# LIOUVILLE THEOREMS AND 1-DIMENSIONAL SYMMETRY FOR SOLUTIONS OF AN ELLIPTIC SYSTEM MODELLING PHASE SEPARATION

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Abstract. We consider solutions of the competitive elliptic system

$$\text{(S)} \qquad \begin{cases} -\Delta u_i = -\sum_{j\neq i} u_i u_j^2 & \text{in } \mathbb{R}^N \\ u_i > 0 & \text{in } \mathbb{R}^N \end{cases} \qquad i = 1, \dots, k.$$

We are concerned with the classification of entire solutions, according with their growth rate. The prototype of our main results is the following: there exists a function $\delta = \delta(k, N) \in \mathbb{N}$, increasing in $k$, such that if $(u_1, \dots, u_k)$ is a solution of (S) and

$$u_1(x) + \cdots + u_k(x) \le C(1 + |x|^d) \qquad \text{for every } x \in \mathbb{R}^N,$$

then $d \ge \delta$. This means that the number of components $k$ of the solution imposes an increasing in $k$ minimal growth on the solution itself. If $N = 2$, the expression of $\delta$ is explicit and optimal, while in higher dimension it can be characterized in terms of an optimal partition problem. We discuss the sharpness of our results and, as a further step, for every $N \ge 2$ we can prove the 1-dimensional symmetry of the solutions of (S) satisfying suitable assumptions, extending known results which are available for $k = 2$. The proofs rest upon a blow-down analysis and on some monotonicity formulae.

## 1. Introduction

This paper concerns the classification of *positive* or *nonnegative* entire solutions with algebraic growth of the competitive elliptic system

$$-\Delta u_i = -\sum_{\substack{j=1 \\ j\neq i}}^{k} u_j^2 u_i \qquad \text{in } \mathbb{R}^N, \text{ for } i = 1, \dots, k, \tag{1}$$

with $k \ge 2$. Here and in the what follow, writing "positive solution" we mean that $u_i > 0$ in $\mathbb{R}^N$ for every $i$, while writing "nonnegative solution" we admit the possibility that some $u_i$ vanish identically, requiring however that at least two components are non-trivial. Note that, by the strong maximum principle, if $u_i \ge 0$ and $u_i \not\equiv 0$, then $u_i > 0$ in $\mathbb{R}^N$. The main result we aim at proving is the following Liouville-type theorem.

**Theorem 1.1.** *Let $N \ge 2$, and let $(u_1, \dots, u_k)$ be a positive solution of* (1) *having algebraic growth, that is, there exists $C, d > 0$ such that*

$$u_1(x) + \cdots + u_k(x) \le C(1 + |x|^d) \qquad \text{for every } x \in \mathbb{R}^N. \tag{2}$$

*Then*

$$d \ge \sqrt{\left(\frac{N-2}{2}\right)^2 + \mathcal{L}_k(\mathbb{S}^{N-1})} - \frac{N-2}{2},$$

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*where* $\mathcal{L}_k(\mathbb{S}^{N-1})$ *is the spectral minimal partition sequence of* $-\Delta_{\mathbb{S}^{N-1}}$ *over* $\mathbb{S}^{N-1}$*, introduced in* [11]*. Note that* $\mathcal{L}_k(\mathbb{S}^1) = k^2/4$ *when* $N = 2$*.*

In one direction, this means that if we consider a positive solution of (1) with a prescribed number $k$ of components, then we have a minimal admissible growth for the solution itself. As we will prove in Lemma 4.1, the minimal growth is strictly increasing in $k$. In the opposite direction, we deduce that a bound on the growth of a positive solution imposes a bound on the number of components $k$ of the solution itself. When $N \geq 3$, the exact value of $\mathcal{L}_k(\mathbb{S}^{N-1})$ is known for $k = 2$, and we will be able to solve it in the special case of $k = 3$ components, thus extending what has been proved in [12] for the two-sphere. As a consequence, we will prove the following.

**Corollary 1.2.** *Let* $N, k \geq 2$*, and let* $(u_1, \dots, u_k)$ *be a positive solution of* (1)*.*
*(i) If the solution has linear growth, that is there exists* $C > 0$ *such that*

$$u_1(x) + \dots + u_k(x) \leq C(1 + |x|) \qquad \text{for every } x \in \mathbb{R}^N, \tag{3}$$

*then* $k = 2$ *and the solution has growth rate* 1*.*
*(ii) If there exists* $C > 0$ *such that*

$$u_1(x) + \dots + u_k(x) \leq C(1 + |x|^{3/2}) \qquad \text{for every } x \in \mathbb{R}^N.$$

*Then either* $k = 2$ *and the solution has linear growth, or or* $k = 3$ *and the solution has growth rate* $3/2$*.*

Here and in the rest of the paper we write that $(u_1, \dots, u_k)$ *has growth rate* $d$ if

$$\lim_{r \to +\infty} \frac{\frac{1}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k u_i^2}{r^{2d'}} = \begin{cases} +\infty & \text{if } d' < d \\ 0 & \text{if } d' > d, \end{cases}$$

where $B_r$ denotes the ball of center 0 and radius $r$. We will observe that any solution of (1) has a growth rate.

As a further step, we address the proof of the validity of some De Giorgi-type conjectures for solutions of (1): under suitable assumptions, we show that a solution of (1) is necessarily 1-dimensional, namely up to a rotation it depends only on 1 variable. In what follows we write that $(u_1, \dots, u_k)$ *has algebraic growth* if it satisfies condition (2) for some $C > 0$ and $d > 1$. If the stronger condition (3) holds, we write that $(u_1, \dots, u_k)$ *has linear growth.*

**Theorem 1.3.** *Let* $N \geq 2$*, let* $(u_1, \dots, u_k)$ *be a nonnegative solution of* (1)*.*

- *(i) If* $(u_1, \dots, u_k)$ *has linear growth, then all the components but two, say* $u_1$ *and* $u_2$*, are identically zero, and* $(u_1, u_2)$ *is* 1*-dimensional.*
- *(ii) If* $(u_1, \dots, u_k)$ *has algebraic growth and for some* $i \neq j$

$$\lim_{x_N \to \pm\infty} (u_i(x', x_N) - u_j(x', x_N)) = \pm\infty,$$

*the limits being uniform in* $x' \in \mathbb{R}^{N-1}$*, then* $(u_1, \dots, u_k)$ *has linear growth, all the components* $u_l$ *with* $l \neq i, j$ *are identically zero, and* $(u_i, u_j)$ *is* 1*-dimensional.*

We postpone a more precise discussion of our main results after a brief review of what is known on existence and qualitative properties of solutions of (1). Such review has to be understood also as a motivation for our study.

The 2 components system

$$\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N \end{cases} \tag{4}$$

has been widely investigated in recent years. It appears in the analysis of phase separation phenomena in a binary mixture of Bose-Einstein condensates with multiple states; we refer to the papers [2] by H. Berestycki, T.-C. Lin, J. Wei and C. Zhao, [3] by H. Berestycki, K. Wang, J. Wei and the second author, and to the references therein for a detailed derivation of the phase separation model. In the quoted papers, the 1-dimensional case has been completely classified: up to translations, scaling and exchange of the components there is only one positive solution $(u, v)$, which has linear growth, has the symmetry property $u(x) = v(-x)$ for every $x \in \mathbb{R}$, and satisfies the monotonicity condition $u' > 0$, $v' < 0$ in $\mathbb{R}$. The linear growth is the least admissible growth rate for positive solutions to (4); indeed in any dimension $N \geq 1$, if $(u, v)$ is a nonnegative solution of (4) and satisfies the sublinear growth condition

$$u(x) + v(x) \leq C(1 + |x|^\alpha) \qquad \text{in } \mathbb{R}^N$$

for some $\alpha \in (0, 1)$ and $C > 0$, then one between $u$ and $v$ is 0, and the other has to be constant. This has been proved by B. Noris, H. Tavares, G. Verzini and the second author in [13], see Proposition 2.6, and together with its counterpart for systems with $k$ components, Proposition 2.7 in the same paper, is the only known example of Liouville-type theorem available for system (1).

The non-existence of positive solutions having sublinear growth, and the existence of a positive solution with linear growth, suggest an analogy between problem (4) and the Laplace equation. This point is made clear in [3], where for every integer $d \in \mathbb{N}$ the authors constructed a positive solution $(u_d, v_d)$ of (4) "modelled on" the homogeneous harmonic polynomial $\Psi_d = \mathfrak{Re}(z^d)$, in the sense that $(u_d, v_d)$ has growth rate $d$ (the same asymptotic growth of $\Psi_d$), and $(u_d, v_d)$ exhibits the symmetry of $(\Psi_d^+, \Psi_d^-)$; in this way the authors associated to any homogeneous harmonic polynomial of two variables a positive solution of (4). Also the converse can be done: to any positive solution to (4) having algebraic growth it is possible to associate a class of homogeneous harmonic polynomials, see the blow-down Theorem 1.4 in [3]. It is worth to point out that the dichotomy "positive solutions to (4)" – "harmonic function" is not an exclusive prerogative of solutions having algebraic growth, as revealed by the existence of solutions with exponential growth which are associated to exponential harmonic functions, for which we refer to the main results in [14] by A. Zilio and the first author.

Most of the quoted achievements admit a natural counterpart for the $k$ components system (1) with $k > 2$. In particular, for any $k > 2$ there exist infinitely many positive solutions having algebraic or exponential growth (see Theorem 1.6 in [3] and Theorem 1.8 in [14]), which are "modelled on" suitable harmonic functions. For this reason, the reader could be tempted to think that the qualitative description of the $k$ components system is essentially the same than that of the 2 component system. As we shall see, when $k > 2$ the picture is more involved. In what follows we restrict our attention to solutions having algebraic growth and, in order to better motivate our study, we report two aforementioned results in [3]. Concerning the notation, here and in the rest of the paper we denote by $B_r(x_0)$ the ball of centre $x_0$ and radius $r$ in $\mathbb{R}^N$, and write simply $B_r$ for $B_r(0)$; we use the complex notation $z = x + iy$ for points of $\mathbb{C} \simeq \mathbb{R}^2$, writing $\bar{z}$ for the complex conjugate of $z$, and we count the indexes $i = 1, \ldots, k, k+1, \ldots$ modulus $k$.

**Theorem 1.4** (Theorem 1.4 in [3]). *Let $N \geq 2$, $(u, v)$ be a positive solution of (1), and let us introduce*

$$(u_R(x), v_R(x)) := \left( \frac{1}{R^{N-1}} \int_{\partial B_R} u^2 + v^2 \right)^{-1/2} (u(Rx), v(Rx)).$$

*Let us assume that*

$$\lim_{r\to+\infty} \frac{r\int_{B_r} |\nabla u|^2 + |\nabla v|^2 + u^2v^2}{\int_{\partial B_r} u^2+v^2} =: d < +\infty. \tag{5}$$

*Then $d$ is a positive integer. Moreover, there exist a subsequence of the blow-down family $\{(u_R, v_R) : R > 0\}$, and a homogeneous harmonic polynomial $\Psi$ of degree $d$, such that $(u_R, v_R) \to (\Psi^+, \Psi^-)$ as $R \to +\infty$ in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$.*

**Theorem 1.5** (Theorem 1.6 in [3])**.** *Let $k \geq 2$ and $d \in \mathbb{N}/2$ such that $2d = hk$ for some $h \in \mathbb{N}$; let $G_{\pi/d}$ denote the rotation of angle $\pi/d$, with order $2d$. There exists a positive solution of system* (1) *in $\mathbb{R}^2$ such that*

$$(i)\quad u_i(z) = u_{i+1}(G_{\pi/d}z) \qquad \text{in } \mathbb{C},\ i = 1, \dots, k$$

$$(ii)\quad u_{k+i-1}(z) = u_i(\bar{z}) \qquad \text{in } \mathbb{C},\ i = 1, \dots, k$$

$$(iii)\quad \lim_{r\to+\infty} \frac{r\int_{B_r} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1\le i<j\le k} u_i^2u_j^2}{\int_{\partial B_r} \sum_{i=1}^k} = d$$

$$(iv)\quad \lim_{r\to+\infty} \frac{1}{r^{1+2d}} \int_{\partial B_r} \sum_{i=1}^k u_i^2 = b$$

*for some $b \in (0, +\infty)$.*

As previously stated, Theorem 1.4 allows us to associate to any positive solution $(u, v)$ of (4) a homogeneous harmonic polynomial. In particular, this implies a quantization of the admissible growth rates at infinity, see the limit (5) and the forthcoming Proposition 1.7. The very same quantization cannot be expected when $k > 2$: indeed, Theorem 1.5 provides solutions with half-integer asymptotic growth for every odd $k$, see point $(iv)$. More important, the presence of more than 2 components prevents the possibility that, if an asymptotic profile exists, has the simple structure $(\Psi^+, \Psi^-)$ for some homogeneous harmonic polynomial $\Psi$. In light of these remarks, an interesting problem is the description of the asymptotic profiles of the solutions of (1). As a further question, we observe that Theorem 1.5 ensures the existence of a positive solution $(u, v)$ to (1) with minimal growth rate $3/2$ when $k = 3$, 2 when $k = 4$, $5/2$ when $k = 5$, ...; we recall that writing "positive solution" we mean that $u_i > 0$ in $\mathbb{R}^N$ for every $i$. It is natural to wonder if these are really the minimal admissible growth rates or not. In the opposite direction, is it true that if a nonnegative solution of (1) has growth rate $d$, then there exists a maximal number of components depending on $d$ and on the dimension $N$ which cannot vanish identically? We recall that in this spirit the non-existence results for positive solutions having sublinear growth holds also when $k > 2$, see Proposition 2.7 in [13].

The aim of this paper is to answer the previous open problems and questions. Moreover, once that such topics are discussed, we will be able to extend some results of 1-dimensional symmetry of solutions in the present setting. The proof of the validity of some De Giorgi's-type conjectures for positive solutions of (4) has been object of an increasing attention in the last years. In dimension $N = 2$, A. Farina proved that if $(u, v)$ has algebraic growth and $\partial_2 u > 0$ in $\mathbb{R}^2$, then $(u, v)$ is 1-dimensional. This enhances a previous result in [2], where the 1-dimensional symmetry of $(u, v)$ was obtained under the linear growth assumption of $(u, v)$ plus the monotonicity condition $\partial_2 u > 0$ and $\partial_2 v < 0$ in $\mathbb{R}^2$. Always in dimension $N = 2$, in [3] it has been proved that if $(u, v)$ is a stable solution of (4) having linear growth, then it is 1-dimensional. Symmetry results in dimension $N = 2$ for systems having a more general form, under either monotonicity or stability assumptions, have been achieved by S.

Dipierro [7]. In the higher dimensional case $N \geq 2$, A. Farina and the first author proved in [9] that if $(u, v)$ has algebraic growth and

$$\lim_{x_N \to \pm\infty} (u(x', x_N) - v(x', x_N)) = \pm\infty,$$

the limit being uniform in $x' \in \mathbb{R}^{N-1}$, then $(u, v)$ depends only on $x_N$. This positively answer to a conjecture formulated in [2]. Furthermore, as product of the main results in [17, 18], K. Wang showed that if $(u, v)$ has linear growth (without other assumptions), then it is 1-dimensional.

As stated in Theorem 1.3, our aim is to extend the two last quoted achievements for solutions of the $k$ components system (1).

In what follows, we introduce convenient notations and state our main results in a precise form.

### 1.1. **Notation and further results.**

- We use the vector notation $\mathbf{u} := (u_1, \ldots, u_k)$.
- Let $A_1, A_2$ be disjoint open subsets of $\mathbb{R}^N$; we write that $A_1$ and $A_2$ are *adjacent* if $\partial A_i \cap \partial A_j$ has positive $(N-1)$-dimensional Hausdorff measure.
- For any continuous function $u$ in $\mathbb{R}^N$, the set $\{u > 0\}$ is called *positivity domain* of $u$, and its connected components are called *nodal domains*.
- For a vector valued function $\mathbf{u}$, we call *nodal set* or *zero level set* $\{\mathbf{u} = \mathbf{0}\}$.
- For any $A \subset \mathbb{R}^N$, we write $\chi_A$ for the characteristic function of $A$.
- For any $A \subset \mathbb{R}^N$, we write $\mathrm{Int}(A)$ for the interior of $A$.
- In the proof of our results we often write u.t.s. instead of "up to a subsequence".
- The notation $\mathcal{H}^m(\Omega)$ is used for the $m$-dimensional Hausdorff measure of $\Omega \subset \mathbb{R}^N$.
- For any $\omega \subset \partial B_1$, the first eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ with Dirichlet boundary condition on $\omega$ is denoted by $\lambda_1(\omega)$.
- We often write

$$f(0^+) := \lim_{r \to 0^+} f(r) \quad \text{and} \quad f(+\infty) = \lim_{r \to +\infty} f(r)$$

  if the limits exist.

In the paper we consider two classes of variational problems: regular ones of type

$$\begin{cases} -\Delta u_i = -\beta \sum_{j \neq i} u_j^2 u_i & \text{in } \mathbb{R}^N \\ u_i \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{6}$$

where $\beta > 0$, and segregated ones of type

$$\begin{cases} -\Delta v_i = 0 & \text{in } \{v_i > 0\} \\ v_i \geq 0 & \text{in } \mathbb{R}^N, \\ v_i v_j \equiv 0 & \text{in } \mathbb{R}^N \text{ for every } i \neq j. \end{cases} \tag{7}$$

We introduce suitable *Almgren frequency functions* according to whether we are considering (6) or (7). If $\mathbf{u}$ is a solution of (6), for $x_0 \in \mathbb{R}^N$ and $r > 0$ we define

(8)

- $$H(\mathbf{u}, x_0, r) := \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} \sum_{i=1}^k u_i^2$$
- $$E(\mathbf{u}, x_0, r) := \frac{1}{r^{N-2}} \int_{B_r(x_0)} \sum_{i=1}^k |\nabla u_i|^2 + \beta \sum_{1 \leq i < j \leq k} u_i^2 u_j^2$$
- $$N(\mathbf{u}, x_0, r) := \frac{E(\mathbf{u}, x_0, r)}{H(\mathbf{u}, x_0, r)} \qquad \text{(Almgren frequency function).}$$

If $\mathbf{v}$ is a solution of (7), for $x_0 \in \mathbb{R}^N$ and $r > 0$ we set

(9)
- $\tilde{E}(\mathbf{v}, x_0, r) := \dfrac{1}{r^{N-2}} \displaystyle\int_{B_r(x_0)} \sum_{i=1}^{k} |\nabla v_i|^2$
- $\tilde{N}(\mathbf{v}, x_0, r) := \dfrac{\tilde{E}(\mathbf{v}, x_0, r)}{H(\mathbf{v}, x_0, r)}$ (Almgren frequency function),

where $H(\mathbf{v}, x_0, r)$ is defined as in the first one of the (8).

Let $\mathbf{u}$ be a solution of (1) (or to (7)). We set, for $x_0 \in \mathbb{R}^N$ and $R > 0$,

$$\mathbf{u}_{x_0,R}(x) := \frac{\mathbf{u}(x_0 + Rx)}{H(\mathbf{u}, x_0, R)^{1/2}}.$$

We are interested in the asymptotic behaviour of the *blow-down family* $\{\mathbf{u}_{x_0,R}\}$ as $R \to +\infty$. We mainly consider the case $x_0 = 0$, writing simply $\mathbf{u}_R$ instead of $\mathbf{u}_{0,R}$ to simplify the notation.

The first of our main results is the extension of the blow-down Theorem 1.4 in the present setting.

**Theorem 1.6.** *Let $N, k \geq 2$, let $\mathbf{u}$ be a nonnegative solution of* (1)*, and let us assume that*

$$\lim_{r \to +\infty} N(\mathbf{u}, 0, r) =: d < +\infty.$$

*Then, up to a subsequence,*

$$\mathbf{u}_R \to \mathbf{u}_\infty = r^d (g_1(\theta), \ldots, g_k(\theta)) \quad \text{as } R \to +\infty$$

*in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$, where $(r, \theta) \in [0, +\infty) \times \mathbb{S}^{N-1}$ is a system of polar coordinates in $\mathbb{R}^N$ centred in $0$. Furthermore:*

- *the components $u_{i,\infty}$ are nonnegative and with disjoint support: $u_{i,\infty} u_{j,\infty} \equiv 0$ for every $i \neq j$;*
- *$\Delta u_{i,\infty} = 0$ in the positivity domain $\{u_{i,\infty} > 0\}$;*
- *if for some $i \neq j$ there exists two adjacent nodal domains $B_i \subset \{u_{i,\infty} > 0\}$ and $B_j \subset \{u_{j,\infty} > 0\}$, then $u_{i,\infty} - u_{j,\infty}$ is harmonic in $\mathrm{Int}(\overline{B_i \cup B_j})$;*
- *the set $\{\mathbf{u}_\infty = \mathbf{0}\} \cap \partial B_1$ has null $(N-1)$-dimensional measure;*
- *$H(\mathbf{u}, 0, R) R^2 \sum_{i<j} u_{i,R}^2 u_{j,R}^2 \to 0$ in $L^1_{loc}(\mathbb{R}^N)$.*

*Let now $N = 2$. Then, in addition, $d$ is a half-integer. Moreover, letting*

$$\Psi_d(r, \theta) := \frac{1}{\sqrt{\pi}} r^d \sin(d\theta),$$

*there exists a partition $(A_1, \ldots, A_k)$ of the positivity domain $\Sigma_{|\Psi_d|} = \{|\Psi_d| > 0\}$, where for every $i$*

*$A_i$ is the union of non-adjacent nodal domains of $\Sigma_{|\Psi_d|}$,*

*such that, up to a subsequence and up to a rotation, $\mathbf{u}_R \to (\chi_{A_1}, \ldots, \chi_{A_k})|\Psi_d|$ as $R \to +\infty$, in $\mathcal{C}^0_{loc}(\mathbb{R}^N)$ and in $H^1_{loc}(\mathbb{R}^N)$.*

**Remark 1.** 1) The same result holds for blow-down sequences centred at $x_0 \neq 0$.
2) By Proposition 5.2 in [3] (reported in Subsection 2.1), it follows that the limit $N(\mathbf{u}, 0, +\infty)$ always exists. Moreover, it is finite if and only if $\mathbf{u}$ has algebraic growth.

Theorem 1.4 with $N = 2$ is a particular case of Theorem 1.6; note that when $k$ is odd we have to take into account the possibility that the homogeneity degree of the limiting profile is a half-integer; this is coherent with Theorem 1.5. Let us also observe that, when $k$ is odd, $\Psi_d$ does not define a harmonic function in $\mathbb{R}^2$ in polar coordinates, since it is not $2\pi$-periodic in $\theta$; it can be seen as a harmonic function in the double covering $\{r \geq 0, 0 \leq \theta < 4\pi\}$.

The blow-down theorem will be the starting point in the derivation of the desired classification results. To this aim, we emphasize the relation between the growth rate of a solution and its Almgren frequency function.

**Proposition 1.7.** *Let* $\mathbf{u}$ *be a nonnegative solution of* (1) *having algebraic growth. Then* $d := N(\mathbf{u}, 0, +\infty) \in (0, +\infty)$ *is the growth rate of* $\mathbf{u}$.

This proposition implies that any solution of (1) having algebraic growth has a growth rate $d := N(\mathbf{u}, 0, +\infty)$. The strategy we shall adopt to prove our Liouville-type theorems rests on the idea that $d$ characterizes the asymptotic profile of the solution by means of Theorem 1.6: in particular, the value $d$ characterizes the maximal number of non-trivial components for a limiting profile, which hopefully should coincide with the maximal number of non-trivial components of the "original" solution. In this perspective, the main difficulty is represented by the lack of uniqueness of the asymptotic profile (the convergence in Theorem 1.6 takes place only up to a subsequence), and in general by the difficulty in deriving rigorous information on the "original" solution starting from the knowledge of the blow-down limit (we remind the interested reader to [9] and [17], where these problems are sources of tremendous complications). We can overcome these obstructions by means of the following intermediate result, which holds in any dimension.

**Proposition 1.8.** *Let* $N \geq 2$*, and let* $\mathbf{u}$ *be a nonnegative solution of* (1) *having algebraic growth. Let us assume that there exists a sequence* $R_n \to +\infty$ *as* $n \to \infty$*, such that* $u_{i,R_n} \to 0$ *in* $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ *for some* $i$*. Then* $u_i \equiv 0$ *in* $\mathbb{R}^N$.

Thanks to Proposition 1.8, we prove the Liouville-type Theorem 1.1 in dimension 2. In terms of the Almgren frequency function, it can be re-phrased as follows.

**Theorem 1.9.** *Let* $N = 2$*,* $k \geq 2$*, and let* $\mathbf{u} = (u_1, \dots, u_k)$ *be a nonnegative solution of* (1) *such that* $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$*. Then at most* $2d$ *components of* $\mathbf{u}$ *do not vanish identically.*
*Equivalently, let* $N = 2$*,* $k \geq 2$*, and let* $\mathbf{u} = (u_1, \dots, u_k)$ *be a positive solution of* (1) *such that* $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$*. Then* $d \geq k/2$.

In light of Theorem 1.5, the result gives a sharp estimate in dimension 2 on the minimal admissible growth rates for positive solutions of (1) with a given number of components. The higher dimensional case is more involved, reflecting the impossibility of deriving a complete description of the admissible limiting profile for solutions of (1), see Theorem 1.6. In connection to this, we point out that all the existence results available in the literature have been achieved in dimension 2 (thus leading to 2-dimensional solutions of (1) in any dimension $N \geq 2$); so far it is still unknown if true $N$-dimensional solutions of (1) with $N \geq 3$ exist and can exhibit different asymptotic behaviour with respect to the 2-dimensional case. Writing "true $N$-dimensional solutions" we refer to solutions in dimension $N$ which cannot be obtained by solutions in dimension $N - 1$ adding the dependence on 1 variable (up to a rotation). Nevertheless even in higher dimension not all is lost: by means of Theorem 1.6 and Proposition 1.8 we can relate the maximal number of nontrivial components of a solution of (1) having a prescribed growth with the solution of an optimal partition problem for the unitary sphere $\mathbb{S}^{N-1}$.

**Definition 1.** Let $1 \leq k \in \mathbb{N}$. A *k-partition* (or, simply, *partition*) of $\mathbb{S}^{N-1}$ is a family $\omega = (\omega_1, \dots, \omega_k)$ of mutually disjoint open and connected subsets $\omega_i \subset \mathbb{S}^{N-1}$. We denote the class of the $k$-partition of $\mathbb{S}^{N-1}$ as $\mathcal{P}_k(\mathbb{S}^{N-1})$.

We define

$$\mathcal{L}_k(\mathbb{S}^{N-1}) := \inf_{\omega \in \mathcal{P}_k(\mathbb{S}^{N-1})} \max_{i=1,\dots,k} \lambda_1(\omega_i) \tag{10}$$

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right). \tag{11}$$

Note that $\gamma$ is monotone increasing and is such that $\gamma(t) \to +\infty$ as $t \to +\infty$. The following is a more convenient statement of Theorem 1.1 holding in any dimension.

**Theorem 1.10.** *Let $N, k \geq 2$, and let $\mathbf{u} = (u_1, \dots, u_k)$ be a nonnegative solution of* (1) *such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. If $m$ is the maximal positive integer such that $\gamma(\mathcal{L}_m(\mathbb{S}^{N-1})) \leq d$, then at most $m$ components of $\mathbf{u}$ do not vanish identically.*
*Equivalently, let $N, k \geq 2$, and let $\mathbf{u}$ be a positive solution of* (1) *such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. Then $d \geq \gamma(\mathcal{L}_k(\mathbb{S}^{N-1}))$.*

This statement is the base point for the proof of Corollary 1.2. Again, we think that the following re-formulation is more suited to describe our result.

**Corollary 1.11.** *Let $N, k \geq 2$, and let $\mathbf{u} = (u_1, \dots, u_k)$ be a nonnegative solution of* (1) *such that $N(\mathbf{u}, 0, +\infty) =: d \in (0, +\infty)$. Then either $d = 1$ or $d \geq 3/2$. Furthermore:*

*(i) if $d = 1$, then $\mathbf{u}$ has exactly 2 non-trivial components;*
*(ii) if $d = 3/2$, then $\mathbf{u}$ has exactly 3 non-trivial components.*

The proof of point $(i)$ is obtained as a particular case of a more general result (see Theorem 1.13 and Proposition 2.3), while the second part and the jump in the admissible values of $N(\mathbf{u}, 0, +\infty)$ require a careful further analysis which can be carried on only for solutions of system (1). We emphasize that, in light of the known existence results for system (1) with $k = 2$ or $k = 3$, Corollary 1.11 is optimal in any dimension. Moreover, in proving point $(ii)$ we can determine the optimal value $\mathcal{L}_3(\mathbb{S}^{N-1})$ for every $N$, partially extending the main result in [12].

**Theorem 1.12.** *In any dimension $N \geq 3$, it results that*

$$\mathcal{L}_3(\mathbb{S}^{N-1}) = \frac{3}{2}\left(\frac{3}{2} + N - 2\right),$$

*and an optimal partition is the extension in dimension $N$ of the so-called $\mathbf{Y}$-partition of $\mathbb{S}^{N-1}$.*

**Remark 2.** For the definition of the $\mathbf{Y}$-partition, we refer to [12]. We point out that we do not prove the uniqueness of the generalized $\mathbf{Y}$-partition as a solution of $\mathcal{L}_3(\mathbb{S}^{N-1})$.

The relation between optimal partition problems and Liouville-type theorems has been already observed e.g. in [1, 4, 5, 13]. We refer in particular to Proposition 7.1 in [5], where the authors related the minimal growth of a positive solution of Lotka-Volterra type systems with the quantity

$$\beta(k, N) := \inf_{(\omega_1, \dots, \omega_k) \in \mathcal{P}_k(\mathbb{S}^{N-1})} \frac{2}{k} \sum_{i=1}^{k} \gamma\left(\lambda_1(\omega_i)\right) \tag{12}$$

We think that it is remarkable to observe that the very same approach leads to a more general result, involving *subsolutions* to a wide class of systems. Let

$$\begin{cases} -\Delta u_i \leq -u_i g_i(x, \mathbf{u}) & \text{in } \mathbb{R}^N \\ u_i \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad i = 1, \dots, k, \tag{13}$$

under the following assumptions on the nonlinear terms $g_i \in \mathcal{C}(\mathbb{R}^N \times [0, +\infty)^k)$:

$(H1)$ $g_i(x,\mathbf{t}) \geq \underline{g_i}(\mathbf{t}) \geq 0$ for every $(x,\mathbf{t}) \in \mathbb{R}^N \times [0,+\infty)^k$, where $\underline{g_i} \in \mathcal{C}([0,+\infty)^k)$;
$(H2)$ if $\underline{g}_i(\mathbf{t}) = 0$, then either $t_j = 0$ for every $j \neq i$, or $t_i = 0$;
$(H3)$ $g_i(x,\mathbf{t})$ is monotone non-decreasing in $t_j$ for every $j$.

As typical example, the reader may think at the case $g_i(x,\mathbf{t}) = \sum_{j\neq i} t_j^2$ defining system (1), but even to more general interaction terms (neither necessarily variational, nor symmetric) such as

$$g_i(x,\mathbf{t}) = \sum_{j\neq i} a_{ij}(x) t_j^{p_j} t_i^{q_j},$$

with $a_{ij}(x) \geq \underline{a}_{ij} > 0$ in $\mathbb{R}^N$ and $p_j > 0$, $q_j \geq 0$.

**Theorem 1.13.** *Let $N,k \geq 2$. Under assumptions (H1)-(H3), let $\mathbf{u} = (u_1,\ldots,u_k)$ satisfy*(13) *and*

$$u_1(x) + \cdots + u_k(x) \leq C(1+|x|^d) \qquad \text{for every } x \in \mathbb{R}^N \tag{14}$$

*for some $C > 0$ and $d \geq 1$. Let $m$ be the maximal positive integer such that $\beta(m,N) \leq 2d$. Then at most $m$ components of $\mathbf{u}$ do not vanish identically.*
*In other words, if $\mathbf{u} = (u_1,\ldots,u_k)$ is a positive solution of* (13) *satisfying* (14)*, then necessarily $\beta(k,N) \leq 2d$.*

System (1) fits in the assumptions of Theorem 1.13. It is then straightforward to obtain the first part of Corollary 1.11 as a particular case of a more general result.

**Corollary 1.14.** *Let $N,k \geq 2$. Let us assume that (H1)-(H3) are satisfied, and let $\mathbf{u} = (u_1,\ldots,u_k)$ satisfy* (13)*.*

*(i) If there exists $C > 0$ such that*

$$u_1(x) + \cdots + u_k(x) \leq C(1+|x|) \qquad \text{for every } x \in \mathbb{R}^N,$$

*then at most 2 components of $\mathbf{u}$ do not vanish identically.*

*(ii) If there exist $C > 0$ and $\alpha \in (0,1)$ such that*

$$u_1(x) + \cdots + u_k(x) \leq C(1+|x|^\alpha) \qquad \text{for every } x \in \mathbb{R}^N,$$

*then at most 1 component of $\mathbf{u}$ do not vanish identically.*

For the proof it is sufficient to recall that $\beta(k,N)$ is monotone non-decreasing in $k$, and such that $\beta(k,N) > \beta(2,N)$ whenever $k \geq 3$ (see the inequality (31) in [5]). Moreover, $\beta(2,N) = 2$ in any dimension $N$ (see [16]).

**Remark 3.** In [4] it has been proved that both $\beta(k,N)$ and $\mathcal{L}_k(\mathbb{S}^{N-1})$ are achieved. Here we used Theorem 1.13 instead of Theorem 1.10 to prove point $(i)$ in Corollary 1.11, but we point out that Theorem 1.10 is stronger in the particular case of system (1). Indeed it is well know that $\gamma(\mathcal{L}_k(\mathbb{S}^{N-1})) \geq \beta(k,N)/2$ for every $k,N$. Moreover, since the optimal value $\beta(2,N)$ is achieved by the equator-cut sphere (see [16]), $\gamma(\mathcal{L}_2(\mathbb{S}^{N-1})) = \beta(2,N)/2 = 1$ for every $N$, which implies directly point $(i)$ in Corollary 1.11.

The last part of the paper is devoted to the 1-dimensional symmetry of solutions of (1). The proof of Theorem 1.3 consists in showing that, under the assumptions of both points $(i)$ and $(ii)$, only two components of the solution can be non-trivial, and thus the solution is 1-dimensional symmetry thanks to the results in [9, 17, 18]. If the solution has linear growth, the fact that $\mathbf{u}$ has at most two non-trivial components follows directly by Corollary 1.11; if $\mathbf{u}$ satisfies the assumption of point $(ii)$, we at first study the asymptotic profile of the blow-down sequences $\{\mathbf{u}_{R_n}\}$, proving that any blow-down limit has two non-trivial components; then, to recover the result for $\mathbf{u}$, we apply the crucial Proposition 1.8.

**Structure of the paper.** In Section 2 we collect some results which will often be employed in the rest of the paper. Section 3 is devoted to the proofs of Theorem 1.6 and of Proposition 1.8. The proofs of the Liouville-type Theorems 1.9 and 1.10, which concern system (1), together with that of Corollary 1.11 and of Theorem 1.12, are the object of Section 4. In Section 5 we consider the general system (13), proving Theorem 1.13. Finally, in Section 6 we address the problem of the 1-dimensional symmetry, proving Theorem 1.3.

## 2. Preliminaries

In what follows we recall some essentially known results which will be useful in the rest of the paper.

2.1. **Almgren monotonicity formulae.** Here we recall some properties of the Almgren frequency function associated to solution of (1), proving in particular Proposition 1.7.

**Proposition 2.1** (Proposition 5.2 in [3])**.** *Let $N \geq 2$, $x_0 \in \mathbb{R}^N$, and let $\mathbf{u}$ be a nonnegative solution to* (1)*. The Almgren frequency function $N(\mathbf{u}, x_0, r)$ is monotone non-decreasing in $r$.*

We infer the following doubling properties.

**Proposition 2.2** (Proposition 5.3 in [3])**.** *Let $\mathbf{u}$ be a nonnegative solution of* (1)*.*

*(i) For every $0 < r_0 \leq r_1 < r_2$ it results that*
$$\frac{H(\mathbf{u}, x_0, r_2)}{r_2^{2N(\mathbf{u},x_0,r_0)}} \geq \frac{H(\mathbf{u}, x_0, r_1)}{r_1^{2N(\mathbf{u},x_0,r_0)}}.$$
*(ii) Assume that $N(\mathbf{u}, x_0, r) \leq d$ for every $r > 0$. Then*
$$\frac{H(\mathbf{u}, x_0, r_2)}{r_2^{2d}} \leq e^d \frac{H(\mathbf{u}, x_0, r_1)}{r_1^{2d}}$$
*for every $0 < r_1 < r_2$.*

The doubling properties allow us to relate the Almgren frequency function with the growth rate of the associated solution, as stated in Proposition 1.7.

*Proof of Proposition 1.7.* If $d' > d$, then by Proposition 2.2-($ii$) we have
$$\frac{H(\mathbf{u}, 0, r)}{r^{2d'}} \leq C\frac{r^{2d}}{r^{2d'}} \to 0 \qquad \text{as } r \to +\infty.$$
If $d' < d$, by monotonicity there exists $\bar{r} > 0$ such that $N(\mathbf{u}, 0, \bar{r}) = d' + \varepsilon < d$ for some $\varepsilon > 0$. Then, by Proposition 2.2-($i$), for every $r > \bar{r}$
$$\frac{H(\mathbf{u}, 0, r)}{r^{2d'}} \geq C\frac{r^{2(d'+\varepsilon)}}{r^{2d'}} \to +\infty \qquad \text{as } r \to +\infty.$$

It is also possible to relate the Almgren quotient with a pointwise upper bound.

**Proposition 2.3.** *$N(\mathbf{u}, 0, r) \leq d$ for every $r > 0$ if and only if there exists $C, d > 0$ such that $\sum_i u_i(x) \leq C(1 + |x|^d)$ in $\mathbb{R}^N$.*

For the proof, see Lemma 2.1 in [8] and Corollary A.8 in [9].

Since the growth rate of a solution $\mathbf{u}$ of (1) coincides with the limit of the Almgren frequency function, it is natural to have the following result.

**Lemma 2.4.** *Let $\mathbf{u}$ be a nonnegative solution of* (1) *having algebraic growth.*
*Then $N(\mathbf{u}, x_0, +\infty)$ is constant as function of $x_0 \in \mathbb{R}^N$.*

*Proof.* Let $N(\mathbf{u},0,+\infty)=d<+\infty$, and let us assume by contradiction that there exists $x_0\neq 0$ such that $N(\mathbf{u},x_0,+\infty)=d'\neq d$. Firstly, $d'<+\infty$ since $\mathbf{u}$ has algebraic growth. Only to fix our minds, we suppose that $d'<d$, so that there exists $\varepsilon>0$ such that $d'+\varepsilon<d$. By Propositions 2.1 and 2.2 there exists $r_0>0$ such that $H(\mathbf{u},x_0,r)\leq Cr^{2d'}$ and $H(\mathbf{u},0,r)\geq Cr^{2(d'+\varepsilon)}$ for $r>r_0$. Therefore on one side

$$\int_{B_{r_1}(x_0)\setminus B_{r_0}(x_0)}\sum_i u_i^2=\int_{r_0}^{r_1}s^{N-1}H(\mathbf{u},x_0,s)\,\mathrm{d}s\leq Cr_1^{2d'+N}$$

for $r_1>r_0$, while on the other side

$$\int_{B_{r_2}(0)\setminus B_{r_0}(0)}\sum_i u_i^2=\int_{r_0}^{r_2}s^{N-1}H(\mathbf{u},0,s)\,\mathrm{d}s\geq Cr_2^{2(d'+\varepsilon)+N}$$

for $r_2>r_0$. As a consequence, for $r>r_0$, we have

$$\begin{aligned}Cr^{2(d'+\varepsilon)+N}&\leq\int_{B_r(0)\setminus B_{r_0}(0)}\sum_i u_i^2\\&\leq\int_{B_{r+|x_0|}(x_0)\setminus B_{r_0}(x_0)}\sum_i u_i^2+\int_{B_{r_0}(x_0)\setminus B_{r_0}(0)}\sum_i u_i^2\\&\leq C(r+|x_0|)^{2d'+N}+C\leq Cr^{2d'+N},\end{aligned}$$

which gives a contradiction for $r$ sufficiently large. □

2.2. **Segregated configurations.** In Definition 1.2 of [15], H. Tavares and the second author introduced the class of functions $\mathcal{G}(\Omega)$. We consider a subclass of particular interest in the present setting.

**Definition 2.** For an open set $\Omega\subset\mathbb{R}^N$, we define the class $\mathcal{G}^*(\Omega)$ of nontrivial functions $\mathbf{0}\neq\mathbf{v}=(v_1,\dots,v_k)$ whose components are nonnegative and locally Lipschitz continuous in $\Omega$, and such that the following properties holds:

- $v_iv_j\equiv 0$ in $\Omega$ for every $i\neq j$;
- for every $i$
$$-\Delta v_i=-\mu_i\qquad\text{in }\Omega\text{ in distributional sense,}$$
where $\mu_i$ is a nonnegative Radon measure supported on the set $\partial\{v_i>0\}$;
- defining for $x_0\in\Omega$ and $r>0$ such that $B_r(x_0)\subset\Omega$ the function $\tilde{E}(\mathbf{v},x_0,r)$ as in (9), we assume that $\tilde{E}$ is absolutely continuous as function of $r$ and
$$\frac{d}{dr}\tilde{E}(\mathbf{v},x_0,r)=\frac{1}{r^{N-2}}\int_{B_r(x_0)}\sum_{i=1}^k(\partial_\nu v_i)^2;$$

We write that $\mathbf{v}\in\mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ if $\mathbf{v}\in\mathcal{G}(B_R)$ for every $R>0$. We write that $\mathbf{v}\in\mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^N)$ if $\mathbf{v}\in\mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ and is homogeneous with respect to some $x_0\in\mathbb{R}^N$, in the sense that there exists $\gamma>0$ such that $\mathbf{v}(r,\theta)=r^\gamma\mathbf{g}(\theta)$, where $(r,\theta)$ is a system of polar coordinates in $\mathbb{R}^N$ centred in $x_0$.

It is possible to introduce an Almgren frequency function associated to any $\mathbf{v}\in\mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^N)$ as in (9), and to prove a monotonicity formula for it (see Theorem 2.2 and Remark 2.4 in [15]).

**Proposition 2.5.** *Let $\mathbf{v}\in\mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$. For $x_0\in\mathbb{R}^N$ and $r>0$, the function $\tilde{N}(\mathbf{v},x_0,r)$ is non-decreasing in $r$. Moreover, $\tilde{N}(\mathbf{v},x_0,r)=const.=\sigma>0$ if and only if $\mathbf{v}(r,\theta)=r^\sigma\mathbf{g}(\theta)$, where $(r,\theta)$ denotes a system of polar coordinates centred in $x_0$.*

As a consequence, it is possible to derive doubling properties similar to those of Proposition 2.2, and to recast the proof of Lemma 2.4 in the present setting, obtaining the following statement.

**Lemma 2.6.** *Let $\mathbf{v} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ having algebraic growth. Then $N(\mathbf{v}, x_0, +\infty)$ is constant as function of $x_0 \in \mathbb{R}^N$.*

We conclude this subsection with a definition.

**Definition 3.** Let $\mathbf{v} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$, and let $x_0 \in \{\mathbf{v} = \mathbf{0}\}$. We define the multiplicity of $x_0$ as

$$\#\{i = 1, \dots, k : \text{ for every } r > 0 \text{ it results } B_r(x_0) \cap \{v_i > 0\} \neq \emptyset\}.$$

2.3. **Decay estimates.** If $(u, v)$ solves (1) and $u$ is large in a ball $B_{2r}(x_0)$, then by comparison principles $v$ has to be exponentially small with respect to $u$ in a smaller ball.

**Lemma 2.7** (Lemma 4.4 in [5]). *Let $x_0 \in \mathbb{R}^N$ and $r > 0$. Let $u \in H^1(B_{2r}(x_0))$ be such that*

$$\begin{cases} -\Delta v \leq -Kv & \text{in } B_{2r}(x_0) \\ v \geq 0 & \text{in } B_{2r}(x_0) \\ v \leq A & \text{on } \partial B_{2r}(x_0), \end{cases}$$

*where $K$ and $A$ are two positive constants. Then there exists $C > 0$ depending only on the dimension $N$ such that*

$$\sup_{x \in B_r(x_0)} v(x) \leq CAe^{-CK^{1/2}r}.$$

## 3. Asymptotic behaviour of positive solution

In this section we prove Theorem 1.6 and Proposition 1.8.

3.1. **Blow-down limits.**

*Proof of Theorems 1.6.* The elements of the blow-down family satisfy

$$\begin{cases} -\Delta u_{i,R} = -\sum_{j \neq i} R^2 H(\mathbf{u}, 0, R) u_{j,R}^2 u_{i,R} & \text{in } \mathbb{R}^N \\ u_{i,R} \geq 0 & \text{in } \mathbb{R}^N \end{cases} \qquad i = 1, \dots, k,$$

so that are well defined functions $H(\mathbf{u}_R, 0, r)$, $E(\mathbf{u}_R, 0, r)$ and $N(\mathbf{u}_R, 0, r)$ as in (8). By direct computations it is easy to check that

$$H(\mathbf{u}_R, 0, r) = \frac{H(\mathbf{u}, 0, rR)}{H(\mathbf{u}, 0, R)}, \quad E(\mathbf{u}_R, 0, r) = \frac{E(\mathbf{u}, 0, rR)}{H(\mathbf{u}, 0, R)}, \quad N(\mathbf{u}_R, 0, r) = N(\mathbf{u}, 0, rR).$$

By the doubling property $(i)$ in Proposition 2.2 $H(\mathbf{u}, 0, R)R^2 \to +\infty$ as $R \to +\infty$. Furthermore, by definition $H(\mathbf{u}_R, 0, 1) = 1$ for every $R > 0$, and by the Almgren monotonicity formula $N_R(r) \leq d$ for every $r, R > 0$. As a consequence, the doubling property $(ii)$ in Proposition 2.2 implies that

$$H(\mathbf{u}_R, 0, r) \leq e^d r^{2d}$$

for every $R, r > 1$. Hence, by subharmonicity, $\{\mathbf{u}_R\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, and we are in position to apply the local version of the main results in [13] (for the local version, we refer to Theorem 2.6 in [17]): up to a subsequence $\mathbf{u}_R \to \mathbf{u}_\infty$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, where by Corollary 8.3 in [15] $\mathbf{u}_\infty \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$. In particular

- $u_{i,\infty} u_{j,\infty} \equiv 0$ in $\mathbb{R}^N$ for every $i \neq j$, and

$$\lim_{R \to +\infty} \int_{B_r} H(\mathbf{u}, 0, R) R^2 \sum_{i<j} u_{i,R}^2 u_{j,R}^2 = 0 \qquad \text{for every } r > 0;$$

- $u_{i,\infty}$ is subharmonic in $\mathbb{R}^N$, and $\Delta u_{i,\infty} = 0$ in $\{u_{i,\infty} > 0\}$, for every $i = 1, \dots, k$.

Let $\tilde{N}(\mathbf{u}_\infty, 0, r)$ be define in (9). For every $r > 0$

$$\tilde{N}(\mathbf{u}_\infty, 0, r) = \lim_{R\to+\infty} N(\mathbf{u}_R, 0, r) = \lim_{R\to+\infty} N(\mathbf{u}, 0, Rr) = d$$

so that as stated in Proposition 2.5 $\mathbf{u}_\infty(r,\theta) = r^d \mathbf{g}(\theta)$. Concerning the functions $g_i$, they have disjoint support and are such that $r^d g_i(\theta)$ is harmonic in $\{u_{i,\infty} > 0\}$. Let us note that if $\{u_{i,\infty} > 0\} \neq \emptyset$, then it is a cone, and by Theorem 1.1 in [15] the zero level set $\{\mathbf{u}_\infty = \mathbf{0}\}$ has null $N$-dimensional measure. By homogeneity, this implies that $\{\mathbf{u}_\infty = \mathbf{0}\} \cap \partial B_1$ has null $(N-1)$-dimensional measure. In what follows, we use the notation

$$\{u_{i,\infty} > 0\} = A_i = B_{1,i} \cup \cdots \cup B_{h_i,i},$$

denoting by $B_{l,i}$ the nodal domains of $u_{i,\infty}$. If $B_{i,l_i}$ and $B_{j,l_j}$ are adjacent, then the reflection law proved in Theorem 1.1 in [15] implies that $u_{i,\infty} - u_{j,\infty}$ is harmonic in $\mathrm{Int}(\overline{B_{i,l_i} \cup B_{j,l_j}})$; the main result in Section 10 of [6] rules out the existence of point of multiplicity 1; in other words, if there exist two non-empty connected components of some $A_i$ $(i = 1, \dots, k)$, then they have to be non-adjacent.

What we proved so far holds in any dimension $N \geq 2$. In what follows, we focus on the case $N = 2$. By Theorem 1.1 in [15] and by homogeneity, the nodal set $\{\mathbf{u}_\infty = \mathbf{0}\}$ is the union of straight lines passing through the origin and meeting with equal angles. For some $i \neq j$, let $B_{i,l_i}$ and $B_{j,l_j}$ be adjacent nodal domains of $u_{i,\infty}$ and $u_{j,\infty}$, respectively. Then, as already noticed, $u_{i,\infty} - u_{j,\infty}$ is harmonic in the cone $\mathrm{Int}(\overline{B_{i,l_i} \cup B_{j,l_j}}) = \{r > 0, \theta_0 < \theta < \theta_1\}$ (where $0 \leq \theta_0 < \theta_1 < 2\pi$); up to a rotation, it is not restrictive to assume that $\theta_0 = 0$, so that $w := g_i - g_j$ satisfies

$$\begin{cases} w'' + d^2 w = 0 & \text{in } (0, \theta_1) \\ w(0) = w(\theta_1/2) = w(\theta_1) = 0 \\ w > 0 \quad \text{in } (0, \theta_1/2), \quad w < 0 & \text{in } (\theta_1/2, \theta_1) \end{cases}$$

for some $\theta_1 \in (0, 2\pi)$. It is straightforward to deduce that for some $C > 0$ we have $w(\theta) = C\sin(d\theta)$ and $\theta_1 = 2\pi/d$. Iterating this line of reasoning for any pair of adjacent nodal domains $B_{i,l_i}$ and $B_{j,l_j}$, and recalling that the functions $g_i$ are segregated and nonnegative, we conclude that $d \in \mathbb{N}/2$, and there exists a unique $C > 0$ such that

$$(g_1(\theta), \dots, g_k(\theta)) = (\chi_{A_1}, \dots, \chi_{A_k}) C \sin(d\theta),$$

which is uniquely determined as $C = 1/\sqrt{\pi}$ by the normalization

$$\int_0^{2\pi} C^2 \sin^2(d\theta)\, \mathrm{d}\theta = 1.$$

This completes the proof. □

**Remark 4.** As observed, any blow-down limit of an arbitrary solution $\mathbf{u}$ of system (1) belongs to $\mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^N)$, see Definition 2. Therefore, the results proved in [15] hold for the blow-down limits. This will be used in Subsection 4.3.

3.2. **Proof of Proposition 1.8.** We aim at showing that if in the blow-down family one component $u_i$ vanishes along one sequence $R_n \to +\infty$, then it is identically zero. We reach this result through a series of lemmas.

We introduce the family $\mathcal{BD}_\mathbf{u}$ of the blow-down limits for a fixed nonnegative solution $\mathbf{u}$ of (1): $\mathbf{w} \in \mathcal{BD}_\mathbf{u}$ if there exists a sequence $R_n \to +\infty$ such that $\mathbf{u}_{R_n} \to \mathbf{w}$ as $n \to \infty$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

In what follows we derive some useful properties of $\mathcal{BD}_{\mathbf{u}}$. Here and in the rest of the section $d$ denotes the limit of the Almgren frequency function of the considered solution: $d := N(\mathbf{u}, 0, +\infty)$, which is finite since $\mathbf{u}$ has algebraic growth. In several cases we consider blow-down sequences $\mathbf{u}_{R_n}$ converging to some limiting profile $\mathbf{u}_\infty \in \mathcal{BD}_{\mathbf{u}}$. Clearly the convergence has to be understood in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, and we often omit this piece of information.

**Lemma 3.1.** *(i) The set $\mathcal{BD}_{\mathbf{u}}$ is locally uniformly bounded in the $1/2$-Hölder norm.*
*(ii) The set $\mathcal{BD}_{\mathbf{u}}$ is closed under locally uniform convergence.*
*(iii) For any $\mathbf{w} \in \mathcal{BD}_{\mathbf{u}}$ it results that*

$$\int_{\partial B_r} \sum_{i=1}^k w_i^2 \le e^d r^{2d+N-1} \qquad \text{for every } r > 1.$$

*Proof.* $(i)$ It is a consequence of the local version of the main results proved in [13], see Theorem 2.6 in [17].
$(ii)$ Let $\{\mathbf{w}_n\} \subset \mathcal{BD}_{\mathbf{u}}$ such that $\mathbf{w}_n \to \mathbf{w}$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$: given $r$ and $\varepsilon > 0$, there exists $\bar n \in \mathbb{N}$ such that

$$n > \bar n \quad \Longrightarrow \quad \sup_{i=1,\dots,k} \sup_{B_r} |w_{i,n} - w_i| < \frac{\varepsilon}{2}. \tag{15}$$

Since $\mathbf{w}_n \in \mathcal{BD}_{\mathbf{u}}$, there exists a sequence $R^n_m \to +\infty$ as $m \to \infty$ such that $\mathbf{u}_{R^n_m} \to \mathbf{w}_n$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $m \to \infty$. Therefore, there exists $\bar m(n) \in \mathbb{N}$ such that

$$m > \bar m(n) \quad \Longrightarrow \quad \sup_{i=1,\dots,k} \sup_{B_r} |u_{i,R^n_m} - w_{i,n}| < \frac{\varepsilon}{2}. \tag{16}$$

Now, for every $n$ let us choose $m_n > \bar m(n)$ so large that $R_n := R^n_{m_n}$ tends to $+\infty$ as $n \to \infty$. Considering the blow-down sequence $\{\mathbf{u}_{R_n}\}$, it is easy to check that it is locally uniformly convergent to $\mathbf{w}$: indeed given $r, \varepsilon > 0$, by (15) and (16) we obtain

$$\sup_{B_r} |u_{i,R_n} - w_i| \le \sup_{B_r} |u_{i,R_n} - w_{i,n}| + \sup_{B_r} |w_{i,n} - w_i| < \varepsilon$$

for every $i = 1, \dots, k$ and $n > \bar n$.
$(iii)$ It follows from the doubling property $(ii)$ in Proposition 2.2, which holds for any element of the blow-down family and is stable under uniform convergence. □

In the next lemma we show that, under the assumptions of Proposition 1.8, for the entire blow-down family the component $u_{i,R} \to 0$ as $R \to +\infty$.

**Lemma 3.2.** *Let us assume that there exists a sequence $R_n \to +\infty$ as $n \to \infty$, such that $u_{i,R_n} \to 0$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then for every sequence $R'_n \to +\infty$ it results that $u_{i,R'_n} \to 0$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$.*

*Proof.* We separate the proof in two steps.
Step 1). *There exists $\bar C > 0$ such that for any $\mathbf{w} \in \mathcal{BD}_{\mathbf{u}}$, for any $i = 1, \dots, k$ and for any (not empty) connected component $\omega_i$ of $\{w_i > 0\} \cap \partial B_1$ it results*

$$\int_{\omega_i} w_i^2 \ge \bar C.$$

Assume by contradiction that the claim is not true. Then there exist a sequence $\{\mathbf{w}_n\} \subset \mathcal{BD}_{\mathbf{u}}$, and a sequence of connected components $\omega_{i_n,n}$ of $\{w_{i_n} > 0\} \cap \partial B_1$, such that

$$\int_{\partial B_1} \chi_{\omega_{i_n,n}} \sum_{j=1}^k w_{j,n}^2 = \int_{\omega_{i_n,n}} w_{i_n}^2 \to 0 \qquad \text{as} \to \infty. \tag{17}$$

Up to a subsequence, we can assume that $i_n = i$. By the properties of the blow-down limits $\mathbf{w}_n(r,\theta) = r^d \mathbf{g}(\theta)$, where $g_{1,n}, \dots, g_{k,n}$ have disjoint supports. Moreover, since $w_{i,n}$ has to be harmonic in its positivity domain

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} g_{i,n} = d(d+N-2) g_{i,n} & \text{in } \omega_{i,n} \\ g_{i,n} > 0 & \text{in } \omega_{i,n} \\ g_{i,n} = 0 & \text{on } \partial\omega_{i,n}. \end{cases}$$

This reveals that $d(d+N-2)$ is the first eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}$ with Dirichlet boundary conditions on $\omega_{i,n}$, with corresponding eigenfunction $g_{i,n}$, and in particular there exists $C > 0$ such that

$$\mathcal{H}^{N-1}(\omega_{i,n}) \geq C \qquad \text{for every } n. \tag{18}$$

This fact follows from the known dependence on $\lambda_1(\omega)$ on the measure of $\omega$, and in particular by the property

$$\lambda_1(\omega) \to +\infty \quad \text{as} \quad \mathcal{H}^{N-1}(\omega) \to 0.$$

Now, point $(iii)$ of Lemma 3.1 implies that

$$\int_{\partial B_r} \sum_{j=1}^k w_{i,n}^2 \leq e^d r^{2d+N-1}$$

for every $r > 1$, for every $n$. By subharmonicity, $\{\mathbf{w}_n\}$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, and by point $(i)$ of Lemma 3.1 it is also equi-continuous. Thus, by the Ascoli-Arzelà theorem it is locally uniformly convergent u.t.s. to some $\mathbf{w}$, still belonging to $\mathcal{BD}_{\mathbf{u}}$ thanks to point $(ii)$ of Lemma 3.1. As a consequence, $\{\mathbf{w} = \mathbf{0}\} \cap \partial B_1$ has null $(N-1)$-dimensional measure, and

$$\sum_{j=1}^k w_{j,n}^2 \not\to 0 \qquad \text{a.e. in } \mathbb{S}^{N-1}. \tag{19}$$

On the other hand, by (17), up to a subsequence

$$\chi_{\omega_{i,n}} \sum_{j=1}^k w_{j,n}^2 \to 0 \qquad \text{a.e. in } \mathbb{S}^{N-1}. \tag{20}$$

A comparison between (19) and (20) implies that $\chi_{\omega_{i,n}} \to 0$ a.e. in $\mathbb{S}^{N-1}$, which by the dominated convergence theorem provides $\mathcal{H}^{N-1}(\omega_n) \to 0$ as $n \to \infty$, in contradiction with (18).

Step 2). *Conclusion of the proof.*

Assume by contradiction that there exist $R'_n \to +\infty$ such that $u_{i,R'_n} \not\to 0$ as $n \to \infty$. Arguing as in the proof of Theorem 1.6, u.t.s. $\mathbf{u}_{R'_n} \to \mathbf{w}' \in \mathcal{BD}_{\mathbf{u}}$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$. By assumption $w'_i \not\equiv 0$, so its support $A_i$ is not empty, and by the first step

$$\lim_{n\to\infty} \int_{\partial B_1} u_{i,R'_n}^2 \geq \frac{1}{2} \int_{\tilde{A}_i} w_i^2 \geq \frac{\bar{C}}{2},$$

where $\tilde{A}_i = A_i \cap \partial B_1$. Extracting if necessary further subsequences, it is possible to sort the terms of $(R_n)$ (the sequence given by the assumption) and $(R'_n)$ in such a way that $R_n \leq R'_n \leq R_{n+1}$ for every $n$. Note that, at least for $n$ sufficiently large,

$$\int_{\partial B_1} u_{i,R_n}^2 < \frac{\bar{C}}{8} \quad \text{and} \quad \int_{\partial B_1} u_{i,R'_n}^2 > \frac{3\bar{C}}{8},$$

so that thanks to the mean value theorem for every $n$ there exists $R''_n \in (R_n, R'_n)$ such that

$$\int_{\partial B_1} u_{i,R''_n}^2 = \frac{\bar{C}}{4}.$$

Clearly, up to a subsequence $\mathbf{u}_{R''_n} \to \mathbf{w}'' \in \mathcal{BD}_{\mathbf{u}}$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$, and

$$\int_{\partial B_1} (w''_i)^2 = \lim_{n\to\infty} \int_{\partial B_1} u_{i,R''_n}^2 = \frac{\bar{C}}{4}.$$

This contradicts what we proved in the first step.

In light of Lemma 3.2, up to relabelling there exists $1 \le h \le k$ such that if $i = 1, \dots, h$, then $u_{i,R_n} \not\to 0$ as $n \to +\infty$ for every $R_n \to +\infty$, while if $i = h+1, \dots, k$, then for the entire blow-down family $u_{i,R} \to 0$ as $R \to +\infty$ (in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$). For every $\varepsilon, R > 0$, we introduce

$$D_{\varepsilon,R} = \left\{ x \in \partial B_1 : \sum_{i=1}^h u_{i,R}^2(x) > \varepsilon \right\}, \tag{21}$$

and its complement $D^c_{\varepsilon,R}$.

**Lemma 3.3.** *It results that*

$$\lim_{\varepsilon\to 0} \left( \lim_{R\to+\infty} \mathcal{H}^{N-1}(D^c_{\varepsilon,R}) \right) = 0.$$

*Proof.* By contradiction, there exists $\bar{\delta} > 0$ and a sequence $\varepsilon_n \to 0$ such that

$$\limsup_{R\to+\infty} \mathcal{H}^{N-1}(D^c_{\varepsilon_n,R}) \ge \bar{\delta} \qquad \text{for every } n.$$

Thus, we can find a sequence of real numbers $R_n > n$ such that

$$\mathcal{H}^{N-1}(D^c_{\varepsilon_n,R_n}) > \frac{\bar{\delta}}{2} \qquad \text{for every } n. \tag{22}$$

Up to a subsequence $\mathbf{u}_{R_n} \to \mathbf{u}_\infty$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, as $n \to \infty$, and by Theorem 1.6 we know that $\mathcal{H}^{N-1}(\{\mathbf{u}_\infty = \mathbf{0}\} \cap \partial B_1) = 0$. Since by Lemma 3.2 we have also $\sum_{i=h+1}^k u_{i,\infty}^2 = 0$ in $\partial B_1$, it is necessary that

$$\mathcal{H}^{N-1}\left( \left\{ x \in \partial B_1 : \sum_{i=1}^h u_{i,\infty}^2(x) = 0 \right\} \right) = 0. \tag{23}$$

We show that this leads to a contradiction with the estimate (22). For any $\rho > 0$, let

$$D_{\rho,\infty} := \left\{ x \in \partial B_1 : \sum_{i=1}^h u_{i,\infty}^2(x) > \rho \right\},$$

By (23) there exists $\bar{\rho} > 0$ sufficiently small such that $\mathcal{H}^{N-1}(\partial B_1 \setminus \overline{D_{\bar{\rho},\infty}}) \le \bar{\delta}/2$; by uniform convergence there exists $\bar{n}$ such that

$$\inf_{x\in D_{\bar{\rho},\infty}} \sum_{i=1}^h u_{i,R_n}^2(x) \ge \frac{\bar{\rho}}{2} \quad \text{and} \quad \varepsilon_n < \frac{\bar{\rho}}{2}$$

for every $n \ge \bar{n}$. Therefore

$$\mathcal{H}^{N-1}(D^c_{\varepsilon_n,R_n}) \le \mathcal{H}^{N-1}\left( \left\{ x \in \partial B_1 : \sum_{i=1}^h u_{i,R_n}^2(x) < \frac{\bar{\rho}}{2} \right\} \right) \le \mathcal{H}^{N-1}(\partial B_1 \setminus \overline{D_{\bar{\rho},\infty}}) \le \frac{\bar{\delta}}{2}$$

whenever $n > \bar{n}$, which is in contradiction with the (22).

Now the core of the proof of Proposition 1.8 begins. Let us assume by contradiction that there exists $\bar{i} = h+1, \dots, k$ such that, although $u_{\bar{i},R} \to 0$ as $R \to +\infty$, it results $u_{\bar{i}} \not\equiv 0$. Without loss of generality, we suppose that $\bar{i} = h+1$. Clearly,

$$\begin{cases} -\Delta u_{h+1} \leq -u_{h+1} \sum_{j=1}^{h} u_j^2 & \text{in } \mathbb{R}^N \\ u_{h+1} > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{24}$$

Let

$$f(r) := \begin{cases} \frac{2-N}{2} r^2 + \frac{N}{2} & \text{if } 0 < r \leq 1 \\ r^{2-N} & \text{if } r > 1. \end{cases} \tag{25}$$

Note that $f \in \mathcal{C}^1((0,+\infty))$ and $\Delta f(|x|) \leq 0$ a.e. in $\mathbb{R}^N$. For $\beta > 0$, let also

$$\Lambda(r) := \frac{r^2 \int_{\partial B_r} |\nabla_\theta u_{h+1}|^2 + u_{h+1}^2 \sum_{j \neq i} u_j^2}{\int_{\partial B_r} u_{h+1}^2}$$

$$I_\beta(r) := \frac{1}{r^\beta} J(r) := \frac{1}{r_\beta} \int_{B_r} f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^{h} u_j^2 \right).$$

**Lemma 3.4.** *For every $r > 1$, it holds*

$$J(r) \leq \frac{r}{2\gamma(\Lambda(r))} \int_{\partial B_r} f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^{h} u_j^2 \right),$$

*where we recall the definition of $\gamma$, see equation* (11).

We remind the reader to the proof of Lemma 5.1, which contains that of Lemma 3.4 as particular case.

In the next lemma we show that the function $I_\beta$ is non-decreasing for sufficiently large radii.

**Lemma 3.5.** *Let $\beta > 2d$. There exists $r_\beta \gg 1$ sufficiently large such that the function $r \mapsto I_\beta(r)$ is monotone non-decreasing for $r > r_\beta$.*

*Proof.* In what follows we consider scaled functions of type

$$v_{j,r}(x) := \frac{u_j(rx)}{\left( \frac{1}{r^{N-1}} \int_{\partial B_r} u_j^2 \right)^{\frac{1}{2}}} \qquad \text{for } r > 0.$$

Note that the $L^2$ norm of $v_{j,r}$ on $\partial B_1$ is normalized to 1 for every $r$ and for every $j$, and moreover

$$\int_{\partial B_1} |\nabla_\theta v_{h+1,r}|^2 \leq \Lambda(r), \tag{26}$$

as one can easily check by direct computations. By Lemma 3.4 it results that

$$\frac{I'_\beta(r)}{I_\beta(r)} = -\frac{\beta}{r} + \frac{\int_{\partial B_r} f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^{h} u_j^2 \right)}{\int_{B_r} f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^{h} u_j^2 \right)} \geq -\frac{\beta}{r} + \frac{2\gamma(\Lambda(r))}{r}$$

for every $r > 1$. Hence to complete the proof of the lemma we wish to show that there exists $r_\beta \gg 1$ such that if $r > r_\beta$, then $2\gamma(\Lambda(r)) > \beta$. Let us assume by contradiction that such a value $r_\beta$ does not exist: then we can find a sequence $R_n \to +\infty$ such that

$$2\gamma(\Lambda(R_n)) \leq \beta \qquad \text{for every } n, \tag{27}$$

and recalling the definition of $\gamma$ this implies that $(\Lambda(R_n))_n$ is bounded. By (26), we deduce that the sequence $\{v_{h+1,R_n}\}$ is bounded in $H^1(\partial B_1)$, so that u.t.s. it is convergent to some $v_{h+1} \in H^1(\partial B_1)$ weakly in $H^1(\partial B_1)$, strongly in $L^2(\partial B_1)$, and a.e. in $\partial B_1$; in particular, $\|v_{h+1}\|_{L^2(\partial B_1)} = 1$. Let $\omega_{h+1} := \operatorname{supp} v_{h+1}$. If we can prove that

$$\gamma(\lambda_1(\omega_{h+1})) > \beta \tag{28}$$

(where we recall that $\lambda_1(\omega_{h+1})$ is the first eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary condition on $\omega_{h+1}$), then we easily reach a contradiction: indeed by the monotonicity of $\gamma$, and by the inequalities (26) and (27), we deduce that

$$\begin{aligned}\beta < \gamma(\lambda_1(\omega_{h+1})) \le \gamma\left(\int_{\partial B_1} |\nabla_\theta v_{h+1}|^2\right) &\le \gamma\left(\liminf_{n\to\infty} \int_{\partial B_1} |\nabla_\theta v_{h+1,R_n}|^2\right) \\ &\le \gamma\left(\liminf_{n\to\infty} \Lambda(R_n)\right) = \liminf_{n\to\infty} \gamma\left(\Lambda(R_n)\right) \le \beta,\end{aligned}$$

a contradiction. To prove the (28), we recall that $\lambda(\omega) \to +\infty$ as $\mathcal{H}^{N-1}(\omega) \to 0$: this implies that there exists $\delta > 0$ such that

$$\mathcal{H}^{N-1}(\omega_{h+1}) < \delta \quad \Longrightarrow \quad \gamma(\lambda_1(\omega_{h+1})) > \beta,$$

where we used the coercivity of $\gamma$. Therefore, the desired result follows if we show that

$$H^{N-1}(\omega_{h+1}) < \delta. \tag{29}$$

As a first step, we observe that for every $n$

$$-\Delta v_{h+1,R_n} \le -R_n^2 H(\mathbf{u}, 0, R_n) \sum_{j=1}^h u_{j,R_n}^2 v_{h+1,R_n} \qquad \text{in } \mathbb{R}^N, \tag{30}$$

where we recall that $u_{j,R_n}(x) = u_j(R_n x)/H(\mathbf{u}, 0, R_n)^{1/2}$ (we emphasize the fact that the normalization of $u_{j,R_n}$ is different with respect to that of $v_{h+1,R_n}$). Moreover, having assumed that $u_{h+1} \not\equiv 0$, by the mean value inequality and Proposition 2.3

$$v_{h+1,R_n}(x) = \frac{u_{h+1}(R_n x)}{\left(\frac{1}{r^{N-1}} \int_{\partial B_r} u_{h+1}^2\right)^{1/2}} \le \frac{C(1 + R_n^d |x|^d)}{u_{h+1}^2(0)} \le C(1 + R_n^d |x|^d) \tag{31}$$

for every $x \in \mathbb{R}^N$, where $C > 0$ is independent of $n$.

Now, by Lemma 3.3 there exists $\bar\varepsilon > 0$ such that for every $0 < \varepsilon < \bar\varepsilon$

$$\mathcal{H}^{N-1}\left(D_{2\varepsilon,r}^c\right) < \delta \qquad \text{provided } r > r_\varepsilon, \tag{32}$$

for some $r_\varepsilon > 0$ sufficiently large. For the reader's convenience, we recall that $D_{2\varepsilon,r}$ has been defined in (21). Clearly, there exists $\bar n_\varepsilon$ such that $R_n > r_\varepsilon$ whenever $n > \bar n_\varepsilon$.
In what follows we fix $\varepsilon \in (0, \bar\varepsilon)$, and we consider the blow-down sequence $\{\mathbf{u}_{R_n} : n > \bar n_\varepsilon\}$. By usual arguments it is uniformly bounded in the 1/2-Hölder norm in compact sets, it is convergent to a limiting profile $\mathbf{u}_\infty \in \mathcal{BD}_{\mathbf{u}}$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$, and thanks to Lemma 3.2 we know that $u_{j,\infty} \equiv 0$ in $\mathbb{R}^N$ for $j = h+1, \dots, k$. For any $\rho > 0$, let

$$D_{\rho,\infty} := \left\{x \in \partial B_1 : \sum_{i=1}^h u_{i,\infty}^2(x) > \rho\right\}.$$

By uniform convergence, there exists $\bar n'_\varepsilon$ such that

$$\inf_{x \in D_{\varepsilon,\infty}} \sum_{j=1}^h u_{j,R_n}^2(x) > \frac{\varepsilon}{2} \qquad \text{for every } n > \bar n'_\varepsilon;$$

furthermore, by the uniform 1/2-Hölder regularity of the sequence $\{\mathbf{u}_{R_n}\}$, there exists $\rho_\varepsilon > 0$ independent of $x_0 \in D_{\varepsilon,\infty}$ and of $n > \bar{n}'_\varepsilon$ such that

$$\sum_{j=1}^{h} u_{j,R_n}^2(x) > \frac{\varepsilon}{4} \qquad \text{for every } x \in B_{\rho_\varepsilon}(x_0). \tag{33}$$

Collecting equations (30), (31) and (33), we deduce that for every $x_0 \in D_{\varepsilon,\infty}$ and for every $n > \bar{n}'_\varepsilon$ we have

$$\begin{cases} -\Delta v_{h+1,R_n} \leq -\frac{\varepsilon}{4} R_n^2 H(\mathbf{u}, 0, R_n) v_{h+1,R_n} & \text{in } B_{\rho_\varepsilon}(x_0) \\ v_{h+1,R_n} \geq 0 & \text{in } B_{\rho_\varepsilon}(x_0) \\ v_{h+1,R_n} \leq C(1 + R_n^d) & \text{in } B_{\rho_\varepsilon}(x_0). \end{cases}$$

As a consequence we can apply Lemma 2.7:

$$v_{h+1,R_n}(x_0) \leq C(1 + R_n^d) e^{-C\varepsilon R_n H(\mathbf{u},0,R_n)^{1/2} \rho_\varepsilon} \leq C(1 + R_n^d) e^{-C\varepsilon R_n \rho_\varepsilon}$$

for every $x_0 \in D_{\varepsilon,\infty}$, for every $n > \bar{n}'_\varepsilon$. Passing to the limit as $n \to \infty$, since $\varepsilon$ is fixed we infer that $v_{h+1}(x_0) = 0$ for every $x_0 \in D_{\varepsilon,\infty}$, that is, $\mathrm{supp}(\omega_{h+1}) \subset D^c_{\varepsilon,\infty}$. Thus, to complete the proof it remains to show that the measure of $D^c_{\varepsilon,\infty}$ in $\partial B_1$ is sufficiently small. By uniform convergence there exists $\bar{n}''_\varepsilon$ such that

$$\sup_{x \in D_{\varepsilon,\infty}} \sum_{j=1}^{h} u_{j,R_n}^2(x) \leq \frac{3\varepsilon}{2} \qquad \text{for every } n > \bar{n}''_\varepsilon;$$

in other worlds, $D^c_{\varepsilon,\infty} \subset D^c_{2\varepsilon,R_n}$ for every $n > \bar{n}''_\varepsilon$, and thanks to the estimate (32) we deduce that $\mathcal{H}^{N-1}(D^c_{\varepsilon,\infty}) < \delta$; in particular the (29) holds, proving the thesis. □

We are ready to complete the proof of Proposition 1.8. The basic idea is that the monotonicity formula proved in the previous lemma imposes a minimal growth rate on the function $J$ for $r$ large, while the algebraic growth of $\mathbf{u}$ gives a maximal growth rate, and having chosen $\beta > 2d$ these two estimates are in contradiction. This kind of argument is by now well understood (see for instance the proofs of Proposition 7.1 in [5] and of Proposition 2.6 in [13]), even though it is usually employed on groups of components rather than on a single one.

*Conclusion of the proof of Proposition 1.8.* By Lemma 3.5, it results that

$$\int_{B_r} f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^{h} u_j^2 \right) \geq C r^\beta \tag{34}$$

for every $r > r_\beta$. On the other hand, let us test the inequality (24) against $\eta^2 f(|x|) u_{h+1}$, where $\eta$ is a smooth cut of function such that $\eta \equiv 1$ in $B_r$, $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2r}$, and $|\nabla \eta| \leq C/r$

in $\mathbb{R}^N$. By means of some integrations by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \eta^2 f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^h u_j^2 \right) \\
&\leq - \int_{\mathbb{R}^N} \left[ 2u_{h+1}\eta f(|x|) \nabla u_{h+1} \cdot \nabla \eta + u_{h+1}\eta^2 \nabla u_{h+1} \cdot \nabla f(|x|) \right] \\
\leq \int_{\mathbb{R}^N} \left[ 2f(|x|) u_{h+1}^2 |\nabla \eta|^2 + \frac{1}{2} f(|x|) \eta^2 |\nabla u_{h+1}|^2 - \eta^2 \nabla \left( \frac{u_{h+1}^2}{2} \right) \cdot \nabla f(|x|) \right] \\
&= \int_{\mathbb{R}^N} \left[ 2f(|x|) u_{h+1}^2 |\nabla \eta|^2 + \frac{1}{2} f(|x|) \eta^2 |\nabla u_{h+1}|^2 \right. \\
&\qquad \left. - \nabla \left( \frac{\eta^2 u_{h+1}^2}{2} \right) \cdot \nabla f(|x|) + u_{h+1}^2 \eta \nabla \eta \cdot \nabla f(|x|) \right].
\end{aligned}$$

Since $f(|x|)$ is superharmonic (recall the definition (25)), it results that

$$- \int_{\mathbb{R}^N} \nabla \left( \eta^2 u_{h+1}^2 \right) \cdot \nabla f(|x|) = \int_{\mathbb{R}^N} \eta^2 u_{h+1}^2 \Delta f(|x|) \leq 0,$$

and as a consequence

$$\int_{\mathbb{R}^N} \eta^2 f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^h u_j^2 \right) \leq 2 \int_{\mathbb{R}^N} \left[ 2f(|x|) u_{h+1}^2 |\nabla \eta|^2 + u_{h+1}^2 \eta \nabla \eta \cdot \nabla f(|x|) \right].$$

By the choice of $\eta$ and the definition of $f$ (25), we infer

(35)

$$\begin{aligned}
\int_{B_r} f(|x|) \left( |\nabla u_{h+1}|^2 + u_{h+1}^2 \sum_{j=1}^h u_j^2 \right) &\leq \int_{B_{2r} \setminus B_r} \left[ 2f(|x|) u_{h+1}^2 |\nabla \eta|^2 + u_{h+1}^2 \eta \nabla \eta \cdot \nabla f(|x|) \right] \\
&\leq \frac{C}{r^2} \int_{B_{2r} \setminus B_r} \frac{u_{h+1}^2}{|x|^{N-2}} + \frac{C}{r} \int_{B_{2r} \setminus B_r} \frac{u_{h+1}^2}{|x|^{N-1}} \\
&\leq \frac{C}{r^2} \int_r^{2r} \rho^{2d+1} \, \mathrm{d}\rho + \frac{C}{r} \int_r^{2r} \rho^{2d} \, \mathrm{d}\rho = C r^{2d},
\end{aligned}$$

where we used the fact that $u_{h+1}(x) \leq C(1 + |x|^d)$ for every $x \in \mathbb{R}^N$. Having chosen $\beta > 2d$, a comparison between (34) and (35) gives a contradiction for $r$ sufficiently large. □

## 4. Liouville-type theorems for system (1)

The aim of this section is to prove Theorems 1.9 and 1.10, and Corollary 1.11.

### 4.1. 2-dimensional case.

*Proof of Theorem 1.9.* Let $\mathbf{u}$ be a positive solution of (1) such that $N(\mathbf{u}, 0, +\infty) = d \in (0, +\infty)$, and let us consider the blow-down family $\{\mathbf{u}_R\}$. By Theorem 1.6, up to a subsequence and up to a rotation

$$\mathbf{u}_R \to (\chi_{A_1}, \dots, \chi_{A_k}) \pi^{-1} r^d \sin(d\theta) \qquad \text{as } R \to +\infty$$

uniformly on compact sets and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, $\chi_{A_i} \neq 0$ for every $i$, since otherwise by Proposition 1.8 we would have $u_i \equiv 0$ in $\mathbb{R}^N$, in contradiction with the positivity of the considered solution. Now, any $A_i$ is the union of non-adjacent nodal domains of the function $r^d \sin(d\theta)$; each nodal domain is a cone of angle $\pi/d$, so that we have exactly $2d$

nodal domains. Since for every $i$ the positivity domain $A_i$ contains at least one cone, we deduce that $k \le 2d$. □

4.2. **Higher dimensional case.** W need a monotonicity result for the dependence of $\mathcal{L}_k(\mathbb{S}^{N-1})$ with respect to $k$.

**Lemma 4.1.** *For every $N, k \ge 2$, it results that $\mathcal{L}_{k+1}(\mathbb{S}^{N-1}) > \mathcal{L}_k(\mathbb{S}^{N-1})$.*

*Proof.* Let $\omega = (\omega_1, \dots, \omega_{k+1})$ be an optimal $(k+1)$-partition for $\mathcal{L}_{k+1}(\mathbb{S}^{N-1})$ (the existence of $\omega$ is given by Theorem 3.4 in [11]). Up to a relabelling, we can assume that $\omega_1$ and $\omega_{k+1}$ are adjacent sets of the partition. We consider the connected set $\omega_1' = \mathrm{Int}(\overline{\omega_1 \cup \omega_{k+1}})$, and the $k$-partition $\omega' = (\omega_1', \omega_2, \dots, \omega_k)$ . Since

$$\max\{\lambda_1(\omega_1'), \lambda_1(\omega_2), \dots, \lambda_k(\omega_k)\} = \max_{i=1,\dots,k+1} \lambda_1(\omega_i) = \mathcal{L}_{k+1}(\mathbb{S}^{N-1}),$$

we deduce that $\mathcal{L}_{k+1}(\mathbb{S}^{N-1}) \ge \mathcal{L}_k(\mathbb{S}^{N-1})$. To show that the strict inequality holds, we assume by contradiction that the values are equal. Arguing as in the first part of the proof, from an optimal $(k+1)$-partition $\omega$ for $\mathcal{L}_{k+1}(\mathbb{S}^{N-1})$ we can construct a $k$-partition $\omega'' = (\omega_1'', \omega_2, \dots, \omega_k)$ such that

$$\max\{\lambda_1(\omega_1''), \lambda_1(\omega_2), \dots, \lambda_k(\omega_k)\} = \max_{i=1,\dots,k+1} \lambda_1(\omega_i) = \mathcal{L}_{k+1}(\mathbb{S}^{N-1}) = \mathcal{L}_k(\mathbb{S}^{N-1}),$$

that is $\omega''$ is optimal for $\mathcal{L}_k(\mathbb{S}^{N-1})$. But $\lambda_1(\omega_1'') \neq \lambda_1(\omega_j)$ for every $j \neq 1$, in contradiction with the fact that for any optimal $k$-partition $\bar\omega$ it holds $\lambda_1(\bar\omega_i) = \lambda_1(\bar\omega_j)$ for every $i \neq j$, see Theorem 3.4 in [11]. □

*Proof of Theorem 1.10.* We consider a positive solution $\mathbf{u}$ of (1). Arguing as in the previous proof, u.t.s. the blow-down family is convergent to a limiting profile $r^d \mathbf{g}(\theta)$ uniformly on compact sets and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, see Theorem 1.6. We point out that thanks to Proposition 1.8 and having assumed that $\mathbf{u}$ is positive, $g_i \not\equiv 0$ in $\mathbb{S}^{N-1}$ for every $i$. Now, since $r^d g_i(\theta)$ is harmonic in its positivity domain, we have

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} g_i = d(d+N-2) g_i & \text{in } \{g_i > 0\} \\ g_i = 0 & \text{on } \partial\{g_i > 0\}. \end{cases}$$

Let $A_i := \{g_i > 0\}$, and let $\omega_{i,1}, \dots, \omega_{i,l_i}$ be the nodal domains of $g_i$. We observe that since the functions $g_i$ have disjoint support

$$\omega := (\omega_{1,1}, \dots, \omega_{l_1}, \omega_{2,1}, \dots, \omega_{l_2}, \omega_{k,1}, \dots, \omega_{k,l_k})$$

is a $\tilde k$-partition of $\mathbb{S}^{N-1}$ for some $\tilde k \ge k$, and is such that

$$d(d+N-2) = \max_{i=1,\dots,\tilde k} \lambda_1(\omega_i) \ge \mathcal{L}_{\tilde k}(\mathbb{S}^{N-1}) \ge \mathcal{L}_k(\mathbb{S}^{N-1}),$$

where we used the definition (10) of $\mathcal{L}_k(\mathbb{S}^{N-1})$ and Lemma 4.1. Recalling the definition (11) of $\gamma$, this is equivalent to $d \ge \gamma(\mathcal{L}_k(\mathbb{S}^{N-1}))$. □

4.3. **Proof of Corollary 1.11.** For point $(i)$, as already observed in the introduction (see Remark 3), we can either recall that $\gamma(\mathcal{L}_2(\mathbb{S}^{N-1})) = 1$ for every $N$ and use Theorem 1.10, or apply Corollary 1.14, whose proof is independent of the argument we developed so far. Thus in what follows we focus on the jump in the admissible values of $N(\mathbf{u}, 0, +\infty)$ and on point $(ii)$ of the thesis.

**Lemma 4.2.** *Let $\mathbf{v} \in \mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^N)$ be homogeneous with respect to $0$. Then*

$$\textit{either} \quad \tilde N(\mathbf{v}, 0, 0^+) = 1, \quad \textit{or} \quad \tilde N(\mathbf{v}, 0, 0^+) \ge \frac{3}{2}.$$

For the reader's convenience, we recall the definitions of $\tilde{N}(\mathbf{v},0,r)$ and of and $\mathcal{G}^*_{\rm loc}(\mathbb{R}^N)$, see (9) and Definition 2.

*Proof.* By homogeneity $0 \in \{\mathbf{v}=\mathbf{0}\}$, and hence by Proposition 4.2 in [15] the assertion holds true in dimension $N=2$. Now we show how to exploit the blow-up analysis tin order to lower the dimension. We proceed by induction assuming that the result holds in dimension $N-1$, and proving that then it holds in dimension $N$. Let $\mathbf{v}=r^d\mathbf{g}(\theta)\in\mathcal{G}^*_{\rm loc}(\mathbb{R}^N)$, and let $\Gamma_{\mathbf{v}} := \{\mathbf{v}=\mathbf{0}\}\cap\partial B_1$.
Case 1). Assume firstly that every $x_0\in\Gamma_{\mathbf{v}}$ is a point of multiplicity 2 (see Definition 3). If $\sigma$ denotes a connected component of $\Gamma_{\mathbf{v}}$ containing $x_0$, there exist two indexes $i\neq j$ such that $\gamma\subset\overline{\omega_i}\cap\overline{\omega_j}$, where $\omega_i$ and $\omega_j$ are connected components of the positivity domains $\{v_i>0\}\cap\partial B_1$ and $\{v_j>0\}\cap\partial B_1$, respectively; here we used the main result in Section 10 of [6] to ensure that $i\neq j$. According to the reflection law in Theorem 1.1 of [15], $v_i-v_j$ is harmonic in $\mathrm{Int}(\overline{\omega_i\cup\omega_j})$. Since any $x_0\in\Gamma_{\mathbf{v}}$ has multiplicity 2, it is possible to iterate this line of reasoning, deducing that there exists a spherical harmonics $\Psi_d$ associated to the eigenvalue $d$ such that

$$\mathbf{v}(r,\theta)=(\chi_{A_1},\dots,\chi_{A_k})\,r^d\Psi_d(\theta),$$

where $(A_1,\dots,A_k)$ is a partition of $\Sigma_{\Psi_d}=\{\Psi_d\neq 0\}$ and $A_i$ is the union of non-adjacent nodal domains of $\Sigma_{\Psi_d}$. We deduce that the Almgren frequency function $\tilde{N}(\mathbf{v},0,r)$ is on one side constant and equal to $d$ by homogeneity (see Proposition 2.5), while, on the other side, it is a positive integer by the harmonicity of $r^d\Psi_d$. So, if it is different from 1, has to be larger than or equal to $2>3/2$.
Case 2). There exists $x_0\in\Gamma_{\mathbf{v}}$ with multiplicity larger than 2. We consider a blow-up in a neighbourhood of $x_0$ by introducing, for any $\rho>0$,

$$v_{i,\rho}(x) := \frac{v_i(x_0+\rho x)}{H(\mathbf{v},x_0,\rho)^{1/2}}.$$

Since $\mathbf{v}\in\mathcal{G}^*_{\rm loc}(\mathbb{R}^N)$, we are in position to apply Theorem 3.3 and Corollary 3.12 in [15]: there exists a sequence $\rho_m\to 0$ such that $\mathbf{v}_m := \mathbf{v}_{\rho_m}\to\bar{\mathbf{v}}$ as $m\to+\infty$ in $\mathcal{C}^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$ for every $0<\alpha<1$ and strongly in $H^1_{\rm loc}(\mathbb{R}^N)$. Furthermore, $\bar{\mathbf{v}}(r,\theta)=r^\sigma\mathbf{g}(\theta)$ belongs to the class $\mathcal{G}^*_{\rm loc}(\mathbb{R}^N)$, see Definition 2. Here $\sigma=\tilde{N}(\mathbf{v},x_0,0^+)$, and necessarily $\sigma>1$ since the multiplicity of $x_0$ is larger than 2 . It is now crucial to observe that, thanks to the homogeneity of $\mathbf{v}$, the function $\bar{\mathbf{v}}$ depends only on $N-1$ variables. To be precise, we claim that

$$\bar{\mathbf{v}}(x+\lambda x_0)=\bar{\mathbf{v}}(x)\qquad\text{for every }\lambda>0\text{ and }x\in\mathbb{R}^N. \tag{36}$$

Since $\mathbf{v}_m\to\bar{\mathbf{v}}$ in $\mathcal{C}^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$, it is sufficient to show that $\mathbf{v}_m(x+\lambda x_0)-\mathbf{v}_m(x)\to 0$ as $m\to\infty$ for any $\lambda>0$ and $x\in\mathbb{R}^N$. Since $\mathbf{v}$ is homogeneous of degree $\sigma$

$$\begin{aligned}\mathbf{v}_m(x+\lambda x_0) &= \frac{\mathbf{v}(x_0+\rho_m(x+\lambda x_0))}{H(\mathbf{v},x_0,\rho_m)^{1/2}} = \frac{\mathbf{v}((1+\lambda\rho_m)x_0+\rho_m x)}{H(\mathbf{v},x_0,\rho_m)^{1/2}}\\ &= \frac{(1+\lambda\rho_m)^\sigma}{H(\mathbf{v},x_0,\rho_m)^{1/2}}\mathbf{v}\left(x_0+\frac{\rho_m}{1+\lambda\rho_m}x\right) = (1+\lambda\rho_m)^\sigma\mathbf{v}_m\left(\frac{x}{1+\lambda\rho_m}\right).\end{aligned}$$

Let $K$ be a compact set containing both $x$ and $x/(1+\lambda\rho_m)$ for $m$ sufficiently large. Then the local $\mathcal{C}^{0,\alpha}$ convergence of $\mathbf{v}_m$ to $\bar{\mathbf{v}}$ implies that

$$\begin{aligned}|\mathbf{v}_m(x+\lambda x_0)-\mathbf{v}_m(x)| &\le \left|(1+\lambda\rho_m)^\sigma\mathbf{v}_m\left(\frac{x}{1+\lambda\rho_m}\right)-\mathbf{v}_m\left(\frac{x}{1+\lambda\rho_m}\right)\right|\\ &+\left|\mathbf{v}_m\left(\frac{x}{1+\lambda\rho_m}\right)-\mathbf{v}_m(x)\right| \le C\,|(1+\lambda\rho_m)^\sigma-1|+C\left|\frac{1}{1+\lambda\rho_m}-1\right|^\alpha|x|^\alpha,\end{aligned}$$

from which the claim (36) follows. Now, up to a rotation we can assume that $x_0 = (0,\dots,0,1)$, so that by the (36) we know that $\bar{\mathbf{v}}$ is independent on $x_N$, and we recall that $\bar{\mathbf{v}} \in \mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^N)$. It is then clear that the restriction $\bar{\mathbf{v}}|_{\mathbb{R}^{N-1}\times\{0\}}$ belongs to $\mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^{N-1})$, so that by inductive assumption and by homogeneity (see Proposition 2.5)

$$\begin{aligned}\sigma = \tilde{N}(\mathbf{v}, x_0, 0^+) &= \lim_{m\to\infty} \tilde{N}(\mathbf{v}, x_0, \rho_m) = \lim_{m\to\infty} \tilde{N}(\mathbf{v}_m, 0, 1)\\ &= \tilde{N}(\bar{\mathbf{v}}, 0, 1) = \tilde{N}(\bar{\mathbf{v}}|_{\mathbb{R}^{N-1}\times\{0\}}, 0, 1) = \tilde{N}(\bar{\mathbf{v}}|_{\mathbb{R}^{N-1}\times\{0\}}, 0, 0^+)\end{aligned}$$

is either 1 or larger than 3/2. Recalling that $\sigma > 1$, we deduce that $\sigma \geq 3/2$. Using again Proposition 2.5 and Lemma 2.6, we conclude that

$$\frac{3}{2} \leq \sigma = \tilde{N}(\mathbf{v}, x_0, 0^+) \leq \tilde{N}(\mathbf{v}, x_0, +\infty) = \tilde{N}(\mathbf{v}, 0, +\infty) = \tilde{N}(\mathbf{v}, 0, 0^+),$$

which completes the proof.

**Remark 5.** In [15] it has been proved that if $\mathbf{v} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$ then either $\tilde{N}(\mathbf{v}, 0, 0^+) = 1$ or $\tilde{N}(\mathbf{v}, 0, 0^+) \geq 1 + \delta_N$ for some $\delta_N > 0$ depending only on the dimension $N$. The extra information which we obtain in Lemma 4.2 is the precise value of $\delta_N$ (which is remarkably independent of $N$) under the extra-homogeneity assumption.

We are now in position to compute the optimal value $\mathcal{L}_3(\mathbb{S}^{N-1})$ in any dimension $N \geq 3$.

*Proof of Theorem 1.12.* We aim at proving that, for every $N \geq 3$, there holds

$$\frac{3}{2}\left(\frac{3}{2} + N - 2\right) = \mathcal{L}_3(\mathbb{S}^{N-1}).$$

By definition it is equivalent to show that $3/2 = \gamma(\mathcal{L}_3(\mathbb{S}^{N-1}))$. The identity is satisfied when $N = 3$ in light of Theorem 1.1 in [12]. We observe that the optimal partition in dimension 3 provides an admissible partition in any dimension, so that $\gamma(\mathcal{L}_3(\mathbb{S}^{N-1})) \leq 3/2$ for every $N$. By Theorem 3.4 in [11] there exists an optimal partition $\omega \in \mathcal{P}_3(\mathbb{S}^{N-1})$ achieving $\mathcal{L}_3(\mathbb{S}^{N-1})$, and such that $\lambda_1(\omega_i) = \lambda_1(\omega_j)$ for every $i \neq j$; moreover, there exists eigenfunctions $\{\varphi_i\}$ corresponding to $\{\lambda_1(\omega_i)\}$ such that

$$-\Delta_{\mathbb{S}^{N-1}}(\varphi_i - \varphi_j) = \mathcal{L}_3(\mathbb{S}^{N-1})(\varphi_i - \varphi_j) \qquad \text{in } \mathrm{Int}(\overline{\omega_{i,k} \cup \omega_{j,h}})$$

where $\omega_{i,k}$ and $\omega_{j,k}$ are adjacent connected components of $\omega_i$ and $\omega_j$, respectively. Let $\mathbf{v} := r^{\gamma(\mathcal{L}_3(\mathbb{S}^{N-1}))}(\varphi_1, \dots, \varphi_k)$. As observed in Section 8 in [15], $\mathbf{v} \in \mathcal{G}_{\mathrm{loc}}(\mathbb{R}^N)$, and by homogeneity we can say even more: $\mathbf{v} \in \mathcal{G}^*_{\mathrm{loc}}(\mathbb{R}^N)$ and is homogeneous with respect to 0. Therefore by Lemma 4.2

$$\text{either} \quad \gamma(\mathcal{L}_3(\mathbb{S}^{N-1})) = \tilde{N}(\mathbf{v}, 0, 0^+) = 1, \quad \text{or} \quad \gamma(\mathcal{L}_3(\mathbb{S}^{N-1})) = \tilde{N}(\mathbf{v}, 0, 0^+) \geq \frac{3}{2}.$$

To obtain the desired result we have to rule out the former alternative. If $N(\mathbf{v}, 0, 0^+) = 1$, then by Lemma 6.1 in [15] the nodal set $\{\mathbf{v} = \mathbf{0}\}$ is a hyper-plane, which implies that only 2 sets $\omega_i$ are not empty. But it is not difficult to see that any partition of type $\omega = (\omega_1, \omega_2, \emptyset)$ cannot be optimal for $\mathcal{L}_3(\mathbb{S}^{N-1})$. Indeed, in such a situation we could replace $\omega$ by $\omega'$ obtained after a splitting of $\omega_2$ in two non-empty sets. This would be another partition achieving $\mathcal{L}_3(\mathbb{S}^{N-1})$, but such that $\lambda_1(\omega'_1) \neq \lambda_1(\omega'_2)$, in contradiction with Theorem 3.4 in [11]. □

*Proof of point (ii) in Corollary 1.11.* If $\mathbf{u}$ has exactly 2 non-trivial components, then $d \in \mathbb{N}$ by Theorem 1.4. If $\mathbf{u}$ has $k \geq 3$ non-trivial components, then by Theorems 1.10 and 1.12 and Lemma 4.1 it is necessary that

$$d \geq \gamma(\mathcal{L}_k(\mathbb{S}^{N-1})) \geq \gamma(\mathcal{L}_3(\mathbb{S}^{N-1}) = 3/2, \tag{37}$$

where we used the monotonicity of $\gamma$ (see (11)). In any case, if $d \not\equiv 1$, then $d \geq 3/2$. Finally, by Lemma 4.1 the inequality (37) is strict whenever $k > 3$, so that if $d = 3/2$, then $k = 3$. □

## 5. Liouville-type theorems for a general class of systems

This section is devoted to the proof of Theorem 1.13. In what follows $\mathbf{u}$ is a subsolution of (13), and $(H1)$-$(H3)$ hold. We use the following notation:

$$\Lambda_i(r) := \frac{r^2 \int_{\partial B_r} |\nabla_\theta u_i|^2 + u_i^2 g_i(x, \mathbf{u})}{\int_{\partial B_r} u_i^2}$$

$$J_i(r) := \int_{B_r} f(|x|) \left(|\nabla u_i|^2 + u_i^2 g_i(x, \mathbf{u})\right).$$

**Lemma 5.1.** *For every $r > 1$ and $i = 1, \dots, k$, it results*

$$J_i(r) \leq \frac{r}{2\gamma(\Lambda_i(r))} \int_{\partial B_r} f(|x|) \left(|\nabla u_i|^2 + u_i^2 g_i(x, \mathbf{u})\right).$$

*Proof.* The proof is essentially contained in the proof of Lemma 7.3 in [5], or in the proof of Lemma 2.5 in [13]. We report the sketch for the sake of completeness.
By testing the $i$-th inequality (13) against $f(|x|)u_i$ in $B_r$ with $r > 1$ and after some computations, we obtain

$$J_i(r) \leq \frac{1}{r^{N-2}} \int_{\partial B_r} u_i \partial_\nu u_i + \frac{N-2}{2r^{N-1}} \int_{\partial B_r} u_i^2,$$

where $\partial_\nu$ denotes the outer normal derivative, as usual. By the Young inequality, it holds

$$\left| \int_{\partial B_r} u_i \partial_\nu u_i \right| \leq \frac{\gamma(\Lambda_i(r))}{2r} \int_{\partial B_r} u_i^2 + \frac{r}{2\gamma(\Lambda_i(r))} \int_{\partial B_r} (\partial_\nu u_i)^2.$$

Hence, using the definition of $\gamma$, we deduce that

$$\begin{aligned} J_i(r) &\leq \frac{1}{2r^{N-1}\gamma(\Lambda_i(r))} \left[ \left(\gamma(\Lambda_i(r))^2 + (N-2)\gamma(\Lambda_i(r))\right) \int_{\partial B_r} u_i^2 + r^2 \int_{\partial B_r} (\partial_\nu u_i)^2 \right] \\ &= \frac{1}{2r^{N-3}\gamma(\Lambda_i(r))} \int_{\partial B_r} |\nabla_\theta u_i|^2 + u_i^2 g_i(x, \mathbf{u}) + (\partial_\nu u_i)^2, \end{aligned}$$

and the thesis follows.

The following Alt-Caffarelli-Friedman monotonicity formula is the natural counterpart of Lemma 7.3 of [5] in the present setting.

**Lemma 5.2.** *Let us assume that for some $1 \leq h \leq k$ it holds $u_i > 0$ in $\mathbb{R}^N$ for every $i = 1, \dots, h$. Let $0 < h' < \beta(h, N)$, and let us consider the function*

$$J(r) := \prod_{i=1}^{h} \frac{1}{r^{h'}} J_i(r).$$

*There exists $r' = r'(h') > 0$ such that $J$ is monotone non-decreasing in $(r', +\infty)$.*

**Remark 6.** The optimal value $\beta(k, N)$ defined in (12) can be characterized as

$$\beta(k,N) = \inf \left\{ \sum_{i=1}^{k} \frac{2}{h} \gamma \left( \int_{\mathbb{S}^{N-1}} |\nabla_\theta u_i|^2 \right) \middle| \begin{array}{l} u_i \in H^1(\mathbb{S}^{N-1}), \int_{\mathbb{S}^{N-1}} u_i^2 = 1, \\ u_i \cdot u_j \equiv 0 \ \forall i \neq j \end{array} \right\}, \tag{38}$$

see Section 2 of [4].

*Proof.* By Lemma 3.4

$$\frac{d}{dr}J(r) = -\frac{h'h}{r} + \sum_{i=1}^{h} \frac{\int_{\partial B_r} f(|x|)\left(|\nabla u_i|^2 + u_i g_i(x,\mathbf{u})\right)}{\int_{B_r} f(|x|)\left(|\nabla u_i|^2 + u_i g_i(x,\mathbf{u})\right)} \geq -\frac{h'h}{r} + \sum_{i=1}^{h}\frac{2\gamma(\Lambda_i(r))}{r}.$$

Therefore, we aim at proving that there exists $r' > 1$ such that

$$\sum_{i=1}^{h} 2\gamma(\Lambda_i(r)) - hh' \geq 0 \qquad \text{for every } r > r'.$$

By contradiction, if this is not true there exists $r_n \to +\infty$ such that the left hand side is smaller than or equal to 0, and by definition of $\gamma$ this implies that $(\Lambda_i(r_n))$ is bounded. Let us define

$$u_i^{(r_n)}(x) := \frac{u_i(r_n x)}{\left(\frac{1}{r_n^{N-1}}\int_{\partial B_{r_n}} u_i^2\right)^{1/2}}.$$

One can easily compute

$$\tag{39} \begin{gathered} \int_{\partial B_1} |\nabla_\theta u_i^{(r_n)}|^2 \leq \Lambda_i(r_n) \\ \int_{\partial B_1} \left(u_i^{(r_n)}\right)^2 g_i\left(r_n x, \mathbf{u}(r_n x)\right) \leq \frac{\Lambda_i(r_n)}{r_n^2}. \end{gathered}$$

By the first one $\mathbf{u}^{(r_n)} \rightharpoonup \tilde{\mathbf{u}}$ weakly in $H^1(\partial B_1)$, strongly in $L^2(\partial B_1)$, and a.e. in $\partial B_1$. We claim that $\tilde{\mathbf{u}}$ is segregated, that is $\tilde{u}_i\tilde{u}_j = 0$ a.e. in $\partial B_1$ for every $i \neq j$. To check this, we note that by subharmonicity (recall $(H1)$) and by the fact that $u_1,\dots,u_h > 0$ in $\mathbb{R}^N$ we have

$$\tag{40} \frac{1}{r_n^{N-1}}\int_{\partial B_{r_n}} u_i^2 \geq u_i^2(0) \geq C_0$$

for every $i = 1,\dots,h$ and for every $n$. Thus by the Fatou lemma

$$\begin{aligned} \int_{\partial B_1} \tilde{u}_i^2 \underline{g}_i\left(C_0^{1/2}\tilde{\mathbf{u}}\right) &\leq \liminf_{n\to+\infty}\int_{\partial B_1}\left(u_i^{(r_n)}\right)^2 \underline{g}_i\left(C_0^{1/2}\mathbf{u}^{(r_n)}\right) \\ &\leq (H1) \leq \liminf_{n\to+\infty}\int_{\partial B_1}\left(u_i^{(r_n)}\right)^2 g_i\left(r_n x, C_0^{1/2}\mathbf{u}^{(r_n)}\right) \\ &\leq ((H3)+(40)) \leq \liminf_{n\to+\infty}\int_{\partial B_1}\left(u_i^{(r_n)}\right)^2 g_i\left(r_n x, \mathbf{u}(r_n x)\right) = (39) = 0, \end{aligned}$$

which in light of assumption $(H2)$ implies $\tilde{u}_j\tilde{u}_i = 0$ a.e. in $\partial B_1$, proving our claim. Now it is not difficult to conclude, by using the absurd assumption, the first estimate in (39), the definition of $\gamma$ and the characterization of $\beta(h,N)$, see the (38):

$$\beta(h,N)h > h'h \geq \liminf_{n\to+\infty}\sum_{i=1}^{h} 2\gamma(\Lambda_i(r_n)) \geq \sum_{i=1}^{h} 2\gamma\left(\int_{\partial B_1}|\nabla_\theta \tilde{u}_i|^2\right) \geq \beta(h,N)h,$$

a contradiction.

*Proof of Theorem 1.13.* Let us assume by contradiction that for some $1 \leq h \leq k$ such that $\beta(h,N) > 2d$ there exists a solution of (1) with at least $h$ positive components, say $u_i > 0$ for $i = 1,\dots,h$. Let $h' \in (2d, \beta(h,N))$. On one side, by Lemma 3.5 it results that

$$\tag{41} \prod_{i=1}^{h}\frac{1}{r^{h'}}J_i(r) \geq C \qquad \forall r \gg 1.$$

On the other side, as in the estimate (35) it results that that for every $i = 1, \dots, h$ and $r > 1$

$$J_i(r) \le C r^{2d}, \tag{42}$$

where we used the growth assumption on $\mathbf{u}$. Comparing (41) and (42), and recalling that $h' > 2d$, we obtain a contradiction for $r$ sufficiently large. □

## 6. Symmetry results

We now pass to the 1-dimensional symmetry of solutions to (1). Thanks to Corollary 1.11 we are in position to extend the main results in [9] and [17, 18] for systems with an arbitrary number of components.

*Proof of Theorem 1.3.* $(i)$ It is a straightforward consequence of point $(i)$ in Corollary 1.11 and of the main results in [17] and [18].
$(ii)$ Only to fix our minds, let $i = 1$ and $j = 2$. We know that $u_1(x', x_N) \to +\infty$ as $x_N \to +\infty$ uniformly in $x' \in \mathbb{R}^{N-1}$. Arguing as in the proof of Corollary 1.2 in [9], we wish to show that as a consequence $u_{l,R}(x) \to 0$ as $R \to +\infty$ in the half-space $\mathbb{R}^{N-1}_+ := \mathbb{R}^{N-1} \times (0, +\infty)$ for every $l \neq 1$. Given $K > 0$, by assumption there exists $M > 0$ such that $u_1 > K$ in $\{x_N > M/2\}$. For an arbitrary $\theta > 1$, if $x \in \{x_N > M,\ |x'| < \theta x_N\}$ the ball $B_x := B_{x_N/100}(x)$ is contained in $\{x_N > M/2,\ |x'| < 2\theta x_N\}$. Consequently, if $x \in \{x_N > M,\ |x'| < \theta x_N\}$ we have

$$u_1(y) \ge K_x := \inf_{z \in B_x} u(z) \ge K \qquad \forall y \in B_x,$$

and since $\mathbf{u}$ has algebraic growth

$$u_l(y) \le C(1 + |y|^d) \le C\left(1 + C(2\theta + 1)^d y_N^d)\right) \le C(1 + x_N^d) =: \delta_x,$$

for every $y \in B_x$, for every $l \neq 1$. Now,

$$\begin{cases} -\Delta u_l \le -K^2 u_l & \text{in } B_x \\ u_l \ge 0 & \text{in } B_x \\ u_l \le \delta_x & \text{in } B_x, \end{cases}$$

so that Lemma 2.7 applies:

$$u_l(x) \le C\delta_x e^{-CKx_N} \le C(1 + x_N^d) e^{-CKx_N} \qquad \forall x \in \{x_N > M,\ |x'| < \theta x_N\},$$

for every $l \neq 1$. Let $x \in \{x_N > 0, |x'| < \theta x_N\}$; there exists $R_x > 0$ such that $Rx \in \{x_N > M,\ |x'| < \theta x_N\}$ for every $R > R_x$. Therefore

$$\lim_{R \to +\infty} u_{l,R}(x) = \lim_{R \to +\infty} \frac{u_l(Rx)}{\sqrt{H(\mathbf{u}, 0, R)}} = 0 \qquad \forall x \in \{x_N > 0, |x'| < \theta x_N\}.$$

As $\theta$ has been arbitrarily chosen, we deduce that $u_{l,R} \to 0$ pointwise in $\mathbb{R}^N_+$ for every $l \neq 1$. On the other hand, by Theorem 1.6, we know that up to a subsequence $\mathbf{u}_R \to \mathbf{u}_\infty$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$, where $\mathbf{u}_\infty$ has the properties described in Theorem 1.6. We infer that $u_{l,\infty} = 0$ in $\mathbb{R}^N_+$ for every $l \neq 1$. Analogously, starting from the fact that $u_2(x', x_N) \to +\infty$ as $x_N \to -\infty$ uniformly in $x' \in \mathbb{R}^{N-1}$, we deduce that $u_{l,\infty} \equiv 0$ in $\mathbb{R}^N_- := \mathbb{R}^{N-1} \times (-\infty, 0)$ for every $l \neq 2$. Since $u_{l,\infty}$ is continuous for every $l$, $u_{l,\infty} \equiv 0$ in $\mathbb{R}^N$ for every $l \neq 1, 2$, and thanks to Proposition 1.8 this implies that $u_l \equiv 0$ in $\mathbb{R}^N$ for any such $l$. Therefore $(u_1, u_2)$ is a solution of the 2-component system (4) such that

$$\lim_{x_N \to \pm\infty} (u_1(x', x_N) - u_2(x', x_N)) = \pm\infty$$

uniformly in $x' \in \mathbb{R}^{N-1}$, and Corollary 1.2 of [9] gives the desired result.

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