Semistrictly quasiconcave approximation and an application to general equilibrium theory

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ABSTRACT

We show how to approximate, in the sense of continuous convergence, a quasiconcave function with a sequence of semistrictly quasiconcave functions. This allows extending former existence results of equilibria for pure exchange economies when the preferences of the agents allow for local points of satiation, and existence results of free disposal equilibria for economies with production.

1. Introduction

Approximating a concave function by a sequence of strictly concave functions is quite simple, since the sum of a concave and a strictly concave function is strictly concave. The problem is more subtle in the case of a quasiconcave function, since the above argument cannot be adapted to this case: the sum of a quasiconcave function with a concave one need not to be quasiconcave. In this paper we show how to approximate a quasiconcave function with a sequence of semistrictly quasiconcave functions. Our result holds in a general Banach space. We consider the case when the domain of the quasiconcave function f and the function itself are bounded. In this case we provide a sequence of semistrictly quasiconcave functions f_n converging uniformly to f. Then we consider the case of an unbounded function f defined on a bounded domain. An adaptation of the construction provided in the above case produces another approximation result. In this case the convergence guaranteed is of a continuous type, in the sense that we produce semistrictly quasiconcave functions f_n, defined on the domain of f, such that, for every x in the domain of f and for every sequence $\{x_n\}$ of elements in its domain with $x_n \to x$, it holds that $f_n(x_n) \to f(x)$.

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Finally, the same convergence mode is guaranteed also in the case when the domain of f is not bounded, but in such a case the approximating functions have bounded domains invading the domain of the limit function f.

In the second part of the paper, we apply our previous results to obtain some general theorems of existence of an equilibrium for market economies. In the classical setting (we refer in particular to the papers [3,6], and [7]), usually it is assumed that consumers have strictly convex preferences, and it is well known that this is equivalent to semistrict quasiconcavity of the associated utility functions. Then a nonsatiation property is assumed, at least when the consumption set of the agents is unbounded, and this means that for every bundle x of goods in the consumption set there must be another bundle which is strictly preferred to x. In the proofs of the existence of an equilibrium, a crucial step is to guarantee that the nonsatiation property holds also locally, in the sense that the bundle preferred to x must be found arbitrarily close to x itself. Our existence theorem avoids this, via an approximation process. In other words, given an economy where the utility functions of the agents are only quasiconcave, by means of our construction we approximate the economy with a sequence of economies where the utility functions are semistrictly quasiconcave; this allows using classical theorems. Then via a standard limit argument we find an equilibrium for the initial economy. Some recent papers deal with the case of bounded consumption sets, and thus obviously there are satiation points for the agents. Thus some theorems were provided in order to have weak requirements about these satiation points (see [1,2,11]). We show that also in this case our approach allows us generalizing these results.

Nowadays the mathematical study of the competitive equilibrium is an active and investigated research topic; an alternative approach to the study of this topic is provided by variational methods (see for instance [4,8,10]). In particular, authors give a new formulation of a competitive equilibrium in terms of a suitable quasivariational inequality involving multivalued maps. This characterization is used to give the existence of the equilibrium when utility functions are semistrictly quasiconcave. To our knowledge, semistrict quasi-concavity is (in a convex setting for preferences of the agents) the weakest condition, present in literature, to guarantee the existence of an equilibrium in an economy (see also [5] and [12] for other existence theorems).

2. Notation and preliminaries

In this section X is a closed and convex subset of a Banach space. Furthermore, we shall use the following notations throughout the paper. For a vector w in some Euclidean space $\mathbb{R}^k$, we denote by w^+ (w^-) the vector whose j-th component is $w_j^+ = \max\{w_j, 0\}$ ($w_j^- = \min\{w_j, 0\}$), so that $w = w^+ + w^-$. Moreover, given two vectors $w, z \in \mathbb{R}^k$ we shall write $w \geq z$ if $z \in w + \mathbb{R}^k_+$, $w > z$ if $z \in w + \mathbb{R}^k_+ \setminus \{0\}$ and $w >> z$ if $z \in w +$ int $\mathbb{R}^k_+$. The closed ball centered at x and with radius r is denoted by $B(x; r)$, while $S(x; r)$ will be its boundary. We recall now a classical definition of set convergence, that will be used in the sequel.

Definition 2.1. Suppose A_n are nonempty subsets of a Banach space. The *lower limit* of the sequence A_n is the set

$$\operatorname{Li} A_n = \{x : x = \lim x_n, x_n \in A_n\}.$$

The *upper limit* of the sequence A_n is the set

$$\operatorname{Ls} A_n = \{x : x = \lim x_k, x_k \in A_{n_k}\}$$

where n_k is a subsequence of the positive integers. Finally, we say the A_n converges to A in *Kuratowski sense* if

$$\operatorname{Ls} A_n \subset A \subset \operatorname{Li} A_n.$$

The set Li A_n is convex provided the sets A_n are convex and it is always closed (see [9, Proposition 8.2.1]). Thus a Kuratowski limit of a sequence of convex sets is always closed and convex. For more about set convergences, see [9, Chapters 8 and B.4].

Given a real valued function f, defined on X, we denote by f^λ and $f^\lambda_>$ the upper level and strict upper level sets of f at height λ:

$$f^\lambda = \{x \in X : \; f(x) \geq \lambda\}, \qquad f^\lambda_> = \{x \in X : \; f(x) > \lambda\}.$$

Definition 2.2. A function $f : X \to \mathbb{R}$ is said to be

- *quasiconcave* if for every $x, y \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\} \, ;$$

- *semistrictly quasiconcave* if for every $x, y \in X$ such that $f(y) \neq f(x)$, and for every $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\};$$

As it is well-known, properties of the upper level sets characterize some properties of the associated functions:

- a function f is quasiconcave if and only if, for every $\lambda \in \mathbb{R}$, the upper level set f^λ is convex, if and only if, for every $\lambda \in \mathbb{R}$, the strict upper level set $f^\lambda_>$ is convex;
- f is upper semicontinuous if and only if, for every $\lambda \in \mathbb{R}$, the upper level set f^λ is closed.

Observe that a concave function (not necessarily strictly concave) is semistrictly quasiconcave. Quasiconcave functions can have local maxima which are not global maxima, as easy examples show. On the contrary, for a semistrictly quasiconcave function a local maximum is automatically a global one, as it is obvious from the definition.

We now define a function which will play a crucial role in our approximation argument. Suppose Q and P are convex closed subsets of a Banach space, with $\emptyset \neq P \subset \operatorname{int} Q$. Let $\mu_{P,Q}(x)$ be the following function

$$\mu_{P,Q}(x) = \inf\{\lambda \geq 0 : \; x \in \lambda Q + (1-\lambda)P\}.$$

In the following remark we summarize some properties of $\mu_{P,Q}$ that will be used in the sequel. Their proofs are straightforward.

Remark 1. For every P, Q as above:

1. $\mu_{P,Q}$ is a real-valued, continuous and convex function;
2. $\mu_{P,Q}(x) = 0$ if $x \in P$, $\mu_{P,Q}(x) < 1$ if $x \in \operatorname{int} Q$ and $\mu_{P,Q}(x) > 1$ if $x \notin Q$;
3. If $x = \lambda q + (1-\lambda)p$ with $q \in \operatorname{int} Q$ and $p \in P$, then $\lambda > \mu_{P,Q}(x)$.

Finally we remind the property, used in the sequel, that if A is convex, $x \in \operatorname{int} A$, $y \in \operatorname{cl} A$, then every z in the segment $[x, y)$ lies in int A (see [9, Proposition 1.1.14]).

3. Approximating quasiconcave functions

As already mentioned, we are interested in approximating quasiconcave functions with semistrictly quasiconcave functions. In order to build our approximating sequence, we introduce some more notation. In particular, given a quasiconcave, bounded and continuous function f defined on a closed convex subset X of a Banach space we introduce, for every $n \in \mathbb{N}$, a partition $\mathcal{P}^n = \{\alpha_k^n\}_{k=0,\ldots,n}$ of the interval $[\inf f, \sup f]$ with the following properties:

1. $\alpha_0^n = \sup f > \alpha_1^n > \cdots > \alpha_n^n = \inf f$;
2. $\alpha_k^n - \alpha_{k+1}^n < \dfrac{2}{n}(\sup f - \inf f)$, for all $k = 0, \ldots, n-1$.

For easy notation, we denote by f_n^k and $f_{n>}^k$, respectively, the upper level sets $f^{\alpha_k^n}$ and $f_>^{\alpha_k^n}$. Since f is quasiconcave and continuous, the upper level sets f_n^k are convex and closed; moreover, since f is continuous, $f_n^k \subset \operatorname{int} f_n^{k+1}$, where the interior is intended in the relative topology of X. We now consider the functions:

$$\mu_k^n(x) = \inf\{\lambda \geq 0 : \ x \in \lambda f_n^{k+1} + (1-\lambda) f_n^k\}.$$

Finally, we define the following maps, for every $n \in \mathbb{N}$, and $k = 1, \ldots, n-1$:

$$f_n(x) = \begin{cases} \alpha_1^n & \text{if } \alpha_0^n \geq f(x) \geq \alpha_1^n, \\ \alpha_k^n + (\alpha_{k+1}^n - \alpha_k^n)\mu_k^n(x) & \text{if } \alpha_k^n > f(x) \geq \alpha_{k+1}^n. \end{cases} \tag{1}$$

The following remark is obvious, but useful for the sequel.

Remark 2. From Remark 1, it follows that:

$$f(x) \in [\alpha_{k+1}^n, \alpha_k^n) \Leftrightarrow f_n(x) \in [\alpha_{k+1}^n, \alpha_k^n).$$

Remark 3. For later purposes, we explicitly observe that, denoting by S (S_n) the set of the points maximizing f (f_n) on X, it holds that $S \subset S_n$ for all n. Moreover S_n is always nonempty while S is nonempty if and only if $\operatorname{Ls} S_n \neq \emptyset$.

Proposition 1. *Suppose X is closed convex bounded; let $f : X \to \mathbb{R}$ be continuous quasiconcave and bounded. Then for every $n \in \mathbb{N}$ the function f_n, as defined in (1), is continuous and semistrictly quasiconcave.*

Proof. Continuity of f_n follows from Remark 1. Now, let $x, y \in X$ be such that $f_n(x) > f_n(y)$, take any $\lambda \in (0, 1)$, and let z be $z = \lambda x + (1 - \lambda)y$. We have to prove that $f_n(z) > f_n(y)$.

Suppose that $\alpha_i^n > f_n(x) \geq \alpha_{i+1}^n$, $\alpha_k^n > f_n(y) \geq \alpha_{k+1}^n$. Then it holds that $i \geq k$. In case $i = k$, the result follows from convexity of μ_k^n. If $i > k$, then the only case to consider is when $\alpha_k^n > f_n(z) \geq \alpha_{k+1}^n$. And one more time the inequality follows from convexity of μ_k^n, since $\mu_k^n(x) = 0$ and thus

$$f_n(z) = \alpha_k^n + (\alpha_{k+1}^n - \alpha_k^n)\mu_k^n(\lambda x + (1-\lambda)y) > \alpha_k^n + (\alpha_{k+1}^n - \alpha_k^n)\mu_k^n(y) = f_n(y).$$

With the help of Proposition 1, we can now state the first approximation result.

Theorem 3.1. *Let X be a closed bounded convex set. Suppose f is a bounded quasiconcave and continuous function on X. Then there exists a sequence (f_n) of continuous and semistrictly quasiconcave functions $f_n : X \to \mathbb{R}$, converging uniformly to f on X:*

$$\lim_{n \to +\infty} \sup_{x \in X} |f_n(x) - f(x)| = 0.$$

Proof. Let f_n be defined as in (1). The only thing we have to prove is that, for every $\varepsilon > 0$, there is N such that, for all $n \geq N$,

$$\sup_{x\in X} |f_n(x) - f(x)| < \varepsilon.$$

Due to Remark 2 it is enough to take N such that $\frac{2}{N}(\sup f - \inf f) < \varepsilon$.

Corollary 3.1. *Let f be a quasiconcave bounded and continuous function on X, let $x_0, x_1, \ldots, x_n, \ldots$ be such that $x_i \in X$ for all $i \geq 0$, and let $\lim_{n\to+\infty} x_n = x_0$. Then $\lim_{n\to+\infty} f_n(x_n) = f(x_0)$.*

We now consider the case when f is unbounded. From Theorem 3.1, we can get:

Theorem 3.2. *Let X be a bounded closed convex set; let $f : X \to \mathbb{R}$ be a continuous quasiconcave function. Then there exists a sequence (f_n) of functions $f_n : X \to \mathbb{R}$, continuous and semistrictly quasiconcave, such that for $x_0, x_1, \ldots, x_n, \ldots$ such that $x_i \in X$ for all $i \geq 0$, and $\lim_{n\to+\infty} x_n = x_0$, it holds $\lim_{n\to+\infty} f_n(x_n) = f(x_0)$.*

Proof. We need to consider only the case of an unbounded function f. For all $n \in \mathbb{N}$, define:

$$g_n(x) = \begin{cases} n & \text{if } f(x) \geq n, \\ f(x) & \text{if } -n < f(x) < n, \\ -n & \text{if } f(x) \leq -n. \end{cases}$$

From Theorem 3.1, we know that for every n there exists f_n such that

$$\sup_{x\in X} |f_n(x) - g_n(x)| < (1/n).$$

Now fix, x_0 and $\varepsilon > 0$. We prove that, if $x_n \to x_0$ as in the statement, then for all large n it is $|f_n(x_n) - f(x_0)| < \varepsilon$. Fix N so large that the following conditions are fulfilled:

- $N > \frac{2}{\varepsilon}$;
- there exists a neighborhood I of x_0 such that, $\forall x \in I$, it holds:

$$|f(x)| < N \quad \wedge \quad |f(x) - f(x_0)| < \varepsilon/2.$$

Then for all $n \geq N$ it holds that

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - g_n(x_n)| + |f(x_n) - f(x_0)| < \varepsilon.$$

We now want to consider the case when X is unbounded.

Theorem 3.3. *Let X be a closed convex set; let $f : X \to \mathbb{R}$ be a continuous, quasiconcave function. Then there exists a sequence (f_n) of functions $f_n : X_n \to \mathbb{R}$, continuous and strictly quasiconcave, such that the sequence (X_n) converges to X in Kuratowski sense*[1] *and such that, for $x_0, x_1, \ldots, x_n, \ldots$ such that $x_i \in X$ for all $i \geq 0$, and $\lim_{n\to+\infty} x_n = x_0$, it holds that $\lim_{n\to+\infty} f_n(x_n) = f(x)$.*

Proof. Let X_n be $X_n = X \cap B(0; n)$. It is quite clear that the sequence (X_n) converges to X in Kuratowski sense. Then the proof is a simple adaptation to this case of the proof of Theorem 3.2, applied to the restriction of f to X_n.

[1] We can observe, for the reader familiar with hypertopologies, that actually we provide convergence also for finer topologies like the Mosco and bounded proximal topologies, and more generally *all* hypertopologies having the lower Vietoris topology as lower part (see [9, Chapter 8]).

4. Some equilibria results

In this section we apply the previous results in order to generalize some theorems stating the existence of an equilibrium for competitive economies. In particular, we shall see how it is possible to relax the assumption of semistrictly quasiconcave utility functions for the agents.

4.1. Free-disposal equilibrium

Our first example deals with a free-disposal economy. The activities considered in the model are: exchange, consumption and production. There are n consumers, indexed by $i \in I = \{1, \ldots, n\}$, m producers, indexed by $j \in J = \{1, \ldots, m\}$ and l different goods indexed by $h \in \{1, \ldots, l\}$. To each commodity h is associated a nonnegative price p_h; then $p = (p_1, \ldots, p_l) \in \mathbb{R}^l_+$ denotes a generic price vector. Each producer $j \in J$ is characterized by a production set $Y_j \subseteq \mathbb{R}^l$ of possible production plans. Y_j represents the technology available to producer j and Y denotes the aggregate production set of the economy: $Y = \sum_j Y_j$. Given a production vector y_j, y_j^+ is a vector of goods produced by j by making use of the vector of goods y_j^-. Taking into account that prices are nonnegative, $\langle p, y_j \rangle$ will be what the producer j gets as income when she offers the production vector y_j to the market, at the price p. Thus, given the price vector $p \in \mathbb{R}^l_+$, the producer j faces the problem of finding a production plan maximizing her profit $\langle p, y_j \rangle$:

$$\text{find } \bar{y}_j \in Y_j \quad \text{such that} \quad \langle p, \bar{y} \rangle = \max_{y_j \in Y_j} \langle p, y_j \rangle.$$

Each consumer $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^l$, and by a binary, reflexive, transitive and complete relation $\succcurlyeq_i$ on X_i, expressing his preferences over the consumption set X_i. In a fairly general setting (more general than that one described here), these preference systems can be characterized by utility functions u_i, defined on X_i: $u_i(x_1) \geq u_i(x_2)$ if and only if $x_1 \succcurlyeq_i x_2$. Each consumer is endowed with an initial endowment $e_i \in X_i$, representing the amount of the various goods that he can consume or trade with other individuals. Each consumer chooses a consumption plan $x_i = (x_i^1, \ldots, x_i^l) \in X_i$, where x_i^h represents the quantity of commodity h consumed by i and $x = (x_1, \ldots, x_n) \in X = \prod_{i \in I} X_i \subseteq \mathbb{R}^{l \times n}$ is the total consumption of market. If x_i belongs to the consumption set of the consumer i, then the x_i^+ represents the consumer's demand for the commodity h, while $-x_i^-$ represents his supply. Moreover, the total production $\displaystyle\sum_{j \in J} y_j^h$ of commodity h is shared among consumers: each consumer i receives the given fraction $\displaystyle\sum_{j \in J} \theta_{ij} y_j^h$, determined by a system of weights $\theta_{ij} \geq 0$ having the property that $\displaystyle\sum_{i \in I} \theta_{ij} = 1$ for all $j \in J$. Hence, each consumer i, relative to commodity h, has at command the quantity $e_i^h + \displaystyle\sum_{j \in J} \theta_{ij} y_j^h$. Thus, if y is the production of the market, the wealth of the i-th consumer, at the current price system p, is $w_i = \langle p, e_i \rangle + \sum_{j \in J} \theta_{ij} \langle p, y_j \rangle$.

Summarizing, each consumer is operating in the market to maximize his utility subject to a natural budget constraint: the value of the consumption plan of consumer i at the current price p, $\langle p, x_i \rangle$, cannot exceed his wealth w_i. Denote by $M_i(p, y)$ the set of the consumption vector available to consumer i at the current price p:

$$M_i(p, y) = \left\{ x_i \in X_i \ : \langle p, x_i \rangle \leq \langle p, e_i \rangle + \sum_{j \in J} \theta_{ij} \langle p, y_j \rangle \right\}.$$

Then the consumer i faces the following maximization problem:

$$\text{find } \bar{x}_i \in M_i(p, y) \quad \text{such that} \quad u_i(\bar{x}_i) = \max_{x_i \in M_i(p,y)} u_i(x_i)$$

where $M_i(p, y)$ represents the budget constraint of the consumer i, at the price p and production y.

The market is usually considered to be in equilibrium when the supply for each commodity equals the demand; but sometimes a weaker condition, called *free-disposal*, is assumed: first of all, demand cannot exceed supply:

$$\sum_{i\in I}(\bar{x}_i - e_i) - \sum_{j\in J}\bar{y}_j \leq 0.$$

Furthermore, the price of a good not saturated by the market must be zero:

$$\langle\sum_{i\in I}(\bar{x}_i - e_i) - \sum_{j\in J}\bar{y}_j, p\rangle = 0.$$

Thus, a competitive economy Ξ is described by the m-list (X_i, u_i, e_i), by the mn-shares (θ_{ij}) and by the n-list (Y_j):

$$\Xi = \Big((X_i, u_i, e_i)_{i\in I}, (\theta_{ij})_{i\in I, j\in J}, (Y_j)_{j\in J}\Big).$$

A *state* of the economy Ξ is an m-list x_i of consumptions of the consumers, an n-list y_j of productions of the producers, and a price vector p. We define an *attainable state* for the economy by the conditions:

(a) for every i, x_i is in X_i
(b) for every j, y_j is in Y_j
(c) $\sum_{i\in I}(x_i - e_i) - \sum_{j\in J} y_j \leq 0.$

The *attainable consumption set* $\widehat{X}_i$ of the i-th consumer is the set of his attainable consumptions and the *attainable production set* $\widehat{Y}_j$ of the j-th producer is the set of her attainable productions:

$$\widehat{X}_i = \{x_i \in X_i : \exists\, x_{i'} \in X_{i'}\ \forall i \neq i',\ \exists y_j \in Y_j\ :\ \sum_{i\in I}(x_i - e_i) - \sum_{j\in J} y_j \leq 0\},$$
$$\widehat{Y}_j = \{y_j \in Y_j : \exists\, y_{j'} \in Y_{j'}\ \forall j' \neq j,\ \exists x_i \in X_i\ :\ \sum_{i\in I}(x_i - e_i) - \sum_{j\in J} y_j \leq 0\}.$$

Definition 4.1. A state $(\bar{p}, \bar{x}, \bar{y})$ is a free-disposal equilibrium of the economy Ξ if $\bar{p} > 0$ and

$$\text{for all } i \in I,\ \ u_i(\bar{x}_i) = \max_{x_i \in M_i(\bar{p},\bar{y})} u_i(x_i) \tag{2}$$

$$\text{for all } j \in J,\ \ \langle\bar{p}, \bar{y}_j\rangle = \max_{y_j\in Y_j}\langle\bar{p}, y_j\rangle \tag{3}$$

$$\sum_{i\in I}(\bar{x}_i - e_i) - \sum_{j\in J}\bar{y}_j \leq 0,\ \ \langle\sum_{i\in I}(\bar{x}_i - e_i) - \sum_{j\in J}\bar{y}_j, \bar{p}\rangle = 0. \tag{4}$$

We now introduce some further notation. S_i represents the set of all *satiation points* of u_i:

$$S_i = \{s_i \in X_i :\ u_i(s_i) \geq u_i(x_i)\, \forall\, x_i \in X_i\}.$$

In different models, S_i is assumed to be either empty, or else a nonempty set, not necessarily reduced to a singleton, but with special features. In order to prove the existence of an equilibrium in the economy, Arrow and Debreu in 1954 introduced the *Nonsatiation assumption*, that is for each $i \in I$, the satiation points are always outside the set of attainable consumption set $\widehat{X}_i$.

We now recall the existence result for a competitive economy, provided by Arrow and Debreu:

Theorem 4.1. *Assume the following:*

(A0) X_i *is closed convex and bounded for all* $i \in I$*;*
(A1) u_i *is continuous and semistrictly quasiconcave for all* $i \in I$*;*
(A2) *(nonsatiation)* $S_i \cap \widehat{X}_i = \emptyset$ *for all* $i \in I$*;*
(A3) *there is* x_i^0 *in* X_i *such that* $x_i^0 \ll e_i$ *for all* $i \in I$*;*
(A4) $0 \in Y_j$ *for all* $j \in J$*;*
(A5) Y_j *is closed and convex for all* $j \in J$*;*
(A6) Y_j *are bounded* $j \in J$*.*

Then there exists a free-disposal equilibrium of economy Ξ*.*

Proof. See Theorem 4 of [7]. □

(A2) is the nonsatiation assumption, as already observed. The meaning of the assumption (A4) is obvious: a producer can decide to produce nothing. In particular this will happen when at a given price all her possible production plans produce nonpositive income. (A6) will be substituted by a more natural one in the main result.

We can now prove our first existence result. We observe that, with respect to Theorem 4.1, we do not assume *semistrict* quasiconcavity of the utility functions. Moreover, the assumption (A6) on the boundedness of X_i, Y_j is substituted by more natural assumptions. However this is classical, the novelty relies on a weaker requirement for the utility functions.

Theorem 4.2. *Assume the following:*

(A0') X_i *is closed convex and bounded from below for all* $i \in I$*;*
(A1') u_i *is continuous and quasiconcave for all* $i \in I$*;*
(A2) *(nonsatiation)* $S_i \cap \widehat{X}_i = \emptyset$ *for all* $i \in I$*;*
(A3) *there is* x_i^0 *in* X_i *such that* $x_i^0 \ll e_i$ *for all* $i \in I$*;*
(A4) $0 \in Y_j$ *for all* $j \in J$*;*
(A5) Y_j *is closed and convex for all* $j \in J$*;*
(A7) $Y \cap (-Y) = \{0\}$*;*
(A8) $Y \supset Y - \mathbb{R}^l_+$*.*

Then there exists a free-disposal equilibrium for the economy Ξ*.*

Proof. First of all, let us point out that under assumptions (A0'), (A4)–(A5)–(A7)–(A8), the sets $\widehat{X}_i$ and $\widehat{Y}_j$ are bounded (see [3, p. 276]). Moreover observe that, without loss of generality, we can confine our attention only to prices that lie on the simplex $\Pi := \{p \in \mathbb{R}^l_+ : \sum_{j \in J} p^j = 1\}$. Next, for all $n \in \mathbb{N}$, we set $X_{i,n} = X_i \cap B(0;n)$ and $Y_{j,n} = Y_j \cap B(0;n)$ and for every u_i we define $u_{i,n}$ on $X_{i,n}$ as the semistrictly quasiconcave functions introduced in Theorem 3.3. Hence, we consider the economy $\Xi_n = \Big((X_{i,n}, u_{i,n}, e_i)_{i \in I}, (\theta_{ij})_{i \in I, j \in J}, (Y_{j,n})_{j \in J}\Big)$. Clearly, Ξ_n satisfies assumptions (A0), (A1), (A3)–(A6). Assumption (A2) holds eventually; to see this suppose, for the sake of contradiction, that there exists $i \in I$ such that assumption (A2) is not satisfied for a subsequence, still labeled with n. This means that $S_{i,n} \cap \widehat{X}_{i,n} \neq \emptyset$. Let $\widetilde{x}_{i,n} \in S_{i,n} \cap \widehat{X}_{i,n}$. Since $\widetilde{x}_{i,n} \in \widehat{X}_{i,n} \subseteq \widehat{X}_i$ for all n, and since $\widehat{X}_i$ is a compact set, we can pass to the limit (again along a suitable subsequence): there is $\widetilde{x}_i \in \widehat{X}_i$ such that $\widetilde{x}_{i,n} \to \widetilde{x}_i$. Let x_i be in X_i, there is $N \in \mathbb{N}$, such that, for all $n > N$, $x_i \in X_{i,n}$. Since $\widetilde{x}_{i,n} \in S_{i,n}$ one has: $u_{i,n}(\widetilde{x}_{i,n}) \geq u_{i,n}(x_i)$ for all $n > N$. Then, from Theorem 3.3, passing to the limit $u_i(\widetilde{x}_i) \geq u_i(x_i)$. But, this contradicts assumption (A2) for economy Ξ.

From Theorem 4.1, for all $n \in \mathbb{N}$, the economy Ξ_n has a free-disposal equilibrium $(\bar{p}_n, \bar{x}_n, \bar{y}_n)$:

$$\text{for all } i \in I,\ u_{i,n}(\bar{x}_{i,n}) = \max_{x_i \in M_i(\bar{p}_n, \bar{y}_n)} u_{i,n}(x_i) \tag{5}$$

$$\text{for all } j \in J,\ \langle \bar{p}_n, \bar{y}_{j,n} \rangle = \max_{y_j \in Y_j} \langle \bar{p}_n, y_j \rangle \tag{6}$$

$$\sum_{i \in I} (\bar{x}^h_{i,n} - e^h_i) - \sum_{j \in J} \bar{y}^h_{j,n} \leq 0,\ \langle \sum_{i \in I} (\bar{x}_{i,n} - e_i) - \sum_{j \in J} \bar{y}_{j,n}, \bar{p}_n \rangle = 0. \tag{7}$$

Since $\{\bar{p}_n\}_{n\in\mathbb{N}} \subseteq \Pi$, $\{\bar{x}_{i,n}\}_{n\in\mathbb{N}} \subseteq \widehat{X}_i$, $\{\bar{y}_{j,n}\}_{n\in\mathbb{N}} \subseteq \widehat{Y}_j$ and the sets Π, $\widehat{X}_i$ and $\widehat{Y}_j$ are compact, we can pass to the limit (along a suitable subsequence): $\bar{p}_n \to \bar{p}$, $\bar{x}_{i,n} \to \bar{x}_{i,n}$ and $\bar{y}_{j,n} \to \bar{y}_j$ for all $i \in I$ and $j \in J$, with $\bar{p} \in \Pi$, $\bar{x}_i \in \widehat{X}_i$ and $\bar{y}_j \in \widehat{Y}_j$. We now have to prove that $(\bar{p}, \bar{x}, \bar{y})$ is a free-disposal equilibrium of the economy Ξ. From (6) and (7), passing to the limit, we easily see that conditions (3) and (4) hold. To conclude, we need to prove (2), i.e.

$$u_i(\bar{x}_i) \geq u_i(x), \qquad \forall x \in M_i(\bar{p}, \bar{y}).$$

Since we know that

$$u_{i,n}(\bar{x}_{i,n}) \geq u_{i,n}(x), \qquad \forall x \in M_i(\bar{p}_n, \bar{y}_n)$$

it is clear that (2) is proved once we prove that

$$\forall x \in M_i(\bar{p}, \bar{y})\, \exists x_n \in M_i(\bar{p}_n, \bar{y}_n) \,:\, x_n \to x.$$

This means that

$$M_i(\bar{p}, \bar{y}) \subset \mathrm{Li} M_i(\bar{p}_n, \bar{y}_n).$$

Now, observe that, if $\langle \bar{p}, x \rangle < \langle \bar{p}, e_i \rangle + \sum_{j \in J} \theta_{ij} \bar{y}_j$, then for all large n it is $\langle \bar{p}_n, x_i \rangle < \langle \bar{p}_n, e_i \rangle + \sum_{j \in J} \theta_{ij} \bar{y}_{j,n}$, showing that $x \in M_{i,n}(\bar{p}_n, \bar{y}_n)$ eventually, and thus $x \in \mathrm{Li}\ M_{i,n}(\bar{p}_n, \bar{y}_n)$. Moreover, from assumption (A3), it is $\langle \bar{p}, x^0_i \rangle < \langle \bar{p}, e_i \rangle + \sum_{j \in J} \theta_{ij} \bar{y}_j$, which means that the relative interior (in X_i) of $M_i(\bar{p}, \bar{y})$ is a nonempty set. Since $M_i(\bar{p}, \bar{y})$ is a closed convex set with nonempty interior (in the relative topology of X_i), then it is $M_i(\bar{p}, \bar{y}) = \mathrm{cl}\ \ \mathrm{int}\ \ M_i(\bar{p}, \bar{y})$. It follows that

$$M_i(\bar{p}, \bar{y}) = \mathrm{cl}\ \ \mathrm{int}\, M_i(\bar{p}, \bar{y}) \subset \mathrm{cl}\, \mathrm{Li}\, M_i(\bar{p}_n, \bar{y}_n) = \mathrm{Li}\, M_i(\bar{p}_n, \bar{y}_n)$$

since the set $\mathrm{Li}\ M_i(\bar{p}_n, \bar{y}_n)$ is a closed set (see [9, Proposition 8.2.1]). This ends the proof.

As a final comment of this part, let us observe that assumption (A7) simply states that if the vector of goods y^+ can be used to produce the vector of goods $-y^-$, then it is not possible to use $-y^-$ to produce y^+: in other words, this is an irreversibility condition. Instead $Y \supset Y - \mathbb{R}^l_+$ is the explicit free-disposal assumption.

4.2. Pure exchange equilibrium

Now we consider a competitive economy without production and without free-disposal, in order to generalize other results concerning existence of an equilibrium. We use the same notation of the previous section, as far as goods, prices, endowments and so on are concerned. The prices p, in this setting, need not to be nonnegative. Thus a price vector p is such that $p = (p_1, \ldots, p_l) \in \mathbb{R}^l \setminus \{0\}$. Negative prices are in principle allowed to consider goods that turn out to be undesirable (such as, for instance, pollution byproducts). Since the wealth of consumer i is $\langle p, e_i \rangle$, the budget constraint set at the current price p is $M_i(p) = \{x_i \in X_i \,:\, \langle p, x_i \rangle \leq \langle p, e_i \rangle\}$. Thus, a competitive economy Ξ is described by the m-list:

$$\Xi = (X_i, u_i, e_i)_{i \in I}.$$

A *state* of the economy Ξ is an n-list $(x_1, \ldots, x_i, \ldots, x_n)$ of the consumptions of the consumers and a price vector p. We now provide the definition of equilibrium in this context.

Definition 4.2. A state $(\bar{p}, \bar{x})$ is an *equilibrium* of the economy Ξ if

$$\text{for all } i \in I, \qquad u_i(\bar{x}_i) = \max_{x_i \in M_i(\bar{p})} u_i(x_i) \tag{8}$$

$$\text{for all } h \in H, \qquad \sum_{i \in I} (\bar{x}_i^h - e_i^h) = 0. \tag{9}$$

As usual, the difference of equilibrium and free-disposal equilibrium relies on the fact that in the first case it is required that the vector price p is nonnull, while in the second usually it is required that the nonnull vector price is also with nonnegative components. Moreover, the so called total demand on the market: $\sum_{i \in I} (\bar{x}_i - e_i)$, that here is assumed to be zero (at the equilibrium) in the free-disposal case must be nonpositive, with some components possibly negative, when the corresponding price is zero.

We now introduce some further notation. For a consumer i, R_i denotes the set of the so called *rational allocations* for i, including those unfeasible:

$$R_i = \{x_i \in X_i : \; u_i(x_i) \geq u_i(e_i)\}.$$

Moreover, let

$$A = \{x \in X : \sum_{i \in I} x_i = \sum_{i \in I} e_i, \; u_i(x_i) \geq u_i(e_i), \; \forall i = 1, \ldots, n\}$$

be the set of *all individually rational feasible allocations*. Denote by A_i the projection of A onto X_i; clearly A_i represents the *individually rational feasible consumption* set of consumer $i \in I$.

Allouch and Le Van (see [1,2]), and Sato (see [11]) in this setting proved some existence theorems for equilibria in the economy under some weak nonsatiation assumptions. In particular, Allouch and Le Van, in 2008, introduced a new condition, called *Weak Nonsatiation*, by requiring that, for each $i \in I$, at least one satiation point lies outside the feasible set:

$$\text{if } \; S_i \neq \emptyset \quad \text{then } \; S_i \cap A_i^c \neq \emptyset.$$

Subsequently, in 2010, Sato considered a weaker condition, called *Boundary satiation*; namely, for each $i \in I$, at least one satiation point lies on the (relative) boundary of the set:

$$\text{if } \; S_i \neq \emptyset \quad \text{then } \; S_i \cap (\mathrm{int}_{R_i} A_i)^c \neq \emptyset$$

where $\mathrm{int}_{R_i} A_i$ denotes the interior of A_i in the relative topology on $R_i \subset \mathbb{R}^l$.

Sato proved the following existence result:

Theorem 4.3. *Assume the following, for all $i \in I$:*

(A0) *X_i is closed convex and bounded;*
(A1) *u_i is continuous and semistrictly quasiconcave;*
(A2') *(boundary satiation) $S_i \cap (\mathrm{int}_{R_i} A_i)^c \neq \emptyset$;*
(A3) *there is x_i^0 in X_i such that $x_i^0 \ll e_i$.*

Then, there exists an equilibrium for the economy Ξ.

Proof. See [11]. □

Here we generalize the Theorem of Sato, in the sense that we do not assume *strict* quasiconcavity; the approach used by him to prove his theorem is similar to that one used here: Sato applies the Theorem of Allouch and Le Van to a sequence of economies fulfilling the more restrictive assumption on satiation given by them, and passing to the limit he gets existence in the case of his more general satiation assumption. Here we build approximating economies in the proof of subsequent Theorem 4.4 that actually fulfill the Allouch and Le Van satiation assumption, thus our approach not only generalizes Sato's theorem, but also provides a much shorter and simpler proof of his result.

Theorem 4.4. *Assume the following, for all $i \in I$:*

(A0) *X_i is closed convex and bounded;*
(A1') *u_i is continuous and quasiconcave;*
(A2') *(boundary satiation) $S_i \cap (\mathrm{int}_{R_i} A_i)^c \neq \emptyset$;*
(A3) *there is x_i^0 in X_i such that $x_i^0 \ll e_i$.*

Then, there exists an equilibrium for the economy Ξ.

Proof. First of all observe that, without loss of generality, we can confine our attention only to prices that lie on the boundary of the unit ball $S(0;1) \subset \mathbb{R}^l$. For all $n \in \mathbb{N}$, we introduce the economy $\Xi_n = (X_i, u_{i,n}, e_i)_{i\in I}$, using Theorem 3.1. We now claim that the economies Ξ_n satisfy assumption (A2').

- Suppose $e_i \in S_{i,n}$ (for a subsequence). Then by definition $A_{i,n} \subseteq S_{i,n}$.
  In this case the claim is proved.
- Now suppose $e_i \notin S_{i,n}$ (for a subsequence). Then, from Remark 3 and boundedness of X_i, it follows that $S_{i,n} \supset S_i$ for all n. Thus $e_i \notin S_i$. By assumption (A2'), there exists $\bar{x} \in S_i \cap (\mathrm{int}_{R_i} A_i)^c$. By continuity of u_i, there exist $\varepsilon, \delta > 0$ such that, for every $x \in B(\bar{x};\varepsilon)$, $u_i(x) > u_i(e_i) - 2\delta$. By uniform convergence of $u_{i,n}$ to u_i, we get that for all large n, and for every $x \in B(\bar{x};\varepsilon)$, $u_{i,n}(x) \geq u_{i,n}(e_i) - \delta$. Thus $B(\bar{x};\varepsilon) \cap X_i \subset R_{i,n}$. Suppose now $\bar{x} \in \mathrm{int}_{R_{i,n}} A_{i,n}$. Then there exists $\eta < \varepsilon$ such that for all $x \in B(\bar{x};\eta) \cap X_i$ there are $x_j \in X_j$, for $j \neq i$ verifying $x + \sum_j x_j = \sum_i e_i$. Thus $x \in A_i$ and since $u_i(x) > u_i(e_i)$ it follows that $\bar{x} \in S_i \cap (\mathrm{int}_{R_i} A_i)$, a contradiction. The claim is proved also in this case.

Thus, the assumptions of Theorem 4.3 hold for the economies Ξ_n. Then for every (large) n there exists a $(\bar{p}_n, \bar{x}_n)$ equilibrium of Ξ_n. Since $\{\bar{p}_n\}$ are norm one vectors, and, since $\{\bar{x}_{i,n}\} \subseteq X_i$ and X_i are bounded, there are $\widetilde{p}$, $\widetilde{x}_i$ limits of some (common) subsequence of $\{\bar{p}_n\}$ $\{\bar{x}_{i,n}\}$: $\lim_{n\to+\infty}(\bar{p}_n, \bar{x}_n) = (\widetilde{p}, \widetilde{x})$. Now the proof that $(\widetilde{p}, \widetilde{x})$ is an equilibrium of Ξ is quite similar to that one of Theorem 4.2, and it is omitted.

As a last result, we can generalize Theorem 4.4 when the consumption sets X_i are unbounded:

Theorem 4.5. *Assume the following, for all $i \in I$:*

(A0') *X_i is closed convex and bounded from below;*
(A1') *u_i is continuous and quasiconcave;*
(A2') *(boundary satiation) if $S_i \neq \emptyset$, then $S_i \cap (\mathrm{int}_{R_i} A_i)^c \neq \emptyset$;*
(A3) *there is x_i^0 in X_i such that $x_i^0 \ll e_i$.*

Then, there exists an equilibrium for the economy Ξ.

Proof. For every i denote by z_i a lower bound for X_i: $x_i \geq z_i$ for every $x_i \in X_i$. Now, set $X_{i,n} = X_i \cap B(0;n)$, and define $u_{i,n}$ on $X_{i,n}$ to be the restriction of u_i to $X_{i,n}$. For every $n \in \mathbb{N}$, we introduce the economy $\Xi =_n (X_{i,n}, u_{i,n}, e_i)_{i\in I}$. We now claim that Ξ_n satisfy the assumption of Theorem 4.4. The only hypothesis to check is (A2') (observe that $S_{i,n}$ is nonempty, due to boundedness of $X_{i,n}$). We need to consider two cases, according to the fact that S_i is either empty or nonempty. If S_i is nonempty, there exists $\hat{x} \in S_i \cap (\mathrm{int}_{R_i} A_i)^c$. Suppose, by contradiction, $\hat{x} \notin S_{i,n} \cap (\mathrm{int}_{R_{i,n}} A_{i,n})^c$ eventually. Since $\hat{x} \in S_{i,n}$ (eventually), this implies $\hat{x} \in (\mathrm{int}_{R_{i,n}} A_{i,n})$, and this is a contradiction, since $A_{i,n} = A_i$ (eventually), because A_i is bounded (see [3, p. 276]). In case $S_i = \emptyset$, then necessarily $S_{i,n}$ does intersect the boundary of $X_{i,n}$, and this in turn implies that (A2') is satisfied for Ξ_n, eventually. Thus there exists an equilibrium for the economy Ξ_n, for all large n. Since, for each n and $i \in I$:

$$z_i \leq \bar{x}_{i,n} \leq \bar{x}_{i,n} + \sum_{j\in I, j\neq i} \left(\bar{x}_{j,n} - z_j\right) \leq \sum_{i\in I} e_i - \sum_{j\in I, j\neq i} z_j, \qquad p_n \in S(0;1)$$

we can pass to the limit, as in the proofs of Theorems 4.2 and 4.4, to find an equilibrium $(x, \bar{p})$ for Ξ.

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