Model predictive control of linear systems with multiplicative unbounded uncertainty and chance constraints☆

Marcello Farina, Riccardo Scattolini

Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, via Ponzio 34/5, 20133 Milan, Italy

This paper presents a novel stochastic Model Predictive Control algorithm for linear systems characterized by multiplicative and possibly unbounded model uncertainty. Probabilistic constraints on the states and inputs are considered, and a quadratic cost function is minimized. The stochastic control problem, and in particular the probabilistic constraints, are reformulated in deterministic terms by means of the Cantelli inequality, so that the on-line computational burden of the algorithm is similar to the one of a standard MPC method. The properties of the algorithm, namely the recursive feasibility and the pointwise convergence of the state, are proven by suitably selecting the terminal cost and the constraints on the mean and the variance of the state at the end of the prediction horizon, and by considering as additional optimization variables also the mean and the covariance of the state at the beginning of the prediction horizon. An extension to deal with the case of expectation, rather than probabilistic, constraints is reported. The numerical issues related to the off-line selection of the algorithm's parameters and its on-line implementation are discussed. Simulation results referred to a system with unbounded uncertainty are shown to compare the performances achievable with probabilistic and expectation constraints.

Keywords: Stochastic systems, Model predictive control, Constrained control

1. Introduction

The development of Stochastic Model Predictive Control (S-MPC) algorithms for systems subject to random noises and probabilistic constraints on the states and the inputs has recently stimulated many research efforts. Two main approaches have been followed so far for addressing optimization problems with probabilistic constraints: analytical methods, see for instance Cannon, Kouvaritakis, and Wu (2009), Farina, Giulioni, Magni, and Scattolini (2015), Geletu, Klöppel, Hoffmann, and Li (2015), Nemirovski and Shapiro (2007) and Primbs and Sung (2009) and randomized, or scenario, techniques, e.g., Blackmore, Ono, Bektassov, and Williams (2010) and Calafiore and Fagiano (2013). In analytical methods the optimization problem, i.e. the cost function to be minimized and the probabilistic constraints on the state and input variables, are reformulated in deterministic terms, so that the resulting algorithm to be implemented on-line is of complexity similar to that of a nominal MPC, often at the price of some degree of conservativeness and of a cumbersome off-line design phase. In scenario-based methods, a properly chosen number of noise realizations is randomly generated at any time instant to compute the optimal solution with a given level of accuracy. Notably, scenario-based algorithms can deal with generic systems, cost functions, and state and control constraints, but they may require a heavier on-line computational load than analytical methods.

A typical assumption in S-MPC methods based on analytical reformulations is that the system under control is linear and is affected by additive or multiplicative uncertainties, see for instance Cannon, Cheng, Kouvaritakis, and Raković (2012), Cannon et al. (2009), Evans, Cannon, and Kouvaritakis (2012), Farina, Giulioni, Magni, and Scattolini (2013), Farina et al. (2015), Paulson, Streif, and Mesbah (2015) and Primbs and Sung (2009). For systems with additive and unbounded noise, a state feedback algorithm has recently been proposed in Farina et al. (2013), while its extension to the output feedback case has been reported in Farina et al. (2015). The method is based on the reformulation of the state and control constraints by means of the Cantelli–Chebyshev inequality and on considering at any time instant the current value of the mean and of the covariance of the state as additional optimization variables to be properly selected. This approach is more conservative than the analytical algorithms discussed, e.g., in Geletu et al. (2015) and Nemirovski and Shapiro (2007) for coping with chance

Article history:
Received 29 April 2015
Received in revised form
28 October 2015
Accepted 29 March 2016
Available online 23 April 2016

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Giancarlo Ferrari-Trecate under the direction of Editor Ian R. Petersen.

E-mail addresses: marcello.farina@polimi.it (M. Farina), riccardo.scattolini@polimi.it (R. Scattolini).

© 2016. This manuscript version is made available under the CC-BY-NC-ND 4.0 license http://creativecommons.org/licenses/by-nc-nd/4.0/
Published Journal Article available at: http://dx.doi.org/10.1016/j.automatica.2016.04.008

constraints; however, it has been selected since it allows to obtain an explicit and linear expression of the reformulated constraints, which is desirable in many engineering problems since it simplifies the problem from the numerical perspective. Notably, recursive feasibility is guaranteed also for unbounded disturbances and mean square convergence of the state. These valuable properties motivate the extension of this approach to the case of multiplicative noises, which can represent model parametric uncertainties of the system, see for example Primbs and Sung (2009), and are popular in many application fields, such as financial optimization, see Primbs (2007) and Shin, Lee, and Primbs (2010). For these reasons, in this paper a new S-MPC algorithm is developed for systems characterized by multiplicative and possibly unbounded model uncertainty, probabilistic constraints on the states and inputs, and a quadratic cost function. The recursive feasibility of the method and the pointwise convergence of the state are proven by suitably selecting the terminal cost and the constraints on the mean and the variance of the state at the end of the prediction horizon. Expectation constraints, rather than probabilistic ones, are also considered, as in Primbs and Sung (2009), and the feasibility and convergence properties of the proposed S-MPC algorithm are extended to this case.

One of the main differences with respect to state-of-the-art algorithms is mostly related to the recursive feasibility issue. More specifically, in Primbs and Sung (2009) (which includes expectation constraints), when the MPC optimization problem results infeasible, an alternative control policy is adopted. A similar approach is adopted in Cannon et al. (2009) (which considers probabilistic constraints), where two alternative control policies must be adopted, depending whether the state of the system lies in the so-called invariant sets with probability p (where probabilistic constraints are guaranteed to be verified) or not. Finally, in Evans et al. (2012), the idea of stochastic tubes (Cannon et al., 2012) is adopted, under the assumption that the noise has bounded support.

The paper is organized as follows. In Section 2 the system to be controlled is introduced together with the considered probabilistic constraints. Section 3 is devoted to define the structure of the control law, to transform the state and control constraints in deterministic terms by means of the Cantelli inequality, to formulate the optimization problem, including the state constraints to be considered at the beginning and at the end of the prediction horizon, and to present the main convergence result. In Section 4 the numerical issues related to the off-line design and the on-line implementation are discussed, and numerical algorithms are provided to compute the main design parameters as the solutions of suitable linear matrix inequalities (LMI's). Section 5 is devoted to extend the previous results to the case of expectation constraints, while in Section 6 a simulation example is reported and commented. Finally, a section of conclusions closes the paper and suggests hints for future research. To improve readability, the proofs of the main results are reported in an Appendix.

2. The system

We consider the following discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t + \sum_{j=1}^{q}(C_j x_t + D_j u_t) w_t^j \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input and $w_t^j \in \mathbb{R}$, for all $j = 1, \ldots, q$, is a zero-mean white noise with unitary variance and possibly unbounded support. Furthermore, we assume that $\mathbb{E}\{w_t^j w_k^i\} = 0$ for all t, k and for all $i \neq j$.

Perfect state information is assumed, together with the stabilizability of the pair (A, B). We also assume that the state and input variables are subject to the following probabilistic constraints, for all t

$$\mathcal{P}\{b_r^T x_t \geq 1\} \leq p_x^r \quad r = 1, \ldots, n_r \tag{2a}$$

$$\mathcal{P}\{c_s^T u_t \geq 1\} \leq p_u^s \quad s = 1, \ldots, n_s \tag{2b}$$

where $\mathcal{P}(\phi)$ denotes the probabilities of ϕ, b_r, c_s are constant vectors, and p_x, p_u are design parameters.

3. MPC algorithm: formulation and properties

According to the standard procedure of MPC, at any time instant t a future prediction horizon of length N is considered and a suitable optimization problem is solved. The main ingredients of the optimization problem are now introduced.

3.1. Control law

Let $\bar{x}_t = \mathbb{E}\{x_t\}$ and consider the state-feedback control law, for $k \geq t$,

$$u_k = \bar{u}_k + K_k(x_k - \bar{x}_k) \tag{3}$$

where the input sequence $\bar{u}_k$ and the gain sequence K_k, $k = t, t+1, \ldots,$ are defined (see later for details) as the result of a suitable optimization problem solved at time t (and therefore independently of the sequences $w_t^j, w_{t+1}^j, \ldots$, for all $j = 1, \ldots, q$).

In view of the fact that w_k^j is a zero-mean white noise, $\bar{x}_k$, for $k > t$, evolves according to

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k. \tag{4}$$

Define $\delta x_k = x_k - \bar{x}_k$ and $X_k = \text{var}\{\delta x_k\} = \mathbb{E}\{\delta x_k \delta x_k^T\}$. Similarly to Primbs and Sung (2009), it is possible to show that X_k evolves according to the equation

$$X_{k+1} = (A + BK_k)X_k(A + BK_k)^T + \sum_{j=1}^{q}(C_j + D_jK_k)X_k(C_j + D_jK_k)^T + \sum_{j=1}^{q}(C_j\bar{x}_k + D_j\bar{u}_k)(C_j\bar{x}_k + D_j\bar{u}_k)^T. \tag{5}$$

The variance of the input is $U_k = \text{var}\{u_k - \bar{u}_k\} = K_k X_k K_k^T$.

3.2. Constraints

In this section we use the Cantelli inequality (Marshall & Olkin, 1960) to cast the probabilistic constraints (2) as deterministic ones, i.e., in terms of variables whose evolution is deterministically defined.

Lemma 1 (*Cantelli Inequality*). *Let y be a (scalar) random variable with mean $\bar{y}$ and variance Y. Then for every $\mathbb{R} \ni \alpha \geq 0$ it holds that $\mathcal{P}(y \geq \bar{y} + \alpha) \leq \frac{Y}{Y+\alpha^2}$.* ■

As in e.g. Farina et al. (2013), constraints (2) are verified, for all time instants $k \geq t$, if

$$b_r^T \bar{x}_k \leq 1 - \sqrt{b_r^T X_k b_r} f(p_x^r) \quad r = 1, \ldots, n_r \tag{6a}$$

$$c_s^T \bar{u}_k \leq 1 - \sqrt{c_s^T U_k c_s} f(p_u^s) \quad s = 1, \ldots, n_s \tag{6b}$$

where $f(p) = \sqrt{(1-p)/p}$, regardless of the specific distribution of the noise w_k^j.

3.3. Cost function

The considered cost function is

$$J = \mathbb{E}\left\{\sum_{k=t}^{t+N-1} \|x_k\|_Q^2 + \|u_k\|_R^2 + \|x_{t+N}\|_P^2\right\} \tag{7}$$

where Q and R are positive definite, symmetric matrices of appropriate size and P is the solution to the algebraic equation

$$(A+B\bar{K})^T P(A+B\bar{K}) + Q + \bar{K}^T R\bar{K} + \sum_{j=1}^{q}(C_j + D_j\bar{K})^T P(C_j + D_j\bar{K}) - P = 0 \tag{8}$$

where $\bar{K}$ is a suitable stabilizing gain for the pair (A, B). As also discussed in Cannon et al. (2009), the computation of suitable matrices P and $\bar{K}$, if possible, can be carried out using linear matrix inequalities, for more details see Section 4. Using standard procedures, we can write $J = J_m + J_v$, where

$$J_m = \sum_{k=t}^{t+N-1} \|\bar{x}_k\|_Q^2 + \|\bar{u}_k\|_R^2 + \|\bar{x}_{t+N}\|_P^2$$

$$J_v = \sum_{k=t}^{t+N-1} tr\{(Q + K_k^T R K_k)X_k\} + tr\{PX_{t+N}\}.$$

Note that J_m depends on $\bar{u}_{t\ldots t+N-1}$ and on the initial condition of the mean value $\bar{x}_t$, while J_v depends on $\bar{u}_{t\ldots t+N-1}$, $K_{t\ldots t+N-1}$, and on the initial conditions of $\bar{x}_t$ and X_t, since the evolution of X_k in (5) depends also on $\bar{x}_k$ and $\bar{u}_k$. As later specified, the pair $(\bar{x}_t, X_t)$ will be also considered as an argument of the MPC optimization problem.

3.4. Terminal constraints

As usual in MPC with guaranteed stability (see e.g. Mayne, Rawlings, Rao, & Scokaert, 2000) and consistently with Farina et al. (2013), constraints are enforced at the end of the prediction horizon on both the mean value $\bar{x}_{t+N}$ and the variance X_{t+N}, i.e.,

$$\bar{x}_{t+N} \in \bar{\mathbb{X}}_f \tag{9}$$

$$X_{t+N} \leq \bar{X}. \tag{10}$$

The set $\bar{\mathbb{X}}_f$, containing the origin, is a positively invariant set for the system (4), with the control law $\bar{u}_k = \bar{K}\bar{x}_k$, see Kolmanovsky and Gilbert (1998), that is

$$(A+B\bar{K})\bar{x} \in \bar{\mathbb{X}}_f \quad \forall \bar{x} \in \bar{\mathbb{X}}_f \tag{11}$$

while $\bar{X}$ verifies the Lyapunov-type equation

$$\bar{X} = (A+B\bar{K})\bar{X}(A+B\bar{K})^T + \sum_{j=1}^{q}(C_j + D_j\bar{K})\bar{X}(C_j + D_j\bar{K})^T + \bar{W} \tag{12}$$

where $\bar{W} = \sum_{j=1}^{q}(C_j + D_j\bar{K})W(C_j + D_j\bar{K})^T$ and W is an arbitrary matrix, defined in such a way that $W > \bar{x}\bar{x}^T$ for all $\bar{x} \in \bar{\mathbb{X}}_f$. The following must also hold for all $\bar{x} \in \bar{\mathbb{X}}_f$.

$$b_r^T\bar{x} \leq 1 - \sqrt{b_r^T\bar{X}b_r f(p_x^r)} \tag{13a}$$

$$c_s^T\bar{K}\bar{x} \leq 1 - \sqrt{c_s^T\bar{K}\bar{X}\bar{K}^T c_s f(p_u^s)}. \tag{13b}$$

Note that, provided that Eq. (12) has a solution for some $\bar{W} > 0$, the key requirement here is the definition of a sufficiently small $\bar{\mathbb{X}}_f$. In fact, the smaller $\bar{\mathbb{X}}_f$, the smaller W, the smaller $\bar{W}$ and hence the smaller $\bar{X}$ resulting from (12). In view of this, it is always possible to define $\bar{\mathbb{X}}_f$ in such a way that inequalities (13) are verified.

3.5. Initial conditions for the mean and the covariance

As in Farina et al. (2015), for feasibility purposes the initial conditions $(\bar{x}_t, X_t)$ at the current time instant must also be accounted for as free variables in the optimization problem. More specifically, the following alternative strategies can be selected.

– *Strategy* 1—*Reset of the initial state*: in the MPC optimization problem set $\bar{x}_{t|t} = x_t, X_{t|t} = 0$

– *Strategy* 2—*Prediction*: in the MPC optimization problem set $\bar{x}_{t|t} = \bar{x}_{t|t-1}, X_{t|t} = X_{t|t-1}$.

This will result in including in the MPC optimization problem the following constraint

$$(\bar{x}_t, X_t) \in \{(x_t, 0), (\bar{x}_{t|t-1}, X_{t|t-1})\}. \tag{14}$$

3.6. MPC problem

The Stochastic MPC problem can now be stated:

$$\min_{\bar{u}_{t\ldots t+N-1}, K_{t\ldots t+N-1}, (\bar{x}_t, X_t)} J \tag{15}$$

subject to the dynamics (4) and (5), to the constraints (6) for all $k = t, \ldots, t+N-1$, to the initial constraint (14), and to the terminal constraints (9), (10). □

Denoting by $\bar{u}_{t\ldots t+N-1|t} = \{\bar{u}_{t|t}, \ldots, \bar{u}_{t+N-1|t}\}$, $K_{t\ldots t+N-1|t} = \{K_{t|t}, \ldots, K_{t+N-1|t}\}$, and $(\bar{x}_{t|t}, X_{t|t})$ the optimal solution of the S-MPC problem, and according to the receding horizon principle, the feedback control law actually used is, consistently with (3), $u_t = \bar{u}_{t|t} + K_{t|t}(x_t - \bar{x}_{t|t})$.

The recursive feasibility and convergence properties of the resulting control system are stated in the following result.

Theorem 1. *If, at time $t = 0$, the S-MPC problem admits a solution, the optimization problem is recursively feasible and $\mathbb{E}\{\|x_t\|_Q^2\} \to 0$ as $t \to +\infty$. Furthermore, the state and input probabilistic constraints (2) are verified for all $t \geq 0$.*

Remark 1. As discussed, the fact that the initial conditions for mean and variance are accounted for, in our approach, as free variables (where only two different solutions are possible) is fundamental to guarantee recursive feasibility of our scheme. This can be done at the price of suitably characterizing the probabilistic constraints (2). In the state-feedback approach presented in Cannon et al. (2009), for example, conditional probability constraints, e.g., of type $\mathcal{P}\{b_r^T x_{t+k} \geq 1|x_t\} \leq p_x^r$ are implicitly enforced in the MPC optimization problem formulated at time step t, when feasible. In view of the receding horizon principle, this may result in $\mathcal{P}\{b_r^T x_{t+1} \geq 1|x_t\} \leq p_x^r$ for all t, meaning that the 1-step forward conditional probability constraint is verified at each time step. In our paper, however, the two possible initial conditions are stated in (14). It is important to remark, indeed, that the two possible initializations imply different probability definitions. Specifically, if the reset Strategy 1 is adopted at time t, we implicitly enforce $\mathcal{P}\{b_r^T x_{t+1} \geq 1|x_t\} \leq p_x^r$ while, if the prediction Strategy 2 is adopted, we verify $\mathcal{P}\{b_r^T x_{t+1} \geq 1|x_{t-\tau}\} \leq p_x^r$, where $t-\tau$ is the most recent past time step when the reset strategy has been adopted. Finally note that our approach allows to enforce $\mathcal{P}\{b_r^T x_t \geq 1|x_0\} \leq p_x^r$, i.e., the fulfillment of the "non-conditional" expectation constraint, by disregarding, at each time step (even if feasible and optimal), the reset strategy, i.e., by setting $(\bar{x}_t, X_t) = (\bar{x}_{t|t-1}, X_{t|t-1})$ for all $t \geq 0$. This would lead to a more straightforward characterization of the probabilistic properties of the proposed approach, but at the price of a reduced optimality of the results. Importantly, the latter choice would not compromise the recursive feasibility and convergence results provided by Theorem 1.

Remark 2. To increase the initial feasibility region, the state constraints (6a) can be enforced also for a different horizon, i.e., for $i = t+1, \ldots, t+N-1$. All guaranteed results can be obtained also in this case.

4. Numerical issues and simplified formulations

In this section the design and implementation issues of the proposed control scheme are discussed.

4.1. Offline design

The main design problem consists in the computation of matrices $\bar{K}$, P, and $\bar{X}$ such that, at the same time, Eqs. (8) and (12) are verified.

4.1.1. Solution with LMI's

Both (8) and (12) can be solved by defining suitable LMI's.

- Concerning (8), if we define $\Pi = P^{-1}$ and $S = \bar{K}P^{-1}$, we can cast the corresponding inequality $P - \{(A+B\bar{K})^T P(A+B\bar{K}) + \sum_{j=1}^{q}(C_j+D_j\bar{K})^T P(C_j+D_j\bar{K}) + Q + \bar{K}^T R\bar{K}\} \geq 0$ as the following LMI

$$\begin{bmatrix} \Pi & [* \;\; * \;\; \dots \;\; * \;\; * \;\; *] \\ \begin{bmatrix} A\Pi + BS \\ C_1\Pi + D_1 S \\ \vdots \\ C_q\Pi + D_q S \\ \Pi \\ S \end{bmatrix} & \begin{bmatrix} \Pi & 0 & \dots & 0 & 0 & 0 \\ 0 & \Pi & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Pi & 0 & 0 \\ 0 & 0 & \dots & 0 & Q^{-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & R^{-1} \end{bmatrix} \end{bmatrix} \geq 0 \quad (16)$$

- regarding (12), it can be cast as an LMI by setting $W = \bar{X}$; note that the latter can be done if $\bar{X}$ satisfies $\bar{X} > \bar{w}I$, where $\bar{w}$ is defined in such a way that $\bar{w}I > \bar{x}\bar{x}^T$ for all $\bar{x} \in \bar{\mathbb{X}}_f$, since W is arbitrary. If we define $Y = \bar{K}\bar{X}$, we can cast the corresponding inequality $\bar{X} - \{(A+B\bar{K})\bar{X}(A+B\bar{K})^T + \sum_{j=1}^{q}(C_j+D_j\bar{K})\bar{X}(C_j+D_j\bar{K})^T + \bar{W}\} \geq 0$ as the following pair of LMI's

$$\begin{bmatrix} \bar{X} & [* \;\; * \;\; \dots \;\; * \;\; *] \\ \begin{bmatrix} (A\bar{X}+BY)^T \\ (C_1\bar{X}+D_1Y)^T \\ \vdots \\ (C_q\bar{X}+D_qY)^T \end{bmatrix} & \begin{bmatrix} \bar{X} & 0 & \dots & 0 \\ 0 & \frac{1}{2}\bar{X} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2}\bar{X} \end{bmatrix} \end{bmatrix} \geq 0 \quad (17)$$

$$\begin{bmatrix} \bar{X} & I \\ I & \frac{1}{\bar{w}}I \end{bmatrix} \geq 0.$$

A problem arises when a unique LMI problem is set, aiming to compute Π, $\bar{X}$, and $\bar{K}$ satisfying both (16) and (17) at the same time with the additional constraint that

$$S\Pi^{-1} = Y\bar{X}^{-1}. \quad (18)$$

In fact, (18) cannot be cast as an LMI and would destroy the convexity of the resulting problem. A solution (although conservative) to this issue consists in setting $\Pi = \bar{X}$ and $S = Y$ by just replacing Π and S in (16) with $\bar{X}$ and Y.

4.1.2. Solution using small gain arguments

An alternative solution, for the computation of matrices P and $\bar{X}$ satisfying (8) and (12), respectively, can be found thanks to the following result.

Proposition 1. *If* (a) *there exists a matrix $\bar{K}$ such that*

$$\frac{\mu^2}{1-\lambda^2}\sum_{j=1}^{q}\|C_j + D_j\bar{K}\|^2 < 1 \quad (19)$$

where $\mu > 0$ and $\lambda \in [0, 1)$ are the positive scalars defined in such a way that $\|(A+B\bar{K})^k\| \leq \mu\lambda^k$, (b) *we set $P(0) = \bar{X}(0) = 0$, and* (c) *we define $P(i)$, $\bar{X}(i)$ according to the following iterative equations*

$$P(i+1) = (A+B\bar{K})^T P(i)(A+B\bar{K}) + \sum_{j=1}^{q}(C_j+D_j\bar{K})^T P(i)(C_j+D_j\bar{K}) + Q + \bar{K}^T R\bar{K} \quad (20a)$$

$$\bar{X}(i+1) = (A+B\bar{K})\bar{X}(i)(A+B\bar{K})^T + \sum_{j=1}^{q}(C_j+D_j\bar{K})\bar{X}(i)(C_j+D_j\bar{K})^T + \bar{W} \quad (20b)$$

then $P(i) \to P$, $\bar{X}(i) \to \bar{X}$ as $i \to \infty$, where P and $\bar{X}$ verify (8) and (12), respectively.

Proposition 1 provides a constructive method for designing matrices P and $\bar{X}$, provided that a suitable gain has been computed, satisfying (19). The problem of computing $\bar{K}$ could be addressed by solving the following nonlinear optimization problem: $\min_{\bar{K}} \gamma$, subject to (i) $\rho(A+B\bar{K}) < 1$, (ii) $\frac{\mu^2}{1-\lambda^2}\sum_{j=1}^{q}\|C_j + D_j\bar{K}\|^2 \leq \gamma$, (iii) $\gamma < 1$, where $\rho(A+B\bar{K})$ is the spectral radius of $A+B\bar{K}$.

4.2. Online implementation

In this section we discuss how to enhance the numerical feasibility of the online implementation of the algorithm, focusing in particular on the optimization problem (15). Similarly to Primbs and Sung (2009), it is possible to show that the update Eq. (5) can be reformulated as a suitable LMI. In fact, defining $Y_k = K_k X_k$ we can write

$$\begin{bmatrix} X_{k+1} & [* \;\; * \;\; \dots \;\; * \;\; * \;\; \dots \;\; *] \\ \begin{bmatrix} (AX_k+BY_k)^T \\ (C_1X_k+D_1Y_k)^T \\ \vdots \\ (C_qX_k+D_qY_k)^T \\ (C_1\bar{x}_k+D_1\bar{u}_k)^T \\ \vdots \\ (C_q\bar{x}_k+D_q\bar{u}_k)^T \end{bmatrix} & \begin{bmatrix} X_k & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & X_k & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & I \end{bmatrix} \end{bmatrix} \geq 0. \quad (21)$$

The constraints (6) are non linear in the optimization variables U_k and X_k, compromising the simplicity of the optimization problem to be solved online. Since we are interested in the computational advantages of the linear constraints, similarly to Farina et al. (2013) linear approximations of (6) can be used. More specifically, the inequalities in (6) are verified if the following are satisfied, for $r = 1, \dots, n_r$, $s = 1, \dots, n_s$.

$$b_r^T\bar{x}_k \leq (1 - 0.5\varepsilon) - \frac{1}{2\varepsilon}b_r^T X_k b_r f(p_x^r)^2 \quad (22a)$$

$$c_s^T\bar{u}_k \leq (1 - 0.5\varepsilon) - \frac{1}{2\varepsilon}c_s^T U_k c_s f(p_u^s)^2 \quad (22b)$$

where $\varepsilon \in (0, 1]$. Therefore, linearity is preserved if constraints (6) are replaced by (22). This implies that also the conditions (13) must be replaced by the following ones.

$$b_r^T \bar{x} \leq (1 - 0.5\varepsilon) - \frac{1}{2\varepsilon} b_r^T \bar{X} b_r f(p_x^r)^2 \tag{23a}$$

$$c_s^T \bar{K}\bar{x} \leq (1 - 0.5\varepsilon) - \frac{1}{2\varepsilon} c_s^T \bar{K}\bar{X}\bar{K}^T c_s f(p_u^s)^2. \tag{23b}$$

Finally, note that, similarly to Farina et al. (2015), the problem can be simplified, from the numerical side, by using constant control gain $K_k = \bar{K}$ for all k. This does not compromise the recursive feasibility and the convergence properties of the control scheme, but significantly reduces the number of degrees of freedom of the control scheme, and consequently the corresponding computational load.

5. Dealing with expectation constraints

In Primbs and Sung (2009) expectation constraints are used instead of probabilistic ones. In line with this, in this paper we consider linear expectation constraints of the type

$$\mathbb{E}\{b_r^T x_k\} \leq 1 \quad r = 1, \ldots, n_r \tag{24a}$$

$$\mathbb{E}\{c_s^T u_k\} \leq 1 \quad s = 1, \ldots, n_s. \tag{24b}$$

Recalling that $\mathbb{E}\{x_k\} = \bar{x}_k$, $\mathbb{E}\{u_k\} = \bar{u}_k$, we rewrite (24) as

$$b_r^T \bar{x}_k \leq 1 \quad r = 1, \ldots, n_r \tag{25a}$$

$$c_s^T \bar{u}_k \leq 1 \quad s = 1, \ldots, n_s. \tag{25b}$$

From the computational perspective, the advantages of (25) with respect to the chance constraints are manifold. For example, (25) is linear with respect to the optimization variables, and there is no need of a further linearization procedure as in (23) to cast the overall constraints as LMI's. Secondly, since the variances X_k and U_k do not appear in (25), for recursive feasibility we do not need to define a terminal region for X_{t+N}. This leads to a less conservative optimization problem, makes the terminal region $\bar{\mathbb{X}}_f$ wider and, even more interestingly, makes it unnecessary to compute $\bar{X}$ in (12). In view of this the offline design phase can be performed in a rather standard form, requiring just to compute the solution to the LMI (16) and allowing to discard the inequalities (17). Overall, the Stochastic MPC problem in this case (denoted avS-MPC) corresponds to

$$\min_{\bar{u}_{t \ldots t+N-1}, K_{t \ldots t+N-1}, (\bar{x}_t, X_t)} J \tag{26}$$

subject to the dynamics (4) and (5), to the constraints (25) for all $k = t, \ldots, t+N-1$, to the initial constraint (14), and to the terminal constraint (9). The set $\bar{\mathbb{X}}_f$ is a positively invariant set for the system (4), with the control law $\bar{u}_k = \bar{K}\bar{x}_k$, such that for all $\bar{x} \in \bar{\mathbb{X}}_f$, $b_r^T \bar{x} \leq 1$, $c_s^T \bar{K}\bar{x} \leq 1$, for all $r = 1, \ldots, n_r$, $s = 1, \ldots, n_s$. Recursive feasibility and convergence of the resulting control system are stated similarly to Theorem 1.

Corollary 1. *If, at time $t = 0$, the avS-MPC problem admits a solution, the optimization problem (26) is recursively feasible and $\mathbb{E}\{\|x_t\|_Q^2\} \to 0$ as $t \to +\infty$. Furthermore, the state and input expectation constraints (24) are verified for all $t \geq 0$.*

Note that similar considerations to those reported in Remark 1 apply when dealing with expectation constraints of type (24).

6. Example

We consider the example illustrated in Primbs and Sung (2009). More specifically, the system (1) is characterized by $A = \begin{bmatrix} 1.02 & -0.1 \\ 0.1 & 0.98 \end{bmatrix}$, $B = \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.01 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0.04 & 0 \\ -0.04 & 0.008 \end{bmatrix}$. The disturbance is $w_t^1 \sim \mathcal{N}(0, 1)$. The weighting matrices are $Q = \text{diag}(2, 1)$ and $R = \text{diag}(5, 20)$. We consider a single constraint on the state variables with $b^T = \begin{bmatrix} -2 & 1 \end{bmatrix}/2.3$. In the remainder of the section two cases are considered, i.e. probabilistic constraints of type (2a) with $p = 0.1$, and expectation constraints of type (24a).

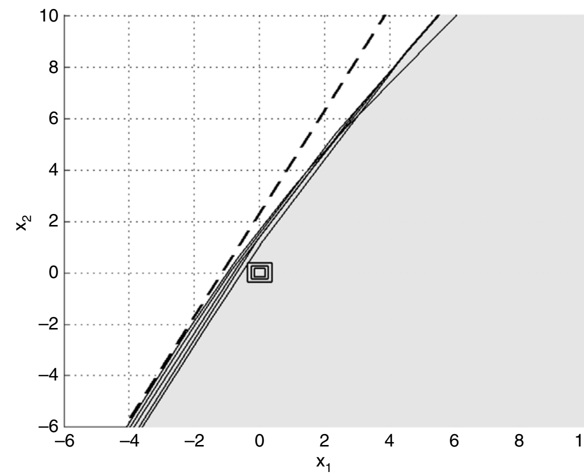

Fig. 1. Feasibility and terminal sets (delimited with solid lines) obtained for various values of parameter $\varepsilon \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. The dashed line denotes the constraint $b^T x \leq 1$.

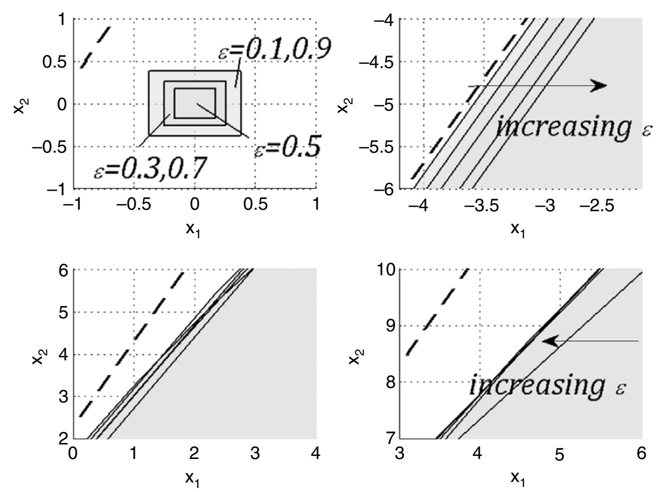

Fig. 2. Feasibility and terminal sets (delimited with solid lines) obtained for various values of the parameter $\varepsilon \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Zooms of four different areas of Fig. 1. The dashed line denotes the constraint $b^T x \leq 1$.

6.1. Feasibility sets in case of probabilistic constraint

The first set of simulation tests (see Figs. 1 and 2) is devoted to evaluating the size of the feasibility sets resulting from the application of the described control scheme, with different values of the arbitrary parameter ε in (22a). Indeed, we consider $\varepsilon \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. As illustrated in Figs. 1 and 2, we cannot derive a clear rule on how to choose the parameter ε in order to enlarge the feasibility set, as it is apparent from the comparison of the top-right and of the bottom-right panels of Fig. 2. Apparently, an acceptable compromise is to take values of ε in the range $[0.3, 0.7]$. It is worth mentioning that, in the case of expectation constraints the feasibility region corresponds to the set of initial conditions such that $b^T x_0 \leq 1$.

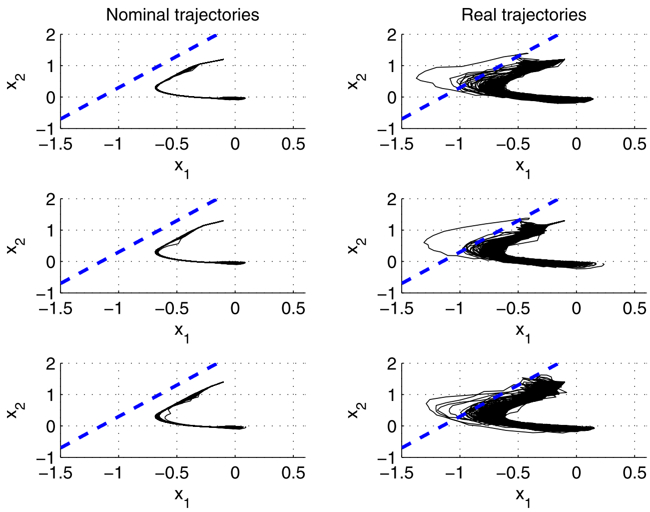

Fig. 3. Probabilistic constraints: results of the Monte Carlo simulation campaign, consisting of 900 runs for each initial condition, with $t \in [0, 40]$ and different noise realizations. Left panels: nominal trajectories $\bar{x}_t$; right panels: real trajectories x_t. The blue dashed line denotes the constraint $b^T x \leq 1$. Initial conditions: (i) $x_0 = [-0.1, 1.2]^T$ (upper panels); (ii) $x_0 = [-0.1, 1.3]^T$ (middle panels); (iii) $x_0 = [-0.1, 1.4]^T$ (lower panels).

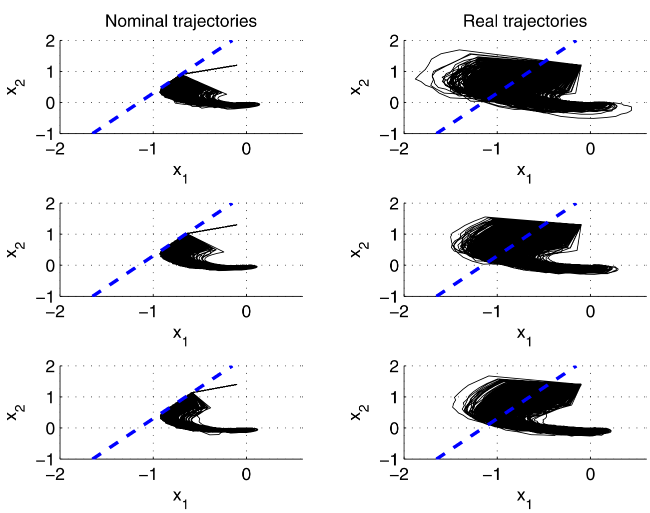

Fig. 4. Expectation constraints: results of the Monte Carlo simulation campaign, consisting of 900 runs for each initial condition, with $t \in [0, 40]$ and different noise realizations. Left panels: nominal trajectories $\bar{x}_t$; right panels: real trajectories x_t. The blue dashed line denotes the constraint $b^T x \leq 1$. Initial conditions: (i) $x_0 = [-0.1, 1.2]^T$ (upper panels); (ii) $x_0 = [-0.1, 1.3]^T$ (middle panels); (iii) $x_0 = [-0.1, 1.4]^T$ (lower panels).

6.2. *Trajectories for probabilistic and expectation constraints*

In this section we show the results of a number of Monte Carlo simulation campaigns, each consisting of 900 runs of the closed-loop controlled system. We want to test the proposed control scheme where we use both probabilistic constraints of type (22a) with $p = 0.1$ and $\varepsilon = 0.5$ and expectation constraints of type (25a). For comparison with Primbs and Sung (2009), we consider different initial conditions, feasible for both approaches, i.e., (i) $x_0 = \begin{bmatrix} -0.1 & 1.2 \end{bmatrix}^T$; (ii) $x_0 = \begin{bmatrix} -0.1 & 1.3 \end{bmatrix}^T$; (iii) $x_0 = \begin{bmatrix} -0.1 & 1.4 \end{bmatrix}^T$. In particular, Figs. 3 and 4 show the behavior of the nominal trajectories $\bar{x}_t$ and of the real ones x_t for all simulated realizations in case of probabilistic and expectation constraints, respectively.

A final analysis has been devoted to the comparison of the different approaches stemming from different initialization

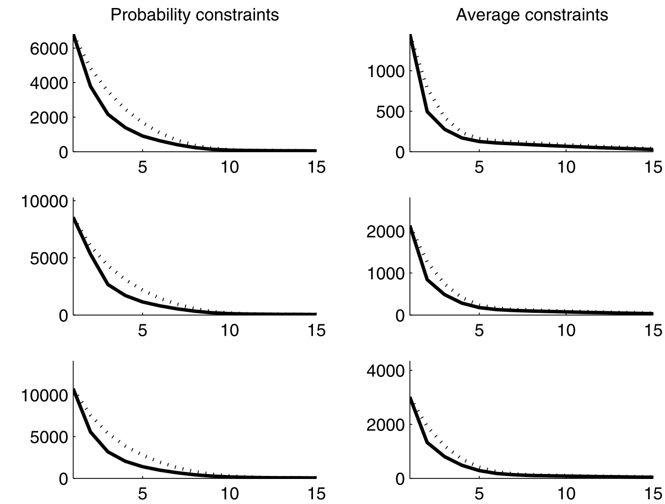

Fig. 5. Average cost function over 900 Monte Carlo realizations for each initial condition, with $t \in [0, 15]$. The left panels are relative to the case when probabilistic constraints are enforced, see Fig. 3, while the right panels are relative to the case when expectation constraints are enforced, see Fig. 4. Solid lines: optimal cost; dashed lines: cost in case the initialization $\bar{x}_{t|t} = \bar{x}_{t|t-1}$ is adopted at each time instant, $t > 1$. Initial conditions: (i) $x_0 = [-0.1, 1.2]^T$ (upper panels); (ii) $x_0 = [-0.1, 1.3]^T$ (middle panels); (iii) $x_0 = [-0.1, 1.4]^T$ (lower panels).

strategies. More specifically, we have carried out Monte Carlo simulation tests in three different cases, i.e., (i) in case (14) is imposed; (ii) in case it is set $\bar{x}_{t|t} = \bar{x}_{t|t-1}$ for all $t > 1$; (iii) in case it is set $\bar{x}_{t|t} = x_t$ for all $t \geq 1$. Subsequent analysis for strategy (iii) is not carried out in view of the fact that recursive feasibility is not guaranteed in this case and feasibility is lost for 70%–90% of realizations (depending on the initial condition) in case of average constraints and for 99.5% of realizations in case of probabilistic constraints. For strategies (i) and (ii), Fig. 5 shows the performance improvements in case (14) is used.

7. Conclusions

This paper describes a stochastic MPC algorithm for linear systems with possibly unbounded multiplicative uncertainty. The design method guarantees recursive feasibility, pointwise convergence of the state, and is characterized by a reduced on-line computational load, which results similar to the one required by nominal MPC algorithms. Possible extensions regard the output feedback case, extensive testing in significant control problems, and its distributed implementation.

Appendix A. Proof of Theorem 1

Assume that, at time instant t, a feasible solution of S-MPC is available, i.e., $(\bar{x}_{t|t}, X_{t|t})$ with optimal sequences $\bar{u}_{t\ldots t+N-1|t}$, $K_{t\ldots t+N-1|t}$. We prove that, at time $t+1$, a feasible solution to S-MPC exists, i.e., $(\bar{x}_{t+1|t}, X_{t+1|t})$ with admissible sequences $\bar{u}^f_{t+1\ldots t+N|t} = \{\bar{u}_{t+1|t}, \ldots, \bar{u}_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}\}$, $K^f_{t+1\ldots t+N|t} = \{K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}\}$.

Constraint (6a) is verified for all pairs $(\bar{x}_{t+1+k|t}, X_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of S-MPC at time t. Furthermore, in view of (9), (10), and the condition (13a), we have that $b^T\bar{x}_{t+N|t} \leq 1 - \sqrt{b_r^T \bar{X} b_r f(p_x^r)} \leq 1 - \sqrt{b_r^T \bar{X}_{t+N|t} b_r f(p_x^r)}$, i.e., constraint (6a) is verified.

Analogously, constraint (6b) is verified for all pairs $(\bar{u}_{t+1+k|t}, U_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of S-MPC at time t. Furthermore, in view of (9), (10), and the condition

(13b), we have that $c^T\bar{K}\bar{x}_{t+N|t} \leq 1-\sqrt{b_r^T\bar{K}\bar{X}\bar{K}^Tb_rf(p_x^r)} \leq 1-\sqrt{b_r^TU_{t+N|t}b_rf(p_x^r)}$, i.e., constraint (6b) is verified.

In view of (9) and of the invariance property (11) it follows that $\bar{x}_{t+N+1|t} = (A+B\bar{K})\bar{x}_{t+N|t} \in \bar{\mathbb{X}}_f$ and, in view of (10) $X_{t+N+1|t} = (A+B\bar{K})X_{t+N|t}(A+B\bar{K})^T + \sum_{j=1}^q(C_j+D_j\bar{K})X_{t+N|t}(C_j+D_j\bar{K})^T + \sum_{j=1}^q(C_j+D_j\bar{K})\bar{x}_{t+N|t}\bar{x}_{t+N|t}^T(C_j+D_j\bar{K})^T \leq (A+B\bar{K})\bar{X}(A+B\bar{K})^T + \sum_{j=1}^q(C_j+D_j\bar{K})\bar{X}(C_j+D_j\bar{K})^T + \sum_{j=1}^q(C_j+D_j\bar{K})W(C_j+D_j\bar{K})^T = \bar{X}$, hence verifying both (9) and (10) at time $t+1$.

In view of the feasibility, at time $t+1$ of the possibly suboptimal solution $\bar{u}^f_{t+1\dots t+N|t}$, $K^f_{t+1\dots t+N|t}$, and $(\bar{x}_{t+1|t}, X_{t+1|t})$, we have that the optimal cost function computed at time $t+1$ is $J^*(t+1) = J^*_m(t+1)+J^*_v(t+1)$. In view of the optimality of $J^*(t+1)$

$$J^*(t+1) \leq J_m(t+1|t)+J_v(t+1|t) \tag{A.1}$$

where $J_m(t+1|t) = J^*_m(t) - \|\bar{x}_{t|t}\|_Q^2 - \|\bar{u}_{t|t}\|_R^2 + \|\bar{x}_{t+N|t}\|_Q^2 + \|\bar{K}\bar{x}_{t+N|t}\|_R^2 - \|\bar{x}_{t+N|t}\|_P^2 + \|(A+B\bar{K})\bar{x}_{t+N|t}\|_P^2$ and

$$\begin{aligned} J_v(t+1|t) = {} & J^*_v(t) - tr\{(Q+K_{t|t}^TRK_{t|t})X_{t|t}\} \\ & + tr\{(Q+\bar{K}^TR\bar{K})X_{t+N|t} \\ & - PX_{t+N|t} + P(A+B\bar{K})X_{t+N|t}(A+B\bar{K})^T \\ & + P\sum_{j=1}^q(C_j+D_j\bar{K})X_{t+N|t}(C_j+D_j\bar{K})^T \\ & + P\sum_{j=1}^q(C_j+D_j\bar{K})\bar{x}_{t+N|t}\bar{x}_{t+N|t}^T(C_j+D_j\bar{K})^T\}. \end{aligned} \tag{A.2}$$

Considering (A.2) and recalling (8), $tr\{(Q+\bar{K}^TR\bar{K})X_{t+N|t} - PX_{t+N|t} + P(A+B\bar{K})X_{t+N|t}(A+B\bar{K})^T + P\sum_{j=1}^q(C_j+D_j\bar{K})X_{t+N|t}(C_j+D_j\bar{K})^T\} = tr\{((Q+\bar{K}^TR\bar{K}) - P + (A+B\bar{K})^TP(A+B\bar{K}) + \sum_{j=1}^q(C_j+D_j\bar{K})^TP(C_j+D_j\bar{K}))X_{t+N|t}\} = 0$. Also, the last line of (A.2) results in

$$\begin{aligned} & tr\left\{P\sum_{j=1}^q(C_j+D_j\bar{K})\bar{x}_{t+N|t}\bar{x}_{t+N|t}^T(C_j+D_j\bar{K})^T\right\} \\ & = \|\bar{x}_{t+N|t}\|^2_{\sum_{j=1}^q(C_j+D_j\bar{K})^TP(C_j+D_j\bar{K})}. \end{aligned} \tag{A.3}$$

From (A.1)–(A.3) and recalling (8) again, we obtain $J^*(t+1) \leq J^*(t) - \mathbb{E}\{\|x_t\|_Q^2 + \|u_t\|_R^2\} + \|\bar{x}_{t+N|t}\|^2_{\tilde{P}} \leq J^*(t) - \mathbb{E}\{\|x_t\|_Q^2\}$, since $\tilde{P} = (Q+\bar{K}^TR\bar{K}) - P + (A+B\bar{K})^TP(A+B\bar{K}) + \sum_{j=1}^q(C_j+D_j\bar{K})^TP(C_j+D_j\bar{K}) = 0$. Using standard arguments we conclude that $\mathbb{E}\{\|x_t\|_Q^2\} \to 0$ as $t \to +\infty$.

Appendix B. Proof of Proposition 1

Proof of Proposition 1. We focus, for brevity, only on the update Eq. (20a) although the same arguments can be applied to (20b). The proof is divided in three steps.

– *Boundedness*. The boundedness of $P(k)$, if computed by iterating (20a), can be proved along the lines of Dashkovskiy, Rüffer, and Wirth (2007). We rewrite (20a) as the following system of coupled equations

$$\begin{aligned} P(i+1) = {} & (A+B\bar{K})^TP(i)(A+B\bar{K}) \\ & + \sum_{j=1}^q(C_j+D_j\bar{K})^T\Delta(i)(C_j+D_j\bar{K}) + Q + \bar{K}^TR\bar{K} \end{aligned} \tag{B.1a}$$

$$\begin{aligned} \Delta(i+1) = {} & (A+B\bar{K})^T\Delta(i)(A+B\bar{K}) \\ & + \sum_{j=1}^q(C_j+D_j\bar{K})^TP(i)(C_j+D_j\bar{K}) + Q + \bar{K}^TR\bar{K} \end{aligned} \tag{B.1b}$$

with $P(0) = \Delta(0)$, from which we derive that $P(k) = ((A+B\bar{K})^T)^kP(0)(A+B\bar{K})^k + \sum_{i=1}^k((A+B\bar{K})^T)^{k-i}\{\sum_{j=1}^q(C_j+D_j\bar{K})^T\Delta(i-1)(C_j+D_j\bar{K}) + Q + \bar{K}^TR\bar{K}\}(A+B\bar{K})^{k-i}$, and that $\Delta(k) = ((A+B\bar{K})^T)^k\Delta(0)(A+B\bar{K})^k + \sum_{i=1}^k((A+B\bar{K})^T)^{k-i}\{\sum_{j=1}^q(C_j+D_j\bar{K})^TP(i-1)(C_j+D_j\bar{K}) + Q + \bar{K}^TR\bar{K}\}(A+B\bar{K})^{k-i}$. It follows that

$$\begin{aligned} \|P(k)\| \leq {} & \|(A+B\bar{K})^k\|^2\|P(0)\| + \sum_{i=1}^k\|(A+B\bar{K})^{k-i}\|^2 \\ & \times\left\{\sum_{j=1}^q\|C_j+D_j\bar{K}\|^2\max_{h\in[0,k]}\|\Delta(h)\| + \|Q+\bar{K}^TR\bar{K}\|\right\} \\ \leq {} & \|(A+B\bar{K})^k\|^2\|P(0)\| + \frac{\mu^2}{1-\lambda^2}\left\{\sum_{j=1}^q\|C_j+D_j\bar{K}\|^2\right. \\ & \left.\times\max_{h\in[0,k]}\|\Delta(h)\| + \|Q+\bar{K}^TR\bar{K}\|\right\} \end{aligned} \tag{B.2a}$$

$$\begin{aligned} \|\Delta(k)\| \leq {} & \|(A+B\bar{K})^k\|^2\|\Delta(0)\| + \frac{\mu^2}{1-\lambda^2}\left\{\sum_{j=1}^q\|C_j+D_j\bar{K}\|^2\right. \\ & \left.\times\max_{h\in[0,k]}\|P(h)\| + \|Q+\bar{K}^TR\bar{K}\|\right\}. \end{aligned} \tag{B.2b}$$

Denote $\delta_k = \max_{h\in[0,k]}\|\Delta(h)\|$ and $p_k = \max_{h\in[0,k]}\|P(h)\|$. From (B.2) we obtain that $p_k \leq \gamma\delta_k + q$, $\delta_k \leq \gamma p_k + q$, where $\gamma = \frac{\mu^2}{1-\lambda^2}\sum_{j=1}^q\|C_j+D_j\bar{K}\|^2$ and $q = \mu^2\|P(0)\| + \frac{\mu^2}{1-\lambda^2}\|Q+\bar{K}^TR\bar{K}\|$ since $P(0) = \Delta(0)$. We define $\Gamma = \begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$ and we write

$$(I-\Gamma)\begin{bmatrix} p_k \\ \delta_k \end{bmatrix} \leq \begin{bmatrix} q \\ q \end{bmatrix}.$$

According to Dashkovskiy et al. (2007), if the spectral radius of Γ is strictly smaller than one, i.e., if $\gamma < 1$, for every initial condition (see, e.g., Lemma 13 for the general nonlinear case), the solution to the system (B.1) exists and is uniformly bounded, since q does not depend on k.

– *Monotonicity*. Define $f(P) = (A+B\bar{K})^TP(i)(A+B\bar{K}) + \sum_{j=1}^q(C_j+D_j\bar{K})^TP(i)(C_j+D_j\bar{K}) + Q + \bar{K}^TR\bar{K}$. To obtain the proof of Proposition 1, we need to show that, if $P_A \geq P_B$ (where both matrices P_A and P_B are symmetric positive semi-definite), then $f(P_A) \geq f(P_B)$. This follows from the fact that $f(P_A) - f(P_B) = (A+B\bar{K})^T(P_A-P_B)(A+B\bar{K}) + \sum_{j=1}^q(C_j+D_j\bar{K})^T(P_A-P_B)(C_j+D_j\bar{K}) \geq 0$.

– *Convergence*. Consider now that $P(0) = 0$. Therefore $P(1) = f(P(0)) \geq 0 = P(0)$. In view of the monotonicity property, $f(P(1)) \geq f(P(0))$. Using recursion arguments, we obtain that, under the stated initialization, $P(k+1) \geq P(k)$ for all $k \geq 0$. In view of the boundedness property, $P(k) \to \bar{P}$ as $k \to \infty$, where $\bar{P}$ results to be the solution to the algebraic equation (8).

Appendix C. Proof of Corollary 1

Assume that, at time instant t, a feasible solution of avS-MPC is available, i.e., $(\bar{x}_{t|t}, X_{t|t})$ with optimal sequences $\bar{u}_{t\dots t+N-1|t}$, $K_{t\dots t+N-1|t}$. Similarly to the proof of Theorem 1, we can prove that, at time $t+1$, a feasible solution to avS-MPC exists, i.e., $(\bar{x}_{t+1|t}, X_{t+1|t})$ with admissible sequences $\bar{u}^f_{t+1\dots t+N|t} = \{\bar{u}_{t+1|t}, \dots, \bar{u}_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}\}$, $K^f_{t+1\dots t+N|t} = \{K_{t+1|t}, \dots, K_{t+N-1|t}, \bar{K}\}$. From this point on, the proof of convergence proceeds as in the proof of Theorem 1, showing that $\mathbb{E}\{\|x_t\|_Q^2\} \to 0$ as $t \to +\infty$.

References

Blackmore, L., Ono, M., Bektassov, A., & Williams, B. C. (2010). A probabilistic particle-control approximation of chance-constrained stochastic predictive control. *IEEE Transactions on Robotics*, *26*(3), 502–517.

Calafiore, G. C., & Fagiano, L. (2013). Robust model predictive control via scenario optimization. *IEEE Transactions on Automatic Control*, *58*(1), 219–224.

Cannon, M., Cheng, Q., Kouvaritakis, B., & Raković, S. V. (2012). Stochastic tube MPC with state estimation. *Automatica*, *48*(3), 536–541.

Cannon, M., Kouvaritakis, B., & Wu, X. (2009). Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints. *Automatica*, *45*(1), 167–172.

Dashkovskiy, S., Rüffer, B. S., & Wirth, F. R. (2007). An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, *19*, 93–122.

Evans, M., Cannon, M., & Kouvaritakis, B. (2012). Linear stochastic MPC under finitely supported multiplicative uncertainty. In *American control conference, ACC* (pp. 442–447).

Farina, M., Giulioni, L., Magni, L., & Scattolini, R. (2013). A probabilistic approach to model predictive control. In *IEEE 52nd annual conference on decision and control, CDC* (pp. 7734–7739).

Farina, M., Giulioni, L., Magni, L., & Scattolini, R. (2015). An approach to output-feedback MPC of stochastic linear discrete-time systems. *Automatica*, *55*, 140–149.

Geletu, A., Klöppel, M., Hoffmann, A., & Li, P. (2015). A tractable approximation of non-convex chance constrained optimization with non-Gaussian uncertainties. *Engineering Optimization*, *47*(4), 495–520.

Kolmanovsky, I., & Gilbert, E. G. (1998). Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, *4*(4), 317–367.

Marshall, A. W., & Olkin, I. (1960). Multivariate Chebychev inequalities. *Annals of Mathematical Statistics*, *34*(4), 1001–1014.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, *36*, 789–814.

Nemirovski, A., & Shapiro, A. (2007). Convex approximations of chance constrained programs. *SIAM Journal on Optimization*, *17*(4), 969–996.

Paulson, J. A., Streif, S., & Mesbah, A. (2015). Stability for receding-horizon stochastic model predictive control. In *American control conference, ACC, 2015* (pp. 937–943), July.

Primbs, J. A. (2007). Portfolio optimization applications of stochastic receding horizon control. In *American control conference, ACC* (pp. 1811–1816).

Primbs, J. A., & Sung, C. H. (2009). Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise. *IEEE Transactions on Automatic Control*, *54*(2), 221–230.

Shin, M., Lee, J. H., & Primbs, J. A. (2010). Constrained stochastic MPC under multiplicative noise for financial applications. In *49th IEEE conference on decision and control, CDC* (pp. 6101–6106).

Marcello Farina received the Laurea degree in Electronic Engineering in 2003 and the Ph.D. degree in Information Engineering in 2007, both from the Politecnico di Milano. In 2005 he was visiting student at the Institute for Systems Theory and Automatic Control, Stuttgart, Germany. He is presently Associate Professor at Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano. His research interests include distributed and decentralized state estimation and control, stochastic model predictive control, and applications, e.g., mobile robots, sensor networks, and energy supply systems.

Riccardo Scattolini was born in Milano, Italy, in 1956. He received the Laurea degree in electronic engineering from the Politecnico di Milano, Italy, in 1979. He is Professor of automatic control with the Politecnico di Milano. From 1984 to 1985, he was a Visiting Researcher at the Department of Engineering Science, Oxford University, U.K. He also spent one year working in industry on the simulation and control of chemical plants. His current research interests include modeling, identification, simulation and control of industrial plants and distribution networks, with emphasis on hierarchical, distributed, and stochastic model predictive control of large-scale systems. Prof. Scattolini received the Heaviside Premium from the Institution of Electrical Engineers, U.K., in 1991. He is currently Associate Editor of Automatica.