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RECENT DEVELOPMENTS OF NEOCASS THE OPEN SOURCE SUITE FOR STRUCTURAL SIZING AND AEROELASTIC ANALYSIS

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Abstract: This paper presents the Matlab-based suite called NeoCASS (Next generation Conceptual Aero-Structural Sizing Suite), developed at the Department of Aerospace Engineering of Politecnico di Milano to be used especially in conceptual design phase. It enables the creation of efficient low-order, medium fidelity models particularly suitable for structural sizing, aeroelastic analysis and optimization at the conceptual design level. The whole methodology is based upon the integration of geometry construction, aerodynamic and structural analysis codes that combine depictive, computational, analytical, and semi-empirical methods, validated in an aircraft design environment. Originally developed in 2010 inside the FP7 EU funded project SimSAC as a module of the CEASIOM environment, since 2012 NeoCASS is a standalone code distributed under the open source scheme. After more than six years of development, and more than 800 downloads, this paper presents the most recent advances in the implementation of a unified aeroelastic and flight mechanics formulation.

1 INTRODUCTION

NeoCASS is a Matlab suite aimed at providing a complete set of tools for conceptual design of aircraft [1] [2]. It has the capability of generating a preliminary sizing of the aircraft starting from geometrical data and also allows aeroelastic analyses on the structural model which is generated after the preliminary sizing phase. Having the possibility to introduce aeroelastic analyses early in the design phase can be very useful since it allows to include aeroelastic stability and structural load analysis in the definition of the aircraft configuration. One additional component that can be considered in the evaluation of the aircraft configuration is the automated flight control system that can either be limited at the control of the motion of the aircraft or can assume additional functions such as load alleviation and flutter suppression.

It is then very important to allow the interface between flight control system and the preliminary aeroelastic models such as the ones generated and used by NeoCASS, and it is also desired that such models can reproduce both the structural dynamic response of the aircraft and its rigid motion, allowing a complete evaluation of the flight control system. This paper describes some additions to the aeroelastic analysis capabilities of NeoCASS that have been introduced in order to allow a complete interface between the aeroelastic model and the flight control system, while still keeping an efficient, low-fidelity formulation suitable for a preliminary evaluation of the aircraft properties.

In order to provide a bridge between the traditional aeroelastic formulation and the linear flight mechanics models used for the evaluation of the handling qualities of an aircraft, a body frame formulation of the equation of motion has been introduced [3]. The body frame formulation has also the advantage of allowing an easy correction of the aerodynamic forces using tabulated coefficients [4], thus providing the possibility to increase the accuracy of the representation of the system dynamics were additional data is available. The linear model thus obtained can be used for standard aeroelastic dynamic analyses, the evaluation of the poles associated with the aircraft flight mechanics and the solution of steady load conditions.

The possibility to couple the linear structural deformation with the full nonlinear rigid motion of the aircraft can be useful to provide the possibility to further increase the accuracy of the simulation, while still keeping a low-fidelity structural and aerodynamic representation. Many formulations have been used for the analysis of nonlinear aeroelastic systems, for example in [5], [6], [7] and [8] the aeroelastic and rigid nonlinear aircraft models are merged in one single model, with the only coupling coming from the aerodynamic forces, having considered a mean axes formulation of the modal basis. It is also possible to couple a nonlinear structural model with the standard linearized aerodynamic modeling [9], allowing the inclusion of geometrical stiffness in the model. Other formulations were proposed that are not based on the nonlinear extension of the finite element method, but are based on a multibody formulation or a nonlinear beam model. A general multibody formulation was for example used in [10] and [11], this formulation allows for an exact representation of the rigid body dynamics and can include complete nonlinear structural models even if in the cited works rigid bodies connected with lumped spring elements were used to represent the flexibility of the structure. A completely different formulation was instead used in [12] and [13], aimed at the aeroelastic simulation of very flexible aircraft such as light High Altitude Long Endurance (HALE) UAVs. The formulation is based on the use of a nonlinear structural model, coupled with an approximated model for nonlinear unsteady subsonic aerodynamic that in [12] is obtained with the use of an unsteady vortex lattice method.

In NeoCASS the inertial forces are formulated starting from a nonlinear kinematic model where the motion of the overall aircraft is fully nonlinear while its deformation is considered as the superposition of linear deformation shapes. The inertial forces are then obtained using the Virtual Work Principle, allowing the generation of all coupling terms between the rigid motion of the aircraft and its elastic deformation.

2 NEOCASS SUITE

The block diagram of NeoCASS suite is presented in Fig. 1. The starting point for the generation of a new model is the definition of its geometrical properties by the use of the Acbuilder module, that allows the generation of a database used for the sizing of the structure. The structural sizing is performed by the GUESS module, and it is based on the database of geometrical properties and on a set of manoeuvre conditions used to compute the ultimate loads the structure must be able to sustain.

Once the structural sizing is completed a simplified structural and aerodynamic model of the aircraft is generated, that can be used by the aeroelastic analysis module of NeoCASS, named SMARTCAD, or can be exported for the use in external simulation software, such as Nastran.

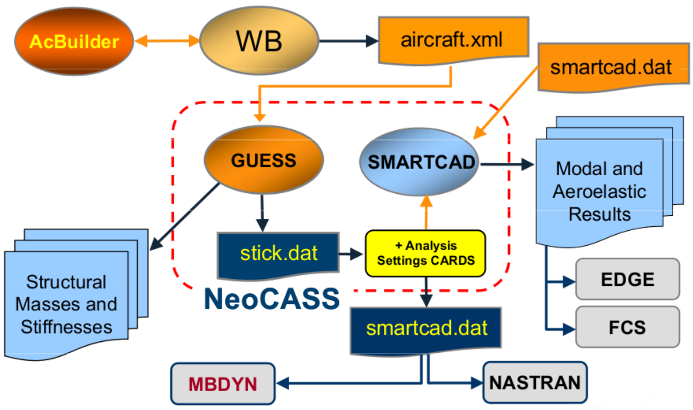


Figure 1: Organization of NeoCASS suite.

3 AEROELASTIC SOLVER

The aeroelastic solver in NeoCASS assumes that the structure is represented by a set of beam elements, with the finite volume formulation presented in [14]. The structural model is coupled to steady and unsteady aerodynamic forces computed using the Vortex Lattice Method (VLM) and the Doublet Lattice Method (DLM) respectively. The DLM is implemented according to the formulation presented in [15]. Three different types of analyses can then be performed: static aeroelastic analysis, aeroelastic stability (flutter) and dynamic response analysis. For the aeroelastic stability and the dynamic response analysis a formulation similar to the one employed in Nastran [16] is used, while a continuation method is used for the evaluation of the frequency and damping of aeroelastic modes in flutter analyses [17].

In addition to the standard aeroelastic analyses in the frequency domain, it is also possible to generate a state-space model of the aeroelastic system in time domain, by using the Matrix Fraction Description method [18] for the definition of unsteady aerodynamic forces in time domain. The state-space formulation is the most convenient for the coupling of the aeroelastic model to the flight control system and then it is the starting point for the additional modifications presented in the following sections.

4 UNIFIED AEROELASTIC AND FLIGHT MECHANICS TREATMENT

The aeroelastic solver in NeoCASS is built in order to have an unified treatment of flight mechanics, static aeroelasticity and dynamic aeroelasticity. This allows to use the same model for all kind of analyses and allows a direct comparison of the results that are obtained in the different analyses. The formulation of the dynamic equations of the system that allows the unified formulation requires two main operations: the first one is the formulation of the equation of motion of the elastic aircraft in body axes; while the second is the generation of a unified definition of steady and unsteady aerodynamic forces. The body axes represents the most convenient frame to formulate the equations of motion of the flexible aircraft since they are traditionally used for the analysis of the flight dynamics of the rigid aircraft and are a convenient reference frame for the definition of aerodynamic forces.

The unified treatment requires also the definition of a common aerodynamic model that is able to define both steady and unsteady aerodynamic forces, with the possibility to increase the fidelity of the modeling with the inclusion of CFD forces or experimental correction factors. This operation is performed in two steps: at first, the steady aerodynamic forces are corrected to allow the inclusion of steady aerodynamic forces from higher-fidelity models namely CFD analyses and databases of aerodynamic coefficients. The corrected steady aerodynamic forces are then used to correct the zero-frequency component of the unsteady aerodynamic forces, thus enforcing a consistency between the results of unsteady and steady analyses.

4.1 Equations of motion in body axes

The equation of motion for the elastic aircraft are traditionally expressed using displacements and rotations defined in a fixed frame that is the rigid motion of an aircraft can be expressed using the linearized displacement $\Delta\boldsymbol{x}_b$ and rotation $\boldsymbol{\varphi}_{b\Delta}$ of a body reference frame, along with the amplitude of the elements of the reduced basis defining the structural deformation $\boldsymbol{q}$ and all their time derivatives. The switch to a body axes formulation is done by using the components of the body velocity in the body reference frame $\Delta\boldsymbol{v}_b$ instead of the time derivative of the body position, and the rotational velocity in body axes $\Delta\boldsymbol{\omega}_b$ instead of the time derivative of the rotation angles. The state vector in inertial axes $\mathbf{x}_i$ and in body axes $\mathbf{x}_b$ can then be defined as:

$$\mathbf{x}_i = \begin{bmatrix} \Delta\boldsymbol{x}_b \\ \boldsymbol{\varphi}_{b\Delta} \\ \boldsymbol{q} \\ \Delta\dot{\boldsymbol{x}}_b \\ \dot{\boldsymbol{\varphi}}_{b\Delta} \\ \dot{\boldsymbol{q}} \end{bmatrix}; \qquad \mathbf{x}_b = \begin{bmatrix} \Delta\boldsymbol{x}_b \\ \boldsymbol{\varphi}_{b\Delta} \\ \boldsymbol{q} \\ \Delta\boldsymbol{v}_b \\ \Delta\boldsymbol{\omega}_b \\ \dot{\boldsymbol{q}} \end{bmatrix}; \tag{1}$$

In the particular case of fully linearised system with no initial rotational velocity, acceleration and rotational acceleration it is possible to switch from the inertial to the body axes formulation using simply a kinematic transformation for the variables and applying the linearised rotation to the equation of motion. This particular linearization configuration allows indeed for the elimination of all inertial force components that would result from the use of a relative frame for the definition of the motion equations. The transformation between $\mathbf{x}_i$ and $\mathbf{x}_b$ can be expressed as [3]:

$$\begin{cases} \mathbf{x}_i = \mathbf{T}_1\mathbf{x}_b \\ \dot{\mathbf{x}}_i = \mathbf{T}_2\dot{\mathbf{x}}_b + \mathbf{T}_3\mathbf{x}_b \end{cases} \tag{2}$$

where

$$\mathbf{T}_1 = \begin{bmatrix} \boldsymbol{R}_{b0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{R}_{b0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 & 0 & 0 \\ 0 & -\bar{\boldsymbol{v}}_{b0}\times & 0 & \boldsymbol{R}_{b0} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{R}_{b0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}; \quad \mathbf{T}_2 = \begin{bmatrix} \boldsymbol{R}_{b0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{R}_{b0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{R}_{b0} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{R}_{b0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}$$
$$\mathbf{T}_3 = \begin{bmatrix} 0 & -\boldsymbol{R}_{b0}\bar{\boldsymbol{v}}_{b0}\times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\boldsymbol{R}_{b0}\bar{\boldsymbol{v}}_{b0}\times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

where $\boldsymbol{R}_{b0}$ represents the rotation matrix that defines the body frame orientation at the reference configuration. The deformation amplitudes $\boldsymbol{q}$ are unaffected by the transformation equations in

Eq. (3), this is due to the fact that the combination of the reference frame rotation and the structural deformation would results in a second order component, which is dropped in the linearization of the transformation [19].

The transformation can then be applied directly to the model in state space form, that relates the dynamic of the aeroelastic state to the system inputs ⊓ and defines the system output vector †:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_b + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{F}\dot{\mathbf{x}}_i + \mathbf{C}\mathbf{x}_i + \mathbf{D}\mathbf{u} \end{cases} \tag{4}$$

leading to

$$\begin{cases} \mathbf{E}\mathbf{T}_2\dot{\mathbf{x}}_b = (\mathbf{A}\mathbf{T}_1 - \mathbf{E}\mathbf{T}_3)\mathbf{x}_b + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{F}\mathbf{T}_2\dot{\mathbf{x}}_b + (\mathbf{A}\mathbf{T}_1 + \mathbf{E}\mathbf{T}_3)\mathbf{x}_b + \mathbf{D}\mathbf{u} \end{cases} \tag{5}$$

A further processing of the state variables can be introduced to switch from the use of the velocity components in body angles to the use of the sideslip angle β and the angle of attack α that define the orientation of the body velocity with respect to the body frame and are defined as

$$\beta = sin^{-1}\left(\frac{v}{\|\boldsymbol{v}_b\|}\right) \qquad \alpha = tan^{-1}\left(\frac{w}{u}\right) \tag{6}$$

equation (6) can be linearized around the reference value for the body velocity $\boldsymbol{v}_{b0}$ in order to obtain a linear transformation for the system state:

$$\begin{bmatrix} \Delta u \\ \Delta\beta \\ \Delta\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{v_0 u_0}{\sqrt{u_0^2+w_0^2}\|\boldsymbol{v}_0\|^2} & \left(1 - \frac{v_0^2}{\|\boldsymbol{v}_0\|^2}\right)\frac{1}{\sqrt{u_0^2+w_0^2}} & -\frac{v_0 w_0}{\sqrt{u_0^2+w_0^2}\|\boldsymbol{v}_0\|^2} \\ -\frac{w_0}{u_0^2+w_0^2} & 0 & \frac{u_0}{u_0^2+w_0^2} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix} \tag{7}$$

4.2 Gravitational forces

When dealing with motion equations for flight mechanics analysis it is convenient to include also the effect of the weight force on the model dynamics, the weight forces acting on the reduced dynamical model can be expressed as

$$\boldsymbol{f}_g = \boldsymbol{U}^T\boldsymbol{M}\boldsymbol{T}\boldsymbol{R}_b^T\boldsymbol{g}^I \tag{8}$$

where $\boldsymbol{M}$ is the mass matrix of the full structural model, with size $[n_{DOF} \times n_{DOF}]$, $\boldsymbol{U}$ is the matrix defining the reduced basis for expressing the motion of the structure and include both the displacements associated with the linearized motion of the body reference frame and the deformation of the structure. The matrix $\boldsymbol{T}$ is a $[n_{DOF} \times 3]$ summation matrix that select from the mass matrix only the columns associated with the three components of displacement of the nodes of the structural model. The intensity and direction of the gravitational field is expressed using components in the inertial frame $\boldsymbol{g}^I$, where they are constant. The linearization of this equation then gives

$$\begin{aligned} \boldsymbol{f}_g &= \boldsymbol{U}^T\boldsymbol{M}\boldsymbol{T}\left[\boldsymbol{R}_{b0}(\boldsymbol{I} + \boldsymbol{\varphi}_{b\Delta}\times)\right]^T\boldsymbol{g}^I \\ &= \boldsymbol{f}_{g0} - \boldsymbol{U}^T\boldsymbol{M}\boldsymbol{T}\boldsymbol{\varphi}_{b\Delta} \times \boldsymbol{R}_{b0}^T\boldsymbol{g}^I \end{aligned} \tag{9}$$

In addition to the weight force in the reference configuration $\boldsymbol{f}_{g0}$ there is also a stiffness contribution related to the rotation of the body reference frame that need to be included in the state matrix $\mathbf{A}$.

4.3 Correction of aerodynamic forces

Two aerodynamic solvers are currently available in NeoCASS, one implementing the Vortex Lattice Method (VLM) for the computation of steady aerodynamic forces and the Doublet Lattice Method (DLM) for the computation of unsteady harmonic aerodynamic forces [15]. Those low-fidelity aerodynamic methods are well suited for preliminary aeroelastic analyses but higher-fidelity methods are usually needed to improve the accuracy of the results. In the contest of the aeroelastic analyses performed in NeoCASS it is required to keep the numerical efficiency of the low-fidelity methods, which have also the additional advantage of not requiring a detailed geometry description of the model, but at the same time it is useful to have the possibility to increase the accuracy of the aerodynamic forces. A series of methods for correcting aerodynamic forces is then used in order to introduce a bridge between the low-fidelity aerodynamic methods and high fidelity Computational Fluid Dynamic (CFD) simulations, and can be used for both steady and unsteady aerodynamic forces. A scheme summarizing the possible corrections is presented in Fig. 2, the correction of the low-fidelity aerodynamic forces is performed with the use of data obtained from CFD simulations or with the use of tabulated aerodynamic coefficients. The CFD data when available represent the most accurate correction available and can define the distribution of aerodynamic forces on the structure; the tabulated coefficients are able to modify only the global forces acting on the rigid motion of the structure, but can be useful in static and dynamic analysis because they allow the proper definition of the trim of the aircraft or the frequency and damping of the flight mechanics modes of the aircraft. The steady aerodynamic forces obtained using the VLM method are linear in the body motion and the structural deformation, and can be expressed as

$$\frac{\boldsymbol{F}_z^a}{q_\infty} = \boldsymbol{F}_0^a + \boldsymbol{K}_u^a \Delta \boldsymbol{v}_b + \boldsymbol{K}_c^a \boldsymbol{\delta}_c + \boldsymbol{K}_z^a \boldsymbol{u}_z \tag{10}$$

where $\boldsymbol{\delta}_c$ contains the deflection of the control surfaces, and $\boldsymbol{u}_z$ is the vector of structural displacements. Aerodynamic forces from CFD analyses must be provided at a reference configuration, then providing a correction for the reference term $\boldsymbol{F}_0^a$, by providing also the same data obtained in perturbed conditions also the correction for the matrices $\boldsymbol{K}_c^a$ and $\boldsymbol{K}_u^a$ can be obtained by finite differences. As shown in Fig. 3 there is no need to provide a full aerodynamic configuration, but the use of CFD computed forces can be limited to some portions of the geometry. The use of aerodynamic coefficients, instead, allows the correction of only the rows of matrices $\boldsymbol{F}_0^a$, $\boldsymbol{K}_u^a$ and $\boldsymbol{K}_c^a$ that are associated with forces on rigid motion of the aircraft, while leaving the distribution of forces on the structure unchanged.

The corrections mentioned above can be applied to the steady aerodynamic forces, and can be used to correct the zero-frequency part of the unsteady response. Unsteady aerodynamic forces are expressed in frequency domain as

$$\Delta \boldsymbol{f}^a(jk) = \boldsymbol{H}^I(k, M)\boldsymbol{u}_a(jk) \tag{11}$$

where k is the reduced frequency and $\boldsymbol{u}^a$ is the vector of inputs for the aerodynamic system, which includes the state $\Delta \mathrm{x}_i$, the control surfaces deflections and the gust input. The use of a fitting procedure allows the definition of the same forces using a dynamical system in time domain

$$\begin{cases} \boldsymbol{x}_a' = \boldsymbol{A}^I \boldsymbol{x}_a + \boldsymbol{B}_{\boldsymbol{0}}^I \boldsymbol{u}_a + \boldsymbol{B}_{\boldsymbol{1}}^I \boldsymbol{u}_a' + \boldsymbol{B}_{\boldsymbol{2}}^I \boldsymbol{u}_a'' \\ \Delta \boldsymbol{f}_a = \boldsymbol{C}^I \boldsymbol{x}_a + \boldsymbol{D}_{\boldsymbol{0}}^I \boldsymbol{u}_a + \boldsymbol{D}_{\boldsymbol{1}}^I \boldsymbol{u}_a' + \boldsymbol{B}_{\boldsymbol{2}}^I \boldsymbol{u}_a'' \end{cases} \tag{12}$$

where the prime ($'$) denotes the derivative with respect to a non-dimensional time $\tau = t/t_a = \frac{tV_\infty}{l_a}$ and l_a is a reference length.

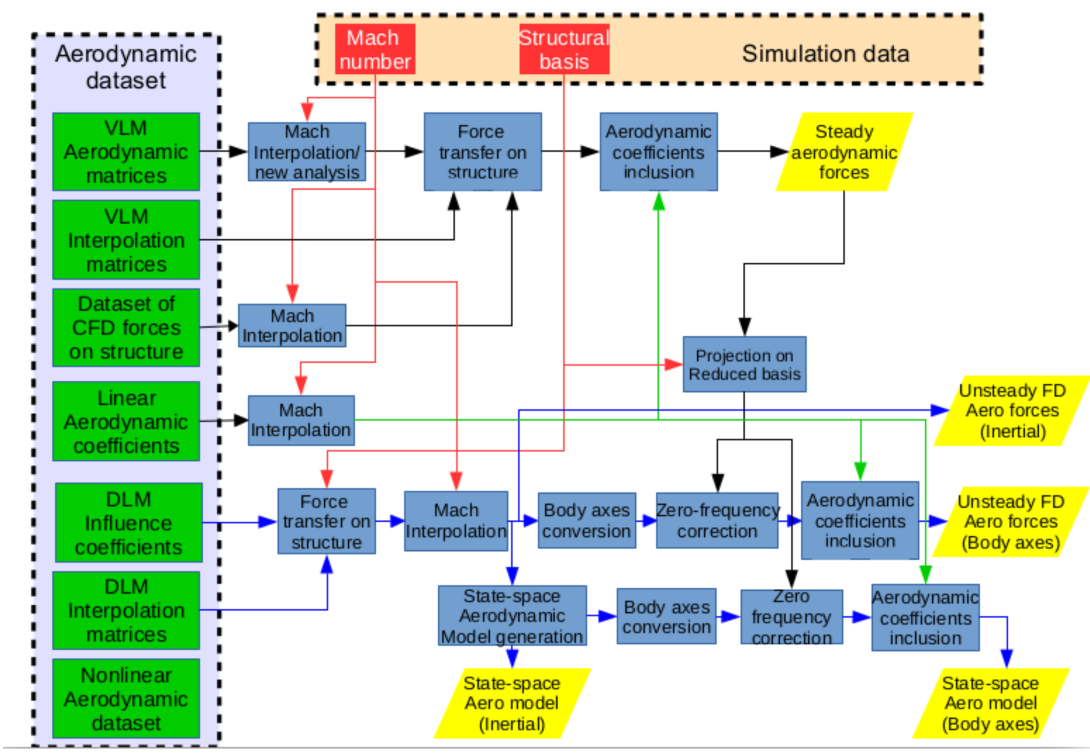


Figure 2: Scheme used for the correction of aerodynamic forces in static and dynamic analyses.

Both the frequency domain and the time domain formulations can be transformed using Eq. (2) in body coordinates, by considering only the structural motion $\Delta \mathrm{x}_b$ as input it results

$$\Delta \boldsymbol{f}^a = \boldsymbol{H}(k, M)\Delta \mathrm{x}_b; \qquad \begin{cases} \boldsymbol{x}_a' = \boldsymbol{A}\boldsymbol{x}_a + \boldsymbol{B_0}\Delta \mathrm{x}_b + \boldsymbol{B_1}\Delta \mathrm{x}_b' + \boldsymbol{B_2}\Delta \mathrm{x}_b'' \\ \Delta \boldsymbol{f}_a = \boldsymbol{C}\boldsymbol{x}_a + \boldsymbol{D_0}\Delta \mathrm{x}_b + \boldsymbol{D_1}\Delta \mathrm{x}_b' + \boldsymbol{B_2}\Delta \mathrm{x}_b'' \end{cases} \tag{13}$$

In order to allow the correction of the steady portion of the aerodynamic forces it is convenient to reshape the system in Eq. (13) to lump all the steady response in $\boldsymbol{D_0}$ and its first derivative in $\boldsymbol{D_1}$, then formulating the system in the form

$$\begin{cases} \boldsymbol{w}_a' = \boldsymbol{A}\boldsymbol{w}_a + \bar{\boldsymbol{B}}_{\boldsymbol{2}}\Delta \mathrm{x}_b'' \\ \Delta \boldsymbol{f}_a = \boldsymbol{C}\boldsymbol{w}_a + \bar{\boldsymbol{D}}_{\boldsymbol{0}}\Delta \mathrm{x}_b + \bar{\boldsymbol{D}}_{\boldsymbol{1}}\Delta \mathrm{x}_b' + \boldsymbol{B_2}\Delta \mathrm{x}_b'' \end{cases} \tag{14}$$

where

$$\bar{\boldsymbol{B}}_{\boldsymbol{2}} = \boldsymbol{B_2} + \boldsymbol{A}^{-1}\left(\boldsymbol{B_1} + \boldsymbol{A}^{-1}\boldsymbol{B_0}\right) \tag{15}$$

$$\bar{\boldsymbol{D}}_{\boldsymbol{0}} = \boldsymbol{D_0} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B_0} \tag{16}$$

$$\bar{\boldsymbol{D}}_{\boldsymbol{1}} = \boldsymbol{D_1} - \boldsymbol{C}\boldsymbol{A}^{-1}\left(\boldsymbol{B_1} + \boldsymbol{A}^{-1}\boldsymbol{B_0}\right) \tag{17}$$

A similar procedure can be applied also for the frequency formulation, leading to the following expression

$$\begin{aligned} \Delta \boldsymbol{f}_a &= \boldsymbol{H}(0)\Delta \mathrm{x}_b - j\left.\frac{\partial \boldsymbol{H}}{\partial k}\right|_0 \Delta \mathrm{x}_b' + \left[\boldsymbol{H}(jk, M) - \boldsymbol{H}(0) - k\left.\frac{\partial \boldsymbol{H}}{\partial k}\right|_0\right]\Delta \mathrm{x}_b \\ &= \bar{\boldsymbol{D}}_{\boldsymbol{0}}\Delta \mathrm{x}_b + \bar{\boldsymbol{D}}_{\boldsymbol{1}}\Delta \mathrm{x}_b' + \tilde{\boldsymbol{H}}(jk, M)\Delta \mathrm{x}_b \end{aligned} \tag{18}$$

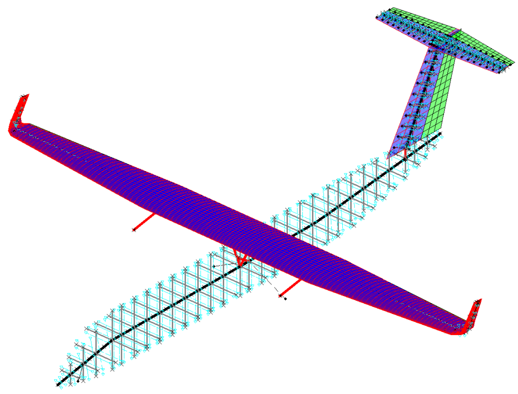


Figure 3: Example of partial definition of CFD forces.

then lumping all the steady response in the $\bar{\boldsymbol{D}}_{\mathbf{0}}$ and $\bar{\boldsymbol{D}}_{\mathbf{1}}$ matrices that can be corrected without modifying the higher frequency contribution in $\tilde{\boldsymbol{H}}$.

As shown in Fig. 2 it is possible to correct the steady forces predicted by the DLM method using the steady aerodynamic forces obtained using the steady VLM, thus recovering the effects of aerodynamic twist and camber. An higher level of fidelity can be obtained if the aerodynamic forces also include CFD force corrections. The aerodynamic matrices $\boldsymbol{F}_0^a$, $\boldsymbol{K}_u^a$, $\boldsymbol{K}_c^a$ and $\boldsymbol{K}_z^a$ can be directly included in the matrices $\bar{\boldsymbol{D}}_{\mathbf{0}}$ and $\bar{\boldsymbol{D}}_{\mathbf{1}}$ leading to the correction of the low-frequency behaviour of the aerodynamic forces. An additional term related to the linearization of the dynamic pressure can be introduced:

$$\Delta \boldsymbol{f}_{aU} = \frac{d}{du}\left(\frac{1}{2}\rho U^2\right)\boldsymbol{f}_{a0} \tag{19}$$

then leading to an additional force coefficient in u

$$\boldsymbol{D}^{\boldsymbol{u}} = -2\boldsymbol{f}_{a0} \tag{20}$$

this coefficient matrix relates the increase in aerodynamic forces due to the variation of flight speed and is required to properly recover the phugoid mode of the aircraft.

It is also possible to introduce aerodynamic coefficients in the aerodynamic force formulation as they are traditionally used in the flight mechanics analysis. The aerodynamic coefficients are used to express the forces and moments in the wind axes (lift L, drag D, side force S, roll moment $\mathcal{L}$, pitch moment $\mathcal{M}$ and yaw moment $\mathcal{N}$) to the aerodynamic parameters (flight speed u, angle of attack α, sideslip angle β, roll velocity p, pitch velocity q, yaw velocity r) and their derivatives.

The symmetry of the aircraft geometry usually lead to a decoupling between the aerodynamic forces associated with the longitudinal and the latero-directional motion, but the more general formulation for the definition of aerodynamic forces using tabulated coefficients is

$$\Delta \boldsymbol{f}_u^{coef} = q_\infty \boldsymbol{S}_1 \begin{bmatrix} C_{X/u} & C_{X/\beta} & C_{X/\alpha} & C_{X/p} & C_{X/q} & C_{X/r} \\ C_{Y/u} & C_{Y/\beta} & C_{Y/\alpha} & C_{Y/p} & C_{Y/q} & C_{Y/r} \\ C_{Z/u} & C_{Z/\beta} & C_{Z/\alpha} & C_{Z/p} & C_{Z/q} & C_{Z/r} \\ C_{l/u} & C_{l/\beta} & C_{l/\alpha} & C_{l/p} & C_{l/q} & C_{l/r} \\ C_{m/u} & C_{m/\beta} & C_{m/\alpha} & C_{m/p} & C_{m/q} & C_{m/r} \\ C_{n/u} & C_{n/\beta} & C_{n/\alpha} & C_{n/p} & C_{n/q} & C_{n/r} \end{bmatrix} \boldsymbol{S}_2 \begin{bmatrix} u \\ \beta \\ \alpha \\ p \\ q \\ r \end{bmatrix} + q_\infty \boldsymbol{S}_1 \begin{bmatrix} C_{X/\dot\beta} & C_{X/\dot\alpha} \\ C_{Y/\dot\beta} & C_{Y/\dot\alpha} \\ C_{Z/\dot\beta} & C_{Z/\dot\alpha} \\ C_{l/\dot\beta} & C_{l/\dot\alpha} \\ C_{m/\dot\beta} & C_{m/\dot\alpha} \\ C_{n/\dot\beta} & C_{n/\dot\alpha} \end{bmatrix} \boldsymbol{S}_3 \begin{bmatrix} \dot\beta \\ \dot\alpha \end{bmatrix} \tag{21}$$

the second matrix has been limited only to the coefficient related to $\dot\alpha$ and $\dot\beta$ since these are the most commonly used. In addition to the linearised forces $\Delta \boldsymbol{f}_a$ the aerodynamic coefficients are also able to define some portions of the steady aerodynamic forces $\boldsymbol{f}_{a0}$, in the same way as they are used to correct the steady aerodynamic force component $\boldsymbol{F}_0^a$ in Eq. (10), using the definition of aerodynamic forces on the reference state:

$$\boldsymbol{f}_0^{coef} = q_\infty \boldsymbol{S}_1 \begin{bmatrix} C_{X0} \\ C_{Y0} \\ C_{Z0} \\ C_{l0} \\ C_{m0} \\ C_{n0} \end{bmatrix} \tag{22}$$

In the equations above $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$ are scaling matrices

$$\boldsymbol{S}_1 = S_{ref} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & b_{ref} & & \\ & & & & c_{ref} & \\ & & & & & b_{ref} \end{bmatrix} \tag{23}$$

$$\boldsymbol{S}_2 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \frac{b_{ref}}{V_\infty} & & \\ & & & & \frac{c_{ref}}{V_\infty} & \\ & & & & & \frac{b_{ref}}{V_\infty} \end{bmatrix} \qquad \boldsymbol{S}_3 = \begin{bmatrix} \frac{b_{ref}}{V_\infty} & 0 \\ 0 & \frac{c_{ref}}{V_\infty} \end{bmatrix} \tag{24}$$

Additional matrices can be defined in order to provide the effect of the control surfaces, for example for the most common case of aileron, elevator and rudder:

$$\Delta \boldsymbol{f}_\delta^{coef} = q_\infty \boldsymbol{S}_1 \begin{bmatrix} C_{X/\delta_a} & C_{X/\delta_e} & C_{X/\delta_r} \\ C_{Y/\delta_a} & C_{Y/\delta_e} & C_{Y/\delta_r} \\ C_{Z/\delta_a} & C_{Z/\delta_e} & C_{Z/\delta_r} \\ C_{l/\delta_a} & C_{l/\delta_e} & C_{l/\delta_r} \\ C_{m/\delta_a} & C_{m/\delta_e} & C_{m/\delta_r} \\ C_{n/\delta_a} & C_{n/\delta_e} & C_{n/\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix} \tag{25}$$

The reference point used for the definition of the aerodynamic moments must be at the origin of the coordinate system, if the aerodynamic coefficients are defined with respect to a different point a transformation is required

$$\left.\boldsymbol{f}_{a1}^{J}\right|_{O}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{r}_{OQ}\times & \boldsymbol{I}\end{bmatrix}\left.\boldsymbol{f}_{a1}^{J}\right|_{Q} \tag{26}$$

where $\boldsymbol{r}_{OQ}$ is the position of the reference point for the aerodynamic coefficients with respect to the body reference frame.

In order to insert these coefficients in the steady aerodynamic force matrix it is necessary to transform them from the wind axes reference frame to body axes, using the transformation matrix in the reference configuration:

$$\boldsymbol{R}_{BA}=\begin{bmatrix}cos(\alpha_0)cos(\beta_0) & -cos(\alpha_0)sin(\beta_0) & -sin(\alpha_0)\\ sin(\beta_0) & cos(\beta_0) & 0\\ sin(\alpha_0)cos(\beta_0) & -sin(\alpha_0)sin(\beta_0) & cos(\alpha_0)\end{bmatrix} \tag{27}$$

it is also necessary to express the body velocity $\Delta\boldsymbol{v}_b$ as function of the aerodynamic angles $\Delta\alpha$, $\Delta\beta$, using the transformation in Eq. (7).

It is possible to have aerodynamic coefficients expressed in the stability reference system, which originates from a rotation of angle α of the body reference system, so that the $\hat{\boldsymbol{s}}_1$ versor of the system is directed as the asymptotic velocity, in this case the rotation matrix includes only the rotation according to the angle of attack and is expressed as

$$\boldsymbol{R}_{BS}=\begin{bmatrix}cos(\alpha_0) & 0 & -sin(\alpha_0)\\ 0 & 1 & 0\\ sin(\alpha_0) & 0 & cos(\alpha_0)\end{bmatrix} \tag{28}$$

The rotation from wind (or stability) reference system to body reference system must be taken in account in the linearization process, then the forces in body axes $\boldsymbol{f}_a^B$ can be defined starting from the forces in wind axes $\boldsymbol{f}_a^A$ as

$$\begin{aligned}\boldsymbol{f}_a^B &= \boldsymbol{R}_{BA}\boldsymbol{f}_{a0}^A+\boldsymbol{R}_{BA}\Delta\boldsymbol{f}_a^A+\Delta\boldsymbol{R}_{BA}\boldsymbol{f}_{a0}^A\\ &= \boldsymbol{f}_{a0}^B+\boldsymbol{R}_{BA}\Delta\boldsymbol{f}_a^A+\boldsymbol{R}_{BA}\boldsymbol{\varphi}_{a\Delta}\times\boldsymbol{f}_{a0}^A\end{aligned} \tag{29}$$

then the linearization introduces and additional contribution to the force coefficients in body axes that depends on the aerodynamic forces in the reference condition in the aerodynamic frame and on the perturbation in the aerodynamic angles, that are contained in $\boldsymbol{\varphi}_{a\Delta}$.

4.4 Mode acceleration method

Dynamic analyses are computationally efficient if a reduced basis is used to represent the deformation of the structure. While the use of a reduced basis can lead to good results in the recovery of internal forces in the structure due to dynamic load conditions, its use for static simulations can lead to poor results due to the fact that the reduced basis may not be able to fully represent the static deformation of the structure. For this reason the mode acceleration method is used to allow a direct comparison of the results obtained with the dynamic model and the analogous results that can be obtained from a static aeroelastic simulation. The forces can be recovered in NeoCASS using the mode acceleration method according to the equation:

$$\boldsymbol{s}=\boldsymbol{S}\boldsymbol{K}^{-1}\left[\boldsymbol{F}_{ext}-\boldsymbol{M}\boldsymbol{U}\ddot{\boldsymbol{q}}+q_\infty\left(\boldsymbol{H}_{am}^*(k,M_\infty)\boldsymbol{q}+\boldsymbol{H}_{ac}^*(k,M_\infty)\boldsymbol{\delta}_c+\boldsymbol{H}_{ag}^*(k,M_\infty)\frac{v_g}{V_\infty}\right)\right] \tag{30}$$

where $\boldsymbol{S}$ is a matrix defining the relationship between the nodal displacement and the selected internal load component, $\boldsymbol{K}$ is the full model stiffness matrix, $\boldsymbol{F}_{ext}$ contains all external forces applied to the structure, $\boldsymbol{U}$ is the array defining the reduced basis used in dynamic simulations and relates the nodal displacements $\boldsymbol{u}$ to the amplitude of the reduced basis elements $\boldsymbol{q}$ as $\boldsymbol{u} = \boldsymbol{U}\boldsymbol{q}$. $\boldsymbol{M}$ is the mass matrix of the system and the unsteady aerodynamic forces are expressed by the matrices $\boldsymbol{H}^*_{am}$, $\boldsymbol{H}^*_{ac}$ and $\boldsymbol{H}^*_{ag}$ that are defined as a function of the frequency and express the relationship between the inputs to the aerodynamic system and the nodal forces on the structure. The allowed aerodynamic inputs are the structural deformations related to the reduced basis $\boldsymbol{q}$, the deflection of the aircraft control surfaces $\boldsymbol{\delta}_c$ and the gust velocity $\boldsymbol{v_g}$. The equation for the recovery of internal loads is defined in frequency domain, thanks to presence of the unsteady aerodynamic forces. An equivalent formulation in time domain can be obtained if a linear state space model is used to fit the nodal aerodynamic forces, using the same methodology used for the fitting of the aerodynamic forces in the motion equations. In order to reduce the dimension of the aerodynamic system that need to be fitted it is possible to condensate the forces as

$$\boldsymbol{H}_\Sigma = \boldsymbol{S}\boldsymbol{K}^{-1}\left[\boldsymbol{H}^*_{am}(k, M_\infty) \quad \boldsymbol{H}^*_{ac}(k, M_\infty) \quad \boldsymbol{H}^*_{ag}(k, M_\infty)\right] \tag{31}$$

the matrix $\boldsymbol{H}_\Sigma$ can then be fitted according to the algorithm described in [18].

5 NONLINEAR RIGID BODY MOTION

In this chapter a formulation of the dynamic equation for a flying aeroelastic vehicle is introduced to increase the accuracy of flight mechanics simulations that include flight control systems and the structural deformation. The formulation assumes an arbitrary, non linearized motion of a reference frame associated with the rigid motion of the aircraft. The linearized structural deformation is then included considering a superposition of modal shapes defining the displacement and rotation of the structural points with respect to the moving reference frame. The formulation is derived from that of the modal element implemented in the free multi-body dynamics software MBDyn [20], specializing it for the analysis of a flying aircraft.

5.1 Kinematics

The position of a point on the structure is given by the composition three different components:

- the position $\boldsymbol{x}_0$ and orientation $\boldsymbol{R}_0 = \boldsymbol{R}(\boldsymbol{\varphi}_0)$ of a reference frame J;
- the position $\tilde{\boldsymbol{r}}$ of the point with respect to the reference frame (the initial orientation is assumed to be zero).
- the displacement $\tilde{\boldsymbol{u}}$ and rotation $\tilde{\boldsymbol{\varphi}}$ relative to the reference frame.

the meaning of the various component is displayed in Fig. 4.

Instead of considering the structure as a continuous here a discretized approach will be followed, where the structure is considered as given by a set of N nodes, each with associated mass, inertia and mass unbalance. While not physical, this approach is directly related to the finite element model formulations that will be used for the generation of the matrices required by this formulation.

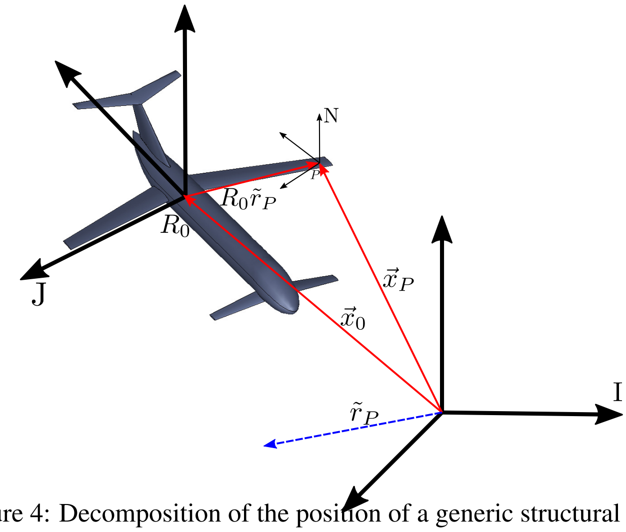


Figure 4: Decomposition of the position of a generic structural point.

The position and rotational velocity of a generic node i can then be expressed as

$$\boldsymbol{x}_i = \boldsymbol{x}_0 + \boldsymbol{R}_0\left(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i\right) \tag{32}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_0 + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{\varphi}}}_i \tag{33}$$

$$\boldsymbol{v}_i = \dot{\boldsymbol{x}} = \boldsymbol{v}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\left(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i\right) + \boldsymbol{R}_0\dot{\tilde{\boldsymbol{u}}}_i \tag{34}$$

$$\boldsymbol{\alpha}_i = \dot{\boldsymbol{\omega}}_i = \boldsymbol{\alpha}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\dot{\tilde{\boldsymbol{\varphi}}}_i + \boldsymbol{R}_0\ddot{\tilde{\boldsymbol{\varphi}}}_i \tag{35}$$

$$\boldsymbol{a}_i = \ddot{\boldsymbol{x}} = \boldsymbol{a}_0 + \boldsymbol{\alpha}_0 \times \boldsymbol{R}_0\left(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i\right) + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \boldsymbol{R}_0\left(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i\right) \tag{36}$$

$$+ 2\boldsymbol{\omega}_0 \times \boldsymbol{R}_0\dot{\tilde{\boldsymbol{u}}}_i + \boldsymbol{R}_0\ddot{\tilde{\boldsymbol{u}}}_i \tag{37}$$

in Eq. (33) a small value of $\tilde{\varphi}_i$ was assumed, allowing the identification of the rotational velocity with the time derivative of the rotation vector associated with the relative rotation of the node with respect to the body reference frame.

The orientation can be expressed combining the orientation of the body frame with the relative orientation $\boldsymbol{R}(\tilde{\varphi}_i)$.

$$\boldsymbol{R}_i = \boldsymbol{R}_0\tilde{\boldsymbol{R}}_i = \boldsymbol{R}(\varphi_0)\boldsymbol{R}(\tilde{\varphi}_i) \tag{38}$$

The change in orientation of reference J is not linearized, then the relation between the time derivative of the rotation vector and the rotational velocity is given by.

$$\boldsymbol{\omega}_0 = \boldsymbol{S}(\varphi_0)\dot{\varphi}_0 \tag{39}$$

Since the principle of virtual work will be used for the formulation of the equation of motion, it is necessary to define the virtual variations of the orientation and position of the point i, the virtual variation of the position is given by

$$\delta\boldsymbol{x}_i = \delta\boldsymbol{x}_0 + \boldsymbol{R}_0\delta\tilde{\boldsymbol{u}}_i + \boldsymbol{\varphi}_{0\delta} \times \boldsymbol{R}_0\left(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i\right) \tag{40}$$

The computation of the virtual variation of orientation is more complex and requires to resort to the definition of a perturbation of the orientation

$$\begin{aligned}
\boldsymbol{\theta}_{\delta i}\times &= \delta \boldsymbol{R}_i \boldsymbol{R}_i^T && (41)\\
&= \delta(\boldsymbol{R}_0\tilde{\boldsymbol{R}}_i)\tilde{\boldsymbol{R}}_i^T\boldsymbol{R}_0^T && (42)\\
&= \delta\boldsymbol{R}_0\tilde{\boldsymbol{R}}_i\tilde{\boldsymbol{R}}_i^T\boldsymbol{R}_0^T + \boldsymbol{R}_0\delta\tilde{\boldsymbol{R}}_i\tilde{\boldsymbol{R}}_i^T\boldsymbol{R}_0^T && (43)\\
&= \boldsymbol{\varphi}_{0\delta}\times + \boldsymbol{R}_0\tilde{\boldsymbol{\varphi}}_{\delta i}\times\boldsymbol{R}_0^T && (44)\\
&= \boldsymbol{\varphi}_{0\delta}\times + (\boldsymbol{R}_0\tilde{\boldsymbol{\varphi}}_{\delta i})\times && (45)
\end{aligned}$$

obtaining then the following expression for the virtual variation of the orientation $\boldsymbol{\theta}_{\delta i}$

$$\boldsymbol{\theta}_{\delta i} = \boldsymbol{\varphi}_{0\delta} + \boldsymbol{R}_0\tilde{\boldsymbol{\varphi}}_{\delta i} \tag{46}$$

The portion of the nodal motion coming from the structural deformation is represented using a superposition of M modal shapes defining its displacement and rotation

$$\tilde{\boldsymbol{u}}_i = \sum_{j=1}^{M}\boldsymbol{U}_{ij}q_j = \boldsymbol{U}_i\boldsymbol{q} \qquad \tilde{\boldsymbol{\varphi}}_i = \sum_{j=1}^{M}\boldsymbol{V}_{ij}q_j = \boldsymbol{V}_i\boldsymbol{q} \tag{47}$$

$$\delta\tilde{\boldsymbol{u}}_i = \sum_{j=1}^{M}\boldsymbol{U}_{ij}\delta q_j \qquad \tilde{\boldsymbol{\varphi}}_{\delta i} = \sum_{j=1}^{M}\boldsymbol{V}_{ij}\delta q_j \tag{48}$$

5.2 Inertial forces

It is assumed that each node on the element has a mass m_i, a static unbalance $\boldsymbol{s}_i = m_i(\boldsymbol{x}_{CGi} - \boldsymbol{x}_i)$ and an inertia tensor $\boldsymbol{J}_i$, the inertial forces and moments acting on the single node are then given by

$$\begin{aligned}
\boldsymbol{f}_i^{in} &= m_i\boldsymbol{a}_i - \boldsymbol{s}_i\times\boldsymbol{\omega}_i - \boldsymbol{\omega}_i\times\boldsymbol{s}_i\times\boldsymbol{\omega}_i && (49)\\
\boldsymbol{m}_i^{in} &= \boldsymbol{s}_i\times\boldsymbol{a}_i + \boldsymbol{J}_i\boldsymbol{\alpha}_i + \boldsymbol{\omega}_i\times\boldsymbol{J}_i\boldsymbol{\omega}_i && (50)\\
& && (51)
\end{aligned}$$

where the moments are evaluated with respect to the node position $\boldsymbol{x}_i$.

It is then possible to formulate the expression for the virtual work of the inertia forces acting on node i

$$\delta\mathcal{L}_i^{in} = \delta\boldsymbol{x}_i\cdot\boldsymbol{f}_i^{in} + \boldsymbol{\theta}_{\delta i}\cdot\boldsymbol{m}_i^{in} \tag{52}$$

Then, by substituting the expression for the inertia forces from Eq. (51) in Eq. (52)

$$\begin{aligned}
\delta\mathcal{L}_i^{in} = {}& \delta\boldsymbol{x}_0\cdot\left[m_i\boldsymbol{a}_i - \boldsymbol{s}_i\times\boldsymbol{\omega}_i - \boldsymbol{\omega}_i\times\boldsymbol{s}_i\times\boldsymbol{\omega}_i\right]\\
&+ \boldsymbol{\varphi}_{0\delta}\cdot\Big[(\boldsymbol{s}_i + m_i\boldsymbol{R}_0(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i))\times\boldsymbol{a}_i + (\boldsymbol{J}_i - \boldsymbol{R}_0(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i)\times\boldsymbol{s}_i\times)\,\boldsymbol{\alpha}_i\\
&\qquad + \boldsymbol{\omega}_i\times\boldsymbol{J}_i\boldsymbol{\omega}_i - \boldsymbol{R}_0(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i)\times\boldsymbol{\omega}_i\times\boldsymbol{s}_i\times\boldsymbol{\omega}_i\Big]\\
&+ \delta\tilde{\boldsymbol{u}}_i\cdot\boldsymbol{R}_0^T\left[m_i\boldsymbol{a}_i - \boldsymbol{s}_i\times\boldsymbol{\omega}_i - \boldsymbol{\omega}_i\times\boldsymbol{s}_i\times\boldsymbol{\omega}_i\right]\\
&+ \tilde{\boldsymbol{\varphi}}_{\delta i}\cdot\boldsymbol{R}_0^T\left[\boldsymbol{s}_i\times\boldsymbol{a}_i + \boldsymbol{J}_i\boldsymbol{\alpha}_i + \boldsymbol{\omega}_i\times\boldsymbol{J}_i\boldsymbol{\omega}_i\right]
\end{aligned} \tag{53}$$

The total virtual work of inertial forces is given by the sum of all the nodal contributions

$$\delta\mathcal{L}^{in} = \sum_{i=1}^{N} \delta\mathcal{L}_i^{in} \tag{54}$$

Before getting an expression for the total inertia forces in the system it is convenient to rotate all the vector and tensor quantities by the rotation tensor $\boldsymbol{R}_0$, in this way the vectorial equations already assume a form analogous to the form obtained when expressed in components in the body frame. The body frame is indeed the preferred reference system for the expression of the equations of motions since it allows a direct comparison with the formulation traditionally used in flight mechanics and is the natural choice for the evaluation of aerodynamic forces.

$$\begin{gathered} \boldsymbol{a}_0 = \boldsymbol{R}_0\bar{\boldsymbol{a}}_0; \quad \boldsymbol{\alpha}_0 = \boldsymbol{R}_0\bar{\boldsymbol{\alpha}}_0; \quad \boldsymbol{\omega}_0 = \boldsymbol{R}_0\bar{\boldsymbol{\omega}}_0; \quad \boldsymbol{s}_i = \bar{\boldsymbol{s}}_i; \\ \boldsymbol{J}_i = \boldsymbol{R}_0\boldsymbol{J}_i\boldsymbol{R}_0^T; \quad \delta\boldsymbol{x}_0 = \boldsymbol{R}_0\delta\bar{\boldsymbol{x}}_0 \quad \boldsymbol{\varphi}_{0\delta} = \boldsymbol{R}_0\bar{\boldsymbol{\varphi}}_{0\delta} \end{gathered} \tag{55}$$

Equation (52) then assumes the form

$$\begin{aligned} \delta\mathcal{L}_i^{in} &= \delta\bar{\boldsymbol{x}}_0 \cdot [m_i\bar{\boldsymbol{a}}_i - \bar{\boldsymbol{s}}_i \times \bar{\boldsymbol{\omega}}_i - \bar{\boldsymbol{\omega}}_i \times \bar{\boldsymbol{s}}_i \times \bar{\boldsymbol{\omega}}_i] \\ &+ \bar{\boldsymbol{\varphi}}_{0\delta} \cdot \Big[(\bar{\boldsymbol{s}}_i + m_i(\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i)) \times \bar{\boldsymbol{a}}_i + \big(\bar{\boldsymbol{J}}_i - (\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i) \times \bar{\boldsymbol{s}}_i \times\big) \bar{\boldsymbol{\alpha}}_i \\ &\qquad + \bar{\boldsymbol{\omega}}_i \times \bar{\boldsymbol{J}}_i\bar{\boldsymbol{\omega}}_i - (\tilde{\boldsymbol{r}}_i + \tilde{\boldsymbol{u}}_i) \times \bar{\boldsymbol{\omega}}_i \times \bar{\boldsymbol{s}}_i \times \bar{\boldsymbol{\omega}}_i \Big] \\ &+ \delta\tilde{\boldsymbol{u}}_i \cdot [m_i\bar{\boldsymbol{a}}_i - \bar{\boldsymbol{s}}_i \times \bar{\boldsymbol{\omega}}_i - \bar{\boldsymbol{\omega}}_i \times \bar{\boldsymbol{s}}_i \times \bar{\boldsymbol{\omega}}_i] \\ &+ \tilde{\boldsymbol{\varphi}}_{\delta i} \cdot \big[\bar{\boldsymbol{s}}_i \times \bar{\boldsymbol{a}}_i + \bar{\boldsymbol{J}}_i\bar{\boldsymbol{\alpha}}_i + \bar{\boldsymbol{\omega}}_i \times \bar{\boldsymbol{J}}_i\bar{\boldsymbol{\omega}}_i\big] \end{aligned} \tag{56}$$

The equations can then be expanded introducing the kinematic relationships described in the previous section. Since a linearized structural deformation is considered all elements quadratic in $\tilde{\boldsymbol{u}}_i$, $\dot{\tilde{\boldsymbol{u}}}_i$, $\ddot{\tilde{\boldsymbol{u}}}_i$ $\tilde{\boldsymbol{\varphi}}_i$, $\dot{\tilde{\boldsymbol{\varphi}}}_i$, $\ddot{\tilde{\boldsymbol{\varphi}}}_i$ are neglected. The inertial forces energetically conjugated to the rigid motion of the body frame J are

$$\begin{aligned} \boldsymbol{f}_x^{in} &= m_i\bar{\boldsymbol{a}}_0 - (\bar{\boldsymbol{s}}_i + m_i\tilde{r}_i) \times \bar{\boldsymbol{\alpha}}_0 + m_i\ddot{\tilde{\boldsymbol{u}}}_i - \bar{\boldsymbol{s}}_i \times \ddot{\tilde{\boldsymbol{\varphi}}}_i - m_i\tilde{\boldsymbol{u}}_i \times \bar{\boldsymbol{\alpha}}_0 \\ &- \bar{\boldsymbol{\omega}}_0 \times (\bar{\boldsymbol{s}}_i + m_i\tilde{r}_i) \times \bar{\boldsymbol{\omega}}_0 - \bar{\boldsymbol{\omega}}_0 \times (m_i\tilde{\boldsymbol{u}}) \times \bar{\boldsymbol{\omega}}_0 \\ &+ 2\bar{\boldsymbol{\omega}}_0 \times (m_i\dot{\tilde{\boldsymbol{u}}}_i - \bar{\boldsymbol{s}}_i \times \dot{\tilde{\boldsymbol{\varphi}}}_i) \end{aligned} \tag{57}$$

while the inertial forces energetically conjugated to the change in orientation of the frame J are

$$\begin{aligned} \boldsymbol{f}_\varphi^{in} &= (\bar{\boldsymbol{s}}_i + m_i\tilde{r}_i) \times \bar{\boldsymbol{a}}_0 + [\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i \times -\bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times -m_i\tilde{\boldsymbol{r}}_i \times \tilde{\boldsymbol{r}}_i\times]\bar{\boldsymbol{\alpha}}_0 \\ &+ m_i\tilde{\boldsymbol{u}}_i \times \bar{\boldsymbol{a}}_0 - [(\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times \tilde{\boldsymbol{u}}_i \times + \tilde{\boldsymbol{u}}_i \times (\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i)\times]\bar{\boldsymbol{\alpha}}_0 \\ &+ (\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times \ddot{\tilde{\boldsymbol{u}}}_i + (\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i\times)\ddot{\tilde{\boldsymbol{\varphi}}}_i \\ &- \bar{\boldsymbol{\omega}}_0 \times [\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i \times -\bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times -m_i\tilde{\boldsymbol{r}}_i \times \tilde{\boldsymbol{r}}_i\times]\bar{\boldsymbol{\omega}}_0 \\ &- \bar{\boldsymbol{\omega}}_0 \times [(\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times \tilde{\boldsymbol{u}}_i \times + \tilde{\boldsymbol{u}}_i \times (\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i)\times]\bar{\boldsymbol{\omega}}_0 \\ &+ [-\dot{\tilde{\boldsymbol{u}}}_i \times (\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times + \dot{\tilde{\boldsymbol{\varphi}}}_i \times (\bar{\boldsymbol{J}}_i - \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i\times)]\bar{\boldsymbol{\omega}}_0 \\ &+ [\bar{\boldsymbol{s}}_i \times \dot{\tilde{\boldsymbol{\varphi}}}_i \times \tilde{\boldsymbol{r}}_i \times -\tilde{\boldsymbol{r}}_i \times \dot{\tilde{\boldsymbol{\varphi}}}_i \times \bar{\boldsymbol{s}}_i\times]\bar{\boldsymbol{\omega}}_0 \\ &+ [-(\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times \dot{\tilde{\boldsymbol{u}}}_i \times -(\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i\times)\dot{\tilde{\boldsymbol{\varphi}}}_i\times]\bar{\boldsymbol{\omega}}_0 \\ &+ \bar{\boldsymbol{\omega}}_0 \times [(\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times \dot{\tilde{\boldsymbol{u}}}_i + (\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i\times)\dot{\tilde{\boldsymbol{\varphi}}}_i] \end{aligned} \tag{58}$$

For the definition of the inertial forces energetically conjugated with the modal coordinates the modal decomposition of $\tilde{\boldsymbol{u}}_i$ and $\tilde{\boldsymbol{\varphi}}_i$ is used

$$\begin{aligned}
\boldsymbol{f}^{in}_{q_k} &= (\boldsymbol{U}^T_{ik} - \boldsymbol{V}^T_{ik}\bar{\boldsymbol{s}}_i)\bar{\boldsymbol{a}}_0 + [\boldsymbol{U}^T_{ik}(\bar{\boldsymbol{s}}_i + m_i\tilde{r}_i) \times + \boldsymbol{V}^T_{ik}(\bar{\boldsymbol{J}}_i - \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times)]\bar{\boldsymbol{\alpha}}_0 \\
&+ \sum_{j=1}^{M}[\boldsymbol{U}^T_{ik} m_i \boldsymbol{U}_{ij} + \boldsymbol{V}^T_{ik}\bar{\boldsymbol{s}}_i \times \boldsymbol{U}_{ij} - \boldsymbol{U}^T_{ik}\bar{\boldsymbol{s}}_i\boldsymbol{V}_{ij} + \boldsymbol{V}^T_{ik}\bar{\boldsymbol{J}}_i\boldsymbol{V}_{ij}]\ddot{q}_j \\
&+ \bar{\boldsymbol{\omega}}^T_0[\boldsymbol{U}_{ik} \times (\bar{\boldsymbol{s}}_i + m_i\tilde{\boldsymbol{r}}_i) \times - \boldsymbol{V}_{ik} \times (\bar{\boldsymbol{J}}_i - \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times)]\bar{\boldsymbol{\omega}}_0 \\
&+ \bar{\boldsymbol{\omega}}^T_0[-\boldsymbol{V}_{ik} \times \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times - \tilde{\boldsymbol{r}}_i \times (\bar{\boldsymbol{s}}_i \times \boldsymbol{V}_{ik})\times]\bar{\boldsymbol{\omega}}_0 \\
&+ \sum_{j=1}^{M}[-\boldsymbol{U}^T_{ik} m_i \boldsymbol{U}_{ij} - \boldsymbol{V}^T_{ik}\bar{\boldsymbol{s}}_i \times \boldsymbol{U}_{ij}]q_j \times \bar{\boldsymbol{\alpha}}_0 \\
&+ 2\sum_{j=1}^{M}(-\boldsymbol{U}^T_{ik} m_i \boldsymbol{U}_{ij} \times - \boldsymbol{V}^T_{ik}\bar{\boldsymbol{s}}_i \times \boldsymbol{U}_{ij} \times + \boldsymbol{U}^T_{ik}\bar{\boldsymbol{s}}_i \times \boldsymbol{V}_{ij} \times - \boldsymbol{V}^T_{ik}\bar{\boldsymbol{J}}_i\boldsymbol{V}_{ij}\times)\dot{q}_j\bar{\boldsymbol{\omega}}_0 \\
&+ \sum_{j=1}^{M}(-2\boldsymbol{U}^T_{ik}\boldsymbol{V}_{ik} \times \bar{\boldsymbol{s}}_i \times + \boldsymbol{V}^T_{ik}\boldsymbol{V}_{ij} \times \bar{\boldsymbol{J}}_i)\dot{q}_j\bar{\boldsymbol{\omega}}_0 \\
&+ \bar{\boldsymbol{\omega}}^T_0\Big[\sum_{j=1}^{M}(\boldsymbol{U}_{ik} \times m_i\boldsymbol{U}_{ij} \times + \boldsymbol{U}_{ij} \times (\boldsymbol{V}_{ik} \times \bar{\boldsymbol{s}}_i)\times)\dot{q}_j\Big]\bar{\boldsymbol{\omega}}_0
\end{aligned} \tag{59}$$

The expression in Eqs. (57), (58) and (59) can be simplified if all the terms depending only on the mass properties of the structure are collected, allowing the definition of the invariants listed in Eqs. (60) and (61). The invariants are not only useful for obtaining a more compact representation of the inertial forces, but they can be computed before running the time simulation, thus reducing the computational cost associated with the solution itself.

$$\begin{aligned}
\boldsymbol{I}_1 &= \sum_i m_i\boldsymbol{I} && (3 \times 3) \\
\boldsymbol{I}_2 &= -\sum_i \bar{\boldsymbol{s}}_i \times + m_i\tilde{\boldsymbol{r}}_i \times && (3 \times 3) \\
\boldsymbol{I}_3 &= \sum_i m_i\boldsymbol{U}_i - \bar{\boldsymbol{s}}_i \times \boldsymbol{V}_i && (3 \times M) \\
\boldsymbol{I}_4 &= \sum_i[(\bar{\boldsymbol{s}}_i + m_i\tilde{r}_i) \times \boldsymbol{U}_i + (\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i\times)\boldsymbol{V}_i] && (3 \times M) \\
\boldsymbol{I}_6 &= \sum_i[\boldsymbol{U}^T_i m_i\boldsymbol{U}_i + \boldsymbol{V}^T_i\bar{\boldsymbol{s}}_i \times \boldsymbol{U}_i - \boldsymbol{U}^T_i\bar{\boldsymbol{s}}_i\boldsymbol{V}_i + \boldsymbol{V}^T_i\bar{\boldsymbol{J}}_i\boldsymbol{V}_i] && (M \times M) \\
\boldsymbol{I}_7 &= \sum_i[\bar{\boldsymbol{J}}_i - \tilde{\boldsymbol{r}}_i \times \bar{\boldsymbol{s}}_i \times - \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times - m_i\tilde{\boldsymbol{r}}_i \times \tilde{\boldsymbol{r}}_i\times] && (3 \times 3)
\end{aligned} \tag{60}$$

$$
\begin{aligned}
\boldsymbol{I}_{8_j} &= -\sum_i (\bar{\boldsymbol{s}}_i + m_i \tilde{\boldsymbol{r}}_i) \times \boldsymbol{U}_{ij} \times && (3 \times M \times 3) \\
\boldsymbol{I}_{12_j} &= -\sum_i m_i \boldsymbol{U}_{ij} \times && (3 \times 3 \times M) \\
\boldsymbol{I}_{13_j} &= -\sum_i \bar{\boldsymbol{s}}_i \times + \boldsymbol{V}_{ij} \times \tilde{\boldsymbol{r}}_i \times && (3 \times 3 \times M) \\
\boldsymbol{I}_{14_j} &= \sum_i [\boldsymbol{U}_{ij} \times (\bar{\boldsymbol{s}}_i + m_i \tilde{\boldsymbol{r}}_i) \times - \boldsymbol{V}_{ij} \times (\bar{\boldsymbol{J}}_i - \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times)] && (3 \times 3 \times M) \\
\boldsymbol{I}_{15_j} &= \sum_i [-\boldsymbol{V}_{ij} \times \bar{\boldsymbol{s}}_i \times \tilde{\boldsymbol{r}}_i \times - \tilde{\boldsymbol{r}}_i \times (\bar{\boldsymbol{s}}_i \times \boldsymbol{V}_{ij}) \times] && (3 \times 3 \times M) \\
\boldsymbol{I}_{16_j} &= \sum_i (-2 \boldsymbol{U}_{ij}^T \boldsymbol{V}_{ij} \times \bar{\boldsymbol{s}}_i \times + \boldsymbol{V}_{ij}^T \boldsymbol{V}_{il} \times \bar{\boldsymbol{J}}_i) && (M \times 3 \times M) \\
\boldsymbol{I}_{17_j} &= \sum_i [-\boldsymbol{U}_{ij}^T m_i \boldsymbol{U}_{il} - \boldsymbol{V}_{ij}^T \bar{\boldsymbol{s}}_i \times \boldsymbol{U}_{il}] && (M \times 3 \times M) \\
\boldsymbol{I}_{18_j} &= -\sum_i (-\boldsymbol{U}_{ij}^T m_i \boldsymbol{U}_{il} \times - \boldsymbol{V}_{ij}^T \bar{\boldsymbol{s}}_i \times \boldsymbol{U}_{il} \times + \boldsymbol{U}_{ij}^T \bar{\boldsymbol{s}}_i \times \boldsymbol{V}_{il} \times - \boldsymbol{V}_{ij}^T \bar{\boldsymbol{J}}_i \boldsymbol{V}_{il} \times) && (M \times 3 \times M) \\
\boldsymbol{I}_{19_j} &= \sum_i (\boldsymbol{U}_{ij} \times m_i \boldsymbol{U}_{il} \times + \boldsymbol{U}_{il} \times (\boldsymbol{V}_{ij} \times \bar{\boldsymbol{s}}_i) \times) && (M \times 3 \times M)
\end{aligned}
\tag{61}
$$

It is then possible to formulate the inertial forces using the definition of the invariants

$$
\begin{aligned}
\boldsymbol{f}_x^{in} &= \boldsymbol{I}_1 \bar{\boldsymbol{a}}_0 + \boldsymbol{I}_2 \bar{\boldsymbol{\alpha}}_0 + \boldsymbol{I}_3 \ddot{\boldsymbol{q}} + \sum_j \boldsymbol{I}_{12_j} q_j \bar{\boldsymbol{\alpha}}_0 + \bar{\boldsymbol{\omega}}_0 \times \boldsymbol{I}_2 \bar{\boldsymbol{\omega}}_0 + \bar{\boldsymbol{\omega}}_0 \times \sum_j \boldsymbol{I}_{12_j} q_j \bar{\boldsymbol{\omega}}_0 \\
\boldsymbol{f}_\varphi^{in} &= \boldsymbol{I}_2^T \bar{\boldsymbol{a}}_0 + \boldsymbol{I}_7 \bar{\boldsymbol{\alpha}}_0 + \sum_j \boldsymbol{I}_{12_j}^T q_j \bar{\boldsymbol{a}}_0 - \sum_j (\boldsymbol{I}_{8_j} q_j + \boldsymbol{I}_{8_j}^T q_j) \bar{\boldsymbol{\alpha}}_0 + \boldsymbol{I}_4 \ddot{q} - \bar{\boldsymbol{\omega}}_0 \times \boldsymbol{I}_7 \bar{\boldsymbol{\omega}}_0 \\
&\quad - \bar{\boldsymbol{\omega}}_0 \times \sum_j (\boldsymbol{I}_{8_j} q_j + \boldsymbol{I}_{8_j}^T q_j) \bar{\boldsymbol{\omega}}_0 + \Big[\sum_j (\boldsymbol{I}_{13_j} q_j + \boldsymbol{I}_{13_j}^T \dot{q}_j) - \sum_j (\boldsymbol{I}_{14_j} q_j + \boldsymbol{I}_{14_j}^T \dot{q}_j) \Big] \bar{\boldsymbol{\omega}}_0 \\
&\quad + \bar{\boldsymbol{\omega}}_0 \times \sum_j (\boldsymbol{I}_{16_j} \dot{q}_j) \\
\boldsymbol{f}_{q_k}^{in} &= \boldsymbol{I}_3^T \bar{\boldsymbol{a}}_0 + \boldsymbol{I}_4^T \bar{\boldsymbol{\alpha}}_0 + \boldsymbol{I}_6 \ddot{q}_l + \bar{\boldsymbol{\omega}}_0^T [\boldsymbol{I}_{14_k} + \boldsymbol{I}_{15_k}] \bar{\boldsymbol{\omega}}_0 + \sum_j (\boldsymbol{I}_{17_j} q_j) \bar{\boldsymbol{\alpha}}_0 \\
&\quad + \Big[-2 \sum_j (\boldsymbol{I}_{18_j} \dot{q}_j) + \sum_j \boldsymbol{I}_{18_j} \dot{q}_j \Big] \bar{\boldsymbol{\omega}}_0 + \sum_{l=1}^{M} (-2 \boldsymbol{U}_{ij}^T \boldsymbol{V}_{ij} \times \bar{\boldsymbol{s}}_i \times + \boldsymbol{V}_{ij}^T \boldsymbol{V}_{il} \times \bar{\boldsymbol{J}}_i) \dot{q}_l \bar{\boldsymbol{\omega}}_0 \\
&\quad + \bar{\boldsymbol{\omega}}_0^T \Big[\sum_j \boldsymbol{I}_{19_{jk}} q_j \Big] \bar{\boldsymbol{\omega}}_0
\end{aligned}
\tag{62}
$$

Some decoupling in the expression of the inertial forces could be obtained by selecting a mean axes formulation [19, 21], that is the modal shapes are defined in order to be orthogonal to the rigid modes with respect to the system mass matrix. In this way it would result

$$
\boldsymbol{I}_3 = 0 \qquad \boldsymbol{I}_4 = 0 \tag{63}
$$

but since this simplification brings a small reduction of the computational cost with respect to the general case it was not included as a requirement of this formulation. Modal shapes obtained from a free structure, however, would naturally be formulated in mean axes, leading to the satisfaction of conditions in Eq. (63).

The equations were so far considered as vectorial equation that are not related to any choice of a reference system used to express them. They are now formulated considering the component in the body reference J and thanks to the transformation in Eq. (55) their expression is not affected by this operation, as can be verified by noticing that

$$\bar{\boldsymbol{a}}_0^I = \boldsymbol{a}_0^J \qquad \bar{\boldsymbol{\omega}}_0^I = \boldsymbol{\omega}_0^J \qquad \bar{\boldsymbol{\alpha}}_0^I = \boldsymbol{\alpha}_0^J \tag{64}$$

where the superscripts ^{J} and ^{I} indicate the components in the references J and I respectively.

It is also convenient to substitute the acceleration of the body frame with the time derivative of the component of the velocity with respect to the body frame, this substitution can be done by using the Poisson formula

$$\boldsymbol{a}_0^J = \dot{\boldsymbol{v}}_0^J + \boldsymbol{\omega}_0^J \times \boldsymbol{v}_0^J \tag{65}$$

The inertial forces can be expressed in terms of $\boldsymbol{v}_0^J$, $\dot{\boldsymbol{v}}_0^J$, $\boldsymbol{\omega}_0^J$, $\dot{\boldsymbol{\omega}}_0^J$.

$$\begin{aligned} \boldsymbol{F}^{in} &= (\boldsymbol{M} + \boldsymbol{M}_q(\boldsymbol{q})) \begin{bmatrix} \dot{\boldsymbol{v}}_0^J \\ \dot{\boldsymbol{\omega}}_0^J \\ \ddot{\boldsymbol{q}} \end{bmatrix} + \boldsymbol{F}_{\boldsymbol{0}}^{in}\left(\boldsymbol{v}_0^J, \boldsymbol{\omega}_0^J, \boldsymbol{\omega}_0^J\right) + \boldsymbol{F}_{\boldsymbol{q}}^{in}\left(\boldsymbol{q}, \boldsymbol{v}_0^J, \boldsymbol{\omega}_0^J, \boldsymbol{\omega}_0^J\right) + \boldsymbol{F}_{\dot{\boldsymbol{q}}}^{in}\left(\dot{\boldsymbol{q}}, \boldsymbol{\omega}_0^J\right) \\ &= (\boldsymbol{M} + \boldsymbol{M}_q(\boldsymbol{q})) \begin{bmatrix} \dot{\boldsymbol{v}}_0^J \\ \dot{\boldsymbol{\omega}}_0^J \\ \ddot{\boldsymbol{q}} \end{bmatrix} + \boldsymbol{F}_{\Delta}^{in}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}_0^J, \boldsymbol{\omega}_0^J\right) \end{aligned} \tag{66}$$

5.3 Weight force

The nodal force and moment associated to gravity vector $\boldsymbol{g}$ and acting on the node i are given by

$$\begin{aligned} \boldsymbol{f}_i^g &= m_i \boldsymbol{g} \\ \boldsymbol{m}_i^g &= \boldsymbol{s}_i \times \boldsymbol{g} \end{aligned} \tag{67}$$

The expression of the virtual work is then

$$\begin{aligned} \delta\mathcal{L}^g &= \sum_{i=1}^{N} \delta\boldsymbol{x}_i \cdot \boldsymbol{f}_i^g + \boldsymbol{\theta}_{\delta i} \cdot \boldsymbol{m}_i^g \\ &= \sum_{i=1}^{N} \Big[\delta\boldsymbol{x}_0 \cdot m_i \boldsymbol{g} + \boldsymbol{\varphi}_{0\delta} \cdot \Big[(\boldsymbol{s}_i + m_i \boldsymbol{R}_0 \tilde{r}_i) \times \boldsymbol{g} + m_i \boldsymbol{R}_0 \tilde{\boldsymbol{u}}_i \times \boldsymbol{g} \Big] \\ &\qquad + \delta\tilde{\boldsymbol{u}}_i \cdot m_i \boldsymbol{R}_0^T \boldsymbol{g} + \tilde{\boldsymbol{\varphi}}_\delta \cdot \bar{\boldsymbol{s}}_i \times \boldsymbol{R}_0^T \boldsymbol{g} \Big] \end{aligned} \tag{68}$$

Leading to the following definition of the generalized forces

$$\boldsymbol{F}^g = \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2^T \\ \boldsymbol{I}_3^T \end{bmatrix} \boldsymbol{R}_0^T \boldsymbol{g}^I + \begin{bmatrix} \boldsymbol{0} \\ \sum_j \boldsymbol{I}_{12_j}^T q_j \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{R}_0^T \boldsymbol{g}^I \tag{69}$$

5.4 Dynamic equations

The formulation of the inertia and weight forces can be completed by introducing the aerodynamic forces. One possibility of introduction of aerodynamic forces consists in the use of a time domain state-space formulation in body axes, as described in the previous chapter, which is however subjected to limitations of small perturbations and subsonic flow. The complete equations of motion can then be formulated including all the inertial, aerodynamic and weight contributions.

6 RESULTS

The presented methodologies are applied here to a model of large transport aircraft, sized using the NeoCASS software using the geometrical data of the B747 aircraft. The aeroelastic model is presented in Fig. 5.

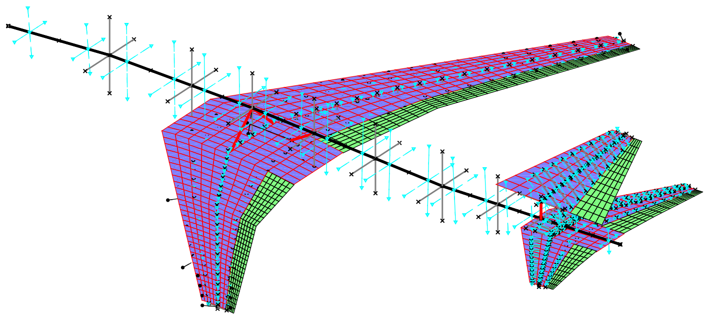


Figure 5: Aeroelastic model

Aerodynamic forces are computed for the model using the VLM and DLM methods, and then they are used for the generation of the state-space model of the aircraft. The aerodynamic forces obtained from the DLM and VLM usually do not allow for the accurate prediction of the low frequency flight mechanics modes of the aircraft. The aerodynamic forces can then be corrected using aerodynamic coefficients in order to provide a model with the expected characteristics. In the present applications the aerodynamic coefficients for a landing configuration were obtained from reference [22] and directly introduced in the aeroelastic model, leading to the flight mechanics modes in Fig. 6. Where also the poles of the system computed directly from the linearized equation of motion of the rigid body aircraft are displayed.

Three different methods were used to compute the flight mechanics modes from the aeroelastic system. In the first case the modes are obtained directly from the aeroelastic state-space model in body axes, without aerodynamic corrections. It can be seen that the high frequency short period and dutch roll modes are recovered with some error in frequency, while the phugoid mode is not recovered, due to the absence of the aerodynamic load in the reference configuration in the DLM formulation. The introduction of VLM matrices to correct the steady portion of the aerodynamic forces in this case does not lead to a significant improvement of the results, this can originate from the different aerodynamic configuration between the NeoCASS model and the actual aircraft in landing configuration that is used as a reference for the aerodynamic coefficients. In addition also the absence of the fuselage model can lead to some inaccuracy in the predicted coefficients. It can be seen, however, that it is now possible to obtain the short period mode. The last analysis is performed by directly introducing the aerodynamic

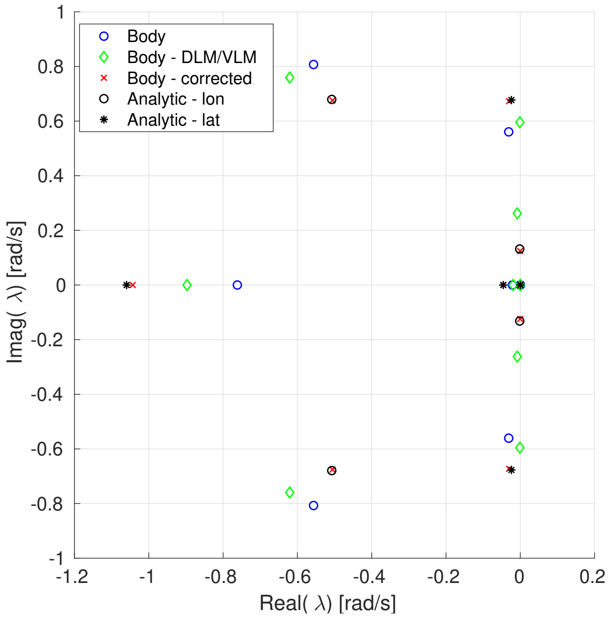


Figure 6: Flight mechanics poles of the B747 aircraft obtained with the corrected aeroelastic model compared with the analytical values.

coefficients in the model equations, in this case the computed frequencies and damping match the values predicted from the longitudinal and latero-directional equations of motion.

A gust response simulation is then performed to evaluate the effect of the nonlinear simulation on the dynamic response. A longitudinal gust with frequency 0.5 Hz and amplitude 20 m/s is applied with the aircraft flying at sea level with Mach number $M = 0.52$. The simulation results are summarized in Fig. 7 where the wing root torsional and bending moment are presented, along with the vertical velocity and the pitch rate.

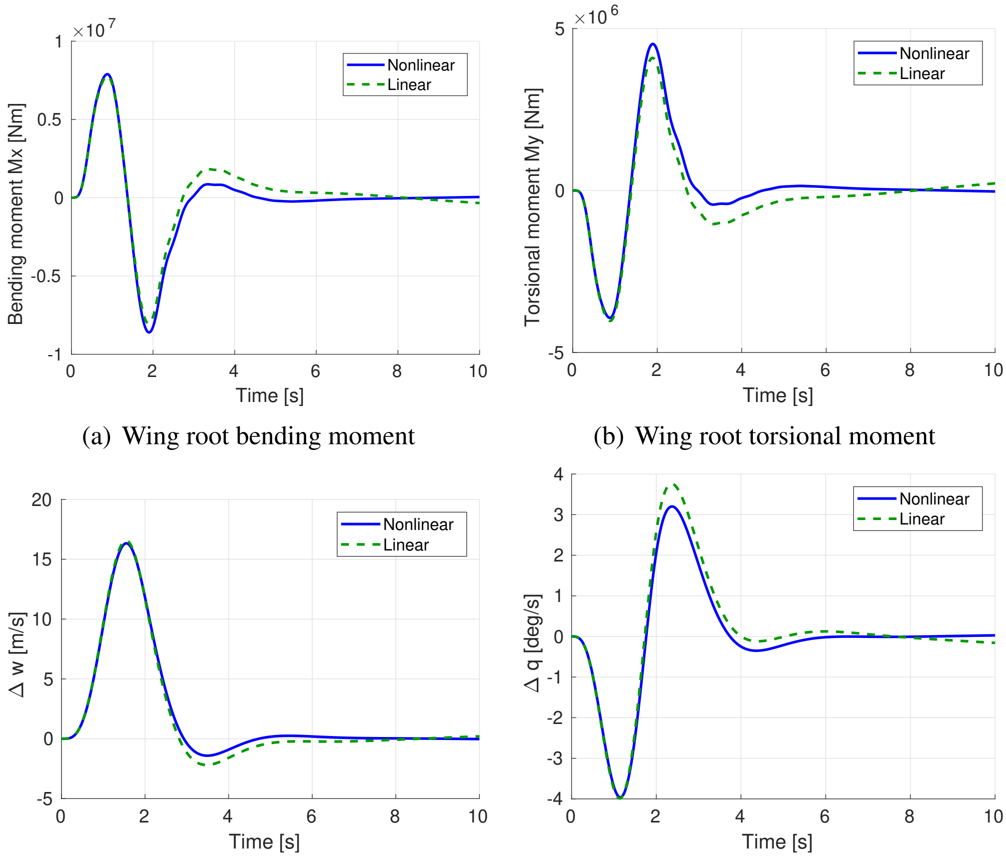
It can be seen that the linear and nonlinear responses are very similar, but the low frequency body motion is behaviour is modified by the introduction of the nonlinear dynamics, as can be seen from the time history of the pitch rate in Fig. 7(d). The different body motion also lead to different internal forces in the second portion of the time response, after the first peak.

7 CONCLUSIONS

The accuracy in the prediction of aircraft motion from aeroelastic models can be increased by defining a unified formulation able to recover both the structural response and the dynamics of the rigid motion. This unified formulation can be expressed in a linearized way or considering a full nonlinear rigid body motion. Both formulations can be easily applied to low-fidelity models such the ones generated by NeoCASS after the preliminary sizing. In most cases the improvement in the recovery of structural loads that can be obtained from the unified formulation is low, as expected by the fact that the corrections operate mostly on the low-frequency rigid modes. It is however useful to have a unified formulation when the effect of automatic control systems need to be evaluated since it allows the proper recovery.

8 REFERENCES

[1] Cavagna, L., Ricci, S., and Riccobene, L. (2011). Structural Sizing, Aeroelastic Analysis, and Optimization in Aircraft Conceptual Design. *Journal of Aircraft*, 48(6), 1840–1855. doi:10.2514/6.2005-1959.



(a) Wing root bending moment (b) Wing root torsional moment

(c) Body vertical velocity (d) Body pitch rate

Figure 7: Simulation results for longitudinal gust encounter

[2] Cavagna, L., Ricci, S., and Travaglini, L. (2011). NeoCASS: An integrated Tool for Structural Sizing, Aeroelastic Analysis and MDO at Conceptual Design Level. *Progress in Aerospace Sciences*, 47, 621–635. doi:http://dx.doi.org/10.1016/j.paerosci.2011.08.006.

[3] Gupta, K. K., Brenner, M. J., and Voelker, L. S. (1991). Development of an integrated aeroservoelastic analysis program and correlation with test data. Tech. rep., NASA, TP–3120.

[4] Baldelli, D. H., Chen, P. C., and Panza, J. (2006). Unified aeroelastic and flight dynamic formulation via rational function approximations. *Journal of Aircraft*, 43(3), 763–772.

[5] Kier, T. (2012). Aerodynamic modelling for integrated loads analysis models. In *Third Symposium Simulation of Wing and Nacelle Stall, 21-22 Jun., Braunschweig, Germany,*. pp. 2–10.

[6] Kier, T. M. and Looye, G. (2009). Unifying manoeuvre and gust loads analysis models. In *International Forum on Aeroelasticity and Structural Dynamics*. IFASD-2009-106.

[7] Kier, T. (2011). An integrated loads analysis model including unsteady aerodynamic effects for position and attitude dependent gust fields. In *International Forum on Aeroelasticity and Structural Dynamics*. IFASD-2011-052.

[8] Looye, G. H. N. (2008). *An integrated approach to aircraft modelling and flight control law design*. Ph.D. thesis, TU Delft, Delft University of Technology.

[9] Demasi, L. and Livne, E. (2009). Dynamic aeroelasticity of structurally nonlinear configurations using linear modally reduced aerodynamic generalized forces. *AIAA Journal*, 47(1), 71–90.

[10] Kruger, W. R. (2007). Multibody dynamics for the coupling of aeroelasticity and flight mechanics of highly flexible structures. In *International Forum on Aeroelasticity and Structural Dynamics*. IFASD-2007-107.

[11] Castellani, M., Cooper, J. E., and Lemmens, Y. (2016). Nonlinear static aeroelasticity of high aspect ratio wing aircraft by fem and multibody methods. In *15th Dynamics Specialists Conference, AIAA SciTech Forum*, vol. AIAA 2016-1573.

[12] Murua, J., Palacios, R., and Graham, J. M. R. (2012). Applications of the unsteady vortex-lattice method in aircraft aeroelasticity and flight dynamics. *Progress in Aerospace Sciences*, 55(Supplement C), 46 – 72. ISSN 0376-0421. doi: https://doi.org/10.1016/j.paerosci.2012.06.001.

[13] Patil, M. J., Hodges, D. H., and Cesnik, C. E. (2000). Nonlinear aeroelastic analysis of complete aircraft in subsonic flow. *Journal of Aircraft*, 37(5), 753–760.

[14] Ghiringhelli, G. L., Masarati, P., and Mantegazza, P. (2000). Multibody implementation of finite volume c^0 beams. *AIAA Journal*, 38(1), 131–138.

[15] Rodden, W. P., Taylor, P. F., and McIntosh, S. C. (1998). Further refinement of the subsonic doublet-lattice method. *Journal of aircraft*, 35(5), 720–727.

[16] Rodden, W. P. and Johnson, E. H. (1994). *MSC/NASTRAN Aeroelastic Analysis Users Guide*. The MacNeal-Schwendler Corp., Los Angeles.

[17] Mantegazza, P. and Cardani, C. (1978). Continuation and direct solution of the flutter equation. *Computers & Structures*, 8(2), 185–192.

[18] Ripepi, M. and Mantegazza, P. (2013). Improved Matrix Fraction Approximation of Aerodynamic Transfer Matrices. *AIAA Journal*, 51(5), 1156–1173. doi:10.2514/1.J052009.

[19] Mantegazza, P. (2011). Tutorial on attached-mean axes and their use in the calculation of deformable static and damped-undamped vibration modes of a free-free structure. *Journal of Aeroelasticity and Structural Dynamics*, 2(1).

[20] Masarati, P., Morandini, M., and Mantegazza, P. (2014). An efficient formulation for general-purpose multibody/multiphysics analysis. *Journal of Computational and Nonlinear Dynamics*, 9(4), 041001.

[21] Milne, R. D. (1964). Dynamics of the deformable airplane. Tech. rep., Ministry of Aviation, Aeronautical Research Council Reports and Memoranda.

[22] Heffley, R. K. and Jewell, W. F. (1972). Aircraft handling qualities data. Tech. rep., NASA, CR–2144.