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NONEXISTENCE RESULTS FOR ELLIPTIC DIFFERENTIAL INEQUALITIES WITH A POTENTIAL ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we are concerned with a class of elliptic differential inequalities with a potential both on $\mathbb{R}^m$ and on Riemannian manifolds. In particular, we investigate the effect of the geometry of the underlying manifold and of the behavior of the potential at infinity on nonexistence of nonnegative solutions.

1. Introduction

One of the most important and well-studied class of elliptic differential inequalities in Global Analysis, due to its ubiquitous presence in many applications, is

$$\Delta u + V(x)u^\sigma \leq 0, \tag{1.1}$$

both on $\mathbb{R}^m$ and on general Riemannian manifolds (M, g), where Δ denotes the Laplace-Beltrami operator associated to the metric and $\sigma > 1$. In particular, in many instances it is also required that the solution u of the problem is positive.

The aim of this paper is to investigate in depth the influence of the geometry of the underlying complete, noncompact Riemannian manifold (M, g) of dimension m and of the potential V on the existence of positive solutions to the class of elliptic differential inequalities

$$\frac{1}{a(x)}\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + V(x)u^\sigma \leq 0, \tag{1.2}$$

thus highlighting the interplay between analysis and geometry in this class of problems, which includes (1.1). Here and in the rest of the paper we assume that $a : M \to \mathbb{R}$ satisfies

$$a > 0, \quad a \in \mathrm{Lip}_{\mathrm{loc}}(M), \tag{1.3}$$

$V > 0$ a.e. on M and $V \in L^1_{\mathrm{loc}}(M)$, and the constants p and σ satisfy $p > 1$, $\sigma > p - 1$. In our results, the geometry of M appears through conditions on the growth of suitably weighted volumes of geodesic balls, involving both the potential V and the function a. We explicitly note that some of the results we find are new also in the specific case of the model equation (1.1).

This class of problems has a very long history, particularly in the Euclidean setting, starting with the seminal works of Gidas [4] and Gidas-Spruck [5]. In those papers the authors show, among other results, that any nonnegative solution of equation (1.1) is in fact identically null if and only if $\sigma \leq \frac{m}{m-2}$, in case $V \equiv 1$ and $m \geq 3$.

We refer to the interesting papers of Mitidieri-Pohozaev [13], [14], [15], [16] for a comprehensive description of results related to these (and also more general) problems on $\mathbb{R}^m$. Note that analogous results have also been obtained for degenerate elliptic equations and inequalities (see, e.g., [3], [17]), and for the parabolic companion problems (see, e.g., [16], [18], [19], [20]). Using nonlinear capacity arguments, which exploit suitably chosen test functions, Mitidieri-Pohozaev prove nonexistence of weak

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or distributional solutions of wide classes of differential inequalities in $\mathbb{R}^m$, which include also many of the examples that we consider here. In particular, they show that equation (1.1) on $\mathbb{R}^m$ does not admit any nontrivial nonnegative solution, provided that

$$\liminf_{R\to+\infty} R^{-\frac{2\sigma}{\sigma-1}} \int_{B_{\sqrt{2}R}\setminus B_R} V^{-\frac{1}{\sigma-1}}\,dx < \infty. \tag{1.4}$$

For the case of equation (1.2) on $\mathbb{R}^m$, the authors show that no nonnegative, nontrivial solution exists in case $V \equiv 1$, $m > p$ and $p - 1 < \sigma \leq \frac{m(p-1)}{m-p}$. This can again be read as a condition relating the volume growth of Euclidean balls, which depends on the dimension m of the space, and the exponent of the nonlinearity in the equation.

The results in the case of a complete Riemannian manifold have a more recent history, in particular we cite the inspiring papers of Grygor'yan-Kondratiev [8] and Grygor'yan-Sun [9], whose approach originates from the work of Kurta [12]. Using a capacity argument which only exploits the gradient of the distance function from a fixed reference point, in particular the authors showed in [8] that equation (1.2), in case $p = 2$, admits a unique nonnegative weak solution provided that there exist positive constants C, C_0 such that for every $R > 0$ sufficiently large and every small enough $\varepsilon > 0$,

$$\int_{B_R} aV^{-\beta+\varepsilon}\,d\mu_0 \leq CR^{\alpha+C_0\varepsilon}(\log R)^k, \tag{1.5}$$

where $d\mu_0$ is the canonical Riemannian measure on M, B_R is the geodesic ball centered at a point $x_0 \in M$ and

$$\alpha = \frac{2\sigma}{\sigma-1}, \quad \beta = \frac{1}{\sigma-1}, \quad 0 \leq k < \beta.$$

Let $r(x)$ denote the geodesic distance of a point $x \in M$ from a fixed origin $o \in M$. Note that condition (1.5) is satisfied if, for instance, for every $R > 0$ large enough one has

$$V(x) \leq C(1 + r(x))^{C_0}$$

and

$$\int_{B_R} V^{-\beta}\,d\mu \leq CR^{\alpha}(\log R)^k$$

for some positive constants C, C_0. The sharpness of the exponent α in this type of results is evident from the Euclidean case (1.1) with $V \equiv 1$ and $m \geq 3$, where $\alpha = m$ and the corresponding critical growth is $\sigma = \frac{m}{m-2}$. The sharpness of the exponent β is definitely a more delicate question and has recently been settled on a general Riemannian manifold, in case $a \equiv 1$ and $V \equiv 1$, in [9]. In that paper, using carefully chosen families of test functions, the authors showed that equation (1.1) with $V \equiv 1$ does not admit any nonnegative weak solution provided (1.5) holds with $k = \beta$.

In this work we intend to further focus our attention on these classes of differential inequalities, with the objective of adding some new results to the already very interesting overall picture. Our results concerning equation (1.1), in their simplest form, are contained in the two following theorems.

Theorem 1.1. *Let (M, g) be a complete Riemannian manifold, $\sigma > 1$, $V \in L^1_{loc}(M)$ with $V > 0$ a.e. on M. Define*

$$\alpha = \frac{2\sigma}{\sigma-1}, \quad \beta = \frac{1}{\sigma-1} \tag{1.6}$$

and assume that there exist C, $C_0 > 0$ such that for every $R > 0$ sufficiently large one has

$$\int_{B_R\setminus B_{R/2}} V^{-\beta}\,d\mu \leq CR^{\alpha}(\log R)^{\beta} \tag{1.7}$$

and

$$\frac{1}{C}(1 + r(x))^{-C_0} \leq V(x) \leq C(1 + r(x))^{C_0} \qquad \text{a.e. on } M. \tag{1.8}$$

Then the only nonnegative weak solution of (1.1) is $u \equiv 0$.

Theorem 1.2. *With the same notation of Theorem 1.1, assume that there exist $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma-1}{\sigma}(k+1), 1\}$ such that for every sufficiently large $R > 0$*

$$\int_{B_R \setminus B_{R/2}} V^{-\beta}\, d\mu \leq CR^{\alpha} (\log R)^k \tag{1.9}$$

and

$$V(x) \leq C(1 + r(x))^{C_0} e^{-\theta(\log r(x))^{\tau}} \qquad a.e.\ on\ M. \tag{1.10}$$

Then the only nonnegative weak solution of (1.1) *is* $u \equiv 0$.

A few remarks are now in order. For equation (1.1) on $\mathbb{R}^m$, condition (1.8) is not required in the results of Mitidieri-Pohozaev. On the other hand, the potential $V(x) = \big(\log(2 + |x|^2)\big)^{-1}$ and the choice of the exponent $\sigma = \frac{m}{m-2}$ with $m \geq 3$ satisfy the conditions of our Theorem 1.1, but they do not satisfy condition (1.4).

Moreover, Theorem 1.1 and Theorem 1.2 extend the result contained in [9] for problem (1.1) on a complete Riemannian manifold (M, g), where only the case of a constant potential is considered, to the case of a nonconstant V.

On the other hand, while (as we have already noted) the exponent $\alpha = \frac{2\sigma}{\sigma-2}$ in the power of R in conditions (1.7) and (1.9) is indeed sharp, Theorem 1.2 also shows that the sharpness of the exponent of the term $\log R$ for this type of results is a notion which is also related to the behavior of the potential V at infinity. In particular, if V decays at infinity faster than any power of $r(x)$, as in condition (1.10), then the critical threshold for the power of the logarithmic term in the volume growth condition (1.9) for the nonexistence of nonnegative, nontrivial solutions of (1.1) correspondingly increases to $\frac{\sigma\tau}{\sigma-1} - 1 > \beta = \frac{1}{\sigma-1}$. We explicitly note that the type of phenomenon described in Theorem 1.2 has not been pointed out before in literature, to the best of our knowledge.

The aforementioned theorems are a consequence of more general results concerning nonnegative weak solutions of inequality (1.2), showing that similar phenomena also occur for this larger class of problems, which includes the case of inequalities involving the p-Laplace operator. We start with a definition describing the weighted volume growth conditions on geodesic balls of (M, g), that will be used in obtaining the nonexistence results for nonnegative solutions of (1.2). We recall that with $d\mu_0$ we denote the canonical Riemannian measure on M, while we define

$$d\mu = a\, d\mu_0 \tag{1.11}$$

the weighted measure on M with density a.

Definition 1.3. *Let $p > 1$, $\sigma > p - 1$, $V > 0$ a.e. on M and $V \in L^1_{loc}(M)$. Define*

$$\alpha = \frac{p\sigma}{\sigma - p + 1}, \qquad \beta = \frac{p-1}{\sigma - p + 1}. \tag{1.12}$$

We introduce the following three weighted volume growth conditions:

i) *We say that* condition (HP1) *holds if there exist $C_0 > 0$, $k \in [0, \beta)$ such that, for every $R > 0$ sufficiently large and every $\varepsilon > 0$ sufficiently small,*

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon}\, d\mu \leq CR^{\alpha + C_0\varepsilon} (\log R)^k. \tag{1.13}$$

ii) *We say that* condition (HP2) *holds if there exists $C_0 > 0$ such that, for every $R > 0$ sufficiently large and every $\varepsilon > 0$ sufficiently small,*

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon}\, d\mu \leq CR^{\alpha + C_0\varepsilon} (\log R)^{\beta} \qquad and \qquad \int_{B_R \setminus B_{R/2}} V^{-\beta-\varepsilon}\, d\mu \leq CR^{\alpha + C_0\varepsilon} (\log R)^{\beta}. \tag{1.14}$$

iii) *We say that* condition (HP3) *holds if there exist* $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma-p+1}{\sigma}(k+1), 1\}$ *such that, for every sufficiently large* $R > 0$ *and every* $\varepsilon > 0$ *sufficiently small,*

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon}\, d\mu \leq C R^{\alpha + C_0\varepsilon} (\log R)^k e^{-\varepsilon\theta(\log R)^\tau}. \tag{1.15}$$

We explicitly note that, when $p = 2$, the definitions of α, β given in (1.6) and (1.12) agree.

Remark 1.4. The following are sufficient conditions that imply the above weighted volume growth conditions for geodesic balls in M.

i) Suppose that there exist $C_0 > 0$, $k > 0$ such that

$$V(x) \leq C(1 + r(x))^{C_0} \tag{1.16}$$

and

$$\int_{B_R \setminus B_{R/2}} V^{-\beta}\, d\mu \leq C R^\alpha (\log R)^k \tag{1.17}$$

for every $R > 0$ sufficiently large; then condition (1.13) holds.

ii) Suppose that there exists $C_0 > 0$ such that

$$\frac{1}{C}(1 + r(x))^{-C_0} \leq V(x) \leq C(1 + r(x))^{C_0} \tag{1.18}$$

and

$$\int_{B_R \setminus B_{R/2}} V^{-\beta}\, d\mu \leq C R^\alpha (\log R)^\beta \tag{1.19}$$

for every $R > 0$ sufficiently large; then condition (1.14) holds.

iii) Suppose there exist $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma-p+1}{\sigma}(k+1), 1\}$ such that

$$V(x) \leq C(r(x))^{C_0} e^{-\theta(\log r(x))^\tau} \tag{1.20}$$

and

$$\int_{B_R \setminus B_{R/2}} V^{-\beta}\, d\mu \leq C R^\alpha (\log R)^k \tag{1.21}$$

for every $R > 0$ sufficiently large; then condition (1.15) holds.

We can now state our main theorem.

Theorem 1.5. *Let* $p > 1$, $\sigma > p - 1$, $V \in L^1_{loc}(M)$ *with* $V > 0$ *a.e. on* M *and* $a \in \mathrm{Lip}_{loc}(M)$ *with* $a > 0$ *on* M*. If* $u \in W^{1,p}_{loc}(M) \cap L^\sigma_{loc}(M, V d\mu_0)$ *is a nonnegative weak solution of* (1.2)*, then* $u \equiv 0$ *on* M *provided that one of the conditions (HP1), (HP2) or (HP3) holds (see Definition 1.3).*

In the particular case $p = 2$, from Theorem 1.5 we can also derive nonexistence criteria for nonnegative weak solutions of the semilinear inequality

$$\frac{1}{a(x)} \operatorname{div}(a(x)\nabla u) + b(x)u + V(x)u^\sigma \leq 0 \quad \text{on } M. \tag{1.22}$$

We refer the reader to Section 4 for a precise description of the results concerning inequality (1.22).

The rest of the paper is organized as follows. In Section 2 we state and prove some preliminary technical results, that we put to use in Section 3, where we give the proof of Theorem 1.5. In Section 4 we describe in more detail nonexistence results for nontrivial nonnegative weak solutions of (1.22). Finally in Section 5 we collect some counterexamples to Theorem 1.5 for the case $p = 2$, showing that the weighted volume growth conditions that we assume on geodesic balls are in many cases sharp.

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2. Preliminary results

For any relatively compact domain $\Omega \subset M$ and $p > 1$, $W^{1,p}(\Omega)$ is the completion of the space of Lipschitz functions $w : \Omega \to \mathbb{R}$ with respect to the norm

$$\|w\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla w|^p \, d\mu_0 + \int_\Omega |w|^p \, d\mu_0 \right)^{\frac{1}{p}}.$$

For any function $u : M \to \mathbb{R}$ we say that $u \in W^{1,p}_{\mathrm{loc}}(M)$ if for every relatively compact domain $\Omega \subset\subset M$ one has $u_{|\Omega} \in W^{1,p}(\Omega)$.

Definition 2.1. *Let $p > 1$, $\sigma > p - 1$, $V > 0$ a.e. on M and $V \in L^1_{loc}(M)$. We say that u is a* weak solution *of equation* (1.2) *if $u \in W^{1,p}_{loc}(M) \cap L^\sigma_{loc}(M, V d\mu_0)$ and for every $\varphi \in W^{1,p}(M) \cap L^\infty(M)$, with $\varphi \geq 0$ a.e. on M and compact support, one has*

$$- \int_M a(x) |\nabla u|^{p-2} \left\langle \nabla u, \nabla \left(\frac{\varphi}{a(x)} \right) \right\rangle d\mu_0 + \int_M V(x) u^\sigma \varphi \, d\mu_0 \leq 0 \qquad \text{on } M. \tag{2.1}$$

Remark 2.2. We note that, by (1.3), $u \in W^{1,p}_{\mathrm{loc}}(M) \cap L^\sigma_{\mathrm{loc}}(M, V d\mu)$ is a weak solution of (1.2) if and only if it is a weak solution of

$$\operatorname{div} \left(a(x) |\nabla u|^{p-2} \nabla u \right) + a(x) V(x) u^\sigma \leq 0 \quad \text{on } M,$$

i.e. if and only if for every $\psi \in W^{1,p}(M) \cap L^\infty(M)$, with $\psi \geq 0$ a.e. on M and compact support, one has

$$- \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle \, d\mu + \int_M V(x) u^\sigma \psi \, d\mu \leq 0 \qquad \text{on } M, \tag{2.2}$$

where $d\mu$ is the measure on M with density a, as defined in (1.11).

Indeed, given any nonnegative $\psi \in W^{1,p}(M) \cap L^\infty(M)$ with compact support, one can choose $\varphi = a\psi$ as a test function in (2.1) in order to obtain (2.2). Similarly, , given any nonnegative $\varphi \in W^{1,p}(M) \cap L^\infty(M)$ with compact support, one can insert $\psi = \frac{\varphi}{a}$ in (2.2) and find (2.1).

The following two lemmas will be crucial ingredients in the proof the Theorem 1.5.

Lemma 2.3. *Let $s \geq \frac{p\sigma}{\sigma - p + 1}$ be fixed. Then there exists a constant $C > 0$ such that for every $t \in (0, \min\{1, p-1\})$, every nonnegative weak solution u of equation* (1.2) *and every function $\varphi \in \mathrm{Lip}(M)$ with compact support and $0 \leq \varphi \leq 1$ one has*

$$\frac{t}{p} \int_M \varphi^s u^{-t-1} |\nabla u|^p \chi_\Omega \, d\mu + \frac{1}{p} \int_M V u^{\sigma - t} \varphi^s \, d\mu \leq C t^{-\frac{(p-1)\sigma}{\sigma - p + 1}} \int_M V^{-\frac{p-t-1}{\sigma - p + 1}} |\nabla \varphi|^{\frac{p(\sigma - t)}{\sigma - p + 1}} \, d\mu, \tag{2.3}$$

where $\Omega = \{x \in M : u(x) > 0\}$, χ_Ω is the characteristic function of Ω and $d\mu$ is the measure on M with density a, as defined in (1.11).

Proof. Let $\eta > 0$ and $u_\eta = u + \eta$, then $u_\eta \in W^{1,p}_{\mathrm{loc}}(M) \cap L^\sigma_{\mathrm{loc}}(M, V d\mu)$. Define $\psi = \varphi^s u_\eta^{-t}$, then ψ is an admissible test function for equation (2.2), with

$$\nabla \psi = s \varphi^{s-1} u_\eta^{-t} \nabla \varphi - t \varphi^s u_\eta^{-t-1} \nabla u \qquad a.e. \text{ on } M.$$

Indeed, $\operatorname{supp} \psi = \operatorname{supp} \varphi$, $0 \leq \psi \leq \eta^{-t}$ so that $\psi \in L^\infty(M)$ and $\psi \in W^{1,p}(M)$ with

$$\begin{aligned} \int_M |\nabla \psi|^p \, d\mu &\leq 2^{p-1} \left[s^p \int_M \varphi^{(s-1)p} u_\eta^{-pt} |\nabla \varphi|^p \, d\mu + t^p \int_M \varphi^{sp} u_\eta^{-(t+1)p} |\nabla u|^p \, d\mu \right] \\ &\leq 2^{p-1} \left[s^p \eta^{-pt} \int_M |\nabla \varphi|^p \, d\mu + t^p \eta^{-(t+1)p} \int_{\operatorname{supp} \varphi} |\nabla u|^p \, d\mu \right] < +\infty. \end{aligned}$$

Equation (2.2) then gives

$$t \int_M \varphi^s u_\eta^{-t-1} |\nabla u|^p \, d\mu + \int_M V u^\sigma u_\eta^{-t} \varphi^s \, d\mu \leq s \int_M \varphi^{s-1} u_\eta^{-t} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, d\mu. \tag{2.4}$$

Now we estimate the right-hand side of (2.4) using Young's inequality, obtaining

$$
\begin{aligned}
s\int_M \varphi^{s-1}u_\eta^{-t}|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,d\mu &\le s\int_M \varphi^{s-1}u_\eta^{-t}|\nabla u|^{p-1}|\nabla\varphi|\,d\mu\\
&\le \int_M \left(t^{\frac{p-1}{p}}\varphi^{s\frac{p-1}{p}}u_\eta^{-(t+1)\frac{p-1}{p}}|\nabla u|^{p-1}\right)\left(st^{-\frac{p-1}{p}}\varphi^{\frac{s}{p}-1}u_\eta^{1-\frac{t+1}{p}}|\nabla\varphi|\right)d\mu\\
&\le \frac{p-1}{p}\int_M t\varphi^s u_\eta^{-t-1}|\nabla u|^p\,d\mu+\frac{1}{p}\int_M s^p t^{-(p-1)}\varphi^{s-p}u_\eta^{p-(t+1)}|\nabla\varphi|^p\,d\mu.
\end{aligned}
$$

From (2.4) we have

$$
\frac{t}{p}\int_M \varphi^s u_\eta^{-t-1}|\nabla u|^p\,d\mu+\int_M Vu^\sigma u_\eta^{-t}\varphi^s\,d\mu\le\frac{1}{p}\int_M s^p t^{-(p-1)}\varphi^{s-p}u_\eta^{p-(t+1)}|\nabla\varphi|^p\,d\mu. \tag{2.5}
$$

We exploit again Young's inequality on the right-hand side of (2.5), with

$$
q=\frac{\sigma-t}{p-t-1},\qquad q'=\frac{q}{q-1}=\frac{\sigma-t}{\sigma-p+1},\qquad \delta=\frac{p-1}{p}
$$

obtaining

$$
\begin{aligned}
\frac{1}{p}\int_M s^p t^{-(p-1)}\varphi^{s-p}u_\eta^{p-(t+1)}|\nabla\varphi|^p\,d\mu &= \int_M\left(\delta^{\frac1q}u_\eta^{p-(t+1)}V^{\frac1q}\varphi^{\frac{s}{q}}\right)\left(\delta^{-\frac1q}\frac{s^p}{pt^{p-1}}\varphi^{\frac{s}{q'}-p}V^{-\frac1q}|\nabla\varphi|^p\right)d\mu\\
&\le\frac{\delta}{q}\int_M u_\eta^{\sigma-t}V\varphi^s\,d\mu+\frac{1}{q'p^{q'}}\delta^{-\frac{q'}{q}}\left(\frac{s^p}{t^{p-1}}\right)^{q'}\int_M\varphi^{s-pq'}V^{-\frac{q'}{q}}|\nabla\varphi|^{pq'}\,d\mu\\
&\le\delta\int_M u_\eta^{\sigma-t}V\varphi^s\,d\mu+\delta^{-\frac{p-1}{\sigma-p+1}}\left(\frac{s^p}{t^{p-1}}\right)^{\frac{\sigma}{\sigma-p+1}}\int_M V^{-\frac{q'}{q}}|\nabla\varphi|^{pq'}\,d\mu\\
&=\frac{p-1}{p}\int_M u_\eta^{\sigma-t}V\varphi^s\,d\mu+Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu.
\end{aligned}
$$

Substituting in (2.5) we have

$$
\begin{aligned}
I=\frac{t}{p}\int_M\varphi^s u_\eta^{-t-1}|\nabla u|^p\,d\mu+\int_M Vu^\sigma u_\eta^{-t}\varphi^s\,d\mu-\frac{p-1}{p}\int_M Vu_\eta^{\sigma-t}\varphi^s\,d\mu&\\
\le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu.&
\end{aligned}\tag{2.6}
$$

Since $\nabla u=0$ a.e. on the set $M\setminus\Omega$, see [6, Lemma 7.7], we have

$$
I=\int_M\left[\frac{t}{p}\varphi^s u_\eta^{-t-1}|\nabla u|^p+Vu^\sigma u_\eta^{-t}\varphi^s-\frac{p-1}{p}Vu_\eta^{\sigma-t}\varphi^s\right]\chi_\Omega\,d\mu-\int_{M\setminus\Omega}\frac{p-1}{p}V\eta^{\sigma-t}\varphi^s\,d\mu.
$$

Now note that $\left[\frac{t}{p}\varphi^s u_\eta^{-t-1}|\nabla u|^p+Vu^\sigma u_\eta^{-t}\varphi^s-\frac{p-1}{p}Vu_\eta^{\sigma-t}\varphi^s\right]\chi_\Omega$ converges a.e. in M to the function

$$
\left[\frac{t}{p}\varphi^s u^{-t-1}|\nabla u|^p+\frac1p Vu^{\sigma-t}\varphi^s\right]\chi_\Omega
$$

as $\eta\to0^+$. By an application of Fatou's lemma and using (2.6) we obtain

$$
\begin{aligned}
\int_M\frac{t}{p}\varphi^s u^{-t-1}|\nabla u|^p\chi_\Omega\,d\mu+\int_M\frac1p Vu^{\sigma-t}\varphi^s\,d\mu&\\
&=\int_M\left[\frac{t}{p}\varphi^s u^{-t-1}|\nabla u|^p+\frac1p Vu^{\sigma-t}\varphi^s\right]\chi_\Omega\,d\mu\\
&\le\liminf_{\eta\to0^+}\int_M\left[\frac{t}{p}\varphi^s u_\eta^{-t-1}|\nabla u|^p+Vu^\sigma u_\eta^{-t}\varphi^s-\frac{p-1}{p}Vu_\eta^{\sigma-t}\varphi^s\right]\chi_\Omega\,d\mu\\
&\le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu+\liminf_{\eta\to0^+}\int_{M\setminus\Omega}\frac{p-1}{p}V\eta^{\sigma-t}\varphi^s\,d\mu\\
&=Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu,
\end{aligned}
$$

that is inequality (2.3).

Lemma 2.4. *Let $s \geq \frac{2p\sigma}{\sigma-p+1}$ be fixed. Then there exists a constant $C > 0$ such that for every nonnegative weak solution u of equation (1.2), every function $\varphi \in \mathrm{Lip}(M)$ with compact support and $0 \leq \varphi \leq 1$ and every $t \in (0, \min\{1, p-1, \frac{\sigma-p+1}{2(p-1)}\})$ one has*

$$\int_M \varphi^s u^\sigma V\,d\mu \leq C t^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}} \left(\int_{M\setminus K} V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}} |\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\,d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}} \left(\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right)^{\frac{p-1}{p}} \left(\int_{M\setminus K}\varphi^s u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}}, \tag{2.7}$$

with $K = \{x \in M : \varphi(x) = 1\}$ and $d\mu$ is the measure on M with density a, as defined in (1.11).

Proof of Lemma 2.4. Under our assumptions $\psi = \varphi^s$ is a feasible test function in equation (2.2). Thus we obtain

$$\int_M \varphi^s u^\sigma V\,d\mu \leq \int_M s\varphi^{s-1}|\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle\,d\mu \leq \int_M s\varphi^{s-1}|\nabla u|^{p-1}|\nabla\varphi|\,d\mu. \tag{2.8}$$

Now let $\Omega = \{x \in M : u(x) > 0\}$ and let χ_Ω be the characteristic function of Ω. Since $\nabla u = 0$ a.e. on the set $M \setminus \Omega$, through an application of Hölder's inequality we obtain

$$\begin{aligned}\int_M s\varphi^{s-1}|\nabla u|^{p-1}|\nabla\varphi|\,d\mu &= \int_M s\varphi^{s-1}|\nabla u|^{p-1}\chi_\Omega|\nabla\varphi|\,d\mu \\ &= s\int_M \left(\varphi^{\frac{p-1}{p}s}|\nabla u|^{p-1}u^{-\frac{p-1}{p}(t+1)}\chi_\Omega\right)\left(\varphi^{\frac{s}{p}-1}u^{\frac{p-1}{p}(t+1)}|\nabla\varphi|\right)d\mu \\ &\leq s\left(\int_M \varphi^s|\nabla u|^p u^{-t-1}\chi_\Omega\,d\mu\right)^{\frac{p-1}{p}}\left(\int_M \varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\,d\mu\right)^{\frac{1}{p}}.\end{aligned} \tag{2.9}$$

Moreover from equation (2.3) we deduce

$$\int_M \varphi^s|\nabla u|^p u^{-t-1}\chi_\Omega\,d\mu \leq Ct^{-1-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu, \tag{2.10}$$

with $C > 0$ depending on s. Thus from (2.8), (2.9) and (2.10) we obtain

$$\int_M \varphi^s u^\sigma V\,d\mu \leq C\left(t^{-1-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_M \varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\,d\mu\right)^{\frac{1}{p}}. \tag{2.11}$$

Now we use again Hölder's inequality with exponents

$$q = \frac{\sigma}{(t+1)(p-1)}, \qquad q' = \frac{q}{q-1} = \frac{\sigma}{\sigma-(t+1)(p-1)}$$

to obtain

$$\begin{aligned}&\int_M \varphi^{s-p}u^{(p-1)(t+1)}|\nabla\varphi|^p\,d\mu \\ &\quad= \int_{M\setminus K}\left(\varphi^{\frac{s}{q}}u^{(p-1)(t+1)}V^{\frac{1}{q}}\right)\left(\varphi^{\frac{s}{q'}-p}V^{-\frac{1}{q}}|\nabla\varphi|^p\right)d\mu \\ &\quad\leq \left(\int_{M\setminus K}\varphi^s u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{\sigma}}\left(\int_{M\setminus K}\varphi^{s-\frac{p\sigma}{\sigma-(t+1)(p-1)}}V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}}|\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\,d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{\sigma}}.\end{aligned}$$

Substituting into (2.11) we get

$$\int_M \varphi^s u^\sigma V\,d\mu \leq Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left(\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_{M\setminus K}\varphi^s u^\sigma V\,d\mu\right)^{\frac{(t+1)(p-1)}{p\sigma}} \left(\int_{M\setminus K}\varphi^{s-\frac{p\sigma}{\sigma-(t+1)(p-1)}}V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}}|\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\,d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}}.$$

Now inequality (2.7) immediately follows from the previous relation, by our assumptions on s, t and since $0 \leq \varphi \leq 1$.

From Lemma 2.4 we immediately deduce

Corollary 2.5. *Under the same assumptions of Lemma 2.4 there exists a constant $C > 0$, independent of u, φ and t, such that*

(2.12)
$$\left(\int_M \varphi^s u^\sigma V\, d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}$$
$$\leq Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left(\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu\right)^{\frac{p-1}{p}}\left(\int_M V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}}|\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\, d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}}.$$

Proof. Inequality (2.12) easily follows form (2.7), since $s \geq \frac{p\sigma}{\sigma-(t+1)(p-1)}$ and $0 \leq \varphi \leq 1$ on M.

3. Proof of Theorem 1.5

We divide the proof of Theorem 1.5 in three cases, depending on which of the conditions (HP1), (HP2) or (HP3) is assumed to hold (see Definition 1.3).

Proof of Theorem 1.5. (a) Assume that condition (HP1) holds (see (1.13)). Let $r(x)$ be the distance of $x \in M$ from a fixed origin o, for any fixed $R > 0$ sufficiently large let $t = \frac{1}{\log R}$ and denote by B_R the metric ball centered at o with radius R. Fix any $C_1 \geq \frac{C_0+p+2}{p\sigma}$ with C_0 as in condition (1.13), define for $x \in M$

$$\varphi(x) = \begin{cases} 1 & \text{for } r(x) < R, \\ \left(\frac{r(x)}{R}\right)^{-C_1 t} & \text{for } r(x) \geq R \end{cases} \tag{3.1}$$

and for $n \in \mathbb{N}$

$$\eta_n(x) = \begin{cases} 1 & \text{for } r(x) < nR, \\ 2 - \frac{r(x)}{nR} & \text{for } nR \leq r(x) \leq 2nR, \\ 0 & \text{for } r(x) \geq 2nR. \end{cases} \tag{3.2}$$

Let

$$\varphi_n(x) = \eta_n(x)\varphi(x) \qquad \text{for } x \in M, \tag{3.3}$$

then $\varphi_n \in \mathrm{Lip}(M)$ with $0 \leq \varphi_n \leq 1$, we have

$$\nabla\varphi_n = \eta_n\nabla\varphi + \varphi\nabla\eta_n \qquad \text{a.e. in } M$$

and for every $a \geq 1$

$$|\nabla\varphi_n|^a \leq 2^{a-1}(|\nabla\varphi|^a + \varphi^a|\nabla\eta_n|^a) \qquad \text{a.e. in } M.$$

Now we use φ_n in formula (2.3) of Lemma 2.3 with any fixed $s \geq \frac{p\sigma}{\sigma-p+1}$ and deduce that, for some positive constant C and for every $n \in \mathbb{N}$ and every small enough $t > 0$, we have

(3.4)
$$\begin{aligned}\int_M Vu^{\sigma-t}\varphi_n^s\, d\mu &\leq Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \\ &= Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\int_M V^{-\beta+\frac{t}{\sigma-p+1}}|\nabla\varphi_n^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \\ &\leq Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}2^{\frac{p(\sigma-t)}{\sigma-p+1}-1}\left[\int_M V^{-\beta+\frac{t}{\sigma-p+1}}|\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu + \int_{B_{2nR}\setminus B_{nR}} V^{-\beta+\frac{t}{\sigma-p+1}}\varphi^{\frac{p(\sigma-t)}{\sigma-p+1}}|\nabla\eta_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu\right] \\ &\leq Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}[I_1 + I_2],\end{aligned}$$

where

$$I_1 := \int_{M\setminus B_R} V^{-\beta+\frac{t}{\sigma-p+1}} |\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}} d\mu,$$

$$I_2 := \int_{B_{2nR}\setminus B_{nR}} \varphi^{\frac{p(\sigma-t)}{\sigma-p+1}} |\nabla\eta_n|^{\frac{p(\sigma-t)}{\sigma-p+1}} V^{-\beta+\frac{t}{\sigma-p+1}} d\mu.$$

By (3.1), (3.2) and assumption (HP1) with $\varepsilon = \frac{t}{\sigma-p+1}$, see equation (1.13), for every $n \in \mathbb{N}$ and every small enough $t > 0$ we have

$$\begin{aligned} I_2 &\le \left(\sup_{B_{2nR}\setminus B_{nR}} \varphi\right)^{\frac{p(\sigma-t)}{\sigma-p+1}} \left(\frac{1}{nR}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_{B_{2nR}\setminus B_{nR}} V^{-\beta+\frac{t}{\sigma-p+1}}\, d\mu \\ &\le C\left(\frac{nR}{R}\right)^{-\frac{p(\sigma-t)}{\sigma-p+1}C_1 t} \left(\frac{1}{nR}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}} (2nR)^{\alpha+\frac{C_0 t}{\sigma-p+1}} [\log(2nR)]^k \\ &\le C n^{\alpha+\frac{C_0 t}{\sigma-p+1}-\frac{p(\sigma-t)}{\sigma-p+1}(C_1 t+1)} R^{\alpha+\frac{C_0 t}{\sigma-p+1}-\frac{p(\sigma-t)}{\sigma-p+1}} [\log(2nR)]^k\,. \end{aligned} \tag{3.5}$$

By our choice of C_1, for every small enough $t > 0$

$$\alpha + \frac{C_0 t}{\sigma-p+1} - \frac{p(\sigma-t)}{\sigma-p+1}(C_1 t+1) = \frac{t(C_0 - p\sigma C_1 + pC_1 t + p)}{\sigma-p+1} \le -\frac{t}{\sigma-p+1} < 0. \tag{3.6}$$

Moreover, since $t = \frac{1}{\log R}$, we have

$$R^{\alpha+\frac{C_0 t}{\sigma-p+1}-\frac{p(\sigma-t)}{\sigma-p+1}} = R^{\frac{C_0+p}{\sigma-p+1}t} = e^{\frac{C_0+p}{\sigma-p+1}t\log R} = e^{\frac{C_0+p}{\sigma-p+1}}$$

In view of (3.5) and (3.6) for $R > 1$ large enough, and thus $t = \frac{1}{\log R}$ small enough, we obtain

$$I_2 \le C n^{-\frac{t}{\sigma-p+1}} [\log(2nR)]^k\,. \tag{3.7}$$

In order to estimate I_1 we recall that if $f : [0,\infty) \to [0,\infty)$ is a nonnegative decreasing function and (1.13) holds, then for any small enough $\varepsilon > 0$ and any sufficiently large $R > 1$ we have

$$\int_{M\setminus B_R} f(r(x))(V(x))^{-\beta+\varepsilon}\, d\mu \le C\int_{\frac{R}{2}}^{+\infty} f(r) r^{\alpha+C_0\varepsilon-1}(\log r)^k\, dr \tag{3.8}$$

for some positive constant C, see [9, formula (2.19)]. Moreover, there holds

$$|\nabla\varphi| \le C_1 t R^{C_1 t} r^{-C_1 t-1}\,. \tag{3.9}$$

Thus, using (3.19)-(3.9),

$$\begin{aligned} I_1 &\le \int_{M\setminus B_R} V^{-\beta+\frac{t}{\sigma-p+1}} (R^{C_1 t} C_1 t r^{-C_1 t-1})^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \\ &\le C\int_{\frac{R}{2}}^{\infty} R^{\frac{p(\sigma-t)}{\sigma-p+1}C_1 t}(1+C_1)^{\frac{p\sigma}{\sigma-p+1}} t^{\frac{p(\sigma-t)}{\sigma-p+1}} r^{-\frac{p(\sigma-t)}{\sigma-p+1}(C_1 t+1)+\alpha+C_0\frac{t}{\sigma-p+1}-1} (\log r)^k\, dr. \end{aligned}$$

Now note that

$$R^{\frac{p(\sigma-t)}{\sigma-p+1}C_1 t} = e^{\frac{p(\sigma-t)}{\sigma-p+1}C_1} < e^{\frac{p\sigma C_1}{\sigma-p+1}}$$

and that by our choice of C_1 we have

$$a := -\frac{p(\sigma-t)}{\sigma-p+1}(C_1 t+1) + \alpha + C_0\frac{t}{\sigma-p+1} = \frac{t}{\sigma-p+1}(pC_1 t - p\sigma C_1 + p + C_0) \le -\frac{t}{\sigma-p+1} < 0\,.$$

Then, by the above inequalities and performing the change of variables $\xi := |a|\log r$, we get

$$
\begin{aligned}
I_1 &\le C t^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_1^{\infty} r^{-\frac{p(\sigma-t)}{\sigma-p+1}(C_1 t+1)+\alpha+C_0\frac{t}{\sigma-p+1}} (\log r)^k \,\frac{dr}{r} \\
&\le C|a|^{-(k+1)} t^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_0^{\infty} e^{-\xi}\xi^k \, d\xi \\
&\le C\left(\frac{t}{\sigma-p+1}\right)^{-k-1} t^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_0^{\infty} e^{-\xi}\xi^k \, d\xi \\
&\le C t^{\frac{p(\sigma-t)}{\sigma-p+1}-k-1} .
\end{aligned} \tag{3.10}
$$

By (3.4), (3.7) and (3.10)

$$
\begin{aligned}
\int_{B_R} V u^{\sigma-t}\, d\mu \le \int_M V u^{\sigma-t}\varphi_n^s \, d\mu \\
\le C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} [n^{-\frac{t}{\sigma-p+1}} (\log(2nR))^k + t^{\frac{p(\sigma-t)}{\sigma-p+1}-k-1}].
\end{aligned} \tag{3.11}
$$

Since $R > 1$ is large and fixed, and thus $t = \frac{1}{\log R} < 1$ is also fixed, taking the lim inf as $n \to \infty$ in (3.11) we obtain

$$
\int_{B_R} V u^{\sigma-t} d\mu \le C t^{\frac{p(\sigma-t)}{\sigma-p+1}-k-1-\frac{(p-1)\sigma}{\sigma-p+1}} . \tag{3.12}
$$

Observe that, for each small enough $t > 0$,

$$
\frac{p(\sigma-t)}{\sigma-p+1} - k - 1 - \frac{(p-1)\sigma}{\sigma-p+1} = \frac{p-1}{\sigma-p+1} - k - \frac{pt}{\sigma-p+1} = \beta - k - \frac{pt}{\sigma-p+1} \ge \delta_* > 0 \, .
$$

Then, for any fixed sufficiently small $t > 0$, we have

$$
\int_M V u^{\sigma-t}\chi_{B_{e^{1/t}}}\, d\mu = \int_{B_R} V u^{\sigma-t}\, d\mu \le C t^{\delta_*} \, .
$$

By Fatou's Lemma, taking the lim inf as $t \to 0^+$ in the previous inequality we obtain

$$
\int_M V u^{\sigma}\, d\mu \le 0,
$$

which implies $u \equiv 0$ in M.

(b) Assume that condition (HP2) holds (see (1.14)). Let the functions φ, η_n and φ_n be defined on M as in formulas (3.1), (3.2) and (3.3), with $R > 1$ large enough, $t = \frac{1}{\log R}$, $C_1 \ge \max\left\{\frac{C_0+p+2}{p\sigma}, \frac{C_0}{\sigma-p+1}\right\}$ and C_0 as in condition (1.14). We now apply formula (2.12), using the family of functions $\varphi_n \in \mathrm{Lip}_0(M)$ and any fixed $s \ge \frac{2p\sigma}{\sigma-p+1}$, and thus we have

$$
\begin{aligned}
&\left(\int_M \varphi_n^s u^{\sigma} V\, d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}} \\
&\quad \le C t^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}} \left(\int_M V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu\right)^{\frac{p-1}{p}} \left(\int_M V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}} |\nabla\varphi_n|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\, d\mu\right)^{\frac{\sigma-(t+1)(p-1)}{p\sigma}} .
\end{aligned}
$$

We now need need to estimate

$$
\int_M V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla\varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \qquad \text{and} \qquad \int_M V^{-\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)}} |\nabla\varphi|^{\frac{p\sigma}{\sigma-(t+1)(p-1)}}\, d\mu. \tag{3.13}
$$

Arguing as in the previous proof of the theorem under the validity of condition (HP1), with the only difference that the condition $k < \beta$ there is replaced here by $k = \beta$, using (1.14) we can deduce that

$$
\int_M V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \le C\Big[n^{-\frac{t}{\sigma-p+1}} (\log(2nR))^{\beta} + t^{\frac{p(\sigma-t)}{\sigma-p+1}-\beta-1}\Big]. \tag{3.14}
$$

In order to estimate the second integral in (3.13) we start by defining $\Lambda = \frac{(p-1)\sigma t}{(\sigma-p+1)[\sigma-(t+1)(p-1)]}$, and we note that

$$\frac{(p-1)\sigma}{(\sigma-p+1)^2}t < \Lambda < \frac{2(p-1)\sigma}{(\sigma-p+1)^2}t < \varepsilon^* \tag{3.15}$$

for every small enough $t > 0$, and that

$$\frac{(t+1)(p-1)}{\sigma-(t+1)(p-1)} = \beta + \Lambda \qquad \text{and} \qquad \frac{p\sigma}{\sigma-(t+1)(p-1)} = \alpha + \Lambda p,$$

with α, β as in Definition 1.3. By our definition of the functions φ_n, for every $n \in \mathbb{N}$ and every small enough $t > 0$ we have

(3.16)

$$\begin{aligned}
\int_M V^{-\beta-\Lambda}|\nabla\varphi_n|^{\alpha+\Lambda p}\,d\mu &\le C\left[\int_M V^{-\beta-\Lambda}{\eta_n}^{\alpha+\Lambda p}|\nabla\varphi|^{\alpha+\Lambda p}\,d\mu + \int_M V^{-\beta-\Lambda}\varphi^{\alpha+\Lambda p}|\nabla\eta_n|^{\alpha+\Lambda p}\,d\mu\right]\\
&\le C\left[\int_{M\setminus B_R} V^{-\beta-\Lambda}|\nabla\varphi|^{\alpha+\Lambda p}\,d\mu + \int_{B_{2nR}\setminus B_{nR}} V^{-\beta-\Lambda}\varphi^{\alpha+\Lambda p}|\nabla\eta_n|^{\alpha+\Lambda p}\,d\mu\right]\\
&:= C(I_1+I_2).
\end{aligned}$$

Now we use condition (1.14) with $\varepsilon = \Lambda$, and we obtain

$$\begin{aligned}
I_2 = \int_{B_{2nR}\setminus B_{nR}} V^{-\beta-\Lambda}\varphi^{\alpha+\Lambda p}|\nabla\eta_n|^{\alpha+\Lambda p}\,d\mu &\le \left(\sup_{B_{2nR}\setminus B_{nR}}\varphi\right)^{\alpha+\Lambda p}\left(\frac{1}{nR}\right)^{\alpha+\Lambda p}\left(\int_{B_{2nR}\setminus B_{nR}} V^{-\beta-\Lambda}\,d\mu\right)\\
&\le Cn^{-(\alpha+\Lambda p)C_1 t}\left(\frac{1}{nR}\right)^{\alpha+\Lambda p}(2nR)^{\alpha+C_0\Lambda}(\log(2nR))^{\beta}\\
&\le Cn^{-(\alpha+\Lambda p)C_1 t - p\Lambda + C_0\Lambda}R^{-p\Lambda+C_0\Lambda}(\log(2nR))^{\beta}.
\end{aligned}$$

By our definition of C_1, Λ and by relation (3.15) we easily find

$$\begin{aligned}
-C_1(\alpha+\Lambda p)t - \Lambda p + \Lambda C_0 &< -\frac{p\sigma t C_1}{\sigma-(t+1)(p-1)} + \frac{\sigma t(p-1)C_0}{[\sigma-(t+1)(p-1)](\sigma-p+1)}\\
&\le -\frac{\sigma t C_0}{[\sigma-(t+1)(p-1)](\sigma-p+1)} < -\frac{\sigma t C_0}{(\sigma-p+1)^2} < 0,
\end{aligned} \tag{3.17}$$

for any small enough $t > 0$. Moreover by (3.15), since $t = \frac{1}{\log R}$, we have

$$R^{-p\Lambda+C_0\Lambda} \le R^{C_0\Lambda} \le R^{\frac{2(p-1)\sigma C_0 t}{(\sigma-p-1)^2}} = e^{\frac{2(p-1)\sigma C_0}{(\sigma-p-1)^2}}.$$

Thus, for any sufficiently large $R > 0$,

$$I_2 \le Cn^{-\frac{\sigma t C_0}{(\sigma-p+1)^2}}(\log(2nR))^{\beta}. \tag{3.18}$$

In order to estimate I_1 we note that if $f : [0,\infty) \to [0,\infty)$ is a nonnegative decreasing function and (1.14) holds, then for any small enough $\varepsilon > 0$ and any sufficiently large $R > 1$ we have

$$\int_{M\setminus B_R} f(r(x))(V(x))^{-\beta-\varepsilon}\,d\mu \le C\int_{\frac{R}{2}}^{+\infty} f(r)r^{\alpha+C_0\varepsilon-1}(\log r)^{\beta}\,dr \tag{3.19}$$

for some positive constant C, see (3.19) and [9, formula (2.19)]. Thus, noting that $|\nabla\varphi| \le C_1 t R^{C_1 t} r^{-C_1 t-1}$ a.e. on M and using (3.15), for every small enough $t > 0$ we have

$$\begin{aligned}
I_1 &\le \int_{M\setminus B_R} V^{-\beta-\Lambda}\left(C_1 t R^{C_1 t} r^{-C_1 t-1}\right)^{\alpha+\Lambda p}\,d\mu\\
&\le C\int_{\frac{R}{2}}^{\infty} R^{(\alpha+\Lambda p)C_1 t}(1+C_1)^{\alpha+\Lambda p}t^{\alpha+\Lambda p}r^{-(\alpha+\Lambda p)(C_1 t+1)+\alpha+C_0\Lambda-1}(\log r)^{\beta}\,dr
\end{aligned}$$

Now, since $t=\frac{1}{\log R}$, by relation (3.15)we have

$$R^{(\alpha+\Lambda p)C_1 t}=e^{(\alpha+\Lambda p)C_1}\leq e^{(\alpha+\varepsilon^* p)C_1};$$

moreover, as we noted already in (3.17), for $t>0$ small enough

$$b=-(\alpha+\Lambda p)(C_1t+1)+\alpha+C_0\Lambda<-\frac{C_0 t\sigma}{(\sigma-p+1)^2}<0.$$

With the change of variables $\xi=|b|\log r$, using the previous relations we find

$$\begin{aligned}I_1&\leq Ct^{\alpha+\Lambda p}\int_1^\infty r^b(\log r)^\beta\,\frac{dr}{r}=Ct^{\alpha+\Lambda p}|b|^{-\beta-1}\left(\int_0^\infty e^{-\xi}\xi^\beta\,d\xi\right)\\&\leq C\left(\frac{(\sigma-p+1)^2}{C_0\sigma}\right)^{\beta+1}t^{\alpha+\Lambda p-\beta-1}=Ct^{\alpha+\Lambda p-\beta-1}.\end{aligned}\tag{3.20}$$

From equations (3.16), (3.18) and (3.20) it follows that

$$\int_M V^{-\beta-\Lambda}|\nabla\varphi_n|^{\alpha+\Lambda p}\,d\mu\leq C\left[t^{\alpha+\Lambda p-\beta-1}+n^{-\frac{\sigma t C_0}{(\sigma-p+1)^2}}(\log(2nR))^\beta\right].\tag{3.21}$$

From (2.12), using (3.14) and (3.21) then we have

$$\begin{aligned}\left(\int_{B_R}u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}&\leq\left(\int_M\varphi_n^s u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}\\&\leq Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left(\int_M V^{-\frac{p-t-1}{\sigma-p+1}}|\nabla\varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu\right)^{\frac{p-1}{p}}\\&\quad\times\left(\int_M V^{-\beta-\Lambda}|\nabla\varphi_n|^{\alpha+\Lambda p}\,d\mu\right)^{\frac{1}{\alpha+\Lambda p}}\\&\leq Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}}\left[n^{-\frac{t}{\sigma-p+1}}(\log(2nR))^\beta+t^{\frac{p(\sigma-t)}{\sigma-p+1}-\beta-1}\right]^{\frac{p-1}{p}}\\&\quad\times\left[t^{\alpha+\Lambda p-\beta-1}+n^{-\frac{\sigma t C_0}{(\sigma-p+1)^2}}(\log(2nR))^\beta\right]^{\frac{1}{\alpha+\Lambda p}}\end{aligned}$$

By taking the lim inf as $n\to+\infty$ we get

$$\left(\int_{B_R}u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}\leq Ct^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}+\frac{(p-1)(\sigma-t)}{\sigma-p+1}-\frac{(\beta+1)(p-1)}{p}+1-\frac{\beta+1}{\alpha+\Lambda p}}$$

for every sufficiently small $t>0$, with $t=\frac{1}{\log R}$. But

$$-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}+\frac{(p-1)(\sigma-t)}{\sigma-p+1}-\frac{(\beta+1)(p-1)}{p}+1-\frac{\beta+1}{\alpha+\Lambda p}=-\frac{(p-1)^2}{p(\sigma-p+1)}t,$$

hence for every small enough $t>0$ we have

$$\left(\int_{B_{e^{1/t}}}u^\sigma V\,d\mu\right)^{1-\frac{(t+1)(p-1)}{p\sigma}}\leq Ct^{-\frac{(p-1)^2}{p(\sigma-p+1)}t}\leq C,$$

that is

$$\int_{B_{e^{1/t}}}u^\sigma V\,d\mu\leq C$$

uniformly in t, for $t>0$ sufficiently small. By taking the limit for $t\to 0^+$ we deduce

$$\int_M u^\sigma V\,d\mu<+\infty,\tag{3.22}$$

and thus $u \in L^\sigma(M, V d\mu)$. Now we exploit inequality (2.7) with the cutoff function φ_n, and using again (3.14) and (3.21) we obtain

$$\int_M \varphi_n^s u^\sigma V\, d\mu \le C t^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}} \left[n^{-\frac{t}{\sigma-p+1}} (\log(2nR))^\beta + t^{\frac{p(\sigma-t)}{\sigma-p+1}-\beta-1} \right]^{\frac{p-1}{p}} \left(\int_{M\setminus B_R} \varphi_n^s u^\sigma V\, d\mu \right)^{\frac{(t+1)(p-1)}{p\sigma}}$$
$$\times \left[t^{\alpha+\Lambda p-\beta-1} + n^{-\frac{\sigma t C_0}{(\sigma-p+1)^2}} (\log(2nR))^\beta \right]^{\frac{1}{\alpha+\Lambda p}}.$$

Since $\varphi_n \equiv 1$ on B_R and $0 < \varphi_n \le 1$ on M, for all $n \in \mathbb{N}$

$$\int_{B_R} u^\sigma V\, d\mu \le \int_M \varphi_n^s u^\sigma V\, d\mu, \qquad \int_{M\setminus B_R} \varphi_n^s u^\sigma V\, d\mu \le \int_{M\setminus B_R} u^\sigma V\, d\mu.$$

Using previous inequalities and taking the lim inf as $n \to +\infty$ we get

$$\int_{B_R} u^\sigma V\, d\mu \le C t^{-\frac{p-1}{p}-\frac{(p-1)^2\sigma}{p(\sigma-p+1)}+\frac{(p-1)(\sigma-t)}{\sigma-p+1}-\frac{(\beta+1)(p-1)}{p}+1-\frac{\beta+1}{\alpha+\Lambda p}} \left(\int_{M\setminus B_R} u^\sigma V\, d\mu \right)^{\frac{(t+1)(p-1)}{p\sigma}}$$
$$= C t^{-\frac{(p-1)^2}{p(\sigma-p+1)}t} \left(\int_{M\setminus B_R} u^\sigma V\, d\mu \right)^{\frac{(t+1)(p-1)}{p\sigma}} \le C \left(\int_{M\setminus B_R} u^\sigma V\, d\mu \right)^{\frac{(t+1)(p-1)}{p\sigma}}$$

uniformly for $t > 0$ sufficiently small, with $t = \frac{1}{\log R}$. Since $u \in L^\sigma(M, V d\mu)$,

$$\int_{M\setminus B_R} u^\sigma V\, d\mu \to 0 \quad \text{as } R \to +\infty.$$

Moreover $\frac{(t+1)(p-1)}{p\sigma} \to \frac{p-1}{p\sigma} > 0$ as $R \to +\infty$. It follows that

$$\int_M u^\sigma V\, d\mu = \lim_{R\to+\infty} \int_{B_R} u^\sigma V\, d\mu = 0,$$

which implies $u \equiv 0$ on M.

(c) Assume that condition (HP3) holds (see (1.15)). Consider the functions φ, η_n and φ_n defined in (3.1), (3.2) and (3.3), with $R > 0$ large enough, $t = \frac{1}{\log R}$, $C_1 \ge \frac{C_0+p+2}{p\sigma}$ and C_0 as in condition (1.15). Arguing as in the previous proof of the theorem under the assumption of the validity of (HP1), by formula (2.3) with any fixed $s \ge \frac{p\sigma}{\sigma-p+1}$, we see that

(3.23)
$$\begin{aligned}\int_M V u^{\sigma-t} \varphi_n^s\, d\mu &\le C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} \int_M V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla \varphi_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \\ &\le C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} \left[\int_M V^{-\frac{p-t-1}{\sigma-p+1}} |\nabla \varphi|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu + \int_{B_{2nR}\setminus B_{nR}} V^{-\frac{p-t-1}{\sigma-p+1}} \varphi^{\frac{p(\sigma-t)}{\sigma-p+1}} |\nabla \eta_n|^{\frac{p(\sigma-t)}{\sigma-p+1}}\, d\mu \right] \\ &:= C t^{-\frac{(p-1)\sigma}{\sigma-p+1}} [I_1 + I_2],\end{aligned}$$

for some positive constant C and for every $n \in \mathbb{N}$ and every small enough $t > 0$. Now, recalling the definitions of φ and η_n, by condition (1.15) with $\varepsilon = \frac{t}{\sigma-p+1}$, for every small enough $t > 0$ we have

$$\begin{aligned} I_2 &\le \left(\sup_{B_{2nR}\setminus B_{nR}} \varphi \right)^{\frac{p(\sigma-t)}{\sigma-p+1}} \left(\frac{1}{nR} \right)^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_{B_{2nR}\setminus B_{nR}} V^{-\beta+\frac{t}{\sigma-p+1}}\, d\mu \\ &\le C n^{-\frac{p(\sigma-t)}{\sigma-p+1}C_1 t} \left(\frac{1}{nR} \right)^{\frac{p(\sigma-t)}{\sigma-p+1}} (2nR)^{\alpha+\frac{C_0 t}{\sigma-p+1}} (\log(2nR))^k e^{-\frac{\theta t}{\sigma-p+1}(\log(2nR))^\tau} \\ &= C n^{\frac{t}{\sigma-p+1}(C_0 - p\sigma C_1 + pC_1 t + p)} R^{\frac{C_0+p}{\sigma-p+1}t} (\log(2nR))^k e^{-\frac{\theta t}{\sigma-p+1}(\log(2nR))^\tau}. \end{aligned}$$

Note that, since $t = \frac{1}{\log R}$, we have

$$R^{\frac{C_0+p}{\sigma-p+1}t} = e^{\frac{C_0+p}{\sigma-p+1}t\log R} = e^{\frac{C_0+p}{\sigma-p+1}}$$

and that by our choice of C_1, if $t > 0$ is sufficiently small, we have $(C_0 - p\sigma C_1 + pC_1 t + p) < -1$. Thus we conclude that

$$I_2 \le C n^{-\frac{t}{\sigma-p+1}} (\log(2nR))^k. \tag{3.24}$$

In order to estimate I_1 we note that if $f : [0,\infty) \to [0,\infty)$ is a nonnegative decreasing function and (1.15) holds, then for any small enough $\varepsilon > 0$ and any sufficiently large $R > 0$ we have

$$\int_{M\setminus B_R} f(r(x))(V(x))^{-\beta+\varepsilon}\,d\mu \le C\int_{\frac{R}{2}}^{+\infty} f(r) r^{\alpha+C_0\varepsilon-1}(\log r)^k e^{-\varepsilon\theta(\log r)^\tau}\,dr \tag{3.25}$$

for some positive fixed constant C. Indeed, by the monotonicity of the involved functions, using condition (1.15) we obtain in a similar way as [9, formula (2.19)]

$$\begin{aligned}
\int_{M\setminus B_R} f(r(x))(V(x))^{-\beta+\varepsilon}\,d\mu &= \sum_{i=0}^{+\infty}\int_{B_{2^{i+1}R}\setminus B_{2^iR}} f(r(x))(V(x))^{-\beta+\varepsilon}\,d\mu \\
&\le \sum_{i=0}^{+\infty} f(2^iR)\int_{B_{2^{i+1}R}\setminus B_{2^iR}} V^{-\beta+\varepsilon}\,d\mu \\
&\le C\sum_{i=0}^{+\infty} f(2^iR)e^{-\varepsilon\theta(\log(2^{i+1}R))^\tau}(2^{i+1}R)^{\alpha+C_0\varepsilon}(\log(2^{i+1}R))^k \\
&\le C\sum_{i=0}^{+\infty} f(2^iR)e^{-\varepsilon\theta(\log(2^{i+1}R))^\tau}(2^{i-1}R)^{\alpha+C_0\varepsilon}(\log(2^{i-1}R))^k \\
&\le C\sum_{i=0}^{+\infty}\int_{2^{i-1}R}^{2^iR} f(r)e^{-\varepsilon\theta(\log r)^\tau} r^{\alpha+C_0\varepsilon-1}(\log r)^k\,dr \\
&= C\int_{\frac{R}{2}}^{+\infty} f(r)e^{-\varepsilon\theta(\log r)^\tau} r^{\alpha+C_0\varepsilon-1}(\log r)^k\,dr.
\end{aligned}$$

Now, since for a.e. $x \in M$ we have

$$|\nabla\varphi(x)| \le C_1 t R^{C_1 t}(r(x))^{-C_1t-1},$$

using (3.25) with $\varepsilon = \frac{t}{\sigma-p+1}$, we obtain that for every small enough $t > 0$ with $t = \frac{1}{\log R}$

$$\begin{aligned}
I_1 &\le \int_{M\setminus B_R} V^{-\beta+\frac{t}{\sigma-p+1}}\left(C_1tR^{C_1t}(r(x))^{-C_1t-1}\right)^{\frac{p(\sigma-t)}{\sigma-p+1}}\,d\mu \\
&\le C\int_{\frac{R}{2}}^{+\infty} R^{\frac{p(\sigma-t)C_1t}{\sigma-p+1}} C_1^{\frac{p(\sigma-t)}{\sigma-p+1}} t^{\frac{p(\sigma-t)}{\sigma-p+1}} r^{-\frac{p(\sigma-t)(C_1t+1)}{\sigma-p+1}+\alpha+\frac{C_0t}{\sigma-p+1}-1}(\log r)^k e^{-\frac{t\theta}{\sigma-p+1}(\log r)^\tau}\,dr.
\end{aligned}$$

Note that $C_1^{\frac{p(\sigma-t)}{\sigma-p+1}} \le (1+C_1)^{\frac{p\sigma}{\sigma-p+1}}$ and that

$$R^{\frac{p(\sigma-t)C_1t}{\sigma-p+1}} = R^{\frac{p(\sigma-t)C_1}{\sigma-p+1}\frac{1}{\log R}} = e^{\frac{p(\sigma-t)C_1}{\sigma-p+1}} \le e^{\frac{p\sigma C_1}{\sigma-p+1}}.$$

Thus, with the change of variable $r = e^\xi$, we deduce

$$\begin{aligned}
I_1 &\le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_1^{+\infty} r^{\frac{t}{\sigma-p+1}(C_0-p\sigma C_1+pC_1t+p)}(\log r)^k e^{-\frac{t\theta}{\sigma-p+1}(\log r)^\tau} r^{-1}\,dr \\
&= Ct^{\frac{p(\sigma-t)}{\sigma-p+1}}\int_0^{+\infty} e^{\frac{t}{\sigma-p+1}(C_0-p\sigma C_1+pC_1t+p)\xi}\xi^k e^{-\frac{t\theta}{\sigma-p+1}\xi^\tau}\,d\xi.
\end{aligned}$$

Now recall that by our choice of C_1, for $t>0$ small enough, we have $(C_0 - p\sigma C_1 + pC_1 t + p) < 0$. Hence, setting $\rho = \left(\frac{t\theta}{\sigma-p+1}\right)^{\frac{1}{\tau}}\xi$, we have

(3.26)
$$I_1 \le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}} \int_0^{+\infty} \xi^k e^{-\frac{t\theta}{\sigma-p+1}\xi^\tau}\, d\xi = Ct^{\frac{p(\sigma-t)}{\sigma-p+1}} \left(\frac{t\theta}{\sigma-p+1}\right)^{-\frac{k+1}{\tau}} \int_0^{+\infty} \rho^k e^{-\rho^\tau}\, d\rho \le Ct^{\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau}}.$$

From (3.23), (3.24) and (3.26) we conclude that for every $n\in\mathbb{N}$ and every small enough $t = \frac{1}{\log R} > 0$ we have

$$\int_{B_R} Vu^{\sigma-t}\, d\mu \le \int_M Vu^{\sigma-t}\varphi_n^s\, d\mu \le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}}\left[t^{\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau}} + n^{-\frac{t}{\sigma-p+1}}(\log(2nR))^k\right]$$

for some fixed positive constant C. Passing to the limit as $n\to+\infty$ in the previous relation yields

$$\int_{B_R} Vu^{\sigma-t}\, d\mu \le Ct^{-\frac{(p-1)\sigma}{\sigma-p+1}+\frac{p(\sigma-t)}{\sigma-p+1}-\frac{k+1}{\tau}}. \tag{3.27}$$

Now note that by our assumptions on τ, k we have

$$-\frac{(p-1)\sigma}{\sigma-p+1} + \frac{p(\sigma-t)}{\sigma-p+1} - \frac{k+1}{\tau} = \frac{\sigma}{\sigma-p+1} - \frac{k+1}{\tau} - \frac{pt}{\sigma-p+1} \ge \frac{1}{2}\left(\frac{\sigma}{\sigma-p+1} - \frac{k+1}{\tau}\right) := \delta_* > 0$$

for every small enough $t = \frac{1}{\log R} > 0$. Thus (3.27) yields

$$\int_{B_{e^{1/t}}} Vu^{\sigma-t}\, d\mu \le Ct^{\delta_*} \tag{3.28}$$

for every small enough $t>0$. Passing to the $\liminf$ as t tends to 0^+ in (3.28), we conclude by an application of Fatou's Lemma that

$$\int_M Vu^\sigma\, d\mu = 0,$$

so that $u\equiv 0$ on M.

4. A problem with lower order terms

In this subsection we consider the semilinear equation

$$\frac{1}{a(x)}\operatorname{div}(a(x)\nabla u) + b(x)u + V(x)u^\sigma \le 0 \quad \text{on } M. \tag{4.1}$$

We start with the following lemma.

Lemma 4.1. *Let $u\in W^{1,2}_{loc}(M)\cap L^\sigma_{loc}(M, V\,d\mu_0)$ be a nonnegative weak solution of (4.1), with a satisfying (1.3), $\sigma>1$, $V>0$ a.e. on M, $V\in L^1_{loc}(M)$ and $b\in L^{\frac{2m}{m+2}}_{loc}(M)$. Assume there exists a weak solution $z>0$, $z\in \mathrm{Lip}_{loc}(M)$ of*

$$\frac{1}{a(x)}\operatorname{div}(a(x)\nabla z) + b(x)z \ge 0 \quad \text{on } M. \tag{4.2}$$

Then $w := \frac{u}{z} \in W^{1,2}_{loc}(M)\cap L^\sigma_{loc}(M, V\,d\mu_0)$ is a nonnegative weak solution of

$$\frac{1}{a(x)z^2(x)}\operatorname{div}\big(a(x)z^2(x)\nabla w\big) + V(x)z^{\sigma-1}(x)w^\sigma \le 0 \quad \text{on } M. \tag{4.3}$$

Proof. By our assumptions, for every $\varphi\in W^{1,2}(M)\cap L^\infty(M)$ with compact support and $\varphi\ge0$ a.e. on M we have

$$-\int_M \langle\nabla u, \nabla\varphi\rangle\, d\mu + \int_M bu\varphi\, d\mu + \int_M Vu^\sigma\varphi\, d\mu \le 0, \tag{4.4}$$

$$-\int_M \langle\nabla z, \nabla\varphi\rangle\, d\mu + \int_M bz\varphi\, d\mu \ge 0. \tag{4.5}$$

We explicitly note that, by our assumptions, all the integrals in (4.4) and (4.5) are finite. Moreover, by a density argument, we easily see that inequality (4.5) also holds for every $\varphi \in W^{1,2}(M)$ with compact support and $\varphi \geq 0$ a.e. on M, not necessarily bounded.

Now we fix $\psi \in W^{1,2}(M) \cap L^\infty(M)$ with compact support and $\psi \geq 0$ a.e. on M, and use $\varphi = z\psi \in W^{1,2}(M) \cap L^\infty(M)$ as a test function in (4.4) and $\varphi = u\psi \in W^{1,2}(M)$ as a test function in (4.5). Subtracting the resulting inequalities one finds

$$-\int_M \langle \nabla u, \nabla \psi \rangle \, z \, d\mu + \int_M \langle \nabla z, \nabla \psi \rangle \, u \, d\mu + \int_M V u^\sigma \psi z \, d\mu \leq 0. \tag{4.6}$$

Since $w = \frac{u}{z} \in W^{1,2}_{\text{loc}}(M) \cap L^\sigma_{\text{loc}}(M, V\, d\mu_0)$ with

$$\nabla w = \frac{1}{z} \nabla u - \frac{u}{z^2} \nabla z \qquad \text{a.e. on } M,$$

inequality (4.6) becomes

$$-\int_M \langle \nabla w, \nabla \psi \rangle \, a z^2 \, d\mu_0 + \int_M \big(V z^{\sigma-1} w^\sigma \psi \big) a z^2 \, d\mu_0 \leq 0.$$

Then, see also Remark 2.2, w is a nonnegative weak solution of (4.3).

Combining Lemma 4.1 with Theorem 1.5, one can easily obtain the following nonexistence results for nontrivial nonnegative weak solutions of equation (4.1).

Proposition 4.2. *Assume there exists a weak solution $z > 0$, $z \in \mathrm{Lip}_{loc}(M)$ of equation (4.2) and let a satisfy (1.3), $\sigma > 1$, $V > 0$ a.e. on M, $V \in L^1_{loc}(M)$ and $b \in L^{\frac{2m}{m+2}}_{loc}(M)$. Then any nonnegative weak solution $u \in W^{1,2}_{loc}(M) \cap L^\sigma_{loc}(M, V\, d\mu_0)$ of (4.1) is identically null, provided one of the following conditions holds:*

i) *there exist $C_0 > 0$, $k \in [0, \frac{1}{\sigma-1})$ such that, for every large enough $R > 0$ and every $\varepsilon > 0$ sufficiently small,*
$$\int_{B_R} V^{-\frac{1}{\sigma-1}+\varepsilon} a z^2 \, d\mu_0 \leq C R^{\frac{2\sigma}{\sigma-1}+C_0\varepsilon} (\log R)^k, \qquad \textit{or}$$

ii) *there exists $C_0 > 0$ such that, for every large enough $R > 0$ and every $\varepsilon > 0$ sufficiently small,*
$$\int_{B_R} V^{-\frac{1}{\sigma-1}+\varepsilon} a z^2 \, d\mu_0 \leq C R^{\frac{2\sigma}{\sigma-1}+C_0\varepsilon} (\log R)^{\frac{1}{\sigma-1}} \qquad \textit{and}$$
$$\int_{B_R} V^{-\frac{1}{\sigma-1}-\varepsilon} a z^2 \, d\mu_0 \leq C R^{\frac{2\sigma}{\sigma-1}+C_0\varepsilon} (\log R)^{\frac{1}{\sigma-1}}, \qquad \textit{or}$$

iii) *there exist $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma-1}{\sigma}(k+1), 1\}$ such that, for every large enough $R > 0$ and every $\varepsilon > 0$ sufficiently small,*
$$\int_{B_{2R} \setminus B_R} V^{-\frac{1}{\sigma-1}+\varepsilon} a z^2 \, d\mu_0 \leq C R^{\frac{2\sigma}{\sigma-1}+C_0\varepsilon} (\log R)^k e^{-\varepsilon\theta(\log R)^\tau}.$$

We now proceed to describe a general setting where one can indeed produce the desired auxiliary function z, in the particular case when $a \equiv 1$ on M.

Let us fix a point $o \in M$ and denote by $\mathrm{Cut}(o)$ the *cut locus* of o. For any $x \in M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$, one can define the *polar coordinates* with respect to o, see e.g. [7]. Namely, for any point $x \in M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$ there correspond a polar radius $r(x) := dist(x, o)$ and a polar angle $\theta \in \mathbb{S}^{m-1}$ such that the shortest geodesics from o to x starts at o with the direction θ in the tangent space T_oM. Since we can identify T_oM with $\mathbb{R}^m$, θ can be regarded as a point of $\mathbb{S}^{m-1}$.

The Riemannian metric in $M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$ in the polar coordinates reads

$$ds^2 = dr^2 + A_{ij}(r, \theta) d\theta^i d\theta^j,$$

where $(\theta^1, \ldots, \theta^{m-1})$ are coordinates in $\mathbb{S}^{m-1}$ and (A_{ij}) is a positive definite matrix. It is not difficult to see that the Laplace-Beltrami operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r,\theta)\frac{\partial}{\partial r} + \Delta_{S_r}, \tag{4.7}$$

where $\mathcal{F}(r,\theta) := \frac{\partial}{\partial r}\big(\log\sqrt{A(r,\theta)}\big)$, $A(r,\theta) := \det(A_{ij}(r,\theta))$, Δ_{S_r} is the Laplace-Beltrami operator on the submanifold $S_r := \partial B(o,r) \setminus \mathrm{Cut}(o)$.

M is a *manifold with a pole*, if it has a point $o \in M$ with $\mathrm{Cut}(o) = \emptyset$. The point o is called *pole* and the polar coordinates (r,θ) are defined in $M \setminus \{o\}$.

A manifold with a pole is a *spherically symmetric manifold* or a *model*, if the Riemannian metric is given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2, \tag{4.8}$$

where $d\theta^2$ is the standard metric in $\mathbb{S}^{m-1}$, and

$$\psi \in \mathcal{A} := \Big\{ f \in C^\infty((0,\infty)) \cap C^1([0,\infty)) : f'(0) = 1,\ f(0) = 0,\ f > 0 \text{ in } (0,\infty) \Big\}. \tag{4.9}$$

In this case, we write $M \equiv M_\psi$; furthermore, we have $\sqrt{A(r,\theta)} = \psi^{m-1}(r)$, so the boundary area of the geodesic sphere ∂S_R is computed by

$$S(R) = \omega_m \psi^{m-1}(R),$$

ω_m being the area of the unit sphere in $\mathbb{R}^m$. Also, the volume of the ball $B_R(o)$ is given by

$$\mu(B_R(o)) = \int_0^R S(\xi)d\xi\,.$$

Moreover we have

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)\frac{\psi'}{\psi}\frac{\partial}{\partial r} + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{m-1}},$$

or equivalently

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{m-1}},$$

where $\Delta_{\mathbb{S}^{m-1}}$ is the Laplace-Beltrami operator in $\mathbb{S}^{m-1}$.

Observe that for $\psi(r) = r$, $M = \mathbb{R}^m$, while for $\psi(r) = \sinh r$, M is the $m-$dimensional hyperbolic space $\mathbb{H}^m$.

Let us recall some useful comparison results for sectional and Ricci curvatures, that will be used in the sequel. For any $x \in M \setminus \big[\mathrm{Cut}(o) \cup \{o\}\big]$, denote by $Ric_o(x)$ the *Ricci curvature* at x in the direction $\frac{\partial}{\partial r}$. Let ω denote any pair of tangent vectors from T_xM having the form $\big(\frac{\partial}{\partial r}, X\big)$, where X is a unit vector orthogonal to $\frac{\partial}{\partial r}$. Denote by $K_\omega(x)$ the *sectional curvature* at the point $x \in M$ of the 2-section determined by ω. If $M \equiv M_\psi$ is a model manifold, then for any $x = (r,\theta) \in M \setminus \{o\}$

$$K_\omega(x) = -\frac{\psi''(r)}{\psi(r)},$$

and

$$Ric_o(x) = -(m-1)\frac{\psi''(r)}{\psi(r)}.$$

Observe moreover that (see [10], [11], [7, Section 15]), if M is a manifold with a pole o and

$$K_\omega(x) \leq -\frac{\psi''(r)}{\psi(r)} \quad \text{for all } \ x = (r,\theta) \in M \setminus \{o\}, \tag{4.10}$$

for some function $\psi \in \mathcal{A}$, then

$$\mathcal{F}(r,\theta) \geq (m-1)\frac{\psi'(r)}{\psi(r)} \quad \text{for all } \ r > 0,\ \theta \in \mathbb{S}^{m-1}\,. \tag{4.11}$$

On the other hand, if M is a manifold with a pole o and

$$Ric_o(x) \geq -(m-1)\frac{\psi''(r)}{\psi(r)} \quad \text{for all } \; x=(r,\theta)\in M\setminus\{o\}, \tag{4.12}$$

for some function $\psi \in \mathcal{A}$, then

$$\mathcal{F}(r,\theta) \leq (m-1)\frac{\psi'(r)}{\psi(r)} \quad \text{for all } \; r>0, \theta\in\mathbb{S}^{m-1}\,. \tag{4.13}$$

We have the following

Lemma 4.3. *Let M be a manifold with a pole o and $b \in L^{\frac{2m}{m+2}}_{loc}(M)$. Let $b_0 : \mathbb{R}^+ \to \mathbb{R}$ be such that*

$$b(r,\theta) \geq b_0(r) \quad \textit{for all } \; x=(r,\theta)\in M\setminus\{o\}. \tag{4.14}$$

Assume that $\psi \in \mathcal{A}$, that $\zeta : \mathbb{R}^+ \to \mathbb{R}$ is a positive weak solution in $\mathrm{Lip}_{loc}(\mathbb{R}^+)$ of

$$\left(\psi^{m-1}\zeta'\right)' + b_0\psi^{m-1}\zeta \geq 0 \qquad \textit{in } \mathbb{R}^+, \tag{4.15}$$

and that either

(A) *ψ satisfies condition* (4.10) *and ζ is nondecreasing,*

or

(B) *ψ satisfies condition* (4.12) *and ζ is nonincreasing.*

Then $z(x) := \zeta(r(x)) \in \mathrm{Lip}_{loc}(M)$ is a positive weak solution of (4.2)*, with $a \equiv 1$ on M.*

Proof. In case condition (A) holds, the result is an easy consequence of (4.7), (4.11), the monotonicity of ζ and condition (4.14). Similarly, when condition (B) holds, the result follows immediately as before, using (4.13) in place of (4.11). □

We refer the interested reader to the stimulating paper of Bianchini, Mari, Rigoli [1] for results concerning the existence of a positive solution of (4.15) and its precise asymptotic behavior as r tends to $+\infty$. These combined with Lemma 4.3 and Proposition 4.2 give somehow explicit nonexistence results for equation (4.1).

5. Counterexamples

In this section, we will produce three counterexamples to the previous nonexistence results, all in the particular case of equation (1.1). Here we follow a similar approach as one finds in [7] and [9]. In the sequel, $\alpha = \frac{2\sigma}{\sigma-1}$ and $\beta = \frac{1}{\sigma-1}$ as in Definition 1.3 with $p=2$, while M will always denote a model manifold with a pole o and metric given by (4.7). Set $B_R \equiv B_R(o)$ and $r \equiv r(x) = dist(x,o)$ for any $x \in M$.

Let $spec(-\Delta)$ be the spectrum in $L^2(M)$ of the operator $-\Delta$. Note that (see [7, Section 10])

$$spec(-\Delta) \subseteq [0,\infty)\,.$$

Denote by $\bar\lambda(M)$ the bottom of $spec(-\Delta)$, that is

$$\bar\lambda(M) := \inf spec(-\Delta)\,.$$

By [2], for each fixed $x \in M$, there holds

$$\bar\lambda(M) \leq \frac{1}{4}\left[\limsup_{R\to+\infty}\frac{\log\mu(B_R(x))}{R}\right]^2 . \tag{5.1}$$

For any $\rho > 0$, let λ_ρ be the first Dirichlet eigenvalue of the Laplace operator in B_ρ, that is the smallest number λ_ρ for which the problem

$$\begin{cases} \Delta u + \lambda u \, = \, 0 & \text{in } B_\rho\,, \\ u \, = \, 0 & \text{on } \partial B_\rho\,. \end{cases} \tag{5.2}$$

has a non-zero solution. Indeed, λ_ρ coincides with the bottom of the spectrum of the operator $-\Delta$ in $L^2(B_\rho)$ with domain $C_0^\infty(B_\rho)$. It is easily checked, see e.g. ****ref uberbook***, that $\lambda_\rho \geq 0$; moreover, $\lambda_{\rho_1} \geq \lambda_{\rho_2}$ if $\rho_1 < \rho_2$, and $\lambda_\rho \to \bar{\lambda}(M)$ as $\rho \to \infty$.

In the sequel we shall make use of the following result (see [9]).

Proposition 5.1. *Let $\sigma > 1$, $r_0 > 0$, $A \in C^1((r_0, \infty))$ with $A > 0$ and $\int_{r_0}^\infty \frac{dr}{A(r)} < \infty$. Let $B \in C((r_0, \infty))$ be such that*

$$\int_{r_0}^\infty [\gamma(r)]^\sigma |B(r)| dr < \infty,$$

where

$$\gamma(r) := \int_r^\infty \frac{d\xi}{A(\xi)} \qquad \text{for } r \geq r_0.$$

Then the equation

$$(A(r)y')' + B(r)y^\sigma \ = \ 0 \qquad \text{for } r > R_0,$$

for $R_0 > r_0$ sufficiently large, admits a positive solution $y(r)$ such that $y(r) \sim \gamma(r)$ as $r \to \infty$.

Example 5.2. Let $\psi \in \mathcal{A}$, see (4.9), with

$$\psi(r) := \begin{cases} r & \text{if } 0 \leq r < 1\,, \\ [r^{\alpha-1}(\log r)^{\beta_0}]^{\frac{1}{m-1}} & \text{if } r > 2\,; \end{cases}$$

here $\beta_0 > \beta$. Let $0 < \delta < \beta_0 - \beta$ and define

$$V(x) \equiv V(r) := (\log(2+r))^{\frac{\delta}{\beta}} \quad \text{for all } \ x \in M\,.$$

For any $R > 0$ sufficiently large we have

$$S(R) = \omega_m R^{\alpha-1}(\log R)^{\beta_0}, \quad \mu(B_R) \simeq CR^\alpha (\log R)^{\beta_0}\,;$$

thus, thanks to (5.1), we have $\bar{\lambda}(M) = 0$. Moreover, there holds

$$\int_{B_R} V^{-\beta}(x)\, d\mu \geq CR^\alpha(\log R)^{\beta_0-\delta}, \tag{5.3}$$

with $\beta_0 - \delta > \beta$. Hence, in view of (5.3), neither condition (1.13) nor condition (1.14) is satisfied. Furthermore, observe that (1.18) holds true, while (1.19) fails. This is essentially due to the choice of ψ.

Note that for any $r_0 > 0$,

$$\int_{r_0}^\infty \frac{d\xi}{S(\xi)} < \infty\,; \tag{5.4}$$

moreover, for $r > 0$ sufficiently large,

$$\gamma(r) := \int_r^\infty \frac{d\xi}{S(\xi)} \simeq \frac{C}{r^{\alpha-2}(\log r)^{\beta_0}}\,.$$

Hence for $r_0 > 0$ large enough

$$\begin{aligned} \int_{r_0}^\infty [\gamma(r)]^\sigma V(r) S(r)\, dr &\leq C \int_{r_0}^\infty \frac{[\log(2+r)]^{\delta/\beta} r^{\alpha-1}(\log r)^{\beta_0}}{r^{\sigma(\alpha-2)}(\log r)^{\beta_0\sigma}}\, dr \\ &\leq C \int_{r_0}^\infty (\log r)^{\beta_0(1-\sigma)+\frac{\delta}{\beta}} \frac{dr}{r} < \infty\,. \end{aligned} \tag{5.5}$$

In view of (5.4)-(5.5), from Proposition 5.1 with $A(r) = S(r)$ and $B(r) = S(r)V(r)$, we have that there exists a solution $y = y(r)$ of

$$y''(r) + \frac{S'(r)}{S(r)} y'(r) + V(r)[y(r)]^\sigma = 0, \quad r > R_0, \tag{5.6}$$

for some $R_0 > r_0$. Furthermore, $y(r) > 0$ for all $r \in [R_0, \infty)$ and $y(r) \sim \gamma(r)$ as $r \to \infty$.

Now for any $\rho > 0$, let v_ρ be the solution to the eigenvalue problem (5.2) with $\lambda = \lambda_\rho$, which we can assume is normalized by setting $v_\rho(o) = 1$. Thus we have that $v_\rho \equiv v_\rho(r)$, that $0 < v_\rho(r) \leq 1$ and that

$v_\rho(r)$ is decreasing for $r \in [0,\rho]$. For any $\rho > R_0$, define $m := \inf_{[R_0,\rho)} \dfrac{y(r)}{v_\rho(r)}$ and for any fixed $\xi \in (R_0,\rho)$ let

$$\widetilde{u}(x) := \begin{cases} m v_\rho(r) & \text{in } B_\xi, \\ y(r) & \text{on } \partial B_\xi^c. \end{cases}$$

Since $\bar{\lambda}(M) = 0$, as in [9] one can prove that for some $\xi \in (R_0,\rho)$ we have $\widetilde{u} \in C^1(M)$, and thus $\widetilde{u} \in W^1_{\mathrm{loc}}(M)$. Moreover

$$\begin{aligned} \Delta(m v_\rho) + \frac{\lambda_\rho}{m^{\sigma-1}}(m v_\rho)^\sigma = 0 \qquad & \text{in } B_\rho, \\ \Delta y + V y^\sigma = 0 \qquad & \text{in } B^c_{R_0}. \end{aligned}$$

Define $M_\rho = \max_{\overline{B}_\rho} V$ and

$$\delta = \min\left\{1, m^{-1}\lambda_\rho^{\frac{1}{\sigma-1}} M_\rho^{-\frac{1}{\sigma-1}}\right\};$$

then, since $V > 0$ on M, $\delta > 0$ and on B_ρ we have

$$\begin{aligned} \Delta(\delta m v_\rho) + V(\delta m v_\rho)^\sigma &\le \Delta(\delta m v_\rho) + M_\rho(\delta m v_\rho)^\sigma \\ &\le \Delta(\delta m v_\rho) + \frac{\lambda_\rho}{(\delta m)^{\sigma-1}}(\delta m v_\rho)^\sigma = 0. \end{aligned}$$

On the other hand on $B^c_{R_0}$ we have

$$\Delta(\delta y) + V(\delta y)^\sigma \le \delta \Delta y + \delta V y^\sigma = 0.$$

Thus we see that the function $u = \delta \widetilde{u}$ is positive and satisfies

$$\Delta u + V u^\sigma \le 0 \qquad \text{on } M.$$

Example 5.3. Let $\psi \in \mathcal{A}$ with

$$\psi(r) := \begin{cases} r & \text{if } 0 \le r < 1\,, \\ [r^{\alpha-1}(\log r)^\beta]^{\frac{1}{m-1}} & \text{if } r > 2\,. \end{cases}$$

Let $\delta > 0$ and define

$$V(x) \equiv V(r) := (\log(2+r))^{-\frac{\delta}{\beta}} \quad \text{for all } \ x \in M\,.$$

For any $R > 0$ sufficiently large we have

$$S(R) = \omega_m R^{\alpha-1}(\log R)^\beta, \quad \mu(B_R) \simeq C R^\alpha (\log R)^\beta\,,$$

thus, thanks to (5.1), we conclude taht $\lambda_1(M) = 0$. Moreover, there holds

$$\int_{B_R} V^{-\beta}(x)\, d\mu \ge C R^\alpha (\log R)^{\beta+\delta}\,. \tag{5.7}$$

Observe that in view of (5.7), neither condition (1.13) nor condition (1.14) is satisfied. Moreover, note that (1.18) holds, while (1.19) fails. This is essentially due to the choice of V.

For any $r_0 > 0$ we have

$$\int_{r_0}^\infty \frac{d\xi}{S(\xi)} < \infty\,; \tag{5.8}$$

moreover, for $r > 0$ large enough,

$$\gamma(r) := \int_r^\infty \frac{d\xi}{S(\xi)} \simeq \frac{C}{r^{\alpha-2}(\log r)^\beta}\,.$$

Hence for $r_0 > 0$ large enough

$$\int_{r_0}^{\infty} [\gamma(r)]^{\sigma} V(r) S(r)\, dr \le C \int_{r_0}^{\infty} \frac{[\log(2+r)]^{-\delta/\beta} r^{\alpha-1} (\log r)^{\beta}}{r^{\sigma(\alpha-2)} (\log r)^{\beta\sigma}}\, dr \le C \int_{r_0}^{\infty} \frac{1}{(\log r)^{\beta(\sigma-1)+\frac{\delta}{\beta}}} \frac{dr}{r} < \infty. \tag{5.9}$$

In view of (5.8)-(5.9), from Proposition 5.1 with $A(r) = S(r)$ and $B(r) = S(r)V(r)$, we have that there exists a solution $y = y(r)$ of (5.6), for some $R_0 > 0$. Furthermore, $y(r) > 0$ for all $r \in [R_0, \infty)$ and $y(r) \sim \gamma(r)$ as $r \to \infty$. Since $\lambda_1(M) = 0$ and $V > 0$ on M, by the same arguments as in the previous Example 5.2, we can construct $u \in C^1(M)$, with $u = u(r) > 0$ on M, which satisfies

$$\Delta u + V u^{\sigma} \le 0 \quad \text{ in } M.$$

Example 5.4. Let $\psi \in \mathcal{A}$ with

$$\psi(r) := \begin{cases} r & \text{if } 0 \le r < 1\,, \\ e^{\sqrt{r}} & \text{if } r > 2\,. \end{cases}$$

For any sufficiently large $R > 0$ we have

$$S(R) = \omega_m e^{(m-1)\sqrt{r}}, \quad \mu(B_R) \simeq C e^{(m-1)\sqrt{R}} [(m-1)\sqrt{R} - 1],$$

for some $C > 0$. Note that $\bar{\lambda}(M) = 0$ by (5.1), since

$$\limsup_{R\to\infty} \frac{\log(\mu(B_R))}{R} = 0.$$

Let

$$\eta = \frac{m-1}{\beta} = (m-1)(\sigma-1), \quad \theta = \frac{\sigma+1}{\sigma-1}$$

and define

$$V(x) \equiv V(r) := e^{\eta\sqrt{r}} (1+r)^{-\frac{\theta}{\beta}} \quad \text{ for all } x \in M.$$

Then for $\varepsilon > 0$ small enough and $R > 0$ sufficiently large we have

$$\int_{B_R} V^{-\beta+\varepsilon}(x)\, d\mu \ge C \int_2^R e^{\varepsilon\eta\sqrt{r}} (1+r)^{\theta(1-\varepsilon/\beta)}\, dr \ge C e^{\varepsilon\eta\sqrt{R}}. \tag{5.10}$$

On the other hand

$$\int_{B_R} V^{-\beta-\varepsilon}(x)\, d\mu \le C \int_0^R e^{-\varepsilon\eta\sqrt{r}} (1+r)^{\theta(1+\varepsilon/\beta)}\, dr \le C R^{\theta+1+\frac{\theta}{\beta}\varepsilon} = C R^{\alpha+\frac{\theta}{\beta}\varepsilon}. \tag{5.11}$$

Observe that, in view of (5.10), neither condition (1.13) nor the first inequality in condition (1.14) is satisfied. On the other hand, by (5.11) the second inequality in (1.14) holds. This is essentially due to the choice of V. Note moreover that only the second inequality in (1.18) is not satisfied, while the first inequality in (1.18) and (1.19) hold.

Note that for any $r_0 > 0$

$$\int_{r_0}^{\infty} \frac{d\xi}{S(\xi)} < \infty\,; \tag{5.12}$$

moreover, for $r > 0$ sufficiently large,

$$\gamma(r) := \int_r^{\infty} \frac{d\xi}{S(\xi)} \simeq C e^{-(m-1)\sqrt{r}} \sqrt{r}.$$

Hence

$$\int_{r_0}^{\infty} [\gamma(r)]^{\sigma} V(r) S(r) dr \le C \int_{r_0}^{\infty} e^{[-(m-1)(\sigma-1)+\eta]\sqrt{r}} r^{\frac{\sigma}{2}-\frac{\theta}{\beta}} < \infty. \tag{5.13}$$

In view of (5.12)-(5.13), from Proposition 5.1 with $A(r) = S(r)$ and $B(r) = S(r)V(r)$, we have that there exists a solution $y = y(r)$ of (5.6), for some $R_0 > 0$. Furthermore, $y(r) > 0$ for all $r \in [R_0, \infty)$ and $y(r) \sim \gamma(r)$ as $r \to \infty$. Since $\bar{\lambda}(M) = 0$ and $V > 0$ on M, by the same arguments as in Example 5.2

above, we can construct $u \in C^1(M)$ with $u = u(r) > 0$ on M, which is a weak solution of

$$\Delta u + V u^\sigma \leq 0 \quad \text{in } M.$$

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