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A tool for form finding using a constrained forced density method

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Abstract

A numerical tool is implemented to cope with the design of arcuated structures through funicular analysis. As investigated in the literature, the force density method can be conveniently implemented to cope with the equilibrium of funicular networks, using independent sets of branches in the case of networks with fixed plan geometry. In this contribution, the minimization of the horizontal thrusts of a spatial network with given plan geometry is formulated not only in terms of an independent set of force densities, but also in the vertical coordinates of the restrained nodes. Constraints are enforced on the height of the nodes, to prescribe the design domain, and on the stress regime in each truss. Due to its peculiar form, this problem can be efficiently solved through techniques of sequential convex programming that were originally conceived to handle multi-constrained formulations in structural optimization. Networks that are fully feasible with respect to the local enforcements on the height of the vertices are retrieved in a limited number of iterations, with no need to initialize the procedure with a feasible starting guess. The same algorithm applies to general networks with any type of geometry, restraints, and loads, including self-weight.

Keywords: funicular analysis, form-finding, force density method, structural optimization, mathematical programming

1. The numerical tool

Funicular analysis is widely adopted to cope with the design of arcuated structures, see e.g. [1, 2, 3, 4]. Following this approach, spatial structures such as three-dimensional trusses and shells can be modelled as statically indeterminate networks of vertices and edges of given topology. Boundary supports are prescribed at the restrained nodes of the network; unrestrained ones are in equilibrium with the applied vertical and horizontal loads.

The equilibrium of funicular networks can be conveniently handled through the force density method, i.e. writing the problem in terms of the ratio of force to length in each branch of the network [5]. As investigated in the literature for the case of vertical loads, independent sets of branches can be detected for networks with fixed plan geometry [3]. However, enforcing the nodes to lie within a prescribed range of heights (the design domain) is not a trivial task from a numerical point of view.

To this goal, a multi-constrained minimization problem has been formulated in [6] to enforce bounds for the vertical coordinates of the vertices of the network.

At first, the equations that link dependent and independent branches in the network with fixed plan projection are derived. The horizontal equilibrium of the nodes with coordinates $\boldsymbol{x}_{s0}$ and $\boldsymbol{y}_{s0}$, under general load conditions, reads:

$$\begin{bmatrix} \boldsymbol{C}^T diag(\boldsymbol{C}_s \boldsymbol{x}_{s0}) \\ \boldsymbol{C}^T diag(\boldsymbol{C}_s \boldsymbol{y}_{s0}) \end{bmatrix} \boldsymbol{q} = \begin{bmatrix} \boldsymbol{p}_x \\ \boldsymbol{p}_y \end{bmatrix}. \tag{1}$$

In the above equations, $\boldsymbol{C}_s$ is the connectivity matrix of the network and $\boldsymbol{C}$ is its subset referring to unrestrained nodes; $\boldsymbol{p}_x$ and $\boldsymbol{p}_y$ are the components along the cartesian axes x and y of the point loads applied at the unrestrained nodes; $\boldsymbol{q}$ the vector of the m force densities, where m is the number of branches in the network. Indeed, by applying Gauss-Jordan elimination to Eqn. (1), see also [3,4], the r dependent force densities $\tilde{\boldsymbol{q}}$ may be re-written in terms of the m-r independent ones $\bar{\boldsymbol{q}}$ as:

$$\tilde{\boldsymbol{q}} = \boldsymbol{B}\bar{\boldsymbol{q}} + \boldsymbol{d}, \tag{2}$$

where $\boldsymbol{B}$ and $\boldsymbol{d}$ have known entries.

The vertical equilibrium of the n unrestrained nodes of the network reads:

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$$\boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C}\boldsymbol{z} + \boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C}_f\boldsymbol{z}_f = \boldsymbol{p}_z, \tag{3}$$

where $\boldsymbol{z}$ and $\boldsymbol{z}_f$ gather the vertical coordinates of the unrestrained and restrained nodes, respectively, $\boldsymbol{C}_f$ is the subset of $\boldsymbol{C}$ for the restrained nodes; $\boldsymbol{p}_z$ are the vertical components of the point loads and $\boldsymbol{Q} = diag(\boldsymbol{q})$.

Hence, a multi-constrained optimization problem is formulated in terms of the independent force densities $\overline{\boldsymbol{q}}$ and the vertical coordinates of the restrained nodes $\boldsymbol{z}_f$ as:

$$\begin{cases} \min\limits_{\overline{\boldsymbol{q}},\ \boldsymbol{z}_f} f(\overline{\boldsymbol{q}}) & & (4.1)\\ s.t.\ \ \tilde{\boldsymbol{q}} = \boldsymbol{B}\overline{\boldsymbol{q}} + \boldsymbol{d} & & (4.2)\\ \boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C}\boldsymbol{z} + \boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C}_f\boldsymbol{z}_f = \boldsymbol{p}_z & & (4.3)\\ z_j(\overline{\boldsymbol{q}}, \boldsymbol{z}_f) \geq z_j^{min} & j = 1 \dots n & (4.4)\\ z_j(\overline{\boldsymbol{q}}, \boldsymbol{z}_f) \leq z_j^{max} & j = 1 \dots n & (4.5)\\ \tilde{q}_k \leq 0 & k = 1 \dots r & (4.6)\\ \overline{q}_i \leq 0 & i = 1 \dots m - r & (4.7)\\ z_{fh}^{min} \leq z_{fh} \leq z_{fh}^{max} & h = 1 \dots n_f & (4.8) \end{cases}$$

A norm of the horizontal thrusts is herein adopted as objective function, i.e. $f(\overline{\boldsymbol{q}}) = \sum \sqrt{R_{xh}^2 + R_{yh}^2}$, where R_{xh} and R_{yh} are the component of the reaction along the x and y direction, respectively, at the h-th of the n_f restrained nodes. By using the vertical equilibrium in Eqn. (4.3) and Eqn. (4.2), the vertical coordinates of the unrestrained nodes may be written in terms of the minimization unkowns, i.e. $\overline{\boldsymbol{q}}$ and $\boldsymbol{z}_f$; suitable constraints can be enforced to prescribe the limits of the design domain, see Eqn. (4.4) and (4.5). Side constraints on $\boldsymbol{z}_f$ are used to enforce similar prescriptions on the restrained nodes, see Eqn. (4.8). Constraints on the sign / magnitude of the force density in each branch of the network may be accounted for: Eqns. (4.6) and (4.7) ask for a compression-only network.

Due to its peculiar form, the optimization problem in Eqn. (4) can be efficiently solved through techniques of sequential convex programming that were originally conceived to handle large scale multi-constrained formulations of size optimization for elastic structures. In a stress-constrained minimum weight problem of truss design, the area of the sections is sought such that the volume is minimized, subject to strength limits. In a statically determinate truss, the objective function is linear in the unknowns, whereas the constrained stress may be written in terms of the inverse of the unknowns. In [6] Eqn. (4) is used to find the funicular polygon of an arch acted upon by vertical loads: it is shown that the thrust is linear in the independent force density $\overline{q}$, whereas the constrained vertical coordinates of the unrestrained nodes $\boldsymbol{z}$ are linear in the vertical coordinate of the abutments z_f and in the reciprocal variable $1/\overline{q}$. Methods of sequential convex programming are available that implement approximations of the objective functions and constraints in the direct or the reciprocal variable depending on the sign of the gradient [8]. These gradient-based methods can be conveneiently adopted to handle the minimization problem in Eqn. (4).

Self-weight, i.e. a design-depend load case, can be straightforwardly included in the optimization, taking full advantage of the direct analytical method to compute sensitivities.

2. A numerical simulation

A preliminary assessment of the proposed tool for form-finding is shown, considering a bay whose four corners and the central point are fully restrained. Symmetry conditions are enforced along each external side of the grid, meaning that additional lateral restraints are prescribed in the relevant nodes. Self-weight is considered, along with constraints enforcing the nodes of the network to lie within a prescribe range of vertical coordinates and all the force densities to be negative. Restrained nodes are not coplanar, being the central node forced to lie at a lower height with respect to the external ones.

Figure 1 shows the grid adopted to perform the optimization. The number of branches in the network is *m=1248*. The independent ones are only 52, the red segments in the figure, meaning that the number of unknowns for the optimization procedure is limited to 57, being 5 the restrained nodes where supports are given.

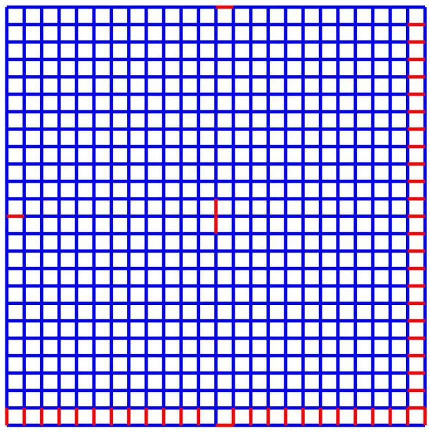

Figure 1. Grid used to investigate optimal networks.

Two simulations are performed, adopting different objective functions. The achieved funicular networks are given in Figure 2. Crosses and circles stand for nodes whose heights match the prescribed upper and lower boundaries of the design domain, respectively. On the left, a network that minimizes the sum of the squared lateral reactions computed for each one of the restrained nodes is shown. On the right, the reactions considered in the objective function do not include the five nodes that are also vertically restrained.

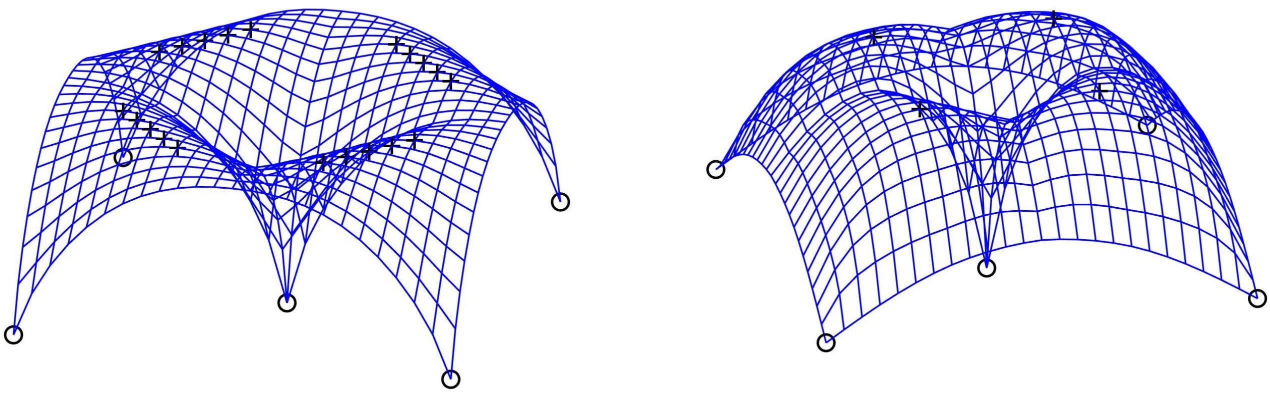

Figure 2. Minimization of the lateral reactions including (left) or excluding (right) the nodes that are also vertically restrained.

Networks that are fully feasible with respect to the local enforcements on the height of the vertices are retrieved in a limited number of iterations, with no need to initialize the procedure with a feasible starting guess. It is remarked that the same algorithm applies to general networks with any type of geometry, loads and restraints. Indeed, applications of assessment are shown in [6] concerning arches, domes and vaults with given shape subject to vertical and horizontal (seismic) loads. The ongoing research on form finding is devoted to testing extensively grids of different topology and to endowing the minimization problem with additional constraints, see e.g. [9].

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