \mathbb{R}} |\varepsilon(t)| \leqslant \varepsilon_\infty \quad \text{and} \quad \sup_{t \in \mathbb{R}} |\varepsilon'(t)| < \infty. \tag{2.6}$$

Besides, we assume that the supremum of ε cannot exceed a limit value related to the form of the nonlinearity, namely,[1]

$$\varepsilon_\infty < \frac{1}{\ell}.$$

[1] Taking into account the physical constants as in (1.2), the condition reads $\varepsilon_\infty < \frac{\alpha}{\ell}$.

In the case when $\ell = 0$, the condition simply means $\varepsilon_\infty < \infty$. Hence, recalling (2.5) and defining the strictly positive constant $\varkappa = 1 - \ell\varepsilon_\infty$, the displacement-dependent damping coefficient $\sigma_\varepsilon(u)$ satisfies

$$\inf_{u\in\mathbb{R}} \sigma_\varepsilon(u) \geqslant \varkappa. \tag{2.7}$$

3. The process on time-dependent spaces

By means of a standard Galerkin scheme, the initial/boundary value problem (2.1) is shown to possess a unique (weak energy) solution. In particular, for every initial datum $z \in \mathcal{H}_\tau$, the solution $Z(t) \in \mathcal{H}_t$ at time $t \geqslant \tau$ can be written in the form

$$Z(t) = \{u(t), u_t(t)\} = U(t,\tau)z,$$

where $U(t,\tau)$ is a *strongly continuous process* on time-dependent spaces (see [6–8]), namely, a two-parameter family of operators depending on $t \geqslant \tau \in \mathbb{R}$ satisfying the following properties:

(i) $U(\tau,\tau)$ is the identity map on $\mathcal{H}_\tau$ for every τ;
(ii) $U(t,\tau)U(\tau,s) = U(t,s)$ for every $t \geqslant \tau \geqslant s$;
(iii) $U(t,\tau) \in \mathcal{C}(\mathcal{H}_\tau, \mathcal{H}_t)$ for every $t \geqslant \tau$.

The next propositions, which are actually part of the proof of the well-posedness result, specify some properties of the process.

Proposition 3.1. *For every $R \geqslant 0$ and every $t \geqslant \tau \in \mathbb{R}$ we have the estimate*

$$\sup_{z\in\mathbb{B}_\tau(R)} \left[\|U(t,\tau)z\|_{\mathcal{H}_t} + \int_\tau^\infty \|u_t(y)\|^2 \,\mathrm{d}y \right] \leqslant \mathcal{Q}(R). \tag{3.1}$$

Proof. For $Z(t) = \{u(t), u_t(t)\} \in \mathcal{H}_t$ define the energy functional $\mathcal{E} = \mathcal{E}(t)$ by

$$\mathcal{E} = \|Z\|^2_{\mathcal{H}_t} + 2\langle F(u), 1\rangle - 2\langle h, u\rangle. \tag{3.2}$$

Since F is continuous, the embedding (1.4) gives

$$2\langle F(u), 1\rangle \leqslant \mathcal{Q}(\|u_x\|).$$

Besides, as $h \in H$,

$$2\langle h, u\rangle \leqslant \frac{\nu}{2}\|u_x\|^2 + C.$$

Hence, recalling (2.3), we end up with

$$\frac{\nu}{2}\|Z\|^2_{\mathcal{H}_t} - C \leqslant \mathcal{E} \leqslant \mathcal{Q}(\|Z\|_{\mathcal{H}_t}). \tag{3.3}$$

A multiplication of the first equation of (2.1) by $2u_t$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + 2\langle \sigma_\varepsilon(u)u_t, u_t\rangle - \varepsilon'\|u_t\|^2 = 0$$

and using (2.7) along with the fact that $\varepsilon' \leqslant 0$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + 2\varkappa\|u_t\|^2 \leqslant 0. \tag{3.4}$$

An integration of (3.4) on (τ, t), with t arbitrarily large, and a subsequent exploitation of (3.3) establish the claim.

Proposition 3.2. *The continuous dependence estimate*

$$\|U(t,\tau)z_1 - U(t,\tau)z_2\|_{\mathcal{H}_t} \leqslant \mathrm{e}^{\mathcal{Q}(R)(t-\tau)}\|z_1 - z_2\|_{\mathcal{H}_\tau}$$

holds for every $R \geqslant 0$ and every pair of initial data $z_i \in \mathcal{H}_\tau$ such that $\|z_i\|_{\mathcal{H}_\tau} \leqslant R$.

Proof. Let us call $\{u_i(t), \partial_t u_i(t)\} = U(t,\tau)z_i$ and denote by $\bar{Z}(t) = \{\bar{u}(t), \bar{u}_t(t)\}$ their difference. Then,

$$\varepsilon\bar{u}_{tt} + \bar{u}_t + \varepsilon[f'(u_1)\partial_t u_1 - f'(u_2)\partial_t u_2] - \bar{u}_{xx} + f(u_1) - f(u_2) = 0.$$

Multiplying by $2\bar{u}_t$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{Z}\|^2_{\mathcal{H}_t} + (2-\varepsilon')\|\bar{u}_t\|^2 = -2\langle f(u_1)-f(u_2),\bar{u}_t\rangle - 2\varepsilon\langle f'(u_1)\partial_t u_1 - f'(u_2)\partial_t u_2,\bar{u}_t\rangle.$$

To estimate the right-hand side, we lean on the continuous embedding (1.4). We have

$$-2\langle f(u_1)-f(u_2),\bar{u}_t\rangle \leqslant C\|\bar{u}_x\|\|\bar{u}_t\| \leqslant 2\|\bar{u}_t\|^2 + C\|\bar{u}_x\|^2,$$

while, observing that

$$\langle f'(u_1)\partial_t u_1 - f'(u_2)\partial_t u_2,\bar{u}_t\rangle = \langle f'(u_1)\bar{u}_t,\bar{u}_t\rangle + \langle [f'(u_1)-f'(u_2)]\partial_t u_2,\bar{u}_t\rangle,$$

we get

$$-2\varepsilon\langle f'(u_1)\bar{u}_t,\bar{u}_t\rangle \leqslant C\varepsilon\|\bar{u}_t\|^2.$$

To control the remaining term, we recall that $\varepsilon\|\partial_t u_2\|^2 \leqslant C$. Hence

$$-2\varepsilon\langle [f'(u_1)-f'(u_2)]\partial_t u_2,\bar{u}_t\rangle| \leqslant C\varepsilon\|\partial_t u_2\|\|\bar{u}_x\|\|\bar{u}_t\| \leqslant C\sqrt{\varepsilon}\|\bar{u}_x\|\|\bar{u}_t\| \leqslant \varepsilon\|\bar{u}_t\|^2 + C\|\bar{u}_x\|^2.$$

In summary, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{Z}(t)\|^2_{\mathcal{H}_t} \leqslant C\|\bar{Z}(t)\|^2_{\mathcal{H}_t}.$$

The sought estimate follows from an application of the Gronwall lemma on $[\tau,t]$.

4. Dissipativity

We now discuss the dissipation properties of the process $U(t,\tau)$ associated with (2.1). We first recall some definitions from [6].

Definition 4.1. A family of sets $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$, with $B_t \subset \mathcal{H}_t$, is said to be *uniformly bounded* if

$$\sup_{t\in\mathbb{R}}\|B_t\|_{\mathcal{H}_t} < \infty.$$

Definition 4.2. A family $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$ is called a *time-dependent absorbing set* if it is uniformly bounded and, for every $R>0$, there exists an entering time $t_{\mathrm{e}} = t_{\mathrm{e}}(R) \geqslant 0$ such that

$$t \geqslant \tau + t_{\mathrm{e}} \quad \Rightarrow \quad U(t,\tau)\mathbb{B}_\tau(R) \subset B_t.$$

We can now state the main result of this section.

Theorem 4.3. *There exists a constant $R_0>0$ such that the family of R_0-balls*

$$\mathfrak{B}_0 = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$$

is a time-dependent absorbing set for the process $U(t,\tau)$.

Proof. Let $\tau\in\mathbb{R}$ and $R>0$ be fixed. For $z\in\mathbb{B}_\tau(R)$, we denote as usual

$$Z(t) = \{u(t),u_t(t)\} = U(t,\tau)z.$$

For $\delta>0$ to be fixed later, define the functional

$$\Lambda(t) = \mathcal{E}(t) + 2\delta\varepsilon(t)\langle u(t),u_t(t)\rangle,$$

where $\mathcal{E}$ is the energy functional given by (3.2). Recalling (3.3), it is easy to see that, for δ small enough,

$$\frac{\nu}{4}\|Z\|^2_{\mathcal{H}_t} - C \leqslant \Lambda \leqslant \mathcal{Q}(\|Z\|_{\mathcal{H}_t}). \tag{4.1}$$

Indeed, in light of (2.6) and the Poincaré inequality, if δ is small enough we have

$$2\delta\varepsilon|\langle u,u_t\rangle| \leqslant \frac{\nu}{4}\|u_x\|^2 + \frac{\nu\varepsilon}{4}\|u_t\|^2 = \frac{\nu}{4}\|Z\|^2_{\mathcal{H}_t}.$$

Then, multiplying the first equation of (2.1) by $2\delta u$ and adding the resulting equality to (3.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + 2\delta\|u_x\|^2 + 2(\varkappa-\delta\varepsilon)\|u_t\|^2 + 2\delta\langle f(u),u\rangle - 2\delta\langle h,u\rangle \leqslant 2\delta\varepsilon'\langle u,u_t\rangle - 2\delta\langle\sigma_\varepsilon(u)u_t,u\rangle.$$

Exploiting (2.6), (3.1) and the continuous embedding (1.4), the right-hand side is easily controlled by

$$\begin{aligned}2\delta\varepsilon'\langle u,u_t\rangle - 2\delta\langle\sigma_\varepsilon(u)u_t,u\rangle &= -2\delta\varepsilon\langle u,u_t\rangle + 2\delta(\varepsilon'+\varepsilon)\langle u,u_t\rangle - 2\delta\langle\sigma_\varepsilon(u)u_t,u\rangle \\ &\leqslant -2\delta\varepsilon\langle u,u_t\rangle + C\delta\|u\|\|u_t\| + C\delta\|f'(u)u\|_{L^\infty}\|u_t\| \leqslant -2\delta\varepsilon\langle u,u_t\rangle + \delta\mathcal{Q}(R)\|u_t\| \\ &\leqslant -2\delta\varepsilon\langle u,u_t\rangle + \delta\mathcal{Q}(R)\|u_t\|^2 + \delta C.\end{aligned}$$

Besides, we learn from (2.4) that

$$2\delta\langle f(u),u\rangle \geqslant 2\delta\langle F(u),1\rangle - \delta(1-\nu)\|u_x\|^2 - \delta C.$$

Collecting the inequalities above, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \delta(1+\nu)\|u_x\|^2 + \delta\varepsilon\|u_t\|^2 + (2\varkappa - 3\delta\varepsilon - \delta\mathcal{Q}(R))\|u_t\|^2 + 2\delta\langle F(u),1\rangle - 2\delta\langle h,u\rangle + 2\delta\varepsilon\langle u,u_t\rangle \leqslant \delta C.$$

At this point, we choose $\delta = \delta(R)$ small enough such that

$$\delta \leqslant \frac{2\varkappa}{3\varepsilon_\infty + \mathcal{Q}(R)}.$$

This entails the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \delta\Lambda \leqslant \delta C$$

and an application of the Gronwall lemma on $[\tau, t)$ together with (4.1) yield

$$\|Z(t)\|^2_{\mathcal{H}_t} \leqslant \mathcal{Q}(R)\mathrm{e}^{-\delta(t-\tau)} + C, \quad \forall t \geqslant \tau.$$

The proof follows by choosing $R_0 = \sqrt{2C}$.

5. The time-dependent global attractor

5.1. Statement of the result

The main object characterizing the asymptotic behavior of a process $U(t,\tau)$ defined on a time-dependent family of spaces is the *time-dependent attractor* [6–8].

Definition 5.1. The time-dependent global attractor for $U(t,\tau)$ is the smallest family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ with the following properties:

- Each A_t is *compact* in $\mathcal{H}_t$.
- $\mathfrak{A}$ is *pullback attracting*, namely, it is uniformly bounded and the limit[2]

  $$\lim_{\tau\to-\infty} \boldsymbol{\delta}_t(U(t,\tau)C_\tau, A_t) = 0$$

  holds for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ and every $t \in \mathbb{R}$.

Let us review some basic facts about the time-dependent attractor. First of all, $\mathfrak{A}$ exists and it is unique if and only if the set

$$\mathbb{K} = \{\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} : K_t \subset \mathcal{H}_t \text{compact}, \ \mathfrak{K} \text{ pullback attracting}\},$$

is not empty (see [6, Theorem 4.2]). Besides, by [6, Theorem 5.6], the attractor is *invariant*, i.e.

$$U(t,\tau)A_\tau = A_t, \quad \forall t \geqslant \tau,$$

as soon as the process satisfies some continuity properties (in fact much weaker than strong continuity). In particular, when the attractor is invariant, it coincides with the set of all complete bounded trajectories (CBT) of the process $U(t,\tau)$ (see [5, Theorem 3.2]), namely,

[2] We denote the Hausdorff semidistance of two (nonempty) sets $B, C \subset \mathcal{H}_t$ by

$$\boldsymbol{\delta}_t(B,C) = \sup_{x\in B} \mathrm{dist}_{\mathcal{H}_t}(x,C) = \sup_{x\in B}\inf_{y\in C}\|x-y\|_{\mathcal{H}_t}.$$

$$\mathfrak{A} = \{Z : t \mapsto Z(t) = \{u(t), u_t(t)\} \in \mathcal{H}_t \text{ with } Z \text{ } cbt \text{ of } U(t,\tau)\}.$$

Recall that a function $t \mapsto Z(t) \in \mathcal{H}_t$ is a CBT of $U(t,\tau)$ if and only if $\sup_{t\in\mathbb{R}} \|Z(t)\|_{\mathcal{H}_t} < \infty$ and

$$Z(t) = U(t,\tau)Z(\tau), \quad \forall t \geqslant \tau \in \mathbb{R}.$$

The existence of the (invariant) time-dependent global attractor for our problem is stated in the following theorem.

Theorem 5.2. *The process $U(t,\tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ generated by (2.1) possesses the time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$, which is also invariant. Besides, A_t is uniformly bounded in $\mathcal{V}_t$, i.e.*

$$\sup_{t\in\mathbb{R}} \sup_{\{u,u_t\}\in\mathfrak{A}} \Big[\|u_{xx}(t)\|^2 + \varepsilon(t)\|u_{xt}(t)\|^2\Big] < \infty. \tag{5.1}$$

Owing to the discussion above, it suffices to prove the existence of $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$, together with the uniform $\mathcal{V}_t$-bound of A_t. Indeed, from Proposition 3.2, the process $U(t, \tau)$ associated to (2.1) is (locally) Lipschitz continuous, so that the attractor is automatically invariant. Accordingly, we will construct a pullback attracting family of (nonvoid) compact sets, proving that $\mathbb{K} \neq \emptyset$, implying in turn the existence of the (invariant) time-dependent global attractor.

5.2. The decomposition

In order to exhibit a pullback attracting family of compact sets, the strategy is to find a decomposition of the process into the sum of a *decaying* part and of a *compact* one. To this aim, let $\mathfrak{B}_0 = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ be the time-dependent absorbing set of Theorem 4.3, and let $\tau \in \mathbb{R}$ be fixed. Then, for any $z \in \mathbb{B}_\tau(R_0)$, we write $Z(t) = U(t, \tau)z$ as

$$Z(t) = \{u(t), u_t(t)\} = Z_0(t) + Z_1(t),$$

where

$$Z_0(t) = \{v(t), v_t(t)\} \quad \text{and} \quad Z_1(t) = \{w(t), w_t(t)\}$$

solve the systems

$$\begin{cases} \varepsilon v_{tt} + \sigma_\varepsilon(u)v_t - v_{xx} = 0, \\ Z_0(\tau) = z \end{cases} \tag{5.2}$$

and

$$\begin{cases} \varepsilon w_{tt} + \sigma_\varepsilon(u)w_t - w_{xx} + f(u) = h, \\ Z_1(\tau) = 0. \end{cases} \tag{5.3}$$

In what follows, the generic constant C will depend only on $\mathfrak{B}_0$.

In a standard way one can prove the uniform decay of the solutions to (5.2).

Lemma 5.3. *There exists $\delta = \delta(\mathfrak{B}_0) > 0$ such that*

$$\|Z_0(t)\|_{\mathcal{H}_t} \leqslant C e^{-\delta(t-\tau)}, \quad \forall t \geqslant \tau.$$

Proof. Since $\varepsilon' \leqslant 0$, a multiplication of (5.2) by $2v_t + 2\delta v$, with $\delta > 0$ small to be fixed later, leads to

$$\frac{d}{dt}\Lambda_0 + 2(\varkappa - \delta\varepsilon)\|v_t\|^2 + 2\delta\|v_x\|^2 \leqslant 2\delta\varepsilon'\langle v_t, v\rangle - 2\delta\langle\sigma_\varepsilon(u)v_t, v\rangle, \tag{5.4}$$

where

$$\Lambda_0(t) = \|Z_0(t)\|^2_{\mathcal{H}_t} + 2\delta\varepsilon(t)\langle v(t), v_t(t)\rangle.$$

The uniform L^∞-bound for $\sigma_\varepsilon(u)$ and the boundedness of ε' allow to estimate the right hand side in (5.4), for δ small, by $\delta\|v_x\|^2 + \varkappa\|v_t\|^2$. This yields

$$\frac{d}{dt}\Lambda_0 + (\varkappa - 2\delta\varepsilon)\|v_t\|^2 + \delta\|v_x\|^2 \leqslant 0.$$

Up to choosing a possibly smaller $\delta > 0$, we get

$$\frac{1}{2}\|Z_0(t)\|^2_{\mathcal{H}_t} \leqslant \Lambda_0(t) \leqslant 2\|Z_0(t)\|^2_{\mathcal{H}_t}$$

and in turn we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_0 + \delta\Lambda_0 + \delta\|v_t\|^2 \leqslant 0.$$

The claim follows by the Gronwall lemma.

An integration of the latter inequality gives

$$\int_\tau^\infty \left(\|v_x(y)\|^2 + \|v_t(y)\|^2\right)\mathrm{d}y \leqslant C. \tag{5.5}$$

In turn, exploiting (3.1),

$$\int_\tau^\infty \|w_t(y)\|^2\,\mathrm{d}y \leqslant C. \tag{5.6}$$

The role of the uniform integral estimates (5.5) and (5.6) will be crucial to prove the compactness of the solutions to (5.3).

Lemma 5.4. *There exists* $M = M(\mathfrak{B}_0) > 0$ *such that*

$$\sup_{t\geqslant\tau}\|Z_1(t)\|_{\mathcal{V}_t} \leqslant M.$$

Proof. For $t \geqslant \tau$, we set

$$\Lambda_1(t) = \|Z_1(t)\|^2_{\mathcal{V}_t} - 2\delta\varepsilon(t)\langle w_t(t), w_{xx}(t)\rangle - 2\langle f(u(t)), w_{xx}(t)\rangle + 2\langle h, w_{xx}(t)\rangle + c$$

for $\delta > 0$ small and some $c \geqslant 0$ (depending on $\|h\|$) large enough to ensure

$$\frac{1}{4}\|Z_1(t)\|^2_{\mathcal{V}_t} \leqslant \Lambda_1(t) \leqslant 2\|Z_1(t)\|^2_{\mathcal{V}_t} + 2c. \tag{5.7}$$

A multiplication of (5.3) by $-2w_{xxt} - 2\delta w_{xx}$, on account of (2.7) and the fact that $\varepsilon' \leqslant 0$, leads to the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + 2(\varkappa - \delta\varepsilon)\|w_{xt}\|^2 + 2\delta\|w_{xx}\|^2 \leqslant -2\delta\langle h, w_{xx}\rangle - 2\delta\varepsilon'\langle w_t, w_{xx}\rangle + 2\delta\langle\sigma_\varepsilon(u)w_t, w_{xx}\rangle \quad - 2\langle[\sigma_\varepsilon(u)]_x w_t, w_{xt}\rangle$$
$$- 2\langle f'(u)u_t, w_{xx}\rangle + 2\delta\langle f(u), w_{xx}\rangle.$$

By straightforward computations, in light of the control $\|u\|_{L^\infty} \leqslant C$, we have

$$-2\delta\langle h, w_{xx}\rangle - 2\langle f'(u)u_t, w_{xx}\rangle + 2\delta\langle f(u), w_{xx}\rangle \leqslant \frac{\delta}{2}\|w_{xx}\|^2 + C(1 + \|u_t\|^2)$$

and, recalling (2.6),

$$-2\delta\varepsilon'\langle w_t, w_{xx}\rangle + 2\delta\langle\sigma_\varepsilon(u)w_t, w_{xx}\rangle \leqslant \frac{\delta}{2}\|w_{xx}\|^2 + C\delta\|w_{xt}\|^2.$$

We now note that $[\sigma_\varepsilon(u)]_x = \varepsilon f''(u)u_x$. Then, the embedding (1.4) gives

$$\varepsilon\langle f''(u)v_x w_t, w_{xt}\rangle \leqslant \varepsilon\|f''(u)\|_{L^\infty}\|v_x\|\,\|w_t\|_{L^\infty}\|w_{xt}\| \leqslant C\varepsilon\|v_x\|\,\|w_{xt}\|^2.$$

Similarly, by exploiting the boundedness of ε,

$$\varepsilon\langle f''(u)w_x w_t, w_{xt}\rangle \leqslant \varepsilon\|f''(u)\|_{L^\infty}\|w_x\|_{L^\infty}\|w_t\|\|w_{xt}\| \leqslant C\varepsilon\|w_t\|\|w_{xx}\|\|w_{xt}\| \leqslant C\|w_t\|(\|w_{xx}\|^2 + \varepsilon\|w_{xt}\|^2).$$

In summary, we obtain

$$-2\langle[\sigma_\varepsilon(u)]_x w_t, w_{xt}\rangle \leqslant C(\|v_x\| + \|w_t\|)\|Z_1\|^2_{\mathcal{V}_t}.$$

Choosing δ sufficiently small and making use of (5.7), collecting the inequalities above we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + 2\rho\Lambda_1 \leqslant C(\|v_x\| + \|w_t\|)\Lambda_1 + C(1 + \|u_t\|^2)$$

for some $\rho > 0$. Finally, noting that

$$C(\|v_x\| + \|w_t\|)\Lambda_1 \leqslant \rho\Lambda_1 + C(\|v_x\|^2 + \|w_t\|^2)\Lambda_1,$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \rho\Lambda_1 \leqslant C(\|v_x\|^2 + \|w_t\|^2)\Lambda_1 + C(1 + \|u_t\|^2).$$

Due to the uniform integrability of $\|u_t\|^2 + \|w_t\|^2 + \|v_x\|^2$ ensured by (5.5) and (5.6), we can apply a modified (but nowadays quite standard) form of the Gronwall Lemma (see e.g. [4]), yielding

$$\Lambda_1(t) \leqslant C, \quad \forall t \geqslant \tau.$$

By a further use of (5.7), the uniform boundedness of $\|Z_1(t)\|_{\mathcal{V}_t}$ is deduced.

5.3. Proof of Theorem 5.2

We are now in the position to conclude the proof of the existence of the attractor. To this aim, we define for all $t \in \mathbb{R}$

$$K_t = \{z \in \mathcal{V}_t : \|z\|_{\mathcal{V}_t} \leqslant M\}.$$

Then, K_t is compact by the compact embedding $\mathcal{V}_t \Subset \mathcal{H}_t$, and since the embedding constants are independent of t, the family

$$\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$$

is uniformly bounded. Besides, Lemma 5.3 and Lemma 5.4 imply that

$$\lim_{\tau \to -\infty} \boldsymbol{\delta}_t(U(t,\tau)\mathbb{B}_\tau(R_0), K_t) = 0, \quad \forall t \in \mathbb{R}.$$

Since $\mathfrak{B}_0 = \{\mathbb{B}_t(R_0)\}_{t \in \mathbb{R}}$ is a time-dependent absorbing set, this means that $\mathfrak{K}$ is pullback attracting. In other words, we proved that $\mathfrak{K}$ is a nontrivial element of $\mathbb{K}$, and the existence of the attractor $\mathfrak{A}$ immediately follows. Finally, since $\mathfrak{A}$ is the minimal element in the class $\mathbb{K}$, then

$$A_t \subset K_t, \quad \forall t \in \mathbb{R}$$

proving that A_t is bounded in $\mathcal{V}_t$ with a bound independent of $t \in \mathbb{R}$.

6. Asymptotic structure of the attractor

We discuss the relationship between the time-dependent Cattaneo model (2.1) and the reaction–diffusion equation subject to homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t - u_{xx} + f(u) = h, \\ u(0,t) = u(L,t) = 0, \end{cases} \tag{6.1}$$

formally corresponding to (2.1) when $t \to \infty$. We recall that, within our assumptions on f and h, problem (6.1) generates a strongly continuous semigroup

$$S(t) : V \to V$$

possessing the (classical) global attractor A_∞, which is also bounded in W (see e.g. [1,23]). Our result establishes the asymptotic closeness of $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$ to A_∞, being $\mathfrak{A}$ the time-dependent global attractor of the process $U(t,\tau)$ generated by (2.1), in the following sense.

Theorem 6.1. *We have the limit*

$$\lim_{t \to \infty} \boldsymbol{\delta}_V(\Pi_t A_t, A_\infty) = 0,$$

where $\Pi_t A_t$ denotes the projection of A_t into its first component, that is,

$$\Pi_t A_t = \{\xi \in V : (\xi, \eta) \in A_t\}.$$

On account of the abstract [5, Theorem 4.2], the conclusion above is a direct consequence of the next lemma.

Lemma 6.2. *For any sequence $z_n = (u_n, \partial_t u_n)$ of* CBT *of the process $U(t,\tau)$ and any $t_n \to \infty$, there exists a* CBT *w of $S(t)$ such that, for every $T > 0$, the convergence*

$$\sup_{t \in [-T,T]} \|u_n(t + t_n) - w(t)\|_V \to 0 \tag{6.2}$$

holds up to a subsequence as $n \to \infty$.

Proof. The proof can be easily obtained by reasoning as in [5, Lemma 6.2]; hence we limit ourselves to describe the main steps. Let $z_n = (u_n, \partial_t u_n)$ CBT of $U(t,\tau)$ and $t_n \to \infty$ be given. Owing to (5.1), for every $T > 0$

$$u_n(\cdot + t_n) \text{ is bounded in } L^\infty(-T,T;W) \cap W^{1,2}(-T,T;H).$$

By the classical compactness result of Simon [22], $u_n(\cdot+t_n)$ is relatively compact in $\mathcal{C}([-T,T],V)$ for every $T>0$. Hence, there exists $w\in\mathcal{C}(\mathbb{R},V)$ such that (6.2) holds up to a subsequence. In particular, by virtue of (5.1),

$$\sup_{t\in\mathbb{R}}\|w_x(t)\|<\infty. \tag{6.3}$$

In order to show that w solves (6.1), we define

$$v_n(t)=u_n(t+t_n) \quad \text{and} \quad \varepsilon_n(t)=\varepsilon(t+t_n).$$

This leads to the equality

$$\partial_t v_n-\partial_{xx}v_n+f(v_n)=-\varepsilon_n\partial_{tt}v_n-\varepsilon_n f'(v_n)\partial_t v_n+h.$$

We first prove that the sequence $\varepsilon_n f'(v_n)\partial_t v_n\to 0$ in the distributional sense. Indeed, for every fixed $T>0$ and every smooth H-valued function φ supported on $(-T,T)$, exploiting again (5.1), we have

$$\left|\int_{-T}^{T}\varepsilon_n(t)\langle f'(v_n(t))\partial_t v_n(t),\varphi(t)\rangle\,\mathrm{d}t\right|\leqslant C\int_{-T}^{T}\sqrt{\varepsilon_n(t)}\sqrt{\varepsilon_n(t)}\|\partial_t v_n(t)\|\,\mathrm{d}t\leqslant C\int_{-T}^{T}\sqrt{\varepsilon_n(t)}\,\mathrm{d}t\leqslant CT\sup_{t\in[-T,T]}\sqrt{\varepsilon_n(t)},$$

where the generic constant $C>0$ now depends on $\mathfrak{A}$ and φ. Thus, the required convergence follows by merely observing that

$$\lim_{n\to\infty}\left[\sup_{t\in[-T,T]}\varepsilon_n(t)\right]=0, \quad \forall T>0.$$

Similarly, after an integration by parts, one can prove that

$$\int_{-T}^{T}\varepsilon_n(t)\langle\partial_{tt}v_n(t),\varphi(t)\rangle\,\mathrm{d}t\to 0.$$

Since the convergence (in the distributional sense)

$$\partial_t v_n-\partial_{xx}v_n+f(v_n)\to w_t-w_{xx}+f(w)$$

is straightforward, this proves that w is a solution of (6.1). In light of (6.3) we conclude that w is a CBT of $S(t)$.

7. Further regularity

In this section, we discuss further regularity properties of the attractor. Indeed, we know from (5.1) that $\|u_{xx}\|$ is uniformly bounded whenever $\{u,u_t\}\in\mathfrak{A}$. Still, we cannot draw the same conclusion for the norm of u_{xt}, since (5.1) alone does not prevent the blow up of $\|u_{xt}(t)\|$ as $t\to\infty$. Here, we show that this situation cannot occur if, besides the previous assumptions, the derivative ε' is not too negative; more precisely, if[3]

$$\liminf_{t\to+\infty}\varepsilon'(t)>-2. \tag{7.1}$$

Accordingly, since $\varepsilon(t)$ vanishes as $t\to\infty$, there exists $\gamma>0$ and $\tau_\star\in\mathbb{R}$ such that

$$2+\varepsilon'(t)-2\ell\varepsilon(t)\geqslant\gamma, \quad \forall t\geqslant\tau_\star. \tag{7.2}$$

Theorem 7.1. *Let $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$ be the invariant global attractor. Then, within the further assumption (7.1),*

$$\sup_{t\in\mathbb{R}}\sup_{\{u,u_t\}\in\mathfrak{A}}\left[\|u_{xt}(t)\|^2+\varepsilon(t)\|u_{tt}(t)\|^2\right]<\infty.$$

The proof relies on the subsequent uniform estimate. In what follows, the generic constant C will depend only on the attractor $\mathfrak{A}$.

Lemma 7.2. *Let $\{u,u_t\}\in\mathfrak{A}$. Then, for every $t\geqslant\tau_\star$, we have*

$$\varepsilon(t)\|u_{tt}(t)\|^2+\|u_{xt}(t)\|^2\leqslant C_\star\mathrm{e}^{-\omega(t-\tau_\star)}+C$$

for some $C_\star>0$ and $\omega>0$, both depending only on $\mathfrak{A}$.

[3] Again, taking into account the physical constants as in (1.2), the condition reads

$$\liminf_{t\to+\infty}\varepsilon'(t)>-2\alpha.$$

Proof. By differentiating the equation with respect to time, calling $\eta = u_t$, we are led to

$$\varepsilon\eta_{tt} + [1 + \varepsilon' + \varepsilon f'(u)]\eta_t - \eta_{xx} = N(u), \tag{7.3}$$

where

$$N(u) = -\left[(1+\varepsilon')f'(u) + \varepsilon f''(u)u_t\right]u_t.$$

Note that

$$\int_{\tau_\star}^{\infty} \|N(u(y))\|^2 \mathrm{d}y \leqslant C. \tag{7.4}$$

Indeed, due to the embedding (1.4) together with (5.1),

$$\|(1+\varepsilon')f'(u)u_t\|^2 \leqslant C\|u_t\|^2$$

and

$$\|\varepsilon f''(u)|u_t|^2\|^2 \leqslant \varepsilon^2\|u_t\|^2_{L^\infty}\|f''(u)\|^2_{L^\infty}\|u_t\|^2 \leqslant C\varepsilon\|u_{xt}\|^2\|u_t\|^2 \leqslant C\|u_t\|^2.$$

Thus, (7.4) follows from the integral estimate in (3.1). Eq. (7.3) is endowed with the initial conditions at the time $t = \tau_\star$

$$\eta(\tau_\star) = b, \quad \eta_t(\tau_\star) = \frac{1}{\varepsilon(\tau_\star)}\left[h - f(a) + a_{xx} - [1 + \varepsilon(\tau_\star)f'(a)]b\right],$$

where we put for short $a = u(\tau_\star)$ and $b = u_t(\tau_\star)$. A multiplication by $2\eta_t$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\varepsilon\|\eta_t\|^2 + \|\eta_x\|^2\right] + \langle[2 + \varepsilon' + 2\varepsilon f'(u)]\eta_t, \eta_t\rangle = 2\langle N(u), \eta_t\rangle.$$

Besides, multiplying by $2\delta\eta$, for $\delta > 0$ to be fixed, we get

$$2\delta\frac{\mathrm{d}}{\mathrm{d}t}\langle\varepsilon\eta, \eta_t\rangle + 2\delta\|\eta_x\|^2 - 2\delta\varepsilon\|\eta_t\|^2 + 2\delta\langle[1 + \varepsilon f'(u)]\eta_t, \eta\rangle = 2\delta\langle N(u), \eta\rangle.$$

By (2.5) and (7.2),

$$2 + \varepsilon' + 2\varepsilon f'(u) \geqslant \gamma, \quad \forall u \in \mathbb{R}.$$

Thus, collecting the two differential equalities, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma + 2\delta\|\eta_x\|^2 + [\gamma - 2\delta\varepsilon]\|\eta_t\|^2 \leqslant -2\delta\langle[1 + \varepsilon f'(u)]\eta_t, \eta\rangle + 2\langle N(u), \eta_t + \delta\eta\rangle,$$

where

$$\Gamma(t) = \|\eta_x(t)\|^2 + \varepsilon(t)\|\eta_t(t)\|^2 + 2\delta\varepsilon(t)\langle\eta(t), \eta_t(t)\rangle.$$

This functional is easily seen to be equivalent to the square norm of $\{\eta(t), \eta_t(t)\}$ in $\mathcal{H}_t$ if $\delta > 0$ is small enough, namely,

$$\frac{1}{2}\|\{\eta(t), \eta_t(t)\}\|^2_{\mathcal{H}_t} \leqslant \Gamma(t) \leq 2\|\{\eta(t), \eta_t(t)\}\|^2_{\mathcal{H}_t}. \tag{7.5}$$

We estimate the right-hand side in a standard way, owing to the boundedness of ε and $\|f'(u)\|_{L^\infty}$, in order to get the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma + \omega\Gamma \leqslant C\|N(u)\|^2$$

for $\delta > 0$ small enough and some $\omega > 0$. Using (7.4), the Gronwall Lemma on $(\tau_\star, t)$ provides

$$\Gamma(t) \leqslant \Gamma(\tau_\star)\mathrm{e}^{-\omega(t-\tau_\star)} + C, \quad \forall t \geqslant \tau_\star.$$

Since by (5.1)

$$\|a_{xx}\|^2 + \varepsilon(\tau_\star)\|b_x\|^2 \leqslant C,$$

we deduce that

$$\|\eta_x(\tau_\star)\|^2 = \|b_x\|^2 \leqslant \frac{C}{\varepsilon(\tau_\star)}$$

and

$$\varepsilon(\tau_\star)\|\eta_t(\tau_\star)\|^2 = \frac{1}{\varepsilon(\tau_\star)}\|h - f(a) + a_{xx} - [1 + \varepsilon(\tau_\star)f'(a)]b\|^2 \leqslant \frac{C}{\varepsilon(\tau_\star)}\left(1 + \frac{1}{\varepsilon(\tau_\star)}\right).$$

A final application of (7.5) completes the argument.

Conclusion of the Proof of Theorem 7.1. Let $\{u, u_t\} \in \mathfrak{A}$. On account of (5.1),

$$\sup_{t\in\mathbb{R}} \varepsilon(t)\|u_{xt}(t)\|^2 \leqslant C.$$

Since ε is decreasing, we get

$$\|u_{xt}(t)\|^2 \leqslant \frac{C}{\varepsilon(\tau_\star)}, \quad \forall t < \tau_\star,$$

while Lemma 7.2 yields in particular

$$\|u_{xt}(t)\|^2 \leqslant C_\star + C, \quad \forall t \geqslant \tau_\star.$$

The claimed uniform bound for $\|u_{xt}\|$ is established. □

8. Concluding remarks

The first construction of an abstract (linear) evolution system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + \mathbb{A}(t)y(t) = 0,$$

where $\{\mathbb{A}(t)\}$ is a family of time-dependent linear operators acting on the *same* Banach space $\mathcal{X}$, is due to Kato [12,13]. The theory has been extended to cover the nonlinear case (see [21, Section 6])

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + \mathbb{A}(t)y(t) = g(t, y(t)).$$

Compared to the previous results, the main novelty of the approach developed in [5–8], and used in the present paper, is that the evolution operator $U(t,\tau)$ acts between spaces *explicitly* depending on time. This is naturally related to the fact that the energy norm of the phase space is somehow dictated by the operator $\mathbb{A}(t)$. In other words, the scale of time-dependent spaces $\mathcal{X}_t$ is adapted to the structure of the equation, namely, to the properties of its time-dependent coefficients. In par-ticular, although for finite time-intervals all the $\mathcal{X}_t$ might be the same geometric space endowed with equivalent norms, such an equivalence can be lost in the longtime. Even more, the (formal) limit $\mathcal{X}_t \to \mathcal{X}_\infty$ can be highly singular. Reason why the classical theory seems not to be apt to capture the asymptotic properties of $U(t, \tau)$, which is exactly the focus of our analysis. It would be however a challenging project trying to extend the ideas of Kato, so to obtain an abstract theory for linear differential equations on time-dependent Banach spaces.

Coming back to the model Eq. (2.1), it would be clearly interesting to extend the analysis in space-dimensions 2 and 3. In the two-dimensional case, thanks to the techniques developed in [14,20], the existence of an invariant time-dependent attractor should be expected for a nonlinearity f of (arbitrary) polynomial growth. The picture is much more complicated in the three-dimensional case, unless f undergoes severe limitations (cf. [19]). The critical case of polynomial growth of order 3 could be treated by introducing the notion of a weak time-dependent attractor. Instead, the existence of a strong attractor is an open problem, even if ε is constant, although a partial answer in that direction (for a constant ε) has been given in [15], but with a $\sigma_\varepsilon(u)$ that does not reflect the structure needed for our purposes.

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