

PROOF OF THE HONEYCOMB ASYMPTOTICS FOR OPTIMAL CHEEGER CLUSTERS

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ABSTRACT. We prove that, in the limit as $k \rightarrow +\infty$, the hexagonal honeycomb solves the optimal partition problem in which the criterion is minimizing the largest among the Cheeger constants of k mutually disjoint cells in a planar domain. As a by-product, the same result holds true when the Cheeger constant is replaced by the first Robin eigenvalue of the Laplacian.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Consider the following optimal partition problem

$$(1) \quad M_k(\Omega) = \inf \left\{ \max_{j=1, \dots, k} h(\Omega_j) : \{\Omega_j\} \in \mathcal{A}_k(\Omega) \right\},$$

where Ω is an open bounded subset of \mathbb{R}^2 with a Lipschitz boundary, $h(\cdot)$ is the Cheeger constant, and $\mathcal{A}_k(\Omega)$ is the class of k -clusters of Ω , meant as families of k Borel sets with finite perimeter which are contained into Ω and have Lebesgue negligible mutual intersections.

Let us recall that the Cheeger constant of Ω is defined by

$$(2) \quad h(\Omega) := \inf \left\{ \frac{\text{Per}(E, \mathbb{R}^2)}{|E|} : E \text{ measurable, } E \subseteq \Omega \right\},$$

where $\text{Per}(E, \mathbb{R}^2)$ denotes the perimeter of E in the sense of De Giorgi. We refer to the review papers [15, 17] and the numerous references therein for an account of the broad literature about the Cheeger constant.

Optimal partitions for the Cheeger constant have been firstly studied by Caroccia in [6], where he gives some existence and regularity results for the similar problem

$$(3) \quad m_k(\Omega) = \inf \left\{ \sum_{j=1, \dots, k} h(\Omega_j) : \{\Omega_j\} \in \mathcal{A}_k(\Omega) \right\}.$$

The main motivation he brings to study problem (3) is finding some bound for the same problem for the first Dirichlet eigenvalue of the Laplacian, $\lambda_1(\Omega)$. (Recall indeed that $\lambda_1(\Omega)$ is bounded from below by $(h(\Omega)/2)^2$, as proved by Cheeger himself in [7].) Actually, for problem (3) with λ_1 in place of the Cheeger constant, a long-standing conjecture by Caffarelli and Lin predicts that, in the limit as $k \rightarrow +\infty$, an optimal configuration is given by a packing of regular hexagons [5]¹.

2010 Mathematics Subject Classification. 52C20, 51M16, 49Q10.

Key words and phrases. Optimal partitions, honeycomb, Cheeger constant, Hales hexagonal inequality, inner Cheeger boundary, Weyl law for higher order Cheeger constant.

¹The question is commonly known in the literature (see *e.g.* [6]) as the Caffarelli-Lin conjecture: in fact a precise mathematical formulation was given in [5], along with the first asymptotic estimates. Nevertheless, the history of the problem seems to be longer. The first predictive formulation of the conjecture appears in a list of open problems proposed by K. Burdzy in a conference in Matrei in 2005 <https://people.kth.se/>

More recently, problems of the kind (1) and (3) have been studied in [3], where it is shown that, under the a priori requirement that all the cells of the partitions are convex, the honeycomb conjecture holds true under the form

$$(4) \quad \lim_{k \rightarrow +\infty} \frac{|\Omega|^{1/2}}{k^{1/2}} M_k(\Omega) = h(H), \quad \lim_{k \rightarrow +\infty} \frac{|\Omega|^{1/2}}{k^{3/2}} m_k(\Omega) = h(H),$$

where H denotes the unit area regular hexagon.

Clearly, the convexity assumption made on the cells in [3] is quite stringent. Nevertheless, as a first approach, it seemed reasonable to attack the problem under this restriction, since, also in the case of perimeter minimizing partitions, the case of convex polygonal cells was much simpler and indeed it was settled a long time before the celebrated result by Hales [10] (see Fejes Tóth [9]).

Goal of this paper is to prove the honeycomb conjecture for the Cheeger constant in full generality, *i.e.* with no convexity constraint on the cells.

We focus our attention on problem (1). Our strategy consists in considering first the case when Ω has a special geometry, that for the sake of simplicity we assume to be that of an equilateral triangle \mathcal{T} (but other shapes, for instance a rectangle, could do the same job), and obtaining an inequality for $M_k(\mathcal{T})$, with k fixed. The choice of treating first the case of a simple geometry comes along with our variational approach of the inequality: we work with an optimal partition and take significant advantage from optimality. For that reason we need to have a complete and simple description of this one. The conjecture will follow in full generality, once this special geometric case is proved. In order to deal with $M_k(\mathcal{T})$, we introduce the auxiliary problems

$$(5) \quad M_{k,p}(\mathcal{T}) = \inf \left\{ \left[\sum_{j=1}^k h^p(\Omega_j) \right]^{1/p} : \{\Omega_j\} \in \mathcal{A}_k(\mathcal{T}) \right\}, \quad p \geq 1,$$

and we set

$$\widetilde{M}_{k,p}(\mathcal{T}) := \max_{j=1,\dots,k} h(\Omega_j^p),$$

being $\{\Omega_1^p, \dots, \Omega_k^p\}$ an optimal cluster for problem (5). Note that there is an abuse of notation, since $\widetilde{M}_{k,p}(\mathcal{T})$ depends on the choice of $\{\Omega_1^p, \dots, \Omega_k^p\}$, but we keep this simple notation as the dependence on the optimal cluster is not important for our purposes.

It is easy to see that $\widetilde{M}_{k,p}(\mathcal{T})$ converges to $M_k(\mathcal{T})$ in the limit as $p \rightarrow +\infty$ (see Section 5). Then we prove that both $\widetilde{M}_{k,p}(\mathcal{T})$ and $M_k(\mathcal{T})$ satisfy the following hexagonal lower bound, being k fixed:

Theorem 1. *Let \mathcal{T} be an equilateral triangle. For every $p \geq 1$, there holds*

$$(6) \quad \frac{|\mathcal{T}|^{\frac{1}{2}}}{k^{\frac{1}{2}}} \widetilde{M}_{k,p}(\mathcal{T}) \geq h(H).$$

Consequently, we have

$$(7) \quad \frac{|\mathcal{T}|^{\frac{1}{2}}}{k^{\frac{1}{2}}} M_k(\mathcal{T}) \geq h(H).$$

[~laptev/ESF/05/Matrei/problems.html](#), and it is motivated by some numerical computations originating in older papers [4, 8] modeling particle systems. It is also worth to notice that a related version of this conjecture, involving the minimization of the maximum among the first eigenvalues of the cells, appears to be mathematically formulated in a paper by B. Helffer, T. Hoffmann-Ostenhof and S. Terracini, who learned the question from M. van den Berg (see [11]).

Theorem 1 is the keystone of our approach. Hereafter is an attempt of enlightening the main ideas upon which our proof is based:

- Optimal clusters satisfy an existence, regularity, and structure result, which is essentially a variant of the one valid in the case $p = 1$ treated by Caroccia (see Proposition 8).
- As a consequence of the structure result, each cell of an optimal cluster is Cheeger of itself and enjoys the following key property. If we call “inner Cheeger boundary” of a cell the inner parallel set at distance to the boundary equal to the inverse of the Cheeger constant, then the oriented area enclosed by such inner Cheeger boundary turns out to be related to the Cheeger constant of the cell itself by a very simple equation (see Proposition 18, eq.(15)). It can be read as the transposition of a well-known relation between the Cheeger set of convex bodies and their inner parallel sets. In turn, this leads to a crucial representation formula for the Cheeger constant of an optimal cell in terms of its area and of the length of its inner Cheeger boundary (see Proposition 18, eq.(16)). Such representation formula can be regarded as the initial seed of our proof.
- Starting from the representation formula, the optimality of the hexagonal honeycomb comes out by combining a *lower bound for the total length of the inner Cheeger boundaries*, with an *upper bound for the total area of the cells*. Both are quite delicate. In particular, the former is obtained by applying Hales’ hexagonal isoperimetric inequality [10, Theorem 4], going through the analysis of the collective behaviour of those inner Cheeger boundaries. The latter requires a careful estimate of the area of the empty chamber, which is carried over through some topological and geometrical arguments (see Proposition 14).

Next, as a consequence of Theorem 1, we are able to consider the case when the equilateral triangle is replaced by k -triangle, that is a region of triangular shape formed by k hexagons. More precisely, for $k = l(l+1)/2$, by k -triangle, we mean a connected set which is obtained as the union of k hexagons lying in a tiling of \mathbb{R}^2 made by a family of copies of a regular hexagon and having the “rough” shape of an equilateral triangle with l cells on each side (precisely, all the centers of those hexagons lie on the boundary and inside an equilateral triangle).

We obtain that, for any fixed k , the energy of a k -triangle (denoted by \mathcal{T}_k), suitably scaled, is precisely that of the regular hexagon:

Theorem 2. *Let \mathcal{T}_k be a k -triangle. There holds*

$$(8) \quad \frac{|\mathcal{T}_k|^{\frac{1}{2}}}{k^{\frac{1}{2}}} M_k(\mathcal{T}_k) = h(H).$$

Finally, relying on Theorem 2 and using a blow-up argument, we obtain that the honeycomb conjecture for the Cheeger constant holds true for every Lipschitz domain Ω in the following asymptotic form (which is exactly the same as in [3], without the convexity assumption on the cells):

Theorem 3. *For every open bounded Lipschitz domain Ω , and every $p \geq 1$, there holds*

$$(9) \quad \lim_{k \rightarrow +\infty} \frac{|\Omega|^{1/2}}{k^{1/2}} M_k(\Omega) = h(H)$$

Remark 4 (Asymptotic behaviour for partitions of the Robin-Laplacian eigenvalues). It is worth noticing that, as a consequence of the above result and Corollary

3 (i) together with Remark 15 in [2], the same result as Theorem 3 holds true if in the definition of $M_k(\Omega)$ the Cheeger constant $h(\Omega_j)$ is replaced by the first eigenvalue of the Laplacian under Robin boundary conditions, $\lambda_1(\Omega_j, \beta)$. Precisely, given $\beta > 0$ (fixed), $\lambda_1(\Omega_j, \beta)$ is the lowest positive number for which the equation

$$\begin{cases} -\Delta u = \lambda_1(\Omega_j, \beta)u & \text{in } \Omega_j \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega_j. \end{cases}$$

has a non trivial solution.

Remark 5 (Weyl asymptotic for the k -th Cheeger constant). The quantity $M_k(\Omega)$ is also called the k -th Cheeger constant of Ω (see the recent paper [18] and references therein), and an equivalent of this notion is intensively studied on graphs, for clustering purposes (see *e.g.* [14]).

Loosely speaking, for the 1-Laplacian operator, the k -th Cheeger constant can be interpreted as a counterpart of the k -th eigenvalue. In this perspective, Theorem 3 can also be interpreted as an asymptotic formula of Weyl type [12] for the k -th Cheeger constant, since it can be rephrased as

$$M_k(\Omega) = \frac{k^{\frac{1}{2}}}{|\Omega|^{\frac{1}{2}}}(\sqrt{\pi} + \sqrt[4]{12}) + o(k^{\frac{1}{2}}).$$

The plan of the paper is the following. In Section 2, we establish all the preparatory results which concern the properties of optimal clusters; the results of this section rely on the work of Caroccia [6]. Next we give the intermediate results of topological nature in Section 3, and the key representation result involving the inner Cheeger boundary in Section 4. The proofs of Theorems 1, 2, and 3, are then given in Section 5. Finally in Section 6, we collect some auxiliary geometrical lemmas needed for the estimate of the area of the empty chamber.

2. ABOUT OPTIMAL CLUSTERS

This section is devoted to the study of optimal clusters for problem (5), in case Ω is an equilateral triangle \mathcal{T} : in Section 2.1 we give a structure result along the same line of the one proved by Caroccia for $p = 1$; in Section 2.2 we fix some important consequences of the structure result; in Section 2.3 we associate with an optimal cluster a planar graph, which will be used as fundamental tool to establish the topological results stated in the next section.

2.1. A structure result for optimal clusters.

Definition 6. We denote by \mathcal{A} the family of Jordan domains Ω of class C^1 contained into \mathcal{T} such that Ω is Cheeger set of itself, and the (positively oriented) boundary $\partial\Omega$ is the union of an even number of nontrivial arcs alternating a *free arc* and a *junction arc*. A free arc is an arc of circle of algebraic curvature $h(\Omega)$ with at least one endpoint in the interior of \mathcal{T} . A junction arc may be either an *inner junction arc* or a *border junction arc*. An inner junction arc is an arc of circle of algebraic curvature strictly less than $h(\Omega)$ (possibly 0), with both the endpoints in the interior of \mathcal{T} . An outer junction arc is a curve, with both the endpoints on $\partial\mathcal{T}$, which is union of segments lying in $\partial\mathcal{T}$ and arcs of circle of curvature $h(\Omega)$.

Remark 7. We point out that \mathcal{A} does not contain any ball, firstly because the number of circular arcs must be even, and also because, if B is a ball of radius R , it holds $h(B) = \frac{2}{R}$, so that the curvature is not equal to $h(B)$. For a similar reason, \mathcal{A} does not contain any stadium-domain (that is, the convex envelope of two balls). As a further example it is easy to check that, among all convex domains obtained from a square by “rounding off” the corners with four circular arcs, only the Cheeger set of the square lies in the class \mathcal{A} .

Proposition 8 (properties of an optimal cluster). *For every fixed $p \geq 1$:*

- (i) *problem (5) admits a solution in which each cell is Cheeger of itself, hereafter denoted by $\{\Omega_1, \dots, \Omega_k\}$;*
- (ii) *each cell Ω_j is a simply connected set of class C^1 ;*
- (iii) *each cell Ω_j belongs to the family \mathcal{A} introduced in Definition 6; moreover:*
 - *any inner junction arc for Ω_j is also an inner junction arc for another set Ω_l , and its curvature, seen from Ω_j , is given by*

$$(10) \quad K_{j,l} = \frac{\frac{h^p(\Omega_j)}{|\Omega_j|} - \frac{h^p(\Omega_l)}{|\Omega_l|}}{\frac{h^{p-1}(\Omega_j)}{|\Omega_j|} + \frac{h^{p-1}(\Omega_l)}{|\Omega_l|}};$$

- *any free arc for Ω_j can intersect $\cup_{l \neq j} \partial\Omega_l$ on at most a finite number of points; moreover, the opening angle of any portion of a free arc which does not contain intersection points with $\cup_{l \neq j} \partial\Omega_l$ is strictly less than π .*

Proof. For $p = 1$ the existence, regularity and structure of optimal clusters of problem (5) have been discussed in [6]. For $p > 1$, the arguments are precisely the same, without any significant difference. For the convenience of the reader, we highlight the main steps, and refer to [6] for details.

- (i) We replace the original problem (5) by the following one

$$(11) \quad \inf \left\{ \left[\sum_{j=1}^k \left(\frac{\mathcal{H}^1(\partial^* \Omega_j)}{|\Omega_j|} \right)^p \right]^{1/p} : \{\Omega_j\} \in \mathcal{A}_k(\mathcal{T}) \right\}.$$

We trivially get an upper bound for the value of the above infimum by referring to some configuration (e.g. k disjoint balls). As a consequence, there exists a constant $M > 0$ such that, for any minimizing sequence $(\Omega_1^n, \dots, \Omega_k^n)$, it holds

$$\sum_{j=1}^k \left(\frac{\mathcal{H}^1(\partial^* \Omega_j^n)}{|\Omega_j^n|} \right)^p \leq M^p.$$

Combined with the isoperimetric inequality, this implies that the measures $|\Omega_j^n|$ remain bounded from below. As well, we get the upper bound

$$\mathcal{H}^1(\partial^* \Omega_j^n) \leq M |\mathcal{T}|.$$

Consequently, the existence of optimal clusters for problem (11) follows by standard compactness/lower semicontinuity arguments in BV . Each set of an optimal configuration is self Cheeger, otherwise this would contradict optimality. Moreover, every solution to problem (11) is also solution to the original problem (5). Let us denote such a solution by $(\Omega_1, \dots, \Omega_k)$.

(ii) All sets Ω_j of the optimal cluster obtained in statement (i) are (locally, inside \mathcal{T}) quasi-minimizers for the perimeter, and hence they are equivalent to open sets with boundary having $C^{1,\alpha}$ regularity (inside \mathcal{T}), with any $\alpha \in (0, \frac{1}{2})$.

For $p = 1$ the quasi minimality argument is given in [6, Theorem 3.6], but the proof does not depend on p . Roughly speaking, together with the minimality in (11), the key points are that each set Ω_j is self Cheeger and has an algebraic curvature (in a distributional sense) not larger than a constant (in our case M). So, we know that each set Ω_j is equivalent to an open set with smooth boundary. A priori, the set Ω_j may not be connected. In case that Ω_j is not connected, two connected components have necessarily to lie at positive distance, and we can choose one of them and replace Ω_j with this component. The energy in (11) does not change. So we know that all sets Ω_j are connected. As a consequence, the vertices of \mathcal{T} do not belong to any of the boundaries $\partial\Omega_j$, since cutting out by a line a piece of Ω_j near the corner, would strictly decrease its Cheeger constant.

Moreover, each set is simply connected. Indeed, if a set Ω_j is not simply connected, we analyze one hole (which is smooth) and translate it inside Ω_j up to a new contact point with $\partial\Omega_j$. This new set is also optimal, contradicting the regularity.

(iii) We analyze now the structure of the boundary. Following the same arguments as for $p = 1$ in [6, Proposition 5.4 and Proposition 5.5], there are no triple points (meaning that a point of $\overline{\mathcal{T}}$ may belong to at most two boundaries $\partial\Omega_j, \partial\Omega_l$), and the boundary of Ω_j is a finite union of arcs of circle. Moreover, looking at a piece of arc of circle which is common to $\partial\Omega_j$ and $\partial\Omega_l$, one can write optimality conditions, which lead precisely to the expression of the algebraic curvature (seen from Ω_j) given by (10). We see from (10) that $K_{j,l}$ is strictly less than $h(\Omega_j)$. If we look now at a piece of arc of circle from $\partial\Omega_j$ lying in a neighborhood of a point which has a positive distance from $\cup_{l \neq j} \partial\Omega_l$, we get from optimality that the curvature has to be equal to $h(\Omega_j)$. As a consequence of the C^1 -regularity, two such pieces of arc from $\partial\Omega_j$ meeting at a point which belongs to $\cup_{l \neq j} \partial\Omega_l$ have to be part of a unique arc of curvature $h(\Omega_j)$. In this way, we identify clearly the boundary of Ω_j as an ordered union of free arcs of circle of algebraic curvature equal to $h(\Omega_j)$ alternating with junction arcs which may be inner or border ones. \square

Remark 9. We point out that part of the information on the properties of an optimal cluster given in Proposition 8 could be obtained in a direct way by applying to each cell a structure result by Ambrosio, Caselles, Masnou and Morel for measurable sets with finite perimeter in two dimensions, which are indecomposable in the sense of geometric measure theory (see [1]).

2.2. Consequences of the structure result. As an outcome of Proposition 8, the structure of an optimal cluster for problem (5) is quite rigid. For later use, it is important to fix in particular the following facts.

– *Connected components of the empty chamber.* By empty chamber, we mean the set $\Omega_0 := \mathcal{T} \setminus \cup_{j=1}^m \overline{\Omega}_j$. Every connected component $c(\Omega_0)$ of the empty chamber is a Jordan domain. If $c(\Omega_0)$ has a positive distance from $\partial\mathcal{T}$, its boundary is a union of free arcs. If $c(\Omega_0)$ touches $\partial\mathcal{T}$, two possibilities may occur: either $\partial c(\Omega_0)$ is union of some free arcs and some segments on $\partial\mathcal{T}$, or $\partial c(\Omega_0)$ is union of two segments lying on consecutive sides of $\partial\mathcal{T}$ and a piece of a border junction arc (and this may occur only around the corners). We point out in particular that, if we endow $\partial c(\Omega_0)$ with a positive orientation, all the arcs of circle have negative curvatures $-h(\Omega_j)$, being Ω_j the neighbouring cells. Thus, as

a consequence of the sign of the curvatures, $\partial c(\Omega_0)$ contains at least three arcs (meant as arcs of circle or segments).

– *Cells sharing several inner junction arcs.* Two cells Ω_j, Ω_l may share several inner junction arcs. In this case, two consecutive inner junction arcs need to enclose another cell. More precisely assume that, following the orientation of $\partial\Omega_j$, we find two consecutive inner junction arcs γ_1 and γ_2 , and let us denote by P the final point of γ_1 and by Q the initial point of γ_2 . Consider the curve γ starting on P , following $\partial\Omega_1$ up to Q and then following $\partial\Omega_2$ up to P (still in the positive sense). Then, γ has necessarily to enclose another cell, different from Ω_j, Ω_l . Indeed, γ does not contain any other inner junction arc between Ω_1 and Ω_2 , because we have chosen two consecutive inner junction arcs. Thus, the only possibility for γ not to enclose another cell would be that on $\partial\Omega_1$ the curve from P to Q is a free arc, and on $\partial\Omega_2$ the curve from Q to P is also a free arc. This is not possible, since the curvature of both free arcs, seen from Ω_j and Ω_l , are positive.

– *Cells sharing several border junction arcs with $\partial\mathcal{T}$.* A cell Ω_j may share several border junction arcs with $\partial\mathcal{T}$. However, in the alternation of free and junction arcs, two border junction arcs cannot be consecutive. Indeed, between two border junction arcs, there is a free arc having one endpoint in the interior of \mathcal{T} , so that an inner junction arc starts at such endpoint. Notice also that a border junction arc may contain different segments lying in $\partial\mathcal{T}$; if this is the case, due to the sign of the curvature of the free arcs, these segments cannot lie on the same side of \mathcal{T} , but belong necessarily to distinct sides of \mathcal{T} ; consequently, they can be either 2, or at most 3.

2.3. Construction of the canonical graph associated with an optimal cluster.

Thanks to the properties of an optimal cluster for problem (5) described so far, we are ready to associate with it a planar graph.

Definition 10. We call *canonical graph* associated with an optimal cluster for problem (5) the planar graph having the following vertices and edges:

– *Vertices:* To each cell Ω_j , $j = 1, \dots, k$ we associate a vertex X_j . Also to the set $\mathbb{R}^2 \setminus \overline{\mathcal{T}}$ we associate a vertex, denoted X_0 . We have thus $k + 1$ vertices. To draw a representation of the graph in the plane, the vertices can be chosen as arbitrary points in the interior of Ω_j and $\mathbb{R}^2 \setminus \overline{\mathcal{T}}$ respectively.

– *Edges.* We distinguish the families \mathcal{E}_{in} and \mathcal{E}_{out} of *inner* and *outer* edges, namely edges of the graph which join two distinct vertices X_j, X_l ($j, l \in \{1, \dots, k\}$), or a vertex X_j with X_0 , respectively. The family \mathcal{E}_{in} is constructed as follows: to every couple (Ω_j, Ω_l) which share an inner junction arc, we associate an edge by joining X_j to X_l through such arc. The family \mathcal{E}_{out} is constructed as follows: to every cell Ω_j having a border junction arc on $\partial\mathcal{T}$, we associate an edge by joining X_j to X_0 through such arc.

Remark 11. Each face of the canonical graph associated with an optimal cluster for problem (5) has at least 3 edges. To prove this claim, it's enough to observe that a face of the graph can be delimited neither by just two inner edges nor by just two outer edges.

Indeed, two cells may share several inner junction arcs, so that two vertices X_j, X_l may be connected by multiple inner edges; however, we know from Section 2.2 that two consecutive inner junction arcs need to enclose another cell, and therefore no face of the graph can be delimited by just two inner edges.

Likewise, a cell may share several border junction arcs with $\partial\mathcal{T}$, but we know from Section 2.2 that they cannot be consecutive, and therefore no face of the graph can be delimited by just two outer edges.

3. INTERMEDIATE TOPOLOGICAL RESULTS

In this section we give two results needed for the proof of Theorem 1, both obtained via the analysis of the canonical graph associated with an optimal cluster: in Proposition 12 we give an upper bound for the average of the number of junction arcs, and in Proposition 14 we provide an estimate from below for the area of the empty chamber.

Proposition 12 (average of number of junction arcs). *For a fixed $p \geq 1$, let $\{\Omega_1, \dots, \Omega_k\}$ be an optimal cluster for problem (5) which satisfies the properties stated in Proposition 8. Let $2\Lambda_j$, $j = 1, \dots, k$ be the number of oriented arcs which compose $\partial\Omega_j$, so that Λ_j is the number of junction arcs in $\partial\Omega_j$. Let E_{out} be the cardinality of the family \mathcal{E}_{out} of outer edges of the canonical graph associated with the cluster $\{\Omega_1, \dots, \Omega_k\}$ according to Definition 10. Then the following inequality holds:*

$$\sum_{j=1}^k \Lambda_j + E_{out} + 6 \leq 6k.$$

Proof. We denote by $V = k + 1$ the number of vertices, E the number of edges, and F the number of sides in the canonical graph associated with the optimal cluster $\{\Omega_1, \dots, \Omega_k\}$. Since every edge borders 2 faces, and each face has at least 3 edges (by Remark 11), there holds

$$2E \geq 3F.$$

Then, using the Euler formula

$$V - E + F = 2,$$

we obtain

$$3k - 3 \geq E.$$

Setting E_{in} and E_{out} the cardinalities of the families \mathcal{E}_{in} and \mathcal{E}_{out} of inner and outer edges according to Definition 10, we get $3k - 3 \geq E_{in} + E_{out}$, or equivalently

$$(12) \quad 6k - 6 \geq 2E_{in} + 2E_{out}.$$

Then the conclusion is obtained by noticing that

$$(13) \quad 2E_{in} + E_{out} = \sum_{j=1}^k \Lambda_j.$$

Indeed, let's count the total number of junction arcs: any inner junction arc is counted twice, and corresponds to an edge in \mathcal{E}_{in} ; any border junction arc is counted once, and corresponds to an edge in \mathcal{E}_{out} . \square

Definition 13. We set:

- Δ_r the curvilinear triangle bounded by three concave arcs of circle with opening angles $\pi/3$ and radius r , pairwise mutually tangent at a common endpoint;
- $\widehat{\Delta}_r$ the region bounded by two concave arcs of circle with opening angles $\pi/2$ and radius r , mutually tangent at a common endpoint, and a line segments tangent to such arcs at their (noncommon) endpoints;
- $\widehat{\widehat{\Delta}}_r$ the region bounded by a concave arc of circle with opening angle $2\pi/3$ and radius r , and two line segments tangent to such arc at its endpoints, forming an angle of $\pi/3$.

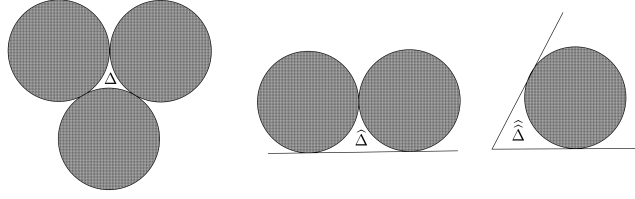


FIGURE 1. The three types of empty regions: $\Delta, \widehat{\Delta}, \widehat{\widehat{\Delta}}$

Proposition 14 (area of the empty chamber). *For a fixed $p \geq 1$, let $\{\Omega_1, \dots, \Omega_k\}$ be an optimal cluster for problem (5) which satisfies the properties stated in Proposition 8. Then the area of the empty chamber $\Omega_0 = \mathcal{T} \setminus \bigcup_{j=1}^k \overline{\Omega}_j$ satisfies*

$$(14) \quad |\Omega_0| \geq (2k - 2)|\Delta_{r_*}| + 3|\widehat{\widehat{\Delta}}_{r_*}|$$

Proof. We call *empty room* a collection of (one or more) connected components of the empty chamber which are enclosed by a face of the canonical graph, and which are *not* of type $\widehat{\widehat{\Delta}}$ (namely are not around a corner of \mathcal{T}).

We observe that the area of the empty chamber can be estimated from below by the global area of all the empty rooms (the inequality may be strict because there may be cells touching $\partial\mathcal{T}$ which are not connected by any outer edge to $\mathbb{R}^2 \setminus \overline{\mathcal{T}}$).

Then, we proceed to minimize the global area of the empty rooms. To that aim, we modify the canonical graph associated to the optimal cluster so that each face has exactly 3 edges. The modification consists in adding a certain number of formal edges for every empty room having on its boundary more than 3 arcs.

Given such an empty room C_0 , there exists a family of $m \geq 3$ disks D_1, \dots, D_m of centers P_1, \dots, P_m and radii r_1, \dots, r_m , with

$$\begin{aligned} d(P_i, P_{i+1}) &= r_i + r_{i+1} & \forall i = 1, \dots, m-1 \\ d(P_i, P_j) &\geq r_i + r_j & \forall i, j \in \{1, \dots, m\}, |i-j| \geq 2, \end{aligned}$$

(where $d(\cdot, \cdot)$ denotes the Euclidean distance and all angles $\angle P_{i-1}P_iP_{i+1}$ are strictly less than π), such that one of the following situation occurs:

- (a) $d(P_1, P_m) = r_1 + r_m$, and $\partial C_0 \subseteq \partial D_1 \cup \dots \cup \partial D_m$;
- (b) $d(P_1, P_m) > r_1 + r_m$, D_1, D_m are tangent to one side S of \mathcal{T} , and $\partial C_0 \subseteq S \cup \partial D_1 \cup \dots \cup \partial D_m$;
- (c) $d(P_1, P_m) > r_1 + r_m$, D_1, D_m are tangent to two consecutive sides S', S'' of \mathcal{T} , and $\partial C_0 \subseteq S' \cup S'' \cup \partial D_1 \cup \dots \cup \partial D_m$.

Now, according to the above cases (a)-(b)-(c), the modification of the graph runs as follows. If we are in situation (a), we label the cells around C_0 by $1, \dots, m$, and then we add edges joining the couples

$$(m, 2), (2, m-1), (m-1, 3), \dots$$

(see Figure 2).

If we are in situation (b) or (c), we do the same kind of procedure starting with the edge $(1, m)$, namely we add edges joining the couples

$$(1, m), (m, 2), (2, m-1), (m-1, 3), \dots$$

(see Figure 3).

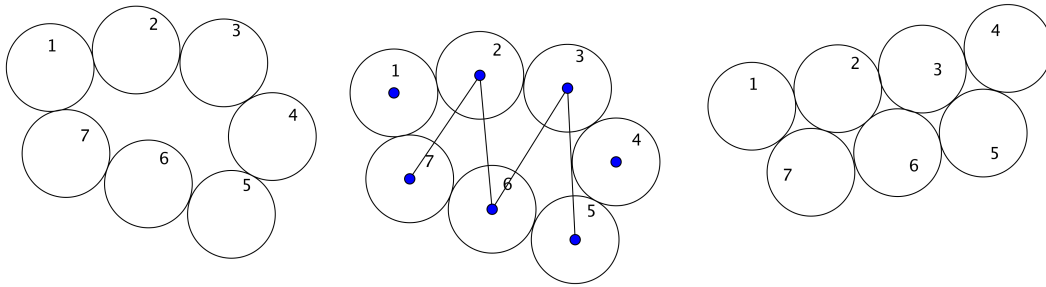


FIGURE 2. Formal topological modification of the graph in situation (a)

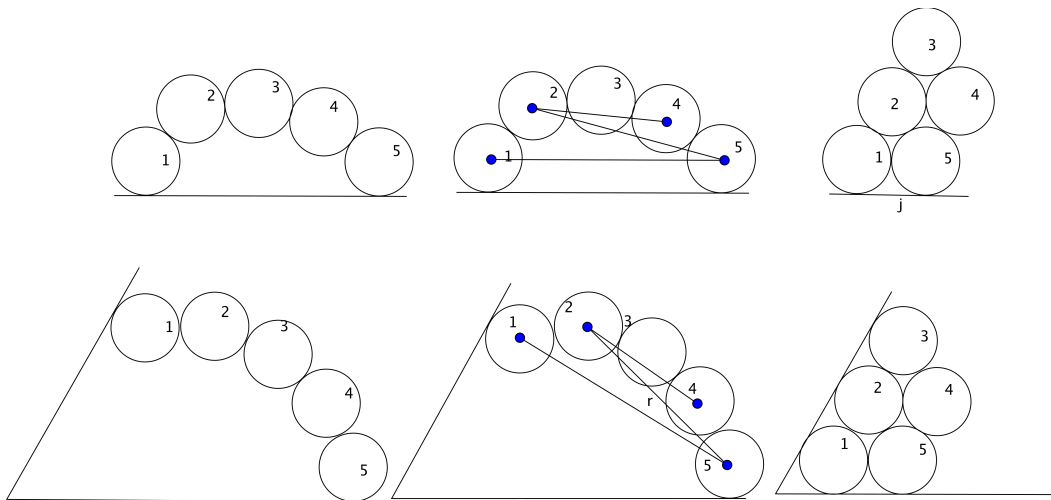


FIGURE 3. Formal topological modification of the graph in situations (b) and (c)

Notice that the fact that the radii of the disks in Figures 2 and 3 are all equal is not relevant to the present topological purposes, and in any case can be a posteriori justified by the results in the Appendix.

By construction, for the graph thus modified, each face has exactly 3 edges. Hence, we have $2E = 3F$. Recalling that $V = k + 1$, the Euler formula $V - E + F = 2$ gives $F = 2k - 2$. Then some easy but lengthy geometrical arguments, that we postpone to the Appendix (see Lemmas 19, 20, and 21), imply that the global area of all the empty rooms is not smaller than $(2k - 2)\Delta_{r,*}$, plus the contribution coming from 3 curvilinear triangles in the corners, each one of area $|\widehat{\Delta}_{r,*}|$.

4. AN INTERMEDIATE RESULT ON THE INNER CHEEGER BOUNDARY

Definition 15. Let Ω belong to the class \mathcal{A} introduced in Definition 6. Let $\Gamma := \partial\Omega$, and let $r := h(\Omega)^{-1}$ be the radius of the free arcs.

We call *inner Cheeger boundary of Ω* the “inner parallel curve at distance r from Γ ” (namely the set of points in Ω lying at distance r from Γ), endowed with the same orientation as Γ .

Remark 16. We can make the following observations.

- (i) The inner Cheeger boundary Γ_r may have self intersection points.
- (ii) If Γ^l , $l = 1, \dots, 2\Lambda$, are the arcs of Γ according to Definition 6, we can decompose Γ_r as $\Gamma_r^1 \cup \dots \cup \Gamma_r^{2\Lambda}$, where Γ_r^l denotes the inner parallel curve at distance r from Γ^l , and the free arcs are labelled with an odd number. Then, for l odd the inner parallel curve Γ_r^l is formally reduced to a point. For l even, Γ_r^l is uniquely determined as follows: if Γ^l is an arc of circle with center C^l and nonzero curvature K^l , Γ_r^l is the arc of circle obtained by applying an homothety of center C^l and ratio $1 - \frac{K(\Gamma^l)}{h(\Omega)}$ to Γ^l ; if Γ^l is a line segment, then Γ_r^l is the line segment obtained by moving Γ^l in the direction of the inner normal to Γ at distance r from its original position.

Definition 17. Let Ω belong to the class \mathcal{A} introduced in Definition 6, and let Γ_r denote its inner Cheeger boundary according to Definition 15. We call *inner Cheeger area of Ω* the oriented area enclosed by Γ_r , and we denote it by $A(\Gamma_r)$. Namely,

$$A(\Gamma_r) = \sum_h m(U_h) |U_h|,$$

where the sum is extended to the bounded connected components U_h of $\mathbb{R}^2 \setminus \Gamma_r$, $|U_h|$ is the Lebesgue measure of U_h , and the number $m(U_h) \in \mathbb{Z}$ is the index of any point of U_h with respect to the oriented curve Γ_r .

The following result is crucial to our purposes. It can be regarded as a transposition, valid within the class \mathcal{A} , of a well-known result for the inner Cheeger set of convex bodies due to Kawohl and Lachand-Robert (see Theorem 1 in [13]); we also refer to [16] for a recent extension to domains ‘without necks’.

Proposition 18 (representation via inner Cheeger set). *Let Ω belong to the class \mathcal{A} introduced in Definition 6. Let Γ_r and $A(\Gamma_r)$ be its inner Cheeger boundary and inner Cheeger area according to Definitions 15 and 17. There holds:*

$$(15) \quad A(\Gamma_r) = \pi r^2$$

$$(16) \quad |\Omega| = r\mathcal{H}^1(\Gamma_r) + 2\pi r^2.$$

Proof. We are going to show the validity of the following Steiner-type formulas:

$$(17) \quad \mathcal{H}^1(\partial\Omega) = \mathcal{H}^1(\Gamma_r) + 2\pi r$$

$$(18) \quad |\Omega| = A(\Gamma_r) + r\mathcal{H}^1(\Gamma_r) + \pi r^2$$

Taking into account that $\frac{\mathcal{H}^1(\partial\Omega)}{|\Omega|} = \frac{1}{r}$, the required equalities (15) and (16) will follow.

To prove (17)-(18), we need to consider the following angles (see Figure 4):

$\theta_i :=$ the opening angles of odd arc of radius r ;

$\alpha_i :=$ the opening angles of even arcs of radius $r_i > r$ and negative curvature;

$\beta_i :=$ the opening angles of even arcs of radius $r_i > r$ and positive curvature.

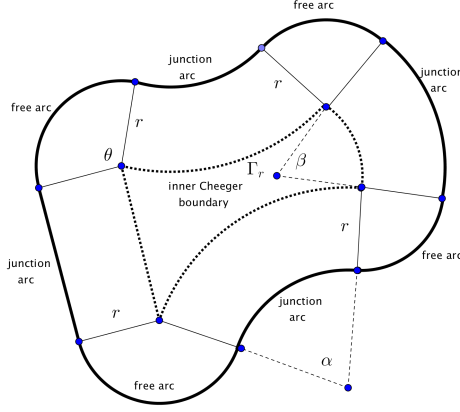


FIGURE 4. The geometry of the inner Cheeger set

We claim that the above angles obeys the following rule:

$$(19) \quad \sum_i \theta_i + \sum_i \beta_i - \sum_i \alpha_i = 2\pi,$$

where the sums are extended to the families of all angles of each type.

In order to prove (19), we consider the oriented polygon P having vertices

$$O_1 T_{1,2} O_2 T_{2,3} \dots O_N T_{\Lambda,1},$$

where $T_{i,i+1}$ is the touching point between the curves Γ^i and Γ^{i+1} , Λ is as in Remark 16, and O_i is defined as follows:

- if Γ^i is a circular arc, O_i is the center of the disk containing Γ^i ;
- if Γ^i is a line segment, O_i is an arbitrary point of Γ^i .

We are going to compute the sum of the inner angles of the polygon P . To that aim, we distinguish the following types of pairs of consecutive curves in $\partial\Omega$:

- Type 1: an odd arc of radius r - an even arc of radius $> r$ and negative curvature;
- Type 2: an odd arc of radius r - an even arc of radius $> r$ and positive curvature;
- Type 3: an odd arc of radius r - an even line segment.

We denote by N_i the number of pairs of type i which are contained in $\partial\Omega$. For $i = 1, 2, 3$, each pair of type i contributes with 4 vertices of P .

Therefore, the sum of the inner angles of P equals

$$(20) \quad [4(N_1 + N_2 + N_3) - 2]\pi.$$

On the other hand, the contribution given to the sum of the inner angles by the pairs of each type is listed below. Setting for brevity $\widehat{T}_{j,j+1} := \angle O_j T_{j,j+1} O_{j+1}$, and $\widehat{O}_j := \angle T_{j-1,j} O_j T_{j,j+1}$, we have

$$\begin{aligned} \text{Type 1:} \quad & \widehat{O}_{2i-1} = 2\pi - \theta, & \widehat{T}_{2i-1,2i} &= \pi, & \widehat{O}_{2i} &= \alpha, & \widehat{T}_{2i,2i+1} &= \pi; \\ \text{Type 2:} \quad & \widehat{O}_{2i-1} = 2\pi - \theta, & \widehat{T}_{2i-1,2i} &= 0, & \widehat{O}_{2i} &= 2\pi - \beta, & \widehat{T}_{2i,2i+1} &= 0; \\ \text{Type 3:} \quad & \widehat{O}_{2i-1} = 2\pi - \theta, & \widehat{T}_{2i-1,2i} &= \frac{\pi}{2}, & \widehat{O}_{2i} &= \pi, & \widehat{T}_{2i,2i+1} &= \frac{\pi}{2}. \end{aligned}$$

From this table we see that the sum of the inner angles of P is:

$$(21) \quad - \sum_i \theta_i + \sum_i \alpha_i - \sum_i \beta_i + 4\pi(N_1 + N_2 + N_3).$$

Imposing the equality between the expressions in (20) and (21), we obtain the required formula (19).

Now, relying on the equality (19), we are ready to prove (17)-(18). We introduce the following notation:

- if Γ^i is a circular arc, we denote by $r(=h(\Omega))$ or $r_i(>r)$ its radius;
- if Γ^i is a line segment, we denote by ℓ_i its length.

By direct computation, recalling Definition 15, we have

$$(22) \quad \mathcal{H}^1(\partial\Omega) = \sum_i \theta_i r + \sum_i \alpha_i r_i + \sum_i \beta_i r_i + \sum_i \ell_i$$

$$(23) \quad \mathcal{H}^1(\Gamma_r) = \sum_i \alpha_i(r_i + r) + \sum_i \beta_i(r_i - r) + \sum_i \ell_i.$$

By subtracting and using (19), we get

$$\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\Gamma_r) = \left[\sum_i \theta_i - \sum_i \alpha_i + \sum_i \beta_i \right] r = 2\pi r,$$

which proves (17).

Now we turn our attention to (18). We observe that

$$(24) \quad |\Omega| = \int_{\Gamma} x dy \quad \text{and} \quad A(\Gamma_r) = \int_{\Gamma_r} x dy.^2$$

To compute the above integrals, we use the decompositions

$$\Gamma = \Gamma^1 \cup \dots \cup \Gamma^{2\Lambda} \quad \text{and} \quad \Gamma_r = \Gamma_r^1 \cup \dots \cup \Gamma_r^{2\Lambda},$$

and we introduce the oriented line segments

$$S_i^r := [T_{i-1,i}^r, T_{i-1,i}],$$

where $T_{i-1,i}$ is the touching point between Γ^{i-1} and Γ^i , and $T_{i-1,i}^r$ is the touching point between Γ_r^{i-1} and Γ_r^i (with the conventions $\Gamma^0 := \Gamma^{2\Lambda}$ and $\Gamma_r^0 := \Gamma_r^{2\Lambda}$). We have

$$(25) \quad \begin{aligned} \int_{\Gamma} x dy - \int_{\Gamma_r} x dy &= \sum_{i=1}^{2\Lambda} \left[\int_{\Gamma^i} x dy - \int_{\Gamma_r^i} x dy \right] \\ &= \sum_{i=1}^{2\Lambda} \left[\int_{\Gamma^i} x dy - \int_{\Gamma_r^i} x dy + \int_{S_i^r} x dy - \int_{S_{i+1}^r} x dy \right]. \end{aligned}$$

By construction, for every $i = 1, \dots, N$, the curve $S_i^r + \Gamma^i - S_{i+1}^r - \Gamma_r^i$ is the positively oriented boundary of a Jordan domain D_i . Thus, each addendum of the last sum in (25) is equal to the Lebesgue measure of D_i , which is easily computed as follows:

² Indeed, by the Gauss-Green Theorem, if U is a Jordan domain with positively oriented, piecewise smooth boundary Γ_U , for any $f \in C^1(\Omega)$ it holds $\iint_U \frac{\partial f}{\partial x} dx dy = \int_{\Gamma_U} f dy$; in particular, taking $f(x, y) = x$, we get $|U| = \int_{\Gamma_U} x dy$. Applying this formula respectively to Ω and to the bounded connected components of $\mathbb{R}^2 \setminus \Gamma_r$, we obtain the equalities in (24).

- if Γ^i is an arc of radius r and opening angle θ_i , then Γ_r^i is a concentric arc of radius 0, so that

$$|D_i| = \frac{\theta_i}{2} r^2;$$

- if Γ^i is a negatively curved arc of radius $r_i > r$ and opening angle α_i , then Γ_r^i is a concentric arc of radius $r_i + r$, so that

$$|D_i| = \frac{\alpha_i}{2} [(r_i + r)^2 - r_i^2] = \alpha_i r_i r + \frac{\alpha_i}{2} r^2;$$

- if Γ^i is a positively curved arc of radius $r_i > r$ and opening angle β_i , then Γ_r^i is a concentric arc of radius $r_i - r$, so that

$$|D_i| = \frac{\beta_i}{2} [r_i^2 - (r_i - r)^2] = \beta_i r_i r - \frac{\beta_i}{2} r^2;$$

- if Γ^i is a line segment of length ℓ_i , then Γ_r^i is a parallel line segment of the same length, so that

$$|D_i| = \ell_i r.$$

Summing up, we obtain

$$\begin{aligned} |\Omega| - A(\Gamma_r) &= \int_{\Gamma} x dy - \int_{\Gamma_r} x dy = \sum_{i=1}^{2\Lambda} |D_i| \\ &= \left[\sum_i \frac{\theta_i}{2} + \sum_i \frac{\alpha_i}{2} - \sum_i \frac{\beta_i}{2} \right] r^2 + \left[\sum_i \alpha_i r_i + \sum_i \beta_i r_i \right] r + \sum_i \ell_i r. \end{aligned}$$

Next, we subtract from the above expression $r\mathcal{H}^1(\Gamma_r)$, that we compute from (23). We get

$$|\Omega| - A(\Gamma_r) - r\mathcal{H}^1(\Gamma_r) = \left[\sum_i \frac{\theta_i}{2} - \sum_i \frac{\alpha_i}{2} + \sum_i \frac{\beta_i}{2} \right] r^2.$$

Eventually, we invoke (19) and we obtain (18).

5. PROOF OF THEOREMS 1, 2, AND 3.

5.1. Proof of Theorem 1. Let us prove inequality (6). We take an optimal partition $\{\Omega_1, \dots, \Omega_k\}$ for problem (5). We set

$$h(\Omega_j) = h_j = r_j^{-1} \quad \forall j = 1, \dots, k$$

$$\max_{j=1, \dots, k} h_j = h_* = r_*^{-1}.$$

We now divide the proof of (6) in 4 steps.

Step 1. For every $j = 1, \dots, k$, we apply Proposition 18 to the cell Ω_j . We denote by Γ_{r_j} the inner Cheeger boundary of Ω_j .

By (16), we have

$$(26) \quad |\Omega_j| = r_j \mathcal{H}^1(\Gamma_{r_j}) + 2\pi r_j^2.$$

multiplying by h_j^2 , we have

$$(27) \quad h_j^2 |\Omega_j| - h_j \mathcal{H}^1(\Gamma_{r_j}) = 2\pi.$$

We look at the polynomial $x \mapsto p_j(x) := |\Omega_j|x^2 - \mathcal{H}^1(\Gamma_{r_j})x$. By (26) we have that h_j is larger than the largest root of p , namely

$$(28) \quad h_j > \frac{\mathcal{H}^1(\Gamma_{r_j})}{|\Omega_j|}.$$

Then, since $h_* \geq h_j$, we have $p_j(h_*) \geq p_j(h_j)$, and we infer from (27) that

$$(29) \quad h_*^2 |\Omega_j| \geq h_* \mathcal{H}^1(\Gamma_{r_j}) + 2\pi.$$

We conclude this step by summing the above inequality over j :

$$(30) \quad h_*^2 \sum_{j=1}^k |\Omega_j| \geq h_* \sum_{j=1}^k \mathcal{H}^1(\Gamma_{r_j}) + 2\pi k.$$

Step 2. In order to estimate from below the r.h.s. of (30), we are going to use Hales hexagonal isoperimetric inequality. According to (15) we have $A(\Gamma_{r_j}) = \pi r_j^2$ for every j . Therefore

$$A\left(\frac{\Gamma_{r_j}}{\sqrt{\pi r_*^2}}\right) = \frac{\pi r_j^2}{\pi r_*^2} \geq 1,$$

so that

$$\min \left\{ A\left(\frac{\Gamma_{r_j}}{\sqrt{\pi r_*^2}}\right), 1 \right\} = 1.$$

On the inner Cheeger boundary Γ_{r_j} , we fix the following family \mathcal{N}_j of nodes: first, we take as nodes all the points which are at distance r_j from an odd free arc of $\partial\Omega_j$ (in equivalent terms, any such node joins two arcs of Γ_{r_j} which are parallel to two consecutive even junction arcs of $\partial\Omega_j$ separated by a free arc); then, we add the following “exceptional nodes”: if a border junction arc of $\partial\Omega_j$ contains different segments lying on $\partial\mathcal{T}$, we take as nodes also the points in Γ_{r_j} which are at distance r_j from the endpoints of all these segments.

Accordingly, we write $\Gamma_{r_j} = \Gamma_{r_j}^1 \cup \dots \cup \Gamma_{r_j}^{N_j}$, where N_j is the cardinality of the family of nodes \mathcal{N}_j , and, for $i = 1, \dots, N_j$, $\Gamma_{r_j}^i$ is the (oriented) portion of Γ_{r_j} delimited by two consecutive nodes n_{i-1}, n_i (with the convention $n_0 = n_{N_j}$).

Now, we set $T(\Gamma_{r_j})$ the (truncated) deficit associated to the oriented curve Γ_{r_j} and the family \mathcal{N}_j . Namely,

$$T(\Gamma_{r_j}) := \sum_{i=1}^{N_j} x(\Gamma_{r_j}^i) \wedge 1 \vee (-1),$$

where $x(\Gamma_{r_j}^i)$ is the signed area enclosed by the oriented curve $\Gamma_{r_j}^i \cup [n_i, n_{i-1}]$.

Then, Hales’ hexagonal isoperimetric inequality [10, Theorem 4] gives

$$(31) \quad \mathcal{H}^1\left(\frac{\Gamma_{r_j}}{\sqrt{\pi r_*^2}}\right) \geq -\frac{1}{\pi r_*^2} T(\Gamma_{r_j}) \sqrt[4]{12} - (N_j - 6)0.0505 + 2 \sqrt[4]{12}.$$

where $2 \sqrt[4]{12}$ is the perimeter of the unit area regular hexagon.

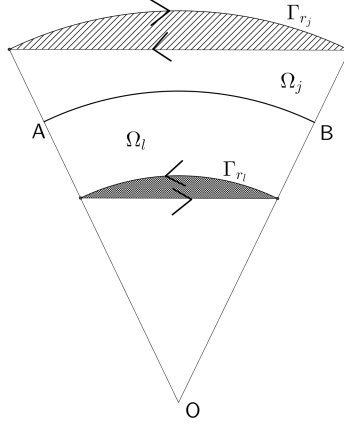


FIGURE 5. The positive contribution of the polygonal deficit

We now multiply (31) by $\sqrt{\pi}$, and we sum over $j = 1, \dots, k$, taking into account that:

$$(32) \quad - \sum_{j=1}^k T(\Gamma_{r_j}) \geq 0$$

$$(33) \quad - \sum_{j=1}^k (N_j - 6) \geq 0$$

To obtain (32), we observe that each piece of curve $\Gamma_{r_j}^i$ is an arc of circle, possibly of zero curvature (in particular, thanks to the addition of the exceptional nodes, $\Gamma_{r_j}^i$ cannot be a broken line). If $\Gamma_{r_j}^i$ has zero curvature, it produces a zero deficit. If $\Gamma_{r_j}^i$ has a nonzero curvature, it is parallel to a junction arc between Ω_j and another cell Ω_l , so that it produces two deficits. Assume the curvature of $\Gamma_{r_j}^i$, seen from Ω_j , has a negative sign. Then the deficit $x(\Gamma_{r_j}^i)$ (the dashed region in Figure 5) has a negative sign, while the deficit $x(\Gamma_{r_l}^i)$ (the black region in Figure 5) has a positive sign, being in absolute value smaller than the previous one. This is simply due to the fact that the two regions which contribute to the deficit are homothetic, with a ratio larger than one. We conclude that (32) holds true.

To obtain (33), we observe that

$$(34) \quad \sum_{j=1}^k N_j \leq \sum_{j=1}^k \Lambda_j + 3 \leq 6k,$$

where the second inequality holds true by Proposition 12. To obtain the first equality in (34), we observe that the arcs of Γ_{r_j} are in bijection with the junction arcs in $\partial\Omega_j$, except for the extra arcs created in Γ_{r_j} by exceptional nodes. Now, the maximal possible number of such extra arcs is 3: in fact, recalling that distinct segments contained into a unique border junction arc must lie on different sides of \mathcal{T} (see end of Section 2.2), we see that the configuration containing the highest number of extra arcs (equal 3) is the one in which there are 3 border junction arcs, each one containing 2 segments (lying on consecutive sides of \mathcal{T}).

By (31), (32), and (33), we get the following lower bound for the r.h.s. of (30)

$$(35) \quad h_* \sum_{j=1}^k \mathcal{H}^1(\Gamma_{r_j}) + 2\pi k \geq k[2\sqrt{\pi} \sqrt[4]{12} + 2\pi].$$

Step 3. By elementary computation, we know that $|\Delta_{r_*}| = \frac{r_*^2}{2}(2\sqrt{3} - \pi)$ and $|\widehat{\Delta}_{r_*}| = \frac{r_*^2}{3}(3\sqrt{3} - \pi)$. In order to estimate from above the l.h.s. of (30), we exploit Proposition 14. Inequality (14), together with the computations above, imply

$$|\Omega_0| \geq 2k|\Delta_{r_*}|.$$

So

$$|\Omega_0| \geq k(2\sqrt{3} - \pi) \frac{1}{h_*^2},$$

yielding

$$(36) \quad h_*^2 \sum_{j=1}^k |\Omega_j| \leq h_*^2(|\mathcal{T}| - |\Omega_0|) \leq [h_*^2|\mathcal{T}| - k(2\sqrt{3} - \pi)].$$

Step 4 (conclusion). We put together the information coming from the previous three steps. By (30), (35), and (36), we have:

$$\frac{|\mathcal{T}|}{k} h_*^2 \geq \pi + 2\sqrt{3} + 2\sqrt{\pi} \sqrt[4]{12} = \left[\frac{12 \sin(\frac{\pi}{6}) + \sqrt{12\pi \sin(\frac{\pi}{3})}}{\sqrt{12 \sin(\frac{\pi}{3})}} \right]^2 = [h(H)]^2.$$

This concludes the proof of (6).

We now turn to the proof of (7). In order to deduce it from (6), it is enough to check that $\widetilde{M}_{k,p}(\Omega)$ converges to $M_k(\Omega)$ in the limit as $p \rightarrow +\infty$. In fact, this follows from the inequalities

$$(37) \quad M_k(\mathcal{T}) \leq \widetilde{M}_{k,p}(\mathcal{T}) \leq k^{1/p} M_k(\mathcal{T}),$$

which are readily obtained as follows. If $\{\Omega_1^p, \dots, \Omega_k^p\}$ is an optimal solution for $M_{k,p}(\mathcal{T})$, and $\{\Omega_1^\infty, \dots, \Omega_k^\infty\}$ is an optimal solution for $M_k(\mathcal{T})$, we have

$$\begin{aligned} M_k(\mathcal{T}) &\leq \max_{j=1, \dots, k} h(\Omega_j^p) = \widetilde{M}_{k,p}(\mathcal{T}) \\ k[M_k(\mathcal{T})]^p &\geq \sum_{i=1}^k h^p(\Omega_i^\infty) \geq \sum_{i=1}^k h^p(\Omega_i^p) \geq \max_{j=1, \dots, k} h^p(\Omega_j^p) = [\widetilde{M}_{k,p}(\mathcal{T})]^p. \end{aligned}$$

5.2. Proof of Theorem 2. Let k be fixed. Clearly, from the definition of $M_k(\mathcal{T}_k)$ and from the geometry of \mathcal{T}_k , precisely since \mathcal{T}_k contains the k -cluster made by k copies of H , it holds

$$\frac{|\mathcal{T}_k|^{\frac{1}{2}}}{k^{\frac{1}{2}}} M_k(\mathcal{T}_k) \leq h(H).$$

Assume by contradiction that the strict inequality holds, namely

$$(38) \quad M_k(\mathcal{T}_k) = (1 - \delta)h(H) \frac{k^{\frac{1}{2}}}{|\mathcal{T}_k|^{\frac{1}{2}}}, \quad \text{with } \delta \in (0, 1).$$

Let \mathcal{T} be the equilateral triangle of fixed area, say equal 1. For every $\eta \in \mathbb{N}$, \mathcal{T} contains a family of mutually disjoint k -triangles $\{\mathcal{T}_k^i\}_{i=1, \dots, \eta}$, having the same area, infinitesimal as η tends to $+\infty$, and such that

$$(39) \quad \lim_{\eta \rightarrow +\infty} |\mathcal{T} \setminus \bigcup_{i=1}^{\eta} \mathcal{T}_k^i| = 0.$$

By Theorem 1, we have

$$(40) \quad M_{\eta k}(\mathcal{T}) \geq h(H)(\eta k)^{1/2}.$$

On the other hand, by using assumption (38) (applied to each of the k -triangles $\{\mathcal{T}_k^i\}$, for $i = 1, \dots, \eta$), we infer that there exists a (ηk) -cluster of \mathcal{T} whose cells have a Cheeger constant not larger than $(1 - \delta)h(H) \frac{k^{1/2}}{|\mathcal{T}_k^i|^{1/2}}$. Thus,

$$(41) \quad M_{\eta k}(\mathcal{T}) \leq (1 - \delta)h(H) \frac{k^{1/2}}{|\mathcal{T}_k^i|^{1/2}}.$$

By combining (40) and (41), we obtain

$$(1 - \delta) \geq (\eta |\mathcal{T}_k^i|)^{1/2}.$$

In the limit as $\eta \rightarrow +\infty$, the above inequality gives a contradiction: indeed, in view of (39), we have $\lim_{\eta \rightarrow +\infty} \eta |\mathcal{T}_k^i| = |\mathcal{T}| = 1$.

5.3. Proof of Theorem 3. Once proved Theorem 2, the way Theorem 3 is deduced is the same as in case of convex cells treated in [3]. Thus we limit ourselves to indicate the strategy, referring to [3] for the detailed arguments.

First, one shows that the equality (8) in Theorem 2 extends to the case in which the k -triangle \mathcal{T}_k is replaced by a “ k -cell” Σ_k , meant as a connected set of arbitrary shape obtained as the union of k hexagons lying in a tiling of \mathbb{R}^2 made by a family of copies of a regular hexagon. The passage from a k -triangle to a k -cell is performed as follows. For simplicity, and without loss of generality, we can assume that $|\Sigma_k| = k$. Since Σ_k contains a k -clusters made by k copies of H , it holds $M_k(\Sigma_k) \leq h(H)$. Assume by contradiction that $M_k(\Sigma_k) < h(H)$. This means that there exists a k -cluster $\{\Omega_j\}$ of Σ_k such that $\max_{j=1, \dots, k} h(\Omega_j) < h(H)$. We can assume (up to shrinking a little bit the sets Ω_j) that each of them is at positive distance from $\partial\Sigma_k$. Then we embed Σ_k into a big k' -triangle $\mathcal{T}_{k'}$, with $k' > k$, and we consider the k' -cluster of $\mathcal{T}_{k'}$ which is made by Ω_j (for $j = 1, \dots, k$) union \tilde{H}_j (for $j = 1, \dots, k' - k$), where \tilde{H}_j are slight deformations of the copies of H contained into $\mathcal{T}_{k'} \setminus \Sigma_k$, constructed so that $h(\tilde{H}_j) < h(H)$ (this can be done by continuity and since we have assumed $\text{dist}(\Omega_j, \partial\Sigma_k) > 0$). We have thus constructed a k' -cluster of $\mathcal{T}_{k'}$ in which each cell has a Cheeger constant strictly less than $h(H)$, against the equality (8).

Now, using the equality (8) for k -cells, it is possible to show separately the inequalities

$$\limsup_{k \rightarrow +\infty} \frac{|\Omega|^{1/2}}{k^{1/2}} M_k(\Omega) \leq h(H) \quad \text{and} \quad \liminf_{k \rightarrow +\infty} \frac{|\Omega|^{1/2}}{k^{1/2}} M_k(\Omega) \geq h(H)$$

via a blow up argument. More precisely, the upper bound inequality is proved by dilating Ω so that it is well approximated from *inside* with a k -cell, and using just the homogeneity and decreasing monotonicity of $M_k(\cdot)$ by domain inclusion. The lower bound inequality is proved by dilating Ω so that it is well approximated from *outside* with a k -cell, and using

now, besides the behaviour of $M_k(\cdot)$ under dilations and inclusions, the crucial information that (8) holds for k -cells.

6. APPENDIX: GEOMETRICAL ESTIMATES FOR THE EMPTY CHAMBER

We give here three geometrical lemmas, in which we estimate from below the area of the region V bounded by a “closed chain” of consecutive tangent disks (Lemma 19), by an “open chain” of consecutive tangent disks and a segment (Lemma 20), and by an “open chain” of consecutive tangent disks and two line segments forming an angle of $\pi/3$ (Lemma 21).

These results are needed in the proof of Proposition 14 in order to estimate from below the global area of all the empty rooms. More precisely, referring to the proof of Proposition 14, Lemma 19, 20 and 21 concern respectively the area of an empty room of type (a), (b), and (c): it turns out to be not smaller than the number of faces associated with the room in the modified graph times the area of a curvilinear triangle Δ_{r_*} , with the addition of an extra curvilinear triangle $\widehat{\Delta}_{r_*}$ in case (c).

As usual, we denote by $d(\cdot, \cdot)$ the Euclidean distance.

Lemma 19. *Let D_1, \dots, D_m be a family of $m \geq 3$ disks of centers P_1, \dots, P_m and radii r_1, \dots, r_m such that*

$$\begin{aligned} d(P_i, P_{i+1}) &= r_i + r_{i+1} & \forall i = 1, \dots, m \\ d(P_i, P_j) &> r_i + r_j & \forall i, j \in \{1, \dots, m\}, |i - j| \geq 2 \\ \angle P_{i-1}P_iP_{i+1} &< \pi & \forall i = 1, \dots, m. \end{aligned}$$

(with the conventions $m + 1 = 1$ and $0 = m$).

Setting V the complement in \mathbb{R}^2 of the unbounded connected component of $\mathbb{R}^2 \setminus \cup_{i=1}^m D_i$, and $r_* := \min\{r_1, \dots, r_m\}$, there holds

$$|V| \geq (m - 2)|\Delta_{r_*}|.$$

Proof. We search for a configuration of the disks D_1, \dots, D_m which minimizes the area of V . The existence of an optimal configuration is immediate, since we deal with a finite-dimensional problem. However, since the constraints are not closed, possibly an optimal configuration is degenerated, meaning it may exhibit some aligned triple of consecutive centers ($\angle P_{i-1}P_iP_{i+1} = \pi$) and/or some touching non-consecutive discs ($d(P_i, P_j) = r_i + r_j$ with $|i - j| \geq 2$).

The statement will be obtained by induction on m .

Initial step. Let $m = 3$. We have to show that the area of a curvilinear triangle bounded by three concave arcs of circle of radii r_1, r_2, r_3 is minimal when the three radii are equal. Let us show that, if one of the three radii, say r_2 , is strictly larger than r_* , we can perturb the configuration of the three disks $\{D_1, D_2, D_3\}$ so to decrease the measure of the bounded connected component V of $\mathbb{R}^2 \setminus (D_1 \cup D_2 \cup D_3)$. The perturbation we consider is the following one: we keep D_1 and D_3 fixed, and we change D_2 into a new disk \widehat{D}_2 which has radius \widehat{r}_2 strictly smaller than r_2 and is tangent to D_1 and D_3 . Denoting by \widehat{V} the bounded connected component of $\mathbb{R}^2 \setminus (D_1 \cup \widehat{D}_2 \cup D_3)$, we claim that the inclusion $\widehat{V} \subset V$ holds. Indeed, if we choose a system of coordinates so that $P_1 = (0, 0)$, and

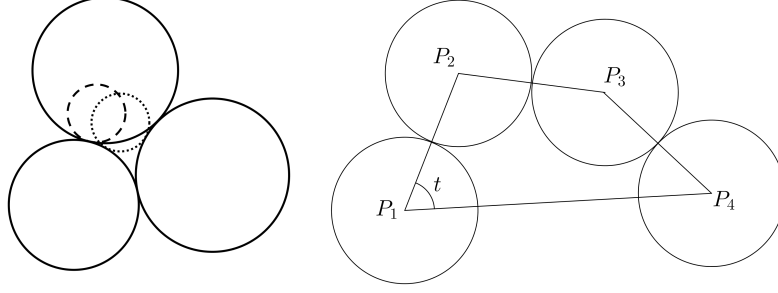


FIGURE 6. Proof of Lemma 19: initial step (left) and induction step (right)

$P_3 = (r_1 + r_3, 0)$, we have $P_2 = (x_0, y_0)$, with

$$(42) \quad x_0 = \frac{r_1^2 + (r_1 + r_3)^2 + 2r_1r_2 - r_3^2 - 2r_3r_2}{2(r_1 + r_3)}$$

$$y_0 = \sqrt{(r_1 + r_2)^2 - \frac{((r_1 - r_3)(r_1 + r_3 + 2r_2) + (r_1 + r_3)^2)^2}{4(r_1 + r_3)^2}}.$$

The geometry is represented in Figure 6, left.

The derivatives of the angles

$$\theta_1 := \angle P_2P_1P_3 = \arctan\left(\frac{y_0}{x_0}\right) \quad \text{and} \quad \theta_3 := \angle P_2P_3P_1 = \arctan\left(\frac{y_0}{r_1 + r_3 - x_0}\right)$$

with respect to r_2 are positive, since they are easily computed as

$$(43) \quad \frac{\partial \theta_1}{\partial r_2} = \frac{2r_1r_3}{(r_1 + r_3)(r_1 + r_2)\sqrt{-\frac{r_1r_3((r_1+r_3)^2 - (r_1+r_3+2r_2)^2)}{(r_1+r_3)^2}}}$$

$$\frac{\partial \theta_3}{\partial r_2} = \frac{2r_1r_3}{(r_1 + r_3)(r_2 + r_3)\sqrt{-\frac{r_1r_3((r_1+r_3)^2 - (r_1+r_3+2r_2)^2)}{(r_1+r_3)^2}}}.$$

The inequalities $\frac{\partial \theta_1}{\partial r_2} > 0$ and $\frac{\partial \theta_3}{\partial r_2} > 0$ imply the inclusion $\widehat{V} \subset V$. In fact, the following simple geometric argument shows that $\partial D_2 \cap \partial \widehat{D}_2 \cap \partial \widehat{V} = \emptyset$. Let a disk of radius \widehat{r}_2 roll from the position when it is externally tangent to D_1 at its tangency point with D_2 , to the final position when it agrees with \widehat{D}_2 . During this movement the intersection points between the boundary of the rolling disk and ∂D_2 are: 1 point at the initial time, then 2 points, and eventually 2, 1 or 0 points at the final time, all these intersections lying outside \widehat{V} . In any case, at the final time no intersection point can belong to $\partial \widehat{V}$, *i.e.* $\partial D_2 \cap \partial \widehat{D}_2 \cap \partial \widehat{V} = \emptyset$.

Induction step. Assume the statement holds true for up to $m - 1$ disks, and let us show it holds true also for m disks. Two cases may occur for an optimal configuration of m disks.

Case 1: $d(P_i, P_j) = r_i + r_j$ for some i, j with $j \neq i + 1$ (equivalently, V is disconnected). With no loss of generality, let $i = 1$ and $2 < j < m$. Consider the two disjoint families of disks $\mathcal{F}' := \{D_1, \dots, D_j\}$ and $\mathcal{F}'' := \{D_j, D_{j+1}, \dots, D_m, D_1\}$. They have cardinalities j and $m + 2 - j$, both strictly smaller than m . Hence, letting V' and V'' be respectively the

complements of the unbounded connected components of $\mathbb{R}^2 \setminus \cup_{D_i \in \mathcal{F}'} D_i$ and $\mathbb{R}^2 \setminus \cup_{D_i \in \mathcal{F}''} D_i$, by induction it holds

$$|V'| \geq (j-2)|\Delta_{r_*}| \quad \text{and} \quad |V''| \geq (m+2-j-2)|\Delta_{r_*}| = (m-j)|\Delta_{r_*}|.$$

Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain that $|V| \geq (m-2)|\Delta_{r_*}|$. This concludes the proof in Case 1.

Case 2: $d(P_i, P_j) > r_i + r_j$ for all i, j with $j \neq i+1$ (equivalently, V is connected). We start by proving the following claim:

$$(44) \quad r_i = r_* \quad \forall i = 1, \dots, m.$$

Namely, let us we show that, if one of the radii r_1, \dots, r_m is strictly larger than r_* , we can perturb the configuration of the disks $\{D_1, \dots, D_m\}$ so to decrease the measure of V . The perturbation we use is similar as the one considered in the initial step: assuming without loss of generality that $r_2 < r_*$, we keep all the disks fixed except D_2 , and we change D_2 into a new disk \widehat{D}_2 which has radius \widehat{r}_2 strictly smaller than r_2 and is tangent to D_1 and D_3 . We remark that such a disk \widehat{D}_2 exists because by assumption the centers P_1, P_2 , and P_3 of the three involved disks are not aligned. Notice also that the perturbation we are considering is admissible because, in the case 2 we are dealing with, it holds $d(P_2, P_j) > r_2 + r_j$ for all $j \neq 1, 3$, which in particular ensures that the new configuration still satisfies the assumptions of the proposition. Denoting by \widehat{V} the bounded connected component of $\mathbb{R}^2 \setminus (D_1 \cup \widehat{D}_2 \cup D_3)$, we claim that the inclusion $\widehat{V} \subset V$ holds. The proof is similar as in the initial step. We choose a system of coordinates so that $P_1 = (0, 0)$, and $P_3 = (l, 0)$, with $l > r_1 + r_3$. Accordingly, equations (42) and (43) are now replaced by

$$(45) \quad x_0 = \frac{l^2 + r_1^2 + 2r_1r_2 - r_3^2 - 2r_3r_2}{2l}$$

$$y_0 = \sqrt{(r_1 + r_2)^2 - \frac{(l^2 + (r_1 - r_3)(r_1 + r_3 + 2r_2))^2}{4l^2}}$$

and

$$(46) \quad \frac{\partial \theta_1}{\partial r_2} = \frac{(l + r_1 - r_3)(l - r_1 + r_3)}{l(r_1 + r_2) \sqrt{-\frac{(l+r_1-r_3)(l-r_1+r_3)(l^2-(r_1+r_3+2r_2)^2)}{l^2}}}$$

$$\frac{\partial \theta_3}{\partial r_2} = \frac{(l + r_1 - r_3)(l - r_1 + r_3)}{l(r_3 + r_2) \sqrt{-\frac{(l+r_1-r_3)(l-r_1+r_3)(l^2-(r_1+r_3+2r_2)^2)}{l^2}}}.$$

Since the above derivative are positive, the inclusion $\widehat{V} \subset V$ can be obtained as in the initial step, and the proof of (44) is concluded.

To achieve our proof in case 2, it remains to show that a contradiction is reached as soon as we have $m \geq 4$. Since we have proved condition (44), we are reduced to show the following assertion: given a number $m \geq 4$ of disks D_1, \dots, D_m with equal radius r_* , and centers P_1, \dots, P_m such that $d(P_i, P_{i+1}) = 2r_*$, $d(P_i, P_j) > 2r_*$ if $j \neq i+1$, it is possible to perturb their configuration so to decrease the area of V . The perturbation we consider consists in keeping D_1 and D_4 fixed, and moving just D_2 and D_3 , so that they remain tangent to each other and to D_1, D_4 respectively. Notice that such perturbation is admissible because $d(P_2, P_j) > 2r_*$ for all $j \neq 1, 3$ and similarly $d(P_3, P_j) > 2r_*$ for all $j \neq 2, 4$. Since (44) holds, showing that the area of V decreases is equivalent to showing that the area of the quadrilateral with vertices P_1, P_2, P_3, P_4 decreases. Assume without loss of generality

that $2r_* = 1$, and let l the distance between P_1 and P_4 . We have $l \geq 1$, with equality if $m = 4$. We name t the angle formed by the side of length l and one of its adjacent sides, see Figure 6, right.

By the assumption on the angles $\angle P_{i-1}P_iP_{i+1}$, our quadrilateral is convex, and its area is given by

$$\varphi(t) := \frac{1}{4}\sqrt{(l^2 - 2l \cos t + 1)(-l^2 + 2l \cos t + 3)} + \frac{1}{2}l \sin t.$$

We have

$$\varphi'(t) = \frac{1}{2}l \left(\frac{\sin t (-l^2 + 2l \cos t + 1)}{\sqrt{(l^2 - 2l \cos t + 1)(-l^2 + 2l \cos t + 3)}} + \cos t \right).$$

Hence the inequality $\varphi'(t) \geq 0$ is equivalent to

$$\cos^2 t (l^2 - 2l \cos t + 1) (-l^2 + 2l \cos t + 3) - \sin^2 t (-l^2 + 2l \cos t + 1)^2 \geq 0,$$

and, in turn, to

$$(1 - l^2) (4 \cos^2 t - 4l \cos t + l^2 - 1) \geq 0.$$

Taking into account that $l \geq 1$, we have $\varphi'(t) \geq 0$ if and only if

$$4 \cos^2 t - 4l \cos t + l^2 - 1 \leq 0.$$

The above inequality is satisfied on the interval $I_l := [0, \arccos(\frac{l-1}{2})]$. Indeed, setting $y := \cos t$, the roots of the polynomial $p_l(y) := 4y^2 - 4ly + l^2 - 1$ are $\frac{l \pm 1}{2}$, so that $p_l(y) \leq 0$ on the interval $[\frac{l-1}{2}, \frac{l+1}{2}]$, which contains I_l (because $1 \leq \frac{l+1}{2}$).

Therefore, the minimum of $\varphi(t)$ is achieved as $t \rightarrow 0$, so that no nondegenerate quadrilateral can be optimal, and our proof is achieved.

Lemma 20. *Let D_1, \dots, D_m be a family of $m \geq 3$ disks of centers P_1, \dots, P_m and radii r_1, \dots, r_m , contained into a half-plane H delimited by a straight line tangent to D_1 and D_m , such that*

$$\begin{aligned} d(P_i, P_{i+1}) &= r_i + r_{i+1} & \forall i = 1, \dots, m-1 \\ d(P_i, P_j) &> r_i + r_j & \forall i, j \in \{1, \dots, m\}, |i-j| \geq 2 \\ \angle P_{i-1}P_iP_{i+1} &< \pi & \forall i = 1, \dots, m \end{aligned}$$

(where P_0, P_{m+1} are the orthogonal projections on $\mathbb{R}^2 \setminus H$ of P_1, P_m).

Setting V the complement in H of the unbounded connected component of $H \setminus \cup_{i=1}^m D_i$, and $r_* := \min\{r_1, \dots, r_m\}$, there holds

$$|V| \geq (m-2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| (\geq (m-1)|\Delta_{r_*}|);$$

Proof. Similarly as in the proof of Lemma 20, we search for a (possibly degenerated) optimal configuration of the disks D_1, \dots, D_m which minimizes the area of V , and we argue by induction on m .

Initial step. Let $m = 3$. Let us show that the area of the region V bounded by three concave arcs lying on disks D_1, D_2, D_3 with (D_1, D_2) and (D_2, D_3) mutually tangent, and a straight line γ tangent to both D_1 and D_3 , is minimal when the three radii are equal to r_* , and V is the (disjoint) union $\Delta_{r_*} \cup \widehat{\Delta}_{r_*}$. The fact that the three radii must be equal can be proved in the very same way as done in the initial step of the proof of Lemma 19. Then we are reduced to minimize the area of the pentagon $P_0P_1P_2P_3P_4$ represented in Figure

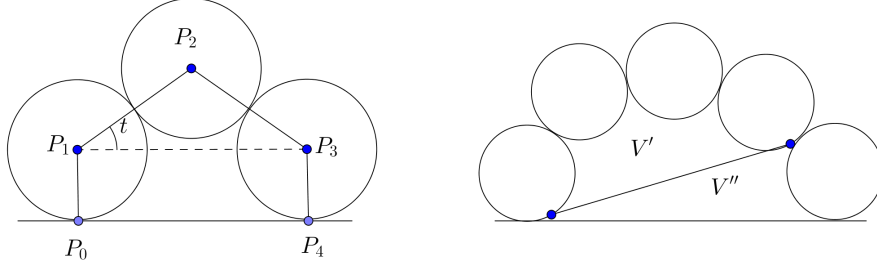


FIGURE 7. Proof of Lemma 20: initial step (left) and induction step (right)

7, left. Assuming without loss of generality that $2r_* = 1$, and setting $t := \angle P_3P_1P_2$, the area of such pentagon is given by

$$\varphi(t) := \cos t (1 + \sin t), \quad t \in [0, \frac{\pi}{3}].$$

Then it is immediate to see that $\varphi'(t) \geq 0$ if and only if $t \in [0, \frac{\pi}{6}]$, so that the minimum of φ on the interval $[0, \frac{\pi}{3}]$ is equal to

$$\min \{ \varphi(0), \varphi(\frac{\pi}{3}) \} = \varphi(\frac{\pi}{3}) = |\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|.$$

Induction step. Assume the statement holds true for up to $m - 1$ disks, and let us show it holds true also for m disks. Two cases may occur for an optimal configuration of m disks.

Case 1: V is disconnected. Two subcases may occur:

Case 1a: The family $\{D_1, \dots, D_k\}$ can be decomposed as the union of two disjoint subfamilies \mathcal{F}' and \mathcal{F}'' , of cardinalities j and $m + 1 - j$ (both strictly smaller than m), both satisfying the assumptions of Lemma 20. In this case, letting V' and V'' be respectively the complements in H of the unbounded connected components of $H \setminus \cup_{D_i \in \mathcal{F}'} D_i$ and $H \setminus \cup_{D_i \in \mathcal{F}''} D_i$, by induction it holds

$$|V'| \geq (j - 2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| \quad \text{and} \quad |V''| \geq (m - j - 1)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|.$$

Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain

$$|V| \geq (m - 3)|\Delta_{r_*}| + 2|\widehat{\Delta}_{r_*}| \geq (m - 2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|.$$

Case 1b: The family $\{D_1, \dots, D_k\}$ can be decomposed as the union of two disjoint subfamilies \mathcal{F}' and \mathcal{F}'' , of cardinalities j and $m + 2 - j$ (both strictly smaller than m), such that one on them, say \mathcal{F}' , satisfies the assumptions of Lemma 19, and the other one satisfies the assumptions of Lemma 20. In this case, letting V' and V'' be respectively the complements in H of the unbounded connected components of $H \setminus \cup_{D_i \in \mathcal{F}'} D_i$ and $H \setminus \cup_{D_i \in \mathcal{F}''} D_i$, by Lemma 20 and induction, it holds

$$|V'| \geq (j - 2)|\Delta_{r_*}| \quad \text{and} \quad |V''| \geq (m - j)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|.$$

Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain

$$|V| \geq (m - 2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|,$$

The proof of the induction step in Case 1 is concluded.

Case 2: V is connected. In this case, we preliminary observe that the equality (44) must be satisfied, otherwise the configuration cannot be optimal (the proof is exactly the same

as in Case 2 of the induction step in the proof of Lemma 19). Then, we consider a straight line tangent to both D_1 and D_{m-1} such that D_1, \dots, D_{m-1} are contained into a half-plane \tilde{H} delimited by such line. We set $V' := V \cap \tilde{H}$ and $V'' := V \cap (\mathbb{R}^2 \setminus \tilde{H})$, see Figure 7, right. By induction, we have $|V'| \geq (m-3)|\Delta_{r_*}| + |\hat{\Delta}_{r_*}|$. On the other hand, we observe that V'' contains a copy of Δ_{r_*} (with strict inclusion, since we are dealing with case 2); hence we have $|V''| \geq |\Delta_{r_*}|$. Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain that $|V| \geq (m-2)|\Delta_{r_*}| + |\hat{\Delta}_{r_*}|$, concluding the proof of the induction step also in Case 2.

Lemma 21. *Let D_1, \dots, D_m be a family of $m \geq 3$ disks of centers P_1, \dots, P_m and radii r_1, \dots, r_m , contained into a sector S of opening angle $\pi/3$ delimited by two half lines tangent respectively to D_1 and D_m , such that*

$$\begin{aligned} d(P_i, P_{i+1}) &= r_i + r_{i+1} & \forall i = 1, \dots, m-1 \\ d(P_i, P_j) &> r_i + r_j & \forall i, j \in \{1, \dots, m\}, |i-j| \geq 2 \\ \angle P_{i-1}P_iP_{i+1} &< \pi & \forall i = 1, \dots, m \end{aligned}$$

(where P_0, P_{m+1} are the orthogonal projections on $\mathbb{R}^2 \setminus Q$ of P_1, P_m).

Setting V the complement in S of the unbounded connected component of $S \setminus \cup_{i=1}^m D_i$, and $r_* := \min\{r_1, \dots, r_m\}$, there holds

$$|V| \geq (m-2)|\Delta_{r_*}| + |\hat{\Delta}_{r_*}| + |\hat{\hat{\Delta}}_{r_*}| \geq (m-1)|\Delta_{r_*}| + |\hat{\hat{\Delta}}_{r_*}|.$$

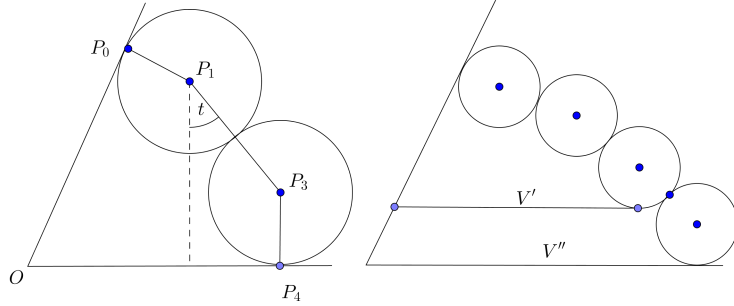


FIGURE 8. Proof of Lemma 21: initial step (left) and induction step (right)

Proof. We argue again by induction on m .

Initial step. Let $m = 3$. Let us show that the area of the region V bounded by three concave arcs lying on disks D_1, D_2, D_3 with (D_1, D_2) and (D_2, D_3) mutually tangent, and two half-lines forming an angle of $\pi/3$ and tangent respectively to D_1 and D_3 , is minimal when the three radii are equal to r_* , and V is the (disjoint) union $\Delta_{r_*} \cup \hat{\Delta}_{r_*} \cup \hat{\hat{\Delta}}_{r_*}$. The fact that the three radii must be equal can be proved in the usual way as Lemmas 19 and 20. Then we observe that, since by assumption the angle $\angle P_1P_2P_3$ is strictly less than π , there exists a straight line γ tangent to both D_1 and D_3 such that D_1, D_2, D_3 are contained into a halfplane delimited by γ . We set V' the region delimited by our three concave arcs and

γ , and $V'' := V \setminus V'$. By Lemma 20, we have $|V'| \geq |\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|$, so that we are reduced to show that $|V''| \geq |\widehat{\widehat{\Delta}}_{r_*}|$. It is not restrictive to prove the latter inequality in the setting when $d(P_1, P_3) = r_1 + r_3$ (because in such setting $|V'| = |\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|$, and $|V''|$ becomes strictly smaller than in the case when $d(P_1, P_3) > r_1 + r_3$). It is readily seen that minimizing $|V''|$ is equivalent to minimizing the area of the pentagon $P_0P_1P_3P_4O$, being O the origin of the two half-lines which delimit S , see Figure 8, left. Setting $t := \angle P_4P_3P_1 - \frac{2\pi}{3}$, by elementary computations the area of such pentagon is given by

$$\varphi(t) := \frac{r_*^2}{3} [6 \sin t + 3 \sin(2t) + 6\sqrt{3} \cos t + \sqrt{3} \cos(2t) + 4\sqrt{3}], \quad t \in [0, \frac{\pi}{2}],$$

and the minimum of the map φ on the interval $[0, \frac{\pi}{2}]$ is attained at $t = \frac{\pi}{2}$. This yields $\varphi(t) \geq (2 + \sqrt{3})r_*^2$, which corresponds to the case $V'' = \widehat{\widehat{\Delta}}_{r_*}$.

Induction step. Assume the statement holds true for up to $m - 1$ disks, and let us show it holds true also for m disks. Two cases may occur for an optimal configuration of m disks.

Case 1: V is disconnected. Two subcases may occur:

Case 1a: The family $\{D_1, \dots, D_k\}$ can be decomposed as the union of two disjoint sub-families \mathcal{F}' and \mathcal{F}'' , of cardinalities j and $m + 1 - j$ (both strictly smaller than m), such that one of them, say \mathcal{F}' , satisfies the assumptions of Lemma 20, and the other one, say \mathcal{F}'' satisfies the assumptions of Lemma 21. Letting V' and V'' be respectively the complements in S of the unbounded connected components of $S \setminus \cup_{D_i \in \mathcal{F}'} D_i$ and $S \setminus \cup_{D_i \in \mathcal{F}''} D_i$, by Lemma 20 and induction, we have

$$|V'| \geq (j - 2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| \quad \text{and} \quad |V''| \geq (m - j - 1)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| + |\widehat{\widehat{\Delta}}_{r_*}|.$$

Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain

$$|V| \geq (m - 3)|\Delta_{r_*}| + 2|\widehat{\Delta}_{r_*}| + |\widehat{\widehat{\Delta}}_{r_*}| \geq (m - 2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| + |\widehat{\widehat{\Delta}}_{r_*}|.$$

Case 1b: The family $\{D_1, \dots, D_k\}$ can be decomposed as the union of two disjoint sub-families \mathcal{F}' and \mathcal{F}'' , of cardinalities j and $m + 2 - j$ (both strictly smaller than m), such that one of them, say \mathcal{F}' , satisfies the assumptions of Lemma 19, and the other one, say \mathcal{F}'' satisfies the assumptions of Lemma 21. Letting V' and V'' be respectively the complements in S of the unbounded connected components of $S \setminus \cup_{D_i \in \mathcal{F}'} D_i$ and $S \setminus \cup_{D_i \in \mathcal{F}''} D_i$, by Lemma 19 and induction, we have

$$|V'| \geq (j - 2)|\Delta_{r_*}| \quad \text{and} \quad |V''| \geq (m - j)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| + |\widehat{\widehat{\Delta}}_{r_*}|.$$

Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain

$$|V| \geq (m - 2)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| + |\widehat{\widehat{\Delta}}_{r_*}|.$$

The proof of the induction step in Case 1 is concluded.

Case 2: no disk D_j , with $j \neq 1, m$, is tangent to a half-line which delimits S (equivalently, V is connected). In this case, we observe that the equality (44) must be satisfied, with the usual proof. Then, we consider a sector \widetilde{S} of opening angle $\pi/3$ delimited by two half-lines: one of them is the same tangent to D_1 which delimits the original sector S , and the other one is tangent to D_{m-1} . We set $V' := V \cap \widetilde{S}$ and $V'' := V \cap (\mathbb{R}^2 \setminus \widetilde{S})$, see Figure 8, right. By induction, we have $|V'| \geq (m - 3)|\Delta_{r_*}| + |\widehat{\Delta}_{r_*}| + |\widehat{\widehat{\Delta}}_{r_*}|$. Moreover, V'' contains a copy of Δ_{r_*} (with strict inclusion, since we are dealing with case 2); hence we have $|V'| \geq |\Delta_{r_*}|$. Since by construction $V' \cap V'' = \emptyset$, and $V = V' \cup V''$, we obtain that

$|V| \geq (m-2)|\Delta_{r_*}| + |\Delta_{r_*}| + |\widehat{\Delta}_{r_*}|$, concluding the proof of the induction step also in Case 2.

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